

Heston's Stochastic Volatility Model Implementation, Calibration and Some Extensions

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1 Introduction

The paper discusses theoretical properties, shows the performance and presents some extensions of Heston's (1993) stochastic volatility model. The model proposed by Heston extends the Black and Scholes (1993) model and includes it as a special case. Heston's setting take into account non-lognormal distribution of the assets returns, leverage effect, important mean-reverting property of volatility and it remains analytically tractable. The Black-Scholes volatility surfaces generated by Heston's model look like empirical implied volatility surfaces. The complication is related to the risk-neutral valuation concept. It is not possible to build a riskless portfolio if we formulate the statement that the volatility of the asset varies stochastically. This is principally because the volatility is not a tradable security.

2 Heston's Stochastic Volatility Model

In this section we specify Heston's stochastic volatility model and provide some details how to compute options prices.

We use the following notations:

$S(t)$	Equity spot price, financial index. ...
$V(t)$	Variance.
C	European call option price.

K	Strike price.
$W_{1,2}$	Standard Brownian movements.
r	Interest rate.
q	Dividend yield.
κ	Mean reversion rate.
θ	Long run variance.
V_0	Initial variance.
σ	Volatility of variance.
ρ	Correlation parameter.
t_0	Current date.
T	Maturity date.

Heston's stochastic volatility model (1993) is specified as followed

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{V(t)} dW_1, \quad (1.1)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2. \quad (1.2)$$

To take into account leverage effect, Wiener stochastic processes W_1, W_2 should be correlated $dW_1 \cdot dW_2 = \rho dt$. The stochastic model (1.2) for the variance is related to the square-root process of Feller (1951) and Cox, Ingersoll and Ross (1985). For the square-root process (1.2) the variance is always positive and if $2\kappa\theta > \sigma^2$ then it cannot reach zero. Note that the deterministic part of process (1.2) is asymptotically stable if $\kappa > 0$. Clearly, that equilibrium point is $V_t = \theta$.

Applying the Ito lemma and standard arbitrage arguments we arrive at Garman's partial differential equation

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{S^2 V}{2} \frac{\partial^2 C}{\partial S^2} + (r - q)S \frac{\partial C}{\partial S} - (r - q)C \\ + [\kappa(\theta - V) - \lambda V] \frac{\partial C}{\partial V} + \frac{\sigma^2 V}{2} \frac{\partial^2 C}{\partial V^2} + \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} = 0, \end{aligned} \quad (1.3)$$

where λ is the market price of volatility risk.

Heston builds the solution of the partial differential equation (1.3) not in the direct way but using the method of characteristic functions. He is looking for the solution in the form corresponding Black and Scholes model

$$C(S_0, K, V_0, t, T) = SP_1 - Ke^{-(r-q)(T-t)}P_2, \quad (1.4)$$

where P_1 is the delta of the European call option and P_2 is the conditional risk neutral probability that the asset price will be greater than K at the maturity. Both probabilities P_1, P_2 also satisfy PDE (1.3) Provided that characteristic functions φ_1, φ_2 are known the terms P_1, P_2 are defined via the inverse Fourier transformation

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-iu \ln K} \varphi_j(S_0, V_0, t, T, u)}{iu} \right] du, \quad j = 1, 2. \quad (1.5)$$

Heston assumes the characteristic functions φ_1, φ_2 having the form

$$\varphi_j(S_0, V_0, \tau; \phi) = \exp\{C_j(\tau; \phi) + D_j(\tau; \phi)V_0 + i\phi S_0\}, \quad \tau = T - t, \quad (1.6)$$

After substitution of φ_1, φ_2 in the Garman equation (1.3) we get the following ordinary differential equations for unknown functions $C_j(\tau; \phi)$ and $D_j(\tau; \phi)$:

$$\frac{dC_j(\tau; \phi)}{d\tau} - \kappa\theta D_j(\tau; \phi) - (r - q)\phi i = 0, \quad (1.7)$$

$$\frac{dD_j(\tau; \phi)}{d\tau} - \frac{\sigma^2 D_j^2(\tau; \phi)}{2} + (b_j - \rho\sigma\phi i)D_j(\tau; \phi) - u_j\phi i + \frac{\phi^2}{2} = 0 \quad (1.8)$$

