Math 331: Applied Mathematics III Lecture Notes

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Part I

Ordinary Differential Equations

Chapter 1

Ordinary Differential Equations of the First Order

Part 1 of this material deals with equations that contain one or more derivatives of a function of a single variable and such equations are called ordinary differential equations, which can be used to model a phenomena of interest in the sciences, engineering, economics, ecological studies, and other areas.

In the first section we will see the basic concepts and ideas and in the remaining sections we will consider equations which involve the first derivative of a given independent variable with respect to an independent variable, which are called Ordinary Differential Equations of the First Order.

1.1 Basic Concepts and Ideas

The derivative dy/dx of a function $y = f(x)$ is itself another function $f'(x)$ found by an appropriate rule of differentiation. For example, the function $y = e^{x^2}$ is differentiable on the interval $(-\infty, \infty)$ and by the Chain Rule its derivative is $dy/dx = 2xe^{x^2}$. If we replace the right-hand expression of the last equation by the symbol y, the equation becomes

$$\frac{dy}{dx} = 2xy. \tag{1.1}$$

In differentiation, the problem was "Given a function $y = f(x)$, find its derivative."

Now, the problem we face here is "If we are given an equation such as (1.1), is there some way or method by which we can find the unknown function $y = f(x)$ that satisfy the given equation, without prior knowledge how it was constructed?" These kind of problems are the ones we are going to focus on in this part of the course.

Definition 1.1.1. *An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a* **Differential Equation** *(***DE***).*

Example 1.1.1.

$$\frac{dy}{dx} + y = x, \quad \frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 3x = \sin t, \quad \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} + 5v = 2. \tag{1.2}$$

are all **Differential Equations**.

Differential equations can be classified by their **type, order,** and in term of **linearity.** We will see these classifications before going to the solution concept.

Classification by Type

- If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it is said to be an **ordinary differential equation (ODE)**.

  For example,

  $$\frac{dy}{dx} + y = x, \qquad \frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0 \qquad \text{and} \qquad \frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 3x = \sin t$$

  are all ordinary differential equations.

- If a function is defined in terms of two or more independent variables, the corresponding derivative will be a partial derivative with respect to each independent variable. An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.

  For example,

  $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} \qquad \text{and} \qquad \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} + 5v = 2.$$

  are both partial differential equations.

In this part we will only consider the case of ordinary differential equations.

Classification by Order

The **order** of a differential equation (either ODE or PDE) is the order of the highest derivative that appear in the equation. For example,

$$4x\frac{dy}{dx} + y = x \qquad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 6y = e^x$$

are first and second-order ordinary differential equations respectively.

The general $n^{\text{th}}-$order ordinary differential equation in one dependent variable is given by the general form

$$F(x, y, y', y'', ..., y^{(n)}) = 0, \tag{1.3}$$

where F is a real-valued function of $n+2$ variables $x, y, y', y'', ..., y^{(n)}$.

Remark 1.1.2. *For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to explicitly solve the differential equation of the form (1.3) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n+1$ variables $x, y, y', y'', ..., y^{(n-1)}$. Then the differential equation (1.3) becomes*

$$\frac{d^n y}{dx^n} = f(x, y, y', ..., y^{(n-1)}), \tag{1.4}$$

where f is a real-valued continuous function and this is referred to as the **normal form** *of (1.3).*

Example 1.1.2. *The normal form of the first-order equation $4xy' + y = x$ is*

$$y' = \frac{x-y}{4x}$$

and the normal form of the second-order equation $y'' - y + 6y = 0$ is

$$y'' = y' - 6y.$$

The first order ordinary differential equation is generally expressed as:

$$F(x, y, y') = 0 \text{ or } y' = f(x, y).$$

For example, the differential equation $y' + y = x$ is equivalent to $y' + y - x = 0$. If $F(x, y, y') = y' + y - x$, then the given differential equation becomes of the form $F(x, y, y') = 0$.

Classification by Linearity

An n^{th}-order ordinary differential equation (1.3) is said to be linear if F is linear in $y, y', ..., y^{(n)}$. This means that an n^{th}-order linear ordinary differential equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - b(x) = 0, \tag{1.5}$$

where $a_n(x) \neq 0$.
If $b(x) \equiv 0$, the equation (1.5) is called a **homogeneous DE** and otherwise it is called **nonhomogeneous**.

Notation

We may equivalently use the notations

$$\frac{d^n y}{dx^n} \qquad \text{and} \qquad y^{(n)}$$

interchangeably for the $n^{th}-$order derivative of y with respect to x. Using this notation, equation (1.5) can be equivalently written as

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

Solution Concept

Consider the equation $y' + 2xy = 0$, which is a first order differential equation for the unknown function $y(x)$. One can easily check that the function $y(x) = e^{-x^2}$ satisfies the given equation on $(-\infty, \infty)$ and we say that e^{-2x} is a solution for the given differential equation.

Definition 1.1.3. *Let $h(x)$ be a real valued function defined on an interval $[a, b]$ and having n^{th} order derivative for all $x \in (a, b)$. If $h(x)$ satisfies the n^{th} order ODE (1.5) on (a, b), that is,*

1. *$F(x, h(x), h'(x), h''(x), \ldots, h^{(n)}(x))$ is defined for all $x \in (a, b)$ and*
2. *$F(x, h(x), h'(x), h''(x), \ldots, h^{(n)}(x)) = 0$, for all $x \in (a, b)$,*

*then $y = h(x)$ is called a (**an Explicit**) solution of the ODE on $[a, b]$.*

Sometimes a solution of a differential equation may appear as an implicit function, i.e. the solution can be expressed implicitly in the form: $h(x, y) = 0$, where h is some continuous function of x and y, and such solution is called an **Implicit Solution** of the DE.

Example 1.1.3. *Consider the differential equation $y'' + y = 0$.*
*Let $h(x) = 2\sin x + 3\cos x$. Then $y = h(x)$, $y' = h'(x) = 2\cos x - 3\sin x$, $y'' = h''(x) = -2\sin x - 3\cos x$ and $y'' + y = (-2\sin x - 3\cos x) + (2\sin x + 3\cos x) = 0$, which means $h(x)$ satisfies the given **DE** and hence it is an explicit solution.*

Example 1.1.4. *Consider the differential equation $yy' = -x$.*
Differentiating both sides of the equation $x^2 + y^2 - 1 = 0, (y > 0)$ with respect to x we get

$$\frac{d}{dx}(x^2 + y^2 - 1) = \frac{d}{dx}(0).$$

That is,

$$2x + 2y\frac{dy}{dx} = 0,$$

which is equivalent to the equation $x + yy' = 0 \iff yy' = -x$.
Hence, $x^2 + y^2 - 1 = 0$ is an implicit solution of the DE $yy' = -x$ on $(-1, 1)$, since $y > 0$.

We are now in a position to solve some differential equations. There are different methods of solving differential equations and one method that works for one DE may not work for another. In this chapter we will consider some of these methods for solving ordinary differential equations of the first order.

1.2 Separable Differential Equations

Consider differential equation

$$\frac{dy}{dx} = f(x). \tag{1.6}$$

Then $dy = f(x)dx$ and it can be solved by integration. If $f(x)$ is a continuous function, then integrating both sides of (1.6) gives

$$y = \int f(x)dx = G(x) + c,$$

where G(x) is an antiderivative (indefinite integral) of $f(x)$.

Example 1.2.1.

1. *If $y' = x$, then $y(x) = \int_0^x t dt + C = \frac{1}{2}x^2 + C$*
2. *If $y' = sin(1 + x^2)$, then $y(x) = \int_0^x sin(1 + t^2)dt + C$. However, it is difficult to find an explicit solution formula for this problem. (In such cases one may use numerical methods to get approximate solutions.)*

Many first-order **ODEs** can be reduced or transformed to the form

$$g(y)y' = f(x),$$

where g and f are continuous functions. Then, from elementary calculus we have:

$$g(y)dy = f(x)dx.$$

Such type of equations are called **separable equations**. Integrating both sides we get:

$$\int g(y)dy = \int f(x)dx + c$$

is the general solution of the given equation.

Example 1.2.2. *Solve the DE* $6yy' + 4x = 0$.

Solution:

The equation $6yy' + 4x = 0$ is equivalent to

$$6y\frac{dy}{dx} = -4x$$

and then $6ydy = -4xdx$. Integrating both sides,

$$\int 6ydy = \int (-4x)dx,$$

gives

$$3y^2 + 2x^2 = C,$$

which is an implicit solution of the given first order differential equation.

Example 1.2.3. *Solve the DE* $y' = y^2e^{-x}$.

First rewrite the equation as

$$\frac{dy}{dx} = y^2e^{2x}.$$

If $y \neq 0$, this has the differential form

$$\frac{1}{y^2}\mathrm{d}y = e^{2x}\mathrm{d}x,$$

where the variables have been separated. Integrating both sides we have

$$\int \frac{1}{y^2}dy = \int e^{2x}dx,$$

which implies

$$\frac{-1}{y} = \frac{e^{2x}}{2} + c,$$

where c is a constant of integration. Then solve for y to get

$$y(x) = \frac{-2}{(e^{2x} + c)},$$

which is an explicit solution of the given first order differential equation.

Remark 1.2.1. *It is recommended to write an explicit solution to the differential equation when ever possible. However, sometimes solving for the dependent variable (in our case y) may not be possible. In those cases one can represent the final solution by an implicit solution of the differential equation.*

1.2.1 Equation Reducible to Separable Form

There are some differential equations which are not separable, but they can be transformed to a separable form by simple change of variables. We will see some of the possible substitutions hereunder.

A. Linear Substitution

Suppose we have a differential equation that can be written in the form:

$$y' = g(ax + by + c) \tag{1.7}$$

Such an equation is not in general separable. However, if we set $u = ax + by + c$, we get

$$\frac{du}{dx} = a + b\frac{dy}{dx}.$$

Or

$$\frac{dy}{dx} = \frac{1}{b}\frac{du}{dx} - \frac{a}{b}.$$

Thus (1.7) will be transformed into

$$\frac{1}{b}\frac{du}{dx} - \frac{a}{b} = g(u),$$

where u and x can be separated.

Example 1.2.4. *Solve the differential equation* $y' = (x + y)^2$.

Solution:

Let $u = x + y$. Then $u' = 1 + y'$ which implies $y' = u' - 1$. With this substitution the equation $y' = (x+y)^2$ is equivalent to

$$u' - 1 = u^2 \iff \frac{du}{dx} = u^2 + 1.$$

Then

$$\frac{du}{u^2+1} = dx$$

and integrate both sides,

$$\int \frac{du}{u^2+1} = \int dx$$

to get $\arctan u = x + c$ for an arbitrary constant c. Substituting back $u = x + y$ in the last equation gives us the general solution of the given DE to be $\arctan(x+y) = x + c$.

Example 1.2.5. *Solve the differential equation* $(2x - 4y + 5)y' + x - 2y + 3 = 0$.

Solution

Let $u = x - 2y$. Then, $u' = 1 - 2y'$ which implies $y' = \frac{1}{2}(1 - u')$. Therefore, the equation $(2x - 4y + 5)y' + x - 2y + 3 = 0$ becomes $(2u + 5)\frac{1}{2}(1 - u') + u + 3 = 0$. Simplifying this we get $(2u + 5) - (2u + 5)u' + 2u + 6 = 0$ which implies

$$(2u + 5)u' = 4u + 11 \iff \left(\frac{2u+5}{4u+11}\right)\frac{du}{dx} = 1.$$

Then

$$\frac{4u+10}{4u+11}du = 2dx \iff (1 - \frac{1}{4u+11})du = 2dx.$$

Now we integrate both sides

$$\int (1 - \frac{1}{4u+11})du = \int 2dx$$

and we get

$$u - \frac{1}{4}\ln|4u + 11| = 2x + c_1.$$

But $u = x - 2y$. Then substituting this in the above equation gives us

$$x - 2y - \frac{1}{4}\ln|4x - 8y + 11| = 2x + c_1$$

for an arbitrary constant c_1, or equivalently $4x + 8y + \ln|4x - 8y + 11| = C$, where $C = -4c_1$.

B. Quotient Substitution

Suppose we have an equation that can be written in the form

$$y' = g(\frac{y}{x}).$$

Let us substitute

$$u = \frac{y}{x}.$$

Then

$$\frac{du}{dx} = \frac{xy' - y}{x^2} = \frac{1}{x}y' - \frac{y}{x^2}.$$

This implies,

$$y' = xu' + \frac{y}{x} = xu' + u.$$

Thus, the differential equation

$$y' = g(\frac{y}{x})$$

is reduced to the equation $xu' = g(u) - u$ which is equivalent to the differential equation

$$\frac{dx}{x} = \frac{du}{g(u) - u}.$$

Then by integrating we obtain a general solution.

Example 1.2.6. *Solve* $x^2y' = x^2 + xy + y^2$.

Solution

For $x \neq 0$, *the differential equation* $x^2y' = x^2 + xy + y^2$ *is equivalent to*

$$y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Let $u = \frac{y}{x}$. *Then* $g(u) = 1 + u + u^2$ *and we get,* $xu' = (1 + u + u^2) - u = 1 + u^2$, *which implies*

$$\frac{du}{1+u^2} = \frac{dx}{x}.$$

We then integrate

$$\int \frac{du}{1+u^2} = \int \frac{dx}{x}$$

and get $\arctan u = \ln|x| + c$.

Now substituting $u = \frac{y}{x}$ *gives us*

$$\arctan(\frac{y}{x}) = \ln|x| + c = \ln|x| + \ln k = \ln k|x|, \ \textit{for some } k > 0.$$

That is,

$$\frac{y}{x} = \tan(\ln k|x|)$$

and solving for y we get $y(x) = x\tan(\ln k|x|)$.

Example 1.2.7. *Solve the DE: $2xyy' = y^2 - x^2$.*

Solution:

Divide both sides by x^2, for $x \neq 0$, to get

$$2\left(\frac{y}{x}\right)y' = \left(\frac{y}{x}\right)^2 - 1.$$

Let $u = \frac{y}{x}$. Then $g(u) = \frac{1}{2}(u - \frac{1}{u})$ and we get

$$xu' = \frac{1}{2}(u - \frac{1}{u}) - u = \frac{-(u^2+1)}{2u}$$

which implies

$$\frac{-2udu}{1+u^2} = \frac{dx}{x}.$$

Then we integrate

$$\int \frac{-2udu}{1+u^2} = \int \frac{dx}{x}$$

and get $\ln(1+u^2) = -\ln x + c$. This implies

$$1 + u^2 = e^{(-\ln x + c)} = Ax, \text{ for a constant A.}$$

Now we substitute $u = \frac{y}{x}$ to get

$$x^2 + y^2 = Ax^3.$$

Notice that the solution of each of the previous examples contains arbitrary constants. To determine the constants in these solutions we need to impose some additional conditions. For example, for the DE equation $6yy' + 4x = 0$, the equation $3y^2 + 2x^2 = C$ represents an implicit solution for an arbitrary constant C. But if $y(0) = 3$ is given in addition, then $C = 27$ and $3x^2 + 2x = 27$ will be a specific solution of the given DE.

Definition 1.2.2. *For the differential equation*

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0,$$

conditions of the form:

$$y(a) = y_0, y'(a) = y_1, \ldots, y^{(n-1)}(a) = y_{(n-1)}$$

are called **initial conditions (IC).**

A Differential Equation $F(x, y, y', \ldots, y^{(n)}) = 0$ together with Initial Conditions is called an **Initial Value Problem (IVP)** *or* **Cauchy's problem.**

Remark 1.2.3. *The number of initial conditions necessary to determine a unique solution equals the order of the differential equation.*

Example 1.2.8. *Solve the IVP $y'' + y = 0$, $y(0) = 3$ and $y'(0) = -4$.*

Solution:

First find the general solution with two unknown constants. Given

$$y(x) = c_1 \cos x + c_2 \sin x,$$

since $y(x)$ satisfy the Differential Equation $y'' + y = 0$, it is a general solution. Now $y(0) = c_1 = 3$ and $y'(x) = -c_1 \sin x + c_2 \cos x$ implies $y'(0) = c_2 = -4$. Hence the particular solution of the equation is $y(x) = 3\cos x - 4\sin x$.

If, in addition, some conditions are imposed at $x = a$ and at $x = b$, where a and b are some real numbers, then the problem is called a **Boundary-Value Problem (BVP).**

Remark 1.2.4. *Total number of conditions that are required to solve the problem uniquely is again equal to the order of the differential equation.*

Example 1.2.9. *Suppose $y'' + y = 0, y(0) = 3$ and $y(\frac{\pi}{2}) = 5$.*

This is a boundary value problem with $y(0) = c_1$ which implies $c_1 = 3$ and $y(\frac{\pi}{2}) = c_2$, which implies $c_2 = 5$.

Hence, the particular solution of this BVP is

$$y(x) = 3\cos x + 5\sin x.$$

Two fundamental questions arise in considering an initial-value problem and these are:

- Does a solution of the problem exist?
- If a solution exists, is it unique?

Getting answer for these questions is crucial before we try find the solutions. The following theorem answers these questions.

Theorem 1.2.5 (Existence and uniqueness of a solution)**.** *If $f(x,y)$ is continuous function on some rectangular region R in the $xy-$ plane containing the point (a,b) in its interior , then the problem*

$$y' = f(x,y), \ \ with\ y(a) = b \tag{1.8}$$

has at least one solution defined on some open interval of x containing $x = a$.

If, in addition, the function

$$\frac{\partial f}{\partial y}$$

is continuous on $\mathbb{R}$, then the solution to the above equation (1.8) is unique on some open interval containing $x = a$.

Remark 1.2.6. *The above condition for uniqueness can be eased by using a condition* piecewise continuous *instead of the condition that "$\frac{\partial f}{\partial y}$ is continuous".*

1.3 Exact Differential Equations

Consider the differential equation:

$$y' = \frac{\sin y}{2y - x\cos y}$$

or equivalently

$$\sin y dx + (x\cos y - 2y)dy = 0.$$

Notice that the left hand side is the **(total) differential** of the function

$$F(x,y) = x\sin y - y^2.$$

Recall that the total differential of a function $F(x,y)$ of two variables is

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

for all (x,y) in the domain of F.

Thus for $F(x,y) = x\sin y - y^2, dF(x,y) = \sin y dx + (x\cos y - 2y)dy = 0$. This implies that $F(x,y) = C$, that is, $x\sin y - y^2 = C$, is the solution of the above DE.

Definition 1.3.1. *The expression*

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact differential equation** *in some domain D (an open connected set of points) if there is a function $F(x,y)$ such that*

$$\frac{\partial F}{\partial x} = M(x,y) \ \ and\ \frac{\partial F}{\partial y} = N(x,y),$$

for all $(x,y) \in D$.

If we can find a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y),$$

then the differential equation $M(x,y)dx + N(x,y)dy = 0$ is just $M(x,y)dx + N(x,y)dy = dF = 0$. But recall that, if $dF = 0$, then $F(x,y) = \text{constant}$. The equation $F(x,y) = c$, where c is an arbitrary constant, implicitly defines the general solution of the deferential equation $M(x,y)dx + N(x,y)dy = 0$.

Now let us ask the following two fundamental questions. Given a Differential Equation

$$M(x,y)dx + N(x,y)dy = 0$$

1. **How can we determine the existence of such a function $F(x,y)$?**
2. **If it exists, how can we find it ?**

The following theorem will answer the first question.

Theorem 1.3.2 (Test for Exactness)**.** *Let $M(x,y), N(x,y), \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be all continuous functions within a rectangle R (or some domain) in the xy-plane. Then*

$$M(x,y)dx + N(x,y)dy$$

is an exact differential in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

every where in R.

Example 1.3.1. *Consider the equation*

$$\frac{dy}{dx} = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}.$$

Then write

$$(2xy^3+2)dx+(3x^2y^2+8e^{4y})dy=0.$$

Let $M(x,y)=2xy^3+2$ *and* $N(x,y)=3x^3y^2+8e^{4y}$. *Then*

$$\frac{\partial M}{\partial y}=6xy=\frac{\partial N}{\partial x}.$$

Therefore, the given differential equation is exact.

Example 1.3.2. *Consider the equation*

$$(y\ln y-e^{-xy})dx+(\frac{1}{y}+x\ln y)dy=0.$$

Let $M(x,y)=y\ln y-e^{-xy}$ *and* $\frac{1}{y}+x\ln y$. *Then* $\frac{\partial M}{\partial y}=\ln y+xe^{-xy}+y$ *and* $\frac{\partial N}{\partial x}=\ln y$, *which implies* $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. *Therefore, the given differential equation is not exact.*

After knowing the exactness of a differential equation, the next question is "How can we solve the given equation?" The method for this is described here below.

Suppose a differential equation $M(x,y)dx+N(x,y)dy=0$ is exact. Then, there exists a function $F(x,y)$ such that

$$M=\frac{\partial F}{\partial x}\text{ and }N=\frac{\partial F}{\partial y}.$$

From $M=\frac{\partial F}{\partial x}$, we have (by integrating with respect to x)

$$F(x,y)=\int Mdx+A(y), \tag{1.9}$$

where $A(y)$ is only a function of y but constant with respect to x.

Now to determine $A(y)$ (the constant of integration), differentiate equation (1.9) with respect to y to get

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\int Mdx+A'(y)$$

which implies

$$N(x,y)=\int\frac{\partial M}{\partial y}dx+A'(y)\text{ and hence }A'(y)=N(x,y)-\int\frac{\partial M}{\partial y}dx$$

by exactness. Therefore,

$$A(y)=\int\left[N(x,y)-\int\frac{\partial M}{\partial y}dx\right]dy.$$

Example 1.3.3. *Solve the differential equation*

$$\sin ydx+(x\cos y-2y)dy=0.$$

Solution

Let $M(x,y)=\sin y$ *and* $N(x,y)=x\cos y-2y$. *Then* $M_y=\cos y=N_x$. *Since* M,N,M_y,N_x *are all continuous in* $\mathbb{R}^2$, *the given equation is exact. Thus, there exists a function* $F(x,y)$ *such that*

$$\frac{\partial F}{\partial x}=\sin y\ \ and\ \frac{\partial F}{\partial y}=x\cos y-2y,$$

which implies $F(x,y)=\int\sin ydx=x\sin y+A(y)$ *and* $\frac{\partial F}{\partial y}=x\cos y+A'(y)$. *That is,* $x\cos y-2y=x\cos y+A'(y)$ *which implies* $A'(y)=-2y$ *and hence*

$$A(y)=\int -2ydy=-y^2+B.$$

Therefore, $F(x,y)=x\sin y-y^2+B=$ *constant, which implies*

$$x\sin y-y^2=C$$

determines $y(x)$ *implicitly.*

Example 1.3.4. *Solve the differential equation*

$$(x^3+3xy^2)dx+(3x^2y+y^3)dy=0.$$

Solution

Step 1: Checking Exactness

Let $M(x,y)=x^3+3xy^2$ and $N(x,y)=3x^2y+y^3$. Then

$$\frac{\partial M}{\partial y}=6xy=\frac{\partial N}{\partial x}.$$

Therefore the given equation is exact.

Step 2: Finding Implicit Solution

Then to find $F(x,y)$ we use

$$F(x,y)=\int Mdx+A(y)=\int(x^3+3xy^2)dx+A(y)=\frac{1}{4}x^4+\frac{3}{2}x^2y^2+A(y)$$

where $A(y)$ is a function of y only. To find A(y);

$$\frac{\partial F}{\partial y}=3x^2y+A'(y)=N=3x^2y+y^3,$$

which implies that $A'(y) = y^3$ and then $A(y) = \int y^3 dy = \frac{1}{4}y^4 + C$.
Therefore,

$$F(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C = \frac{1}{4}(x^4 + 6x^2y^2 + y^4) + C \tag{1.10}$$

Step 3: Checking

Differentiate implicitly to check for y':

$$\frac{1}{4}(4x^3 + 12xy^2 + 12x^2yy' + 4y^3y') = 0$$

which implies

$$x^3 + 3xy^2 + (3x^2y + y^3)y' = 0$$

and then

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$$

Exercise 1.3.3. *Solve each of the following differential equations.*

1. $(y + e^y)dx + x(1 + e^y)dy = 0; \qquad y = 0$ *when* $x = 1$.
2. $\dfrac{dy}{dx} = \dfrac{2x+1}{2y+1}; \qquad y(0) = 0.$
3. $\sin hx \cos y dx = \cos hx \sin y dy$.

Integrating Factors

The differential equation $ydx + 2xdy = 0$ is not exact. But if we multiply this equation by y, the equation is changed to exact equation. That is,

$$y^2dx + 2xydy = 0$$

is exact, since

$$\frac{\partial y^2}{\partial y} = 2y = \frac{\partial(2xy)}{\partial x}.$$

Definition 1.3.4. *If the differential equation* $M(x,y)dx + N(x,y)dy = 0$ *is not exact but the differential equation*

$$\mu(x,y)M(x,)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, then the multiplicative function $\mu(x,y)$ *is called an* **integrating factor** *of the DE.*

(Of course $\mu(x,y) \neq 0$ so that the two equations are equivalent.)

Example 1.3.5. *Consider the differential equation*

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0.$$

Let $M(x,y) = 3y + 4xy^2$ *and* $N(x,y) = 2x + 3x^2y$. *Then*

$$\frac{\partial M}{\partial y} = 3 + 8xy \qquad \textit{and} \qquad \frac{\partial N}{\partial x} = 2 + 6xy,$$

which implies

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence the DE is not exact.
But if $\mu(x,y) = x^2y$ *then* $\mu(x,y)Mdx + \mu(x,y)Ndy = 0$ *is exact, since*

$$\frac{\partial(\mu(x,y)M)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial(\mu(x,y)N)}{\partial x}.$$

Suppose we have a differential equation which is not exact but it can be made exact by an integrating factor. Then we can ask the following fundamental questions.

1. **How can we find the integrating factor μ ?**
2. **Given μ, how can we solve the problem?**

The method is described below.

Clearly $\mu(x,y)$ is any (non-zero) solution of the equation

$$\frac{\partial}{\partial y}(\mu N) = \frac{\partial}{\partial x}(\mu N) \tag{1.11}$$

which is equivalent to the equation

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

This is a first-order partial differential equation in μ. However the integrating factor μ can be found to be a function of x alone $\mu(x)$ (or a function of y alone $\mu(y)$).

Then in this case equation (1.11) will be reduced to

$$\mu M_y = \frac{d\mu}{dx}N + \mu N_x \text{ or equivalently } \frac{d\mu}{dx} = \mu(\frac{M_y - N_x}{N}) \tag{1.12}$$

which is a separable differential equation.

This idea works correctly if the ratio

$$\frac{M_y - N_x}{N}$$

is a function of x only, that is,

$$p(x) = \frac{M_y - N_x}{N} = \text{ a function of } x.$$

In this case

$$\frac{d\mu}{\mu} = \left(\frac{M_y - N_x}{N}\right)dx,$$

which implies

$$\mu(x) = e^{\int p(x)dx}.$$

If the quotient $\frac{M_y - N_x}{N}$ is not a function of x alone, then the integrating factor μ can not be obtained using the above procedure, but we can try to find μ as a function of y alone, $\mu(y)$.

Then when $\mu(y)$ is only a function of y, equation (1.11) will be reduced to

$$\frac{d\mu}{dy}M + \mu M_y = \mu N_x$$

which implies

$$\frac{d\mu}{dy} = -\mu\left(\frac{M_y - N_x}{M}\right), \text{ which is a separable differential equation.}$$

If the fraction

$$\frac{M_y - N_x}{M}$$

is a function of y alone, then

$$\mu(y) = e^{-\int q(y)dy}.$$

Example 1.3.6. *Consider the equation*

$$dx + (3x - e^{-2y})dy = 0. \tag{1.13}$$

Let $M = 1$ and $N = 3x - e^{-2y}$. Then $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 3$ and hence $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ which implies that the given differential equation is not exact.

Assume that the given equation has an integrating factor. But

$$\frac{M_y - N_x}{N} = \frac{0-3}{3x - e^{-2y}} = \frac{-3e^{2y}}{3xe^{2y}}$$

which is not a function of x alone. Hence obtaining $\mu(x)$ is not possible.

However,

$$\frac{M_y - N_x}{M} = \frac{0-3}{1} = -3$$

can be considered as a function of y alone. Therefore, it is possible to solve for $\mu(y)$ and is given by

$$\mu(y) = e^{-\int(-3)dy} = e^{3y}.$$

Now to solve the problem in (1.13), multiplying the given equation by $\mu(y) = e^{3y}$ we get the equation

$$e^{3y}dx + (3x - e^{-2y})e^{3y}dy = 0,$$

which is an exact differential equation. Thus, there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = e^{3y} \text{ and } \frac{\partial F}{\partial y} = (3x - e^{-2y})e^{3y}$$

which implies that

$$F(x,y) = \int \frac{\partial F}{\partial x}dx = \int e^{3y}dx = xe^{3y} + A(y).$$

To determine $A(y)$ we use $F(x,y)$ which is obtained above and differentiate it with respect to y and equate the result with $(3x - e^{-2y})e^{3y}$. Hence we have

$$(3x - e^{-2y})e^{3y} = \frac{\partial F}{\partial y} = 3xe^{3y} + A'(y).$$

Then $3xe^{3y} - e = 3xe^{3y} + A'(y)$ which implies that

$$A'(y) = -e^y \text{ and hence } A(y) = -\int e^y dy = -e^y + B.$$

Therefore, $F(x,y) = xe^{3y} - e^y + B = constant$. That means $xe^{3y} - e^y = C$, where C is an arbitrary constant, defines the implicit solution of the Differential Equation in (1.13).

1.4 Linear First Order Differential Equations

Consider the general first-order linear differential equation

$$a_1(x)y' + a_0(x)y = f(x),\ a_1(x) \neq 0 \tag{1.14}$$

By dividing both sides by $a_1(x) \neq 0$, we get $y' + p(x)y = q(x)$, where

$$p(x) = \frac{a_0(x)}{a_1(x)} \quad \text{and} \quad q(x) = \frac{f(x)}{a_1(x)}.$$

Here we assume that $p(x)$ and $q(x)$ are continuous.

There is a general approach to solve linear equations. To solve for $y(x)$ from the given equation we start with the simplest case, when $q(x)=0$. That is, (1.14) becomes

$$y'+p(x)y=0. \tag{1.15}$$

This problem is called a homogeneous version of (1.14). Now to solve (1.15) first we get $y'=-p(x)y$ and we divide both sides by y and get

$$\frac{y'}{y}=-p(x).$$

Then by integrating

$$\int\frac{dy}{y}=-\int p(x)dx$$

we get

$$\ln|y|=-\int p(x)dx+C,$$

which implies

$$|y|=e^{c-\int p(x)dx}=Be^{-\int p(x)dx},\ \text{for } B>0.$$

Therefore,

$$y(x)=Ae^{-\int p(x)dx},\ \text{where } A \text{ is an arbitrary constant,}$$

is a general solution of (1.15).

Example 1.4.1. *Solve the following differential equations.*

1. $y'+2xy=0$
2. $(x+2)y'-xy=0$

Solution:

1. If $y'+2xy=0$, then $\frac{y'}{y}=-2x$. We integrate

$$\int\frac{dy}{y}=-2\int xdx$$

to get $\ln|y|=x^2+C$ and hence $y(x)=Ae^{-x^2}$ is the general solution.

2. If $(x+2)y'-xy=0$, then

$$\frac{y'}{y}=\frac{x}{x+2}.$$

We integrate

$$\int\frac{dy}{y}=\int\frac{x}{x+2}dx$$

to get $y(x)=Ae^{(x-2\ln(x+2))}$, or $y(x)=\frac{A}{(x+2)^2}e^x$ which is the general solution of the given equation.

Now we want to solve the general first - order linear ordinary differential equation

$$y'+p(x)y=q(x) \tag{1.16}$$

This can be done in two steps.

Step 1.

Consider the homogeneous version of (1.16) and find the solution to be $y_h(x)=Ae^{-\int p(x)dx}$, where h indicate the general solution for the homogeneous part of the equation

Step 2.

To get the solution for the non-homogeneous part of the equation we vary the constant A with different value of x.

Hence we assume that

$$y(x)=A(x)e^{-\int p(x)dx} \tag{1.17}$$

is a solution for (1.16). Then (1.17) must satisfy (1.16). i.e.

$$\big(A(x)e^{-\int p(x)dx}\big)'+p(x)\big(A(x)e^{-\int p(x)dx}\big)=q(x),$$

which implies

$$A'(x)e^{-\int p(x)dx}+A(x)(-p(x))e^{-\int p(x)dx}+p(x)A(x)e^{-\int p(x)dx}=q(x).$$

Simplifying this gives us,

$$A'(x)=q(x)e^{\int p(x)dx}.$$

Now integrate both sides $\int A'(x)dx = \int q(x)e^{\int p(x)dx}dx$ to get

$$A(x) = \int q(x)e^{\int p(x)dx}dx + C.$$

Hence the general solution of the non-homogeneous ODE (1.16) is given by:

$$\begin{aligned} y(x) &= A(x)e^{-\int p(x)dx} = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right) \\ &= Ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx \\ &= y_h(x) + y_p(x) \end{aligned}$$

Remark 1.4.1. *It may not be necessary to memorize this long formula for $y(x)$. Instead, we can carry out the following procedure.*

Step 1. *If the differential equation is linear, $y' + p(x)y = q(x)$, then first compute*

$$e^{\int p(x)dx}.$$

This is called an integrating factor *for the linear equation.*

Step 2. *Multiply both sides of the differential equation by the integrating factor.*

Step 3. *Write the left side of the resulting equation as the derivative of the product of y and the integrating factor. The integrating factor is designed to make this possible. The right side is a function of just x.*

Step 4. *Integrate both sides of this equation with respect to x and solve the resulting equation for y, obtaining the general solution.*

Example 1.4.2. *Solve the differential equation $y' + 3y = 6$.*
The given equation is linear with $p(x) = 3$ and $q(x) = 6$.

Step 1. *We compute the integrating factor*

$$e^{\int p(x)dx} = e^{\int 3dx} = e^{3x}$$

Step 2. *We multiply $y' + 3y = 6$ by e^{3x} to get $y'e^{3x} + 3ye^{3x} = 6e^{3x}$.*

Step 3. *The above equation is equivalent to*

$$\frac{d(ye^{3x})}{dx} = 6e^{3x}.$$

Step 4. *We integrate*

$$\int \frac{d(ye^{3x})}{dx}dx = \int 6e^{3x}$$

and get $y(x)e^{3x} = 2e^{3x} + C$. Then solve for $y(x)$ to get the general solution $y(x) = Ce^{-3x} + 2$ for an arbitrary constant C.

1.5 *Nonlinear Differential Equations of the First Order

Some nonlinear differential equations can be reduced to linear form. In this section we will consider three famous nonlinear equations: **Bernoulli Equation**, **Riccati Equation** and **Clairuat Equation**

1.5.1 The Bernoulli Equation

A differential equation of the form

$$y' + p(x)y = q(x)y^{\alpha},$$

where α is a constant, is called **Bernoulli** Equation.
If $\alpha = 0$, then the equation is linear and if $\alpha = 1$, then the equation is separable. We have seen these two cases in the previous section.

For $\alpha \neq 1$, use the change of variable $u = y^{1-\alpha}$. Then by differentiating with respect to x, we get $u' = (1-\alpha)y^{-\alpha}y'$. But, from $y' + p(x)y = q(x)y^{\alpha}$, we get $y' = q(x)y^{\alpha} - p(x)y$. Then

$$\begin{aligned} u' &= (1-\alpha)y^{-\alpha}y' \\ &= (1-\alpha)y^{-\alpha}(qy^{\alpha} - py) \\ &= (1-\alpha)(q - py^{-1-\alpha}) \\ &= (1-\alpha)(q - pu), \text{ since } u = y^{1-\alpha} \end{aligned}$$

This implies that $u' + (1-\alpha)u = (1-\alpha)q$, which is a linear differential equation of first order and hence we can solve it using one of the methods we have seen in the previous sections.

Example 1.5.1. *Solve the Bernoulli Equation $y' - Ay = -By^2$, where A and B are positive constants. This equation is called Varhulst Equation.*

Solution

Since $\alpha = 2$, let $u = y^{1-2} = y^{-1}$. Then $u' = -y^{-2}y'$ and substituting $Ay - By^2$ for y' we get $u' = -y^{-2}(Ay - By^2) = B - Ay^{-1} = B - Au$. Then we get the differential equation $u' + Au = B$ which is equivalent to the equation $du - (B - Au)dx = 0$. This implies

$$\frac{du}{B - Au} = dx$$

and we integrate

$$\int \frac{du}{B - Au} = \int dx$$

and get $u = \frac{B}{A} + Ce^{-Ax}$.

Therefore, the general solution of the original differential equation is

$$y = \frac{1}{u} = \frac{1}{\frac{B}{A} + Ce^{-Ax}}.$$

1.5.2 The Riccati Equation.

A differential equation of the form

$$y' = p(x)y^2 + q(x)y + r(x)$$

is called **Riccati** equation. If $p(x) \equiv 0$, then the equation is linear.

If we can obtain one particular solution $s(x)$ of the **Riccati** equation, then the change of variables

$$y = s(x) + \frac{1}{z}$$

transforms the **Riccati** equation in to a linear equation in x and z. Then we find the general solution of this linear equation and we use it to write the general solution of the original **Riccati** equation.

Example 1.5.2. *Solve the* **Riccati** *equation*

$$y' = \frac{1}{x^2}y^2 - \frac{1}{x}y + 1.$$

(Hint: $y = x$ is one solution.)

1.5.3 The Clairuat Equation

A nonlinear differential equation of the form

$$y = xy' + g(y'),$$

is called **Clairuat** equation.

To solve such equation, let us differentiate both sides of the equation with respect to x. Then we get, $y' = y' + xy'' + g'(y')y''$,

which implies that $y''(x + g'(y')) = 0$ and hence $y'' = 0$ or $x + g'(y') = 0$.

Solving $y'' = 0$ gives us the general solution $y = ax + b$ and solving $x + g'(y') = 0$ gives us a singular solution (include definition of singular solution in the basic section).

Example 1.5.3. *Solve the* **Clairuat** *equation*

$$y = xy' + \frac{1}{y'}.$$

Solution

Differentiating both sides with respect to x to get

$$y' = y' + x'' - \frac{y''}{(y')^2}.$$

This implies

$$y''\left(x - \frac{1}{(y')^2}\right) = 0$$

and then solving $y'' = 0$ gives us a general solution $y = ax + b$ and solving

$$x - \frac{1}{1 - (y')^2} = 0$$

gives us a singular solution. Then $(y')^2 = \frac{1}{x}$ which implies $y' = \frac{1}{\pm\sqrt{x}}$. Hence $y = 2\sqrt{x} + c$ is a singular solution.

Remark 1.5.1. *The general solution of the* **Clairuat** *equation is $y = ax + b$. Therefore, our main focus for such equation is the singular solution.*

1.6 Exercises

Exercise 1.6.1. *Solve each of the following differential equations.*

1. $xy' + y = 6x^2$
2. $xy' + 2y = x + 2$ *with inital condition* $y(0) = 1$.

Chapter 2

Ordinary Differential Equations of The Second and Higher Order

A second-order differential equation is a differential equation containing a second derivative of a dependent variable with respect to the independent variable but no higher derivative. The theory of second-order differential equations is vast, and here we will focus on linear second-order equations, which have many important uses. Most of the results are given for a higher order ODEs and second order ODEs are special cases, but most of our examples are for second order ODEs.

2.1 Basic Theory

In this section, we will focus on the general theory of linear ordinary differential equations before we start to discuss about solving such problems.

Definition 2.1.1. *A linear ordinary differential equation of order n in the dependent variable y and independent variable x is an equation which can be expressed in the form:*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = f(x), \tag{2.1}$$

where $a_n(x) \not\equiv 0$ and the functions $a_0, \ldots, a_n$ are continuous real- valued functions of $x \in [a, b]$. The function $f(x)$ is called the non-homogeneous term and all the points $x_\epsilon \in [a, b]$ in which $a_n(x_\epsilon) = 0$ are called singular points of the DE (2.1).

If $f(x) \equiv 0$, then (2.1) is reduced to:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = 0 \tag{2.2}$$

This equation is known as homogeneous Linear ODE of order n.

Example 2.1.1. *The equation $y'' + 3xy' + x^3y = e^x$ is a non homogeneous linear ordinary differential equation of the 2^{nd} order, whereas $y'' + 3xy' + x^3y = 0$ is a homogeneous linear ordinary differential equation of the 2^{nd} order.*

Theorem 2.1.2 (Basic Existence Theorem for IVP)**.** *Consider the linear ODE given in (2.1), where $a_0, a_1, \ldots, a_{n-1}, a_n$ and f are continuous functions on the interval $[a,b]$ and $a_n(x) \neq 0, \forall x \in [a,b]$. Furthermore, let x_0 be any point in $[a,b]$ and let $c_0, c_1 \ldots c_{n-1}$ be arbitrary real constants. Then there exists a unique solution function $g(x)$ of (2.1) on $[a,b]$ satisfying the initial conditions,*

$$g(x_0) = c_0, g'(x_0) = c_1, \ldots, g^{n-1}(x_0) = c_{n-1}.$$

2.2 General Solution of Homogeneous Linear ODEs

Consider the linear differential equation

$$y'' + y = 0. \tag{2.3}$$

Then, $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of the differential equation (2.3). Let c_1 and c_2 be arbitrary constants. Then

$$y = c_1y_1 + c_2y_2 = c_1\cos x + c_2\sin x$$

is also a solution of (2.3). Indeed, $y' = -c_1\sin x + c_2\cos x$, and $y'' = -c_1\cos x - c_2\sin x$ which implies that

$$y'' + y = (-c_1\cos x - c_2\sin x) + (c_1\cos x + c_2\sin x) = 0$$

for all x. Therefore, any linear combination of the functions $y_1 = \cos x$ and $y_2 = \cos x$ is a solution for the given differential equation.

This condition can be generalized for any homogenous linear differential equation in the following theorem.

Theorem 2.2.1 (Linear Combination of Solutions)**.** *If $y_1, y_2, \ldots, y_k$ are solutions of the homogeneous linear ODE (2.1) and if $c_1, c_2, \ldots c_k$ are arbitrary constants, then the linear combination*

$$y = c_1y_1 + c_2y_2 + \ldots + c_ky_k = \sum_{i=1}^{k} c_iy_i$$

is also a solution of (2.1). That is, any linear combination of solutions of a linear homogeneous differential equation is also a solution.

Definition 2.2.2 (Linearly Dependent and Linearly Independent Functions)**.**

1. *The functions $f_1, f_2, \ldots, f_n$ are said to be Linearly Dependent (LD) on some interval $[a,b]$ if there are constants $c_1, c_2, \ldots, c_n$, not all zero, such that*

$$c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0 \tag{2.4}$$

   for all $x \in [a,b]$.

2. *If the relation (2.4) is true only when $c_1 = c_2 = \ldots = c_n = 0$, then the functions $f_1, f_2, \ldots, f_n$ are said to be Linearly Independent (LI) on $[a,b]$.*

Example 2.2.1. *Examples of LD and LI functions.*

1. *The functions $f_1(x) = e^x$ and $f_2(x) = 4e^x$ are Linearly Dependent on $\mathbb{R}$ since*

$$-4f_1(x) + f_2(x) = -4e^x + 4e^x = 0, \text{ for all } x \in \mathbb{R}.$$

2. *The functions*

$$f_1(x) = e^x, f_2(x) = e^{-x}, f_3(x) = \sinh x$$

   are linearly dependent on $\mathbb{R}$ since

$$f_3(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

   and $(1)f_1(x) + (-1)f_2(x) + (-2)f_3(x) = 0, \forall x \in \mathbb{R}$.

3. *The two functions $f_1(x) = x$ and $f_2(x) = x^3$ are Linearly Independent on $\mathbb{R}$, since for $c_1, c_2 \in \mathbb{R}$,*

$$c_1f_1(x) + c_2f_2(x) = c_1x + c_2x^3 = 0, \forall x \in \mathbb{R}\backslash\{0\}$$

   implies $c_1 = 0$ and $c_2 = 0$.

The following theorem guarantees that any n^{th} order Linear Homogenous Ordinary Differential Equation has n linearly independent solutions.

Theorem 2.2.3 (Existence of Linearly Independent Solutions for a LHODE)**.** *The Linear Homogenous Differential Equation (LHODE) (2.2) always has n Linearly Independent (LI) solutions. Furthermore, if $f_1(x), f_2(x), \ldots, f_n(x)$ are n LI solutions of (2.2), then every solution of (2.2) can be expressed as a linear combination of these solution functions. i.e. If y is a solution for (2.2), then*

$$y(x) = \sum_{i=1}^{n} c_i f_i(x)$$

for some $c_1, \ldots, c_n \in \mathbb{R}$..

Example 2.2.2. *Consider the second order linear homogenous DE*

$$y'' + y = 0.$$

Then $f_1(x) = \sin x, f_2(x) = \cos x$ are LI solutions of the given equation. Then $\{\sin x, \cos x\}$ is the fundamental set of solutions of the given DE and hence the general solution of the DE is given by $y(x) = c_1 \sin x + c_2 \cos x$, for constants $c_1, c_2 \in \mathbb{R}$.

Definition 2.2.4. *If $f_1(x), f_2(x), \ldots, f_n(x)$ are n linearly independent solutions of (2.2) on $[a, b]$, then the set $\{f_1(x), f_2(x), \ldots f_n(x)\}$ is called the* **Fundamental Set of Solutions** *of (2.2) and the function*

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), x \in [a, b],$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants is called a **General Solution** *of (2.2) on $[a, b]$. and each $f_1, f_2, \ldots, f_n$ are called particular solutions.*

Example 2.2.3. *Consider the third order linear homogenous DE*

$$y''' - 2y'' - y' + 2y = 0.$$

a) The functions e^x, e^{-x}, e^{2x} are (particular) solutions (check!)

b) e^x, e^{-x} and e^{2x} are LI (check!)

c) Therefore, the general solution of the given equation is given by:

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

There is a simple test to determine whether a given set of functions is linearly independent or dependent on an open interval $I = (a, b)$, for some real numbers a, b, by using the idea of determinant of a matrix.

Definition 2.2.5. *Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n real valued functions each of which has an $(n-1)^{th}$ derivative on the interval $[a, b]$. The determinant:*

$$\mathbf{W}[\mathbf{f_1}, \mathbf{f_2} \ldots, \mathbf{f_n}] = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \vdots & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix} = \mathbf{W}(x)$$

is called the **Wronskian** *of these n functions.*

Example 2.2.4. *The function $y_1(x) = e^{2x}$ and $y_2(x) = xe^{4x}$ are solutions of the second order linear homogenous differential equation $y'' - 4y' + 4y = 0$. Then the Wronskian, $\mathbf{W}(\mathbf{x})$ of y_1 and y_2 is*

$$\mathbf{W}(\mathbf{x}) = \begin{vmatrix} e^{2x} & xe^{4x} \\ 2e^{2x} & e^{4x} + 4xe^{4x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

Question: Are the two functions $y_1(x) = e^{2x}$ and $y_2(x) = xe^{4x}$ linearly independent?

The above question can be easily answered using the following theorem.

Theorem 2.2.6 (Wronskian Test for Linearly Independence)**.** *The n functions $f_1, f_2, \ldots, f_n$ are Linearly Independent on an interval $[a, b]$ if and only if the* **Wronskian** *of $f_1, f_2, \ldots, f_n$ is different from zero for some $x \in [a, b]$. That is, $f_1, f_2, \ldots, f_n$ are LI if and only if there exists $x \in [a, b]$ such that $\mathbf{W}(x) \neq 0$.*

Example 2.2.5. *1. Show that x and x^2 are Linearly Independent.*

Solution

Consider the **Wronskian** *of x and x^2,*

$$\mathbf{W}(\mathbf{x}, \mathbf{x^2}) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

This implies $\mathbf{W}(x, x^2) = x^2 \neq 0, \forall x \neq 0$ and hence x and x^2 are LI.

2. Show that e^x, e^{-x}, e^{2x} are Linearly Independent.

Solution

Consider the **Wronskian** *of* e^x, e^{-x} *and* e^{2x}

$$\mathbf{W}(\mathbf{x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix},$$

which is equal to

$$e^x(-4e^x + 2e^x) - e^{-x}(4e^{3x} - 2e^{3x}) + e^{2x}(1+1)$$

$$= -2e^{2x} - 2e^{2x} + 2e^{2x}$$

$$= -2e^{2x} \neq 0, \forall x \in \mathbb{R}.$$

Hence, e^x, e^{-x} *and* e^{2x} *are Linearly Independent.*

3. *Show that the functions*

$$f_1(x) = e^x, f_2(x) = e^{-x}, f_3(x) = \sinh x$$

are linearly dependent on $\mathbb{R}$*, since*

$$f_3(x) = \sinh x = \frac{e^x - e^{-x}}{2}.$$

Solution

Consider the **Wronskian** *of* e^x, e^{-x} *and* $\sinh x$

$$\mathbf{W}(\mathbf{x}) = \begin{vmatrix} e^x & e^{-x} & \frac{e^x - e^{-x}}{2} \\ e^x & -e^{-x} & \frac{e^x + e^{-x}}{2} \\ e^x & e^{-x} & \frac{e^x - e^{-x}}{2} \end{vmatrix} = 0, \forall x \in \mathbb{R}.$$

Hence e^x, e^{-x} *and* $\sinh x$ *are linearly dependent.*

Remark 2.2.7. *The Wronskian of* n *solutions* $f_1, f_2, \ldots, f_n$ *of the DE (2.2) is either identically zero on* $[a, b]$ *or else is never zero on* $[a, b]$*. That is, if* $f_1, f_2, \ldots, f_n$ *are solutions of the DE (2.2), then* $W(x) = 0, \forall x \in [a, b]$ *if* $f_1, f_2, \ldots, f_n$ *are LD. or* $W(x) \neq 0, \forall x \in [a, b]$ *if* $f_1, f_2, \ldots, f_n$ *are LI.*

2.2.1 Reduction of Order

In the preceding section we saw that the general solution of a homogeneous linear second-order differential equation

$$y'' + p(x)y + q(x)y = 0 \tag{2.5}$$

is a linear combination $y(x) = c_1y_1(x) + c_2y_2(x)$, where y_1 and y_2 are linearly independent solutions on some interval I.

In this method we can construct a second solution y_2 of a homogeneous equation (2.5) (even when the coefficients in (2.5) are variable) provided that we know a nontrivial solution y_1 of the DE. The basic idea described in this section is that equation (2.5) can be reduced to a linear first-order DE by means of a substitution involving the known solution y_1. A second solution y_2 of (2.5) is apparent after this first-order differential equation is solved.

The method is described bellow.

Suppose that y_1 denotes a nontrivial solution of (2.5) and that y_1 is defined on an interval I. We want to find a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I.

The quotient y_2/y_1 is nonconstant on I, that is,

$$\frac{y_2(x)}{y_1(x)} = u(x)$$

or $y_2(x) = u(x)y_1(x)$. The function $u(x)$ can be found by substituting

$$y_2(x) = u(x)y_1(x)$$

into the given differential equation.

Consider the derivatives $y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + 2u'y_1' + uy_1''$ and substituting these in (2.5) we get

$$(u''y_1 + 2u'y_1' + uy_1'') + p(x)(u'y_1 + uy_1') + q(x)uy_1 = 0$$

and simplifying this gives us

$$u''y_1 + u'(2y_1' + p(x)y_1) + u(y_1'' + p(x)y_1' + q(x)y_1) = 0.$$

But y_1, by assumption, is a solution of (2.5) and hence

$$u(y_1'' + p(x)y_1' + q(x)y_1) = 0$$

which implies

$$u''y_1 + u'(2y_1' + p(x)y_1) = 0.$$

This is a second order DE in u. Let $u' = z$. Then $u'' = z'$. Using separation of variables we get

$$\frac{z'}{z} = \frac{-2y_1'}{y_1} - p$$

which is a first order DE and hence the name **reduction of order** and integrating and taking the constant zero gives us $\ln z = -2\ln y_1 - \int p dx$. This implies

$$z = \frac{1}{y_1^2} e^{-\int p dx}.$$

But $z = u'$. Then

$$u' = \frac{1}{y_1^2} e^{-\int p dx}$$

and then

$$\frac{y_2}{y_1} = u = \int \left(\frac{1}{y_1^2} e^{-\int p dx}\right) dx.$$

Therefore, the second solution for the given equation is

$$y_2 = y_1 \int \left(\frac{1}{y_1^2} e^{-\int p dx}\right) dx.$$

Example 2.2.6. *The function $y_1(x) = x$ is a solution of the homogenous DE*

$$(x^2 - 1)y'' - 2xy' + 2y = 0.$$

Solve the given DE.

Solution

The given equation is equivalent to $y'' + p(x)y' + q(x)y = 0$, where

$$p(x) = \frac{-2x}{x^2 - 1} \qquad \textit{and} \qquad q(x) = \frac{2}{x^2 - 1}.$$

If y_2 is a second solution of the given equation then $y_2(x) = u(x)y_1(x)$, where

$$u(x) = \int \left(\frac{1}{x^2} e^{-\int \left(\frac{-2x}{x^2-1}\right) dx}\right) dx = \int \left(\frac{1}{x^2} e^{\ln |x^2 - 1|}\right) dx = \int (1 - \frac{1}{x^2}) dx = x + \frac{1}{x}.$$

Therefore, $y_1(x) = x$ and $y_2(x) = u(x)y_1(x) = x^2 + 1$ are two linearly independent solutions of the given equation and hence the general solution of the given equation is $y(x) = c_1 x + c_2(x^2 + 1)$, where c_1 and c_2 are constants.

.... Other possible reduction methods, such as Bernoli or Ricati (one of the two) should be mentioned here and one should be indicated in the exercises ....

2.3 Homogeneous LODE with Constant Coefficients

Definition 2.3.1. *A Differential Equation*

$$b_n y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0, \tag{2.6}$$

where $b_0, b_1, \ldots, b_n$ are all real constants, is called a Homogenous Linear Differential Equation of constant coefficients.

Let $f(x)$ be any solution of (2.6) in $[a, b]$. Then

$$b_n f^{(n)}(x) + b_{n-1} f^{(n-1)}(x) + \cdots + b_1 f'(x) + b_0 f(x) = 0 \text{ for all } x \in [a, b].$$

Hence the derivatives of f are linearly dependent since at least one of the coefficients $b_0, b_1, \ldots, b_n$ is different from zero.

The simplest case with this property is a function f such that

$$f^{(k)}(x) = cf(x), \forall x \in [a, b]$$

for some constant c.

Let $f(x) = e^{\lambda x}$. Then $f^k(x) = \lambda^k f(x) = \lambda^k e^{\lambda x}$ which implies $c = \lambda^k$.

Thus we will look for the solution of (2.6) in the form $y = e^{\lambda x}$ where the constant λ will be chosen so that $y = e^{\lambda x}$ does satisfy the equation (2.6).

Now insert $y = e^{\lambda x}$ into (2.6) to get;

$$(b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + b_0) e^{\lambda x} = 0.$$

Hence, if $e^{\lambda x}$ is a solution of the equation in (2.6), then λ should satisfy:

$$b_0 \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \lambda + b_n = 0, \tag{2.7}$$

since $e^{\lambda x} \neq 0$ for all $x \in \mathbb{R}$.

Definition 2.3.2. *The algebraic equation (2.7) is called an* **Auxiliary equation** *or* **characteristic equation** *of the given differential equation in (2.6).*

There are 3 different cases of the roots of (2.7).

Case 1. Distinct Real Roots

Suppose that (2.7) has n distinct roots, $\lambda_1, \lambda_2, \dots \lambda_n$ where $\lambda_i \neq \lambda_j$, for $i \neq j$. Then, the solutions $e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_n x}$ are linearly independent. (Use the Wronskian to prove this.)

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the n distinct real roots of (2.7), then the general solution of (2.6) is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x} = \sum_{i=1}^{n} c_i e^{\lambda_i x},$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Example 2.3.1. 1. *For the differential equation $y'' - 3y' + 2y = 0$, the characteristic equation is: $\lambda^2 - 3\lambda + 2 = 0$, and $\lambda_1 = 2$ and $\lambda_2 = 1$ are the two distinct real roots of this characteristic equation. Hence, the general solution of the given differential equation is $y(x) = c_1 e^{2x} + c_2 e^{x}$.*

2. *For the differential equation $y''' - 4y'' + y' + 6y = 0$, the corresponding characteristic equation is: $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$ with distinct real roots $\lambda_1 = 2, \lambda_2 = 3$ and $\lambda_3 = -1$. Therefore, the general solution of the give equation is $y(x) = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{-x}$.*

Case 2. Repeated Real Roots

To understand the situation let us consider the following example.

Example 2.3.2. *Consider the DE $y'' - 6y' + 9y = 0$. Then, its characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$, which implies $(\lambda - 3)^2 = 0$. Therefore, $\lambda_1 = \lambda_2 = 3$, which is a repeated real root. One of the solutions of the given linear differential equation is e^{3x}.*

Let $y_1(x) = e^{3x}$. The given equation will have two linearly independent solutions and the second solution can be found by using the method of **reduction of order.** Let y_2 be another solution so that y_1 and y_2 are linearly independent. Then $y_2 = uy_1$, where

$$u(x) = \int \left(\frac{e^{-\int -6dx}}{\left(e^{3x}\right)^2} \right) dx = \int \left(\frac{e^{6x}}{e^{6x}} \right) dx = \int 1 dx = x.$$

Therefore $y_2(x) = xe^{3x}$ and $y(x) = c_1 e^{3x} + c_2 x e^{3x}$ is a general solution for constants c_1 and c_2.

Remark 2.3.3. *Given a differential equation:*

1. *if the characteristic equation has double real root λ, then $e^{\lambda x}$ and $xe^{\lambda x}$ are two linearly independent solutions and;*

2. *if the characteristic equation has triple root λ, then the corresponding linearly independent solutions are $e^{\lambda x}, xe^{\lambda x}$ and $x^2 e^{\lambda x}$.*

Let us proof the first part of the above remark for a second order linear homogenous differential equation .

If the given DE is $ay'' + by' + cy = 0$, then its characteristic equation is $a\lambda^2 + b\lambda + c = 0$ and then $\lambda = \lambda_1 = \lambda_2 = \frac{-b}{2a}$. One of the solution of the given DE is $y_1 = e^{\lambda x}$. Then we can use the method of reduction of order to find a second solution y_2 so that y_1 and y_2 are linearly independent.

The given equation is equivalent to

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

and $y_2 = uy_1$, where

$$u = \int \left(\frac{e^{-\int \frac{b}{a} dx}}{\left(e^{\lambda x}\right)^2} \right) dx = \int \frac{e^{\frac{-b}{a} x}}{e^{2\lambda x}} = \int 1 dx = x,$$

since $2\lambda = \frac{-b}{a}$ and hence $y_2 = xe^{\lambda x}$.

The following theorem is a generalization for the above remark.

Theorem 2.3.4.

1. *If the characteristic equation (2.7) has the real root λ occurring k times $(i.e. \lambda_1 = \lambda_2 = \cdots = \lambda_k)$ where $k \leq n$, then the part of the general solution for (2.6) corresponding to this k fold repeated root is*

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{\lambda x}$$

2. *If further, the remaining roots are the distinct real roots $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$, the general solution of (2.6) will be:*

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} + c_3 x^2 e^{\lambda x} + \cdots + c_k x^{k-1} e^{\lambda x} + c_{k+1} e^{\lambda_{k+1} x} + \cdots + c_n e^{\lambda_n x}.$$

Example 2.3.3.

1. *Consider the Differential Equation*

$$y^{(4)} - 5y''' + 6y'' + 4y' - 8y = 0.$$

The corresponding characteristic equation is $\lambda^4 - 5\lambda^3 + 6\lambda^2 - 8 = 0$ and the roots are $\lambda_1 = \lambda_2 = \lambda_3 = 2, \lambda_4 = -1$.

Therefore, the general solution is given by $y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 e^{-x}$, where c_1, c_2, c_3 and c_4 are constants.

2. *Consider the Differential Equation* $y''' - 4y'' - 3y' + 18y = 0$ *The roots of the characteristic equation are,* $\lambda_1 = \lambda_2 = 3$ *and* $\lambda_3 = -2$ *and hence the general solution of the equation is:* $y(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-2x}$, *where* c_1, c_2 *and* c_3 *are constants.*

Case 3. Conjugate Complex Roots

Suppose the equation (2.7) has a complex root $\lambda = a + ib, a, b \in \mathbb{R}$. Then (we know from the theory of algebraic equations that) the conjugate $\bar{\lambda} = a - ib$ is also a root of (2.7) and the corresponding part of the general solution of (2.6) will be:

$$k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x}.$$

But $e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$, (by applying Euler's formula) and then

$$\begin{aligned} k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x} &= k_1 e^{ax}(\cos bx + i \sin bx) + k_2 e^{ax}(\cos b^x - i \sin bx) \\ &= e^{ax}[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx] \\ &= e^{ax}(c_1 \cos bx + c_2 \sin bx), \end{aligned}$$

where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$ are arbitrary constants from the set of complex numbers $\mathbb{C}$.

On the other hand if $a + ib = \lambda$ and $a - ib = \bar{\lambda}$ are each k fold roots of (2.7), then the part of the general solution that corresponds to this part is

$$e^{ax}[\left(c_1 + c_2 x + \cdots + c_k x^{k-1}\right)\cos bx + i\left(c_{k+1} + c_{k+2}x + \cdots c_{2k}x^{k-1}\right)\sin bx].$$

Example 2.3.4. *Solve* $y'' - 2y' + 10y = 0.$

Solution

The characteristic equation of the given equation is $\lambda^2 - 2\lambda + 10 = 0$ with roots $\lambda_1 = 1 + 3i$ and $\lambda_2 = 1 - 3i$. Then $y_1 = e^{1+3i}$ and $y_2 = e^{1-3i}$ are two independent solutions of the given equation. Therefore, $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are constants, is a general solution of the given equation. That means

$$y(x) = e^x(c_1 \cos 3x + c_2 \sin 3x).$$

2.3.1 Exercises

Exercise 2.3.5. *Solve each of the following Differential Equation.*

1. $y'' + y = 0.$
2. $y'' - 6y' + 25y = 0.$
3. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = 0$, *where* $\lambda_1 = \lambda_2 = 1 + 2i$ *and* $\lambda_3 = \lambda_4 = 1 - 2i$.

2.4 Nonhomogeneous Equations with Constant Coefficients

Consider the equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t). \tag{2.8}$$

which governs the displacement $x(t)$ of a mechanical oscillator. Here $F(t)$ is the forcing function and the equation is a non-homogeneous linear ODE with constant coefficients. There are several practical problems which can be modeled in this form.

Recall that differential equations of the form

$$b_n(x)y^{(n)} + \cdots + b_1(x)y' + b_0(x)y = f(x), \text{ where } f(x) \neq 0 \tag{2.9}$$

are called nonhomogeneous differential equations. In the previous sections we have seen how to solve homogeneous differential equations. In this section we are going to see how to solve differential equations of the form

$$b_n y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y' + b_0 y = f(x), \tag{2.10}$$

where $b_n, ..., b_0$ are constants is called a nonhomogeneous differential equation with constant coefficients. The following theorem is very important in such cases.

Theorem 2.4.1 (Homogeneous-Nonhomogeneous Solution Relation)**.** *Consider the nonhomogeneous differential equation*

$$b_n(x)y^{(n)} + \cdots + b_1(x)y' + b_0(x)y = f(x), \text{ where } f(x) \not\equiv 0.$$

If $f(x) \equiv 0$, *then the equation becomes a homogeneous equation.*

1. *If* y_1 *and* y_2 *are solutions of the nonhomogeneous equation on an interval I, then* $y_1 - y_2$ *is also a solution of the homogeneous equation in the interval I.*

2. *If y_1 is a solution of the nonhomogeneous equation and y_2 is a solution of the homogeneous equation in an interval I, then $y_1 + y_2$ is a solution of the nonhomogeneous equation in the interval I.*

The following remark follows directly from the theorem given above.

Remark 2.4.2. *Suppose $y_h(x)$ denote the general solution of the homogeneous part and $y_p(x)$ denote the particular solution of the DE:Then the general solution of (2.10) is given by $y(x) = y_h(x) + y_p(x)$.*

Theorem 2.4.3 (Superposition Principle)**.** *If $y_h(x)$ is a general solution of the homogeneous part of (2.10) on an interval $[a, b]$ and $y_{p_1}(x), y_{p_2}(x), \ldots, y_{p_k}(x)$ are particular solutions of (2.10) corresponding to $f_1(x), f_2(x), \ldots, f_k(x)$ respectively on the right hand side, then the general solution of (2.10) where, $f(x) = f_1(x) + \cdots + f_k(x)$ on $[a, b]$, is*

$$y(x) = y_h(x) + y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x).$$

The above result is called a **Superposition Principle.** It tells us that the response y_p to a superposition of inputs (the forcing functions $f_1 + f_2 + \cdots + f_k$) is the superposition of their individual outputs $(y_{p_1}, \ldots, y_{p_k})$.

We are going to use these results in solving nonhomogeneous differential equations with constant coefficients in the coming sections.

2.4.1 The undetermined coefficient method

Definition 2.4.4.

1. *A function is called an* **undetermined coefficient** *function (UC function) if it is either:*
   a) *a function defined by (a linear combination) of the following*
      i) $x^n, n = 0, 1, 2, \ldots,$
      ii) e^{ax}, *where a is any non-zero constant*
      iii) $\sin(bx + c)$, *where b, c are constants, such that $b \neq 0$.*
      iv) $\cos(bx + c)$, *where b, c are constants, such that $b \neq 0$.*

      or
   b) *a function which is defined as a finite product of two or more functions of the above 4 types.*

2. *Let f be an UC function. The set S of functions consisting of f and all the derivatives of f which are mutually LI UC functions is said to be the UC set of function f, if S is a finite set and we shall denote it by S.*

Example 2.4.1.

1. *Let $f(x) = x^3$. Then f is UC function.*

$$f'(x) = 3x^2, \qquad x^2 \textit{ is UC function.}$$

$$f''(x) = 6x, \qquad x \textit{ is UC function.}$$

$$f'''(x) = 6, \qquad 1 \textit{ is UC function.}$$

Therefore, $S = \{1, x, x^2, x^3\}$.

2. *Let $f(x) = \sin(2x)$. Then f is an UC function.*

$$f'(x) = 2\cos(2x), \qquad \cos(2x) \textit{ is UC function.}$$

$$f''(x) = -4\sin(2x), \qquad \sin(2x) = f(x).$$

Therefore, $S = \{\sin(2x), \cos(2x)\}$.

3. *Let $g(x) = 2xe^{-x}$. g is an UC function (as a product of UC function).*

$$g'(x) = 2e^{-x} - 2xe^{-x} \textit{ and } e^{-x}, xe^{-x} \textit{ are UC functions.}$$

$$g''(x) = -4e^{-x} + 2xe^{-x} \textit{ and } e^{-x}, xe^{-x} \textit{ are UC functions.}$$

Therefore, $S = \{e^{-x}, xe^{-x}\}$.

4. *The function*

$$f(x) = \frac{1}{x}$$

is not a UC function.

We outline the method by using the following example.

Example 2.4.2. *Consider the differential equation*

$$y^{(4)} - y'' = 3x^2 - \sin 2x \tag{2.11}$$

1) *First find the general solution to the homogeneous part*

$$y^{(4)} - y'' = 0.$$

The characteristic equation of the given equation is $\lambda^4 - \lambda^2 = 0$. *Then* $\lambda^2(\lambda^2 - 1) = 0$ *and hence* $\lambda_1 = \lambda_2 = 0$ *and* $\lambda_3 = 1, \lambda_4 = -1$. *Therefore, the general solution is:*

$$y_h(x) = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}.$$

2) *The forcing function (non- homogeneous term) is a combination of* x^2 *and* $\sin 2x$.

Next find the set of UC functions corresponding to the component functions

$$f_1(x) = 3x^2$$

with $S_1 = \{x^2, x, 1\}$ *and*

$$f_2(x) = -\sin 2x$$

with $S_2 = \{\sin 2x, \cos 2x\}$.

To find a particular solution $y_{p_1}(x)$ *corresponding to* $f_1(x)$, *tentatively we seek it to be a linear combination of the functions in* S_1, *i.e.*

$$y_{p_1}(x) = Ax^2 + Bx + C,$$

where A,B and C are called the **undetermined constants.**

- *Check each term in* $y_{p_1}(x)$ *for duplication with terms in* $y_h(x)$. *Here the* Bx *and C terms are constant multiples of* $c_2 x$ *and* c_1 *respectively.*
- *If there is any duplicate, then successively multiply each member of* S_i *by the lowest positive integral power of* x, *until (so that) the resulting revised set contains no duplicate of the terms in the homogeneous (and previously found particular* y_{p_i}*'s) solutions.*
  - $y_{p_1}(x) = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$ *still a duplicate is there,*
  - $y_{p_1}(x) = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2$, *no more duplicate.*
- *Substitute the final revised form into the equation and determine the coefficients A,B and C.*

- $y_{p_1}^{(4)} - y_{p_1}''(x) = 3x^2$ *which implies* $24A - 12Ax^2 - 6Bx - 2C = 3x^2$.

  Equating the coefficients of like terms we get:

$$\begin{cases} -12A = 3 \\ -6B = 0 \\ 24A - 2C = 0 \end{cases}$$

This implies, $A = \frac{-1}{4}$, $B = 0$ *and* $C = -3$.

Therefore,

$$y_{p_1}(x) = -\frac{1}{4}x^4 - 3x^2.$$

Next, we need to find $y_{p_2}(x)$ *which corresponds to* $f_2(x) = -\sin 2x$. *We seek* $y_{p_2}(x)$ *to be a linear combination of the elements of* S_2, *that is,*

$$y_{p_2}(x) = D\sin 2x + E\cos 2x.$$

- *Check for a duplicate both in* $y_h(x)$ *and* $y_{p_1}(x)$. *– No duplicate.*
- *Then substitute in* $y^{(4)} - y'' = -\sin 2x$ *which implies*

$$y_{p_2}^{(4)}(x) - y_{p_2}''(x) = -\sin 2x.$$

Hence,

$$(2^4 D\sin 2x + 2^4 E\cos 2x) - (-2^2 D\sin 2x - 2^2\cos 2x) = -\sin 2x.$$

Therefore, $20D\sin 2x + 20E\cos 2x = -\sin 2x$ *and then*

$$20D = -1 \iff D = -\frac{1}{20}$$

and $20E = 0$. *Therefore,*

$$y_{p_2}(x) = -\frac{1}{20}\sin 2x.$$

Hence the general solution of (2.11) is:

$$y(x) = y_h(x) + y_{p_1}(x) + y_{p_2}(x) = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} - \frac{1}{4}x^4 - 3x^2 - \frac{1}{20}\sin 2x.$$

Example 2.4.3. *Solve* $y'' - 2y' - 3y = 2e^{-x} - 10\sin x$.

Let $F(x) = 2e^{-x} - 10\sin x$, $f_1 = 2e^{-x}$, $f_2 = -10\sin x$. *Then* $S_1 = \{e^{-x}\}$ *and* $S_2 = \{\sin x, \cos x\}$

- *Solution of the homogeneous part* $y'' - 2y' - 3y = 0$.

  Then $\lambda^2 - 2\lambda - 3 = 0$ *and hence* $\lambda_1 = 3, \lambda_2 = -1$. *Therefore,* $y_h(x) = C_1e^{3x} + C_2e^{-x}$.

- *Particular solution corresponding to* $f_1(x) = 2e^{-x}$: $y_{p_1}(x) = Be^{-x}$ *which duplicates with* C_2e^{-x}.

  This implies, $y_{p_1}(x) = Bxe^{-x}$, *– no more duplicate.*

- *Insert this into* $y'' - 2y' - 3y = 2e^{-x}$ *to get*

$$y''_{p_1} - 2y'_{p_1} - 3y_{p_1} = 2e^{-x}.$$

  This implies, $(-2Be^{-x} + Bxe^{-x}) - 2(Be^{-x} - Bxe^{-x}) - 3Bxe^{-x} = 2e^{-x}$.
  Hence $-4Be^{-x} = 2e^{-x} \iff B = -\frac{1}{2}$. *Therefore,*

$$y_{p_1}(x) = -\frac{1}{2}xe^{-x}.$$

- *Particular solution corresponding to* $f_2(x) = -10\sin x$.
  Let $y_{p_2}(x) = D\sin x + E\cos x$ *(No duplicate both in* y_h *and* y_{p_1}*).*
  Then, $y''_{p_2} - 2y'_{p_2} - 3y_{p_2} = -10\sin x$. *This implies,*

$$(-D\sin x - E\cos x) - 2(D\cos x - E\sin x) - 3(D\sin x + E\cos x) = -10\sin x.$$

  Simplifying this gives us, $(2E - 4D)\sin x + (-2D - 4E)\cos x = -10\sin x$.
  Therefore,

$$\begin{cases} 2E - 4D = -10 \\ -2D - 4E = 0 \end{cases}$$

  which implies $D = \dfrac{20}{3}$ *and* $E = \dfrac{10}{3}$. *Then,* $y_{p_1}(x) = \dfrac{20}{3}\sin x + \dfrac{10}{3}\cos x$.
  Therefore, the general solution is

$$y(x) = C_1e^{3x} + C_2e^{-x} - \frac{1}{2}xe^{-x} + \frac{20}{3}\sin x + \frac{10}{3}\sin x + \frac{10}{3}\cos x.$$

Exercise 2.4.5. *Solve each of the following DEs.*

1. $y'' - gy = 4 + 5\sinh 3x$
2. $y'' - 2y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

2.4.2 Variation of Parameters

The Undetermined Coefficient method is easier to apply, but works only for constant coefficients and certain types of non-homogeneous terms (or forcing functions). If the forcing function is, for example, of the form $f(x) = \tan x$ or $f(x) = \dfrac{x+1}{x^2+1}$, then both of them are not UC functions. and hence we can not employ the method of undetermined coefficients in those cases. Hence, we need another method which works for more general set of problems. In this subsection we will consider the method of **Variation of Parameters** for a second order linear ordinary differential equation.

Consider the following second order linear differential equation.

$$y'' + b_1(x)y' + b_2(x)y = f(x), \tag{2.12}$$

where b_1, b_2 and f are continuous functions. Suppose that the general solution for the homogeneous part of (2.12) is

$$y_h(x) = c_1y_1(x) + c_2y_2(x).$$

Now we want to get a particular solution corresponding to $f(x)$ and this can be done by varying the constants, c_1 and c_2 with respect to x. If y_p is a particular solution corresponding to $f(x)$, then

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

We differentiate and substitute it in (2.12) to get

$$y''_p(x) + b_1(x)y'_p + b_2(x)y_p = f(x).$$

But $y'_p = c_1y'_1 + c_2y'_2 + c'_1y_1 + c'_2y_2$.

Since we are going to have only one equation with two variable functions c_1 and c_2, we are free to choose a condition which simplifies the equation. Therefore, we take the condition

$$c'_1y_1 + c'_2y_2 = 0.$$

This will simplify the equation as

$$y''_p = c_1y''_1 + c'_1y'_1 + c'_2y'_2 + c_2y''_2$$

and after simplification, the equation (2.12) becomes

$$c_1(y''_1 + b_1y'_1 + b_2y_1) + c_2(y''_2 + b_1y'_2 + b_2y_2) + c'_1y'_1 + c'_2y'_2 = f.$$

Since y_1 and y_2 are linearly independent solutions for the homogeneous part of equation (2.12) we have the following system of equations:

$$\begin{cases} c_1'y_1' + c_2'y_2' = f \\ c_1'y_1 + c_2'y_2 = 0, \end{cases} \tag{2.13}$$

which is a system of two algebraic equations in c_1' and c_2'. Then (2.13) has a unique solution if the determinant of the coefficient matrix is non-zero, that is,

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

However, the above determinant is the Wronskian of the functions y_1 and y_2. Since y_1 and y_2 are LI functions, then

$$\mathbf{W}_{[\mathbf{y_1},\mathbf{y_2}]}(x) \neq 0.$$

Hence by **Cramer's** rule we have:

$$\mathbf{c_1'(x)} = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W(x)} = \frac{W_1(x)}{W(x)}$$

and

$$\mathbf{c_2'(x)} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1 & f \end{vmatrix}}{W(x)} = \frac{W_2(x)}{W(x)}$$

By integrating both sides we will get:

$$y_p(x) = \left[\int \frac{\mathbf{W_1}(x)}{\mathbf{W}(x)}dx\right] y_1(x) + \left[\int \frac{\mathbf{W_2}(x)}{\mathbf{W}(x)}dx\right] y_2(x).$$

Example 2.4.4. *Solve the differential equation*

$$y'' - 4y = 8x.$$

Solution

First solve the homogeneous equation $y'' - 4y = 0$.

Then the characteristic equation is $\lambda^2 - 4 = 0$, *which implies* $\lambda = \pm 2$. *If* y_1 *and* y_2 *are two linearly independent solutions of the equation* $y'' - 4y = 0$, *then* $y_1 = e^{2x}, y_2 = e^{-2x}$. *Therefore,*

the general solution of the homogeneous equation is $y_h(x) = c_1e^{2x} + c_2e^{-2x}$ *and*

$$\mathbf{W(x)} = \begin{vmatrix} y_1 & y_2 \\ `y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4,$$

$$\mathbf{W_1(x)} = \begin{vmatrix} 0 & y_2 \\ 8x & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{2x} \\ 8x & -2e^{-2x} \end{vmatrix} = -8xe^{-2x}$$

Therefore,

$$c_1(x) = \int \frac{W_1(x)}{W(x)}dx = \int \frac{-8xe^{-2x}}{-4}dx = 2\int xe^{-2x}dx = -xe^{-2x} + e^{-2x}$$

and similarly

$$c_2(x) = \int \frac{W_2(x)}{W(x)}dx = \int \frac{8xe^{2x}}{-4}dx = -2\int xe^{2x}dx = xe^{-2x} - e^{-2x}.$$

Therefore, $y_p(x) = c_1(x)e^{2x} + c_2(x)e^{-2x}$ *a particular solution and the general solution for the problem is* $y(x) = y_h(x) + y_p(x)$.

Remark 2.4.6. *This method looks easier when the integrads (or the quotients of the Wronskian) are simple. However, it could be very difficult to get the particular solution when the integrand is complicated.*

2.5 The Laplace Transform Method to Solve ODEs

In the previous sections we have discussed how to solve differential equations of the form:

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x) \tag{2.14}$$

by finding the general solutions and then evaluate the arbitrary constants in accordance with the given initial conditions. However, the solution methods mainly dependent on the structure of the forcing function $f(x)$. Moreover, all the coefficients are assumed to be constants. To address problems with more general forcing function and some form of variable coefficients, we discuss the use of Laplace transform as possible alternative.

Definition 2.5.1 (Laplace Transform)**.** *The Laplace Transform of a function* $f(t)$, *if it exists, is denoted by* $\mathcal{L}\{f(t)\}$ *is given by,*

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt,$$

where s is a real number called a **parameter of the transform**. *For short we may write,*

$$\mathcal{F}(s) \text{ to denote } \int_0^\infty e^{-st} f(t)dt. \text{ i.e., } \mathcal{F}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Example 2.5.1. *Find the Laplace Transform of the constant function $f(t) = 1$.*

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \times 1dt.$$

Solution

$$\begin{aligned} \mathcal{L}\{1\} &= \int_0^\infty e^{-st} \times 1dt \\ &= \lim_{T\to\infty} \int_0^T e^{-st}dt \\ &= \lim_{T\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^T \\ &= \lim_{T\to\infty} \left[\frac{-1}{s}e^{-sT} + \frac{1}{s}\right] \\ &= \begin{cases} \frac{1}{s}; & \text{if } s > 0 \\ \infty; & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, $\mathcal{L}\{1\} = \dfrac{1}{s}$, if $s > 0$.

Table of some basic Laplace Transforms

Function $(f(t))$	Laplace Trnsf. $(F(s))$
1	$\frac{1}{s}, s > 0$
$t^n, n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, s > 0$
e^{tk}	$\frac{1}{s-k}, s > k$
$t^n e^{kt}$	$\frac{n!}{(s-k)^{n+1}}, s > k$
$\sin(kt)$	$\frac{k}{s^2+k^2}, s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}, s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2}, s > \lvert k\rvert$
$\cosh(kt)$	$\frac{s}{s^2-k^2}, s > \lvert k\rvert$

From the table above we have

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \text{ for } s > a.$$

Thus the inverse operator applied on $\frac{1}{s-a}$ will give us back the function e^{at}

$$\text{i.e., } \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at} \text{ for } s > a.$$

In general, $\mathcal{L}^{-1}$, the inverse Laplace Operator, is given by

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds,$$

(where γ is a positive real number), which is a complex improper integral.

Properties

Here below we state some important properties of the transform in a serious of theorems without proof.

Theorem 2.5.2 (Linearity)**.**

(a) If $u(t), v(t)$ are functions and α, β are any constants, then

$$\mathcal{L}\{\alpha u(t) + \beta v(t)\} = \alpha\mathcal{L}\{u(t)\} + \beta\mathcal{L}\{v(t)\}.$$

(b) For any functions $U(s), V(s)$ and any given scalars α, β, we have

$$\mathcal{L}^{-1}\{\alpha U(s) + \beta V(s)\} = \alpha\mathcal{L}^{-1}\{U(s)\} + \beta\mathcal{L}^{-1}\{V(s)\}.$$

Example 2.5.2. *Evaluate the following transforms*

1. $\mathcal{L}\{3t + 5e^{-2t}\}$.
2. $\mathcal{L}\{\cos^2 3t\}$.
3. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\}$.

Solutions

1. $\mathcal{L}\{3t+5e^{-2t}\}$; Applying the liniarity property we get,

$$\begin{aligned}\mathcal{L}\{3t+5e^{-2t}\} &= 3\mathcal{L}\{t\}+5\mathcal{L}\{e^{-2t}\}\\ &= 3\left(\frac{1}{s^2}\right)+5\left(\frac{1}{s+2}\right)\\ &= \frac{3}{s^2}+\frac{5}{s+2}=\frac{5s^2+3s+6}{s^2(s+2)}\end{aligned}$$

2. $\mathcal{L}\{\cos^2 3t\}$; Using half angle formula we can get

$$\mathcal{L}\{\cos^2 3t\}=\mathcal{L}\{\frac{1+\cos 6t}{2}\}=\mathcal{L}\{\frac{1}{2}+\frac{1}{2}\cos 6t\}.$$

Then by linearity we have $\mathcal{L}\{\cos^2 3t\}=\frac{1}{2}\mathcal{L}\{1\}+\frac{1}{2}\mathcal{L}\{\cos 6t\}$.

Then we are now able to read the transforms of 1 and $\cos 6t$ from the table and get,

$$\mathcal{L}\{\cos^2 3t\}=\frac{1}{2}\left(\frac{1}{s}\right)+\frac{1}{2}\left(\frac{s}{s^2+6^2}\right)=\frac{s^2+18}{s(s^2+36)}.$$

3. Using partial fractions we have

$$\frac{s^2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

$$\Rightarrow s^2=A(s+1)^2+B(s+1)+C=As^2+(2A+B)s+(A+B+C).$$

$$\Rightarrow A=1,\quad B=-2,\quad C=1.$$

Hence we can rewrite the inverse transform and apply linearity to get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{-2}{(s+1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}\\ &= e^{-t}-2te^{-t}+\frac{1}{2}t^2e^{-t}\\ &= (1-2t+\frac{1}{2}t^2)e^{-t}.\end{aligned}$$

The other important property that leads us to use the Laplace transform in solving ordinary differential equation is how the transform performs on the derivative.

Theorem 2.5.3 (Transform of the derivative)**.** *Let $f(t)$ be continuous and $f'(t)$ be piecewise continuous on some interval $[0,t_o]$ for every finite t_o, and let $|f(t)|<Ke^{ct}$ for some constants K,T, and c and for all $t>T$. Then the transform $\mathcal{L}\{f'(t)\}$ exists for all $s>c$ and*

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0).$$

Example 2.5.3. *Use the Laplace transform method to solve the initial-value problem.*

$$y'+2y=0 \text{ with } y(0)=1.$$

Solution

Applying Laplace transform on both sides of the equation we have

$$\mathcal{L}\{y'+2y\}=\mathcal{L}\{0\}\quad\Rightarrow\quad\mathcal{L}\{y'(t)\}+2\mathcal{L}\{y(t)\}.$$

Now, letting $\mathcal{L}\{y(t)\}:=Y(s)$, we get the algebraic equation,

$$sY(s)-y(0)+2Y(s)=0\quad\Rightarrow\quad Y(s)=\frac{1}{s+2}.$$

Therefore, reading from the transform table we get

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\{\frac{1}{s+2}\}=e^{-2t}.$$

i.e., $y(t)=e^{-2t}$ is the solution for the differential equation. $\boxdot$

We can also use the Laplace method to solve higher order equations with constant coefficients.

The following property of the transform, which is the continuouation of the above theorem, is required.

Theorem 2.5.4. *Let $f(t)$ be continuous and $f^{(n)}(t)$ be piecewise continuous on some interval $[0,t_o]$ for every finite t_o, and let $|f(t)|<Ke^{ct}$ for some constants K,T, and c and for all $t>T$. Then we have*

$$\mathcal{L}\{f^{(n)}(t)\}=s^n\mathcal{L}\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0).$$

Theorem 2.5.5 (First shifting theorem)**.** *If $\mathcal{L}\{f(t)\}=\mathcal{F}(s)$ for $Re(s)>b$, then $\mathcal{L}\{e^{at}f(t)\}=\mathcal{F}(s-a)$ for $Re(s)>a+b$.*

The proof of this theorem is easy to see using the definition.

Example 2.5.4. *Find the Laplace transform for the function $f(t)=e^{3t}\cos 4t$.*

Solution

Recall that $\mathcal{L}\{\cos 4t\} = \dfrac{s}{s^2+4^2}$.
Then using the first shifting theorem we get

$$\mathcal{L}\{e^{3t}\cos 4t\} = \frac{s-3}{(s-3)^2+4^2}.$$

Example 2.5.5. *Find the inverse Laplace transform for the function* $\mathcal{F}(s) = \dfrac{s}{s^2+s+1}$.

Solution

First let us rewrite the function $\mathcal{F}(s)$ as

$$\mathcal{F}(s) = \frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

and hence,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}.$$

Then using the first shifting theorem, we have,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-t/2}\cos\frac{\sqrt{3}t}{2} - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}t}{2}.$$

Consider the general Laplace transform formula

$$\mathcal{F}(s) = \int_0^\infty e^{-st}f(t)dt.$$

Taking the derivative with respect to s on both sides we get,

$$\mathcal{F}'(s) = \int_0^\infty (-t)e^{-st}f(t)dt = \mathcal{L}\{-tf(t)\}.$$

By further differentiating the above equation with respect to s, we get

$$\mathcal{F}''(s) = \mathcal{L}\{t^2 f(t)\}.$$

In general we have

Theorem 2.5.6 (Derivative of the transform)**.** *For a piecewise continuous function $f(t)$ and for any positive integer n, it holds that*

$$\mathcal{L}\{(-1)^n t^n f(t)\} = \mathcal{F}^{(n)}(s).$$

The formula in this theorem can be used to find transforms of functions of the form $x^n f(x)$ when the Laplace transform of $f(t)$ is known.

Exercise 2.5.7. *Use the Laplace transform method to solve*

$$xy'' + (2x+3)y' + (x+3)y = 3e^{-x}; \quad y(0) = 0, y'(0) = 1.$$

Remark 2.5.8. *The main idea in using Laplace transform in solving ODEs is that, it transforms the differential equation into an algebraic equation. Once the transformation is completed, we seek for a solution to $\mathcal{L}\{y(t)\}$ algebraically. Then the final step will be to get back the value of $y(t)$ using the inverse Laplace transform.*

2.6 The Cauchy-Euler Equation

In this section we are going to consider linear differential equations where the coefficients are variables with some special forms.

Definition 2.6.1. *The linear differential equation with variable coefficient of the form:*

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = F(x) \tag{2.15}$$

where $a_0, a_1, \ldots, a_n$ are constants is called the **Cauchy-Euler Equation.**

Example 2.6.1. *The linear differential equation $3x^2y'' - 11xy' + 2y = \sin x$ is a Cauchy- Euler equation.*

To solve Cauchy-Euler DEs first we reduce the given DE into a linear differential equation with constant coefficients and solve the given equation with the methods derived in the previous sections.

Theorem 2.6.2. *The transformation $x = e^t, t \in \mathbb{R}$ reduces the Cauchy-Euler DE to a linear DE with constant coefficients.*

Let us consider the case when $n = 2$. In this case the equation is:

$$a_2x^2y'' + a_1xy' + a_0y = F(x) \tag{2.16}$$

Let $x = e^t$. Then by solving for t we get $t = \ln x$ for $x > 0$ (or $x = -e^t$ if $x < 0$) and

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}.\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\frac{d^2y}{dt^2}.\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting into (2.15) we get:

$$a_2x^2\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1x.\frac{1}{x}\frac{dy}{dt} + a_2y = F(e^t).$$

This implies,

$$a_2\frac{d^2y}{dt^2} + (a_1 - a_2)\frac{dy}{dt} + a_0y = F(e^t).$$

Then

$$A_2\frac{d^2y}{dt^2} + A_1\frac{dy}{dt} + A_0y = G(t), \tag{2.17}$$

where $A_2 = a_2, A_1 = a_1 - a_2, A_0 = a_0$ and $F(e^t) = G(t)$, which is a second order linear differential equation with constant coefficients.

Example 2.6.2. *Solve each of the following DEs.*

1. $x^2y'' - 2xy' + 2y = 0.$
2. $3x^2y'' - 11xy' + 2y = \sin x.$
3. $x^2y'' - 2xy' + 2y = x^3.$

Solution

1. Let $x = e^t$. Since $a_2 = 1, a_1 = -2$ and $a_0 = 2$ we have $A_2 = a_2 = 1$, $A_1 = a_1 - a_2 = -3$ and $A_0 = a_0 = 2$ which reduces the given equation to $y'' - 3y' + 2y = 0$ which is a homogenous second order linear differential equation with constant coefficients. Then the characteristic equation of the equation is $\lambda^2 - 3\lambda + 2 = 0$ which has eigenvalues $\lambda_1 = 1, \lambda_2 = 2$.

Therefore, the differential equation $y'' - 3y' + 2y = 0$ has a general solution $y(t) = c_1e^t + c_2e^{2t}$ and since $x = e^t$ the DE $x^2y'' - 2xy' + 2y = 0$ has a general solution

$$y(x) = c_1x + c_2x^2,$$

where c_1 and c_2 are arbitrary constants.

2. Let $x = e^t$. Since $a_2 = 3, a_1 = -11, a_0 = 2$, we have $A_2 = a_2 = 3$, $A_1 = a_1 - a_2 = -11 - 3 = -14$, and $A_0 = a_0 = 2$, which reduce the given equation to $3y'' - 14y' + 2y = \sin(e^t)$ which is a DE with constant coefficients.
3. The given equation is transformed into $y'' - 3y' + 2y = e^{3t}$, with the substitution $x = e^t$, which is a DE with constant coefficients.

Example 2.6.3 (Application)**.** *Consider a mechanical oscillator.*

Let $F_G = mg$, where m is mass of the object on the spring and g is the gravity and $|\hat{F_R}| = kx$, (Hook's law) where k is the spring stiffness constant and x is the the distance moved by the mass m.

If F_D is a damping force for small velocity of mass, then $|F_D| = C\left|\frac{dx}{dt}\right|$, where $C > 0$ is called a damping constant.

Therefore, the final form of our governing equation of motion is of the form:

$$mx'' + Cx' + kx = mg = F(t).$$

If F is a variable force then the problem will be a second order nonhomogeneous differential equation, and can be solved using one of the previously discussed methods.

2.7 *The Power Series Solution Method

Recall that, the Taylor series of a given smooth function f about a point a_o is:

$$TS\ f|_{a_o} = \sum_{n=0}^{\infty}\frac{f^{(n)}(a_o)}{n!}(x - a_o)^n;$$

If this series converges in some interval $|x - a_o| < r$, and is equal to f, then we call f is *analytic at* a_o and r is the radius of convergence.

If f is not analytic at a_o, we call it is *singular at* a_o.

The method mainly uses the following theorem

Theorem 2.7.1 (Power Series Solution). *If the functions p and q are analytic at a point c_o, then every solution of the DE*

$$y'' + p(x)y' + q(x)y = 0 \tag{2.18}$$

is also analytic at c_o, and can be found in the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - c_o)^n.$$

Moreover, the radius of convergence of every solution is at least as large as the smaller of the radii of convergence of $TS\ p|_{c_o}$ and $TS\ q|_{c_o}$.

Example 2.7.1. *Solve the DE*

$$(x-1)y'' + y' + 2(x-1)y = 0 \tag{2.19}$$

on the interval $[4, \infty)$ with initial conditions $y(4) = 5$ and $y'(4) = 0$.

Solution Procedure:

- Convert the problem to the form of equation (2.18) in the above Theorem
- Check analyticity of the coefficient functions $p(x)$ and $q(x)$ at the point x_o (the given initial point).
- Substitute into the equation (2.19) the general solution

  $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

  and determine the values of the coefficients a_n, for each $n = 0, 1, 2, \ldots$.

Frobenius Method

Consider again a second order equation with variable coefficients,

$$h(x)y'' + p(x)y' + q(x)y = 0 \tag{2.20}$$

If $h(x) \neq 0$ for some x we can equivalently have

$$y'' + \frac{p(x)}{h(x)}y' + \frac{q(x)}{h(x)}y = 0, \text{ for } h(x) \neq 0.$$

If $h(x) \neq 0$ for all x we can simply apply the power series solution method. But if $h(x) = 0$ for some x the resulting equation will be different from the original one at those points x where $h(x) = 0$.

Definition 2.7.2.

1. *A point x_o is said to be an* ordinary point *of equation (2.20) if $h(x_o) \neq 0$ and $\frac{p(x)}{h(x)}, \frac{q(x)}{h(x)}$ are analytic at x_o. Otherwise, it is called a* singular point *of equation (2.20).*
2. *A* singular *point x_o is said to be a* regular singular point *of equation (2.20) if the function*

   $$(x - x_o)\frac{p(x)}{h(x)} \text{ and } (x - x_o)^2\frac{q(x)}{h(x)}$$

   are analytic at x_o. A non regular singular point is called an irregular singular point *of equation (2.20).*

If equation (2.20) has a regular singular point at x_o, then use the power series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^{n+r}$$

and determine the values of r and $a_n, n = 0, 1, 2, \ldots$

This last series is called a **Frobenius series** solution.

Example 2.7.2. *Use Frobenius method to solve*

$$x^2y'' + 5xy' + (x+4)y = 0$$

Ans.: $y(x) = a_o \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^{n-2}$

The general solution (also the number of different solutions) of differential equations using Frobenius Method depend upon the solution to the equation (which is called **the indicial equation**)

$$r(r-1) + b_o r + c_o = 0,$$

which forces the coefficient of x^r to be zero.

2.8 Systems of ODE of the First Order

A system of n linear first-order equations in the n unknowns $x_1(t), x_2(t), \ldots, x_n(t)$ is a system that can be written in the form:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t), \end{aligned} \tag{2.21}$$

which is called **the normal form**. In vector form this system becomes:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t),$$

where

$$\mathbf{X} = (x_1, x_2, \dots, x_n)^T, \ \mathbf{A} = (a_{ij})_{n\times n}, \text{ and } \mathbf{F}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

The system (2.21) is called homogeneous if $\mathbf{F}(t) \equiv 0$, so that $\mathbf{X}' = \mathbf{A}\mathbf{X}$ and if $\mathbf{F}(t) \not\equiv 0$ for some t, the system is nonhomogeneous.

Definition 2.8.1. *A solution vector of the system of differential equation in (2.21) over some interval I is a vector $(x_1(t), x_2(t), \dots, x_n(t))^T$ whose entries are differentiable functions that satisfies the system in (2.21) on the interval I.*

In this section, we are going to see how to solve such systems of equations. Two methods are going to be considered, the **Eigenvalue Method** and **Elimination Method.**

2.8.1 Eigenvalue Method

A. Homogeneous Systems with Constant Coefficients

Consider the system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \tag{2.22}$$

with $\mathbf{A} = (a_{ij})_{n\times n}$ be a constant matrix, that is, all the entries of $\mathbf{A}$ are constants.

Recall that, in the scalar case if $y' = ky$, then $y = ce^{kt}$, where c is a constant (by integration).

Let $\mathbf{y} = \mathbf{x}e^{\lambda t}$, where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$. Substituting this into (2.22) we get:

$$\lambda \mathbf{x} e^{\lambda t} = \mathbf{y}' = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t} \text{ and } e^{\lambda t} \neq 0.$$

This implies,

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x},$$

which is an eigenvalue problem. Once we find the eigenvalues λ_i and a corresponding eigenvector x_i, the general solution will be

$$\mathbf{y}(t) = c_1 x_1 e^{\lambda_1 t} + \cdots + c_n x_n e^{\lambda_n t},$$

where $c_1, \dots, c_n$ are constants.

Example 2.8.1. *Solve each of the following systems of linear differential equations.*

1. $\begin{cases} y_1' = -3y_1 + y_2 \\ y_2' = y_1 - 3y_2 \end{cases}$ **2.** $\begin{cases} y_1' = 2y_1 + y_2 + y_3 \\ y_2' = y_1 + 2y_2 + y_3 \\ y_3' = y_1 + y_2 + 2y_3 \end{cases}$

Solution:

1. The system can be written as:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let $\mathbf{y}(t) = \mathbf{x}e^{\lambda t}$. Then the corresponding eigenvalue problem will be:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \iff \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is equivalent to:

$$\begin{pmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which has characteristic equation

$$\left| \ A - \lambda I \ \right| = \mathbf{0} \iff \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (3+\lambda)^2 - 1 = 0.$$

This implies $\lambda^2 + 6\lambda + 8 = 0 \iff (\lambda+2)(\lambda+4) = 0$. Therefore, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -4$

a) Now, let us find an eigenvector corresponding to $\lambda_1 = -2$.

$$\begin{pmatrix} -3-(-2) & 1 \\ 1 & -3-(-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x_1 - x_2 = 0,$$

which implies $x_1 = x_2$ and hence we have $(x_1, x_2)^T = x_1(1,1)^T$. Therefore, the vector $(1,1)^T$ is an eigenvector corresponding to $\lambda_1 = -2$.

b) Next, let us find an eigenvector corresponding to $\lambda_2 = -4$.

$$\begin{pmatrix} -3-(-4) & 1 \\ 1 & -3-(-4) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies $x_1 + x_2 = 0$ and hence $x_1 = -x_1$ which implies $(x_1, x_2)^T = x_1(1, -1)^T$. Therefore, the vector $(1, -1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -4$

Hence, the general solution of the given system is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

which is equivalent to

$$y_1(t) = c_1 e^{-2t} + c_2 e^{-4t}$$
$$y_2(t) = c_1 e^{-2t} - c_2 e^{-4t}$$

2. The given system is equivalent to the system

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

That is $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where

$$\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad and \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Let $\mathbf{y}(t) = \mathbf{x}e^{\lambda t}$, $\mathbf{y}, \mathbf{x} \in \mathbb{R}^3$. Then corresponding eigenvalue problem is $\mathbf{AX} = \lambda\mathbf{X}$ with characteristic equation

$$\left| \mathbf{A} - \lambda\mathbf{I} \right| = \mathbf{0}$$

which is equivalent to the equation

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

This equation is reduced to $(\lambda-1)^2(\lambda-4) = 0$. Therefore, $\lambda_1 = 1, \lambda_2 = 4$ are the eigenvalues.

Eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ can be found as follows.

$$\begin{pmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This implies, $x_1 + x_2 + x_3 = 0$.
i.e., $(x_1, x_2, x_3)^T = (-x_2 - x_3, x_2, x_3)^T = x_2(-1, 1, 0)^T + x_3(-1, 0, 1)^T$.

Similarly, eigenvector corresponding to $\lambda_2 = 4$ is obtained to be $(1, 1, 1)^T$.

Thus, the general solution will be

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} te^t + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^4 t.$$

B. Nonhomogeneous Systems

Suppose a non homogeneous system of linear ODEs is given by:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{F}(t), \qquad \text{with} \qquad \mathbf{F}(t) \not\equiv 0.$$

The general solution for this system takes the form $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$, where $\mathbf{y}_h$ is the general solution of the corresponding homogeneous system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and $\mathbf{y}_p$ is any particular solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{F}$.

Now to find a particular solution vector $\mathbf{y}_p$ we use the method of Undetermined Coefficients. As in the scalar case, first assuming that $\mathbf{y}_p$ has the same general form as F and then find the constants. We will illustrate the method by the following example.

Example 2.8.2. *Solve the following system of DEs.*

$$\begin{aligned} y_1' &= y_1 + f_1(t) \\ y_2' &= 6y_1 - y_2 + f_2(t), \end{aligned}$$

where

a) $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, **b)** $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ *and* **c)** $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solution

The system is $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{F}(t)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}.$$

Then, the corresponding homogeneous system $y' = Ay$ has characteristic equation

$$\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 1 - \lambda^2 = 0,$$

with eigenvalues $\lambda = \pm 1$.
An eigenvector corresponding to the eigenvalue $\lambda = 1$ is $(1,3)^T$ and an eigenvector corresponding to the eigenvalue $\lambda = -1$ is $(0,1)^T$ and hence the solution to homogenous system is given by

$$\mathbf{y}_h = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}.$$

Then for each given case, we are going to find $\mathbf{y}_p$

a) Since $\mathbf{F}(t) = (2,1)^T$ which is a constant vector, then $\mathbf{y}_p$ will take the form $(p_1,p_2)^T$ which is also a constant vector. Now, substituting into the equation $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{F}$ we have:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This implies that $p_1 + 2 = 0$ and $6p_1 - p_2 + 1 = 0$ and solving this gives us $p_1 = -2$ and $p_2 = -11$. Therefore, the particular solution is $\mathbf{y_p} = (-2,-11)^T = -(2,11)^T$ and the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} - \begin{pmatrix} 2 \\ 11 \end{pmatrix}.$$

b) Here we have

$$\mathbf{F}(t) = \mathbf{p}e^{2t} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e^{2t}.$$

This implies

$$2\mathbf{p}e^{2t} = \mathbf{A}\mathbf{p}e^{2t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 2p_1 \\ 2p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ 6p_1 - p_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Since $e^t \neq 0$, by simplifying the given equation we get $2p_1 = p_1 + 2$, $2p_2 = 6p_1 - p_2 + 1$ and solving for p_1 and p_2 gives us $p_1 = 2$ and $p_2 = \frac{13}{2}$.

c) For the given DE we have

$$\mathbf{F}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

Then we get

$$\mathbf{y}_{p_1} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \sin t + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \cos t, \qquad \textit{and} \qquad y_{p_2} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} e^t$$

and solve for the constants.

2.8.2 The Method of Elimination:

In some cases it could be preferable to consider a higher order differential equation in stead of a system of first order equations (especially when the characteristic polynomial is easier to solve). It is possible to transform a system of n first order linear ODE to an n^{th} order linear ODE. The transformation requires the idea of the differential operators.
The operator $\frac{d^n}{dx^n}$, which is denoted by D^n, is called a *differential operator* and the natural number n is the power of the operator.

Example 2.8.3.

1. $D^2(x^3 + 3x) = D(3x^2 + 3) = 6x.$

2. $D^2(2x^3 - 2x^2 + 3) = 12x - 4.$

A linear combination of differential operators of the form

$$a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + D_0,$$

where $a_0, a_1, \ldots, a_n$ are constants is called an n^{th} order polynomial operator and is denoted $P(D)$ and

$$P(D)y = (a_n D^n + \cdots + a_1 D + a_1)y = a_n \frac{d^n y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 y$$

Example 2.8.4.

1. $y'' + 3y' - y = 0$ *implies* $(D^2 + 3D - 1)y = 0$

2. $y''' - 4y' = \cos x$ *implies* $(D^3 - 4D)y = \cos x.$

Definition 2.8.2.

1. *Two polynomial operators $P_1(D)$ and $P_2(D)$ are equal if and only if $P_1(D)y = P_2(D)y$ for all functions y.*

2. *The sum $P_1(D) + P_2(D)$ is obtained by first expressing P_1 and P_2 as linear combinations of the operator D and adding the coefficients of like powers of D*

3. *The product $P_1(D)P_2(D)$ is obtained by using the operator $P_2(D)$ followed by $P_1(D)$, i.e.*

$$[P_1(D)P_2(D)]y = P_1(D)[P_2(D)y].$$

Example 2.8.5. *Let us illustrate the sum and product of operators*

1. *If $P_1(D) = 3D^2 + 7D - 5, P_2(D) = D^3 + 6D^2 - 2D - 3$, then*

$$P_1(D) + P_2(D) = D^3 + gD^2 + 5D - 8.$$

2. *If $P_1(D) = 2D + 3, P_2(D) = D - 5$, then*

$$P_1(D)P_2(D) = (2D + 3)(D - 5) = 2D^2 - 7D - 15$$

Basic Properties

If $P(D)$ a differential operator, y_1, y_2 and y are functions and c is a constant then:

a. $P(D)(y_1 + y_2) = P(D)y_1 + P(D)y_2$ and

b. $P(D)(cy) = cP(D)y$.

To solve any system of linear ODE with constant coefficients by elimination method, we first write each equation using polynomial operators and treat the operators as simple constants and solve the system using linear algebraic solution methods.

Example 2.8.6. *Solve the following systems.*

1. $\begin{aligned} y_1' - y_2 &= x^2 \\ y_2' + 4y_1 &= x \end{aligned}$ **2.** $\begin{aligned} y_1' - 2y_1 + 2y_2' &= 2 - 4e^{2x} \\ 2y_1' - 3y_1 + 3y_2' - y_2 &= 0. \end{aligned}$

Solution

1. The system is equivalent to

$$\left.\begin{aligned} Dy_1 - y_2 &= x^2 \\ Dy_2 + 4y_1 &= x \end{aligned}\right\} \iff \begin{cases} Dy_1 - y_2 = x^2 \\ 4y_1 + Dy_2 = x \end{cases}$$

To eliminate y_2, apply D on the first equation. Then the equation is equivalent to:

$$\begin{aligned} D^2y_1 - Dy_2 &= 2x \\ 4y_1 + Dy_2 &= x \end{aligned}$$

Adding the two equations gives $D^2y_1 + 4y_1 = 3x \iff (D^2 + 4)y_1 = 3x$ and the characteristic equation for the homogenous part is $\lambda^2 + 4 = 0$, which implies $\lambda = \pm 2i$ and $y_{1_h} = C_1 \cos 2x + C_2 \sin 2x$
Then using undetermined constants we get: $y_{1_p} = Ax + B$ which implies

$$A = \frac{3}{4}, B = 0$$

Therefore, $y_1 = C_1 \cos 2x + C_2 \sin 2x + \frac{3}{4}x$ and from $4y_1 + Dy_2 = x$ we have:

$$y_2' = x - 4y_1 = x - 4C_1 \cos 2x - 4C_2 \sin 2x - 3x,$$

which implies $y_2' = -4C_1 \cos 2x - 4C_2 \sin 2x - 2x$.

By integrating both sides we get

$$y_2 = -2C_1 \sin 2x + 2C_2 \cos 2x - x^2 + C_3.$$

Then substituting y_1 and y_2 into the first equation we obtain $C_3 = \frac{3}{4}$. Therefore,

$$y_1 = C_1 \cos 2x + C_2 \sin 2x + \frac{3}{4}x$$

and

$$y_2 = -2C_1 \sin 2x + 2C_2 \cos 2x - x^2 + \frac{3}{4}$$

Remark 2.8.3. *It is possible, and may be easier, to use Crammer's rule in solving such non homogeneous equation systems.*

2. The system is equivalent to

$$\begin{aligned} Dy_1 - 2y_1 + 2Dy_2 &= 2 - 4e^{2x} \\ 2Dy_1 - 3y_1 + 3Dy_2 - y_2 &= 0 \end{aligned} \iff \begin{aligned} (D-2)y_1 + 2Dy_2 &= 2 - 4e^{2x} \\ (2D-3)y_1 + (3D-1)y_2 &= 0 \end{aligned}$$

And this can be written in matrix form as

$$\begin{pmatrix} D-2 & 2D \\ 2D-3 & 3D-1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2-4e^{2x} \\ 0 \end{pmatrix}$$

Then by Crammer's rule

$$y_1 = \frac{\begin{vmatrix} 2-4e^{2x} & 2D \\ 0 & 3D-1 \end{vmatrix}}{\begin{vmatrix} D-2 & 2D \\ 2D-3 & 3D-1 \end{vmatrix}} = \frac{(3D-1)(2-4e^{2x})}{(D-2)(3D-1)-(2D-3)2D}$$

This implies $[\langle 3D^2-7D+2\rangle - \langle 4D^2-6D\rangle]y_1 = (3D-1)(2-4e^{2x})$. Simplifying this gives us $(-D^2-D+2)y_1 = -12x2e^{2x} - 2 + 4e^{2x}$ and then $-y_1'' - y_1' + 2y_1 = -20e^{2x} - 2$ which is reduced to a second order linear DE in y_1 (Solve this equation.)

and

$$y_2 = \frac{\begin{vmatrix} D-2 & 2-4e^{2x} \\ 2D-3 & 0 \end{vmatrix}}{\begin{vmatrix} D-2 & 2D \\ 2D-3 & 3D-1 \end{vmatrix}} = \frac{(2D-3)(2-4e^{2x})}{(-D^2-D+2)}$$

This implies

$$(-D^2-D+2)y_2 = -8(2e^{2x}) - 6 + 12e^{2x}$$

and then $-y_2'' - y_2' + 2y_2 = -4e^{2x} - 6$ which is reduced to a second order linear DE in y_2 (Solve this equation.)

Therefore, the solution is:

$$y_1 = C_1e^{-2x} + C_2e^{x} + 5e^{2x} - 1$$
$$y_2 = -C_1e^{-2x} + \tfrac{1}{2}C_2e^{x} - e^{2x} + 3$$

2.8.3 Reduction of higher order ODEs to systems of ODE of the first order

In the previous sections we used the characteristics equation to solve higher order ODEs. The characteristic equations are polynomials of degree n, where n is the order of the ODE. However, solving polynomials is a challenging task when the degree gets larger. Because of the techniques developed in linear algebra to reduce matrices, it is preferable to solve eigenvalue problems when the order of the ODE is higher.

There is an equivalence between an n^{th} order linear ODE and a system of n ODEs of first order. In subsection 2.8.2 we have seen how to transform a system of n first order ODEs to an n^{th} order linear ODE.

It is also possible to convert a higher order equation into a system of first order equations using new variable definitions. To see this, consider a homogeneous n^{th} order linear ODE:

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0, \quad \text{where } t \text{ is the independent variable.}$$

Then redefine the variables as follows,

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ x_3(t) &= y''(t) \\ \vdots &= \vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

Then the equation will be equivalent to the system

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ x_3'(t) &= x_4(t) \\ \vdots &= \vdots \\ x_{n-1}'(t) &= x_n(t) \\ x_n'(t) &= -\frac{a_{n-1}}{a_n}x_n - \frac{a_{n-2}}{a_n}x_{n-1} - \cdots - \frac{a_1}{a_n}x_2 - \frac{a_0}{a_n}x_1 \end{aligned}$$

Or in matrix notation:

$$X' = AX,$$

where the coefficient matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_{n-4}}{a_n} & -\frac{a_{n-3}}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix},$$

which is the so called the companient matrix of the n^{th} degree charachteristic equation of the differential equation. Such matrices have special future in matrix theory and the eigenvalue problem could be solved by employing Jordan form of the matrix.

2.9 Numerical Methods to Solve ODEs

It could be impossible to analytically solve many practical problems. But an approximate solution can be obtained using quantitative methods.

2.9.1 Euler's Method

Consider a first order initial - value problem:

$$y' = f(x,y); \quad y(a) = b.$$

Since $y'(x) = \lim_{\triangle x \to 0} \dfrac{\triangle y}{\triangle x}$, we can approximate y' by the ratio $\dfrac{\triangle y}{\triangle x}$

Hence we have $\triangle y \simeq f(x,y)\triangle x$

Let us denote the y values at different points as $y_o, y_1, y_2, \ldots$, where $y_o = y(x_o)$ is the initial value. And let $\triangle x = h$ denote the increment in x, called the step size.

Then we have:

$$\begin{aligned} y_1 &= y_o + f(x_o, y_o)h \\ y_2 &= y_1 + f(x_1, y_1)h; \quad x_1 = x_o + h \\ \vdots & \quad \vdots \end{aligned}$$

In general, the nth iteration will be

$$y_{n+1} = y_n + f(x_n, y_n)h; \quad x_n = x_{n-1} + h, n = 0, 1, 2, \ldots$$

This iterative method is known as Euler's method. Since Euler's method is based on first order approximation, it may work for very small step size h, which makes the iterative scheme very slow.

2.9.2 Runge-Kutta Method

To improve the drawback in Euler's method, it is better to take the mid-point of Δy_n and Δy_{n+1} as an increment for y instead of simply Δy_n alone. Hence we have

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2), \\ \text{where } k_1 &= hf(x_n, y_n), \quad k_2 = hf(x_{n+1}, y_n + k_1) \end{aligned}$$

which is called the Runge-Kutta method of second order.

The Runge-Kutta method works by using a weighted average of slopes in the basic Euler formula to estimate $y(x_o + h)$ [or in general $y(x_k + h)$.]

The fourth-order Runge-Kutta method is given by

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h\,(m_1 + 2m_2 + 2m_3 + m_4) \\ \text{where } m_1 &= f(x_n, y_n) \\ m_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hm_1) \\ m_3 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hm_2) \\ m_4 &= f(x_n + h, y_n + hm_3) \end{aligned}$$

This method is surprisingly accurate for values of $h < 1$.

2.10 Exercises

Chapter 3

*Nonlinear ODEs and Qualitative Analysis

- Many Nonlinear equations cannot be solved in closed form.
- Hence we need to develop qualitative methods to determine properties of solutions without having them explicitly in hand.
- The properties tell us the way how all the trajectories (solution curves) behave near to some points.

Read on Phase portrait and Phase plane Analysis for ODEs.

3.1 Critical Points and Stability

- Consider the autonomous Nonlinear system in two variables

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned} \tag{3.1}$$

If we assume that y is dependent on x, we can equivalently get

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

- Any point (x_o, y_o) such that both P and Q vanishes is called a critical (or singular or equilibrium) point of the system (3.1).
- At a singular point (x_o, y_o), since $\dot{x} = 0$ and $\dot{y} = 0$, a particular solution of equation (3.1) is simply the constant values $x(t) = x_o, y(t) = y_o$.
- An equilibrium point $X_o = (x_o, y_o)$ of system (3.1) is said to be stable if motions (or trajectories) that start sufficiently close to X_o remain close to X_o.

  Mathematically:

  Definition 3.1.1. *Let $d(P_1, P_2)$ denote the distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ and let $P(t) = (x(t), y(t))$ denote the representative point in the phase plane corresponding to system (3.1). Then a singular (or an equilibrium) point $X_o = (x_o, y_o)$ is stable if for any given $\epsilon > 0$, there is a $\delta > 0$ such that*

$$d(P(0), X_o) < \delta \Rightarrow d(P(t), X_o) < \epsilon, \forall t > 0$$

  Otherwise, the equilibrium point X_o is called unstable.

- A singular point X_o is called asymptotically stable if motions (or trajectories) that start out sufficiently close to X_o not only stay close to X_o but actually approach X_o as $t \to \infty$

$$\text{i.e., } \exists \delta > 0 \text{ s.t. } d(P(o), X_o) < \delta \Rightarrow \lim_{t\to\infty} d(P(t), X_o) = 0$$

Definition 3.1.2. *A singular point is called:*

1) *a Center if it is surrounded by closed orbits (paths) corresponding to periodic motions.*

   A center is stable but not asymptotically stable.

2) *a Focus (or Spiral) if all trajectories around X_o "focus" towards (or outward) it as $t \to \infty$.*

   A fucus can be asymptotically stable or unstable.

3) *a Node if there are infinitely many trajectories entering (or leaving) the point X_o.*

   There are four cases → Proper or Improper nodes with each could be stable or unstable.

4) a Saddle if all trajectories (paths) approach to X_o in one direction and move away from it in the other direction.

A saddle is always unstable.

– *The two straight-line trajectories through the saddle (along which the flow is attracted and repelled) are called the stable and unstable manifolds respectively.*

- In many practical problems we will be interested in the stability of equilibrium points. That means, if we take an initial point near to an equilibrium point $X_o = (x_o, y_o)$, does the point $(x(t), y(t))$ on the solution curve (trajectory) remain near X_o?
- To study this, we approximate the nonlinear system (equation (3.1)) by its linear terms in the Taylor series expansion in the neighborhood of each singular point.

3.1.1 Stability for linear systems

3.1.2 Stability for nonlinear systems

- Consider again the system:

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned}$$

- From the Taylor series we have

$$\begin{aligned} P(x,y) &\approx P_x(x_o,y_o)(x-x_o) + P_y(x_o,y_o)(y-y_o) \\ Q(x,y) &\approx Q_x(x_o,y_o)(x-x_o) + Q_y(x_o,y_o)(y-y_o) \end{aligned}$$

- Now letting $a = P_x(x_o,y_o), b = P_y(x_o,y_o), c = Q_x(x_o,y_o)$ and $d = Q_y(x_o,y_o)$ we have

$$\begin{aligned} \dot{X} &= aX + bY \\ \dot{Y} &= cX + dY, \end{aligned} \tag{3.2}$$

where $X = x - x_o$ and $Y = y - y_o$.

- The above process is called a linearization process.

- System (3.2) can be rewritten as

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

- Clearly $(0,0)$ is a critical point for the linear system (3.2) [and hence the point (x_o, y_o) is a critical point for the system (3.1)]
- Let λ_1 and λ_2 be the two eigenvalues of the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Then the nature of the critical point $(0,0)$ of the system (3.2) depends upon the nature of the eigenvalues λ_1 and λ_2.

1. If λ_1 and λ_2 are real, unequal and of the same sign, then the critical point $(0,0)$ of the linear system (3.2) is a node.
   - If, in addition, both λ_1 and λ_2 are positive, then the critical point is an unstable node.
   - If, both λ_1 and λ_2 are negative, then the critical point is a stable node.
2. If λ_1 and λ_2 are real and of opposite sign, then the critical point $(0,0)$ of the linear system (3.2) is a saddle point.
3. If λ_1 and λ_2 are real and equal, then the critical point $(0,0)$ of the linear system (3.2) is a node.
   - If, in addition, $\lambda_1 = \lambda_2 < 0$, then it is a stable node and if $\lambda_1 = \lambda_2 > 0$, then it is an unstable node.
   - If, $a = d \neq 0$ and $b = c = 0$, then it is a proper node, otherwise an improper node.
4. If λ_1 and λ_2 are complex conjugates with the real part not zero, then the equilibrium point $(0,0)$ of the linear system (3.2) is a focus or spiral.
   - If, in addition, the real part is negative, then the critical point is a stable focus.
   - If, the real part is positive then it is an unstable focus.
5. If λ_1 and λ_2 are pure imaginary, then the equilibrium point $(0,0)$ of the linear system (3.2) is a center.

   A center is always stable even though it is not asymptotically stable.

Remark:

1. In the linearization process it was assumed that
   - the constants $a, b, c,$ and d are real numbers;
   - the functions P and Q have continuous first partial derivatives in the neighborhood of the critical points.

   The above two requirements will be met, if the Jacobian

$$\left.\frac{\partial(P,Q)}{\partial(x,y)}\right|_{(x_o,y_o)} \neq 0.$$

2. The constant terms in the linearized system are missing because $P(x_o, y_o) = Q(x_o, y_o) = 0$.
3. The nature of the equilibrium points of the nonlinear system (3.1) can be determined from that of the linearized system (3.2) as in the following Theorems

Theorem 3.1.3 (Poincare's Result)**.** *The classification of all singular points of the non-linear system (3.1) correspond in both type and stability with the results obtained by considering the linearized system (3.2) except for a center and a proper node.*

In these exceptional cases

(i) a center of the linearized system could be either a focus or a center for the nonlinear system.

(ii) a proper node could also be either a spiral or a node for the nonlinear system (3.1).

To determine these exceptional cases one requires to study further the original nonlinear system itself.

The above procedure (the linearization process) can also be used to solve a second order non-linear ODE. This can be done by using substitution of variables $y = \dot{x}$, which imply that $\dot{y} = \ddot{x}$. This will result in a nonlinear system of two first order equations.

However, if such an equation has no term in $\dot{x}$, we need the following theorem.

Theorem 3.1.4. *If the nonlinear equation $\ddot{x} + f(x) = 0$ has a singular point in the $x\dot{x}$ plane (phase plane), where the linearized system indicates a center or a proper node, the nonlinear equation also has the same property.*

Example 3.1.1. *1. The pair of differential equations*

$$\begin{aligned} \dot{x} &= \frac{1}{2}x - xy \\ \dot{y} &= -2y + xy, \qquad x, y \geq 0, \end{aligned}$$

occur in a study of interacting populations. Find the equilibrium points and determine their nature.

Ans.: $(0,0)$ is a saddle equilibrium (not stable)
and the equilibrium point $(2, 1/2)$ is a cenetr (stable).

2. The equation $\ddot{x} + \epsilon\dot{x}^3 + x = 0$ models a harmonic oscillator with cubic damping - that is, with a damping term proportional to the velocity cubed. Find the critical point(s) and determine their nature.

Ans.: the point $(0,0)$ is the only critical point and it is a center for the nonlinear equation.

3.2 Stability by Lyapunav's Method

3.3 Exercises

Part II

Vector Analysis

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Chapter 4

Vector Differential Calculus

4.1 Vector Calculus

In the previous Applied Mathematics courses, specifically in the linear algebra part, we have been discussing about constant vectors, but the most interesting applications of vectors involve also vector functions.

The simplest example is a position vector that depends on time. We can differentiate such a function with respect to time and the first derivative of such function is the velocity and its second derivative is the acceleration of the particle whose position is given by the position vector. In this case, the coordinates of the tip of the position vector are functions of time.

Therefore, it is worth to talk about such functions and in this course, specially in this chapter we are going to address the calculus of vector fields (vector valued functions).

4.1.1 Vector Functions of One Variable in Space

First recall the definition of a function, that is, a function is a rule that assigns to each element in the domain an element in the range.

Definition 4.1.1. *A vector-valued function, or vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.*

In this course, we are most interested in vector functions whose values are three-dimensional vectors. This means that for every number t in the domain of there is a unique vector in $\mathbb{R}^3$

denoted by $\mathbf{r}(t)$. If $f(t), g(t)$ and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g and h are real-valued functions called the component functions of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$$

Example 4.1.1. *The function $\mathbf{r}(t) = t^3 i + e^{-t} j + \sin tk$ is a vector valued function and the component functions of $\mathbf{r}$ are t^3, e^{-t} and $\sin t$.*

Remark 4.1.2. *The domain of a vector valued function $\mathbf{r}$ consists of all values of t for which the expression $\mathbf{r}(t)$ is defined, that is the values of t for which all the component functions are defined.*

For example, if

$$\mathbf{r}(t) = \sqrt{t}i + \ln(t-2)j + 3t,$$

then the domain of $\mathbf{r}(t)$ is the set of points in $\mathbb{R}$, where $\sqrt{t}, \ln(t-2)$ and $3t$ are defined. That is, $t \geq 0$ and $t - 2 > 0$ and hence the domain of $\mathbf{r}$ is (t, ∞).

For each t, where $\mathbf{r}$ is defined, draw $\mathbf{r}(t)$ as as a vector from the origin to the point $(f(t), g(t), h(t))$. The end points of these vectors traces out a curve C as t varies.

Example 4.1.2. *The function $\mathbf{r}(t) = (1+t)i + tj + (3-t)k$ is a vector valued function of one variable. The curve that is traced out by the heads of the position vectors this vector valued function is a line that passes through the point $(1, 0, 3)$ and with directional vector $(1, 1, -1)$.*

4.1.2 Limit of A Vector Valued Function

Definition 4.1.3. *A vector valued function $V(t)$ is said to have the limit l as t approaches t_0, if $v(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and*

$$\lim_{t \to t_0} \|V(t) - l\| = 0.$$

Then we write

$$\lim_{t \to t_0} V(t) = l.$$

A vector function $v(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} V(t) = v(t_0).$$

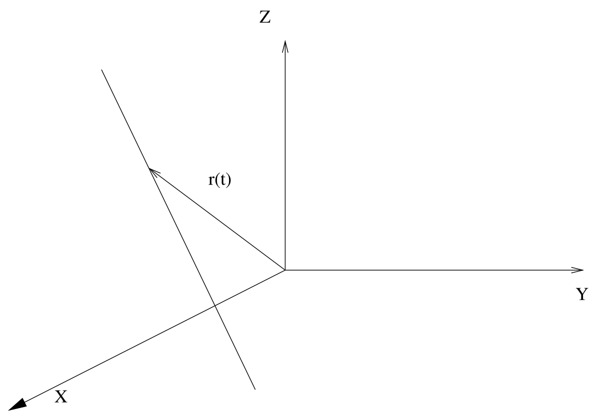

Figure 4.1: Graph of the line in Example 4.1.2

The following theorem is used as an alternative definition of limit of a vector valued function.

Theorem 4.1.4. *If $\mathbf{r}(t) = (f(t), g(t), h(t))$, then*

$$\lim_{t\to t_0} \mathbf{r}(t) = l$$

if and only if

$$\big(\lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\big) = l.$$

Example 4.1.3. *Find $\lim_{t\to 0}\mathbf{r}(t)$, if*

$$\mathbf{r}(t) = t^3 i + e^{-t} j + \big(\frac{\sin t}{t}\big)k.$$

Solution

By Theorem 4.1.4 we have ,

$$\lim_{t\to 0}\mathbf{r}(t) = \big(\lim_{t\to 0} t^3\big)i + \big(\lim_{t\to 0} e^{-t}\big)j + \big(\lim_{t\to 0}\frac{\sin t}{t}\big)k = j + k.$$

Remark 4.1.5. *If $\mathbf{r}(t) = \big(f(t), g(t), g(t)\big) = f(t)i + g(t)j + h(t)k$ a vector valued function all t in the domain of $\mathbf{r}$, then $\mathbf{r}$ is continuous at t_0 if and only if its (three) component functions f, g and h are continuous at t_0.*

Vector valued functions and curves in space have close connection. Suppose that f, g and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space

where

$$x = f(t), \qquad y = g(t) \qquad \text{and} \qquad z = h(t) \tag{4.1}$$

and t varies in I is called a **space curve**.

The equations in 4.1 are called **parametric equations** of the curve C and the variable t is called the **parameter.**

Example 4.1.4. *The components of the vector valued function $\mathbf{r}(t) = (2\cos t, 2\sin t, 0)$ are parametric equations of a circle with center at the origin and radius 2 in space.*

4.1.3 Derivative of a Vector Function

Recall that, if f is a real-valued function of one variable, then the derivative of f at any point t in the domain of f is

$$f'(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h},$$

provided that the limit exists. Now, let us define the derivative of a vector valued function of one variable.

Definition 4.1.6. *A vector function $V(t)$ is said to be differentiable at a point t in the domain of V if the limit*

$$\lim_{h\to 0}\frac{V(t+h) - V(t)}{h}$$

exists and if the limit exists then it is denoted by $V'(t)$. That is,

$$V'(t) = \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}$$

Remark 4.1.7. *If the function $V(t) = (V_1(t), V_2(t), V_3(t))$ is a vector field, then $V'(t) = (v_1'(t), V_2'(t), V'3(t))$*

Example 4.1.5. *Consider the following functions.*

1. *If $V(x) = (\cos x, \sin x)$ then $V'(x) = (-\sin x, \cos x)$.*

2. *If $V(t) = (a\cos t, a\sin t, ct)$ then $V'(t) = (-a\sin t, a\cos t, c)$*

Differentiation Rules

Let $U(t)$ and $V(t)$ be a vector valued functions in space and c be any constant. Then

a) $(cV)' = cV'$

b) $(U+V)' = U' + V'$

c) $(U.V)' = U'.V + U.V'$

d) $(U \times V) = U' \times V + U \times V'$

Remark 4.1.8. *Let V(t) be a vector function of constant norm. i.e. $\|V(t)\| = c$ for a constant c or $V.V = c^2$ Then $(V.V)' = (c^2)' = 0$ which implies $2V'.V = 0$. Then, either $V' = 0$ or $V' \perp V$. Therefore, a nonzero vector field with constant norm is perpendicular to its derivative.*

4.1.4 Vector and Scalar Fields

Now, let us consider vector valued functions, called vector fields, of several variables. Vector valued functions of one variable are also called vector fields.

Definition 4.1.9. *A function f whose value is a scalar (or a real number), say $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, is called a* **scalar field**.
A function v whose value is a vector, say $v : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$, is called a **vector field.** *That is a vector field is a vector valued function.*

Example 4.1.6.

1. *The function $T : X \to \mathbb{R}$ given by*
$$T(x,y) = \frac{100}{(x+1)^2 + (y+1)^2}$$
where $X = \{(x,y) | 0 \le x \le 1, 0 \le y \le 1\}$, which is a Temperature Field of a square plate is a scalar field.

2. *The function $f : X \to \mathbb{R}^3$ given by $f(x,y) = (x^2 + y, \ln(x^2+y^2), \sin(x+3y))$, where $X = \mathbb{R}^2 \backslash \{(0,0)\}$ is a vector field.*

3. *If (x_0, y_0, z_0) is a point in $\mathbb{R}^3$, then the function $d : \mathbb{R}^3 \to \mathbb{R}$ given by $d(x,y,z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is a scalar field. (d is called the Euclidean Distance.)*

Definition 4.1.10. *Let $V : \mathbb{R}^n \to \mathbb{R}^3, V = (V_i, V_2, V_3)$ where each V_i is a function of n variables, $t_1, t_2, \ldots, t_n$. Then the partial derivative of V with respect to t_i is denoted by $\frac{\partial V}{\partial t_i}$ and is defined as the vector function*
$$\frac{\partial V}{\partial t_i} = (\frac{\partial V_1}{\partial t_i}, \frac{\partial V_2}{\partial t_i}, \frac{\partial V_3}{\partial t_i})$$

Example 4.1.7. *If $f(x,y) = \big((x^2+y), \ln(x^2+y^2), \sin(x+3y)\big)$, then*
$$\frac{\partial f}{\partial x} = \left(2x, \frac{2x}{x^2+y^2}, \cos(x+3y)\right) \qquad \textit{and} \qquad \frac{\partial f}{\partial y} = \left(y, \frac{2y}{x^2+y^2}, 3\cos(x+3y)\right).$$

4.2 The Gradient Field

Let $F(x,y,z)$ be a real valued functions of three variables (i.e. F is a scalar field defined from $X \subset \mathbb{R}^3$ into $\mathbb{R}$.) The gradient of F , denoted by ∇F, is a vector field defined by
$$\nabla F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$
and if P is a point in the domain of F, the gradient of F evaluated at P is denoted by $\nabla F(P)$ and also if f is a function of two variables, then the the gradient of f, denoted by ∇f, is defined by
$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$
But in this section we will focus on the gradient of functions of three variables.

Example 4.2.1. *If $F(x,y,z) = 2x + xy - yz^2$, then F is a scalar field and*
$$\nabla F(x,y,z) = (2+y)i + (x-z^2)j - (2yz)k = (2+y, x-z^2, -2yz).$$
The point $(2,-1,0)$ is a point in the domain of F and $\nabla F(2,-1,0) = i + j$.

Example 4.2.2. *The gradient of $f(x,y) = xy + 2x^3$ is*
$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial x} j = (y + 6x^2) i + xj.$$

Remark 4.2.1. *Let F and G be scalar fields of three variables and c be a constant. Then*

1. *$\nabla(F+G) = \nabla F + \nabla G$ and*
2. *$\nabla(cF) = c\nabla F$.*

Let $P(x_0, y_0, z_0)$ be a point and $u = ai+bj+ck$ be a unit vector, i.e. $a^2+b^2+c^2 = 1$. Then the directional derivative of a scalar field F at the point P in the direction of u, denoted by $D_uF(P)$, is defined by

$$D_uF(P) = a\frac{\partial F}{\partial x}(x_0, y_0, z_0) + b\frac{\partial F}{\partial y}(x_0, y_0, z_0) + c\frac{\partial F}{\partial z}(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0).u,$$

the scalar product of the vectors $\nabla F(x_0, y_0, z_0)$ and u.

Example 4.2.3. *Given $F(x, y, z) = 2x + xy - yz^2$, the directional derivative of F at the point $(1, 2, 2)$ in the direction of the unit vector $u = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ is*

$$D_uF(1, 2, 2) = \frac{1}{3}\left(\frac{\partial F}{\partial x}(1, 2, 2)\right) + \frac{2}{3}\left(\frac{\partial F}{\partial y}(1, 2, 2)\right) + \frac{1}{3}\left(\frac{\partial F}{\partial z}(1, 2, 2)\right) = \frac{-7}{3}.$$

Remark 4.2.2. *If F is a scalar field of three variables and v is any nonzero vector then the directional derivative of F at a point P in the direction of v is given by $D_uF(P)$, where $u = \frac{1}{\|v\|}v$.*

Let F be a scalar field and F and its partial derivatives be continuous in some sphere about a point P and u be a unit vector. Then

$$D_uF(P) = \nabla F(P).u = \|\nabla F(P)\|\|u\| \cos\theta = \|\nabla F(P)\| \cos\theta,$$

where θ is the angle between u and $\nabla F(P)$.

Therefore $D_uF(P)$ has its maximum when $\cos\theta = 1$, which occurs when $\theta = 0$, that is, u is in the same direction as $\nabla F(P)$ and $D_uF(P)$ has its minimum when $\cos\theta = -1$, that is ,$\theta = \pi$ and hence $\nabla F(P)$ and u are in opposite directions.

Therefore we have proved the following theorem.

Theorem 4.2.3. *Let F be a scalar field and F and its partial derivatives be continuous in some sphere about a point P and suppose that $\nabla F(P) \neq \mathbf{0}$. Then*

1. *At P, F has its maximum rate of change in the direction of $\nabla F(P)$ and this maximum rate of change is $\|\nabla F(P)\|$.*
2. *At P, F has its minimum rate of change in the direction of $-\nabla F(P)$ and this minimum rate of change is $-\|\nabla F(P)\|$.*

Example 4.2.4. *Let $F(x, y, z) = 2xz + yz^2$ and $P(1, 1, 2)$. The gradient of F is*

$$\nabla F(x, y, z) = 2zi + z^2j + (2x + 2yz)k$$

and then

$$\nabla F(2, 1, 1) = 2i + j + 6k.$$

The maximum rate of change of F at $(2, 1, 1)$ is in the direction of $2i+j+6k$ and this maximum rate of change is $\sqrt{2^2 + 1^2 + 6^2} = \sqrt{41}$.

4.2.1 Level Surfaces, Tangent Planes and Normal Lines

The gradient of a scalar field can be use to find equations of tangent planes and equations of normal lines of a level surface defined by the scalar field at a given point.

Let F be a function of three variables and c be a number. The set of points (x, y, z) such that $F(x, y, z) = c$ is called a level surface of F.

Example 4.2.5. *Let $F(x, y, z) = x^2 + y^2 + z^2$. If $c > 0$, then the level surface $F(x, y, z) = c$ is a sphere with radius $\sqrt{c}$; if $c = 0$, then the level surface $F(x, y, z) = c$ is just the point $(0, 0, 0)$ and if $c < 0$, then the level surface $F(x, y, z) = c$ is empty set.*

For example, if $c = 9$, then the surface $F(x, y, z) = 9$ is a sphere $x^2 + y^2 + z^2 = 9$ with radius 3 and center at the origin.

Let F be a scalar function of three variables, c be a constant and S be the level surface given by $F(x, y, z) = c$. Let $P_0 = (x_0, y_0, z_0)$ be a point on S. Assume that there are smooth curves on the surface S passing through P_0. Then each such curve has a tangent vector at P_0.

The plane containing these tangent vectors is called the **tangent plane** to the surface S at P_0 and a vector orthogonal to this tangent plane at P_0 is called a **normal vector**, or **normal**, to the surface S at P_0. The line through P_0 in the direction of the normal vector is called a **normal line** to the surface S at the point P_0.

Therefore, to determine equation of the tangent plane and normal line to a surface S at a given point P, we need to have a normal vector to the tangent plane and for this purpose we have the following theorem.

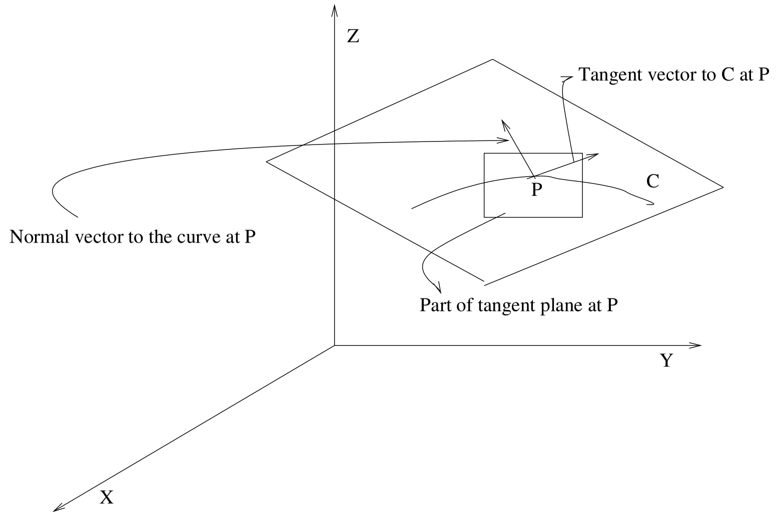

Figure 4.2: Normal vector to a surface.

Theorem 4.2.4. *Let F be a function of three variables and suppose that F and its first partial derivatives are continuous at a point P on the level surface S given by $F(x, y, z) = c$. Suppose that $\nabla F(P) \neq 0$. Then $\nabla F(P)$ is normal to the level surface S at the point P.*

Example 4.2.6. *Find the equation of the tangent plane and normal line to the surface*

$$3x^4 + 3y^4 + 6z^4 = 12$$

at the point $(1, 1, 1)$.

Solution

Let $F(x, y, z) = 3x^4 + 3y^4 + 6z^4$. Then

$$\frac{\partial F}{\partial x}(x, y, z) = 12x^3, \qquad \frac{\partial F}{\partial x}(x, y, z) = 12y^3 \qquad \text{and} \qquad \frac{\partial F}{\partial x}(x, y, z) = 24z^3$$

which are continuous at the point $(1, 1, 1)$ and hence $\nabla F(1, 1, 1) = 12i + 12j + 24k$.

Since $\nabla F(1, 1, 1)$ is normal to the plane, equation of the plane is given by

$$12x + 12y + 24z = 12 + 12 + 24 = 48$$

which is equivalent to $x + y + 3z = 4$ and equation of the normal line is

$$(x, y, z) = (1, 1, 1) + t(1, 1, 3), t \in \mathbb{R}.$$

4.3 Curves and Arc length

Let

$$x = x(t), y = y(t) \text{ and } z = z(t) \tag{4.2}$$

be continuous functions of a real parameter t over a closed interval $[a, b]$. The points

$$\mathbf{r}(t) = (x(t), y(t), z(t)),$$

for $a \leq t \leq b$ are said to constitute a curve C joining the endpoints $\mathbf{r}(a)$ and $\mathbf{r}(b)$ and (4.2) is called a parametrization of the curve. We call the functions x, y and z, coordinate functions.

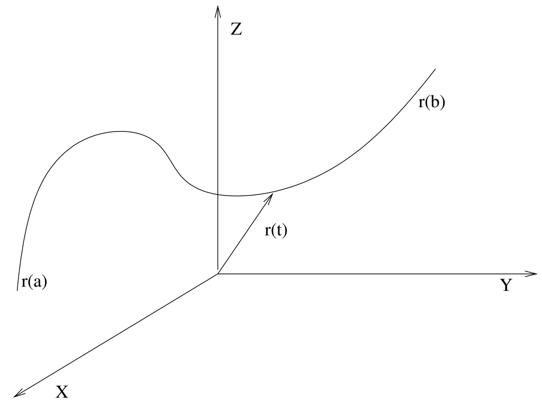

Figure 4.3: A curve with initial point r(a) and terminal point r(b).

We call a curve C that is parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$:

- continuous if each coordinate function is continuous;
- differentiable if each coordinate function is differentiable;
- closed if the initial and terminal points coincide, that is,

$$(x(a), y(a), z(a)) = (x(b), y(b), z(b))$$

and if a curve is not closed it is called an **arc**;

- simple if $a < t_1 < t_2 < b$ implies that $(x(t_1), y)t_1), z(t_1)) \neq (x(t_2), y(t_2), z(t_2))$, in other words, if it does not intersect itself;
- smooth if the coordinate functions have continuous derivatives which are never all zero for the same value of t, that is, it possesses a tangent vector that varies continuously along the length of C.
- piecewise smooth if it has continuous tangent at all but finitely many points. Such a curve is a curve with a finite number of corner at which there is no tangent.

If C is a curve which is divided into smooth curves $C_1, C_2, ..., C_n$ such C begins with C_1 , C_2 begins where C_1 ends and so on, but at the point where C_i and C_{i+1} join, there may be no tangent in the resulting curve , then C is piecewise smooth curve and we write such a curve as

$$C = C_1 \oplus C_2 \oplus ... \oplus C_n.$$

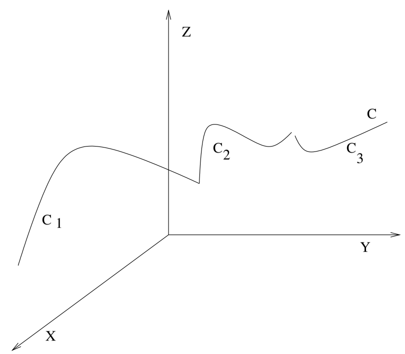

Figure 4.4: A piecewise smooth curve.

If the tail of the position vector

$$\mathbf{r}(t) = x(t)i + y(t)j + z(t)k \tag{4.3}$$

is fixed at the origin, then the head of $r(t)$ generates the curve as t varies from a to b.

Example 4.3.1. *The following are examples of curves.*

1. **Straight line:**
   A straight line L through a point P with position vector in the direction of a constant vector A can be represented as

$$r(t) = P + tA = (v_1 + ta_1, v_2 + ta_2, v_3 + ta_3), \ for\ all\ t \in \mathbb{R},$$

where $P = (v_1, v_2, v_3), A = (a_1, a_2, a_3)$.

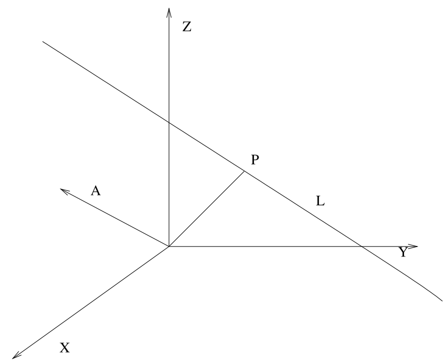

Figure 4.5: A line passing through a point and parallel to a given vector.

2. **Ellipse, circle:**
   The vector function:

$$r(t) = (a\cos t, b\sin t, 0) \tag{4.4}$$

represents an ellipse and is a circle if $a = b$.

3. **Circular helix:**
   The twisted curve represented by the vector function:

$$r(t) = (a\cos t, a\sin t, ct), \tag{4.5}$$

$c \neq 0$ *is a* **circular helix.**

Consider a curve C that is parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$. If it exists, the derivative of $\mathbf{r}(t_0)$ for $t_0 \in (a, b)$ is given by

$$\mathbf{r}'(\mathbf{t_0}) = \lim_{h\to 0} \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h}$$

which is equal to

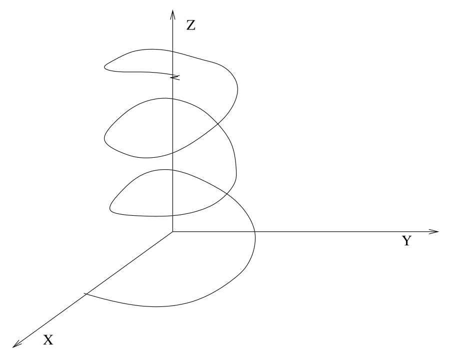

Figure 4.6: A circular helix in three dimensional space.

$$\lim_{h\to 0}\frac{\mathbf{x}(t_0+h)-\mathbf{x}(t_0)}{h}i+\lim_{h\to 0}\frac{\mathbf{y}(t_0+h)-\mathbf{y}(t_0)}{h}j+\lim_{h\to 0}\frac{\mathbf{z}(t_0+h)-\mathbf{z}(t_0)}{h}k = x'(t_0)i+y'(t_0)j+z'(t_0)k.$$

That is, $\mathbf{r}(t_0) = x'(t_0)i + y'(t_0)j + z'(t_0)k$.

Recall from calculus that, the derivative of a function at a point is the slope of a tangent line to the graph of the function at the point.

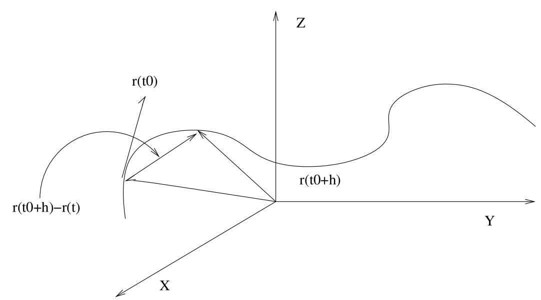

Figure 4.7: Derivative in diagram.

Consider the figure above. As $h \to 0$, the vector $\mathbf{r(t_0+h)} - \mathbf{r(t_0)}$ moves toward $\mathbf{r}(t_0)$ along the curve C and the vector

$$\frac{1}{h}(\mathbf{r(t_0+h)} - \mathbf{r(t_0)})$$

moves into a tangent vector to C at the point $(\mathbf{x}(t_0), \mathbf{y}(t_0), \mathbf{z}(t_0))$.

Hence the derivative $r'(t)$ (if it exists) of the curve is called the **tangent vector** to the curve at the point $r(t)$ and the equation of the tangent line to the curve C at point P is

$$q(s) = r + sr' \tag{4.6}$$

Example 4.3.2. *For the curve C given by $F(t) = 2ti - t^2 j + 4tk$, the vector $F'(t) = 2i - 2tk + 4k$ is tangent to the curve at the point $(2t, -t^2, 4t)$.*

Definition 4.3.1. *The length ℓ of the curve C which is given by the parametrization $\mathbf{r}(t) = x(t)i + y(t)j + z(t)k$ on $[a, b]$ is defined by*

$$\ell = \int_a^b \sqrt{\mathbf{r}'(t).\mathbf{r}'(t)}dt.$$

If we replace b (the fixed upper limit of integration) by a variable t, $a \le t \le b$, the integral becomes a function of t.

$$s(t) = \int_a^t \sqrt{\mathbf{r}'.\mathbf{r}'}d\tau, \text{ where } \mathbf{r}' = \frac{dr}{d\tau}$$

and is called the **arc length** function.
Differentiating the arc length function gives us

$$\frac{ds}{dt} = \sqrt{\mathbf{r}'.\mathbf{r}'} = \|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\|.$$

Example 4.3.3. *Let $\mathbf{r}(t) = (a\cos t, a\sin t, ct), c \neq 0$. represent circular helix.*
Then $\mathbf{r}'(t) = (-a\sin t, a\cos t, c)$ and

$$\mathbf{r}'.\mathbf{r}' = (-a\sin t, a\cos t, c).(-a\sin t, a\cos t, c) = a\sin^2 t + a\cos^2 t + c^2 = a^2 + c^2.$$

Hence the are length of the circular helix is:

$$s(t) = \int_o^t \sqrt{\mathbf{r}'.\mathbf{r}'}dt = \int_0^t \sqrt{a^2+c^2}ds = t\sqrt{a^2+c^2}.$$

4.4 Tangent, Curvature and Torsion

Let $\mathrm{F}(t) = x(t)i + y(t)j + z(t)k$ be the position vector of a curve C for $a \le t \le b$. Assume that the coordinate functions x, y and z are twice continuously differentiable.

If a particle is moving along the curve C with a position vector $\mathrm{F}(t) = x(t)i + y(t)j + z(t)k$, then the the velocity $\mathbf{v}(t)$ of the particle at time t is:

$$\mathbf{v}(t) = \mathbf{F}'(t)$$

and the speed $v(t)$ of of the particle is the norm of the velocity, i.e

$$v(t) = \|\mathbf{v}(t)\| = \|\mathbf{F}'(t)\|,$$

which is the rate of change of the distance covered by the particle along the curve with respect to the time and the acceleration $\mathbf{a}(t)$ of the moving particle is the rate of change of the velocity with respect to time, i.e.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{F}''(t).$$

If $\mathbf{F}'(t) \neq \mathbf{0}$, then the vector $\mathbf{F}'(t)$ is tangent to the curve C. Let $\mathbf{T}(t)$ be a unit vector in the direction of $\mathbf{F}'(t)$, i.e.

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{F}'(t)\|}\mathbf{F}'(t)$$

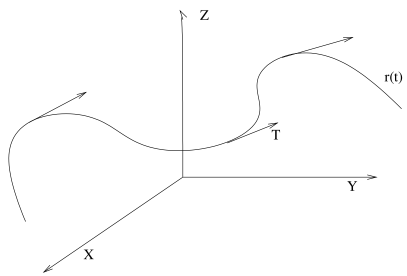

Figure 4.8: Tangent unit vectors

Then as t varies, T turns with bending of the curve, but T is a unit vector and hence the length of T remains constant.

Definition 4.4.1. *Let C be a smooth curve with parametrization $\mathbf{F}(t)$ such that $\mathbf{F}(t)$ is differentiable. The norm of the rate of change of the unit vector $\mathbf{T}(t)$ with respect to the arc length function S is called the curvature K of the curve C. That is,*

$$K(S) = \left\|\frac{dT}{dS}\right\|.$$

Consider the relation

$$\frac{dT}{dS} = \frac{dT}{dt}.\frac{dt}{dS} = \frac{dT/dt}{dS/dt}.$$

But $dS/dt = \|\mathbf{F}'(t)\|$ and hence we get

$$K(t) = \frac{1}{\|\mathbf{F}'(t)\|}\|\mathbf{T}'(t)\|$$

which is a function of t.

Example 4.4.1. *Curvature of a line at any point is zero.*
To see this, let l be a line that passes through a point $P(x_0, y_0, z_0)$ with directional vector $A = (a, b, c)$. Then the parametric equation of l is given by

$$\mathbf{F}(t) = (x_0 + ta)i + (y_0 + tb)j + (z_0 + tc)k, \qquad t \in \mathbb{R}.$$

Then $\mathbf{F}'(t) = ai + bj + ck$ and

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{F}'(t)\|}\mathbf{F}'(t) = \frac{1}{\sqrt{a^2 + b^2 + c^2}}(ai + bj + ck).$$

This implies that, $\mathbf{T}'(t) = \mathbf{0}$ for all t and hence $K(t) = 0$ for all t. This is clear from the fact that a particle moving on a straight line does not change its direction.

Example 4.4.2 (Curvatures of ellipses and circles)**.** *Recall that the vector function*

$$\mathbf{r}(t) = (a\cos t, b\sin t, 0), \qquad t \in \mathbb{R}$$

represents an ellipse and it represents a circle if $a = b$ in space. Then $\mathbf{r}'(t) = (-a\sin t, b\cos t, 0)$ and $\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$. Therefore

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, b\cos t, 0).$$

If $a = b$, then $\mathbf{T}(t) = (-\sin t, \cos t, 0)$. This implies $\mathbf{T}'(t) = (-\cos t, -\sin t, 0)$ and hence the curvature of the circle is $K(t) = \frac{1}{a}$.

Remark 4.4.2. *If C is a curve that is traced-out by a vector field $\mathbf{r}(t)$, then the curvature K of the curve C is given by*

$$K(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Example 4.4.3. *Find the curvature of the helix*

$$\mathbf{r}(t) = (a\cos t)i + (a\sin t)j + btk,$$

where $a, b \geq 0$ *and* $a^2 + b^2 \neq 0$.

First $\mathbf{r}'(t) = (-a\sin t)i + (a\cos t)j + bk$ *and* $\mathbf{r}''(t) = (-a\cos t)i - (a\sin t)j$. *Then*

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\cos^2 t + a^2\sin^2 t + b^2} = \sqrt{a^2+b^2}$$

and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t i + ab\cos t j + a^2 k,$$

which implies $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = a\sqrt{a^2+b^2}$. *Therefore the curvature* $K(t)$ *of the helix is*

$$K(t) = \frac{\|\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a}{a^2+b^2}.$$

In the case of plane curves, that is, graph of functions of the form $y = f(x)$ can be considered as curves traced out by a vector field $\mathbf{r}(t) = ti + f(t)j$. Here the k^{th} component is considered to be zero. Therefore $\mathbf{r}'(t) = i + f'(t)j$ and $\mathbf{r}''(t) = f''(t)j$ and then the curvature of this curve is given by

$$K(t) = \frac{|f''(t)|}{(1+(f'(t))^2)^{\frac{3}{2}}},$$

since $\mathbf{r}'(t) \times \mathbf{r}''(t) = f''(t)k$.

Example 4.4.4. *Find the curvature of the parabola* $y = ax^2 + bx + c$, *where* $a \neq 0$.
The vector field that traces out the parabola is given by $\mathbf{r}(t) = ti + f(t)j$, *where* $f(x) = x^2+bx+c$. *Then* $f'(t) = 2at + b$ *and* $f''(t) = 2a$. *Therefore*

$$K(t) = \frac{|a|}{(1+(2at+b)^2)^{\frac{3}{2}}}.$$

Given a curve C which is parameterized by the the position vector $\mathbf{r}(t)$, we have a unit tangent vector T at a point where the coordinate functions are differentiable. Now we are looking to get a unit normal vector to the curve at a point where the coordinate functions are differentiable.

Since T has constant length, $\frac{d\mathbf{T}}{ds}$ is orthogonal to T. At a point where $K(S) \neq 0$, the vector

$$N = \frac{1}{K}.\frac{dT}{dS} = \frac{T'(t)}{\|T'(t)\|} = \frac{r''(S)}{\|r(S)\|}$$

is a unit vector parallel to $T'(t)$ and hence normal to the curve and it is called **principal unit normal vector** for a curve C.

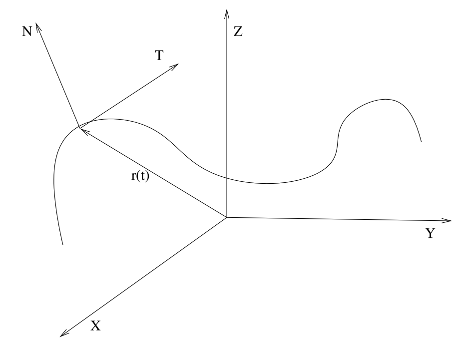

Figure 4.9: Unit tangent vector and principal unit normal vector to a curve.

Definition 4.4.3. *The unit vector* $\mathbf{B} = T \times N$ *is called the* **binomial vector** *of the curve* C *trace out by the vector field* $\mathbf{r}(t)$.

Now, by using the rule of differentiation we have

$$\frac{d\mathbf{B}}{dS} = \frac{d}{dS}(\mathbf{T} \times \mathbf{N}) = \left(\frac{d\mathbf{T}}{dS} \times \mathbf{N}\right) + \left(\mathbf{T} \times \frac{d\mathbf{N}}{dS}\right).$$

But

$$\frac{d\mathbf{T}}{dS} \times \mathbf{N} = \mathbf{0},$$

since they are of the same direction. This implies

$$\frac{d\mathbf{B}}{dS} = \mathbf{T} \times \frac{d\mathbf{N}}{dS},$$

which implies $\frac{d\mathbf{B}}{dS} \perp T$ and since B is vector of constant norm we get that $\frac{d\mathbf{B}}{dS} \perp \mathbf{B}$.

Hence, the vector $\frac{d\mathbf{B}}{dS}$ is parallel to N. This implies that $\frac{d\mathbf{B}}{dS} = -\tau N$ for some constant τ (the negative sign is traditional). Here the scalar τ is called the **torsion** along the curve and from $\mathbf{N}.\frac{d\mathbf{B}}{dS} = -\tau\mathbf{N}.\mathbf{N} = -\tau.1$ we have

$$\tau = -\frac{d\mathbf{B}}{dS}.\mathbf{N}.$$

Unlike K, which is always positive, τ can be positive, negative or zero.

Since $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ are mutually orthogonal, they are linearly independent. Hence any vector in $\mathbb{R}^3$ can be represented as a linear combination of these vectors.

If $\mathbf{B}'$, $\mathbf{T}'$ and $\mathbf{N}'$ exist, then we get the following:

$$\mathbf{T}' = K\mathbf{N}$$
$$\mathbf{N}' = -K\mathbf{T} + \tau\mathbf{B}$$
$$\mathbf{B}' = -\tau\mathbf{N}$$

and this formula is called **Frenet** formula.

Example 4.4.5. *Let $\mathbf{F}(t) = t^2 i - 2tj + tk$. Find the curvature, principal unit vector, binomial vector of the curve C with position vector F and the torsion along the curve.*

Solution

$\mathbf{F}'(t) = 2ti - 2j + k$ and then $\|\mathbf{F}'(t)\| = \sqrt{4t^2+5}$. This implies

$$T(t) = \frac{1}{\|F'(t)\|}F'(t) = \frac{1}{\sqrt{4t^2+5}}(2ti - 2j + k).$$

4.5 Divergence and Curl

Recall that, the gradient operator produces a vector field from a scalar field. We will discuss two other important vector operations. One produces a scalar field from a vector field and the other produces a vector field from a vector field.

Definition 4.5.1. *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field given by*

$$F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k.$$

1. *The* **divergence** *of F, denoted by $divF$, is the scalar field defined by*
$$divF = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$
2. *The* **curl** *of F, denoted by $curlF$, is the vector field defined by*
$$curlF = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k.$$

Example 4.5.1. *Let $F(x, y, z) = 3xyi - 2yzj + x^2k$. Then $f = 3xy, g = -2yz, h = z$ and hence $\frac{\partial f}{\partial x} = 3y$, $\frac{\partial f}{\partial y} = 3x$, $\frac{\partial f}{\partial z} = 0$, $\frac{\partial g}{\partial x} = 0$, $\frac{\partial g}{\partial y} = -2z$, $\frac{\partial g}{\partial z} = -2y$, $\frac{\partial h}{\partial x} = 2x$, $\frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial z} = 0$. Therefore*

1. *$divF = 3y - 2z + 2xz$ which is a scalar in $\mathbb{R}$.*

2. *$curlF = 2zi - 2xj - 3xk$, which is a vector in $\mathbb{R}^3$.*

Let ∇ be the operator defined from the set of scalar fields of three variables into the set of vectors in $\mathbb{R}^3$ by

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k.$$

If F is a scalar field of three variables, then the products $\frac{\partial}{\partial x}(F)$, $\frac{\partial}{\partial y}(F)$ and $\frac{\partial}{\partial z}(F)$ are defined to be $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ respectively.

Remark 4.5.2. *The ∇ operator and gradient, divergence and curl.*

1. *The product of ∇ and a scalar field F in the given order is the gradient of F, that is,*
$$\nabla F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = gradF.$$
.
2. *The product of ∇ and a vector field F in the given order is the divergence of F , that is, if $F = fi + gj + hk$, then*
$$\nabla.F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(fi + gj + hk) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = divF.$$
Here even though $\nabla.\mathbf{F} = div\mathbf{F}$ this notation is not directly equivalent to the scalar (dot) product. This is because
$$\nabla.\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$
which is a real number depending on values of f, g and h, whereas
$$\mathbf{F}.\nabla = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$$
which is an operator.
3. *The cross product of ∇ and a vector field F is the curl of F, that is, if $F = fi + gj + hk$, then*
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k = crulF.$$
4. *The physical significance of $div\mathbf{F}$ at point P is that it describes the out flow of $\mathbf{F}$ per unit volume at P.*

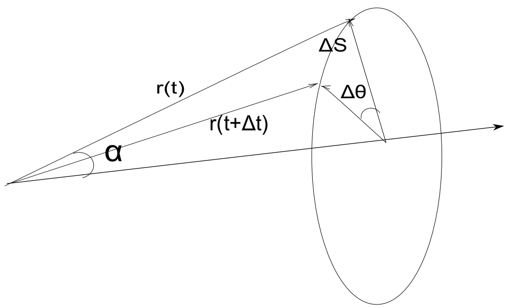

Figure 4.10: Geometrical Interpretation of curl.

Consider a particle moving around a circle. The rate of change the angular position of the particle is called the angular speed ω.

Consider the figure above. Then

$$\omega = \frac{d\theta}{dt} = \lim_{\triangle t \to 0} \frac{\triangle \theta}{\triangle t}.$$

For any point (x, y, z) on the rotating object, let $\mathbf{r} = xi + yj + zk$. Then the speed v of the particle is, by definition,

$$v = \lim_{\triangle t \to 0} \frac{\triangle S}{\triangle t} = \lim_{\triangle t \to 0} \frac{R \triangle \theta}{\triangle t} = Rw,$$

where R is the radius of the circle. But $R = \|\mathbf{r}\| \sin\theta$. and the angular velocity Ω of the moving particle has magnitude ω and is in the same direction as a vector drawn from the origin to the center of the circle, pointing in the positive direction of advance of a right-hand screw when turned in the same sense as the rotation of the particle as shown above in the figure.

Therefore, the tangential velocity $\mathbf{v}$ is given by $\mathbf{v} = \Omega \times \mathbf{r}$. But, if $\Omega = ai + bj + ck$, then

$$\mathbf{v} = \Omega \times \mathbf{r} = (bz - cy)i + (cx - az)j + (ay - bx)k.$$

But

$$\nabla \times \mathbf{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2ai + 2bj + 2ck = 2\Omega.$$

This implies

$$\Omega = \frac{1}{2}(\nabla \times \mathbf{v}) = \frac{1}{2} crul\mathbf{v} \iff crul\mathbf{v} = 2\Omega.$$

Hence, the Curl of the velocity of the particle is two times its angular velocity.

Let G be a continuous scalar field with continuous first and second partial derivatives. Then

$$\mathrm{grad} G = \frac{\partial F}{\partial x} i + \frac{\partial G}{\partial y} j + \frac{\partial G}{\partial z} k$$

and

$$\nabla \times (\nabla G) = \nabla \times \left(\frac{\partial F}{\partial x} i \frac{\partial G}{\partial y} j + \frac{\partial G}{\partial z} k \right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 G}{\partial y \partial z} - \frac{\partial^2 G}{\partial z \partial y} \right) i + \left(\frac{\partial^2 G}{\partial z \partial x} - \frac{\partial^2 G}{\partial x \partial z} \right) j + \left(\frac{\partial^2 G}{\partial x \partial y} - \frac{\partial^2 G}{\partial y \partial x} \right) k.$$

But by assumption the function G is continuous with continuous first and second partial derivatives and hence the mixed partial derivatives are equal, that is,

$$\frac{\partial^2 G}{\partial y \partial z} = \frac{\partial^2 G}{\partial z \partial y}, \qquad \frac{\partial^2 G}{\partial z \partial x} = \frac{\partial^2 G}{\partial x \partial z} \qquad \text{and} \qquad \frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x}.$$

Therefore $\nabla \times (\nabla G) = \mathbf{0}$.

Let F be a continuous vector field given by $F = fi + gj + hk$ such that f, g and h have continuous first and second partial derivatives. Then

$$\nabla.(\nabla \times F) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) . \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

$$= \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 h}{\partial y \partial x} + \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y} = 0,$$

since the component functions f, g and h have continuous first and second partial derivatives.

Therefore we have proved the following relationships between gradient, divergence and curl that are fundamental to vector analysis.

Theorem 4.5.3. *Let F be a continuous vector field given by $F = fi + gj + hk$ such that f, g and h have continuous first and second partial derivatives and G be a continuous scalar field with continuous first and second partial derivatives. Then $curl\ grad\ G = \mathbf{0}$, the zero vector and $div\ curl\ F = 0$, the number zero.*

4.5.1 Potential

Recall that, if a scalar field f is differentiable at every point D of its domain, then $V(P) = \nabla f(P)$ defines a vector field V on D.

Example 4.5.2. *If $f(x,y) = 3x^2 + xy + y^3$, then $\nabla f(x,y) = V(x,y) = (6x+y, x+3y^2)$ is a vector field. Here the function f is called a potential of the vector V.*

However, not every vector field has a potential, that is, not every vector field ia a gradient of some scalar field.

Example 4.5.3. *The velocity field*

$$V(x,y) = (-cy, cx), c \in \mathbb{R}\backslash\{0\}$$

is not the gradient of any function f.

Suppose the contrary, i.e. suppose that there exists a function

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad \textit{such that} \qquad \nabla f = V.$$

That is, $(f_x, f_y) = V$. But $f_x(x,y) = -cy$ and $f_y(x,y) = cx$. This implies $f(x,y) = -cxy + A(y)$, where A is a function of y only and $f_y(x,y) = -cx + A'(y) = cx$. This gives us $-2cx + A'(y) = 0$ and hence $A'(y) = 2cx$. But this is a contradiction since A is a function of y only.

Let V be a vector field. Let us ask the following two questions.

1. **How do we check whether V has a potential or not?**
2. **How do we determine the potential f, given V?**

Now let us answer the first question. How do we check whether a vector field has a potential or not? The following proposition will answer this question.

Proposition 4.5.4. *(Test for Existence of a Potential)*

1. *Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by $V(p) = (V_1(p), V_2(p))$. Then V has a potential function if and only if*

$$\frac{\partial V_1}{\partial y}(p) = \frac{\partial V_2}{\partial x}(p) \qquad \textit{for all } p \textit{ in the Domain of } V.$$

2. *Let $V : \mathbb{R}^n \to \mathbb{R}^3, n = 2,3$ be given by $V(p) = (V_1(p), V_2(p), V_3(p))$. Then V has a potential function if and only if $curV = \nabla \times V = \mathbf{0} = (0,0,0)$.*

Example 4.5.4. *1. Let $V(x,y,z) = (2+y, x-z^2, -2yz)$. Then*

$$\nabla \times V = \begin{vmatrix} i & i & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)i + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)k$$

$$= (-2z - (-2z))i + (0-0)j + (1-1)k = (0,0,0).$$

Therefore, V has a potential.

2. *Let $V(x,y) = (-cy, cx), c \in \mathbb{R}\backslash 0$. Then since*

$$\frac{\partial V_1}{\partial y}(x,y) = -c \qquad \textit{and} \qquad \frac{\partial V_2}{\partial x}(x,y) = c,$$

but $-c \neq c, \forall c \neq 0$ and hence $V(x,y)$ has no potential.

For the second question we illustrate the procedure by the considering the following two examples.

1. Let $V(x,y) = (6x+y, x+3y^2)$ be given. Then if there is a potential function f for V it must satisfy

$$f_x(x,y) = 6x + y \qquad \text{and} \qquad f_y(x,y) = x + 3y^2.$$

This implies

$$f(x,y) = \int (6x+y)dx = 3x^2 + xy + A(y),$$

were $A(y)$ is constant with respect to x(or, it is a function of y only).
Then from $f_y(x,y) = x + 3y^2$, we get $f_y(x,y) = x + A'(y) = x + 3y^2$, which implies that $A'(y) = 3y^2$ and hence

$$A(y) = \int 3y^2 dy = y^3 + C, \qquad \text{where C is a constant.}$$

Therefore the scalar field $f(x,y) = 3x^2 + xy + y^3 + C$ is the potential of the vector field

$$V(x,y) = (6x+y, x+3y^2).$$

2. Let V be a vector field given by $V(x,y,z) = (2+y, x-z^2, -2yz)$.
Then if f is a potential, we must have $f_x(x,y,z) = 2+y$, $f_y(x,y,z) = x - z^2$ and $f_z(x,y,z) = -2yz$. We integrate $f_x(x,y,z) = 2+y$ with respect to x and get

$$f(x,y,z) = \int (2+y)dx = 2x + yx + A(y,z).$$

But $f_y(x,y,z) = x - z^2$ and

$$f_y(x,y,z) = x + \frac{\partial A}{\partial y}(y,z)$$

which implies that

$$x + \frac{\partial A}{\partial y}(y,z) = x - z^2$$

and hence

$$\frac{\partial A(y,z)}{\partial y} = -z^2.$$

We integrate this with respect to y to get $A(y,z) = -z^2y + B(z)$, where B is a function of z only and hence $f(x,y,z) = 2x + y - z^2y + B(z)$.

Since $f_z(x,y,z) = -2yz$, we get $f_z(x,y,z) = -2zy + B'(z) = -2yz$ which implies that $B'(z) = 0$, which means $B(z) = C$, where C is a constant.

Therefore, $f(x,y,z) = 2x + yx - z^2y + C$.

4.6 Exercises

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Chapter 5

Line and Surface Integrals

5.1 Line Integrals

Recall that, the integral
$$\int_a^b f(x)dx$$
of a continuous function f represents the definite integral of the function f over a closed interval $[a, b] = \{x : a \leq x \leq b\}$.

In the line integral we shall integrate a given function F along a curve, say C. So we need some preliminaries about curves. First, recall the points that we have raised about curves in Section 4.3.

Suppose C is a curve with parametrization

$$\mathbf{x} = \mathbf{x}(t), \qquad \mathbf{y} = \mathbf{y}(t), \qquad \mathbf{z} = \mathbf{z}(t)) \qquad \text{for} \qquad a \leq t \leq b.$$

Here the functions $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are the coordinate functions, $\mathbf{r}(a)$ is the initial point and $\mathbf{r}(b)$ is the terminal point of C. In this case C is known as an oriented curve and the orientation is indicated by putting arrows along the path on the curve.

A curve C with parametrization

$$\mathbf{r}(t) = (\mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t)) = \mathbf{x}(t)i + \mathbf{y}(t)j + \mathbf{z}(t)k$$

is called a smooth curve, if it has a unique tangent at each of its points. i.e. if $\mathbf{r}(t)$ is differentiable and
$$\mathbf{r}'(t) = \frac{dr}{dt}$$

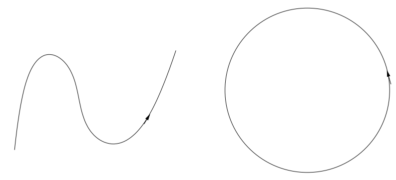

Figure 5.1: Oriented Curves.

is continuous and nonzero at every point of C.

Definition 5.1.1. *The* **Line Integral** *of a vector valued function $F(r)$ over a curve C parameterized by $\mathbf{r}(t) = \mathbf{x}(t)i + \mathbf{y}(t)j + \mathbf{z}(t)k$ for $a \leq t \leq b$ is defined by*
$$\int_C F(r).dr = \int_a^b F(r(t)).\frac{dr}{dt}dt, \qquad \textit{where } dr = (dx, dy, dz).$$
When we write it componentwise, that is, if $F = (F_1, F_2, F_3)$, then it becomes:
$$\int_C F(r).dr = \int_C (F_1dx + F_2dy + F_3dz) = \int_a^b (F_1x' + F_2y' + F_3z')dt.$$
The curve C is called path of integration.

Example 5.1.1. *Let $F(r) = (y, x, z)$ and C be the helix $r(t) = (\cos t, \sin t, 3t)$ for $0 \leq t \leq 2\pi$. Then $r'(t) = (-\sin t, \cos t, 3)$ and*
$$\int_C F(r).dr = \int_0^{2\pi} F(r).r'(t)dt = \int_0^{2\pi} (\sin t, \cos t, 3t)(-\sin t, \cos t, 3)dt$$
$$= \int_0^{2\pi} (-\sin^2 t \cos^2 t + 9t)dt = -\int_0^{2\pi} \sin^2 tdt + \int_0^{2\pi} \cos^2 tdt + \int_0^{2\pi} 9tdt$$
$$= \frac{1}{4}\sin 2t|_0^{2\pi} - \frac{1}{2}t|_0^{2\pi} + \frac{1}{4}\sin 2t|_0^{2\pi} + \frac{1}{2}t|_0^{2\pi} + \frac{9}{2}t^2|_0^{2\pi} = 18\pi^2$$

Example 5.1.2. *Evaluate*
$$\int_L (xyzdx - \cos(xy)dy + ydz),$$
where L is the line segment from $(0, 1, 1)$ to $(2, 1, -3)$.

Solution

Parametric equation of L is given by $x = 2t, y = 1, z = 1 - 4t$ for $0 \leq t \leq 1$ and $dx = 2dt$, $dy = 0$, $dz = -4dt$. Then the line integral is
$$\int_L (xyzdx - \cos(xy)dy + ydz) = \int_0^1 \big(2t(1-4t)(2) - \cos(2t)(0) + 1(-4)\big)dt$$

$$= \int_0^1 (4t - 16t^2 - 4)dt = \left(2t^2 - \frac{16}{3}t^3 - 4t\right)\Big|_0^1 = 2 - \frac{16}{3} - 4 = -\frac{22}{3}.$$

Remark 5.1.2. *Let C be a curve with parametrization* $\mathbf{r}$ *on the closed interval* $[a, b]$ *and F be a vector valued function defined on C.*

1. *The integrand in the line integral is a scalar not a vector, because we take a dot (scalar) product of two vectors,* $F(r(t)).r'(t)$.
2. *If the integrand function F is a scalar valued function, the line integral will take the following form.*
$$\int_C f(x, y, z,)dS = \int_a^b f(x(t), y(t), z(t))\sqrt{(r'(t))^2}.dt = \int_a^b f(x(t), y(t), z(t))\sqrt{r'(t).r'(t)}dt$$
3. *If the path of integration C is a closed curve, that is, if* $r(a) = r(b)$, *then the line integral will be denoted by*
$$\oint_C \qquad instead\ of \qquad \int_C .$$

Example 5.1.3 (Mass of a Helical wire). *Determine the mass M of a wire that is in the shape of a helical curve* $C : r(t) = (a\cos t, a\sin t, bt)$ $0 \le t \le 2n\pi, n \in \mathbb{N}$ *and that has a mass density* $\sigma = ct$ *that varies along C.*

Solution

Recall that the mass M of the wire is given by

$$M = \int_C \sigma dS, \qquad where \qquad dS = \sqrt{r'(t).r'(t)}dt.$$

But $r'(t) = (-a\sin t, a\cos t, b)$ *and* $r'(t).r'(t) = a^2\sin^2 t + a^2\cos^2 t + b^2 = a^2 + b^2$. *Therefore,*

$$M = \int_C \sigma dS = \int_0^{2n\pi} ct.\sqrt{r'(t).r'(t)}dt = \int_0^{2n\pi} ct\sqrt{a^2 + b^2}dt = 2cn^2\pi^2\sqrt{a^2 + b^2}.$$

5.2 Line Integrals Independent of Path

Consider the line integral

$$\int_c F(r).dr = \int_C (F_1dx + F_2dy + F_3dz), \tag{5.1}$$

where C is the path of integration.

The line integral (5.1) is said to be **independent of path of integration in a domain D** if for every pair of endpoints A and B in D the integral (5.1) has the same value for all paths in D that begin at A and end at B.

Question: Which line integrals are independent of paths?

The following theorem has an answer for this question.

Theorem 5.2.1. *Let* F_1, F_2 *and* F_3 *be continuous functions in a set D and let* $F = (F_1, F_2, F_3)$. *A line integral (5.1) is independent of path in D if and only if* $F = (F_1, F_2, F_3)$ *is the gradient of some potential function f in D, i.e., if there exists a function f in D such that* $F = \nabla f$, *which is equivalent to saying that*

$$F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y} \qquad and \qquad F_3 = \frac{\partial f}{\partial z}.$$

Let F be a conservative vector valued function with potential function f and let C be a curve with coordinate function $x = x(t), y = y(t), z = z(t)$ for $a \le t \le b$. Then

$$\begin{aligned}\int_C F.dr &= \int_C \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt\\ &= \int_a^b \frac{d}{dt}\Big(f(x(t), y(t), z(t)\Big)dt\\ &= f(B) - f(A),\end{aligned}$$

where $f(B) = f(x(a), y(b), z(b)), f(A) = f(x(a), y(a), z(a))$, the end points of the curve C. Therefore, we have the following theorem and some times it is called **Fundamental Theorem of Line Integrals**

Theorem 5.2.2. *If the integral (5.1) is independent of path in D, then*

$$\int_A^B (F_1dx + F_2dy + F_3dz) = f(B) - f(A), \qquad where \qquad F = \nabla f.$$

Example 5.2.1. *Evaluate:*

1. *The integral*
$$\int_C (2xdx + 2ydy + 4zdz),$$
if C is a curve with initial point $A = (0,0,0)$ and terminal point $B = (2,2,2)$.

2. *The integral*
$$\int_C (e^z dx + 2ydy + xe^z dz),$$
if C is a curve with with initial point $A = (0,1,0)$ and terminal point $B = (-2,1,0)$.

Solution

1. *Let $F_1 = 2x, F_2 = 2y$ and $F_3 = 4z$. Then $F = (F_1, F_2, F_3)$ and*
$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_C (2xdx + 2ydy + 4zdz),$$
where C is a curve with initial point $A = (0,0,0)$ and terminal point $B = (2,2,2)$.

But
$$\nabla \times F = \begin{vmatrix} i & i & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})k = \mathbf{0}.$$
Then there exists a function f such that $F = \nabla f$ which implies
$$F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y} \qquad and \qquad F_3 = \frac{\partial f}{\partial z}$$
and hence by Fundamental Theorem of Line Integrals, we have
$$\int_C (2xdx + 2ydy + 4zdz) = \int_C (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A).$$
Now, by using the same procedure as in Section **??** *of the previous chapter, we can get $f(x,y,z) = x^2 + y^2 + 2z^2 + k$, where k is a constant.*

Therefore,
$$\int_C (2xdx + 2ydy + 4zdz) = f(2,2,2) - f(0,0,0) = 24.$$

2. *Let $F_1 = e^z, F_2 = 2y$ and $F_3 = xe^z$. $F = (F_1, F_2, F_3)$ and*
$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_C (e^z dx + 2ydy + xe^z dz),$$
where C is a curve with initial point $A = (0,1,0)$ and terminal point $B = (-2,1,0)$.

But But
$$\nabla \times F = \begin{vmatrix} i & i & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})k = \mathbf{0}.$$
Then there exists a function f such that $F = \nabla f$ which implies
$$F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y} \qquad and \qquad F_3 = \frac{\partial f}{\partial z}$$
and hence by Fundamental Theorem of Line Integrals, we have
$$\int_C (2xdx + 2ydy + 4zdz) = \int_C (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A).$$
Now, by using the same procedure as in Section **??** *of the previous chapter, we can get $f(x,y,z) = xe^z + y^2 + k$, where k is a constant.*

Therefore,
$$\int_C (2xdx + 2ydy + 4zdz) = f(-2,1,0) - f(0,1,0) = -2.$$

The following remark is an immediate consequence of the Fundamental Theorem of Line Integrals.

Remark 5.2.3. 1. *The line integral (5.1) is independent of path in a domain D if and only if its value around every closed path in D is zero.*

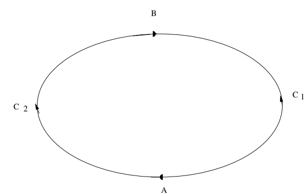

Figure 5.2: Two different paths between two points

2. *The line integral usually represents the work done by force F in the displacement of a body along path C. Hence if F has a potential function f, the line integral of F for displacement around any closed path is zero.*

   In this case, the vector field F is called **conservative**, *otherwise it is called* **nonconservative.**

Another way of checking the independence of path of a line integral is using exactness of a differential form. Recall that a differential form

$$F_1 dx + F_2 dy + F_3 dz$$

is said to be exact in a domain D if there is a differentiable function f such that

$$F_1 = \frac{\partial f}{\partial x}, \qquad F_2 = \frac{\partial f}{\partial y} \qquad \text{and} \qquad F_3 = \frac{\partial f}{\partial z} \qquad \text{in D.}$$

Definition 5.2.4. *A domain D is said to be simply connected if every closed curve in D can be shrunk to any point in D.*

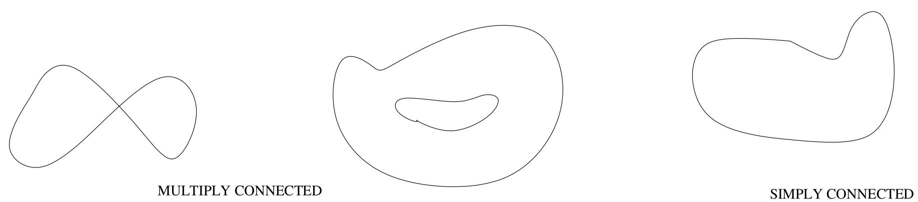

Figure 5.3: Multiply and Simply connected regions.

Theorem 5.2.5. *Suppose F_1, F_2 and F_3 are continuous and having continuous first order partial derivatives in a domain D and consider the line integral*

$$\int_c F(r).dr = \int_c (F_1 dx + F_2 dy + F_3 dz) \tag{5.2}$$

1. *If the line integral (5.2) is independent of path in D, then $CurlF = 0$. i.e.*

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad and \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

2. *If $CurlF = 0$ in D and if D is simply connected then the line integral (5.2) is independent of path in D.*

Remark 5.2.6. *In the plane, $\mathbb{R}^2$, the line integral $\int_C F(r).dr = \int_C (F_1 dx + F_2 dy)$ and $curlF = 0$ means*

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Example 5.2.2. *Show that thee following integrands are exact and evaluate the integrals.*

1.
$$\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy)$$

2.
$$\int_A^B \left(2xyz^2 dx + (x^2x^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right),$$

where $A = (0,0,1,)$ and $B = (1, \frac{\pi}{4}, 2)$.

Solution

1. *Let $F_1 = e^x \cos y$ and $F_2 = -e^x \sin y$. Then*

$$\frac{\partial F_1}{\partial y} = -e^x \sin y = \frac{\partial F_2}{\partial x}$$

and hence the differential in the integral is exact.

Then let us find the function f.

$$f(x,y) = \int F_2 dy = e^x \cos y + A(x)$$

and $f_x = e^x \cos y + A_x = e^x \cos y = F_1$, which implies that $A = C$, a constant. Therefore, the potential function is $f(x,y) = e^x \cos y + C$ and hence

$$\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy) = f(3, \frac{\pi}{2}) - f(0,\pi) = 0 + \pi = \pi.$$

2. *Let $F_1 = 2xyz^2$, $F_2 = x^2z^2 + z\cos yz$ and $F_3 = 2x^2yz + y\cos yz$. Then since we have $(F_3)_y = 2x^2z + \cos yz \sin yz = (F_2)_z$, $(F_1)_z = 4xyz = (F_3)_x$ and $(F_2)_x = 2xz^2 = (F_1)_y$, the differential in the integral is exact.*

Then let us find the function f.

$$f(x,y,z) = \int F_2 dy = \int (x^2z^2 + z\cos yz)dy = x^2z^2y + z\sin yz + A(x,z)$$

and $f_x = 2xz^2y + A_x(x,z) = 2xyz^2 = F_1$, which implies $A_x = 0$ and hence $A = B(z)$. Therefore $f(x,y,z) = x^2z^2y + \sin yz + B(x)$ which means $f_z = 2x^2yz + y\cos yz + B'(z) =$

$2x^2yz + y\cos yz$ that implies $B'(z) = 0$ and hence $B = C$, a constant.
Therefore the potential function is $f(x,y,z) = x^2z^2y + \sin y + C$ and hence

$$\int_A^B (2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz)$$

$$= f(B) - f(A) = 1.\frac{\pi}{4}.2^2 + (\sin(\frac{\pi}{4}\times 2) - 0 + \sin 0$$
$$= \pi + 1 - 0 = 1 + \pi.$$

General Properties of Line integrals

Let F and G be continuous vector fields, C be a path joining points A and B. Furthermore, suppose that C is subdivided into two arcs C_1 and C_2 that have the same orientation as C. Then

1. $$\int_C kF.dr = k\int_C F.dr$$

   for any constant k.

2. $$\int_C (F+G).dt = \int_C F.dr + \int_C G.dr$$

3. $$\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr$$

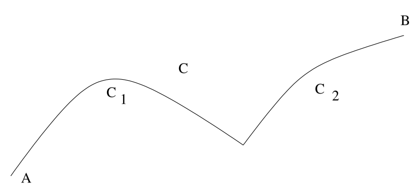

Figure 5.4: Subdivisions of a curve

Example 5.2.3. *Let C be a curve consisting of portion of a parabola $y = x^2$ in the $xy-$plane from $(0,0)$ to $(2,4)$ and a horizontal line from $(2,4)$ to $(4,4)$. Evaluate*

$$\int_C (y^2dx + x^2dy).$$

Write $C = C_1 \oplus C_2$, where C_1 is the portion of the parabola and C_2 is the line segment. Parameterize C_1 by $x = t, y = t^2$ for $0 \le t \le 2$ and on C_1, $dx = dt, dy = 2tdt$. Therefore,

$$\int_{C_1} (y^2dx + x^2dy) = \int_0^2 (t^4 + 2t^3)dt = (\frac{t^5}{5} + \frac{t^4}{2})\Big|_0^2 = \frac{32}{5} + 8 = \frac{72}{5}.$$

Parameterize C_2 by $x = t, y = 2$ for $2 \le t \le 4$ and on C_2, $dx = dt, dy = 0$. Therefore,

$$\int_{C_2} (y^2dx + x^2dy) = \int_2^4 (4+0)dt = 4t\Big|_2^4 = 8.$$

Hence

$$\int_C (y^2dx + x^2dy) = \frac{72}{5} + 8 = \frac{112}{5}.$$

4. If C' has an opposite orientation to that of C, then

$$\int_C F.dr = -\int_{C'} F.dr.$$

5.3 Green's Theorem

Over a plane region, double integrals can be transformed into line integrals over the boundary of the regions and conversely. This can be done using Green's Theorem which is stated stated below.

Theorem 5.3.1 (Green's Theorem)**.** *Let R be a closed bounded region in the $xy-$plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives every where in some domain containing R, (i.e.$(F_1)_y$ and $(F_2)_x$ are continuous in R.) Then*

$$\iint_R (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy = \oint_C (F_1dx + F_2dy), \tag{5.3}$$

where C is the boundary of R.

Example 5.3.1. *1. Use Green's Theorem to evaluate*

$$\int_C (x^2ydx + xdy)$$

over the triangular path in the figure.

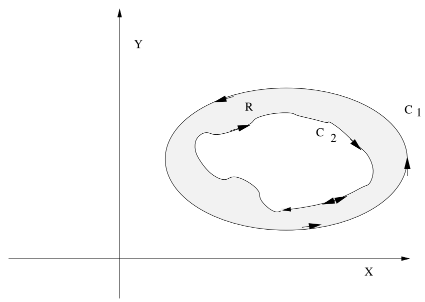

Figure 5.5: Two boundaries of a closed region

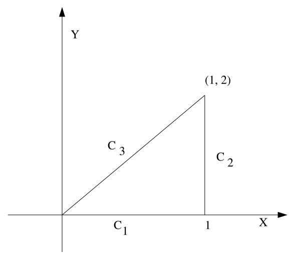

Figure 5.6: Triangular path

Solution

The curves are parameterized as: $C_1 : r(t) = (t, 0)$ *for* $0 \le t \le 1$, $C_2 : r(t) = (1, 2t)$ *for* $0 \le t \le 1$ *and* $C_3 : r(t) = (1-t, 2-2t)$ *for* $0 \le t \le t$.
Since $F_1 = x^2y$ *and* $F_2 = x$*, we have from Green's Theorem that:*

$$\oint_C (x^2ydx + xdy) = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2y)\right)dA = \int_0^1 \int_0^{2x} (1-x^2)dydx$$

$$= \int_0^1 (1-x^2)(2x)dx = \left(x^2 - \frac{x^4}{2}\right)\Big|_0^1 = \frac{1}{2}.$$

2. *Find the work done by the force field* $F(x,y) = (e^x - y^3)i + (\cos y + x^3)j$ *on a particle that travels once around the unit circle* $x^2+y^2=1$ *in the counterclockwise direction.*

Solution

Clearly the work W done by the field F is given by

$$W = \oint_C F.dr = \oint_C \big((e^x - y^3)dx + (\cos y + x^3)dy\big).$$

But since F is not conservative (verify) W need not be zero,though C is a simple closed curve. If we parameterize the circle by $x = \cos t, y = \sin t$ *on* $0 \le t \le 2\pi$*, the integral will be complicated and difficult to solve.*
However if we use Green's Theorem we have:

$$\begin{aligned} W &= \oint_C (e^x - y^3)dx + (\cos y + x^3)dy \\ &= \iint_R [\frac{2}{2x}(\cos y + x^3) - \frac{2}{2y}(e^x - y^3)]dA. \end{aligned}$$

$$\begin{aligned} &= \iint_R (3x^2+3y^2)dA = 3\iint_R (x^2+y^2)dA. \\ &= 3\int_0^{2\pi}\int_0^1 r^2 r dr d\theta \ (using\ polar\ coordinates) \\ &= \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3\pi}{2}. \end{aligned}$$

Here R is region bounded by a unit circle.

Remark 5.3.2. *Green's Theorem can be used to find areas of a plane region. Let R be a plane region with boundary C.*

1. *Area of the region R in cartesian coordinates.*
First choose $F_1 = 0$ *and* $F_2 = x$*. Then, as in* 5.3.1 *above,*

$$\iint_R dxdy = \oint_C xdy$$

and then choose $F_1 = -y$ *and* $F_2 = 0$ *to get*

$$\int\int_R dxdy = -\oint_C yxdy$$

By adding up the two we get

$$2\iint_R dxdy = \oint_c xd - \oint_c ydy = \oint_c (xdy - ydx)$$

Therefore, the area $A(R)$ of the region bounded by the curve C is given by:

$$A(R) = \int\int_R dxdy = \frac{1}{2}\oint_c (xdy - ydx). \tag{5.4}$$

For example, to find the area of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we write $x = a\cos t, y = b\sin t$, $0 \le t \le 2\pi$. Then $x' = -a\sin t, y' = b\cos t$ Then area of region bounded by the ellipse is:

$$\begin{aligned} A(R) &= \frac{1}{2}\oint(xdy - ydx) = \frac{1}{2}\oint(xy' - yx')dt \\ &= \frac{1}{2}\int^{2\pi}(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)dt \\ &= \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t + ab\sin^2 t)dt \\ &= \frac{1}{2}\int_0^{2\pi}(ab)dt = \frac{ab}{2}t\Big|_0^{2\pi} = ab\pi \end{aligned}$$

2. *Area of a plane region in polar coordinates.*
   Let $x = r\cos\theta$ and $y = r\sin\theta$, where (r,θ) is the polar coordinate of point (x,y). Then $dx = \cos\theta dr - r\sin\theta d\theta, dy = \sin\theta dr + r\cos\theta d\theta$. Hence (5.4) becomes

$$\begin{aligned} A(R) &= \frac{1}{2}\oint_c(xdy - udx) \\ &= \frac{1}{2}\oint_c[r\cos\theta(\sin\theta dr + r\cos\theta d\theta) - r\sin\theta(\cos\theta dr - r\sin\theta d\theta)] \\ &= \frac{1}{2}\oint_c[(r\cos\theta dr - r\cos\theta\sin\theta dr) + (r^2\cos^2\theta d\theta + r^2\sin^2\theta d\theta)] \\ &= \frac{1}{2}\oint_c r^2 d\theta. \end{aligned}$$

Therefore, the area $A(R)$ of the region bounded by the curve C is given in polar form by:

$$A(R) = \frac{1}{2}\oint_C r^2 d\theta.$$

For example, to find the area of the region bounded by the cardioid $r = a(1-\cos\theta)$, where $0 \le \theta \le 2\pi$ and a is a positive constant.

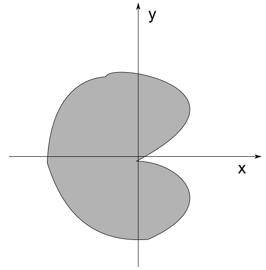

Figure 5.7: A cardioid $r = a(1-\cos\theta)$, where $0 \le \theta \le 2\pi$ and a is a positive constant.

$$\begin{aligned} A(R) &= \frac{1}{2}\oint_c r^2 d\theta = \int^{2\pi}[a(1-\cos\theta)]^2 d\theta = \frac{a^2}{2}\int_0^{2\pi}(1 - 2\cos\theta + cos^2\theta)d\theta \\ &= \frac{a^2}{2}\int_0^{2\pi}d\theta - a^2\int_0^{2\pi}\cos\theta + \frac{a^2}{4}\int_0^{2\pi}(\cos 2\theta + 1)d\theta \\ &= \frac{a^2}{2}x2\pi + \frac{a^2}{4}x2\pi + 0 = a^2\pi + a^2\frac{\pi}{2} = \frac{3a^2\pi}{2} \end{aligned}$$

Therefore $A(R) = \frac{3a^2\pi}{2}$.

Graphs in Polar Coordinates

1. **Circles**

   A circle of radius a that is centered at the origin consists of all point of the form $P(a,\theta)$. Thus such circles in polar coordinate has the equation $r = a$.
   The equation of a circle that is centered on the $x - axis$ and passes through the origin has an equation of the form $r = 2a\cos\theta$ or $r = -2a\cos\theta$.

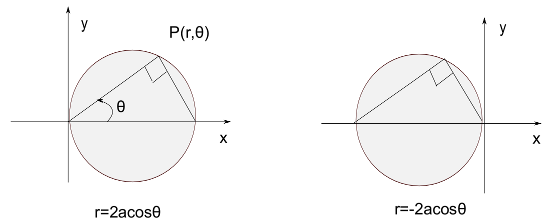

Figure 5.8: Circles centered on $x-$axis.

The equation of a circle that is centered on the $y-axis$ and passes through the origin has an equation of the form $r = 2a\sin\theta$ or $r = -2a\sin\theta$.

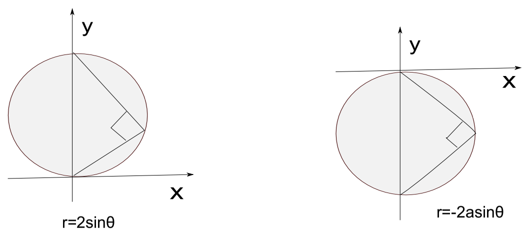

Figure 5.9: Circles centered on $y-$axis.

2. **Cardioid and Limacons**
   Equations of the form

$$r = a + \sin\theta, \qquad r = a - b\sin\theta r = a + b\cos\theta \qquad \text{or} \qquad r = a - b\cos\theta$$

produce polar curves called limacons.
Depending on the ratio $c = \frac{a}{b}$ we get four categories.

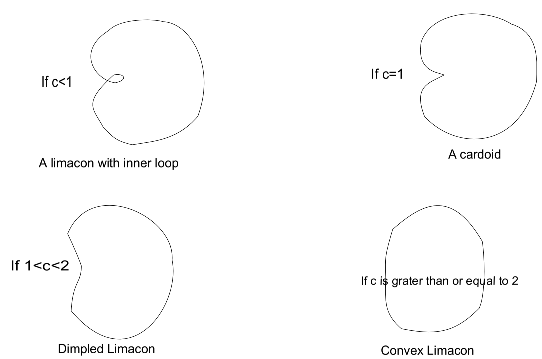

Figure 5.10: Cardioid and Limacons.

3. **Rose Curves.** Equations of the form

$$r = a\sin n\theta \qquad \text{and} \qquad r = a\cos n\theta$$

represents a flower shaped curves called **roses.**

When we graph r verses θ in the cartesian (r, θ) plane, we ignore the points where r is imaginary but plot positive and negative parts from the points where r^2 is positive.

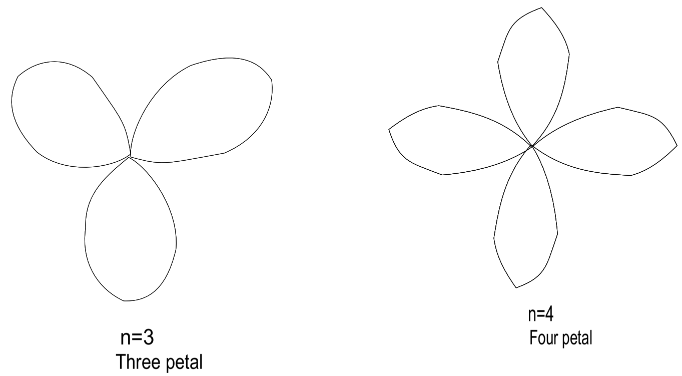

Figure 5.11: Rose Curves.

5.3.1 Green's Theorem for Multiply Connected Regions

Recall that, a region in $\mathbb{R}^2$ is called simply connected if it is connected and has no holes, and is called **multiply connected** if it is connected but it has finitely many holes.

Now consider the vector field $F = (F_1, F_2)$ which is continuously differentiable over the plane region R, which is multiply connected as shown in Figure 5.3.1.

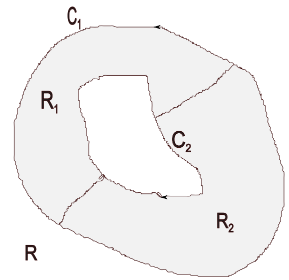

Figure 5.12: Dividing a region in to regions

First divide R into simply connected regions R_1 and R_2. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$
$$= \oint_{C_1} (F_1 dx + F_2 dy) + \oint_{C_2} (F_1 dx + F_2 dy),$$

where the curves C_1 and C_2 are the boundaries of the regions R_1 and R_2 respectively. The orientation of the curves should be in such a way that when traveling along the curves the region should be to the left.

Example 5.3.2. *Evaluate the integral*

$$\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2} \right)$$

if C is a piecewise smooth simply closed curve oriented counterclockwise such that C incloses the origin. Consider Figure 5.3.2.

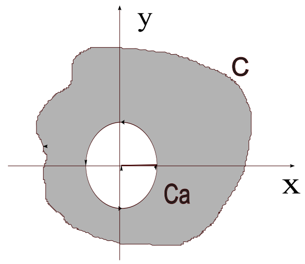

Figure 5.13: A curve that encloses the origin

Solution

Let $F = (F_1, F_2)$ *such that* $F_1 = \frac{-y}{x^2+y^2}$ *and* $F_1 = \frac{x}{x^2+y^2}$. *Since* F_1 *and* F_2 *are undefined at the origin, we can not apply Green's Theorem.*

Thus construct a circle C_a *with sufficiently small radius a and oriented counterclockwise as C. Thus*

$$\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2} \right) + \oint_{-C_a} \left(\frac{-ydx + xdy}{x^2 + y^2} \right) = \iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) dA,$$

by Green's Theorem, where R is the region bounded by the curves $-C_a$ *and* C. *But*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2} \qquad \textit{which implies that} \iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) dA = 0.$$

This implies

$$\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2} \right) + \oint_{-C_a} \left(\frac{-ydx + xdy}{x^2 + y^2} \right) = 0$$

and hence

$$\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2} \right) = \oint_{C_a} \left(\frac{-ydx + xdy}{x^2 + y^2} \right).$$

Now let $x = a\cos t$ *and* $y = a\sin t$ *for* $0 \le t \le 2\pi$ *on* C_a *implies* $dx = -a\sin t dt$ *and* $dy = a\cos t dt$.

$$\begin{aligned} \oint_C \frac{-ydx + xdy}{x^2 + y^2} &= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)dt + (a\cos t)(a\cos t)dt}{(a\cos t)^2 + (a\sin t)^2} \\ & \int_0^{2\pi} \frac{(a^2\sin^2 t + a^2\cos^2 t)dt}{(a^2\cos^2 t + a^2\sin^2 t)} = \int_0^{2\pi} dt = 2\pi \end{aligned}$$

for any small radius a *and hence*

$$\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2} \right) = 2\pi.$$

5.4 Surface Integrals

In the previous sections we have been working on integral of vector fields over curves. Now are going to consider integrals of vector fields over surfaces. Let us start by discussing some facts about surfaces.
In the case of line integrals we represented a curve in $\mathbb{R}^3$ perimetrically as

$$x = x(t), y = y(t), z = z(t),$$

for $a \le t \le b$. That is, a curve is given by coordinate functions of one variable, where as, a surface is defined by parametric functions of two variables

$$x = x(u, v), y = y(u, v), z = z(u, v),$$

for (u, v) in some set in the $uv-$plane and the variables u and v are called parameters.

Example 5.4.1. *Parametrization of some surfaces.*

1. *The parametric representation of a cylinder*

$$x^2 + y^2 = a^2, -1 \le z \le 1 \qquad \textit{is} \qquad r(u, v) = a\cos u i + a\sin u j + v k.$$

2. *The parametric representation of a sphere*

$$x^2 + y^2 + z^2 = a^2 \qquad \textit{is} \qquad r(u, v) = a\cos v\cos u i + a\cos v\sin u j + a\sin v k$$

3. *The parametric representation of a cone*

$$z = \sqrt{x^2+y^2}, 0 \le z \le T \qquad is \qquad r(u,v) = u\cos v i + u \sin v j + uk$$

where $0 \le u \le T$ *and* $0 \le v \le 2\pi$.

For a surface we write a position vector as

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

and $r(u,v)$ can be considered as a vector in $\mathbb{R}^3$ with initial point the origin and terminal point $\big(x(u,v), y(u,v), z(u,v)\big)$ which is on the surface.

A surface with parametrization $\mathbf{r}$ is simple if it does not fold over and intersect itself. This means $r(u_1,v_1) = r(u_2,v_2)$ can occur only when $u_1 = u_2$ and $v_1 = v_2$.

5.4.1 Normal Vector and Tangent plane to a Surface

Recall that: if C is a curve with coordinate functions $x(t), y(t), z(t)$, then

$$T = x'(t_0) + y'(t_0)j + z'(t_0)k$$

is a vector that is tangent to the curve at a point $P_0 = \big(x(t_0), y(t_0), z(t_0)\big)$.

Let Ω be a surface in $\mathbb{R}^3$ with coordinate functions $x(u,v), y(u,v), z(u,v)$ and let P_0 be the point $\big(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)\big)$ on the surface Ω. We want to find a normal vector N to the surface at P_0.

Let Ω_u be the curve with coordinate functions

$$x(u,v_0), y(u,v_0), z(u,v_0).$$

Then the tangent vector

$$T_{v_0} = \frac{\partial x}{\partial u}(u,v_0)i + \frac{\partial y}{\partial u}(u,v_0)j + \frac{\partial z}{\partial u}(u,v_0)k$$

is a tangent vector to the curve Ω_u at P_0. Similarly, the vector

$$T_{v_0} = \frac{\partial x}{\partial u}(u_0,v))i + \frac{\partial y}{\partial u}(u_0,v)j + \frac{\partial z}{\partial u}(u_0,v)k$$

is a tangent vector to the curve Ω_v with coordinate functions

$$x(u_0,v), y(u_0,v), z(u_0,v).$$

Assume that these two vectors are not zero. Then these two vectors lie in a plane tangent to the surface Ω at the point P_0 and hence the vectors

$$N(P_0) = T_{u_0} \times T_{v_0}$$

is a normal vector to the tangent plane and hence the surface to at the point P_0. But

$$T_{u_0} \times T_{v_0} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)i + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)j + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)k,$$

in which all the partial derivatives are evaluated at (u_0,v_0).
From the previous courses, recall that, the Jacobian of two functions f and g is defined to be

$$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}.$$

Then the normal vector

$$N(P_0) = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}j + \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}j + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}k,$$

where the partial derivatives are evaluated at (u_0,v_0).

For an arbitrary point (u,v) on the surface, the normal line to the tangent plane is given by $N = r_u \times r_v$ and we denote the corresponding unit vector in the direction of N by $\mathbf{n}$ and it is given by

$$\mathbf{n} = \frac{1}{\|N\|}N = \frac{1}{\|r_u \times r_v\|}(r_u \times r_v).$$

Example 5.4.2. *Find the equation of the tangent plane to the surface given by*

$$r(u,v) = ui + (u+v)j + (u+v^2)k$$

at a point$(2,4,6)$.

Solution

Here $x(u,v) = u, y(u,v) = u+v$ and $z(u,v) = u+v^2$. First let us find the normal vector $N(2,4,6)$ to the plane tangent to the surface at the given point which is

$$N(2,4,6) = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}i + \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}j + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}k = 8i + k.$$

Therefore, equation of the plane Π tangent to the given surface at the point $(2,4,6)$ is

$$\Pi : 8x + z = 22.$$

Remark 5.4.1. *If the surface Ω is a surface represented by the equation $g(x,y,z) = 0$, then the unit normal vector is given by*

$$\mathbf{n} = \frac{1}{\|\nabla g\|}\nabla g.$$

Example 5.4.3. *If Ω is the sphere $f(x,y,z) = x^2+y^2+z^2-a^2 = 0$ and $a \neq 0$, then*

$$\begin{aligned} n &= \frac{1}{\|\nabla f\|}\nabla f = \frac{1}{\sqrt{\nabla f.\nabla f}}(2x,2y,2z) \\ &= \frac{1}{\sqrt{4(x^2+y^2z^2)}}(2x,2y,2z) \\ &= \frac{1}{2a}(2x,2y,2z) = (\frac{x}{a},\frac{y}{a},\frac{z}{a}) \end{aligned}$$

Recall the following from calculus.
Suppose Ω represents a surface in $\mathbb{R}^3$ with equation $z = g(x,y)$ and let R be its projection on the $xy-$plane. If g has continuous first partial derivatives on R, then the surface area of Ω is

$$\text{Area of } \Omega = \iint_R \sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)}dA.$$

But the integrant is the norm of the normal vector $N(x,y)$ to the surface, that is,

$$\text{Area of } \Omega = \iint_R ||N(x,y)||dA.$$

Definition 5.4.2. *A surface S is called a smooth surface if the unit normal vector $\mathbf{n}$ is continuous on S and surface S is called piecewise smooth if it consists of finitely many smooth portions.*

Example 5.4.4. *Examples of smooth and piecewise smooth surfaces.*

1. *A sphere is a smooth surface, since at any point on the sphere, there is a continuous tangent normal.*

2. *A cube is a piecewise smooth since all the six faces are smooth, but the eight sides do not have tangents.*

Now we are in a position to define the surface integral of a vector field over a piecewise smooth surface.

Definition 5.4.3. *Suppose S is a smooth surface parameterized by $\mathbf{r}(u,v)$ with normal vector $N(u,v) = r_u \times r_v$. Let F be a continuous function on S. Then the surface integral of F over S is denoted by*

$$\iint_S F(x,y,z)d\sigma$$

and is defined by

$$\iint_S F(x,y,z)d\sigma = \iint_R F(r(u,v))\|N(u,v)\|dudv$$

and if F is a vector field then the surface integral of F over S

$$\iint_S F(x,y,z)d\sigma$$

is defined by

$$\iint_S F(x,y,z)d\sigma = \iint_R F(r(u,v)).N(u,v)dudv.$$

Example 5.4.5. *Evaluate*

$$\iint_S (x+y)d\sigma$$

where S is the portion of the cylinder $x^2+y^2 = 3$ between the planes $z = 0$ and $z = 6$.

Solution

The parametrization of the cylinder is

$$r(u,v) = \sqrt{3}\cos ui + \sqrt{3}\sin uj + vk,$$

for $0 \le u \le 2\pi$ and $0 \le v \le 6$.
Then $r_u = -\sqrt{3}\sin ui + \sqrt{3}\cos uj$ and $r_v = k$ and

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -\sqrt{3}\sin u & \sqrt{3}\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \sqrt{3}\cos ui + \sqrt{3}\sin uj$$

which implies $\|r_u \times r_v\| = \sqrt{3}$.
Therefore,

$$\begin{aligned}\iint_S (x+y)d\sigma &= \iint_R (\sqrt{3}(\cos u + \sin u))\|r_u \times r_v\| dA \\ &= 3\int_0^{2\pi}\int_0^6 (\cos u + \sin u)dvdu \\ &= 18\int_0^{2\pi}(\cos u + \sin u)du \\ &= 18\Big[\sin u - \cos u\Big]_0^{2\pi} = 0.\end{aligned}$$

Hence, $\iint_S (x+y)d\sigma = 0$.

5.4.2 Applications of Surface Integrals

Flux of A fluid Across a Surface

Suppose a fluid moves in some region of the space with velocity. The volume of fluid crossing a certain surface S per unit time is known as the **flux** across the surface S and the surface integral of a vector function F over a surface S describes the flux across S, when $F = \rho v$, ρ the density of fluid, v velocity of the flow.

Hence the above surface integral is known as the flux integral and if $F = (F_1, F_2, F_3)$ and $N = (N_1, N_2, N_3)$, then

$$\iint_s F.ndA = \iint_R (F_1N_1 + F_2N_2 + F_3N_3)dudv.$$

Similarly, for surface S in surface integrals we parameterize the surfaces. But since surfaces are two dimensional; S can be represented as

$$\Upsilon(u,v) = x(u,v)i + y(u,v)j + z(u,v)k, \qquad (u,v) \in R,$$

where R is some region in uv-plane.

A normal vector N of a surface S whose parametric form is

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

at the point P is

$$N = r_u \times r_v.$$

We denote the corresponding unit vector in the direction of N by n,

$$n = \frac{1}{\|N\|}N = \frac{1}{\|r_u \times r_v\|}(r_u \times r_v).$$

If a surface S is represented by a the equation $g(x,y,z) = 0$, then

$$n = \frac{1}{\|\nabla g\|}\nabla g.$$

Example 5.4.6. *Let S be the portion of the surface* $z = 1 - x^2 - y^2$ *that lie above the* xy*-plane, and suppose that S is oriented upward(i.e. n is in the upward direction at all points of S). Find the flux* Φ *of the flow field* $F(x,y,z) = (x,y,z)$ *across S.*

Solution

Here S is described by $g(x,y,z) = z - 1 + x^2 + y^2$ *and* $N = \nabla g = (2x, 2y, 1)$.
Returning back to the definition of surface integrals:

$$\iint_S F.ndA = \iint_R F(r(u,v)).N(u,v)dudv,$$

$N = n\|N\|$ *and* $n = (\cos\alpha, \cos\beta, \cos\gamma)$*, the direction cosines of N,* $F = (F_1, F_2, F_3)$.
Then

$$\begin{aligned}\iint_S F.ndA &= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA \\ &= \iint_S (F_1dydz + F_2dxdz + F_3dxdy)\end{aligned}$$

(This is similar to the formulation in line integrals.)

Surface Area

If Ω *is a piecewise smooth surface, then the area of the surface* Ω *is given by*

$$Area\ of\ \Omega = \iint_\Omega dA.$$

But $\|N\| = \|r_u \times r_v\|$ *represents the area of a parallelogram with adjacent side vectors* r_u *and* r_v*. Therefore, we can write* dA *as* $dA = \|r_u \times r_v\|dudv$*. Hence*

$$Area\ of\ \Omega = \iint_\Omega dA = \iint_R \|r_u \times r_v\|dudv,$$

where R is the projection on the $uv-$*plane of the surface* Ω.

Example 5.4.7. *Find the area of the surface of the torus given Figure 5.4.7.*

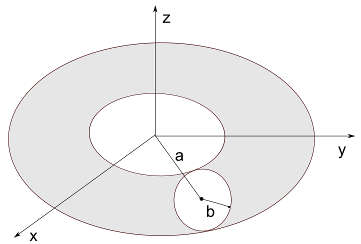

Figure 5.14: A torus

Here $\gamma(u,v) = (a + b\cos v)\cos ui + (a + b\cos v)\sin uj + b\sin vk$.
Thus $r_u = -(a + b\cos v)\sin ui + (a + b\cos v)\cos uj + 0k$,
$r_v = -b\sin v\cos ui - b\sin v\sin uj + b\cos vk$ *and and hence*

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -(a + b\cos v)\sin u & (a + b\cos v)\cos u & 0 \\ -b\sin v \ \cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$= b(a + b\cos v)(\cos u\cos vi + \sin u\cos vj + \sin vk).$$

Which implies $\|r_u \times r_v\| = b(a + bb\cos v)$ *and hence*

$$\begin{aligned} A(S) &= \int\int_R \|r_u \times r_v\| dudv = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)dudv \\ &= \int_0^{2\pi}\int_0^{2\pi} badudv + \int_0^{2\pi}\int_0^{2\pi} b^2\cos vdudv. \\ &= 4\pi^2 ab + b^2\int_0^{2\pi} odv \\ &= 4\pi^2 ab \end{aligned}$$

Then

$$\Phi = \iint_S F.ndA = \iint_R (2x^2 + 2y^2 + z)dA.$$

But since $z = 1 - x^2 - y^2$, *we have*

$$\begin{aligned} \Phi &= \int\int_R (x^2 + y^2 + 1)dA \\ &= \int_0^{2\pi}\int_0^1 (\gamma^2 + 1)\gamma d\gamma d\theta \\ &= \int_0^{2\pi}\left(\frac{3}{4}\right)d\theta = \frac{3\pi}{2} \end{aligned}$$

Mass and Center of Mass of a Shell

Consider a shell of negligible thickness in the shape of piecewise smooth surface Ω. Let $\delta(x,y,z)$ be the density of the material of the shell at point (x,y,z).
Let $x(u,v), y(u,v)$ and $z(u,v)$ be the coordinate functions of Ω for $(u,v) \in R$, where R is the projection of the surface in the $xy-$plane. Then the mass of Ω is given by

$$\text{Mass of } \Omega = \iint_\Omega \delta(x,y,z)d\sigma$$

and the center of mass of the shell is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_\Omega x\delta(x,y,z)\sigma, \qquad \bar{y} = \frac{1}{m}\iint_\Omega y\delta(x,y,z)\sigma \qquad \text{and} \qquad \bar{z} = \frac{1}{m}\iint_\Omega z\delta(x,y,z)\sigma,$$

where m is the mass of the shell.

If the surface is given by $z = f(x,y)$ for $(x,y) \in R$, then the mass is given by

$$m = \iint_\Omega \delta(x,y,z)\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2}\, dydx.$$

Example 5.4.8. *Find the center of mass of the sphere* Ω, $x^2 + y^2 + z^2 = a^2$, *in the first octant, if it has constant density* μ_0.

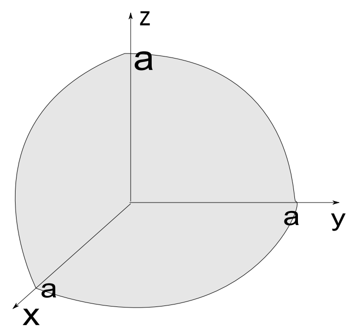

Figure 5.15: A sphere of radius a in the first octant.

Solution

The mass m *of the sphere is*

$$m = \iint_\Omega \mu_0 d\sigma.$$

In spherical coordinates we know that the equation of a sphere of radius a is given by, $\rho = a$. Now we change the cartesian coordinates in to spherical coordinates to get

$$x = a\cos\theta\sin\phi, \qquad y = a\sin\theta\sin\phi \qquad and \qquad z = a\cos\phi$$

for $0 \le \theta \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le \phi \le \pi$.
Therefore, the parametrization of the sphere in the first octant is :

$$r(\theta,\phi) = a\cos\theta\sin\phi i + a\sin\theta\sin\phi j + a\cos\phi k$$

for $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$.
Then $r_\theta = -a\sin\theta\sin\phi i + a\cos\theta\sin\phi j$ and $r_\phi = a\cos\theta\cos\phi i + a\sin\theta\cos\phi j - a\sin\phi k$.
Therefore,

$$r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \end{vmatrix}.$$

This implies

$$r_\theta \times r_\phi = -a^2\sin^2\phi\cos\theta i - a^2\sin^2\phi\sin\theta j - a^2\sin\phi\cos\phi k$$

and $\|r_\theta \times r_\phi\| = a^2\sin\phi$.
Therefore

$$m = \iint_\Omega \mu_0 a^2 \sin\phi d\theta d\phi = \mu_0 a^2 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\phi d\theta d\phi = \mu_0 \frac{a^2\pi}{2}.$$

Then let us find the coordinates of center of mass which is given by

$$\overline{x} = \frac{1}{m}\iint_\Omega x\mu_0 d\sigma, \qquad ,\overline{y} = \frac{1}{m}\iint_\Omega y\mu_0 d\sigma \qquad and \qquad \overline{z} = \frac{1}{m}\iint_\Omega z\mu_0 d\sigma.$$

$$\overline{x} = \frac{1}{m}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \mu_0 a^3\cos\theta\sin^2\phi d\theta d\phi = 2a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos\theta\sin^2\phi d\theta d\phi = \frac{a}{2}$$

and in a similar fashion we can find find $\overline{y} = \frac{a}{2}$ and $\overline{z} = \frac{a}{2}$. Therefore, the center of mass of the portion of the sphere is

$$(\overline{x},\overline{y},\overline{z}) = (\frac{a}{2},\frac{a}{2},\frac{a}{2}).$$

5.5 Divergence and Stock's Theorems

If a surface S is smooth and P is any point in S we can choose a unit normal vector n of S at P. Then we can take the direction of n as the positive normal direction of S at P(two possibilities).

A smooth surface is said to be **orientable** if the positive normal direction, given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface.

A smooth surface is said to be **piecewise orientable** if we can orient each smooth piece of the surface S in such a manner that along each curve C^* which is a common boundary of two pieces S_1 and S_2 the positive direction of C^* relative to S_1 is opposite to the positive direction of C^* relative to S_2.

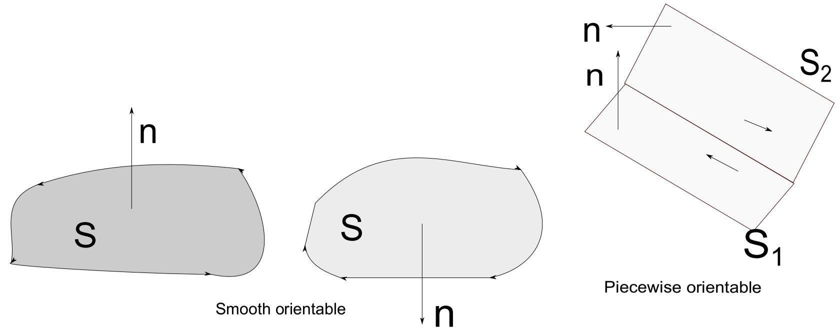

Figure 5.16: Smooth orientable and pieceorientable surfaces.

There are also non-orientable surfaces. Möbius strip [no inward and no outward directions once in once out word.]

Consider a boundary surface of a solid region D in 3-space. Such surfaces are called **closed.**

If a closed surface is orientable or piecewise orientable, then there are only two possible orientations: inward (to ward the solid) and outward (away from the solid).

Let $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ be vector field defined on a solid D. Then $divF = \frac{2F_1}{2x} + \frac{2F_2}{2y} + \frac{2F_3}{2z}$

Theorem 5.5.1 (Divergence Theorem of Gauss)**.** *Let D be a solid in $\mathbb{R}^3$ with surface S oriented outward. If $F = F_1 i + F_2 j + F_3 k$, where F_1, F_2 and F_3 have continuous first and second partial derivatives on some open set containing D, then*

$$\int\int_S F.ndA, = \iiint_D divF dv,$$

that is,

$$\iiint_D (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z})dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy).$$

Example 5.5.1. *Let S be the sphere $x^2+y^2+z^2 = a^2$ oriented outward. Find the flux of the vector function $F(x,y,z) = zk$ across S.*

Solution

Here

$$divF = \frac{\partial z}{\partial z} = 1.$$

If D is the spherical solid enclosed by S by Divergence Theorem, the flux Φ across S is

$$\Phi = \iint_S F.ndA = \iiint_D dV = volume\ of\ D = \frac{4\pi a^3}{3}.$$

Example 5.5.2. *Let S be the surface of the solid enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$, oriented outward. use the Divergence theorem to find the flux Φ of the vector field*

$$F(x, y, z) = x^3 i + y^3 j + z^2 k$$

across S.

Solution

We have $divF = 3x^2 + 3y^2 + 2z$. Thus if D is the cylindrical solid enclosed by S, we have:

$$\Phi = \iint_S F.ndA = \iiint_D divFdv = \iiint_D (3x^2 + 3y^2 + 2z)dv.$$

Let $x = \gamma\cos\theta$ for $0 \le \theta \le 2\pi$, $y = \gamma\sin\theta$ for $0 \le \gamma \le 3$ and $z = z$ for $0 \le z \le 2$. Then

$$\begin{aligned}\Phi &= \int_0^{2\pi}\int_0^3\int_0^2 (3\gamma^2 + 2z)\gamma dz d\gamma d\theta \\ &= \int_0^{2\pi}\int_0^3\int_0^2 (r^3 + 2rz)dzdrd\theta \\ &= \int_0^{2\pi}\int_0^3 (6\gamma^3 + 4\gamma)drd\theta = 279\pi\end{aligned}$$

Physical interpretation of the Divergence of a Vector Field

The Divergence Theorems leads us to the following physical interpretation of the divergence of a vector field F. Suppose D is a small spherical region centered at the point P_0 and that its surface S is oriented outward. Let $V(D)$ denote its volume and $\Phi(D)$ denote the flux of F across S.
If $divF$ is continuous on D, then it will not vary much from $divF(P_0)$ over a small region D. Hence

$$\Phi(D) = \iint_s F.ndA = \iiint_D divFdv \cong divF(P_0)\iiint_D dv = divF(P_0)V(D).$$

This implies,

$$divF(P_0) \cong \frac{\Phi(D)}{V(D)}.$$

The ratio $\frac{\Phi(D)}{V(D)}$ is called the **flux density of F** over D. If the radius of the sphere approach zero,(i.e. if $V(D) \to 0$) then the approximate value will be exact.

Hence

$$divF(P_0) = \lim_{v(D)\to 0}\frac{\Phi(D)}{v(D)} \quad \text{or} \quad divF(P_0) = \lim_{v(D)\to 0}\frac{1}{v(D)}\iint_S F.ndA.$$

This limit is called the **flux density of F** at the point P_0 and some times is taken as the definition for divergence.

In an incompressible fluid:

- Points P_0 at which div $F(_0) > 0$ are called sources (because $\Phi(D) > 0$, out flow).
- Points P_0 at which div$F(P_0) < 0$ are called **sinks** (b/c $\Phi(D) < 0$, in flow).
- Fluid enters the flow at a source and drains out at a sink.

If an incompressible fluid is without sources or sinks, we must have: $divF(P) = 0$ for all P point and in hydrodynamics, this is called the **continuity equation for incompressible fluids.**

Up to this point we were looking at the application of $divF$ in 3- space, We will now go to the $curlF$ in 3-space which helps us in generalizing Green's Theorem to a 3-dimensional object.

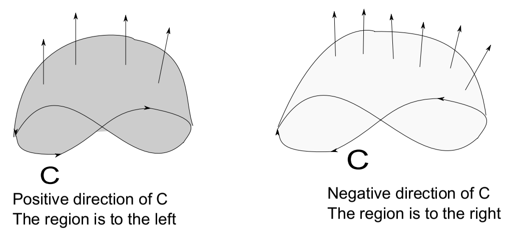

Figure 5.17: Orientated Curves.

Consider an oriented surface S with boundary C. If

$$F(x, y, z) = F_1(x, y, z)i + F_2(x, y, z)j + F_3(x, y, z)k,$$

then recall that

$$\mathbf{CurlF} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

Theorem 5.5.2 (Stoke's Theorem). *Let S be a piecewise smooth orientable surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the components of $F = (F_1, F_2, F_3,)$ are continuous and have continuous first partial derivatives on some open set containing S, and if T is the unit tangent vector of C, then.*

$$\oint_C F.TdS = \iint_S (curlF).ndA.$$

Recall that, $T = \frac{dr}{dS}$ which implies $dr = TdS$. Hence the above formula takes the following form

$$\oint_c F.dr = \iint_S (curlF).ndA$$

Example 5.5.3. *Let S be the portion of the paraboloid*

$$z = 4 - x^2 - y^2$$

for which $z \geq 0$, and let the vector field

$$F(x, y, z) = 2zi + 3xj + 5yk$$

is defined on S. Verify Stoke's Theorem , if S is oriented upward.

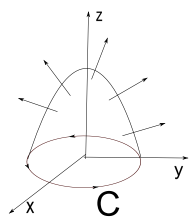

Figure 5.18: The paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$.

Solution

Here C is the circle $x^2 + y^2 = 4$ and is oriented counterclockwise, since S is oriented upward. Hence C can be parameterized as $r(t) = (2\cos t, 2\sin t), z = 0$, for $(0 \leq t \leq 2\pi)$ and we have

$dr = (-2\sin t, 2\cos t)dt$. *Therefore*

$$\begin{aligned}\oint_C F.dr &= \oint_C (2zdx + 3xdy + 5ydz) \\ &= \int_0^{2\pi} \big(2 \times 0 \times dx + 3(2\cos t)([2\cos tdt) + 5(2\sin t)dz\big) \\ &= \int_0^{2\pi} 12\cos s^2 tdt = [\frac{1}{2}t + \frac{1}{4}\sin 2t]_0^{2\pi} = 12\pi.\end{aligned}$$

On the other hand:

$$\mathbf{curlF} = \begin{vmatrix} i & j & k \\ \frac{2}{2x} & \frac{2}{2y} & \frac{2}{2z} \\ 2z & 3x & 5y \end{vmatrix} = 5i + 2j + 3k = (5, 2, 3)$$

Since $z = 4 - x^2 - y^2$, we have $g(x, y, z) = x^2 + y^2 + z - 4$ and $N = \nabla g = (2x, 2y1)$ Then:

$$\begin{aligned}\iint_S (curlF).ndA &= \iint_R (5,2,3).(2x,2y,1)dA = \iint_R (10x + 4y + 3)dA \\ &= \int_0^{2\pi}\int_0^2 (10r\cos\theta + 4r\sin\theta + 3)rdrd\theta \\ &= \int_0^{2\pi}\int_0^2 (10r^2 cos\theta + 4r^2 sin\theta + 3r)drd\theta \\ &= \int_0^{2\pi} \left(\frac{10}{3} \times 8\cos + \frac{4}{3}x8\sin\theta + \frac{3}{2} \times 4\right)d\theta \\ &= \left[\frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta\right]_0^{2\pi} = 12\pi,\end{aligned}$$

which agrees with the line integral value.

Remark 5.5.3. *1. If S_1 and S_2 have the same boundary C which is oriented positively, then for any vector function F that satisfy the hypotheses in Stoke's Theorem, we have:*

$$\iint_{S_1} (curlF).ndA = \iint_{S_2} (curlF).ndA.$$

2. If $F = (F_1, F_2)$ is a vector function that is continuously differentiable in a domain in the $xy-$plane containing a simply connected domain S whose boundary C is a piecewise smooth simple closed curve, then

$$(curlF).n = (curF).k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

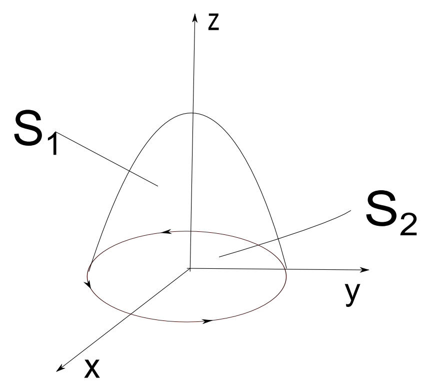

Figure 5.19: Surfaces with the same boundary.

Hence from Stoke's Theorem we have:

$$\iint_S \left(\frac{2F_2}{2x} - \frac{2F_1}{2y} \right) dA = \oint_C (F_1 dx + F_2 dy),$$

which is the result of Green's Theorem.

5.6 Exercises

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Part III

Complex Analysis

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Chapter 6

COMPLEX ANALYTIC FUNCTIONS

6.1 Complex Numbers

In this section we are going to revise the set of complex numbers on which we are going to work with in the coming chapters.

A complex number z is a symbol of the form $x+yi$ or $x+iy$, where x and y are real numbers and $i^2=-1$. Let $a+bi$ and $c+di$ be two complex numbers. The four basic arithmetic operations are defined as follows.

1. Equality: $a+bi=c+di$ if and only if $a=c$ and $b=d$.
2. Addition: $(a+bi)(c+di)=(a+c)+(b+d)i$.
3. Multiplication: $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$.
4. Division: Let $z=a+bi$ and $w=c+di$ be complex numbers and $z\neq 0$. Then

$$\frac{1}{z}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

and hence

$$\frac{w}{z}=w.\frac{1}{z}.$$

Example 6.1.1. *Suppose we want write*

$$\frac{3+i}{2-4i}$$

in the form of $a+bi$. But

$$\frac{3+i}{2-4i}=\frac{1}{2-4i}(3+i)$$

and

$$\frac{1}{2-4i}=\frac{2}{2^2+(-4)^2}+\frac{4}{2^2+(-4)^2}i=\frac{2}{20}+\frac{4}{20}i=\frac{1}{10}+\frac{1}{5}i.$$

Therefore,

$$\frac{3+i}{2-4i}=\left(\frac{1}{10}+\frac{1}{5}i\right)(3+i)=\frac{1}{10}+\frac{7}{10}i.$$

For a complex number $z=a+bi$, the number a is called the real part of z and denoted by $Re(z)$ and b is called the imaginary part of z and denoted by $Im(z)$.

Remark 6.1.1. *Some basic points about complex numbers.*

1. *The real and imaginary parts of any complex number are real numbers.*
2. *Any real number a can be considered as a complex number $a+0i$. Therefore, the set of complex numbers is an extension of the set of real numbers.*
3. *The set of complex numbers is denoted by $\mathbb{C}$.*
4. *If x,y and z are complex numbers, then:*

   4.1. $x+y=y+x$ *(Addition is commutative.)*

   4.2. $xy=yx$ *(Multiplication is commutative.)*

   4.3. $x+(y+z)=(x+y)+z$ *(Associative law for addition.)*

   4.4. $x(yz)=(xy)z$ *(Associative law for multiplication.)*

   4.5. $x(y+z)=xy+xz$ *(Distributive law.)*

   4.6. $x+0=0+x=x$ *(0 is identity element for addition.)*

   4.7. $x.1=1.x=x$ *(1 is an identity element for multiplication.)*

Any complex number $z=a+bi$ can be represented by the point (a,b) in the cartesian coordinate plane. In this case the coordinate plane is called the complex plane and the horizontal and the vertical axes are called the real axis and the imaginary axis respectively.

Remark 6.1.2. *The representation of a complex $z=a+bi$ by the point (a,b) gives us a one to one correspondence between the set of complex numbers $\mathbb{C}$ and the set of ordered pairs of real numbers $\mathbb{R}\times\mathbb{R}$.*

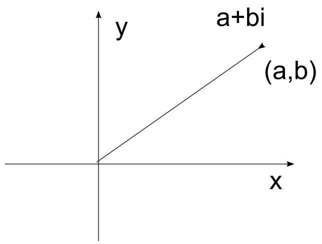

Figure 6.1: The complex number a+bi as a point in the cartesian coordinate plane.

Definition 6.1.3. *Let $z = a + bi$ be a complex number.*

1. *The magnitude (modules)of z is the real number $mod(z) = |z| = \sqrt{a^2 + b^2}$.*
2. *The point (a, b) has polar coordinates (r, θ), where $r = |z|$ and $\theta = \arctan(\frac{b}{a})$. Then θ is called an argument of z, denoted by $arg(z)$.*
3. *If (r, θ) is a polar coordinate of (a, b), then*

$$z = a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

and using Euler's formula we can write $r(\cos\theta + i\sin\theta) = re^{i\theta}$. The expression $z = re^{i\theta}$ is called the polar form of z.

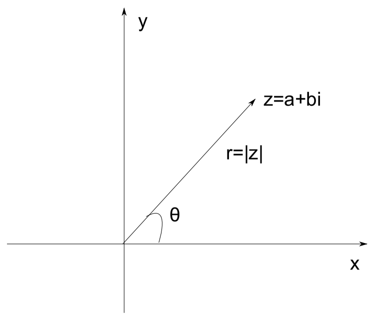

Figure 6.2: Polar coordinates of a complex number $z = a + bi$.

Example 6.1.2. *Let $z = -1 + i$. Then $|z| = r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and*

$$arg(z) = \arctan -1 = \frac{3\pi}{4}.$$

Therefore the polar form of z is ,

$$z = \sqrt{2}e^{i\frac{3\pi}{4}}$$

and in the polar coordinates (r, θ)

$$z = \sqrt{2}\Big(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\Big).$$

Definition 6.1.4. *Let $z = a + bi$ be a complex number. The conjugate of z is the complex number $\bar{z} = a - bi$.*

On the complex plane, the conjugate of a complex number $z = a + bi$ is the reflection of z on the real axis.

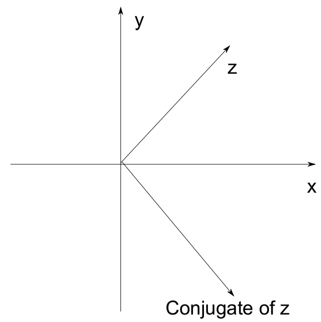

Figure 6.3: A complex number and its conjugate in the complex plane.

Remark 6.1.5. *Let z and w be complex numbers. Then*

1. *$\overline{\overline{z}} = z$ and $\overline{z} = z$ if and only if z is a real number.*
2. *$\overline{z \pm w} = \overline{z} \pm \overline{w}$, $\overline{zw} = \overline{z}\overline{w}$ and $\overline{z/w} = \overline{z}/\overline{w}$ if $w \neq 0$.*
3. *$|z| = |\overline{z}|$ and $|z|^2 = z\overline{z}$.*
4. *$Re(z) = \frac{1}{2}(z + \overline{z})$ and $Im(z) = \frac{1}{2i}(z - \overline{z})$.*

Disks Open Sets and Closed Sets in the Complex Plane

Let $z = x + iy$ and $z_0 = x_0 + iy_0$ be complex numbers and r be a positive real number.

1. Then $|z - z_0| = r$ if and only if

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

which is a circle with center (x_0, y_0) and radius r.

2. The set $\{z \in \mathbb{C} : |z - z_0| < r\}$ is an open disk of radius r about z_0 and it contains all points enclosed by the circle but does not contain the boundary.

3. The set $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ is a closed disk about z_0 and it contains all points enclosed by the circle the boundary points on the circle.

4. The S be a set of complex numbers and w be a complex number.

   4.1 w is an the interior of S is there is some open disk about z which is contained in S.

   4.2 w is a boundary point of S if every open disk about w contains at least one point of S and at leats one point ont in S.

   4.3 The set S is an open set if every point of S is an interior point of S.

   4.4 The set S is a closed set if S contains all the boundary points.

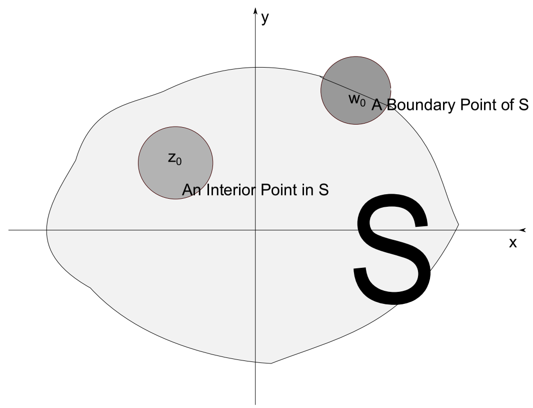

Figure 6.4: Interior and boundary points of a set in the complex plane.

6.2 Complex Functions, Differential Calculus and Analyticity

In the next subsequent sections we are going to consider functions from a subset of the set of complex numbers to the set of complex numbers.

Definition 6.2.1. *A function w of a complex variable z is a rule that assigns a unique value w(z) to each point z in some set D in the complex plane. If w is a complex function and $z = x + iy$,*

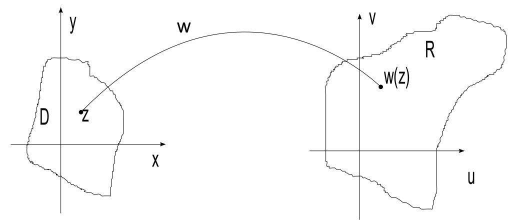

Figure 6.5: A complex function.

then we can always write

$$w(z) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions of x and y and $u(x, y) = Re(f(z))$ and $v(x, y) = Im(f(z.))$

If $z = x + iy$, then $w(z) = u(x, y) + iv(x, y)$. That is, the real and imaginary parts of $w(z)$ are functions of x and y.

Example 6.2.1. *Let w be a complex function defined by $w(z) = z^2 .= (x + iy)^2 = (x^2 - y^2) + i2xy$. Hence $Re(w(z)) = u(x, y) = x^2 - y^2$ and $Im(w(z)) = v(x, y) = 2xy$.*

Example 6.2.2. *Let $f(z) = \frac{1}{z}$, for $z \neq 0$. Then*

$$f(z) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Then $Re(f(z)) = u(x, y) = \frac{x}{x^2+y^2}$ and $Im(f(z)) = v(x, y) = \frac{-y}{x^2+y^2}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function. Then clearly f maps $\mathbb{R}^2$ into $\mathbb{R}^2$ and hence all the concept of limit and derivatives that are defined for vector functions of two variables also apply here with the notations modified in terms of the complex numbers notation.

6.2.1 Limit

Let z_0 be an interior point in the domain of definition of a function $f : \mathbb{C} \to \mathbb{C}$. We say that the limit of $f(z)$ as z approaches to z_0 is L and write

$$\lim_{z \to z_0} f(z) = L$$

if to each $\epsilon > 0$ (no matter how small it is), there corresponds a $\delta > 0$ such that $|f(z) - L| < \epsilon$ for all z satisfying $0 < |z - z_0| < \delta$.

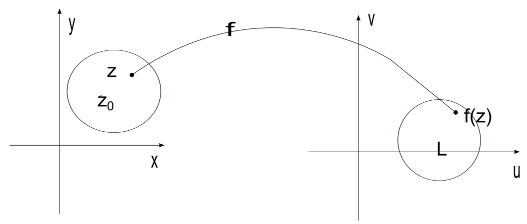

Figure 6.6: Limit of a complex function at a given point.

In the above definition, $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Moreover $|z - z_0|$ means the modulus of the complex number $z - z_0$ and $|z - z_0| < \delta$ represents an open circle centered z_0.

Recall that, in the calculus of real variables, the limit of sum(product, quotient) of two functions is sum(product, quotient) of the limits whenever the limits are defined and the limit of the denominator is nonzero. The same is true for complex functions, which is summarized below.

Remark 6.2.2. *Let f and g be complex functions and z_0 and c be complex numbers such that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z)) = M$.*

1.
$$\lim_{z \to z_0} (f \pm g)(z) = L + M.$$

2.
$$\lim_{z \to z_0} (f.g)(z) = LM.$$

3. *If $M \neq 0$, then*
$$\lim_{z \to z_0} \left(\frac{f}{g}\right)(z) = \frac{L}{M}.$$

4.
$$\lim_{z \to z_0} (cf)(z) = cL.$$

Definition 6.2.3. *For a complex function f, if*

$$\lim_{z \to z_0} f(z) = f(z_0),$$

then we say that f is continuous at z_0 and a function is continuous in a set if it is continuous at each point of the set.

6.2.2 Derivatives

Let f be a complex function. The derivative of f at the point z_0, denoted by $f'(z_0)$, is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z)}{\triangle z},$$

if the limit exists and is a complex number.

Here as well the limit value should be unique and independent of the way z approaches z_0.

Example 6.2.3. *Find the derivative of each of the following functions if it exists.*

1. *If $f(z) = z^2$, then*

$$f'(z) = \lim_{\triangle z \to} \frac{f(z + \triangle z) - f(z)}{\triangle z} = \lim_{\triangle z \to 0} \frac{(z + \triangle z)^2 - z^2}{\triangle z}$$
$$= \lim_{\triangle z \to 0} \frac{z^2 + 2z\triangle z + \triangle z^2 - z^2}{\triangle z} = \lim_{\triangle z \to 0} (2z + \triangle z) = 2z.$$

Therefore $(z^2)' = 2z$ as in the real case

2. *$f(z) = \bar{z}$ (the complex conjugate function) is not differentiable any where in $\mathbb{C}$.*
To see this Let $z = x + iy$ be any complex number. Then

$$\frac{f(z + \triangle z) - f(z)}{\triangle z} = \frac{\overline{(z + \triangle z)} - \bar{z}}{\triangle z} = \frac{\bar{z} + \overline{\triangle z} - \bar{z}}{\triangle z} = \frac{\overline{\triangle z}}{\triangle z} = \frac{\overline{(\triangle x + i\triangle y)}}{\triangle x + i\triangle y} = \frac{\triangle x - i\triangle y}{\triangle x + i\triangle y}.$$

Now if we approach to 0 in the horizontal direction, then $\triangle y = 0$. Hence

$$\lim_{\triangle z \to 0} \frac{f(z + \triangle z) - f(z)}{\triangle z} = \lim_{\triangle x + i\triangle y \to 0} \frac{\triangle x - i\triangle y}{\triangle x + i\triangle y} = \lim_{\triangle x \to 0} \frac{\triangle x}{\triangle x} = 1$$

On the other hand if we approach 0 in the vertical direction, we set $\nabla x = 0$. In this case

$$\lim_{\triangle z \to 0} \frac{f(z + \triangle z) - f(z)}{\triangle z} = \lim_{\triangle x + i\triangle y \to 0} \frac{\triangle x - i\triangle y}{\triangle x + i\triangle y} = \lim_{\triangle y \to 0} \frac{i\triangle y}{i\triangle y} = -1$$

Hence

$$\lim_{\triangle z\to 0}\frac{f(z+\triangle z)-f(z)}{\triangle z}$$

does not exist for all $z \in \mathbb{C}$. That is, $f(z)=\overline{z}$ is not differentiable any where in the complex plane.

Remark 6.2.4. *By induction it can be shown that $(z^n)' = nz^{n-1}$, for all $n \in \mathbb{N}$.*

As in the real functions case we use the definition of derivatives for complex functions very rarely, since we have the following rules of differentiation of complex functions.

Rules of Differentiation for Complex Functions

Let f and g be complex functions and c be a complex number.

1. Sum(Difference) Rule: $(f \pm g)'(z) = f'(z) \pm g'(z)$.
2. Constant Multiple Rule: $(cf)'(z) = cf'(z)$.
3. Product Rule: $(f.g)'(z) = f'(z)g(z) + f(z)g'(z)$.
4. Quotient Rule:
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{\big(g(z)\big)^2}.$$
5. The complex version of the Chain Rule: $(fog)'(z) = f'(g(z))g'(z)$.

Example 6.2.4. *Let $f(z)=\frac{1}{z}$. Then*

$$f'(z) = \frac{(1)'z - 1.(z)'}{z^2} = \frac{-1}{z^2}.$$

Therefore, f is differentiable every where in $\mathbb{C}$ except at $z=0$.

Suppose a complex function f is differentiable at z_0. Consider the equation

$$f(z)-f(z_0) = (z-z_0)\bigg(\frac{f(z)-f(z_0)}{z-z_0}\bigg).$$

Then

$$\lim_{z\to z_0}\Big(f(z)-f(z_0)\Big) = \lim_{z\to z_0}\bigg((z-z_0)\bigg(\frac{f(z)-f(z_0)}{z-z_0}\bigg)\bigg).$$

But

$$\lim_{z\to z_0}\bigg((z-z_0)\bigg(\frac{f(z)-f(z_0)}{z-z_0}\bigg)\bigg) = \lim_{z\to z_0}(z-z_0).\lim_{z\to z_0}\bigg(\frac{f(z)-f(z_0)}{z-z_0}\bigg) = 0.f'(z_0) = 0.$$

Therefore, $\lim_{z\to z_0} f(z) = f(z_0)$ and hence we have the following theorem.

Theorem 6.2.5. *If f is a differentiable complex function at z_0, then f is continuous at z_0.*

6.3 The Cauchy - Riemann Equation

Since differentiability of a complex function (together with analyticity) plays a crucial role in the study of complex variables theory, we need to answer the question that: **When is a complex function differentiable?** The answer will be given in this section.

Definition 6.3.1. *Let f be a complex function. Then*

1. *f is said to be* **analytic in a domain D** *if $f(z)$ is defined and differentiable at all points of D.*
2. *f is said to be* **analytic at a point** *$z_0 \in D$ if is analytic in some neighborhood of z_0.*
3. *f is said to be (simply)* **an analytic function** *if it is analytic in some domain (open connected subset of $\mathbb{C}$.)*

6.3.1 Test for Analyticity

Recall that, if f is a complex function and $z = x+iy$, then we can always write

$$f(z) = u(x,y) + iv(x,y),$$

where u and v are real-valued functions of x and y.

Consider a complex function $f(z) = u(x,y)+iv(x,y)$ with $z=x+iy$. If f is analytic in some domain D (and hence differentiable in D), then the partial derivatives exist and for $z_0 = x_0+iy_0$ and $z = x_0+iy$, we have

$$\begin{aligned}
\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} &= \lim_{z\to z_0}\frac{\big(u(x_0,y)+iv(x_0,y)\big)-\big(u(x_0,y_0)+iv(x_0,y_0)\big)}{(x_0+iy)-(x_0+iy_0)}\\
&= \lim_{z\to z_0}\frac{\big(u(x_0,y)+iv(x_0,y)\big)-\big(u(x_0,y_0)+iv(x_0,y_0)\big)}{i(y-y_0)}\\
&= \frac{1}{i}\lim_{y\to y_0}\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0} + \frac{i}{i}\lim_{y\to y_0}\frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}\\
&= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{1}{i}u_y + v_y = \frac{i}{i}\frac{1}{i}u_y + v_y\\
&= \frac{i}{-1}u_y + v_y = v_y - iu_y = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.
\end{aligned}$$

Similarly, if we set $\triangle y = 0$ and $\triangle x \to 0$, that is, if $z = x + iy_0$ and $z_0 = x_0 + iy_0$, then we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\big(u(x, y_0) + iv(x, y_0)\big) - \big(u(x_0, y_0) + iv(x_0, y_0)\big)}{(x + iy_0) - (x_0 + iy_0)}$$
$$= \lim_{x \to x_0} \frac{\big(u(x, y_0) - u(x_0, y_0)\big) - \big(u(x_0, y_0) + iv(x_0, y_0)\big)}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Since f is differentiable at z_0, the two partial derivatives must be equal. That is, we must have

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and this implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.1}$$

The above equation (6.1) is called the **Cauchy-Riemann** equations (and is only the necessary condition for analyticity of f at z_0.) Hence we proved the first part of the following Theorem.

Theorem 6.3.2 (Necessary and Sufficient Conditions for Analyticity)**.** *Let $f(z) = u(x, y) + iv(x, y)$ be a function that is defined throughout some neighborhood of a point $z_0 = x_0 + iy_0$.*

1. *(**Necessary Condition**) If f is differentiable at z_0, then the Cauchy-Riemann equation is satisfied.*

2. *(**Sufficient Condition**) If the Cauchy-Riemann equations are satisfied at z_0 and if u and v are continuously differentiable (real valued functions of two variables) in some neighborhood of z_0, then f is analytic at z_0*

If f is differentiable at $z = x + iy$, then due to the Cauchy-Riemann equations, $f'(z)$ can be given in any one of the following four equivalent expressions.

$$f'(z) = u_x(x, y) + iv_x(x, y)$$

or

$$f'(z) = v_y(x, y) - iu_y(x, y)$$

or

$$f'(z) = u_x(x, y) - iu_y(x, y)$$

or

$$f'(z) = v_y(x, y) + iv_x(x, y).$$

Example 6.3.1. *Consider $f(z) = |z|^2 = z\bar{z}$. Then $f(z) = (x^2 + y^2) + 0i$ and hence we have $u(x, y) = x^2 + y^2$ and $v = 0$.*
Since $u_x = 2x, u_y = 2y, v_x = 0$ and $v_y = 0$, all u, v, u_x, u_y, v_x, v_y are continuous in $\mathbb{R}^2$ and hence u and v are continuously differentiable everywhere in $\mathbb{R}^2$.
But $u_x = v_y$ only if $x = 0$, that is on the y-axis and $v_x = -u_y$ only if $y = 0$, that is, on the x-axis. Thus the Cauchy-Riemann equations holds true only at the origin and hence $f(z) = |z|^2$ is differentiable only at $z = 0$ and it is analytic nowhere.

Example 6.3.2. *Let $f(z) = z^2 - 8z + 3$. If $z = x + iy$, then*

$$f(z) = f(x + iy) = (x^2 - y^2) + 2xyi - 8x - 8yi + 3 = (x^2 - 8x - y^2 + 3) + (2xy - y)i$$

and hence $u(x, y) = x^2 - 8x - y^2 + 3$ and $v(x, y) = 2xy - 8y$. Then $u_x = 2x - 8, u_y = -2y, v_x = 2y$ and $v_y = 2x - 8$ and we also have

$$u_x = 2x - 8 = v_y \text{ and } v_x = 2y = -u_y$$

for all $(x, y) \in \mathbb{R}^2$, that is, the Cauchy-Riemann equations are satisfied everywhere in $\mathbb{R}^2$ and all u_x, u_y, v_x and v_y are continuous in $\mathbb{R}^2$ and hence u and v are continuously differentiable everywhere in $\mathbb{R}^2$.
Therefore, f is differentiable for all z and $f'(x + iy) = u_x(x, y) + iv_x(x, y) = (2x - 8) + 2yi$.

Remark 6.3.3. *Let $f = u + iv$ be a differentiable complex function on an open disk D such that $f'(z) = 0$ on D. Suppose that u and v are continuous with continues first and second derivatives and satisfy the Cauchy-Riemann equations on D. Then*

$$0 = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

This implies $u_x = v_x = 0$ and $u_y = v_y = 0$ and hence u and v are constant functions.
Therefore f is a constant function on D.

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain D, then clearly

$$u_x = v_y \text{ and } u_y = -v_x$$

on D and the partial derivatives of the component functions of all order exist and are continuous in D. Then taking the derivative with respect to x and y respectively we have: $u_{xx} = v_{yx}$ and $u_{xy} = v_{yy}$ and also $u_{yy} = -v_{xy}$ and $u_{yx} = -v_{xx}$.
Hence $v_{yx} = v_{xy}$ implies $u_{xx} + u_{yy} = 0$ in D and $u_{xy} = u_{yx}$ implies $v_{yy} + v_{xx} = 0$ in D.

Definition 6.3.4. *A real-valued function $u(x, y)$ of two variables satisfy Laplace's equation, that is,*

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

and all various first and second - order partial derivatives of its component functions with respect to x and y are continuous (in this case these functions are called C^2 functions) is called **a harmonic function**.

Then we have proved the following theorem.

Theorem 6.3.5 (Harmonic Functions)**.** *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harnonic in D. That is, they are $C^{(2)}$ functions and satisfy the Laplace's equation:*

$$\nabla^2 u = u_{xx} + v_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

Since f is analytic, u and v are related by the Cauchy-Riemann equation and to refer this relationship we such functions are called **conjugate harmonic functions.**

Example 6.3.3. *Show that $u = x^2 - y^2 - y$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.*

Solution

Clearly $\nabla^2 u = u_{xx} + u_{yy} = 2 - 2 = 0$ and hence u is harmonic. To find the conjugate harmonic function v, first we have $u_x = 2x$ and $u_y = -2y - 1$ and by the Cauchy-Riemann equations v must satisfy $v_y = u_x = 2x$ and $v_x = -u_y = 2y + 1$.
Integrating the first part with respect to y, we get $v = 2xy + h(x)$, where $h(x)$ is a function of x only and we differentiate v with respect to x to get $v_x = 2y + h'(x) = 2y + 1$. This implies $h'(x) = 1$ and hence $h(x) = x + c$ for some constant c.
Therefore $v(x, y) = 2xy + x + c$ and hence

$$f(z) = u(x, y) + iv(x, y) = (x^2 - y^2 - y) + i(2xy + x + c)$$

$$= \big(x^2 - y^2 + i2xy\big) + \big(-y + ix + ic\big) = z^2 + i(z + c) = z^2 + iz + c_1,$$

where $c_1 = ic$.

6.4 Elementary Functions

6.4.1 Exponential Functions

For a complex number $z = x + iy$, the complex exponential function e^z is defined by

$$e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i \sin y)$$

and $e^{iy} = \cos y + i \sin y$ is Euler's formula.
We also have $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$ and hence $|e^z| = |e^x e^{iy}| = e^x|e^{iy}| = e^x$, for all $z = x + iy$.

Example 6.4.1. $|e^{-2+4i}| = e^{-2}$ *and* $|e^{3-5i}| = e^3$.

Example 6.4.2. *If $e^z = 2i$, then $e^x \cos y + ie^x \sin y = 0 + 2i$. This implies $e^x \cos y = 0$ and $e^x \sin y = 2$. Squaring these equations adding the results will give us*

$$e^{2x}(\cos^2 y + \sin^2 y) = 4$$

which implies $e^{2x} = 4$ and then $x = \ln 2$ and

$$\frac{e^x \cos y}{e^x \sin y} = \cot y = 0$$

which gives $y = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$. Therefore, the solutions of $e^z = 2i$ are

$$z = \ln 2 + i2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}.$$

Remark 6.4.1. *For a complex number $z = x + iy$, we have $e^z \neq 0$ for all $z \in \mathbb{C}$ since $e^x \neq 0$ for all (finite) x and $\cos y$ and $\sin y$ do not vanish simultaneously for any value of y.*

For $z = x+iy$, let $f(z) = e^z$. Then $f(z) = e^x(\cos y + i \sin y)$ and the functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are continuous with continuous first partial derivatives.
We also have $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$ and hence u and v satisfy the Cauchy-Riemann equations. Therefore, the complex exponential function $f(z) = e^z$ is differentiable for all z and

$$f'(z) = \big(e^z\big)' = u_x + iv_x = e^z.$$

6.4.2 Trigonometric and Hyperbolic Functions

From Euler's formula we have that: $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. By adding these two equations we get: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ which implies

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and by subtracting the second from the first we get: $e^{i\theta} - e^{i\theta} = 2i\sin\theta$ which implies

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Using these two formulae we can now define the trigonometric complex functions as follows.

For any complex number z

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{6.2}$$

All other trigonometric functions can be derived from these two basic definitions. For example

$$\tan z = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \text{ and } \sec z = \frac{2}{e^{z} + e^{-iz}}.$$

Recall that:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \text{ and } \cosh x = \frac{e^{x} + e^{-x}}{2}$$

for any $x \in \mathbb{R}$. Similarly for complex numbers z we define

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \text{ and } \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{6.3}$$

From (6.2) and (6.3) it follows that

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z$$

and

$$\sin(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i.\frac{e^{z} - e^{-z}}{2} = i\sinh z.$$

Therefore, we have proved the relations

$$\cos(iz) = \cosh z \text{ and } \sin(iz) = i\sinh z.$$

Similarly we can show that

$$\cos(iz) = \cos z \text{ and } \sin(iz) = i\sin z.$$

Another basic relation of the complex trigonometric functions is derived as follows.

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{(e^{iz} - e^{-iz})^2}{-4} + \frac{(e^{iz} + e^{-iz})^2}{4}$$

$$= \frac{1}{4}[-e^{2iz} + 2e^{iz-iz} - e^{-ziz} + e^{2iz} + 2e^{iz-iz} + e^{-2iz}] = \frac{1}{4}(2+2) = 1.$$

Therefore $\sin^2 z + \cos^2 z = 1$.

For a complex number z show that

i) $\sin(-z) = -\sin z$

ii) $\cos(-z) = \cos z$

iii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

iv) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

v) $\cos(z + 2\pi) = \cos z$ and $\sin(z + 2\pi) = \sin z$.

vi) $\cosh^2 z - \sinh^2 z = 1$.

Example 6.4.3. *Let* $z = x + iy$ *and* $f(z) = \sin z$. *Then* $f(z) = \sin(x + iy) = \sin x \cosh y + i\cos x \sinh y$. *This implies* $u(x, y) = \sin x \cosh y$ *and* $v(x, y) = \cos x \sinh y$ *and we also have* $u_x = \cos x \cosh y$, $u_y = \sin x \sinh y$, $v_x = -\sin x \sinh y$ *and* $v_y = \cos x \cosh y$.
Since u, v, u_x, u_y, v_x, v_y, *are all continuous everywhere in* $\mathbb{R}^2$ *and* $u_x = v_y$ *and* $u_y = -v_x$, *f is differentiable and* $f'(z) = \cos x \cosh y - i\sin x \sinh y$. *But*

$$\cos x \cosh y - i\sin x \sinh y = \cos x \cosh y - \sin x(i\sinh y)$$

$$= \cos x \cos(iy) - \sin x \sin(iy) = \cos(x + iy) = \cos z.$$

Therefore $(\sin z)' = \cos z$.

Example 6.4.4. *Let* $f(z) = e^{\sin z}$. *Then by using chain rule we get* $f'(z) = e^{\sin z}\cos z$.

6.4.3 Polar form and Multi-Valuedness.

The Polar form of a complex number z is

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arctan\frac{y}{x} = argz$. However, the angle $\theta = argz$ for $z \neq 0)$ can be determined only to within an arbitrary integer multiple of 2π The angle θ with $-\pi < \theta < \pi$ is called the **Principal argument** of z and denoted by $argz$. That is,

$$\theta = argz + 2k\pi, k \in \mathbb{Z}.$$

The expression z^k is single valued only if the exponent k is an integer. If k is a rational number $\frac{m}{n}$ (in its reduced form), then the map

$$f(z) = z^k$$

is n - valued, (since there are exactly n n^{th} roots of a complex number z.)

Example 6.4.5. *Let* $z = 1 + i$. *Then* $r = \sqrt{1+1} = \sqrt{2}$ *and* $argz = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$.
Therefore

$$z^{1/3} = (re^{i\theta})^{1/3} = 2^{1/6}e^{i\pi/3})$$

for $k = 0, 1, 2$. *Then* $F_k = e^{ik(\frac{2\pi}{2})}$ *for* $k = 0, 1, 2$ *which implies* $F_0 = e^0 = 1, F_1 = e^{i2\pi/3,}$ *and* $F_2 = e^{i4\pi/3}$ *which correspond to the three points on the unit circle.*
Therefore,

$$z^{1/3} = 2^{1/6}e^{i\pi/12}, 2^{1/6}e^{i\frac{9\pi}{12}}, 2^{1/6}e^{i\frac{17\pi}{12}}.$$

6.4.4 The Logarithmic Functions

Recall that: in the calculus of real function, the natural logarithm is the inverse of the exponential function.

$$y = \ln x \text{ if and only if } x = e^y,$$

for $x > 0$. Hence the real logarithm is a solution to the equation $x = e^y$.
Here we want to develop the natural logarithm, that is, for a complex number $z \neq 0$ we want to find a solution for $z = e^w$.
Let $z = re^{i\theta}$ in its polar form and $w = a + bi$. Then $z = re^{i\theta} = e^{a+bi} = e^a e^{ib}$ and we also have $r = |z| = e^a$, which implies that $a = \ln r$.

From the equation $re^{i\theta} = e^{a+bi} = e^a e^{ib}$ we get $e^{i(b-\theta)} = 1 = e^{2k\pi i}$ for $k \in \mathbb{Z}$, which implies that $i(b-\theta) = 2k\pi i$ and hence $b = \theta + 2k\pi$ for $k \in \mathbb{Z}$. Therefore, for $z \neq 0$ there are infinitely many numbers

$$w = \ln r + i(\theta + 2k\pi), k \in \mathbb{Z}$$

such that $z = e^w$.
Now we are in a position to define the logarithm of a nonzero complex number z as

$$\log(z) = \ln|z| + i(\arg(z) + 2k\pi), k \in \mathbb{Z}$$

which is infinite valued.

Example 6.4.6. *Compute* $\log(1 + i)$.
Let $z = 1 + i$. *Then* $r = \sqrt{1+1} = \sqrt{2}$ $arg(z) = \arctan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$.
Therefore

$$\log(1 + i) = \ln\sqrt{2} + i(\frac{\pi}{4} + 2k\pi), k \in \mathbb{Z}.$$

Let f be a complex function which is differentiable at z. If f is expressed in polar form as: $f(z) = u(r,\theta) + iv(r,\theta)$, the Cauchy-Rieman equations can be calculated (using definition along constant θ and along constant r or using change of variables $x = r\cos\theta$ and $y = r\sin\theta$.)
By using change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and from chain rule we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{\frac{\partial x}{\partial r}} = \frac{\partial u}{\partial r}.\frac{1}{\cos\theta}$$

which implies that

$$\frac{\partial u}{\partial x} = \frac{1}{cos\theta}\frac{\partial u}{\partial r}, \tag{6.4}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial \theta}.\frac{1}{\frac{\partial x}{\partial \theta}} = \frac{\partial u}{\partial \theta}.\frac{1}{r\sin\theta}$$

which implies that

$$\frac{\partial u}{\partial x} = -\frac{1}{r\sin\theta}\frac{\partial u}{\partial \theta}, \tag{6.5}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} = \frac{\partial v}{\partial r}.\frac{1}{\frac{\partial y}{\partial r}} = \frac{\partial v}{\partial r}.\frac{1}{\sin\theta}$$

which implies that

$$\frac{\partial v}{\partial y} = \frac{1}{\sin\theta}\frac{\partial v}{\partial r} \tag{6.6}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial \theta}.\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial \theta}.\frac{1}{\frac{\partial y}{\partial \theta}} = \frac{\partial v}{\partial \theta}.\frac{1}{r\cos\theta}$$

which implies that

$$\frac{\partial v}{\partial y} = \frac{1}{r\cos\theta}.\frac{\partial v}{\partial \theta}. \tag{6.7}$$

From the relation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and from (6.4) and (6.7) we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and from from (6.5) and (6.6) we have

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

and thus

$$f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{e^{-i\theta}}{r}(v_\theta - iu_\theta) = e^{-i\theta}(u_r - \frac{i}{r}u\theta) = e^{-i\theta}(\frac{1}{r}v_\theta + iv_r).$$

Example 6.4.7. *Let* $f(z) = \log z = \ln r + i\theta$, *with* $\theta = arg(z)$ *is the principal argument* $(i.e. 0 < r < \infty$ *and* $-\pi < \theta < \pi$ *Then find* $f'(z)$ *in terms of z.*

Here $u(r,\theta) = \ln r, v(r,\theta) = \theta$ *and* $u_r = \frac{1}{r}, u_\theta = 0, u_r = 0, v_\theta = 1.$

Since $u, v, u_r, v_r, u_\theta, v_\theta$ *are all continuous in the plane where* $\log z$ *is defined and since the Cauchy-Riemann equations are satisfied,* $\log z$ *is analytic everywhere in the domain of* $\log z$.

Hence

$$f'(z) = (\log z)' = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{1}{r} + i \times 0) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

as in the real case.

6.5 Exercises

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Chapter 7

COMPLEX INTEGRAL CALCULUS

7.1 Complex Integration:

Recall that: there is a one=to-once correspondence between the set of Complex numbers $\mathbb{C}$ and the set of points in the Euclidean real plane $\mathbb{R} \times \mathbb{R}$.. Hence the natural generalization of the Riemann integral

$$\int_a^b f(x)dx$$

of a real valued function f on a real $x-$axis the line integral in $\mathbb{R}^2$ or in $\mathbb{R}^3$. Following this fact we define the integral of a complex valued of a complex variable as a line integral of the function along a given oriented curve C in the complex plane.

i.e. The complex integral of a complex function f on a curve C is given by:

$$I = \int_C f(z)dz.$$

Here we assume that C is an oriented curve in the complex plane, wich is piecewise smooth and simple.

In the complex plane, a curve C be perimetrically represented as:

$$z(t) = x(t) + iy(t).$$

The direction in which t is increasing is called the positive sense of C. we can now state:

Definition 7.1.1. *Let C be a smooth curve, represented by $z = z(t)$, where $a \leq t \leq b$. Let $f(z)$ be a continuous complex function on C.*

Then the integral of f over C is, denoted by

$$\int_c f(z)dz,$$

is defined by

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt, \ where\ \dot{z}(t) = \frac{dz}{dt}.$$

Now the question is how can we evaluate this integral. One other possible way to evaluate the complex integral is to write the line integral into one or more real line integrals. To see this, let $f(z) = u + iv$ and $z = x + iy$. Then $dz = dx + idy$.

Hence

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i \int_C (vdx + udy)$$

Example 7.1.1. *Evaluate*

$$\oint_C \frac{dz}{z},$$

where C is a unit circle around the origin.

Solution

Here C is parameterized by $z(t) = \cos t + i \sin t = e^{it}, 0 \leq t \leq 2\pi$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it}.ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i.$$

Example 7.1.2. *Evaluate $I = \int_C z^2 dz$, where C is the parabolic arc given in the figure below.*

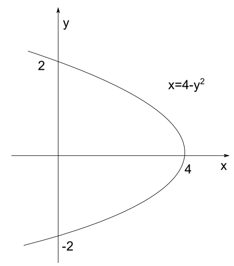


Figure 7.1: The parabola $x = 4 - y^2$.

Solution

First $z^2 = (x+iy)^2 = (x^2-y^2) + i(2x.y)$ *which implies* $u(x,y) = x^2 - y^2, v = 2xy$. *Then*

$$I = \int_C z^2 dz = \int_C \big((x^2-y^2)dx - (2xy)dy\big) + i\int_C \big(2xydx + (x^2-y^2)dy\big).$$

Parameterizing C according to $t = y$, *we have* $x = 4 - t^2$ *and* $dy = dt, dx = -2tdt, -2 \le t \le 2$. *Therefore,*

$$I = \int_2^{-2}\Big(((4-t^2)^2-t^2)(-2tdt)-(2(4-t^2)t)dt\Big)+i\int_{+2}^{-2}\Big(2(4-t^2)+(-2tdt)+((4-t^2)^2-t^2))dt\Big)$$
$$= \int_{+2}^{-2}[(16-9t^2+t^4)(-2t)-(8t-2t^3)]dt + i\int_{+2}^{-2}[((-16t^2+4t^4)+(16-9t^2+t^4)]dt$$
$$= \int_{+2}^{-2}\big(-2t^5+20t^3-40t\big)dt + i\int_{+2}^{-2}\big(5t^4-25t^2+16\big)dt = 0+i[-64+\frac{25}{3}\times 16-64] = 0+i\frac{16}{3} = \frac{16}{3}i.$$

Example 7.1.3. *Evaluate*

$$\oint_C (z-a)^n dz,$$

where a *is any given complex number, n is any integer and C is a circle centered at* a *and with radius* r.

Solution

Here the curve is parameterized by $z - a = re^{it}$ *for* $0 \le t \le 2\pi$ *which implies* $dz = ire^{it}dt$. *Therefore,*

$$\oint_C (z-a)^n dz = \int_0^{2\pi}\big(re^{it}\big)^n .ire^{it}dt = \int_0^{2\pi} ir^{n+1}e^{i(n+1)t}dt = r^{n+1}\int_0^{2\pi} e^{i(n+1)t}(idt).$$

But

$$r^{n+1}\int_0^{2\pi} e^{i(n+1)t}(idt) = \begin{cases} \frac{r^{n+1}}{n+1}\left[e^{i(n+1)t}\right]_0^{2\pi} = 0 \ if\ n \ne -1 \\ i\int_0^{2\pi} r^0 e^{i(0)t}dt = 2\pi i \ if \ = -1. \end{cases}$$

Therefore

$$\oint_C (z-a)^n dz = \begin{cases} 2\pi i \ if\ n = -1 \\ 0 \ if\ n \ne -1. \end{cases}$$

In the previous examples, we have been integrating over smooth curves. Let C be a piecewise smooth curve. That is, C is a curve made up of smooth curves $C_1, C_2, ..., C_n$ such that the terminal point of C_i is the initial point of C_{i+1} and in this case we write $C = C_1 \oplus ... \oplus C_n$.

Definition 7.1.2. *Let C be a piecewise smooth curve such that* $C = C_1 \oplus ... \oplus C_n$ *and* $f(z)$ *be a continuous complex function on C. Then we define*

$$\int_C f(z)dz = \sum_{i=1}^n \int_{C_i} f(z)dz.$$

Example 7.1.4. *Let C be a curve consisting of portion of a parabola* $y = x^2$ *in the* $xy-$*plane from* $(0,0)$ *to* $(2,4)$ *and a horizontal line from* $(2,4)$ *to* $(4,4)$. *If* $f(z) = Im(z)$, *then evaluate*

$$\int_C f(z)dz.$$

Solution

First we write $C = C_1 \oplus C_2$, *where* C_1 *is the portion of the parabola and* C_2 *is the line segment. Parameterize* C_1 *by* $z(t) = t + it^2$ *for* $0 \le t \le 2$ *and on* C_1, $dz = (1+2ti)dt$ *and* $f(z(t)) = Im(z(t)) = t^2$. *Therefore,*

$$\int_{C_1} Im(z(t))dz = \int_0^2 t^2(1+2ti)dt = \int_0^2 t^2dt + i\int_0^2 2t^3dt = \left(\frac{t^3}{3} + i\frac{t^4}{2}\right)\Big|_0^2 = \frac{8}{3} + 8i.$$

Parameterize C_2 *by* $z(t) = t + 2i$ *for* $2 \le t \le 4$ *and on* C_2, $dz = dt$ *and* $f(z(t)) = Im(z(t)) = 2$. *Therefore,*

$$\int_{C_2} Im(z)dz = \int_2^4 2dt = 2t\big|_2^4 = 4.$$

Hence

$$\int_C (y^2dx + x^2dy) = \left(\frac{8}{3} + 8i\right) + 8 = \frac{32}{3} + 8i.$$

Remark 7.1.3. *As in the line integrals we have the following generalizations.*
Let f and g be continuous complex functions on a piecewise smooth curve C and k *be a constant. Then*

1. $$\int_C (kf)(z)dz.dr = k\int_C f(z)dz.$$

2. $$\int_C (f+g)(z)dz = \int_C f(z)dz + \int_C g(z)dz.$$

3. *If C' has an opposite orientation to that of C, then*

$$\int_C f(z)dz = -\int_{C'} f(z)dz.$$

7.2 Cauchy's Integral Theorem.

Let C be a piecewise-smooth simple closed curve in the complex plane (and hence in $\mathbb{R}^2$). Then C encloses some simply connected region R. Let $f(z) = u(x,y) + iv(x,y)$ be continuous in a simply connected domain D containing the curve C. Then

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy) \tag{7.1}$$

Now assume that f is analytic and that $f'(z)$ is continuous in D so that u and v are continuously differentiable. Then by Green's Theorem on u and v we can write (7.1) as:

$$\oint_C f(z)dz = \iint_R \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right)dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dA$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dA.$$

But since f is analytic in D, by Cauchy-Riemann equations we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This implies

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \text{ and } \frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{ in D}.$$

Therefore

$$\oint_C f(z)dz = \iint_R 0dA + \iint_R 0dA = 0$$

and hence we have proved the following theorem (called Cauchy's Theorem.)

Theorem 7.2.1 (Cauchy's Theorem). *If $f(z)$ is analytic in a simply connected domain D,then*

$$\oint_C f(z)dz = 0,$$

for every piecewise smooth simple closed curve C in D.

Remark 7.2.2. *In the Cauchy's Theorem above, the continuity of $f'(z)$ is omitted. This is done intentionally because as we can show it later, if f is analytic at a point z_0, then the derivatives of all order of f at z_0 exists. and hence $f'(z)$ is continuous*

Example 7.2.1. *Consider the integral*

$$\oint_C \frac{dz}{z^2 - 5z + 6} = \oint_C \frac{dz}{(z-2)(z-3)},$$

where C is the unit circle centered at the origin and traverse counterclockwise. Then

$$f(z) = \frac{1}{(z-2)(z-3)}$$

is analytic everywhere except at $z = 2$ and $z = 3$. But $z = 2$ and $z = 3$ are out of the region enclosed in C. Hence f is analytic in the region enclosed by C. Then by Cauchy's Theorem

$$\oint_C \frac{dz}{z^2 - 5z + 6} = 0.$$

Let C_1 and C_2 be closed paths in the complex plane with C_2 is in the interior of C_1. Suppose that a complex function f is analytic in an open set containing both paths and all points between them. Now let L be the line segment as shown in the Figure 7.4. Then the region D is a simply connected region bounded by the curve C, where $C = C_1 \oplus C_2' \oplus L \oplus L'$, where L' is the line segment which is oriented opposite to that of L and C_2' is the curve C_2 but in opposite orientation.

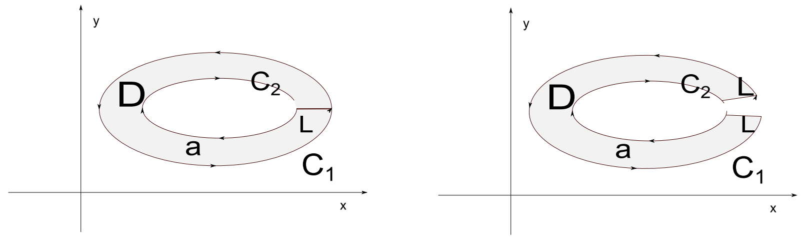


Figure 7.2: Multiply and Simply Connected Regions.

Then. since f is analytic in D , by Cauchy's Theorem, we have

$$\int_C f(z)dz = 0.$$

But since C is a piecewise smooth curve, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_L f(z)dz + \int_{L'} f(z)dz$$

and

$$\int_{L'} f(z)dz = -\int_L f(z)dz \text{ and } \int_{C_2'} f(z)dz = -\int_{C_2} f(z)dz$$

Therefore, we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

and hence

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Therefore, we have proved the following theorem.

Theorem 7.2.3 (The Deformation Theorem). *Let C_1 and C_2 be closed paths in the complex plane with C_2 is in the interior of C_1. Suppose that a complex function f is analytic in an open set containing both paths and all points between them. Then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Remark 7.2.4. *If f is analytic in a simply connected domain D, then the integral $\int_c f(z)dz$is independent of path in D. That is, if C_1 and C_2 are open curves with the same initial and terminal points, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Hence we can deform C_1 into C_2 without changing the value of the integral.
However, if f is not analytic in D, then Cauchy's Theorem does not hold true in general.

Example 7.2.2. *Consider the integral*

$$\int_c \frac{dz}{z-a}$$

where C is any piecewise smooth simple closed curve, oriented counterclockwise and containing a inside. Since

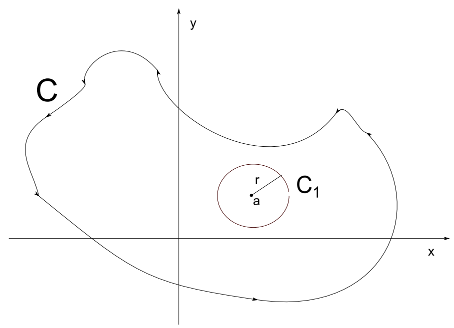


Figure 7.3: A curve C containing a inside.

$$f(z) = \frac{1}{z-a}$$

is analytic in the region bounded by C except in some neighborhood of $z = a$, we can conclude that f is analytic in every domain not containing a inside.
Thus, because of path deformation, we can assume without loss of generality that C_1 is a circular

path with radius r and centered at a.
Then

$$\oint_c \frac{dz}{z-a} = \oint_{C_1} \frac{dz}{z-a}.$$

Set $z - a = re^{i\theta}$. Then $dz = rie^{i\theta}d\theta$ and hence

$$\int_{C_1} \frac{rie^{i\theta}}{re^{i\theta}}d\theta = i\oint_{C_1} d\theta = i\int_0^{2\pi} d\theta = 2\pi i \neq 0.$$

7.3 Cauchy's Integral Formula and The Derivative of Analytic Functions.

In the last example of the previous section we have seen that

$$\oint_C \frac{dz}{z-a} = 2\pi i,$$

where C is any piecewise smooth, simple closed curve oriented counterclockwise and containing a in the interior. During the evaluation of the integral we used the idea of path deformation and a circle C_1 with center a and radius r.

Now let $f(z)$ be analytic in a simply- connected domain D containing C inside. Then

$$I = \oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz,$$

where C_1 is a sufficiently small circle with radius r and centered at a.
Since this last integral is independent of r, provided C_1 stays inside we will let $r \to 0$. Hence

$$I = \oint_{C_1} \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(a)}{z-a}za + \oint_{C_1} \left(\frac{f(z)-f(a)}{z-a}\right)dz = f(a)2\pi i + \oint_{C_1} \left(\frac{f(z)-f(a)}{z-a}\right)dz.$$

At this final step letting $r \to 0$ we have $|f(z) - f(a)| \to 0$.(The deviation of integrand goes to zero).
Thus

$$I = \oint_C \frac{f(z)}{z-a}dz = f(a)2\pi i.$$

Definition 7.3.1. *A Complex function g is said to be singular at a point, say $z = z_0$, if it is not analytic at that point.*

We have proved the following theorem

Theorem 7.3.2 (Cauchy Integral Formula). *Let $f(z)$ be analytic in a simply - connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then*

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

for all a in D. This implies

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz.$$

Example 7.3.1. *Evaluate*

$$\oint_C \left(\frac{z^3-6}{2z-i}\right)dz,$$

where C is any closed simple piecewise smooth curve containing $a = \frac{i}{2}$ in its interior.

Then

$$\oint_C \left(\frac{z^3-6}{2z-i}\right)dz = \oint_C \left(\frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}\right)dz = \oint_C \left((\frac{f(z)}{z-\frac{1}{2}i}\right)dz$$

where $f(z) = \frac{1}{2}z^3 - 3$ and f is analytic every where in $\mathbb{C}$. Thus

$$\oint_C \left(\frac{z^3-6}{2z-i}\right)dz = 2\pi i.f(\frac{1}{2}i) = 2\pi i\left(\frac{1}{20\left(\frac{1}{2}i\right)^3}-3\right) = 2\pi i\left(-\frac{1}{16}i-3\right) = \frac{\pi}{8} - 6\pi i.$$

Example 7.3.2. *Evaluate*

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right)dz,$$

where C is a unit circle centered at $z=1$.

Here

$$\frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z-1)(z+1)} = \left(\frac{z^2+1}{z+1}\right).\left(\frac{1}{z-1}\right) = \frac{f(z)}{z-1},$$

where $f(z) = \frac{z^2+1}{z+1}$. Therefore,

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right)dz = 2\pi i f(1) = 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i \times \frac{2}{2} = 2\pi i.$$

If C is a unit circle containing -1 in its interior, then we can write

$$\frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)} = \left(\frac{z^2+1}{z-1}\right).\left(\frac{1}{z+1}\right) = \frac{f(z)}{z+1}.$$

where $f(z) = \frac{z^2+1}{z-1}$. Hence

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right) = 2\pi i f(-1) = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

A very striking result in complex analysis is that, if f is analytic in a domain D (once it is differentiable at a point of D), then it has derivatives of all orders in D. We have the following theorem that can be used to find higher order derivatives of an analytic at a given point and evaluate integrals.

Theorem 7.3.3 (Cauchy Integral Formula for Higher Derivatives). *Let $f(z)$ be analytic in a simply - connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then for all a in D*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$$

for any nonnegative integer n.

Example 7.3.3. *By using Cauchy Integral Formula for Higher Derivatives we have*

$$\oint_C \frac{\sin z}{(z-\pi i)^2}dz = 2\pi i(\sin z)'\Big|_{z=\pi i} = 2\pi i\cos(\pi i) = 2\pi i\cosh\pi$$

for any simple closed path C containing πi in its interior and oriented in counterclockwise direction.

Example 7.3.4. *Evaluate*

$$\oint_C \frac{dz}{z^2(z-2)(z-4)},$$

where C is the rectangle in Figure 7.3.4.

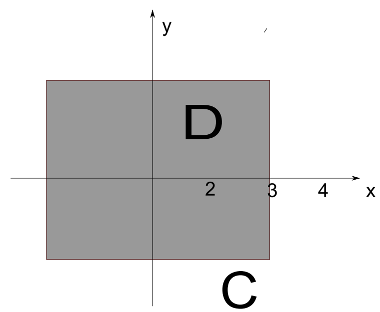


Figure 7.4: A rectangular region for Example 7.3.4.

Solution

Expanding the integrand

$$\frac{1}{z^2(z-2)(z-4)}$$

in partial fractions we have:

$$\frac{1}{z^2(z-2)(z-4)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-2} + \frac{D}{z-4}$$

which implies $A = \frac{3}{32}, B = \frac{1}{8}, C = -\frac{1}{8}, D = \frac{1}{32}$. *Therefore*

$$\oint_C \frac{dz}{z^2(z-2)(z-4)} = \frac{3}{32}\oint_C \frac{dz}{z} + \frac{1}{8}\oint_C \frac{dz}{z^2} - \frac{1}{8}\oint_C \frac{dz}{z-2} + \frac{1}{32}\oint_C \frac{dz}{z-4}.$$

But $\frac{1}{8}\oint_C \frac{dz}{z^2} = 0$ *since the exponent is 2 and* $\frac{1}{32}\oint_C \frac{dz}{z-4}$ *since* $frac1z-2$ *is analytic in D. Hence*

$$\oint_C \frac{dz}{z^2(z-2)(z-4)} = \left(\frac{3}{32}\times 2\pi i\right) + \left(\frac{1}{8}\times 0\right) + \left(\frac{1}{8}\times 2\pi i\right) + \left(\frac{1}{32}\times 0\right) = \frac{-\pi}{16}i.$$

7.4 Cauchy's Theorem for Multiply Connected Domains

Now let us extend the the Cauchy's theorem for multiply connected regions.

Suppose f is analytic on C_1 and C_2 and in the annulus domain D bounded by C_1 and C_2 counterclockwise and clockwise respectively, and a is in the interior of the domain as shown in Figure 7.4.

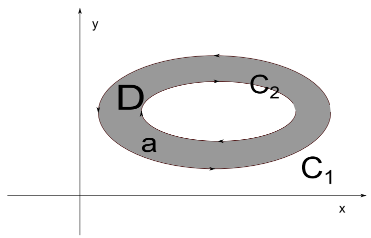


Figure 7.5: Annulus.

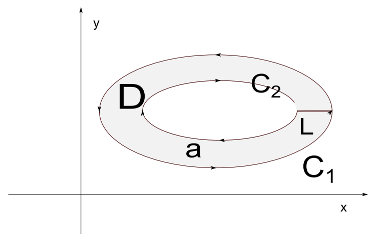
Now let L be the line segment as shown in the Figure 7.4. Then the region D is a simple bounded by the curve C, where $C = C_1 \oplus C_2 \oplus L \oplus L'$, where L' is the line segment which is oriented opposite to that of L.

Then. since f is analytic in D and a is D, by Cauchy Integral Formula, we have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

But C is a piecewise smooth curve, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz + \int_{C_2} \frac{f(z)}{z-a}dz + \int_L \frac{f(z)}{z-a}dz + \int_{L'} \frac{f(z)}{z-a}dz$$



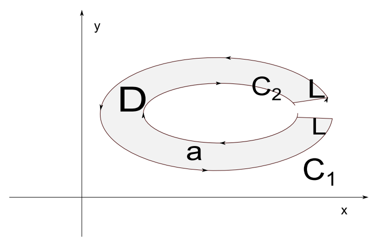


Figure 7.6: Multiply and Simply Connected Regions.

and

$$\int_{L'} \frac{f(z)}{z-a}dz = -\int_L \frac{f(z)}{z-a}dz.$$

Therefore, we get

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz + \int_{C_2} \frac{f(z)}{z-a}dz$$

and hence

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\left(\int_{C_1} \frac{f(z)}{z-a}dz + \int_{C_2} \frac{f(z)}{z-a}dz\right).$$

In general if D is bound by $C_1, C_2, C_3, \ldots C_m$, where C_1 is oriented counterclockwise and all the others are oriented clockwise and each of the $C_i's$ are closed, simple paths, a is in the interior of D, then

$$\oint_{C_1} \frac{f(z)}{z-a}dz + \oint_{C_2} \frac{f(z)}{z-a}dz + \ldots + \oint_{C_m} \frac{f(z)}{z-a}dz = 2\pi i f(a).$$

Theorem 7.4.1. *Let C be a closed path and* $C_1, ..., C_n$ *be closed paths enclosed by C. Assume that any two of* $C, C_1, ..., C_n$ *intersect and no interior point to any* C_i *is interior to any other* C_k. *Let* f *be analytic on an open set containing C and each* C_i *and all the points that are both interior to C an exterior to each* C_i. *Then*

$$\int_C f(z)dz = \sum_{i=1}^{n}\int_{C_i} f(z)dz.$$

Example 7.4.1. *Evaluate*

$$\oint_C \frac{dz}{z(z-1)},$$

where C is the circle $|z| = 3$ *counterclockwise.*

Solution

Let C_1 *and* C_2 *be the circles as in Figure 7.4.1.*

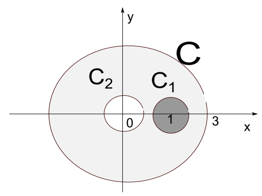


Figure 7.7: The curves in Example 7.4.1.

Therefore,

$$\int_C \frac{dz}{z(z-1)} = \int_C \frac{dz}{z(z-1)} + \int_C \frac{dz}{z(z-1)} = \big(2\pi i \times f_1(1)\big) + \big(2\pi i \times f_2(0)\big),$$

where $f_1(z) = \frac{1}{z}$ *and* $f_2(z) = \frac{1}{z-1}$. *Therefore*

$$\int_C \frac{dz}{z(z-1)} = \big(2\pi i \times 1\big) + \big(2\pi i \times (-1)\big) = 0.$$

Example 7.4.2. *Evaluate*

$$\oint_c \frac{z+1}{z(z-2)(z-4)^3}dz,$$

where C is the counterclockwise circle $|z-3| = 2$.

Solution

Here the integrand has singularities at $z = 0, 2$ *and* 4. *of which 2 and 4 lie inside C. It is easier to deform C into two closed curves and to evaluate each of the integrand using generalized Cauchy's formula,*

$$\begin{aligned}\oint_C \frac{(z+1)}{z(z-2)(z-4)^3}dz &= \oint_{C_1}\left[\frac{z+1}{z(z-4)^3}\right]\frac{dz}{z-2} + \oint_{C_2}\left[\frac{z+1}{z(z-2)}\right]\frac{dz}{(z-4)^3} \\ &= 2\pi i\left[\frac{z+1}{z(z-4)^3}\right]_{z=2} + \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[\frac{z+1}{z(z-2)}\right]_{z=4} \\ &= -\frac{3\pi i}{8} + \frac{23\pi i}{64} = \frac{\pi i}{64}.\end{aligned}$$

7.5 Fundamental Theorem of Complex Integral Calculus

Suppose f is complex continuous function on an open set D and F be a function defined on D with the property that $F'(z) = f(z)$ for all $z \in D$. Then any function $F(z)$ satisfying $F'(z) = f(z)$ is called an **anti-derivative** or a **primitive** of $f(z)$.

If z_0 is any fixed point in D, then the integral

$$\int_L f(\zeta)d\zeta \text{ denoted by } \int_{z_0}^{z} f(\zeta)d\zeta$$

where L is the line segment with initial point z_0 and terminal point z is path independent and hence defines a single- valued function of z, provided that f is analytic in the domain D. Thus we can define

$$G(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

and thus it follows that $G'(z) = f(z)$.

If $F(z)$ is any particular primitive of $f(z)$, then

$$G(z) = \int_{z_0}^{z} f(\zeta)d\zeta = F(z) + k$$

where k is arbitrary constant. Suppose the line segment is parameterized by $z(t)$ for $a \le t \le b$. First write $F(z) = U(x,y) + iV(x,y)$. Then we get

$$\begin{aligned}\int_L f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_a^b \frac{d}{dt}F(z(t))dt \\ &= \int_a^b \frac{d}{dt}U(x(t),y(t))dt + i\int_a^b \frac{d}{dt}V(x(t),y(t))dt \\ &= U(x(b),y(b)) + iV(x(b),y(b)) - U(x(a),y(a)) - iV(x(a),y(a)) \\ &= \int_a^b F'(z(t))z'(t)dt \\ &= F(z) - F(z_0).\end{aligned}$$

Hence we have the following theorem.

Theorem 7.5.1 (Fundamental Theorem of the complex Integral Calculus)**.** *Let* $f(z)$ *be analytic in a simply - connected domain D and let* z_0 *be any fixed point in D. Then*

(i) The function

$$G(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

is analytic in D and $G'(z) = f(z)$

(ii) if $F(z)$ is any primitive of $f(z)$, then

$$\int_{z_0}^{z} f(\zeta)d\zeta = F(z) - F(z_0).$$

Example 7.5.1. *Evaluate:*

1. *the integral*

$$\int_{2i}^{3} \sin z dz$$

2. *the integral*

$$\int_{1+i}^{-i} \frac{dz}{z}$$

Solution

1. *Since $\sin z$ is analytic on the segment joining the points $2i$ and3, we have*

$$\int_{2i}^{3} \sin z dz = \big[-\cos z\big]_{2i}^{3} = -\cos 3 + \cos 2i = \cosh 2 - \cos 3$$

2. *Since $\frac{1}{z}$ is analytic every where except at $z = 0$, it is analytic on the line segment joining $-i$ and $1+i$. Hence*

$$\int_{1+i}^{-i} \frac{dz}{z} = \Big[\log z\Big]_{1+i}^{-i} = \Big[\ln r + i\theta\Big]_{r=\sqrt{2},\frac{3\pi}{2}}^{r=1,\theta=\pi/2}$$
$$= (\ln 1 + i\frac{3\pi}{2}) - (\ln\sqrt{2} + i\frac{\pi}{4})$$
$$= -\frac{\ln 2}{2} + i\frac{5\pi}{4} = -\frac{\ln 2}{2} - i\frac{3\pi}{4}.$$

Therefore we have

$$\int_{1+i}^{-i} \frac{dz}{z} = -\frac{\ln 2}{2} - i\frac{3\pi}{4}.$$

7.6 Exercises

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Chapter 8

TAYLOR AND LAURENT SERIES

8.1 Sequence and Series of Complex Numbers

For all the discussions in this chapter, the reader is assumed to be familiar with real sequences and series, which are discussed in the previous courses.

A sequence $\{z_n\}$ whose terms are complex numbers is a complex sequence. A complex sequence is a complex valued function whose domain is a subset of the set of integers which is bounded from below (or $\mathbb{N}$.) That is, a sequence z_n is a function $f : \mathbb{N} \to \mathbb{C}$, and $f(1) = z_1, ..., f(n) = z_n$.

Remark 8.1.1. *Suppose that $\{z_n\}$ is a complex sequence. If $z_n = x_n + iy_n$, then*

$$\lim_{n\to\infty} = l, \qquad l = a + bi$$

if and only if

$$\lim_{n\to\infty} x_n = a \qquad and \qquad \lim_{n\to\infty} y_n = b.$$

Therefore limits and convergence properties of complex sequences is exactly similar (if not identical) to that of real sequences. So, we omit the discussion here.

Any sum of the form

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \dots,$$

where the terms $c_n's$ are complex numbers is called a **Complex series**.

As in the case of the real series, a complex series $\sum_{n=1}^{\infty} c_n$ is convergent, if and only if its partial sum $s_n = \sum_{k=1}^{n} c_k$ is a cauchy sequence, that is, if and only if to each $\epsilon > 0$ there corresponds an integer $N(\epsilon)$ such that $|s_m - s_n| < \epsilon$ for all integers m and n greater than $N(\epsilon)$.

However, since this state is difficult to apply, we use, in practice, array of standard convergence theorems (comparison, integral test, ratio test and others) which are easier to apply.

Theorem 8.1.2. *Consider a complex series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{n}(a_k + ib_k)$.*

1. *The series $\sum_{k=1}^{\infty} c_k$, converges to $u + iv$ if and only $\sum_{k=1}^{\infty} a_k = u$ and $\sum_{k=1}^{\infty} b_k = v$.*
2. *We say that the series $\sum_{k=1}^{\infty} c_k$ converges absolutely if the series $\sum_{k=1}^{\infty} |c_k|$ converges and if the series $\sum_{k=1}^{\infty} c_k$ converges absolutely, then it is convergent.*
   All absolute convergence tests that apply for real series(i.e. comparison test, ratio test and root test) hold also for complex series with the necessary notational adjustments.

Example 8.1.1. *Determine the convergence or divergence of the series*

$$1. \sum_{n=0}^{\infty} \frac{(1+i)^n}{n!} \qquad 2. \sum_{k=2}^{\infty} e^{-(2+3i)k} \qquad 3. \sum_{n=1}^{\infty} \left(\frac{(-1)^n + i}{n}\right).$$

Solution

1. *Here $c_n = \frac{(1+i)^n}{n!}$. Then by ratio test if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$, the series converges and $\lim_{n\to\infty} \left|\frac{c_{n+1}}{n_n}\right| > 1$, the series diverges.*
   But $\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(1+i)^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+i)^n}\right| = \left|\frac{1+i}{n+1}\right| = \frac{1}{n+1}\left|1+i\right| = \frac{1}{n+1}\sqrt{2}$ and by evaluating the limit we get $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to 0} \left(\frac{1}{n+1}\sqrt{2}\right) = 0$.
   Therefore, the series $\sum_{n=0}^{\infty} \frac{(1+i)^n}{n!}$ converges.
2. *Here $c_k = e^{-(2+3i)k}$ and by root test, if $\lim_{k\to\infty} |(c_k)^{1/k}| < 1$, the series converges and if $\lim_{k\to\infty} |(c_k)^{1/k}| > 1$, the series diverges.*
   But $\left|(c_k)^{1/k}\right| = \left|(e^{-(2+3i)k})^{1/k}\right| = \left|e^{-(2+3i)}\right| = \left|e^{-2}.e^{-3i}\right| = \left|e^{-2}\right|.\left|e^{-3i}\right| = e^{-2} < 1$. and $\lim_{k\to\infty} |(c_k)^{1/k}| = \lim_{k\to\infty} e^{-2} = e^{-2} < 1$. Hence the series converges.
3. *In this case we have $(\frac{(-1)^n + i}{n} = \frac{(-1)^n}{n} + \frac{i}{n}$ and we know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is divergent. Hence the series $\sum_{n=1}^{\infty} \left(\frac{(-1)^n + i}{n}\right)$ is divergent.*

8.2 Complex Taylor Series.

A series of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots, \tag{8.1}$$

where the terms are complex numbers, is known as a **power series** in powers of $z-a$. In the power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots,$$

- $c_n's$ are the coefficients, (real or complex constants)
- z is the variable (complex variable)
- a is the center (real or complex constant) of the series.

Remark 8.2.1. *The given power series (8.1) converges at $z = z_0$. If the series (8.1) converges at $z_1 \neq z_0$, then the series converges for all z, $|z - z_0| \leq |z_1 - z_0|$.*

If we apply the ratio test on (8.1) we get: $\left|\frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n}\right| = \left|\frac{c_{n+1}(z-a)}{c_n}\right| = \left|\frac{c_{n+1}}{c_n}\right| |z-a|$.
If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$, the power series (8.1) converges in the disk $|z-a| < \frac{1}{L}$ and diverges in the set $|z-a| > \frac{1}{L}$. If $L = \infty$, then the series converges only at $z = a$, and if $L = 0$, then it converges for all z.

We have proved the following theorem.

Theorem 8.2.2. *Given a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$, there is a number R such that:*

1. *$\sum_{n=0}^{\infty} c_n(z-a)^n$ converges if $|z-a| < R$ and*
2. *$\sum_{n=0}^{\infty} c_n(z-a)^n$ diverges if $|z-a| > R$.*

The number R is called the radius of convergence.

Example 8.2.1. *For the series:*

1. *$\sum_{n=0}^{\infty} z^n$, the radius of convergence is $R = 1$.*
2. *$\sum_{n=0}^{\infty} \frac{1}{n} z^n$, the radius of convergence is $R = 1$.*

3. *$\sum_{n=0}^{\infty} \frac{1}{n!} z^n$, the radius of convergence is $R = \infty$.*
4. *$\sum_{n=0}^{\infty} n^n z^n$, the radius of convergence is $R = 0$.*

In (8.1) if $a = 0$ and the radius of convergence is $R > 0$, the function:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < R$$

is a power series representation of f.

Remark 8.2.3. *If the power series representations*

$$\sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k z^k$$

both converge for $|z| < R$ to the same value for all z such that $|z| < R$, then the two series are identical. That is, $a_n = b_n$ for all $n = 0, 1, \dots$. Thus if a complex function f has a power series representation with any center a, then the representation

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

is unique.

Theorem 8.2.4. *If a power series function*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

converges for all z in D, where $D = \{z \in \mathbb{C} : |z-a| < R\}$ where $R > 0$ is the radius of convergence, then

1. *by termwise differentiation , $f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ for all z in D, where $D = \{z \in \mathbb{C} : |z-a| < R\}$ and $f'(z)$ has the series has the same radius of convergence as $f(z)$.*
2. *If C is any path in $D = \{z \in \mathbb{C} : |z-a| < R\}$, then by termwise integration, we have*

$$\int_C f(z)dz = \sum_{n=1}^{\infty} \int_C (z-a)^n dz$$

and $\int_C f(z)dz$ has the same radius of convergence as $f(z)$.

Recall that, unlike to real valued functions, for a complex function f, if f is analytic in some domain D, then it admits derivatives of all orders in D. If a is in the interior of D, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

is well defined and is known as the **Taylor series** (or Expansion) of f about the point a and if $a = 0$, then the Taylor series is also known as the **Maclaurin Series**

Theorem 8.2.5 (Taylor Expansion)**.** *If the disk $|z-a| < R$ lie entirely within D, then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

and by Cauchy's integral formula, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}$$

where C is a curve that is contained in D that contains a in its interior.
Thus the Taylor series of an analytic function f is :

$$f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$$

where

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta.$$

(Here b_n is called the n^{th} Taylor coefficient of f at a.)
That is, every analytic function can be represented by a Taylor's series.

Example 8.2.2 (Geometric series)**.** *Let $f(z) = \frac{1}{1-z}$. Then $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ and $f^n(0) = n!$. Therefore,*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \qquad for \qquad |z| < 1.$$

Differentiating this gives us

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1} == 1 + z + z^2 + \dots \qquad for \qquad |z| < 1.$$

Differentiating this gives us

$$\frac{2}{(1-z)^3} = \sum_{n=2}^{\infty} n(n-1)z^{n-2} \qquad for \qquad |z| < 1.$$

On the other hand, if we replace z by $-z$ in $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ we obtain

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad for \qquad |z| < 1$$

and differentiating this yields

$$\frac{-1}{(1+z)^2} = \sum_{n=1}^{\infty} n(-1)^n z^{n-1} \qquad for \qquad |z| < 1.$$

Example 8.2.3 (Exponential function)**.** *Let $f(z) = e^z$. Then $f^{(n)}(z) = e^z$. Therefore,*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Replacing z by z^2 we get

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}.$$

Example 8.2.4 (Trigonometric and hyperbolic functions.)**.** *Since we have $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2!}$, $\cosh z = \cos(iz)$ and $\sinh z = i\sin(iz)$, we have*

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n!)}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \cos hz = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

and

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

Example 8.2.5. *Consider $g(z) = \frac{1}{1+z^2}$. Then g is analytic everywhere except at $z = \pm i$. The Maclaurin's series of g is:*

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

converges for $|z| < 1$.

Example 8.2.6 (Undetermined coefficients)**.** *Find the Maclaurin's series for*

$$f(z) = \frac{e^z}{\cos z}.$$

Clearly $f(z)$ is analytic in the neighborhood of 0. Hence f has the Taylor's series representation at $z = 0$, say ;

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad for \qquad |z| < \frac{\pi}{2}.$$

But the successive derivatives get tedious to find the coefficients a_n. *Then*

$$\frac{e^z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \dots$$

implies $e^z = (a_0 + a_1 z)\cos z$. *Thus*

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = (a_0 + a_1 z + a_2 z^2 + \dots)(1 - \frac{z^2}{2} + \frac{z^4}{24} + \dots)$$

which implies

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots = a_0 + a_1 z + (a_2 - \frac{1}{2}a_0)z^2 + (a_3 - \frac{1}{2}a,)z^3$$

Equating coefficients we get: $a_0 = 1_1 a_1 = 1, a_2 - \frac{1}{2}x1 = \frac{1}{2} \Rightarrow a_2 = 1, a_3 = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}, \dots$
Hence

$$\frac{e^z}{\cos z} = 1 + z + z^2 + \frac{2}{3}z^3 + \dots$$

for $|z| < \frac{\pi}{2}$.

In the above example we used the product of two power series term by term. This kind of manipulation can be generalized in the following theorem.

Theorem 8.2.6 (Termwise Product of Power series). *If*

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

converges to $f(z)$ *in* $|z-a| < R_1$ *and*

$$\sum_{n=0}^{\infty} b_n (z-a)^n$$

converges to $g(z)$ *in* $|z-a| < R_2$,

then the termwise product

$$\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-a)^n\right) = \sum_{n=0}^{\infty}(a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)(z-a)^n.$$

converges to the product $f(z)g(z)$ *in* $|z-a| < min\{R_1, R_2\}$.

Example 8.2.7. *Find the Maclaurin series of:*

$$\frac{1}{(1-z)(1+2z)}.$$

Here

$$\frac{1}{(1-z)(1+2z)} = \left(\frac{1}{1-z}\right)\left(\frac{1}{1+2z}\right)$$

and

$$.\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

in $|z| < 1$.

$$.\frac{1}{1+2z} = \sum_{n=0}^{\infty}(-2z)^n = \sum_{n=0}^{\infty}(-1)^n 2^n z^n = 1 - 2z^1 + 4z^2 - 8z^3 + \dots$$

in $|z| < \frac{1}{2}$.
Then the product

$$\begin{aligned}
\left(\frac{1}{1-z}\right)\left(\frac{1}{1+2z}\right) &= \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{n=0}^{\infty}(-1^n(2z)^n)\right) \\
&= \sum_{n=0}^{\infty}(1.(-1)^n 2^n 0 + 1.(-1)^{n-1}2^{n-1} + \dots + 1)z^n \\
&= 1 - z + 3z^2 + \dots.
\end{aligned}$$

in $|z| < \frac{1}{2}$.

Remark 8.2.7. *From the Fundamental Theorem of Complex Integration, we know that, if* f *is analytic in an open disk D about* a, *then there exists a function* F *such that* $F'(z) = f(z)$ *for all* z *in D. Then we can construct an antiderivative F of* f *from the power series expansion of* f *about* a *in D. If* $f(z) = \sum_{n=1}^{\infty} c_n (z-a)^n$, *then* $F(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} c_n (z-a)^{n+1}$ *is an antiderivative of* f.

8.3 Laurent Series

In the previous part we have seen that if a function f is analytic in some domain D containing a point a in its interior, then f admits the Taylor series representation at a and this representation is unique.

However, if f is not analytic at $z = a$, then the Taylor series about a do not have a unique-representation. Hence we need the following result.

Theorem 8.3.1 (Laurent's Theorem). *Let D be the closed region between, and including concentric circles* C_1 *and* C_0 *with their centers at* $z = a$.

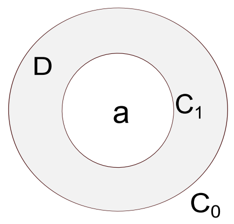

Figure 8.1: Annulus.

If f is analytic in D, then it admits a **Laurent series representation** *given by*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n$$

in D, with the coefficients $b_n's$ are calculated from:

$$b_n = \frac{1}{2\pi i} \oint_c \frac{f(w)}{(w-a)^{n+1}} dw, \tag{8.2}$$

where C is a piecewise smooth simple closed counterclockwise curve in D.

Remark 8.3.2. *Note that:*

1. *If f is analytic on or in the interior of C_1 then $\frac{1}{(w-a)^{n+1}}$ is also analytic for all $n < 0, n \in \mathbb{Z}$. Hence $b_n = 0 \forall n \epsilon z and n < 0 in (*)$. Therefore we have the Taylor series expansion.*

2. *The Laurent series representation depends on the choice of the annulus D with the same center a.*
   Therefore, a Laurent series is not in general a unique representation (unlike the Taylor series of analytic functions).

But do we really need to evaluate (9.1) to get the coefficients $b_n's$? No, practically (9.1) will not be calculated. This can be seen in the next examples

Example 8.3.1. *Expand*

$$f(z) = \frac{1}{z+i}$$

about $z = 0$

(a) *Since f is analytic every where except at $z = -i$, the Taylor series expansion $at z = 0$ is*

$$\begin{aligned} \frac{1}{z+i} &= \frac{1}{i(1+\frac{z}{i})} = -i\frac{1}{1-iz} = -i\sum_{n=0}^{\infty}(iz)^n; |z| < 1 \\ &= -i[1 + iz + (iz)^2 + \ldots] = -i[1 + iz - z^2 + \ldots] \\ &= -i + z + iz^2 - \ldots \end{aligned}$$

for $|z| < 1$.

(b) *However, in the annulus $1 < |z| < \infty$, we can expand it using Laurent series. This time we write*

$$\frac{1}{z+i} = \frac{1}{z(1+\frac{i}{z})} = \left(\frac{1}{z}\right)\left(\frac{1}{1+i/z}\right)$$

to extract out the singularity point of f. Since we are expanding in the annulus $1 < |z| < \infty$ about $z = 0, \frac{1}{z}$ is already in the required form.

Now put $t = \frac{i}{z}$. Then $|t| = |\frac{i}{z}| = \frac{1}{|z|} < 1$, since $|z| > 1$. Thus we can use Taylor series expansion on :

$$\begin{aligned} \frac{1}{1+\frac{i}{z}} &= \frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n \qquad for |t| < 1. \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{z}\right)^n \\ &= 1 - \frac{i}{z} - \frac{1}{z^2} + \ldots \end{aligned}$$

in $1 < |z| < \infty$. Therefore

$$\begin{aligned} f(z) = \frac{1}{z+i} &= \frac{1}{z}\left(\frac{1}{1+i/z}\right) = \frac{1}{z}\left(1 - \frac{i}{z} - \frac{1}{z^2} + \ldots\right) \\ &= \frac{1}{z} - \frac{i}{z^2} - \frac{1}{z^3}\ldots \\ &= \sum_{n=-\infty}^{-1}(i)^{(n+1)}z^n \end{aligned}$$

Example 8.3.2. *Derive the Laurent expansion of*

$$f(z) = \frac{1}{\sin z}$$

about $z = \pi$, in the annulus $0 < |z - \pi| < \pi$.

Since $f(z)$ has singularity at $z = \pi$, it does not admit Taylor expansion. Now let $t = z - \pi$. Then $z = t + \pi$. and we have:

$$\frac{1}{\sin z} = \frac{1}{\sin(t+\pi)} = -\frac{1}{\sin t} = -\frac{1}{t}\frac{t}{\sin t}.$$

Here since

$$\begin{aligned}\sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \\ &= t\Big(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots\Big),\end{aligned}$$

$$\frac{1}{\sin t} = \frac{1}{t(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots)}$$

has singularity at $t = 0$. *So, the factor* $\frac{1}{t}$ *contributes to the non-singularity of* $\frac{1}{\sin t}$, *hence we factored it out. But*

$$\frac{t}{\sin t} = \frac{1}{1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots}$$

is not singular at $t = 0$. *That is,* $\frac{t}{\sin t}$ *is analytic at* $t = 0$, *hence admits a Taylor series expansion in some neighborhood of* $t = 0$. *Therefore, for* $0 \leq |t| < \pi$, *the Taylor series will be of the form, say;*

$$\frac{t}{\sin t} = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

(Though $-\frac{1}{t}$ *is singular at* $t = 0$, *it is already a one - term Laurent series about* $t = 0$. *Thus*

$$\frac{t}{sint}$$

has been " desingularized " at $t = 0$ *by t at the numerator.)*

Now to find the coefficients $a_0, a_1, \ldots$, *we use the "undetermined coefficients - method ":*

$$\begin{aligned}t &= \frac{t}{\sin t}.\sin t \\ &= (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots)(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots) \\ &= 0 + a_0 t + a_1 t^2 + (a_2 - \frac{a_0}{3!})t^3 + (a_3 - \frac{a_1}{3!})t^4 + \ldots\end{aligned}$$

This implies $a_0 = 1, a_1 = 0, a_2 = \frac{1}{6}, a_3 = 0$ *and hence* $\frac{t}{\sin t} = 1 + \frac{1}{6}t^2 + \frac{7}{360}t^4 + \ldots$

Thus

$$\begin{aligned}\frac{1}{\sin z} &= -(\frac{1}{z-\pi})(1 + \frac{1}{6}(z-\pi)^2 + \frac{7}{360}(z-\pi)^4 + \ldots) \\ &= -\frac{1}{z-\pi} - \frac{1}{6}(z-\pi) - \frac{7}{360}(z-4)^3 - \ldots.\end{aligned}$$

is the desired Laurent series in the annulus $0 < |z - \pi| < \pi$.

Remark 8.3.3. *In the Laurent series expansion of a function* f *about* a *we have two parts:*

$$\begin{aligned}f(z) &= \sum_{n=-\infty}^{\infty} c_n(z-a)^n \\ &= \sum_{n=-\infty}^{-1} c_n(z-a)^n + \sum_{n=0}^{\infty} c_n(z-a)^n \\ &= \sum_{m=1}^{\infty} b_m(z-a)^{-m} + \sum_{n=0}^{\infty} c_n(z-a)^n \\ &= \sum_{m=1}^{\infty} \frac{b_m}{(z-a)^m} + \sum_{n=0}^{\infty} c_n(z-a)^n\end{aligned}$$

where $b_m = c_{-n}$ *and* $m = -n$. *In this last expression the sum*

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

is part of the Taylor series if the function is analytic and the sum

$$\sum_{m=1}^{\infty} \frac{b_m}{(z-a)^m}$$

is known as $the\ principal\ part\ of\ the\ Laurent\ Series.$ *of* $f(z)$

Exercise 8.3.4. *Expand* $f(z) = e^{1/z}$ *about* $z = 0$.

8.4 Exercises

Chapter 9

INTEGRATION BY THE METHOD OF RESIDUE.

9.1 Zeros and Classification of Singularities.

Recall that: A complex function f is said to be singular at a point $z = a$, if f is not analytic at a, that is, f is differentiable in an open disk about a.

There are different kinds of singularities. Suppose $z = a$ is a singularity point for f, that is, f is not analytic at a. If f is analytic in an annulus $0 < |z - a| < p$, for some $p > 0$, then $z = a$ is called an isolated singularity point of f, otherwise (i.e. there does not exist such $p > 0$) it is known as a **non-isolated singular point.**

Example 9.1.1. *Let $f(z) = \frac{1}{\sin z}$. Then $z = 0, \pm\pi, \pm 2\pi, \dots$ are all singularity points of f. Moreover all of them are isolated singularity points. For instance, to show at $z = \pi$, consider the annulus $0 < |z - \pi| < \pi$. clearly f is analytic on the annulus. To see this: with $t = z - \pi$, we have:*

$$\frac{1}{\sin z} = \frac{1}{\sin(t+\pi)} = -\frac{1}{\sin t} = -\frac{1}{t}\frac{t}{\sin t}$$

for $z \neq \pi$ and hence $t \neq 0$,

$$\frac{t}{\sin t}$$

is analytic in the annulus.

Example 9.1.2. *The function*

$$g(z) = \frac{1}{\sin(\frac{1}{z})}$$

is singular at $z = \frac{1}{k\pi}, k = \pm, 1, \pm 2, \dots$ and at $z = 0$. The singularity points $z = \frac{1}{k\pi}$, $k \in \mathbb{Z}\backslash\{0\}$ are isolated (as we can find a $\rho > 0$,) while the point $z = 0$ is not isolated because every annulus $0 < |z| < \rho$ inevitably contains at least one singular point (in fact, infinitely many of them) no matter how small we choose $\rho > 0$. (Since $\frac{1}{k\pi} \to 0$ as $k \to \infty$, 0 is the limit point or accumulation point of non-singularities.)

Assume that f has an isolated singularity at $z = a$ That is, there exists $\rho > 0$ such that $0 < |z - a| < \rho$ in which f has a Laurent series of the form:

$$f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n = \sum_{m=1}^{\infty} \frac{b_m}{(z-a)^m} + \sum_{n=0}^{\infty} c_n(z-a)^n.$$

where $m = -n$ and $b_m = c_{-n}$.
Recall that the sum

$$\sum_{m=1}^{\infty} \frac{b_m}{(z-a)^m}$$

is the principal part of the Laurent series for f.

1. If $b_m = 0$ for every integer in the principal expansion, then $z = a$ is called a removable singularity.

2. If the expansion of this principal part terminates at some point, say after $N < \infty$ terms, then the singularity point $z = a$ of f is known as an N^{th}**–order pole**. Here, a is the pole and N is the order.
   If $N = 1$, then the pole $z = a$ is called a **simple** pole, and the Laurent series will take the form
   $$f(z) = \frac{b_1}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$

3. If the principal part has infinitely many terms, then f is said to have an **isolated essential singularity** at $z = a$.

Example 9.1.3. *Let*

$$f(z) = \frac{\cos z}{z^2}.$$

Then f is differentiable at all $z \neq 0$ and not defined at $z = 0$. Using the Maclaurin series of $\cos z$, we can get the Laurent expansion of $f(z)$ around zero is

$$f(z) = \frac{\cos z}{z^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}\right) = \frac{1}{z^2} - 1 + \frac{1}{26}z^2 - + ...$$

Then the highest power of $\frac{1}{z}$ in the expansion is 2, so f has a pole of order 2 at 0.

For

$$f(z) = \frac{\cos z}{z^2},$$

consider $\lim_{z\to 0} z^2 f(z) = 1 \neq 0$. We have the following theorem that relates a pole of order m at $z = a$ and pole and $\lim_{z\to a} z^m f(z)$.

Theorem 9.1.1. *Let f be differentiable at $0 < |z-a| < p$. Then f has a pole of order m at a if and only if $\lim_{z\to a} z^m f(z)$ is a non zero number.*

Now, the question is: **how to (find) determined the order of a pole?** We use a zero of a function.

Suppose that a function f is analytic at $z = a$. We say that a function f has a zero at $a \in D$ if $f(a) = 0$. A zero a of f is said to have $\text{order } m$ if $f(a) = f'(a) = f''(a) = \ldots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$.

Example 9.1.4. *Let $f(z) = z^3$. Then $f'(z) = 3z^2, f''(z) = 6z$ and $f'''(z) = 6$.*

Here $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) = 6 \neq 0$ and hence f has a$a\ zero,\ z = 0,\ of\ order\ 2$.

Remark 9.1.2. *If f is analytic at a and $f(a) \neq 0$, we say for convenience that f has a zero, $z = a$, of order 0, or a $zeroth\text{-}order\ of\ zero$ at $z = a$.*

Now we state a theorem which help us to determine the different kind of singularities of a function.

Theorem 9.1.3. *Let p and q be analytic functions at $z = a$, and have zero of order P and Q respectively at a. Then*

1. *$f(z) = \frac{1}{p(z)}$ has a pole of order P at $z = a$.*
2. *$f(z) = \frac{p(z)}{q(z)}$ has a pole of order $N = Q - P$ at $z = a$, if $Q - P > 0$, and f is analytic at a if $Q \leq P$.*

Example 9.1.5. *Find and classify all the singularities of:*

$$f(z) = \frac{(\pi - z)(z^4 - 3z^2)}{\sin^2 z}$$

Solution

Let $p(z) = (\pi - z)(z^4 - 3z^2) = (\pi - z)(z^2 - 3)z^2)$ and $q(z) = \sin^2 z$. Then p has 1^{st} order zero at $z = \pi, z = \pm\sqrt{3}$ and a $2^{nd}-$ order zero at $z = 0$ and q has 2^{nd} order zero at $z = 0$ and also

it has 2^{nd} order zero at $z = n\pi, n \neq 0$.

Hence by the above Theorem, f is analytic at $z = 0$, has a pole of order 1 at $z = \pi$ and of order 2 at all $z = n\pi, n \neq 0, 1$.

Suppose that f has a pole at $z = a$, say of second-order. Then from the Laurent series we have:

$$f(z) = \frac{b_2}{(z-a)^2} + \frac{b_1}{(z-a)} + \overbrace{c_0 + c_1(z-a) + c_2(z-a)^2}^{g(z)+\ldots}, \qquad \text{with } b_2 = c_{-2} \neq 0$$

for $0 < |z-a| < \rho$ for some $\rho > 0$.

By multiplying both sides by $(z-a)^2$, we get:

$$(z-a)^2 f(z) = b_2 + b_1(z-a) + (z-a)^2 g(z).$$

Since $g(z)$ is analytic at a, it is continuous at a. Hence

$$\lim_{z\to a}(z-a)^2 f(z) = b_2 + (b_1 \times 0) + (0 \times g(a)) = b_2.$$

This implies $|(z-a)^2 f(z)| = |z-a|^2|f(z)| \to |b_2| \neq 0$ which implies

$$\lim_{z\to a}|f(z)| = \lim_{z\to a}\frac{|b_2|}{|z-a|^2} = \infty.$$

Hence we have the following theorem for the general case.

Theorem 9.1.4 (Behavior of a function near its pole)**.** *If $f(z)$ has a pole at $z = a$, then*

$$\lim_{n\to z}|f(z)| = \infty.$$

However, if f has an essential singularity at $z = a$, the above theorem does not hold in general.

9.2 The Residue Theorem

Consider a complex function f which has 2 isolated singularity points say a and b, inside a simple closed path C (called contour) and analytic elsewhere in C.

Then by Cauchy's Theorem, the integral

$$I = \int_C f(z)dz$$

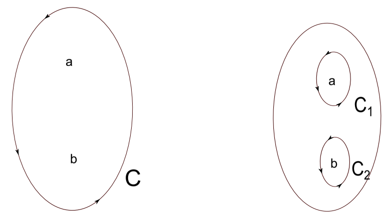

Figure 9.1: The Contour C.

can be decomposed into

$$I = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,$$

where C_1 and C_2 be the curves as in Figure 9.2. Since a and b are singularity points f, the Laurent series expansion of f will be:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(1)}(z-a)^n \text{ in } 0 < |z-a| < p_1 \text{ and } f(z) = \sum_{n=-\infty}^{\infty} c_n^{(2)}(z-b)^n \text{ in } 0 < |z-a| < p_2$$

for some $p_1 > 0, p_2 > 0$.

Assume that C_1 is in the annulus $0 < |z-a| < p_1$ and C_2 is in the annulus $0 < |z-b| < p_2$ Then:

$$\begin{aligned} I = \oint_C f(z)dz &= \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz \\ &= \oint_{C_1} \sum_{n=-\infty}^{\infty} c_n^{(1)}(z-a)^n dz + \oint_{C_2} \sum_{n=-\infty}^{\infty} c_n^{(2)}(z-b)^n dz \\ &= \sum_{n=-\infty}^{\infty} \oint_{C_1} c_n^{(1)}(z-a)^n dz + \sum_{n=-\infty}^{\infty} \oint_{C_2} c_n^{(2)}(z-b)^n dz \\ &= \left(2\pi i \times c_{-1}^{(1)}\right) + \left(2\pi i \times c_{-1}^{(2)}\right) = 2\pi i\left(c_{-1}^{(1)} + c_{-1}^{(2)}\right) \end{aligned}$$

In the above integral, the surviving coefficients $c_{-1}^{(1)}$ and $c_{-1}^{(2)}$ are called **the Residue** of $f(z)$ at a and b respectively. The residue of a function f at a is is denoted by $Res(f,a)$. We can generalize the above result in the following theorem.

Theorem 9.2.1 (Residue - Theorem). *Let C be a piecewise smooth simple closed curve oriented counterclockwise and let $f(z)$ be analytic inside and on C except at finitely many isolated singularity points $a_1, a_2, \ldots, a_k$ in the interior of C and c_{-1}^i denotes the residue of at a_j, then*

$$I = \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} c_{-1}^j$$

That is, the integral I is equal to $2\pi i$ times the sum of the residues of f in C.

Example 9.2.1. *Calculate the residue and evaluate*

$$I = \oint_C z^3 \cos\left(\frac{1}{z}\right) dz,$$

where C is a circle $|z| = 1$ oriented counterclockwise.

Solution

The only singular point is $z = 0$. and the Laurent series about $z = 0$ is

$$\begin{aligned} z^3 \cos\left(\frac{1}{z}\right) &= z^3\left(\sum_{n=0}^{\infty}(-1)^n \frac{(1/z)^{2n}}{(2n)!}\right) \\ &= z^3\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \ldots\right) \\ &= z^3 - \frac{z}{2!} + \frac{1}{4!z} - \frac{1}{6!z^3} + \ldots \ \textit{for } 0 < |z| < \infty. \end{aligned}$$

Thus the residue is $\frac{1}{4!} = \frac{1}{24} = c_{-1}$ and hence

$$I = \oint_C z^3 \cos\frac{1}{z} dz = 2\pi i \times \frac{1}{24} = \frac{\pi}{12} i.$$

Question: How to calculate the residue in the general case?

To start with, suppose $f(z)$ has a simple (or first-order) pole at $z = a$, so that

$$f(z) = c_{-1}\frac{1}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots$$

in the annulus $0 < |z-a| < p$ for some $p > 0$. Then

$$(z-a)f(z) = c_{-1} + c_0(z-a) + c_1(z-a)^2 + c_2(z-a)^3 + \ldots$$

and $\lim_{z\to a}[(z-a)f(z)] = c_{-1}$ which is the residue of f at $z = a$.
Next suppose that f has an N^{th} order pole at $z = a$, that is,

$$f(z) = c_{-N}\frac{1}{(z-a)^N} + c_{-N+1}\frac{1}{(z-a)^{N-1}} + \ldots + c_0 + c_1(z-a) + \ldots$$

and

$$(z-a)^N f(z) = c_{-N} + c_{-N+1}(z-a) + \ldots + c_0(z-a)^N + c_1(z-a)^{N+1} + \ldots \tag{9.1}$$

However, unfortunately

$$\lim_{z\to 0}[(z-a)^N f(z)] = c_{-N}$$

is not the residue c_{-1} of f.

Now to find the residue c_{-1} of f at $z=a$, observe that the right hand side of (9.1) is the Taylor series expansion of $g(z)=(z-a)^N f(z)$ at $z=a$ and the coefficients c_{-N+j} of $(z-a)^j$ is the j^{th} derivative of $g(z)$ at $z=a$ divided by $j!$. That is,

$$c_{-N+j} = \frac{1}{j!}\frac{d^j}{dz^j}g(a).$$

When $i=N-1$, $-N+j=-1$ and c_{-N+j} becomes c_{-1}, the residue of f at $z=a$.

Therefore,

$$c_{-1} = \frac{1}{(N-1)!}\lim_{z\to a} g^{(N-1)}(z),$$

where $g(z)=(z-a)^N f(z)$. This holds true only if the singularity at $z=a$ is not essential.

Theorem 9.2.2 (Residue at a Pole of Order m.). *Let f be a function having a pole of order m at $z=a$. Then*

$$Res(f,a) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\Big((z-a)^m f(z)\Big).$$

Example 9.2.2. *Evaluate all residues of $f(z)=\frac{1}{(z+2)(z-1)^3}$*

Solution

The denominator of f has first - order zero at $z=-2$ and 3^{rd} order zero at $z=1$. Since the numerator 1 has no zeros, f has first - order pole at $z=-2$ and a third - order pole at $z=1$ (N = 3). Thus

$$Resf = \frac{1}{0!}\lim_{z\to -2}\left((z+2).\frac{1}{(z+2)(z-1)^3}\right) = \lim_{z\to -2}\frac{1}{(z-1)^3} = -\frac{1}{3^3} = -\frac{1}{27}$$

and

$$Resf = \frac{1}{(3-1)!}\lim_{z\to 1}\left((z-1)^3.\frac{1}{(z+2)(z-1)^3}\right)^{(N-1)=2} = \frac{1}{2!}\lim_{z\to 1}(\frac{1}{z+2})''$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{(-1).(-2)}{(z+2)^3} = \lim_{z\to 1}\frac{1}{(z+2)^3} = \frac{1}{3^3} = \frac{1}{27}.$$

Thus

$$\oint_C f(z)dz = \oint_c \frac{dz}{(z+2)(z-1)^3} = 2\pi i(-\frac{1}{27}+\frac{1}{27}) = 0,$$

for a counterclockwise C containing both - 2 and 1 in the interior.

If C contains only, say $z=-2$, then

$$\oint_C f(z)dz = 2\pi i.(-\frac{1}{27}) = -\frac{2\pi}{27}i.$$

If C contains neither of them, then f is analytic in C, and hence $\oint_C f(z)dz = 0$.

Example 9.2.3. *Let $f(z)=\frac{ze^{\pi z}}{(z-2)^2(z^2+4)}$. Evaluate the integral I of f over the ellipse $9x^2+y^2=9$ counterclockwise.*

Solution

Since the denominator $(z-2)^2(z^2+4)$ has zeros at $z=2$ of order 2 and $z=\pm 2i$ each of order 1, f has poles at $z=2$ of order 2 and at $z=2i$ and at $z=-2i$ of order 1 (as the numerator $ze^{\pi z}$ has no zeros). But since $z=2$ is not inside the ellipse C, it has no relevance for integration. Hence we consider only $z=-2i$ and $z=2i$. Their respective residues are:

$$Res_{atz=-2i}\frac{ze^{\pi z}}{(z-2)^2(z^2+4)} = \lim_{z\to -2i}\frac{ze^{\pi z}}{(z-2)^2(z-2i)(z+2i)}\times(z+2i)$$
$$= \frac{(-2i)e^{-2\pi i}}{(-2-2i)^2(-4i)} = \frac{1}{2(-2-2i)^2} = \frac{1}{16i} = \frac{-i}{16}$$

and

$$Res_{atz=2i}\frac{ze^{\pi z}}{(z-2)^2(z^2+4)} = \lim_{z\to +2i}\frac{ze^{\pi z}}{(z-2)^2(z-2i)(z+2i)}(z-2i)$$
$$= \frac{(2i)e^{2\pi i}}{(2i-2)^2(2i+2i)} = \frac{2i}{4(-1+i)^2(4^2 i)}$$
$$= \frac{1}{8\times(-2i)} = \frac{i}{16}$$

Therefore,

$$\oint_C f(z)dz = 2\pi i\left(\frac{-i}{16}+\frac{i}{16}\right) = 0.$$

9.3 Evaluation of Real Integrals.

Consider the class of real integrals of the general form

$$I = \int_0^{2\pi} F(\cos\theta,\sin\theta)d\theta$$

where F is a rational function of $\cos\theta$ and $\sin\theta$.

Example 9.3.1. *The functions*

$$F_1(\cos\theta, \sin\theta) = \frac{2 - \cos^2\theta \sin\theta}{1 + \cos\theta}$$

and

$$F_2(\cos\theta, \sin\theta) = \frac{\sin a\theta}{(1 + \cos b\theta)^2},$$

for $a, b \in \mathbb{R}$, *rational function of* $\cos\theta$ *and* $\sin\theta$.

To evaluate integrals of the above form, use the change of variables $z = e^{i\theta}$. This change of variables will transform the real integral into a closed path complex integral. If $\theta_1 = 0$ then $z_1 = 1$ and if $\theta_2 = 2\pi$ then $z_2 = e^{2\pi i} = 1$.

Here $dz = ie^{i\theta}d\theta$, which implies that $d\theta = \frac{dz}{iz}$ and with this change of variable we get: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i} = \frac{z^2-1}{2zi}$.

Hence

$$I = \int_0^{2\pi} F(cos\theta, sin\theta)d\theta = \oint_C F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}.$$

Then we can use the Residue Theorem to evaluate the final integral.

Example 9.3.2. *Evaluate:*

$$I = \int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}.$$

Solution

Let $z = e^{i\theta}$. *Then* $d\theta = \frac{dz}{iz}$. *As* θ *goes from* 0 *to* $2\pi, z$ *traverses through a complete circular revolution with radius* $r = 1$. *Thus*

$$\begin{aligned} I = \int_0^{2\pi} \frac{d\theta}{2 - sin\theta} &= \oint_c \frac{1}{2 - \frac{z^2-1}{2iz}}.\frac{dz}{iz} \\ &= -2\oint_c \frac{dz}{z^2 - 4iz - 1} \\ &= -2\oint_c \frac{dz}{(z-z_1)(z-z_2)}, \end{aligned}$$

where $z_1 = (2+\sqrt{5})i$ *and* $z_2 = (2-\sqrt{5})i$

since z_1 *lies outside of C, its contribution to the integral is zero, Thus we've:*

$$I = \oint_c \frac{-2dz}{(z-z_1)(z-z_2)} = 2\pi ix Res_{atz=z_2}[\frac{-2}{(z-z_1)(z-z_2)}]$$

$$= 2\pi ix \lim_{z\to z_2}[(z-z_2).\frac{-2}{(z-z_1)(z-z_2)}] \tag{9.2}$$

$$= 2\pi ix \lim_{z\to z_2}(\frac{-2}{z-z_1}) \tag{9.3}$$

$$= 2\pi ix\frac{-2}{z_2-z_1} = 2\pi ix - \frac{1}{\sqrt{5}i} \tag{9.4}$$

$$= \frac{2}{\sqrt{5}}\pi or \frac{2\sqrt{5}}{5}\pi. \tag{9.5}$$

(9.6)

$\therefore \int_0^{2\pi} \frac{d\theta}{2-sin\theta} = \frac{2}{\sqrt{5}}\pi$

<u>Exercise.</u> Evaluate using Residue Theorem:

$I = \int_0^{\pi} \frac{cos\theta}{17-8cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{cos\theta}{17-8cos\theta}d\theta.$

Ans. $\frac{\pi}{60}$

9.3.1 Improper Integrals:

. Consider real Integrals of type:

$\int_{-\infty}^{\infty} f(x)dx$

Clearly the improper integral can be written as:

$\int_{-\infty}^{\infty} f(x)dx = \lim_{A\to\infty}\int_{-A}^{0} f(x)dx + \lim_{B\to\infty}\int_0^B f(x)dx.$

If both limits exists, then the improper integral is said to be convergent and can be expressed in the form:

$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx.\dots\dots(*)$

Now assume that $f(x) = \frac{p(x)}{q(x)}s.t.q(x) \neq 0 \forall x \epsilon \Re$ and deg. $q(x) - deg.p(x) \geq 2$. Then clearly (*) is convergent and we can use the expression in (**) without any further remark.

Consider the upper semicircle C and define a complex $fnf(z)withrealpartf(x).Then.$

$\oint_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_{c_R} f(z)dz = 2\pi i\sum Resf(z)$

$\Rightarrow \int_{-R}^{R} f(x)dx = 2\pi i\sum Resf(2) - \int_{c_R} f(z)dz.$

Now as $R \to \infty$, we consider the integral $\int_{c_R} f(z)dz$. using the substitution $z = Re^{i\theta}$. then C_R can be represented parametrically that $|z| = R \Rightarrow$ and hence $R =$ const. will be the $eqnofC_R.asz$ ranges along C_R, θ varies from$0to\pi$.

Thus $\quad |f(z)| < \frac{k}{|z|^2}$ for $|z| = R > R_0$ sufficiently large k. constant

$\Rightarrow |\int_{c_R} f(x)dz| \leq \int_{c_R} |f(z)|dz < \frac{k}{R^2}\pi R = \frac{k\pi}{R} for R > R_0$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} [2\pi i \sum Resf(z) - \int_{c_R} f(z)dz]$$

$$= 2\pi i \sum Resf(z) - \lim_{R\to\infty} \int_{c_R} f(z)dz \tag{9.7}$$

$$2\pi i \sum Resf(z) - 0 \tag{9.8}$$

(9.9)

$\therefore \boxed{\int_{-R}^{\infty} f(x)dx = 2\pi i \sim Resf(z)}$

where sum is over all the residues of f over the upper half plane.

Examples. 1. Evaluate:

$\int_0^\infty \frac{cosax}{x^2+1}dx \qquad a > o.$

soln.

$I = \int_0^\infty \frac{cosax}{x^2+1}dx = \frac{1}{2}\int^\infty -\infty \frac{cosax}{x^2+1}dx$

Now consider the function $f(z) = \frac{e^{iaz}}{z^2+1}$

.Clearly f is analytic everywhere except at $z = \pm i$. At these two points, f has simple poles.

Thus $\quad \oint_c \frac{e^{iaz}}{z^21}dz = 2\pi i Res_{atz=i}.f(z)(attheupperhalfplane).$

$= 2\pi i \frac{e^{iaxi}}{2i} = \pi e^{-a}$

$$i.e.\pi e^{-a} = \oint_c \frac{e^{iaz}}{z^2+1}dz = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx + \lim_{R\to\infty}\int_{c_R} \frac{e^{iaz}}{z^2+1}dz$$

$$= \lim_{R\to\infty}\oint_{-R}^{R} \frac{cosax + isinax}{x^2+1}dx + \lim_{R\to\infty}\int_{c_R} \frac{e^{iaz}}{z^2+1}dz \tag{9.10}$$

$$= \int_{-\infty}^{\infty} \frac{cosax}{x^2+1}dx + i\int_{-\infty}^{\infty} \frac{sinax}{x^2+1}dx |\frac{e^{iaz}}{z^2+1}| \leq |\frac{1}{z^2+1}| \leq \frac{1}{(R-1)\sqrt{R^2+1}} \tag{9.11}$$

$\therefore \int_{-\infty}^{\infty} \frac{cosax}{x^2+1}dx = \pi e^{-a} and \int_{-\infty}^{\infty} \frac{sinax}{x^2+1}dx = 0$

$\Rightarrow I = \int_0^\infty \frac{cosax}{x^2+1}dx\frac{1}{2}\int^{\infty-\infty} \frac{cosax}{x^2+1}dx\frac{1}{2}\pi e^{-a} = \frac{\pi}{2}e^{-a}$

Exercise! Evaluate

$\int_D^\infty \frac{x^{1/3}}{(x+1)^2}dx$

using Residue Theorem.