Applied Mathematics III
Unit 5
Complex Analytic Functions

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Complex Numbers

A complex number z is a symbol of the form $x + yi$ or $x + iy$, where x and y are real numbers and $i^2 = -1$. Let $a + bi$ and $c + di$ be two complex numbers. The four basic arithmetic operations are defined as follows.

1. Equality: $a + bi = c + di$ if and only if $a = c$ and $b = d$.
2. Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
3. Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
4. Division: Let $z = a + bi$ and $w = c + di$ be complex numbers and $z \neq 0$. Then
$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$
and hence
$$\frac{w}{z} = w.\frac{1}{z}.$$

Example

Write $\dfrac{3 + i}{2 - 2i}$ in the form of $a + bi$.

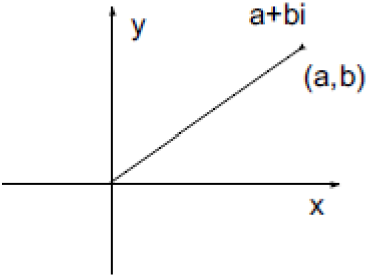
For a complex number $z = a + bi$, the number a is called the **real part** of z and denoted by $Re(z)$ and b is called the **imaginary part** of z and denoted by $Im(z)$.

Remark

Some basic points about complex numbers.

1. The real and imaginary parts of any complex number are real numbers.
2. Any real number a can be considered as a complex number $a + 0i$. Therefore, the set of complex numbers is an extension of the set of real numbers.
3. The set of complex numbers is denoted by $\mathbb{C}$.
4. If x, y and z are complex numbers, then: Addition is commutative $(x + y = y + x)$, multiplication is commutative $(xy = yx)$, associative law for addition $(x + (y + z) = (x + y) + z)$, associative law for multiplication $(x(yz) = (xy)z)$, distributive law $(x(y + z) = xy + xz)$, 0 is identity element for addition $(x + 0 = 0 + x = x)$ and 1 is an identity element for multiplication $(x.1 = 1.x = x)$.

- Any complex number $z = a + bi$ can be represented by the point (a, b) in the cartesian coordinate plane. In this case the coordinate plane is called the **complex plane** and the horizontal and the vertical axes are called the **real axis** and the **imaginary axis** respectively.
- The representation of a complex $z = a + bi$ by the point (a, b) gives us a one to one correspondence between the set of complex numbers $\mathbb{C}$ and the set of ordered pairs of real numbers $\mathbb{R} \times \mathbb{R}$.



Let $z = a + bi$ be a complex number.

1. The magnitude (modules)of z is the real number $mod(z) = |z| = \sqrt{a^2 + b^2}$.
2. The point (a, b) has polar coordinates (r, θ), where $r = |z|$ and $\theta = \arctan(b/a)$. Then θ is called an argument of z, denoted by $arg(z)$.
3. If (r, θ) is a polar coordinate of (a, b), then $z = a + bi = r\cos\theta + ir\sin\theta$ and using Euler's formula we can write $r(\cos\theta + i\sin\theta) = re^{i\theta}$. The expression $z = re^{i\theta}$ is called the polar form of z.

Example

Find the polar form of $z = 1 - i$.

Solution:

Definition

Let $z = a + bi$ be a complex number. The conjugate of z is the complex number $\bar{z} = a - bi$. On the complex plane, the conjugate of a complex number $z = a + bi$ is the reflection of z on the real axis.

Remark

Let z and w be complex numbers. Then

1. $\bar{\bar{z}} = z$ and $\bar{z} = z$ if and only if z is a real number.
2. $\overline{z \pm w} = \bar{z} \pm \bar{w}, \overline{zw} = \bar{z}\bar{w}$ and $\overline{z/w} = \bar{z}/\bar{w}$ if $w \neq 0$.
3. $\overline{|z|} = |\bar{z}|$ and $|z|^2 = z\bar{z}$.
4. $Re(z) = \frac{1}{z}(z + \bar{z})$ and $Im(z) = \frac{1}{2i}(z - \bar{z})$.

Let $z = x + iy$ and $z_0 = x_0 + iy_0$ be complex numbers and r be a $\mathbb{R}^+$.

