The Chaotic Pendulum

Luna Greenberg and *Hamza Yasin*

11017146 and *11020964*

School of Physics and Astronomy

University of Manchester

Second year computational project report

April 2024

Abstract

The goal and aims of our project was via the use of numerical methods like the RK4 to provide simulation of the forced,damped single pendulum and to investigate this behavior using various techniques namely Poincare plots and Lyapunov exponents against time to show or offer analysis of categorizing behavior shown by system under the varying forcing term. It can be found principally through the use of the Lyapunov exponents that when the system is made to mimic the simple case (under small angle approximation) the system showns a zero, static Lyapunov exponent. When such assumptions are not made, then through the additional use of bifurcation diagrams and Poincare plots, not only will the sensitivity to initial conditions be found to indicate chaotic behavior but with the rest of the tools the density of periodic orbits and topological mixing conditions can be met.

1 Introduction

A single, forced, and damped pendulum is one where the original simple pendulum - with the assumption of a massless rod - is subject to a damping force and lone variable forcing term, made by permutations of the forcing frequency and a driving force. This gives us the complicated nature of the single 2nd order ordinary differential equation which arises when the small angle approximation is not made. This system is shown to exhibit chaos under certain conditions - 'chaos is a type of motion that lies between the regular deterministic trajectories arising from integrable equations and a state unpredictable stochastic behavior characterized by complete randomness' [1]. For this equation an exact solution does not exist so we must proceed with the numerical methods like: Euler's forward method, which is too computationally intense for too little reward, the RK4 method strikes the perfect balance between accuracy and computational intensity.

2 Theory

Chaos can generally be seen through

- Sensitivity to initial conditions,
- Topological mixing, and
- Dense periodic orbits.

There are multiple maps and plots that can be made to examine these prperties. The most common of these is the Poincaré section. This is a plot of the phase space of the system, with position on the x-axis and velocity on the y-axis. The Poincaré section is a useful tool for examining the periodic orbits of the system, and the general method of using the state space of the system is useful in analyzing the other two properties, topological mixing and initial condition sensitivity. Other tools include the bifurcation diagram, which shows the states of the system as a function of the driving force, as well as the Poincaré plot, which is a useful tool in examining the underlying structure of the system. [2]

3 Methodology

For simulating a system such as the chaotic pendulum, care must be taken to ensure the numerical methods used are very precise, and to use a small enough time step to ensure that the system is accurately represented. Euler's method is simple and can be used to simulate the system, but it is not very accurate when contrasted to the Runge-Kutta method, which uses half steps in the integration to give a better stepwise estimate. Although more computationally expensive, the Runge-Kutta method is a good choice for this project, specifically RK4 [2]. RK4 was used to calculate the steps of angular velocity using the equation of motion for the forced, damped pendulum, and Euler's method was used to calculate the angle from the angular velocity, as since the angular velocities are calculated discretely with RK4 there is no way to take the half step which is required for another RK4 iteration in a way that would provide a benefit to the numerical computation. The equation of motion for this system [3] is given by

$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\sin(\theta) - \frac{b}{M}\frac{d\theta}{dt} + F_d \sin(\Omega_d t) \tag{1}$$

where θ is the angle of the pendulum, t is the time since the initial condition, g is the acceleration due to gravity, R is the length of the pendulum, b is the damping coefficient, M is the mass of the pendulum, F_d is the driving force, and Ω_d is the frequency of the driving force. This is a second order differential equation, requiring a mesh of 2d initial conditions. For our analysis, we take $g = 9.81\text{ms}^{-2}$, $R = 10\text{m}$, $b = 0.1\text{kgs}^{-1}$, and $M = 10\text{kg}$ to all be constant, and vary the driving force, F_d, and the frequency of the driving force, Ω_d. The tools we will use to determine chaotic behavior as outlined in the theory section are Poincaré sections [4], bifurcation diagrams [5], Poincaré plots [6], and the Lyapunov exponent [7], as well as a qualitative analysis of the phase space evolution of the system.

