Question

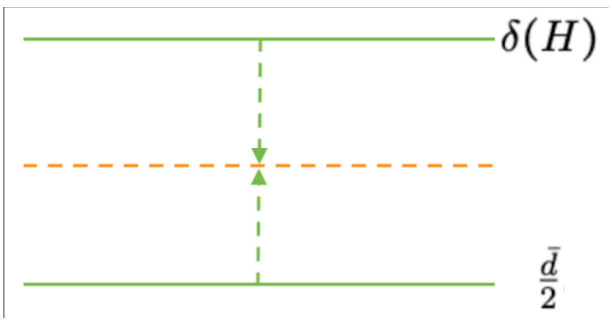
Let G be a graph of average vertex degree $\bar{d}$ (so $\bar{d} = \frac{1}{n}\sum_{v\in V(G)} d(v)$).

a) Determine a necessary and sufficient condition on $d(x)$ so that $G\backslash\{x\}$ has average degree at least $\bar{d}$. Ans: $d(x) \leq \frac{\bar{d}}{2}$

b) Using item a), prove that G contains a subgraph with minimum degree at least $\frac{\bar{d}}{2}$

c) How close is item b) to being tight?

We'll discuss part c). The inequality in b) is $\delta(H) \geq \frac{\bar{d}}{2}$.



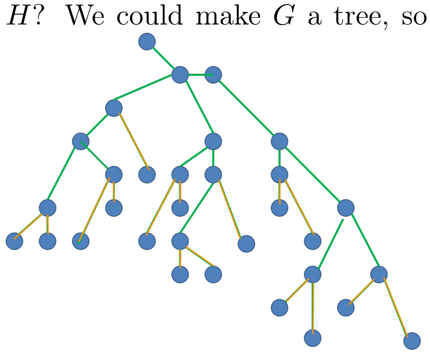
Since $\delta(H)$ depends on G, which we can't adjust. We want to see if there is still room for us to adjuste $\frac{\bar{d}}{2}$ upward. So the question in part c) can be rephrased to

What is the least upper bound of λ in the statement

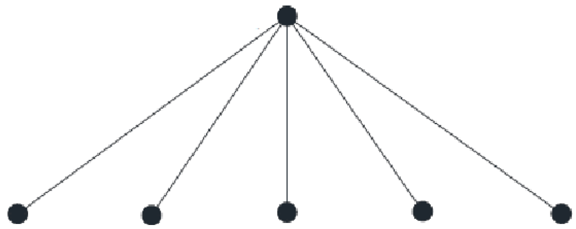
'Graph G has a subgraph H with $\delta(H) \geq \lambda\bar{d}$'?

Since we want to maximize a minimum quantity, we want to consider some extreme scenarios where the minimum quantity $\delta(H)$ is meaningfully minimized[1]. Making $\delta(H) = 0$ is not very meaningful since that is true for any non-empty graph G. So let's consider the scenarios with $\delta(H) = 1$. Note any vertex with degree 1 is a leaf. So we want to bound scenarios where there are leaves in our subgraph H.

Now the question is in which scenarios can we consistently conveniently get some leaves in H? We could make G a tree, so any subgraph of a tree would guarantee to have some leaves.



The other important quantity is $\bar{d}$. For a random tree, it's hard to count the average degree. So we want to restrict our attention even more to focus on trees that are "regular"[2]. Since in this context the important feature of the tree are the leaves, let's consider the scenarios where I only have a root and some leaves.



Now I can easily calculate $\bar{d} = \frac{2n}{n+1}$. And for a subgraph H, $\delta(H)$ is either 1 or 0. Again, we don't use those subgraphs with $\delta(H) = 0$. So we have a scenario here where the subgraph H has $\delta(H) = 1$. So we know we have have

$$\delta(H) = 1 \geq \lambda \frac{2n}{n+1} = \lambda \bar{d}$$

Rearranging the inequality, we get

$$\lambda \leq \frac{1}{2} + \frac{1}{2n}$$

So we can see that the least upper bound for λ is $\frac{1}{2}$, so the bound given in part b) is tight.

Moral of the story:

1. When trying to tightly bound something, consider extreme cases.

2. When something doesn't work, find a work around by simplifying the situation