1. The set $\{z \in \mathbb{C} : |z - z_0 < r\}$ is an open disk of radius r about z_0 and it contains all points enclosed by the circle but does not contain the boundary.
2. The set $\{z \in \mathbb{C} : |z - z_0 \leq r\}$ is a closed disk about z_0 and it contains all points enclosed by the circle and the boundary points on the circle.

Definition

A function w of a complex variable z is a rule that assigns a unique value $w(z)$ to each point z in some set D in the complex plane. If w is a complex function and $z = x + iy$, then we can always write

$$w(z) = u(x, y) + iv(x, y),$$

where u and v are real valued functions of x and y and $u(x, y) = Re(f(z))$ and $v(x, y) = Im(f(z))$. If $z = x + yi$, then $w(z) = u(x, y) + iv(x, y)$. That is, the real and imaginary parts of $w(z)$ are functions of x and y.

Example

Let w be a complex function defined by
$w(z) = z^2 = (x + yi)^2 = (x^2 - y^2) + 2xyi$. Hence
$Re(w(z)) = u(x, y) = x^2 - y^2$ and
$Im(w(z)) = v(x, y) = 2xy$.

Limit

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function. Then clearly f maps $\mathbb{R}^2$ into $\mathbb{R}^2$ and hence all the concept of limit and derivatives that are defined for vector functions of two variables also apply here with the notations modified in terms of the complex numbers notation.

Definition

Let z_0 be an interior point in the domain of definition of a function $f : \mathbb{C} \to \mathbb{C}$. We say that the limit of $f(z)$ as z approaches to z_0 is L and write

$$\lim_{z \to z_0} f(z) = L$$

if to each $\epsilon > 0$ (no matter how small it is), there corresponds a $\delta > 0$ such that $|f(z) - L| < \epsilon$ for all z satisfying $0 < |z - z_0| < \delta$.

In the above definition, $z = x + yi$ and $f(z) = u(x, y) + iv(x, y)$. Moreover $|z - z_0|$ means the modulus of the complex number $z - z_0$ and $|z - z_0| < \delta$ represents an open circle centered z_0.

Remark

Let f and g be complex functions and z_0 and c be complex numbers such that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$.

1. $\lim\limits_{z\to z_0} (f \pm g)(z) = L + M.$
2. $\lim\limits_{z\to z_0} (fg)(z) = LM.$
3. $\lim\limits_{z\to z_0} (f/g)(z) = L/M$ if $M \neq 0$.
4. $\lim\limits_{z\to z_0} (cf)(z) = cL.$

Definition

For a complex function f, if

$$\lim_{z\to z_0} f(z) = f(z_0),$$

then we say that f is continuous at z_0 and a function is continuous in a set if it is continuous at each point of the set.

Derivatives

Definition

Let f be a complex function. The derivative of f at the point z_0, denoted by $f'(z_0)$, is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z)}{\triangle z},$$

if the limit exists and is a complex number. Here as well the limit value should be unique and independent of the way z approaches z_0.

Example

Find the derivative of each of the following functions if it exists.

1. $f(z) = z^2$
2. $f(z) = \bar{z}$.

Solution:

Rules of Differentiation for Complex Functions

Let f and g be complex functions and c be a complex number.

1. Sum(Difference) Rule: $(f \pm g)'(z) = f'(z) \pm g'(z)$.
2. Constant Multiple Rule: $(cf)'(z) = cf'(z)$.
3. Product Rule: $(f.g)'(z) = f'(z)g(z) + f(z)g'(z)$.
4. Quotient Rule: $\left(\dfrac{f}{g}\right)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$.
5. The complex version of the Chain Rule: $(fog)'(z) = f'(g(z))g'(z)$.

Theorem

If f is a differentiable complex function at z_0, then f is continuous at z_0.

Example

Let $f(z) = \dfrac{1}{z}$, then find $f'(z)$.

Solution:

Since differentiability of a complex function (together with analyticity) plays a crucial role in the study of complex variables theory, we need to answer the question that: When is a complex function differentiable?

Definition

Let f be a complex function. Then

1. f is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.
2. f is said to be analytic at a point $z_0 \in D$ if is analytic in some neighborhood of z_0.
3. f is said to be (simply) an analytic function if it is analytic in some domain (open connected subset of C.)

Test for Analyticity

Recall that, if f is a complex function and $z = x + yi$, then we can always write

$$f(z) = u(x,y) + iv(x,y),$$

where u and v are real-valued functions of x and y. Consider a complex function $f(z) = u(x,y) + iv(x,y)$ with $z = x + yi$. If f is analytic in some domain D (and hence differentiable in D), then the partial derivatives exist and for $z_0 = x_0 + y_0 i$ and $z = x_0 + yi$, we have

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{(u(x_0,y) + iv(x_0,y)) - (u(x_0,y_0) + iv(x_0,y_0))}{(x_0 + yi) - (x_0 + y_0 i)} \\
&= \lim_{y\to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{i(y - y_0)} + i \lim_{y\to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{i(y - y_0)} \\
&= \frac{1}{i} u_y + v_y \\
&= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.
\end{aligned}$$

Similarly, if we set $\triangle y = 0$ and $\triangle x \to 0$, that is, if $z = x + y_0 i$ and $z_0 = x_0 + y_0 i$, then we have

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{(u(x, y_0) + iv(x, y_0)) - (u(x_0, y_0) + iv(x_0, y_0))}{(x + y_0 i) - (x_0 + y_0 i)} \\
&= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{(x - x_0)} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{i(x - x_0)} \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\end{aligned}$$

Since f is differentiable at z_0, the two partial derivatives must be equal. That is, we must have

$$\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and this implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1}$$

The above equation (1) is called the **Cauchy-Riemann equations** (and is only the necessary condition for analyticity of f at z_0.) Hence we proved the first part of the following Theorem

Theorem (Necessary and Sufficient Conditions for Analyticity.)

Let $f(z)u(x, y) + iv(x, y)$ be a function that is defined throughout some neighborhood of a point $z_0 = x_0 + iy_0$.

1. *(Necessary Condition) If f is differentiable at z_0, then the Cauchy-Riemann equation is satisfied.*
2. *(Sufficient Condition) If the Cauchy-Riemann equations are satisfied at z_0 and if u and v are continuously differentiable (real valued functions of two variables) in some neighborhood of z_0, then f is analytic at z_0*

If f is differentiable at $z = x + yi$, then due to the Cauchy-Riemann equations, $f'(z)$ can be given in any one of the following four equivalent expressions. $f'(z) = u_x(x, y) + iv_x(x, y)$ or $f'(z) = u_y(x, y) + iv_y(x, y)$ or $f'(z) = u_x(x, y) + iv_y(x, y)$ or $f'(z) = u_y(x, y) + iv_x(x, y)$.

Example

Show that $f(z) = |z|^2$ is differentiable only at $z = 0$ and it is analytic nowhere.

Example

Let $f(z) = z^2 - 8z + 3$. If $z = x + yi$, then show that f is differentiable for all z and find $f'(x + yi)$.

Definition

A real-valued function $u(x, y)$ of two variables satisfy Laplace's equation,

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

and all various first and second order partial derivatives of its component functions w.r.t. x and y are continuous is called a **harmonic function**.

Theorem (Harmonic Functions)

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harnonic in D. That is, they satisfy the Laplace's equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

Since f is analytic, u and v are related by the Cauchy-Riemann equation and such functions are called **conjugate harmonic functions**.

Example

Show that $u = x^2 - y^2 - y$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.
Solution:

Example

Show that $u(x, y) = x^3 - 3xy^2 + 3x + 1$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.
Solution:

Elementary Functions

Exponential Functions

For a complex number $z = x + yi$, the complex exponential function e^z is defined by

$$e^z = e^{x+yi} = e^x(\cos y + i \sin y)$$

and $e^{yi} = \cos y + i \sin y$ is Euler's formula.

We also have $|e^{yi}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y = 1}$ and hence $|e^z| = |e^x e^{yi}| = e^x |e^{yi}| = e^x$, for all $z = x + yi$.