3.1 Poincaré Sections

Poincaré sections for this project were made by analyzing individual initial conditions and plotting the position and velocity of the pendulum until the system reached a periodic orbit. Determining what makes a periodic orbit in a chaotic system with numerical methods is not completely obvious due to floating point precision - the error in the numerical method was tracked for each initial condition, and the periodic orbit was determined to be when the point returned to within the error of the initial condition. The error was determined by the step size of the Runge-Kutta method and Euler method and then propagated forward through the equation of motion.

The Poincaré section will look different depending on whether or not the system is chaotic. If the system is not chaotic, which is ideally recovered in the case of a small driving force, the Poincaré section will look like an ellipse or a circle. If the system is chaotic, the Poincaré section will look like a dense cloud of points, visually not appearing to have any structure. We can use qualitative analysis of the structure of the graph and whether or not such graphs are even calculable in the given time frame to determine the chaotic behavior of the system, since a chaotic system should have dense periodic orbits that don't closely resemble ellipses.

3.2 Bifurcation Diagrams

Bifurcation diagrams were made for this system that plotted angle and angular velocity as a function of the driving force. The system was simulated for a spectrum of driving forces, and the pendulum was plotted for many initial conditions over a long time frame. The diagrams were made using a 2d histogram on a log scale of counts.

A spread bifurcation diagram implies a chaotic system, while a rigid bifurcation diagram implies a non-chaotic system. We predict that the system will become chaotic at a certain driving force, and thus we can use bifurcation diagrams to determine that critical force, which can inform what selection of parameters we use for the other methods.

3.3 Poincaré Plots

Poincaré plots were made plotting the position of the pendulum at a given point on the x axis and the next position of the pendulum on the y axis to get a sense of the underlying structure. Structure in the Poincare plot is indicative of underlying patterns of behavior, though the Poincaré plot is not as useful for determining the presence of chaos as other methods, and would require more in-depth analysis to reach conclusions much more easily reached by the Poincaré sections and bifurcation diagrams.

3.4 Lyapunov Exponent

The Lyapunov exponent is a measure of the sensitivity to initial conditions of a system that arises from the rate at which adjacent phase space points diverge. The Lyapunov exponent applies locally to two adjacent points in phase space, but may not reflect the overall behavior for the system for cases such as this one where the system is modular. Therefore, it may be more useful in our case to analyze the different graphs produced by taking the differences between adjacent points over long periods of time and contrasting chaotic and non-chaotic behavior qualitatively rather than calculating the Lyapunov exponent quantitatively.

3.5 Phase Space Analysis

The phase space is a 2d plot of the angle of the pendulum on the x-axis and the angular velocity of the pendulum on the y-axis. The phase space is a useful tool for analyzing the behavior of the system, and can be used qualitatively to assess sensitivity to initial conditions and topological mixing. A full GUI was created for analyzing the system qualitatively, however the format of the lab report means that only certain snapshots can be shown, but the project can be found at https://www.github.com/solalunara/chaos which can be run with python to show the GUI.

4 Results

4.1 Poincaré Sections

The forces and frequencies analyzed for the Poincaré sections were chosen qualitatively based on the bifurcation diagrams. Below are three groups of Poincaré sections for driving forces of 0.5, 1.0, and 2.0 Newtons, organized into columns of uniform driving frequency.

