Applied Mathematics III
Unit 6
Complex Integral Calculus

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Complex Integration:

Integral of a Complex Valued Function of Real Variable

Definition

Let $f(t) = u(t) + iv(t)$ be a continuous complex function, then u and v are also continuous. Define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

If $U' = u, \quad V' = v$ and $F(t) = U(t) + iV(t)$, then by fundamental theorem of the complex integral calculus

$$\int_a^b f(t)dt = F(b) - F(a).$$

Example

1. $\displaystyle\int_0^1 (1+ti)^2 dt = \frac{2}{3} + i.$
2. $\displaystyle\int_{2i}^3 \sin z dz = -cos3 + \cos 2i = \cosh 2 - \cos 3$

Contour Integral

Definition

A curve in complex analysis is a continuous function $\sigma(t) = x(t) + iy(t)$ with x and y are real valued functions and $t \in [a, b]$.

- A curve σ is called a **smooth curve** if σ is differentiable and σ' is continuous and nonzero for all t.
- A **contour/piecewise smooth** curve is a curve that is obtained by joining finitely many smooth curves end to end.
- A curve σ is **simple** if it does not intersect itself except possibly at end points. That means $\sigma(t_1) \neq \sigma(t_2)$ when $a < t - 1 < t_2 < b$.
- A curve σ is said to be a **closed curve** if $\sigma(a) = \sigma(b)$.
- A curve σ is simple and closed the we say that σ is a **simple closed curve** or **Jordan curve**.
- Let σ be a simple closed contour with parametrization $\sigma(t),\ t \in [a, b]$. As t moves from a to b, the curve σ moves in a specific direction called the orientation of the curve induced by the parametrization. In this case we say the orientation is in the **positive sense** (counter clockwise or anticlockwise sense). Otherwise σ is oriented **negatively** (clockwise direction).

Definition

Let σ be a piecewise smooth curve defined on $[a, b]$. The length of σ is given by

$$L(\sigma) = \int_a^b |\sigma'(t)|dt.$$

Definition

Let C be a contour parametrically represented by $\sigma(t)$; $t \in [a, b]$ and f be complex valued continuous function defined on C then the line integral or the contour integral of f along the curve C is defined by

$$\int_C f(z)dz = \int_a^b f(\sigma(t))\sigma'(t)dt \quad \text{where} \quad \sigma'(t) = \frac{d\sigma}{dt}$$

Example

1. Evaluate $\displaystyle\oint_C f(z)dz$ where C is a unit circle around the origin.
2. Evaluate $\oint_C \overline{z}dz$ where $C : \sigma(t) = e^{it}, t \in [0, \pi]$

Solution:

Example

1. Evaluate $I = \int_C z^2 dz$ where C is the parabolic arc given by $x = 4 - y^2$ and $-2 \leq y \leq 2$.
   Solution:

2. Evaluate $\oint_C (z-a)^n dz$, where a is any given complex number, n is any integer and C is a circle centered at a and with radius r.
   Solution:

Definition

Let C be a piecewise smooth curve such that $C = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ and $f(z)$ be a continuous complex function on C. Then we define

$$\int_C f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz.$$

Example

Let C be a curve consisting of portion of a parabola $y = x^2$ in the $xy-$plane from $(0,0)$ to $(2,4)$ and a horizontal line from $(2,4)$ to $(4,4)$. If $f(z) = Im(z)$, then evaluate $\int_C f(z)dz$.

Remark

1. Let f, g be piecewise continuous complex valued functions then

$$\int_C [kf + g](z)dz = k\int_C f(z)dz + \int_C g(z)dz \quad \text{where } k \text{ is aconstant.}$$

2. If C' has an opposite orientation to that of C, then $\int_C f(z)dz = -\int_{C'} f(z)dz$.

Cauchy's Integral Theorem.

Definition

1. A domain D is called **simply connected** if every simple closed contour (within it) encloses points of D only.
2. A domain D is called **multiply connected** if it is not simply connected. For example $\mathbb{C}' = \mathbb{C}/\{0\}$ and the annulus $A(a, b) = \{z \in \mathbb{C} : a < |z| < b\}$.

Theorem (Cauchy's Theorem)

If a function f is analytic on a simply connected domain D and C is a simple closed contour lying in D then

$$\oint_C f(z)dz = 0.$$

Proof Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and $C : \sigma(t) = x(t) + iy(t)$; $a \leq t \leq b$ is the curve C. Then

$$\oint_C f(z)dz = \int_a^b f(\sigma(t))\sigma'(t)dt$$

$$
\begin{aligned}
&= \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))][x'(t) + iy'(t)]dt \\
&= \int_a^b (ux' + vy')dt + i\int_a^b (vx' + uy')dt \\
&= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \\
&= \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy, \quad \text{(by Greens Theorem)} \\
&= 0 \quad \text{(by CR equations} \quad u_x = v_y \;\; and \;\; u_y = v_x).
\end{aligned}
$$

Example

Let C be a unit circle given by $\sigma(t) = e^{it}, -\pi \leq t \leq \pi$.

1. It follows from Cauchy's theorem that $\int_C f(z)dz = 0$, if $f(z) = e^{z^n}, f(z) = \cos z$ or $f(z) = \sin z$.
2. $\int_C f(z)dz = 0$ if $f(z) = \frac{1}{z^2}$ or $f(z) = \csc^2 z$ from the fundamental theorem as $\frac{d}{dz}\left(-\frac{1}{z}\right) = \frac{1}{z^2}$ and $\frac{d}{dz}(-\cot z) = \csc^2 z$. Note that here Cauchy's theorem cannot be applied as the integrands are not analytic at zero.