with zero initial conditions

$$C_j(0, \phi) = D_j(0, \phi) = 0. \quad (1.9)$$

The solution of the system (1.7) (1.9) is given by

$$\begin{aligned} C(\tau, \phi) &= (r - q)\phi i\tau + \frac{\kappa\theta}{\sigma^2} \left\{ (b_j - \rho\sigma\phi i + d)\tau - 2 \ln \left[\frac{1 - ge^{d\tau}}{1 - g} \right] \right\}, \\ D(\tau; \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2} \left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}} \right], \end{aligned} \quad (1.10)$$

where

$$\begin{aligned} g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \quad d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}. \\ u_1 &= 0.5, \quad u_2 = -0.5, \quad a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \\ b_2 &= \kappa + \lambda. \end{aligned} \quad (1.11)$$

3 Realization of Heston's Stochastic Volatility Model

3.1 How to use the model

Implementing such a model consists of different parts that can be divided under a lot of people:

- The first thing is to implement the closed-form solutions for a standard call for the Heston model and the Heston model with jump diffusion, trying to optimize the numerics for speed, such that the calibration can be done as fast as possible.
- The closed-form solution should be verified with a Monte-Carlo (MC) simulation and by directly solving the resulting PDE's using the Finite Difference Method (FDM).
- With the closed-form solutions a suitable set-up should be established to calibrate the models to traded standard calls.
- With the now calibrated model we finally should be able to calculate the price and the greeks of volatility sensitive products such as cliquets using again Monte-Carlo simulation and the Finite Difference Method.

Everything should be done in C++ and be usable as a DLL in Microsoft Excel.

3.2 Implementing the Fourier integral

Inverse Fourier transformation (1.5) is the main point in numerical implementation of the option valuation algorithm provided that characteristic function is known.

The complex numbers can be conveniently implemented by using the complex <> class from the C++ Standard Library. Because the integral should be computed with a high precision for a wide range of parameters (parameters of the stochastic vol process, different strikes and maturities) we decided to use an adaptive quadrature for the first try. Then the algorithm can adjust to changes in the integrand on its



own, saving us from the need to do so. We use an adaptive Simpson and an adaptive Gauss-Lobatto quadrature which both give good results, where the Gauss-Lobatto one uses less computation time for the same precision. But after some experience with the model, we ended up with a special optimized fixed stepwidth Gauss quadrature for faster computation.

3.3 The pitfalls of the complex logarithm

Due to the fact, that the complex logarithm is multiple valued (see Figure 1)

$$\log z = \log |z| + i(\arg(z) + 2\pi n) \quad (1.13)$$

with n being an integer, one usually restricts the logarithm to its principle branch by restricting $\arg(z) \in [-\pi, \pi]$ and setting $n = 0$. This choice is used by the standard C++ log-function and it is necessarily discontinuous at the cut along the negative real axis. At first we had problems with numerical implementation of complex logarithm. Fortunately, we found help at www.wilmott.com in the thread on stochastic volatility models. After implementing the code with a complex logarithm function that maintains continuous over the cut (thanks Roger for the instruction), the results of our three different numerical approaches (Monte-Carlo simulations, Finite Difference method and closed-form solution) agreed nicely and this gave us the confidence to continue our work.

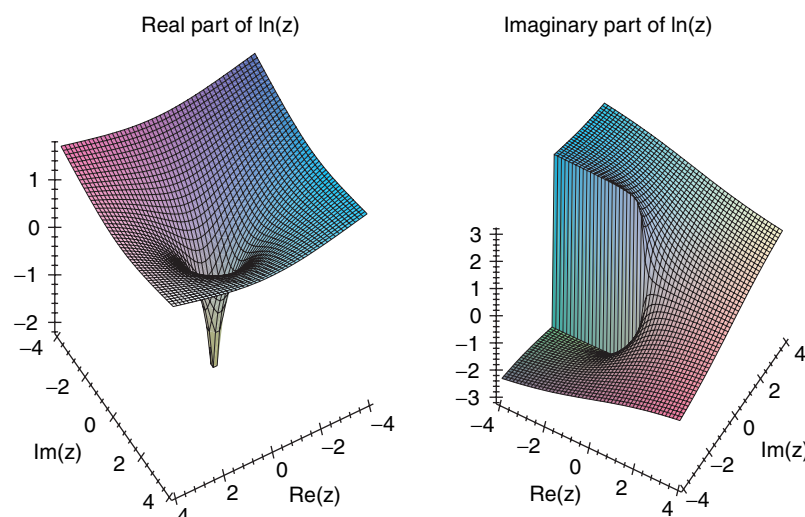


FIGURE 1: Shows the real part and principle branch of imaginary part of complex logarithm

4 Calibration of Heston's Model to Market Data

With the now stable implementation of the closed-form solution we are able to calibrate the models to some traded plain vanilla calls.

4.1 Calibration scheme

We decide to do a least squared error fit in the following way.

Let $\tau_1, \tau_2, \dots, \tau_M$ be some times to maturities with $fw d_1, fw d_2, \dots, fw d_M$ being the corresponding forwards and $dfs_1, dfs_2, \dots, dfs_M$ the corresponding discount factors. Let X_1, X_2, \dots, X_N be a set of strikes and σ_{ij}^{imp} the corresponding market implied volatility. The aim of the calibration is to minimize the least squared error

$$SqErr(\theta) = \sum_{i=1}^N \sum_{j=1}^M w_{ij} [C_{MP}(X_i, \tau_j) - C_{SV}(S(t), X_i, fw d_j, dfs_j, \tau_j, \theta)]^2 + Penalty(\Theta, \Theta_0) \quad (1.14)$$

where $C_{MP}(X_i, \tau_j)$ denotes the market price for a call with strike X_i and maturity τ_j . C_{SV} is the price calculated with the stochastic volatility model which depends on the vector of model parameters $\Theta = (\kappa, \theta, \sigma, \rho, V_0, \lambda)$ for the Heston model. Further α , typically $\alpha = 2n$, $n = 1, 2, \dots$. The penalty function may be e. g. the distance to the initial parameter vector $Penalty(\Theta, \Theta_0) = \|\Theta - \Theta_0\|^2$ and may be used to give the calibration some additional stability.

As it turns out, the suitable choice of the weight factors w_{ij} is crucial for good calibration results.

4.2 Local vs. global optimization

Minimizing the objective function (1.14) is clearly a nonlinear programming (NLP) problem with the nonlinear constrain $2\kappa\theta - \sigma^2 > 0$. This condition ensures that the volatility process cannot reach zero. Unfortunately the objective function is far from being convex and it turned out, that usually there exist many local extrema. As a consequence we decide to try both local and global optimizers:

- **Local (deterministic) algorithms**

Within these types of algorithms one has to choose an initial guess (hopefully a good one) for the parameter vector $\Theta_0 \in \mathbb{R}^d$. The algorithm then determines the optimal direction and the stepsize and is moving downhill on the parameter manifold to the minimum of the objective function. There are a lot of algorithms available both for unconstrained and constrained problems and are usually based on simplex or some kind of gradient method. Most of these algorithms work reasonably fast, but one always has the risk to end up in a local minimum. As a consequence a good initial guess is crucial.

- **Stochastic algorithms**

In contrast to the local optimizers the initial guess is (hopefully) irrelevant in the concept of stochastic optimization. The simulated annealing algorithm chooses the direction and stepsize randomly, it “searches everywhere”. It moves always downhill but may accept an uphill move with a certain probability p_T which depends on the annealing parameter T . This parameter is called “the temperature” for historical reasons. During the optimization process the temperature is gradually reduced. There exist some convergence theorems, which state that the algorithm always ends up in the global minimum if the annealing process is sufficiently slow. There are different variants (e.g. FA, VFSRA, ASA) available which differ from the original simulated annealing (SA) in the annealing scheme, but in general these stochastic algorithms are computationally more burdensome than the local optimizers.

4.3 Results

We tested different local optimizer and surprisingly the built-in Excel solver, which comes with Excel for free, turned out to be very robust and reliable. It is based on the Generalized Reduced Gradient (GRG) method (s. www.solver.com for details) and is our favored optimizer when we have some “good” initial guess for our parameter vector, e.g. if one has to recalibrate the model every day and the volatility surface has not changed much. We were able to calibrate the Heston model to the S&P 500 index with an maximum error of less than 0.15% for ATM calls (s. Figures 2 and 3). The

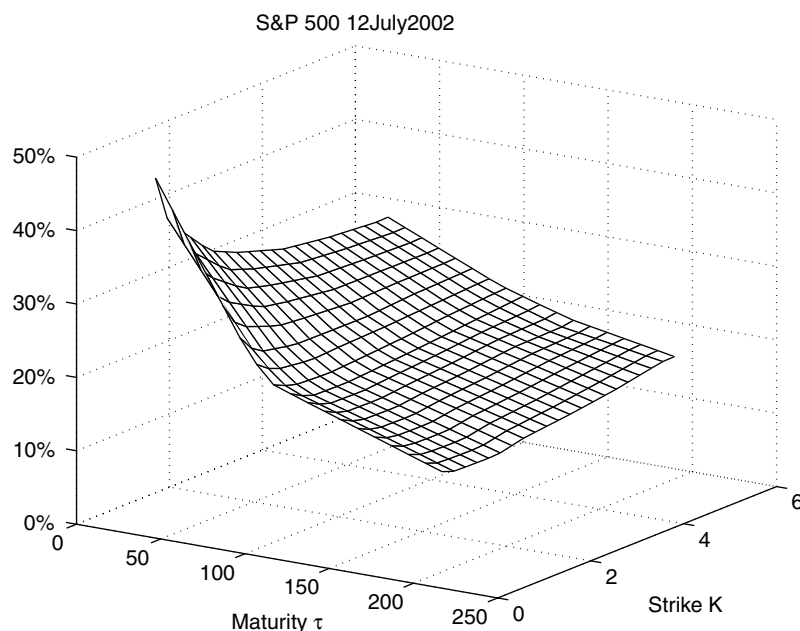


FIGURE 2: Volatility surface for the S&P 500 index

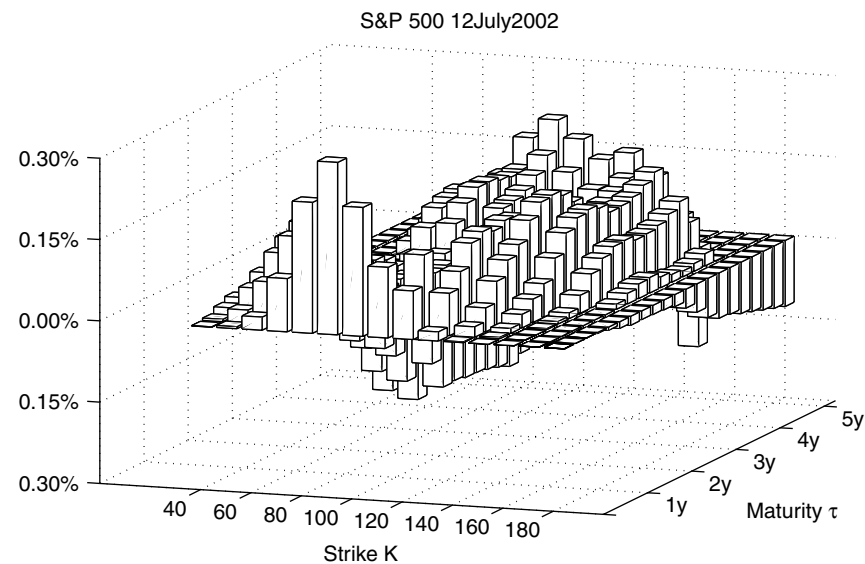


FIGURE 3: Errors after calibration the Heston model to the S&P 500 index

Excel solver may however sometimes end up in a local minimum instead of reaching the global minimum. In such cases or when there is no good initial guess available, we use the adaptive simulated annealing (ASA) algorithm (www.ingberg.com), which allows a faster annealing scheme than the standard SA. It further turned out, that adding jump-diffusion to the Heston model often does not improve the quality of the calibration any more. This may be due to the fact, that the market now frequently shows an inverted yield curve and the model is simply overtaxed with this situation.

5 Stochastic Volatility Model with Time-dependent Parameters

Why are more complex stochastic models required? The answer is simple—because the prices from stochastic engines are not supported by market prices. As a result financial engineers have to recalibrate model parameters every day to new market data. It is not consistent with an accurate description of the dynamics. The next (but not the last) step in the stochastic volatility models history is to models with time-dependent parameters. Since the Riccati differential equation (1.8) is non-linear, the generalization of Heston model for variable parameters is not straightforward.

5.1 Analytical solutions to the Riccati equation

We rewrite the Riccati equation (1.8) in the standard form

$$\frac{dx(t)}{dt} + a(t)x^2(t) + b(t)x(t) + c(t) = 0. \quad (1.15)$$

Recall, that the general solution of a Riccati equation (1.15) cannot be expressed by means of quadratures except in some particular cases.

The simplest case is $a(t) \equiv 0$. In this case we have a linear differential equation with variable parameters that has an analytical solution.

After change of variable $y(t) = -1/x(t)$ we arrive again at Riccati equation

$$\frac{dy(t)}{dt} + c(t)y^2(t) + b(t)y(t) + a(t) = 0. \quad (1.16)$$

Therefore if $c(t) \equiv 0$ in the original Riccati equation then after transformation we obtain again the linear equation with analytical solution.

The general solution of the Riccati equation can be written by means of two quadratures if one particular solution of a Riccati equation is known.

For the Heston stochastic volatility model the ordinary extension for the time-dependent coefficients is long run variance θ . Since this parameter does not appear in the Riccati equation (1.8) the analytical solution for arbitrary $\theta(t)$ can be constructed. For the other Heston models coefficients κ, ρ, σ the generalization to the time-dependent model is not so straightforward. Some analytical solutions are possible. For example if $\kappa(t) = at + b$, or $\kappa(t) = ae^{-\alpha t}$. In this case the Riccati equation (1.8) has closed form solutions expressed by means of hypergeometric function. The drawback—numerical implementation of this analytical solution might be more time consuming than direct numerical integration of equations (1.7), (1.8).

5.2 Asymptotic solution to Riccati equation

As to find the general solution of the Riccati equation with time-variable coefficient is not possible. Natural approach is to apply asymptotic methods. Let for simplicity all Heston model parameters but the correlation coefficient are constant. The approximate solution to the Riccati equation can be found in the form of the asymptotic expansion

$$\begin{aligned} \rho(t) &= \rho_0 + \varepsilon \rho_1(t) + \varepsilon^2 \rho_2(t) + \dots, \\ D(t) &= D_0(t) + \varepsilon D_1(t) + \varepsilon^2 D_2(t) + \dots, \quad \varepsilon \ll 1. \end{aligned} \quad (1.17)$$

In the first approximation we arrive at a linear equation with time-variable coefficients. To obtain the solution of this ODE is straightforward

$$\begin{aligned} D_1(t) &= -\sigma u i \int_0^t \rho_1(\tau) D_0(\tau) \exp \left\{ \int_0^\tau D_0(\xi) d\xi - (-\rho_0 \sigma u i + b_j) \tau \right\} d\tau \\ &\quad \times \exp \left(- \int_0^t D_0(\tau) d\tau + (-\rho_0 \sigma u i + b_j) t \right) \end{aligned} \quad (1.18)$$

The alternative to the above-discussed approach is asymptotic analysis of the systems with slow varying parameters.

5.3 Analytical solution to Riccati with piece-wise constant parameters

The second extension of standard Heston stochastic volatility model to time-dependent coefficients is the setting with piecewise-constant parameters. We can define the solution of the Riccati equation (1.8) with piecewise-constant coefficients by means of adjusting of initial conditions.

At first we need a solution of the equations (1.7), (1.8) with arbitrary initial conditions

$$C_j(0, \phi) = C_j^0, D_j(0, \phi) = D_j^0. \quad (1.19)$$

The solution was build by means of computer-algebra system Maple.

$$\begin{aligned} C_j(\tau, \phi) &= (r - q)\phi\tau + \frac{\kappa\theta}{\sigma^2} \\ &\quad \times \left((b_j - \rho\sigma\phi i + d)\tau - 2 \ln \left(\frac{1 - ge^{\tau d}}{1 - g} \right) \right) \end{aligned} \quad (1.20)$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma\phi i + d - (b_j - \rho\sigma\phi i - d)ge^{\tau d}}{(1 - ge^{\tau d})\sigma^2} \quad (1.21)$$

where

$$g = \frac{b_j - \rho\sigma\phi i + d - D_j^0\sigma^2}{b_j - \rho\sigma\phi i - d - D_j^0\sigma^2}, \quad d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}. \quad (1.22)$$

The solution is close to the Heston one (1.10), (1.11). The time interval to maturity $[t, T]$ is divided into n subintervals $[t, t_1], \dots, [t_i, t_j], \dots, [t_{n-1}, T]$ where $t_k, k = 1, \dots, n-1$ is the time of model parameters jumps. Model parameters are constant during $[t_i, t_j]$ but different for the each subinterval. Further on it is convenient to use the inverse time $\tau = T - t$. The initial condition for the first subinterval from the end $[0, \tau_1]$ where $\tau_k = T - t_{n-k}, k = 1, \dots, n-1$ is zero. Therefore we can use Heston's solution (1.10), (1.11). For the second subinterval $[\tau_1, \tau_2]$ we employ the general solution (1.20)–(1.22) with arbitrary initial conditions (1.19). Provided that functions $C_j(\tau, \phi), D_j(\tau, \phi)$ are continuous in the time of parameters jump τ_1 the initial conditions for the second subinterval can be found from the following condition

$$C_j(0, \phi) = C_j^0 = C_j^H(\tau_1, \phi), D_j(0, \phi) = D_j^0 = D_j^H(\tau_1, \phi) \quad (1.23)$$

where $C_j^H(\tau_1, \phi), D_j^H(\tau_1, \phi)$ are Heston's solutions with zero initial conditions, Solving the above equations relative to C_j^0, D_j^0 we obtain the initial values for the second time interval. The same procedure is repeated at each time moment $\tau_k, k = 2, \dots, n-1$ of the parameters jumps.

Thus the calculation of the option price for the model with piecewise-constant parameters consists of two phase:

Firstly we determine the initial conditions for the each time interval in accordance with formulas (1.23)

Secondly we calculate the functions $C_j(\tau, \phi), D_j(\tau, \phi)$ using the solution (1.20)–(1.22) with initial conditions (1.22)

For the numerical realization this solution is closed to the Heston one. Additionally we have to calculate initial conditions for the second time interval.

6 Numerical Verification of the Model with Time-dependent Parameters

Here we compare options prices calculated according to techniques described in section 5.3 and options prices from a Monte Carlo engine. The algorithm was implemented in C/C++ code. We assume that mean reversion parameter κ time-dependent and all other model parameters are constant. Opening price S_0 of the underlying asset is 1, the maturity of the option considered is 5 years, interest rate is 0, start value for volatility V_0 is 0.1, the long run variance θ is 0.1 volatility of variance σ is 0.2, correlation coefficient ρ is -0.3 , market price of volatility risk λ is 0.

TABLE I. COMPARISON OF ANALYTIC SOLUTION WITH MONTE CARLO SIMULATIONS

$\kappa = \{4, 2, 1\}, T = 5$					
	Monte-Carlo		Analytical solution		
	N = 150000, n_b = 150				
K	Value	StdDev	Value	Abs Err	Rel Err
0.5	0.545298	0.001	0.543017	0.002281	0.004201
0.75	0.387548	0.001048	0.385109	0.002439	0.006333
1	0.275695	0.001021	0.273303	0.002392	0.008752
1.25	0.197629	0.000949	0.195434	0.002195	0.011231
1.5	0.143341	0.00086	0.14121	0.002131	0.015091

The results of the numerical simulations for various strikes K are presented in the table 1.

7 Conclusions

The attractive features of the Heston stochastic volatility model are:

- its volatility updating structure permits analytical solutions to be generated for standard plain vanilla European options and thus the model allows a fast calibration to given market data
- the form of the Heston stochastic process used to model price dynamics allows for non-lognormal probability distributions
- Heston stochastic model takes into account the leverage effect
- The Heston and the HJD model are able to nicely reproduce a wide range of the volatility surfaces implied from option prices in the market

On the other hand there remain some disadvantages and open questions:

- the integrals needed for the computation of the option prices do not always converge nicely
- To perform well across a large time interval of maturities further extensions of the model are necessary (such as time-dependent parameters)
- Heston’s model implicitly takes systematic volatility risk into account by means of a linear specification for the volatility risk premium.
- The standard Heston model usually fails to create a short term skew as strong as the one given by the market, the HJD model is often unable to fit an inverse yield curve.

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