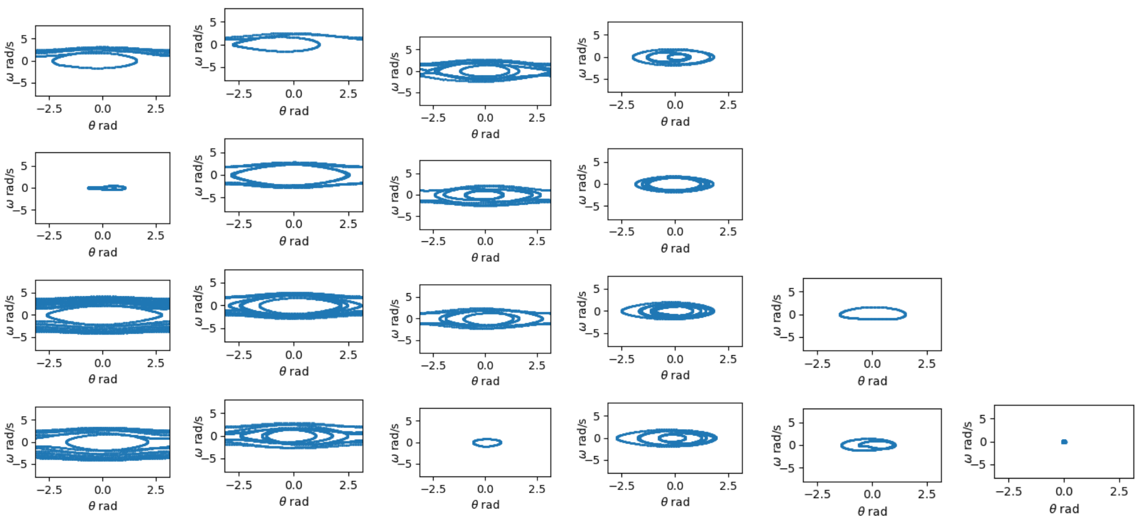

Figure 1: Poincaré sections of the chaotic pendulum for a driving force of a half Newton, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

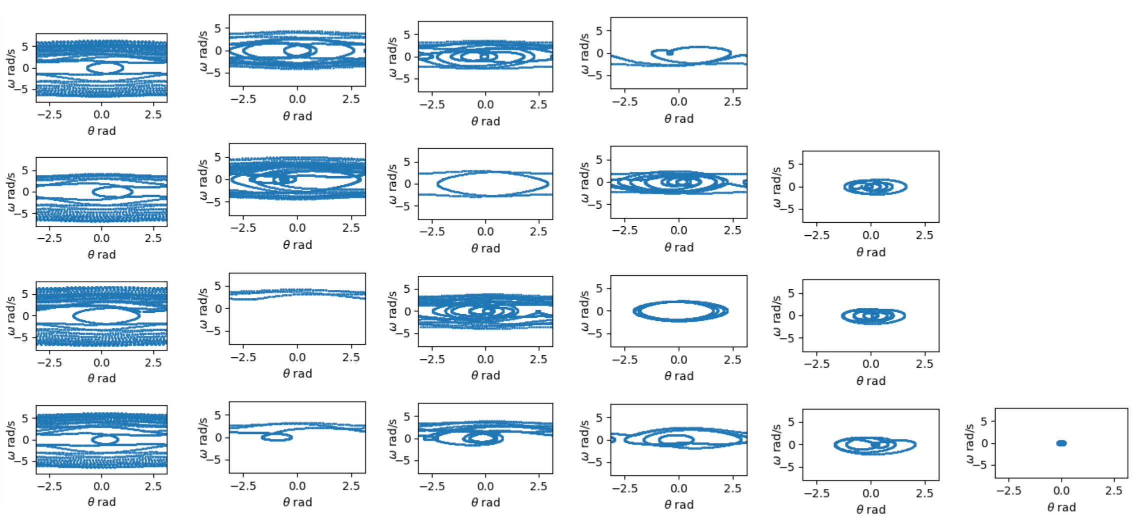

Figure 2: Poincaré sections of the chaotic pendulum for a driving force of one Newton, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