Example

Let C be a unit circle given by $\sigma(t) = e^{it}, -\pi \leq t \leq \pi$.

1. $\displaystyle\int_C \frac{e^{(iz)^2}}{z^2+4} dz = 0$ by Cauchy's theorem. Note that the integrand is not analytic at $z = 2$ but that does not bother us as these points are not enclosed by C.
2. If $f(z) = (Imz)^2$, then $\displaystyle\int_C f(z)dz = 0$ (check this). As f is not analytic anywhere in C Cauchy's theorem can not be applied to prove this.

Theorem (The Deformation Theorem)

Let C_1 and C_2 be closed paths in the complex plane with C_2 is in the interior of C_1. Suppose that a complex function f is analytic in an open set containing both paths and all points between them. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Remark

If f is analytic in a simply connected domain D, then the integral $\int_C f(z)dz$ is independent of path in D. That is, if C_1 and C_2 are open curves with the same initial and terminal points, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Hence we can deform C_1 into C_2 without changing the value of the integral. However, if F is not analytic in D, then Cauchy's Theorem does not hold true in general.

Consider the integral $\displaystyle\int_C \frac{dz}{z-a}$ where C is any piecewise smooth simple closed curve, oriented counterclockwise and containing a inside. Since $f(z) = \dfrac{1}{z-a}$ is analytic in the region bounded by C except in some neighborhood of $z = a$, we can conclude that f is analytic in every domain not containing a inside. Thus, because of path deformation, we can assume without loss of generality that C_1 is a circular path with radius r and centered at a. Then

$$\oint_C \frac{dz}{z-a} = \oint_{C_1} \frac{dz}{z-a}.$$

Set $z - a = re^{i\theta}$. Then $dz = rie^{i\theta}d\theta$ and hence

$$\oint_{C_1} \frac{rie^{i\theta}}{re^{i\theta}}d\theta = i\oint_{C_1} d\theta = i\int_0^{2\pi} d\theta = 2\pi i \neq 0.$$

Theorem

Let $C, C_1, C_2, \cdots, C_n$ be simple closed positively oriented contours such that C_k lies interior to C for $k = 1, 2, \cdots, n$ and C_k has no point in common with the interior of C_j if $k \neq j$. Let f be analytic on a domain D that contains all the contour and the region between C and $C_1 + C_2 + \cdots + C_n$. Then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz.$$

Cauchy's Integral Formula

Definition

A Complex function g is said to be singular at a point, say $z = z_0$, if it is not analytic at that point.

Theorem (Cauchy Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then

$$\oint_C \frac{f(z)}{z-a} dz = 2i\pi f(a)$$

for all a in D. This implies

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

Example

Evaluate $\oint_C \left(\frac{z^2+1}{z^2-1} \right)$,where C is a unit circle centered at $z = 1$.

Example

1. Evaluate $\oint_C \left(\frac{z^3 - 6}{2z - i}\right)$, where C is any closed simple piecewise smooth curve containing $a = \frac{i}{2}$ in its interior.
   Solution:

2. Show that
   a) $\oint_C \frac{\cos z}{z} dz = 2\pi i$, where C is the cirle $|z - 4| = 5$.
   b) $\oint_C \frac{z^2}{z^2 + 1} dz = -\pi$, where C is the cirle $|z - i| = 1$.
   c) $\oint_C \frac{e^z}{z(z - 1)} dz = 2\pi i(e - 1)$, where C is a cirle centered at $z = 0$ and radius 2 units.

Theorem (Cauchy Integral Formula for Higher Derivatives)

Let $f(z)$ be analytic in a simply connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then for all a in D

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

for any nonnegative integer n.

Example

By using Cauchy Integral Formula for Higher Derivatives evaluate $\displaystyle\oint_C \frac{\sin z}{(z-\pi i)^2} dz$, where C is any simple closed path containing πi in its interior and oriented in counterclockwise direction.

Solution:

Example

Show that

1. $\oint_C e^z z^{-3} dz = i\pi$, where C is the circle $|z| = 1$.
2. $\oint_C \frac{1}{(z-4)(z+1)^4} dz = \frac{-2i\pi}{81}$, where C is the circle $|z-1| = \frac{5}{2}$.

Solution:

Summery

Let C be a simple closed curve contained in a simply connected domain D and f is an analytic function defined on D. Then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \begin{cases} 2i\pi f(a), & \text{if } n = 0 \text{ and } a \text{ is enclosed by } C. \\ \frac{2i\pi}{n!} f(a), & \text{if } n \geq 1 \text{ and } a \text{ is enclosed by } C. \\ 0, & a \text{ lies outside of the region enclosed by } C. \end{cases}$$

Cauchy's Theorem for Multiply Connected Domains

Theorem

Let C be a closed path and $C_1, C_2, \cdots, C_n$ be closed paths enclosed by C. Assume that any two of $C, C_1, C_2, \cdots, C_n$ intersect and no interior point to any C_i is interior to any other C_k. Let f be analytic on an open set containing C and each C_i and all the points that are both interior to C an exterior to each C_i. Then

$$\oint_C f(z)dz = \sum_{i=1}^{n} \oint_{C_i} f(z)dz.$$

Example

1. Evaluate $\displaystyle\oint_C \frac{dz}{z(z-1)}$, where C is the circle $|z| = 3$ counterclockwise.
2. Evaluate $\displaystyle\oint_C \frac{z+1}{z(z-2)(z-4)^3}dz$, where C is the circle $|z-3| = 2$ counterclockwise.

Solution: