Applied Mathematics III

Unit 2

Ordinary Differential Equations of the Second Order

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Homogeneous Linear Equation of the Second Order

A second-order differential equation is a differential equation containing a second derivative of a dependent variable with respect to the independent variable. In this unit we will focus on linear second-order equations, which have many important uses.

Definition

A linear ordinary differential equation of order n in the dependent variable y and independent variable x is an equation which can be expressed as:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = f(x), \qquad (1)$$

where $a_n(x) \neq 0$ and the functions $a_0, a_1, \cdots, a_n$ are continuous real-valued functions of $x \in [a, b]$.
The function $f(x)$ is called the **non-homogeneous term** and all the points $x_i \in [a, b]$ in which $a_n(x_i) = 0$ are called **singular points** of the DE (1).
If $f(x) = 0$ in equation (1), then the equation is said to be **homogeneous Linear ODE of order** n.

Example

The equation $y'' + 7y' - y = 2x^3$ is a non homogeneous linear ordinary differential equation of the 2^{nd} order, whereas $y'' + 7y' - y = 0$ is a homogeneous linear ordinary differential equation of the 2^{nd} order.

Theorem (Basic Existence Theorem for IVP.)

Consider the linear ODE given in (1), where $a_0, a_1, \cdots, a_{n-1}, a_n$ *and* f *are continuous functions on the interval* $[a, b]$ *and* $a_n \neq 0, \forall x \in [a, b]$*. Furthermore, let* x_0 *be any point in* $[a, b]$ *and let* $c_0, c_1, \cdots, c_{n-1}$ *be arbitrary real constants. Then there exists a unique solution function* $g(x)$ *of (1) on* $[a, b]$ *satisfying the initial conditions,*

$$g(x_0) = c_0, \quad g'(x_0) = c_1, \cdots, g_{n-1}(x_0) = c_{n-1}.$$

Consider the linear differential equation

$$y'' + y = 0. \tag{2}$$

Then, $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of the differential equation (2). Let c_1 and c_2 be arbitrary constants. Then

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos x + c_2 \sin x$$

is also a solution of (2). Indeed, $y' = -c_1 \sin x + c_2 \cos x$, and $y'' = -c_1 \cos x - c_2 \sin x$ which implies that

$$y'' + y = (-c_1 \sin x + c_2 \cos x) + (-c_1 \cos x - c_2 \sin x) = 0$$

for all x. Therefore, any linear combination of the functions $y_1 = \cos x$ and $y_2 = \sin x$ is a solution for the given differential equation.

Theorem (Linear Combination of Solutions)

If $y_1, y_2, \cdots, y_k$ are solutions of the homogeneous linear ODE (1) and if $c_1, c_2, \cdots, c_k$ are arbitrary constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k = \sum_{i=1}^{k} c_i y_i$$

is also a solution of (1). That is, any linear combination of solutions of a linear homogeneous differential equation is also a solution.

Theorem (Existence of Linearly Independent Solutions for a LHODE)

The Linear Homogenous Differential Equation (LHODE) of order n always has n Linearly Independent (LI) solutions. Furthermore, if $f_1(x), f_2(x), \cdots, f_n(x)$ are n LI solutions, then every solution y of LHODE can be expressed as a linear combination of these solution functions. i.e

$$y = \sum_{i=1}^{n} c_i f_i(x), \qquad \text{for some } \; c_1, c_2, \cdots, c_n \in \mathbb{R}.$$

Definition

Let $f_1(x), f_2(x), \cdots, f_n(x)$ be n real valued functions each of which has an $(n-1)^{th}$ derivative on the interval $[a, b]$. The determinant:

$$W[f_1, f_2, \cdots, f_n] = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} = W(x)$$

is called the **Wronskian** of these n functions.
The n functions $f_1, f_2, \cdots, f_n$ are Linearly Independent on an interval $[a, b]$ if and only if the Wronskian of $f_1, f_2, \cdots, f_n$ is different from zero for some $x \in [a, b]$. Otherwise the n functions are said to be Linearly Dependent.

Example

1. Show that x and x^2 are linearly independent.
2. Show that e^x, e^{-x} and $\sinh x$ are linearly dependent.

Definition

If $f_1(x), f_2(x), \cdots, f_n(x)$ are n linearly independent solutions of LHODE on $[a, b]$, then the set $\{f_1(x), f_2(x), \cdots, f_n(x)\}$ is called **the Fundamental Set of Solutions** of LHODE and the function

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \quad x \in [a, b],$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants is called a **General Solution** of LHODE on $[a, b]$ and each $f_1, f_2, \cdots, f_n$ are called particular solutions.

Example

Consider the second order linear homogenous DE $y'' - 4y' + 4y = 0$.

a) The functions $y_1(x) = e^{2x}$ and $y_2(x) = xe^{4x}$ are particular solutions.

b) e^{2x} and xe^{4x} are linearly independent.

c) Therefore, the general solution of the given equation is

$$y(x) = c_1 e^{2x} + c_2 xe^{4x}.$$

Definition

A Differential Equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{3}$$

where $a_0, a_1, \cdots, a_n$ are all real constants, is called a **Homogenous Linear Differential Equation of constant coefficients**.

Now we will look for the solution of HLODE (3) in the form $y = e^{\lambda x}$ where the constant λ will be chosen so that $y = e^{\lambda x}$ does satisfy the equation (3). Now insert $y = e^{\lambda x}$ into (3) to get;

$$(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0.$$

Hence, if $e^{\lambda x}$ is a solution of the equation in (3), then λ should satisfy:

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0 \tag{4}$$

since $e^{\lambda x} \neq 0$ for all $x \in \mathbb{R}$.

Definition

The algebraic equation (4) is called an **Auxiliary equation** or **characteristic equation** of the given differential equation in (3).

There are 3 different cases of the roots of (4).

Case 1. Distinict Real Roots

Suppose that (4) has n distinct roots, $\lambda_1, \lambda_2, \cdots, \lambda_n$ where $\lambda_i \neq \lambda_j$, for $i \neq j$. Then, the solutions $e^{\lambda_1 x}, e^{\lambda_2 x}, \cdots, e^{\lambda_n x}$ are linearly independent. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the n distinct real roots of (4), then the general solution of (3) is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x} = \sum_{i=1}^{n} c_i e^{\lambda_i x},$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Example

Solve

1. $y'' - 3y' + 2y = 0$
2. $y''' - 4y'' + y' + 6y = 0$

Solution:

Case 2. Repeated Real Roots

Given a differential equation:

1. if the characteristic equation has double real root λ, then $e^{\lambda x}$ and $xe^{\lambda x}$ are two linearly independent solutions and;
2. if the characteristic equation has triple root λ, then the corresponding linearly independent solutions are $e^{\lambda x}$, $xe^{\lambda x}$ and $x^2 e^{\lambda x}$.

If the given DE is $ay'' + by' + cy = 0$, then its characteristic equation is $a\lambda^2 + b\lambda + c = 0$ and then $\lambda = \lambda_1 = \lambda_2 = -\frac{b}{2a}$. One of the solution of the given DE is $y_1 = e^{\lambda x}$. Then we can use the method of reduction of order to find a second solution y_2 so that y_1 and y_2 are linearly independent. The given equation is equivalent to

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

and $y_2 = uy_1$, where

$$u = \int \left(\frac{e^{-\int \frac{b}{a}dx}}{(e^{\lambda x})^2} \right) dx = \int \frac{e^{-\frac{b}{a}x}}{e^{2\lambda x}} = \int 1 dx = x$$

since $2\lambda = -\frac{b}{a}$ and hence $y_2 = xe^{\lambda x}$.

Example

Solve

1. $y'' - 4y' + 4y = 0$
2. $y''' - 4y'' - 3y' + 18y = 0$

Case 3. Conjugate Complex Roots

Suppose the equation (4) has a complex root $\lambda = a + ib, \quad a, b \in \mathbb{R}$. Then (we know from the theory of algebraic equations that) the conjugate $\bar{\lambda} = a - ib$ is also a root of (4) and the corresponding part of the general solution of (3) will be: $k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x}$.
By applying Euler's formula

$$k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x} = e^{ax}(c_1 \cos bx + c_2 \sin bx),$$

where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$ are arbitrary constants from the set of complex numbers $\mathbb{C}$.
On the other hand if $a + ib = \lambda$ and $a - ib = \bar{\lambda}$ are each k fold roots of (4), then the part of the general solution that corresponds to this part is

$$e^{ax}[(c_1 + c_2 x + \cdots + c_k x^{k-1}) \cos bx + i(c_{k+1} + c_{k+2} x + \cdots c_{2k} x^{k-1}) \sin bx].$$

Example

Solve $y'' - 2y' + 10y = 0$.

Solution:

Activity

Solve each of the following differential equation.

1. $y'' + y = 0$
2. $y'' - 6y' + 25y = 0$
3. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = 0$, where $\lambda_1 = \lambda_2 = 1 + 2i$ and $\lambda_3 = \lambda_4 = 1 - 2i$.

Nonhomogeneous Equations with Constant Coefficients

Consider the nonhomogeneous differential equation

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=f(x), \text{ where } f(x)\neq 0.$$

If $f(x)=0$, then the equation becomes a homogeneous equation.

1. If y_1 and y_2 are solutions of the nonhomogeneous equation on an interval I, then y_1-y_2 is also a solution of the homogeneous equation in the interval I.
2. If y_1 is a solution of the nonhomogeneous equation and y_2 is a solution of the homogeneous equation in an interval I, then y_1+y_2 is a solution of the nonhomogeneous equation in the interval I.

The following remark follows directly from the above.

Remark

Suppose $y_h(x)$ denote the general solution of the homogeneous part and $y_p(x)$ denote the particular solution of the DE: Then the general solution of the nonhomogenous DE is given by $y(x)=y_h(x)+y_p(x)$.

The Undetermined Coefficient Method

Definition

1. A function is called an **undetermined coefficient function (UC function)** if it is either:
   a) a function defined by (a linear combination) of the following
      i) $x^n, \quad n = 0, 1, 2, \cdots,$
      ii) e^{ax}, where a is any non-zero constant
      iii) $\sin(bx + c)$, where b, c are constants, such that $b \neq 0$.
      iv) $\cos(bx + c)$, where b, c are constants, such that $b \neq 0$.
      OR
   b) a function which is defined as a finite product of two or more functions of the above 4 types.
2. Let f be an UC function. The set S of functions consisting of f and all the derivatives of f which are mutually LI UC functions is said to be the UC set of function f, if S is a finite set and we shall denote it by S.

Example

1. Let $f(x) = x^3$. Then f is UC function.
   $f'(x) = 3x^2$, x^2 is UC function. $f''(x) = 6x$, x is UC function.
   $f'''(x) = 6$, 1 is UC function.
   Therefore, $S = \{1, x, x^2, x^3\}$.
2. Let $f(x) = \sin(2x)$. Then f is an UC function.
   $f'(x) = 2\cos(2x)$, $\cos(2x)$ is UC function.
   $f''(x) = -4\sin(2x)$, $\sin(2x) = f(x)$.
   Therefore, $S = \{\sin(2x), \cos(2x)\}$.
3. Let $g(x) = 2xe^{-x}$. g is an UC function (as a product of UC function).

$$g'(x) = 2e^{-x} - 2xe^{-x}, \quad e^{-x}, xe^{-x} \text{ are UC functions.}$$
$$g''(x) = -4e^{-x} + 2xe^{-x}, \quad e^{-x}, xe^{-x} \text{ are UC functions.}$$

   Therefore, $S = \{e^{-x}, xe^{-x}\}$.
4. The function $f(x) = \frac{1}{x}$ is not a UC function.

We outline the method by using the following example.

Example

Solve $y'' - 2y' - 3y = 2e^{-x} - 10\sin x$.

Solution: Let $F(x) = 2e^{-x} - \sin x$, $f_1 = 2e^{-x}$, $f_2 = -10\sin x$. Then $S_1 = \{e^{-x}\}$ and $S_2 = \{\sin x, \cos x\}$

- Solution of the homogeneous part $y'' - 2y - 3y = 0$ is $y_h(x) = C_1e^{3x} + C_2e^{-x}$.
- Particular solution corresponding to $f_1(x) = 2e^{-x} : y_{p_1}(x) = Be^{-x}$ which duplicates with C_2e^{-x}. This implies, $y_{p_1}(x) = Bxe^{-x}$, no more duplicate.
- Insert this into $y'' - 2y' - 3y = 2e^{-x}$ to get $y''_{p_1} - 2y'_{p_1} - 3y_{p_1} = 2e^{-x}$. This implies, $(-2Be^{-x} + Bxe^{-x}) - 2(Be^{-x} - Bxe^{-x}) - 3Bxe^{-x} = 2e^{-x}$. Hence $-4Be^{-x} = 2e^{-x}$ so $B = -\frac{1}{2}$. Therefore, $y_{p_1}(x) = -\frac{1}{2}xe^{-x}$.
- Similarly particular solution corresponding to $f_2(x) = -10\sin x$ is $y_{p_2} = D\sin(2x) + E\cos(2x) = \frac{20}{3}\sin(2x) + \frac{10}{3}\cos(2x)$.
  Therefore the general solution is
  $$y(x) = C_1e^{3x} + C_2e^{-x} - \frac{1}{2}xe^{-x} + \frac{20}{3}\sin(2x) + \frac{10}{3}\cos(2x).$$

Example

Solve $y'' - y' = 3x^2 - \sin(2x)$

Solution:

Activity

Solve

1. $y'' - 2y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$
2. $y'' - 9y = 4 + 5\sinh 3x$

Variation of Parameters

The Undetermined Coefficient method is easier to apply, but works only for constant coefficients and certain types of non-homogeneous terms (or forcing functions). Hence, we need another method which works for more general set of problems. In this subsection we will consider the method of Variation of Parameters for a second order linear ordinary differential equation. Consider the following second order linear differential equation.

$$y'' + a_1(x)y' + a_2(x)y = f(x) \tag{5}$$

where a_1, a_2 and f are continuous functions. Suppose that the general solution for the homogeneous part of (5) is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Now we want to get a particular solution corresponding to $f(x)$ and this can be done by varying the constants, c_1 and c_2 with respect to x. If y_p is a particular solution corresponding to $f(x)$, then

$$y_h(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

We differentiate and substitute it in (5) to get

$$y_p'' + a_1(x)y_p' + a_2(x)y_p = f(x).$$

But

$$y_p' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2$$

. Since we are going to have only one equation with two variable functions c_1 and c_2, we are free to choose a condition which simplifies the equation. Therefore, we take the condition

$$c_1'y_1 + c_2'y_2 = 0.$$

This will simplify the equation as $y_p'' = c_1y_1'' + c_1'y_1' + c_2'y_2' + c_2y_2''$ and after simplification, the equation (5) becomes

$$c_1(y_1'' + a_1y_1' + b_2y_1) + c_2(y_2'' + b_1y_2' + b_2y_2) + c_1'y_1' + c_2'y_2' = f.$$

Since y_1 and y_2 are linearly independent solutions for the homogeneous part of equation (5) we have the following system of equations:

$$\begin{cases} c_1'y_1 + c_2'y_2 = 0 \\ c_1'y_1' + c_2'y_2' = f. \end{cases} \tag{6}$$

which is a system of two algebraic equations in c_1' and c_2'. Then (6) has a unique solution if the determinant of the coefficient matrix is non-zero, that is,

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

However, the above determinant is the Wronskian of the functions y_1 and y_2. Since y_1 and y_2 are LI functions, then

$$W_{[y_1,y_2]}(x) \neq 0$$

Hence by Cramer's rule we have:

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ f & y_2'(x) \end{vmatrix}}{W(x)} = \frac{W_1(x)}{W(x)} \quad \text{and} \quad c_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f \end{vmatrix}}{W(x)} = \frac{W_2(x)}{W(x)}.$$

By integrating both sides we will get:

$$y_p(x) = \left[\int \frac{W_1(x)}{W(x)} dx\right] y_1(x) + \left[\int \frac{W_2(x)}{W(x)} dx\right] y_2(x).$$

Example

Solve the following differential equations

a) $y'' - 4y = 8x$

b) $y'' + y = \sec x$

.

Solution:

Remark

This method looks easier when the integrads (or the quotients of the Wronskian) are simple. However, it could be very difficult to get the particular solution when the integrand is complicated.

The Cauchy-Euler Equation

Definition

The linear differential equation with variable coefficient of the form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = F(x) \tag{7}$$

where $a_0, a_1, \cdots, a_n$ are constants is called the **Cauchy-Euler Equation**.

Example

The linear differential equation $3x^2 y'' - 11xy' + 2y = \sin x$ is a Cauchy-Euler equation.

To solve Cauchy-Euler DEs first we reduce the given DE into a linear differential equation with constant coefficients and solve the given equation with the methods derived in the previous sections.

Theorem

The transformation $x = e^t, t \in \mathbb{R}$ reduces the Cauchy-Euler DE to a linear DE with constant coefficients.

Let us consider the case when $n = 2$. In this case the equation is:

$$a_2x^2y'' + a_1xy' + a_0y = F(x) \tag{8}$$

Let $x = e^t$. Then by solving for t we get $t = \ln x$ for $x > 0$ (or $x = e^{-t}$ if $x < 0$) and

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}.\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\frac{d^2y}{dt^2}.\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting into equation (8) we get:

$$a_2x^2\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1x\frac{1}{x}\frac{dy}{dt} + a_0y = F(e^t)$$

This implies,

$$a_2\frac{d^2y}{dt^2}+(a_1-a_2)\frac{dy}{dt}+a_0y=F(e^t), \text{ then } A_2\frac{d^2y}{dt^2}+A_1\frac{dy}{dt}+A_0y=G(t)$$

where $A_2=a_2, A_1=a_1-a_2, A_0=a_0$ and $F(e^t)=G(t)$, which is a second order linear differential equation with constant coefficients.

Example

Solve each of the following differential equations.

1. $x^2y''-2xy'+2y=0$
2. $3x^2y''-11xy'+2y=\sin x$
3. $x^2y''-2xy'+2y=x^3$

Solution: 1. Let $x=e^t$. Since $a_2=1, a_1=-2$ and $a_0=2$ we have $A_2=a_2=1, A_1=a_1-a_2=-3$ and $A_0=a_0=2$ which reduces the given equation to $y''-3y'+2y=0$ which is a homogenous second order linear differential equation with constant coefficients. Then the general solution $y(t)=c_1e^t+c_2e^{2t}$ and since $x=e^t$ the DE $x^2y''-2xy'+2y=0$ has a general solution $y(t)=c_1x+c_2x^2$, where c_1 and c_2 are arbitrary constants.