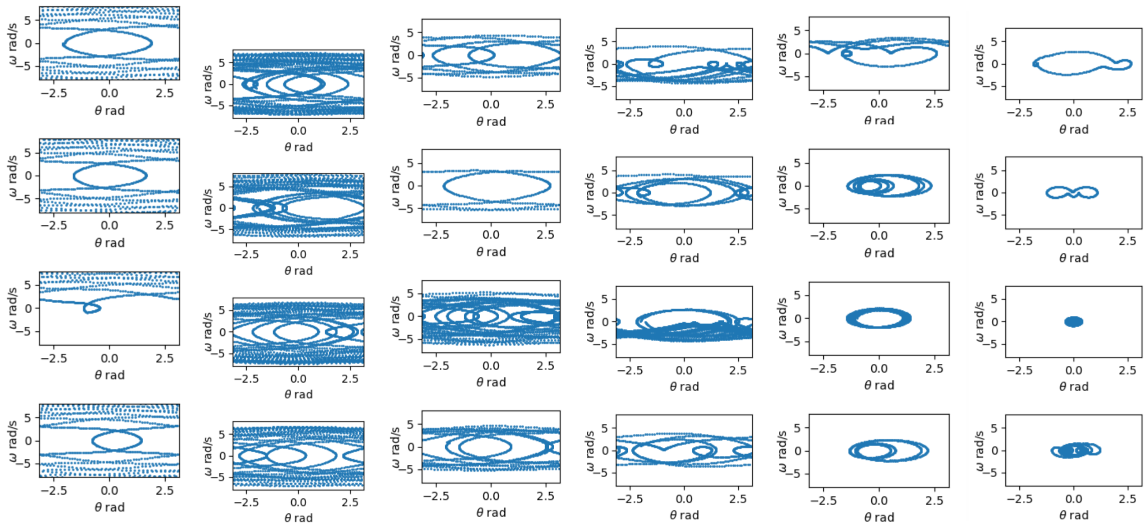

Figure 3: Poincaré sections of the chaotic pendulum for a driving force of two Newtons, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

Two main qualitative observations can be made from the Poincaré sections - the first is that at higher values of the driving frequencies, the iterator has a harder time finding 4 periodic orbits, indicating that the periodic orbits are less dense at those values, and thus the system is less chaotic. The second qualitative observation is that the system is more chaotic at higher driving forces, as can be seen by the density of the points and lack of ellipse structure in the Poincaré sections for higher forces that is well observed at lower force values. These qualitative differences help demonstrate the chaotic behavior of the system at higher force and lower frequency values.

4.2 Bifurcation Diagrams

Below are a selection of bifurcation diagrams from the dynamic program that was created to analyze the chaotic pendulum. Due to the amount of data points involved and the limitations of scatter plots on performance, a log-scaled 2d histogram was used to represent the data. The values of the driving frequency were chosen based on qualitative analysis of the system.

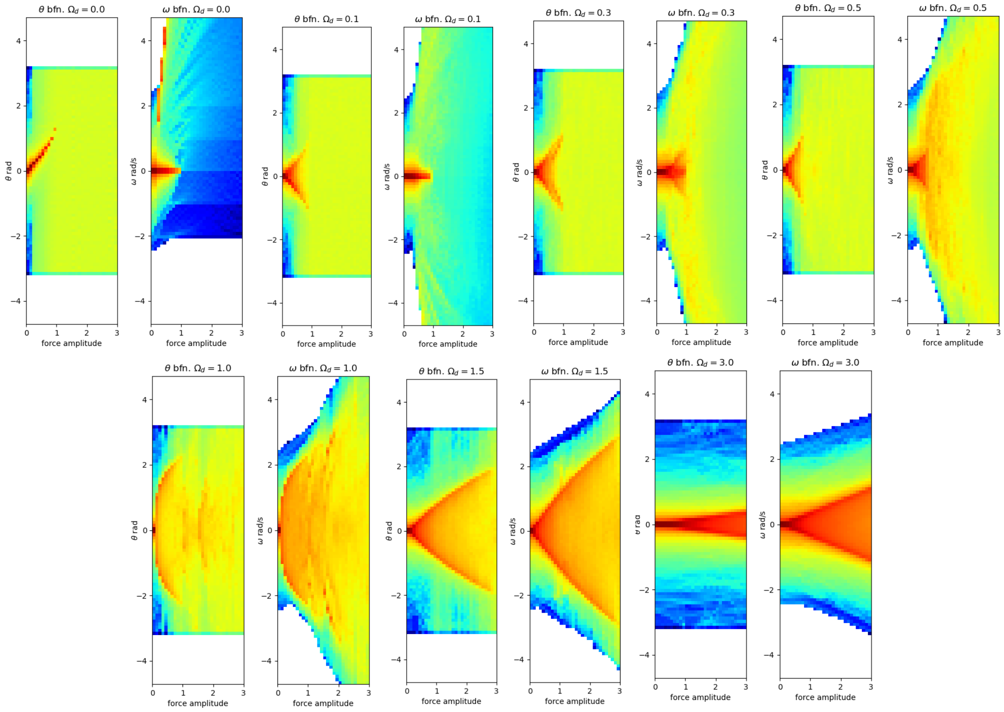

Figure 4: Bifurcation diagrams of the angle and angular velocity of the pendulum as a function of the driving force

Using these bifurcation diagrams, we observed the system to generally be highly chaotic above a driving force of one Newton, and to generally be moderately to lightly chaotic below a driving force of one Newton, becoming non-chaotic closer to zero. This result was used to inform the selection of driving forces for the Poincaré sections. The chaos of the system can be seen through the bifurcation diagrams based on their spread, and as can be seen in every case the diagram becomes more spread for higher force values, with the most rapid spreading for values of the driving frequency between 0.15 and 1.5 Hz. Below this range the frequency is so low that the force becomes either dominant in the motion of the pendulums or negligible depending on its magnitude, and above the range the force is acting too quickly for many chaotic effects to occur, and a dense collection of points can be seen at a smaller spread of angles and angular velocities.

4.3 Poincaré Plots

We constructed Poincaré plots for forces and frequencies of interest determined by the bifurcation diagrams. While in the case of bifircation diagrams it was important to analyze the spread of the data, in this case the structure was the most relevant detail, and as such a 2d histogram was not neccesary, and instead scatter plots were used.

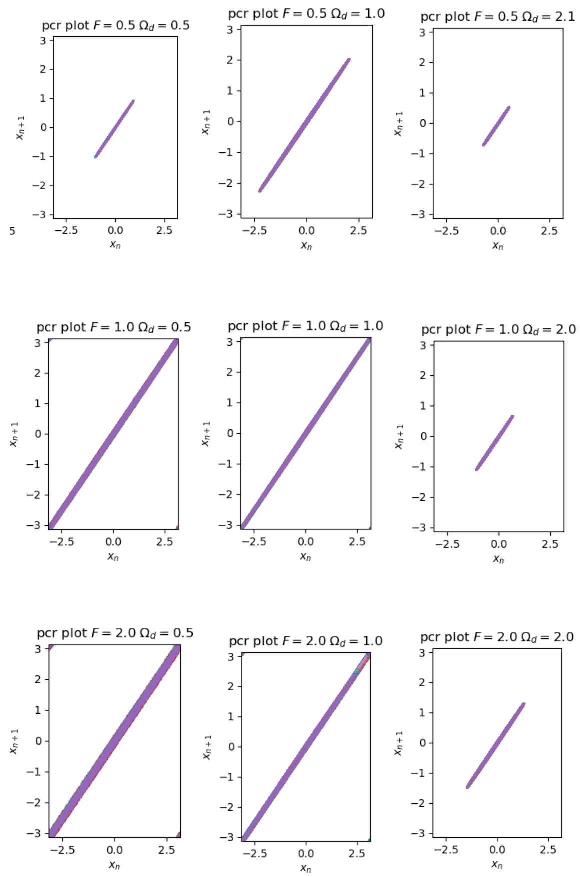

Figure 5: Poincaré plots of the chaotic pendulum for different driving frequencies (0.5, 1.0, and 2.0 left to right) and different forces (0.5, 1.0, and 2.0 top to bottom).

It can be seen that the Poincaré plots are the most spread for higher values of the force and lower values of the frequency, which can be an indicator of chaotic behavior. What's most interesting about the Poincaré plots is that they all follow a similar structure, a line of some width from the bottom left to the top right of the graph. This tells us that for each angle, the system will either move to the left or to the right with some relatively predictable velocity. These graphs aren't very useful for determining the chaotic behavior of the system, but they are useful for determining the underlying structure of the system, and showing that chaos does not necessarily mean that the system is unpredictable.

4.4 Lyapunov Exponent

While the Lyapunov Exponent could be calculated numerically, we thought it more appropriate to use a qualitative analysis of the system and the difference data to assess whether or not the system was chaotic and its sensitivity to initial conditions. The reason for this is that the graph that would be used to determine the Lyapunov exponent contains far more information about the behavior of different initial conditions and about the growth or decay of the differences than a single number would. Due to space concerns on the GUI the force and frequency were not included in the titles of the graphs, but they follow the same layout with the same values as the Poincaré plots

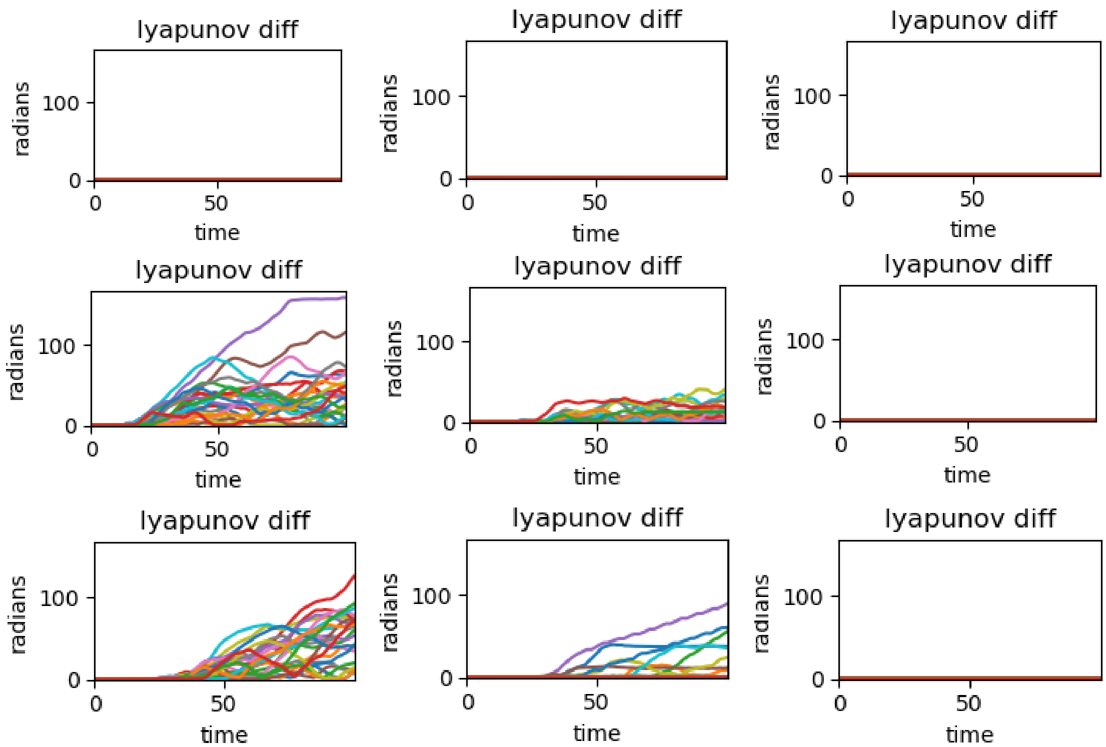

Figure 6: Lyapunov difference plots for different driving frequencies (0.5, 1.0, and 2.0 left to right) and different forces (0.5, 1.0, and 2.0 top to bottom).

The Lyapunov exponent graphs are a much better qualitative indicator of chaos than the Poincaré plots for the same force and frequency magnitudes. The graphs that aren't all within one rotation (2π) of each other show an extreme sensitivity to initial conditions, which tends to happen for the lower frequency and higher force values. The graphs themselves aren't particularly exponential over large scales, which becomes even more relevant when you consider that these are the un-modded difference graphs - if you were to observe the experiment physically where $2\pi = 0$, it becomes impossible to see an exponential difference in initial conditions. The choice to use qualitative analysis with regard to the Lyapunov exponent becomes apparent - the point at which local exponential differences become outdone by global patterns is arbitrary, and quantitative analysis would have little use.

4.5 Phase Space Analysis

Our above analysis has suggested certain systems that are likely chaotic and certain systems that are likely not chaotic, however, to prove that the system is, in fact, chaotic, it is neccesary to establish the condition of topological transitivity. To analyze the phase space of the system to show that it is chaotic for all forces and frequencies, however, is both extremely computationally expensive and not particularly enlightening, as we have established so far a general pattern that the system tends to be chaotic for higher forces and lower frequencies. The phase space evolution is shown below for the system of interest merely as a suppliment to the other analysis, and is shown for a chaotic ($F_d = 1.0$ N, $\Omega_d = 0.5$ Hz) and a non-chaotic ($F_d = 0.5$ N, $\Omega_d = 1.0$ Hz) configuration.

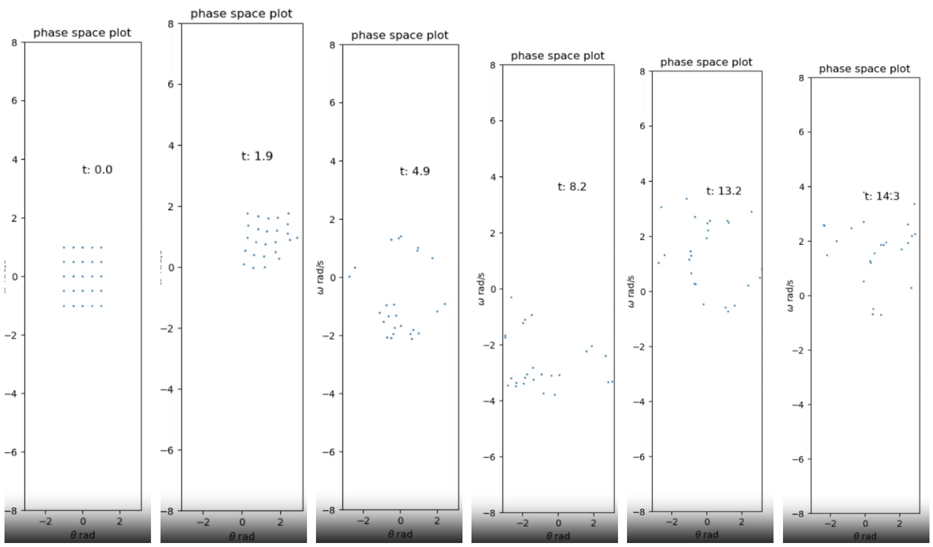

Figure 7: Phase space of the chaotic pendulum in the chaotic configuration evolution through time.

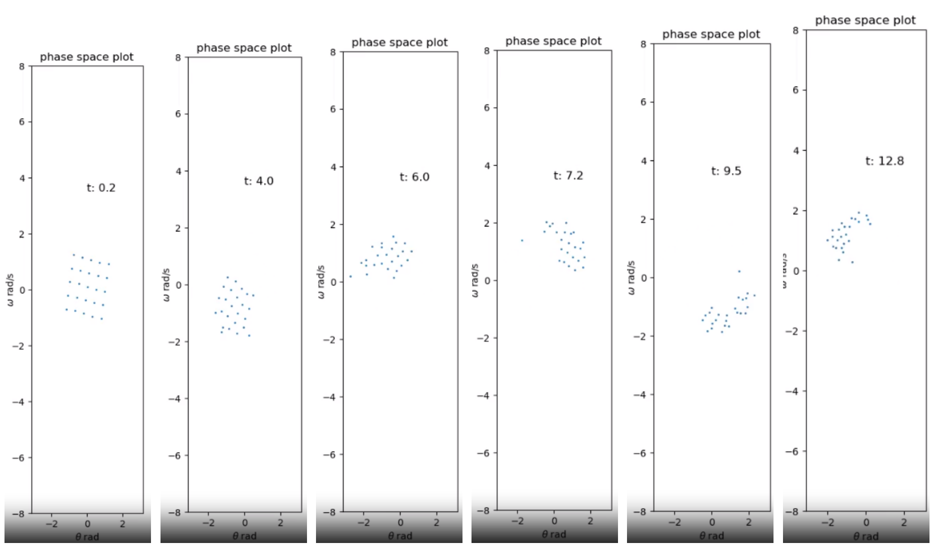

Figure 8: Phase space of the chaotic pendulum in the non-chaotic configuration evolution through time.

For our analysis we had access to a GUI that allowed us to view the phase space of the system in real time, but due to the limitations of the lab report format we must instead attempt to show the evolution of the system through time in a series of snapshots. What can be seen here is that when the system is chaotic, the phase space points tend to spread out and mix amongst each other, retaining little total structure, while when the system is not chaotic the points tend to form a more rigid structure. This therefore corroborates our previous analysis of when the system is chaotic versus not chaotic.

5 Conclusions

To conclude this project,we initially found the equation of motion for our system and got to work trying to find a valid numerical solver.The RK4 was found to be perfect as it captured the essence between computational intensity and minimizing orders of error,whereas the inaccurate Euler method was aptly rejected. After this step, the independent variable namely the forcing term was varied to show its relationship with the nature/behavior of the then system that arises: this can be achieved via taking all possible permutations of both the forcing frequency and the force.Moreover, all other variables associated with the system, i.e. the damping factor, the length of the pendulum and g were all kept constant. A group of premises had to be met to quantifiably state this specific configuration exhibits chaotic behavior, which were as follows: sensitivity to initial conditions, topological mixing, and dense periodic orbits. Leading to our various tools to deal with the previous set of premises we used Poincare sections, which we would look like a circle/ellipse if the system was non chaotic but a dense cloud of points when the system was characterized to be chaotic. Our takeaway from this part of the analysis was that as the driving frequency increased solely the periodic orbits get less and less dense highlighting how the system is exhibiting non chaotic behavior. Our final takeaway for this section was that when the driving force was solely increased was the system was more chaotic concluded by the dissipation of the elliptical structure in favor of the dense fog of points. Bifurcation diagrams were of great use investigating the effect of the driving force and using this information to inform our selection of Poincare sections. The more rigid the diagram was the less chaotic and a spread out diagram indicated chaotic behavior and using this the graphs indicated that above the driving force of 1.0 N the system was highly chaotic, decreasing totally as the driving force equalled zero with this range showing moderately chaotic behavior. Through this same principle a range of 0.15 to 1.5 Hz was selected to be of the most use in terms of the forcing frequency. Unfortunately, poincare plots were of little use so not much about the relationship between driving force and frequency could be deduced. Lyapunov exponents were to deal with a qualitative assessment of a particular configurations sensitivity to initial conditions and it was found that the systems with a lower driving frequency, simultaneously increasing the driving force led to chaotic behavior.The unilateral increase of the driving frequency yields it impossible to see a difference initial conditions as they aren't exponential over large distances. We have to move out of the realm of a particular configuration being likely chaotic and establish as chaotic via the 3rd and final condition of topological mixing/transitivity, which can't be done for all configurations due to being too computationally expensive. We use systems that configure to our well established pattern of a system being chaotic as driving force increases and driving frequency decreases; it can be seen apparently for the system that is chaotic the phase points in time spread out losing most structure. The case of the non chaotic system shows the points forming a more compact structure with time.

References

[1] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics.* Addison Wesley, 2002.

[2] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering.* Westview Press, 2000.

[3] S.-Y. Kim and K. Lee, "Multiple transitions to chaos in a damped parametrically forced pendulum," *Phys. Rev. E*, vol. 53, pp. 1579–1586, Feb 1996.

[4] G. Teschl, *Ordinary Differential Equations and Dynamical Systems.* Graduate Studies in Mathematics, AMER MATHEMATICAL SOCIETY, 2024.

[5] J. D. Crawford, "Introduction to bifurcation theory," *Rev. Mod. Phys.*, vol. 63, pp. 991–1037, Oct 1991.

[6] P. W. Kamen, H. Krum, and A. M. Tonkin, "Poincare plot of heart rate variability allows quantitative display of parasympathetic nervous activity in humans," *Clinical science*, vol. 91, no. 2, pp. 201–208, 1996.

[7] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining lyapunov exponents from a time series," *Physica D: Nonlinear Phenomena*, vol. 16, no. 3, pp. 285–317, 1985.