Example

1. $|e^{-2+4i}| = e^{-2}$ and $|e^{3-5i} = e^3|$.
2. If $e^z = 2i$, then find z.
   Solution:

Remark

For a complex number $z = x + yi$, we have $e^z \neq 0$ for all $z \in \mathbb{C}$ since $e^x \neq 0$ for all (finite) x and $\cos y$ and $\sin y$ do not vanish simultaneously for any value of y.

For $z = x + yi$, let $f(z) = e^z$. Then $f(z) = e^x(\cos y + i \sin y)$ and the functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are continuous with continuous first partial derivatives. We also have

$$u_x = e^x \cos y = v_y$$

and

$$u_y = -e^x \sin y = -v_x$$

and hence u and v satisfy the Cauchy- Riemann equations. Therefore, the complex exponential function $f(z) = e^z$ is differentiable for all z and

$$f'(z) = (e^z)' = u_x + iv_x = e^z.$$

Trigonometric and Hyperbolic Functions

From Euler's formula we have that: $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. By adding and subtracting these two equations we can get $\cos\theta$ and $\sin\theta$ respectively.
For any complex number z

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2} \tag{2}$$

All other trigonometric functions can be derived from these two basic definitions. For example

$$\tan z = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \quad \text{and} \quad \sec z = \frac{2}{e^{iz} + e^{-iz}}.$$

Recall that:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

for any $x \in \mathbb{R}$.

Similarly for complex numbers z we define

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2} \tag{3}$$

From (2) and (3) it follows that

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^z}{2} = \cosh z \quad \text{and}$$
$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -i\frac{e^{-z} - e^z}{2} = i \sinh z.$$

Therefore, we have proved the relations

$$\cos(iz) = \cosh z \quad \text{and} \quad \sin(iz) = i \sinh z.$$

Similarly we can show that

$$\cos(iz) = \cos z \quad \text{and} \quad \sin(iz = i \sin z).$$

For a complex number z show that

i) $\sin(-z) = -\sin z$

ii) $\cos(-z) = \cos z$

iii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

iv) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

v) $\cos(z + 2\pi) = \cos z$ and $\sin(z + 2\pi) = \sin z$

vi) $\cos^2 z + \sin^2 z = 1$

vii) $\cosh^2 z - \sinh^2 z = 1.$

Example

1. Let $z = x + yi$ and $f(z) = \sin z$. Then show that $f(z)$ is differentiable and $f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos z$.
   Solution:

2. Let $f(z) = e^{\sin z}$. Then find $f'(z)$.
   Solution:

Polar form and Multi-Valuedness

The Polar form of a complex number z is

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arctan \frac{y}{x} = \arg z$. However, the angle $\theta = \arg z$ for $z \neq 0$ can be determined only to within an arbitrary integer multiple of 2π. The angle θ with $-\pi < \theta < \pi$ is called the Principal argument of z and denoted by $\arg z$. That is,

$$\theta = \arg z + 2k\pi, \quad k \in \mathbb{Z}.$$

The expression z^k is single valued only if the exponent k is an integer. If k is a rational number $\frac{m}{n}$ (in its reduced form), then the map $f(z) = z^k$ is n valued, (since there are exactly n n^{th} roots of a complex number z.)

Example

Let $z = 1 + i$, then find $z^{\frac{1}{3}}$.
Solution:

The Logarithmic Functions

Let $z = re^{i\theta}$ in its polar form and $w = a + bi$. Then $z = re^{i\theta} = e^{a+bi} = e^a e^{bi}$ and we also have $r = |z| = e^a$, which implies that $a = \ln r$. From the equation $re^{i\theta} = e^{a+bi} = e^a e^{bi}$ we get $e^{i(b-\theta)} = 1 = e^{2k\pi i}$ for $k \in \mathbb{Z}$, which implies that $i(b - \theta) = 2k\pi i$ and hence $b = \theta + 2k\pi$ for $k \in \mathbb{Z}$. Therefore, for $z \neq 0$ there are infinitely many numbers $w = \ln r + i(\theta + 2k\pi), k \in \mathbb{Z}$ such that $z = e^w$. Now we are in a position to define the logarithm of a nonzero complex number z as

$$\log(z) = \ln|z| + i(\arg(z) + 2k\pi), k \in \mathbb{Z}$$

which is infinite valued.

Example

Compute $\log(1 + i)$.

Solution: