Wolkite University
Department of Mathematics
Linear Algebra Worksheet 2
For Physics Students

Vectors

1. Let $A = (0, 1, 5)$ and $B = (-\sqrt{14}, 5, 1)$. Find the angle between A and B.
2. Find a non-zero vector orthogonal to $(1, 2, -1)$.
3. Find a unit vector in the direction of $(3, -1, 2, 4)$.
4. Let $u_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$, $u_2 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ and $u_3 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$,
   (a) Show that each u_1, u_2, u_3 is orthogonal to the other two and that each is a unit vector.
   (b) Find the projection of E_1 on each of u_1, u_2, u_3.
   (c) Find the projection of $A = (a_1, a_2, a_3)$ on u_1.
5. In the following cases compute $(A \times B).C$
   (a) $A = (1, 2, 0)$, $B = (-3, 1, 0)$ and $C = (4, 9, -3)$
   (b) $A = (-3, 1, -2)$, $B = (2, 0, 4)$ and $C = (1, 1, 1)$
6. If A, B and C be vectors and $A + B + C = 0$, then show that $A \times B = B \times C = C \times A$.
7. Find parametric equations of lines through
   (a) $(-5, -6, 8)$ and $(1, 3, 7)$
   (b) $(10, 3, 1)$ and $(6, -2, -3)$
8. Find equation of a plane through points $(0, 1, 0), (0, -1, -1)$ and $(1, 2, 1)$.
9. Find a point of intersection of the lines $\{p : p = (1, -5, 2) + t(-1, 1, 0)\}$ and $\{p : p = (3, -3, 1) + t(4, 0, -1)\}$.
10. Find all points of intersection of the line $\{p : p = t(1, -3, 6)\}$ and the plane $\{p : x + 3y + z = 2\}$.
11. Find a line through (x_o, y_o, z_o) and normal to the plane $\{(x, y, z) : ax + by + cz = d\}$.
12. Let l be the line $x = 1 + 2t, \quad y = -1 + 3t, \quad z = -5 + 7t$. Find the two points on l at a distance 3 units from the plane $2(x - 1) + 2(y + 3) - z = 0$.
13. The set of all points equidistant from $(0, 1, 5)$ and $(5, -1, 3)$ is a plane. Find the equation of a plane.

Vector Spaces

1. Let K be the set of all numbers which can be written in the form $a + b\sqrt{2}$, where a and b are rational numbers. Show that K is a field.

2. Show that the set $x + y = 3z$ is the subspace of all (x, y, z) in $\mathbb{R}^3$.

3. Show that the set $x = y$ and $z = 2y$ is the subspace of all (x, y, z) in $\mathbb{R}^3$.

4. If U and W are subspaces of a vector space V, show that $U \cap W$ and $U \cup W$ are subspaces.

5. Identify whether the following vectors are linearly independent or not (on $\mathbb{R}$)
   (a) $(\pi, 0)$ and $(0, 1)$
   (b) $(-1, 1, 0)$ and $(0, 1, 2)$
   (c) $(0, 1, 1), (0, 2, 1)$ and $(1, 5, 3)$

6. Find the coordinates of X with respect to the vectors A, B and C
   (a) $X = (1, 0, 0), \quad A = (1, 1, 1), B = (-1, 1, 0), C = (1, 0, -1)$
   (b) $X = (1, 1, 1), \quad A = (0, 1, -1), B = (1, 1, 0), C = (1, 0, 2)$

7. Find a basis and the dimension of the subspace of $\mathbb{R}$ generated by $\{(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)\}$.

8. Let W be the space generated by the polynomials $x^3 + 3x^2 - x + 4$ and $2x^3 + x^2 - 7x - 7$. Find a basis and the dimension of W.

Linear Transformation

1. Determine whether or not each of the following mappings is linear transformation.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = xy$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x_1, x_2, x_3) = (1 + x_1, x2)$.

   (c) $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by $L(x, y, z) = (z + x, y)$.

2. Let M_2 denote the vector space of 2×2 matrices over $\mathbb{R}$. Let $T : M_2 \to M_2$ be given by $T \begin{bmatrix} a & b \\ d & d \end{bmatrix} = \begin{bmatrix} a+b & c \\ -3a & c+d \end{bmatrix}$. Is T a linear mapping? Justify.

3. Show that the mapping $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(a, b) = |a–b|$ is not a linear transformation.

4. Let U, V, and W be vector spaces over the same field K. If $g : U \to V$ and $f : V \to W$ are linear transformations show that fog is also a linear transformation form U in to W.

5. Find a linear transformation

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1, 2) = (3, 0)$ and $T(2, 1) = (1, 2)$.

   (b) $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that $L(2, -5) = (-1, 2, 3)$ and $L(3, 4) = (0, 1, 5)$.

   (c) $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(3, 1, 1) = (1, 0), L(2, -1, 5) = (0, 1)$ and $L(4, 0, -3) = (-1, 1)$.

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping, such that $L(3, 1) = (1, 2)$ and $L(-1, 0) = (1, 1)$, then compute $L(1, 0)$.

7. For each of the following linear transformation, find a basis and dimension of its image and kernel.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, x + 2y, y)$.

   (b) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(a, b, c) = (a, c)$.

   (c) $L : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $L(x, y, z, t) = (x–y + z + t, x + 2z–t, x + y + 3z - 3t)$

8. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\{(x, y, z) : 4x - 3y + z = 0\}$ is the
   .   i) Kernel of T ii) image of T

9. Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(u, v, w) = (u + v - 2w, u + 2v - w, 2u + v)$. Find the rank and nullity of T.