Applied Mathematics III
Unit 1
Ordinary Differential Equations of the First Order

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Table of Content

Basic Concepts and Ideas

In this section we will see the basic concepts and ideas and in the remaining sections we will consider equations which involve the first derivative of a given independent variable with respect to an independent variable, which are called Ordinary Differential Equations of First Order.

Definition

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **Differential Equation (DE)**.

Example

$$\frac{dy}{dx} + y = x, \quad \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} = v, \quad \frac{d^5 x}{dt^5} + 4\frac{d^3}{dx^3} - \frac{dx}{dt} = 5x^3$$

are all Differential Equations.

Differential equations can be classified by their **type**, **order**, and in term of **linearity**. We will see these classifications before going to the solution

Classification by Type

- If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it is said to be an **ordinary differential equation (ODE)**.
- If a function is defined in terms of two or more independent variables, the corresponding derivative will be a partial derivative with respect to each independent variable. An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.

Example

1. $\frac{dy}{dx} + y = x^2$, $\frac{d^4x}{dt^4} + xt\frac{dx}{dt} = x^3y^2$ & $\left(\frac{dy}{dx}\right)^4 + xy = 0$ are all ordinary differential equations.
2. $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = v$ & $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$ are partial differential equations.

In this part we will only consider the case of ordinary differential equations.

Classification by Order

The order of a differential equation (either ODE or PDE) is the order of the highest derivative that appear in the equation. For example,

$$3x\frac{dy}{dx} + y = 4x \quad \text{and} \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = \sin x$$

are first and second-order ordinary differential equations respectively.

The general n^{th}-order ordinary differential equation in one dependent variable is given by the general form

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0 \tag{1}$$

where F is a real-valued function of $n+2$ variables $x, y, y', y'', \cdots, y^{(n)}$

$$\frac{d^ny}{dx^n} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

where f is a real-valued continuous function and this is referred to as the normal form of (1).

Classification by Linearity

An n^{th}-order ordinary differential equation (1) is said to be linear if F is linear in $y, y', \cdots, y^{(n)}$. This means that an n^{th}-order linear ordinary differential equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - b(x) = 0$$

where $a_n(x) \neq 0$.
If $b(x) \equiv 0$, the above equation is called a **homogeneous DE** and otherwise it is called **nonhomogeneous**.

Definition

Let $h(x)$ be a real valued function defined on an interval $[a, b]$ and having n^{th} order derivative for all $x \in (a, b)$. If $h(x)$ satisfies the n^{th} order ODE on (a, b). $y = h(x)$ is called an **Explicit solution** of the ODE on $[a, b]$.
Sometimes a solution of a differential equation may appear as an implicit function, i.e. the solution can be expressed implicitly in the form: $h(x, y) = 0$, where h is some continuous function of x and y, and such solution is called an **Implicit Solution** of the DE.

Example

1. Show that $h(x) = 2\sin x + 3\cos x$ is an explicit solution of the differential equation $y'' + y = 0$.
2. Show that $x^2 + y^2 = 1$ is an implicit solution of the differential equation $yy' + x = 0$.

Solution:

Separable Differential Equations

Consider differential equation

$$\frac{dy}{dx} = f(x)$$

Then $dy = f(x)dx$ and it can be solved by integration. If $f(x)$ is a continuous function, then integrating both sides gives

$$y = \int f(x)dx = G(x) + c,$$

where $G(x)$ is an antiderivative (indefinite integral) of $f(x)$.

Example

If $y' = x$, then $y(x) = \int_0^x tdt = \frac{1}{2}x^2 + c$.

Many first order ODEs can be reduced or transformed to the form

$$g(y)y' = f(x),$$

where g and f are continuous functions.

Then, from elementary calculus we have:

$$g(y)dy = f(x)dx.$$

Such type of equations are called **separable equations**. Integrating both sides we get:

$$\int g(y)dy = \int f(x)dx + c$$

is the general solution of the given equation.

Example

Solve the following DEs

a) $6yy' + 4x = 0$

b) $y' = y^2 e^{-x}$

Solution:

Remark

It is recommended to write an explicit solution to the differential equation when ever possible.

Equation Reducible to Separable Form

There are some differential equations which are not separable, but they can be transformed to a separable form by simple change of variables.

A. Linear Substitution

Suppose we have a differential equation that can be written in the form:

$$y' = g(ax + by + c)$$

Such an equation is not in general separable. However, if we set $u = ax + by + c$, we get

$$\frac{du}{dx} = a + b\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\frac{du}{dx} - \frac{a}{b}.$$

Thus $y' = g(ax + by + c)$ will be transformed into

$$\frac{1}{b}\frac{du}{dx} - \frac{a}{b} = g(u),$$

where u and x can be separated.

Example

Solve the following DEs.

a) $y' = (x+y)^2$.

b) $(2x-4y+5)y' + x - 2y + 3 = 0$

Solution:

B. Quotient Substitution

Suppose we have an equation that can be written in the form

$$y' = g(\frac{y}{x}).$$

Let us substitute

$$u = \frac{y}{x}. \quad \text{Then} \quad \frac{du}{dx} = \frac{xy' - y}{x^2} = \frac{1}{x}y' - \frac{y}{x^2}.$$

This implies,

$$y' = xu' + \frac{y}{x} = xu' + u.$$

Thus, the differential equation

$$y' = g(\frac{y}{x})$$

is reduced to the equation $xu' = g(u) - u$ which is equivalent to the differential equation

$$\frac{dx}{x} = \frac{du}{g(u) - u}$$

Then by integrating we obtain a general solution.

Example

Solve

a) $x^2y' = x^2 + xy + y^2$
b) $2xyy' = y^2 - x^2$

Solution:

Exact Differential Equations

Recall that the total differential of a function $F(x, y)$ of two variables is

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

for all x, y in the domain of F.

Definition

The expression

$$M(x,y)dx + N(x,y)dy = 0$$

is called an exact differential equation in some domain D (an open connected set of points) if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

for all $x, y \in D$.

If we can find a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y)$$

then the differential equation $M(x, y)dx + N(x, y)dy = 0$ is just $M(x, y)dx + N(x, y)dy = dF = 0$. But recall that, if $dF = 0$, then $F(x, y) =$ constant. The equation $F(x, y) = c$, where c is an arbitrary constant, implicitly defines the general solution of the deferential equation $M(x, y)dx + N(x, y)dy = 0$.

Theorem (Test for Exactness)

Let $M(x, y), N(x, y), \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be all continuous functions within a rectangle R (or some domain) in the $xy-$plane. Then $M(x, y)dx + N(x, y)dy = 0$ is an exact differential in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

every where in R.

Example

Test the exactness of the following DEs.

a) $\dfrac{dy}{dx} = \dfrac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}$

b) $(y \ln y - e^{-xy})dx + (\frac{1}{y} + x \ln y)dy = 0$

Solution:

Suppose a differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact. Then, there exists a function $F(x, y)$ such that

$$M = \frac{\partial F}{\partial x} \quad \text{and} \quad N = \frac{\partial F}{\partial y}$$

From $M = \dfrac{\partial F}{\partial x}$, we have (by integrating with respect to x)

$$F(x, y) = \int M dx + A(y),$$

where $A(y)$ is only a function of y but constant with respect to x.

Now to determine $A(y)$ (the constant of integration), differentiate equation the above equation with respect to y to get

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int M dx + A'(y)$$

which implies

$$N(x,y) = \int \frac{\partial M}{\partial y} dx + A'(y) \text{ and hence } A'(y) = N(x,y) - \int \frac{\partial M}{\partial y} dx$$

by exactness. Therefore

$$A(y) = \int \left[N(x,y) - \int \frac{\partial M}{\partial y} dx \right] dy.$$

Example

Solve the following DEs.

a) $\sin y dx + (x \cos y - 2y) dy = 0$

b) $(x^3 + 3xy^2)dx + (3x^2 y + y^3)dy = 0.$

Integrating Factors

Definition

If the differential equation $M(x,y)dx + N(x,y)dy = 0$ is not exact but the differential equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, then the multiplicative function $\mu(x,y)$ is called an **integrating factor** of the DE.

Suppose we have a differential equation which is not exact but it can be made exact by an integrating factor. Then we can ask the following fundamental questions.

1. How can we find the integrating factor μ?
2. Given μ, how can we solve the problem?

The method is described below. Clearly $\mu(x, y)$ is any (non-zero) solution of the equation

$$\frac{\partial}{\partial y}(\mu N) = \frac{\partial}{\partial x}(\mu N) \tag{2}$$

which is equivalent to the equation

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

This is a first-order partial differential equation in μ. However the integrating factor μ can be found to be a function of x alone $\mu(x)$ (or a function of y alone $\mu(y)$). Then in this case equation (2) will be reduced to

$$\mu M_y = \frac{d\mu}{dx} N + \mu N_x \quad \text{equivalently} \quad \frac{d\mu}{dx} = \mu \left(\frac{M_y - N_x}{N} \right)$$

which is a separable differential equation.

This idea works correctly if the ratio $\dfrac{M_y - N_x}{N}$ is a function of x only, that is,

$$p(x) = \frac{M_y - N_x}{N} = \text{is a function of } x.$$

In this case

$$\frac{d\mu}{\mu} = \left(\frac{M_y - N_x}{N}\right) dx,$$

which implies

$$\mu(x) = e^{\int p(x)dx}.$$

If the quotient

$$\frac{M_y - N_x}{N}$$

is not a function of x alone, then the integrating factor μ can not be obtained using the above procedure, but we can try to find μ as a function of y alone, $\mu(y)$. Then when $\mu(y)$ is only a function of y, equation (2) will be reduced to

$$\frac{d\mu}{dy} M + \mu M_y = \mu N_x$$

which implies

$$\frac{d\mu}{y} = -\mu \left(\frac{M_y - N_x}{M}\right),$$

which is a separable differential equation.

If the fraction $\dfrac{M_y - N_x}{M}$ is a function of y alone, then

$$\mu(y) = e^{-\int q(y)dy}.$$

Example

Solve

$$dx + (3x - e^{-2y})dy = 0.$$

Solution:

Linear First Order Differential Equations

Consider the general first-order linear differential equation

$$a_1(x)y' + a_0(x)y = f(x), \quad a_1(x) \neq 0 \tag{3}$$

By dividing both sides by $a_1(x) \neq 0$, we get $y' + p(x)y = q(x)$, where

$$p(x) = \frac{a_0(x)}{a_1(x)} \text{ and } q(x) = \frac{f(x)}{a_1(x)}.$$

Here we assume that $p(x)$ and $q(x)$ are continuous.
There is a general approach to solve linear equations. To solve for $y(x)$ from the given equation we start with the simplest case, when $q(x) = 0$. That is, equation (3) becomes

$$y' + p(x)y = 0.$$

This problem is called a homogeneous version of (3). Now to solve the above equation first we get $y' = -p(x)y$ and we divide both sides by y and get

$$\frac{y'}{y} = -p(x).$$

Then by integerating we get

$$\ln|y| = -\int p(x)dx + C.$$

Therefore,

$$y(x) = Ae^{-\int p(x)dx} \quad \text{where } A \text{ is an arbitrary constant}$$

is a general solution of $y' + p(x)y = 0$.

Example

Solve the following differential equations.

a) $y' + 2xy = 0$

b) $(x+2)y' - xy = 0$

Solution:

Now we want to solve the general first order linear ordinary differential equation

$$y' + p(x)y = q(x) \tag{4}$$

This can be done in two steps.

Step 1. Consider the homogeneous version of (4) and find the solution to be $y_h = Ae^{-\int p(x)dx}$, where h indicate the general solution for the homogeneous part of the equation.

Step 2. To get the solution for the non-homogeneous part of the equation we vary the constant A with different value of x. Hence we assume that

$$y(x) = A(x)e^{-\int p(x)dx} \tag{5}$$

is a solution for (4). Then (5) must satisfy (4). i.e.

$$(A(x)e^{-\int p(x)dx})' + p(x)(A(x)e^{-\int p(x)dx}) = q(x),$$

which implies

$$A'(x)e^{-\int p(x)dx} + A(x)(-p(x))e^{-\int p(x)dx} + p(x)A(x)e^{-\int p(x)dx} = q(x).$$

Simplifying this gives us,

$$A'(x) = q(x)e^{\int p(x)dx}.$$

Now integrate both sides to get

$$A(x) = \int q(x)e^{\int p(x)dx}dx + C$$

Hence the general solution of the non-homogeneous ODE (1.16) is given by:

$$\begin{aligned} y(x) &= A(x)e^{-\int p(x)dx} = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right) \\ &= Ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx \\ &= y_h(x) + y_p(x) \end{aligned}$$

Example

Solve the differential equation $y' + 3y = 6$.
Solution:

Bernoulli Equation

Definition

A differential equation of the form

$$y' + p(x)y = q(x)y'\alpha$$

where α is a constant, is called **Bernoulli Equation**.

If $\alpha = 0$, then the equation is linear and if $\alpha = 1$, then the equation is separable. We have seen these two cases in the previous section.
For $\alpha \neq 1$, use the change of variable $u = y^{1-\alpha}$. Then by differentiating with respect to x, we get $u' = (1-\alpha)y^{-\alpha}y'$. But, from $y' + p(x)y = q(x)y^{\alpha}$, we get $y' = q(x)y^{\alpha} - p(x)y$. Then

$$\begin{aligned} u' &= (1-\alpha)y^{-\alpha}y' \\ &= (1-\alpha)y^{-\alpha}(qy^{\alpha} - py) \\ &= (1-\alpha)(q - py^{-1-\alpha}) \\ &= (1-\alpha)(q - pu), \quad \text{since } u = y^{1-\alpha} \end{aligned}$$

This implies that $u' + (1-\alpha)u = (1-\alpha)q$, which is a linear differential equation of first order and hence we can solve it using one of the methods we have seen in the previous sections.

Example

Solve the Bernoulli Equation $y' - 2y = -6y^2$.

Solution: