Standard option pricing model: Black Scholes Merton

Ito process

$dX_t = \mu_t dt + \sigma_t dZ_t\ (SDE)$

$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$ constant mean and variance

$dS_t = \mu S_t dt + \sigma S_t dZ_t$, can go negative

How to go from dSt to $dlogSt$?

Ito's Lemma

$$df(zt,t) = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(dz)^2 + \text{ higher order terms can be ignored}$$

$$df(z_t,t) = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}dt \text{ RECALL } (dz)^2 = dt$$

$$df(s_t,t) = \frac{\partial f(s_t,t)}{\partial s_t}ds_t + \frac{\partial f(s_t,t)}{\partial t}dt + \frac{1}{2}\frac{\partial f(s_t,t)}{\partial {s_t}^2}(ds_t)^2$$

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2}$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ$$

$$\frac{d(S_t)^2}{S_t^2} = \mu^2 dt^2 + \sigma^2 dz^2 + 2\mu\sigma dt dZ$$

$$d\log S_t = \mu dt + \sigma dZ - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ$$

$$\log S_T - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma Z_T$$

$$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma z_T} GBM \text{ with drift}$$

Under risk neutral pricing, passing from P to Q (risk neutral world), u=r. Therefore, GBM with drift u=r

Derivation of BS PDE

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

Define derivative price as $f(S_t, t)$

$$df(S_t, t) = \frac{\partial f(S_t, t)}{\partial S_t} dS_t + \frac{\partial f(S_t, t)}{\partial t} dt + \frac{1}{2} \frac{\partial f(S_t, t)}{\partial {S_t}^2} (dS_t)^2$$

can be rewritten as

$$\left[\frac{\partial f(S_t, t)}{\partial S} uS + \frac{\partial f(S_t, t)}{\partial t} + \frac{1}{2} \frac{\partial f(S_t, t)}{\partial S^2} \sigma^2 S\right] dt + \frac{\partial f(S_t, t)}{\partial S} \sigma S dZ_t$$

Replicating portfolio that is long one option and short $\frac{\partial f}{\partial S}$ stocks

$$P = f - \frac{\partial f}{\partial S} S$$

$$dP = df - \frac{\partial f}{\partial S} dS = \left[\frac{\partial f(S, t)}{\partial t} + \frac{1}{2} \frac{\partial f(S, t)}{\partial S^2} \sigma^2 S^2\right] dt$$

Since this portfolio is riskless

$$\frac{dP}{P} = rdt \text{ or } dP = rPdt$$

$$\frac{\partial f(S_t, t)}{\partial t} + \frac{1}{2} \frac{\partial f(S_t, t)}{\partial S^2} \sigma^2 s^2 = r\left(f - \frac{\partial f}{\partial s} S\right)$$

$f(S, t)$ must solve the PDE

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial f}{\partial S^2} \sigma^2 S^2 = rf$$

NOTE: PDE does not depend on u

What is the problem with Black Scholes?

The assumptions:

- Constant volatility (very problematic!)
- Constant interest rates
- Log-normality distributed stock prices
- Constant dividend yield

Is volatility constant? Obviously no. Empirical evidence of skew.

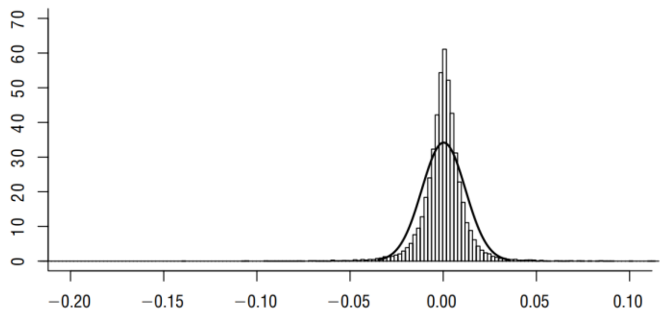

FIGURE 1.2 Frequency distribution of (77 years of) SPX daily log returns compared with the normal distribution. Although the −22.9% return on October 19, 1987, is not directly visible, the x-axis has been extended to the left to accommodate it!

Why people keep using Black Scholes? Because it is very convenient

Black Scholes needs the following input: Stock price, Strike price, Maturity, Volatility, Interest Rate. The only input that is not observable from the market is volatility. Therefore, Black Scholes becomes a tool for traders to back out implied vols from option market prices. Black Scholes is so used among derivatives traders that they trade options in terms of implied vols rather than option prices (US dollars).

In other words, ***"Implied volatility is the wrong number you put in the wrong formula to get the right price."*** **(Make sure to understand fully this quote)**

Nevertheless, we need alternative models to improve option pricing. How can we improve an option pricing model? By making volatility non-constant, i.e. stochastic.

Among the stochastic volatility models we have:

Stochastic volatility models (Heston, SABR)

Stochastic volatility with jumps in stock price process

Stochastic volatility with jumps in stock price and volatility processes

For the scope of this project, we will focus on stochastic volatility models. In particular, the Heston model.

Another advantage of stochastic volatility models is that they can model the skew observed in the market, i.e. option prices (implied vol) is higher for option with lower strikes. This is the so called leverage effect.

Assume first the correlation between volatility and the stock price is negative.
Then stock price goes up, vol down → high Stock prices less likely → Black Scholes Merton overestimates price of OTM calls
If stock price goes down, vol up -> low stock prices more likely → Black Scholes Merton underestimates OTM puts

$$dS_t = rS_t dt + S_t \sigma_t d\widehat{W_t^S}$$
$$d\sigma_t = \alpha(\sigma_t, t)dt + \gamma(\sigma_t, t)d\widehat{W_t^V}$$

Where ρ = Corr($d\widehat{W_t^S}, d\widehat{W_t^V}$)

Imagine $\rho < 0$ Then as Vol increases ($d\sigma_t$ goes up), stock price goes down (dS_t goes down). As stock price goes down, more negative news $d\widehat{W_t^S}$ negative but since $\rho < 0,\ d\widehat{W_t^V}$ positive. Which leads to even more vol and repeat the process.

Let's set $\alpha(\sigma_t, t) = \kappa(\theta - v_t)$ and $\gamma(\sigma_t, t) = \sigma\sqrt{v_t}$ where v_t is the variance

Heston model

$$dS_t = rS_t dt + S_t\sqrt{v_t}d\widehat{W_t^S}$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}d\widehat{W_t^V}$$

ρ = $d\widehat{W_t^S} \cdot d\widehat{W_t^V}$

θ is the long run variance

$\kappa(\theta - v_t)$ is the mean reversion component

σ can be interpreted as volatility of volatility

The Heston model has a semi-closed form solution that can be found in the Gatheral book which involves the use of Fourier transformation. For practical purposes, I will not go through the whole derivation here.

Useful resources for Heston:

Gatheral Chapter 1, 2
Original Heston 1993: [A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options (jstor.org)]
[Calibration of Heston Model with Keras (sorbonne-universite.fr)]

Purpose of the project

The project consists in applying a stochastic volatility model (Heston) and subsequently a stochastic volatility with jumps (SVJ) model to the cryptocurrency market. The first part consists in calibrating the Heston model using options data gathered in the last months.