Applied Mathematics III
Unit 3
Vector Differential Calculus

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In Applied Mathematics I, specifically in the linear algebra part, we have been discussing about constant vectors, but the most important applications of vectors involve also vector functions.

The simplest example is a position vector that depends on time. We can differentiate such a function with respect to time and the first derivative of such function is the velocity and its second derivative is the acceleration of the particle whose position is given by the position vector.

In this case, the coordinates of the tip of the position vector are functions of time. Therefore, it is worth to talk about such functions and in this course, specially in this unit we are going to address the calculus of vector fields (vector valued functions).

Vector Functions of One Variable in Space

Definition

A vector-valued function, or vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.

In this course, we are most interested in vector functions whose values are three-dimensional vectors. This means that for every number t in the domain of there is a unique vector in $\mathbb{R}^3$ denoted by $r(t)$. If $f(t), g(t)$ and $h(t)$ are the components of the vector $r(t)$, then f, g and h are real-valued functions called the component functions of r and we can write

$$r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$$

Example

The function $r(t) = t^3 i + e^{-t} j + \sin tk$ is a vector valued function and the component functions of r are t^3, e^{-t} and $\sin t$.

Remark

The domain of a vector valued function r consists of all values of t for which the expression $r(t)$ is defined, that is the values of t for which all the component functions are defined.

For example, if $r(t) = \sqrt{t}i + \ln(t-2)j + 3tk$, then the domain of $r(t)$ is the set of points in $\mathbb{R}$, where $\sqrt{t}, \ln(t-2)$ and $3t$ are defined. That is, $t \geq 0$ and $t - 2 > 0$ and hence the domain of r is $(2, \infty)$.

For each t, where r is defined, draw $r(t)$ as as a vector from the origin to the point $(f(t), g(t), h(t))$. The end points of these vectors traces out a curve C as t varies.

Example

The function $r(t) = (1+t)i + tj + (3-t)k$ is a vector valued function of one variable. The curve that is traced out by the heads of the position vectors of this vector valued function is a line that passes through the point $(1, 0, 3)$ and with directional vector $(1, 1, -1)$.

Limit of a Vector Valued Function

Definition

A vector valued function $v(t)$ is said to have the limit l as t approaches t_0, if $v(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t \to t_0} \|v(t) - l\| = 0.$$

Then we write

$$\lim_{t \to t_0} v(t) = l.$$

A vector function $v(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} v(t) = v(t_0).$$

Theorem

If $r(t) = (f(t), g(t), h(t))$, *then*

$$\lim_{t \to t_0} r(t) = l$$

if and only if

$$(\lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t)) = l.$$

Example

Find $\lim_{t \to 0} r(t)$, if $r(t) = t^2 i + e^t j + \sin tk$.

Solution:

Remark

If $r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$ a vector valued function all t in the domain of r, then r is continuous at t_0 if and only if its (three) component functions f, g and h are continuous at t_0.

Derivative of a Vector Function

Definition

A vector function $V(t)$ is said to be differentiable at a point t in the domain of V if the limit

$$\lim_{h \to 0} \frac{V(t+h) - V(t)}{h}$$

exists and if the limit exists then it is denoted by $V'(t)$. That is,

$$V'(t) = \lim_{h \to 0} \frac{V(t+h) - V(t)}{h}.$$

Remark

If the function $V(t) = (V_1(t), V_2(t), V_3(t))$ is a vector field, then $V'(t) = (V_1'(t), V_2'(t), V_3'(t))$.

Example

Consider the following functions.

1. If $V(t) = (\sin t, \cos t)$, then $V'(t) = (\cos t, -\sin t)$.
2. If $V(t) = (t^3, 3\cos t, 23)$, then $V'(t) = (3t^2, -3\sin t, 0)$.

Differentiation Rules

Let $U(t)$ and $V(t)$ be a vector valued functions in space and c be any constant. Then

1. $(cV)' = cV'$
2. $(U + V)' = U' + V'$
3. $(U.V) = U'.V + U.V'$
4. $(U \times V)' = U' \times V + U \times V'$

Let $V(t)$ be a vector function of constant norm. i.e. $\|V(t)\| = c$ for a constant c or $V.V = c^2$. Then $(V.V)' = (c^2)' = 0$ which implies $2V'.V = 0$. Then, either $V' = 0$ or $V' \perp V$. Therefore, a nonzero vector field with constant norm is perpendicular to its derivative.

Vector and Scalar Fields

Definition

A function f whose value is a scalar (or a real number), say $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, is called a **scalar field**.
A function v whose value is a vector, say $v : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$, is called a **vector field**. That is a vector field is a vector valued function.

Example

1. If (x_0, y_0, z_0) is a point in $\mathbb{R}^3$, then the function $d : \mathbb{R}^3 \to \mathbb{R}$ given by $d(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ is a scalar field. (d is called the Euclidean Distance.)
2. The function $f : X \to \mathbb{R}^3$ given by $f(x, y) = (x^2 + y, \ln(x^2 + y^2), \sin(x + 3y))$, where $X = \mathbb{R}^2 \backslash \{(0, 0)\}$ is a vector field.

Definition

Let $v : \mathbb{R}^n \to \mathbb{R}^3,\ v = (v_1, v_2, v_3)$ where each v_i is a function of n variables, $t_1, t_2, \cdot, t_n$. Then the partial derivative of v with respect to t_i is denoted by $\dfrac{\partial v}{\partial t_i}$ and is defined as the vector function

$$\frac{\partial v}{\partial t_i} = \left(\frac{\partial v_1}{\partial t_i}, \frac{\partial v_2}{\partial t_i}, \frac{\partial v_3}{\partial t_i} \right)$$

Example

If $f(x, y) = ((x^2 + y^2), \ln(x + y), \sin(x + y))$, then

$$\frac{\partial f}{\partial x} = \left(2x, \frac{1}{x + y}, \cos(x + y) \right)$$
$$\frac{\partial f}{\partial y} = \left((2y, \frac{1}{x + y}, \cos(x + y)) \right)$$

The Gradient Field

Definition

Let $F(x, y, z)$ be a real valued functions of three variables (i.e. F is a scalar field defined from $X \subset \mathbb{R}^3$ into $\mathbb{R}$.) The gradient of F , denoted by ∇F, is a vector field defined by

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$

and if P is a point in the domain of F, the gradient of F evaluated at P is denoted by $\nabla F(P)$ and also if f is a function of two variables, then the the gradient of f, denoted by ∇f, is defined by $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$.

Example

If $F(x, y, z) = 2x + xy - yz^2$, then find ∇F and $\nabla F(1, 2, 3)$.

Solution:

Definition

Let $P(x_0, y_0, z_0)$ be a point and $u = ai + bj + ck$ be a unit vector, i.e. $a^2 + b^2 + c^2 = 1$. Then the directional derivative of a scalar field F at the point P in the direction of u, denoted by $D_uF(p)$, is defined by

$$\begin{aligned} D_uF(P) &= a\frac{\partial F}{\partial x}(x_0, y_0, z_0) + b\frac{\partial F}{\partial y}(x_0, y_0, z_0) + c\frac{\partial F}{\partial z}(x_0, y_0, z_0) \\ &= \nabla F(x_0, y_0, z_0).u \end{aligned}$$

the scalar product of the vectors $F(x_0, y_0, z_0)$ and u.

Example

Given $F(x, y, z) = 2x + xy - yz^2$, find the directional derivative of F at the point $(1, 2, 2)$ in the direction of the unit vector $u = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Solution:

Remark

If F is a scalar field of three variables and v is any nonzero vector then the directional derivative of F at a point P in the direction of v is given by $D_u F(P)$, where $u = \frac{1}{\|v\|} v$.

Theorem

Let F be a scalar field and F and its partial derivatives be continuous in some sphere about a point P and suppose that $\nabla F(P) \neq 0$. Then

1. *At P, F has its maximum rate of change in the direction of $\nabla F(P)$ and this maximum rate of change is $\|\nabla F(P)\|$.*
2. *At P, F has its minimum rate of change in the direction of $-\nabla F(P)$ and this minimum rate of change is $-\|\nabla F(P)\|$.*

Proof: Use $D_u F(P) = \nabla F(P).u = \|\nabla F(P)\| \|u\| \cos\theta = \|\nabla F(P)\| \cos\theta$.

Example

Let $F(x, y, z) = 2xz + yz^2$, then find the maximum rate of change of F at $(2, 1, 1)$.

Level Surfaces, Tangent Planes and Normal Lines

Definition

Let F be a function of three variables and c be a number. The set of points (x, y, z) such that $F(x, y, z) = c$ is called a level surface of F.

Example

Let $F(x, y, z) = x^2 + y^2 + z^2$ and $c = 9$. The level surface $F(x, y, z) = 9$ is a sphere $x^2 + y^2 + z^2 = 9$ with radius 3 and center at the origin.

Let F be a scalar function of three variables, c be a constant and S be the level surface given by $F(x, y, z) = c$. Let $P_0 = (x_0, y_0, z_0)$ be a point on S. Assume that there are smooth curves on the surface S passing through P_0. Then each such curve has a tangent vector at P_0. The plane containing these tangent vectors is called the **tangent plane** to the surface S at P_0 and a vector orthogonal to this tangent plane at P_0 is called a normal vector to the surface S at P_0. The line through P_0 in the direction of the normal vector is called a **normal line** to the surface S at the point P_0.

To determine equation of the tangent plane and normal line to a surface S at a given point P, we need to have a normal vector to the tangent plane and for this purpose we have the following theorem.

Theorem

Let F be a function of three variables and suppose that F and its first partial derivatives are continuous at a point P on the level surface S given by $F(x, y, z) = c$. Suppose that $\nabla F(P) \neq 0$. Then $\nabla F(P)$ is normal to the level surface S at the point P.

Example

Find the equation of the tangent plane and normal line to the surface $3x^4 + 3y^4 + 6z^4 = 12$ at the point $(1, 1, 1)$.

Solution:

Curves and Arc length

Let $x = x(t), y = y(t)$ and $z = z(t)$ be continuous functions of a real parameter t over a closed interval $[a, b]$. The points $r(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$ are said to constitute a curve C joining the endpoints $r(a)$ and $r(b)$) and $r(t) = (x(t), y(t), z(t))$ is called a parametrization of the curve. We call the functions x, y and z, coordinate functions.

We call a curve C that is parameterized by $r(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$:

- continuous if each coordinate function is continuous;
- differentiable if each coordinate function is differentiable;
- closed if the initial and terminal points coincide, that is, $(x(a), y(a), z(a)) = (x(b), y(b), z(b))$ and if a curve is not closed it is called an arc;
- simple if $a < t_1 < t_2 < b$ implies that $(x(t_1), y(t_1), z(t_1)) \neq (x(t_2), y(t_2), z(t_2))$, in other words, if it does not intersect itself;

- smooth if the coordinate functions have continuous derivatives which are never all zero for the same value of t, that is, it possesses a tangent vector that varies continuously along the length of C.
- piecewise smooth if it has continuous tangent at all but finitely many points. Such a curve is a curve with a finite number of corner at which there is no tangent.

If C is a curve which is divided into smooth curves $C_1, C_2, \cdots, C_n$ such C begins with C_1, C_2 begins where C_1 ends and so on, but at the point where C_i and C_{i+1} join, there may be no tangent in the resulting curve , then C is piecewise smooth curve and we write such a curve as

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_n.$$

The following are examples of Curves.

1) **Stright Line:** A straight line L through a point P with position vector in the direction of a constant vector A can be represented as $r(t) = P + tA = (v_1 + ta_1, v_2 + ta_2, v_3 + ta_3)$, for all $t \in \mathbb{R}$, where $P = (v_1, v_2, v_3)$, $A = (a_1, a_2, a_3)$.

The following are examples of Curves.

2) **Ellipse, circle:** The vector function: $r(t) = (a\cos t, b\sin t, 0)$ represents an ellipse and is a circle if $a = b$.

3) **Circular helix:** The twisted curve represented by the vector function: $r(t) = (a\cos t, a\sin t, ct)$, $c \neq 0$ is a circular helix.

Consider a curve C that is parameterized by $r(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$. If it exists, the derivative of $r(t_0)$ for $t_0 \in (a, b)$ is given by

$$r'(t_0) = x'(t_0)i + y'(t_0)j + z'(t_0)k.$$

The derivative of a function at a point is the slope of a tangent line to the graph of the function at the point. Hence the derivative $r'(t)$ (if it exists) of the curve is called the tangent vector to the curve at the point $r(t)$ and the equation of the tangent line to the curve C at point P is

$$q(s)r + sr'$$

Example

For the curve C given by $F(t) = 2ti - t^2j + 4tk$, the vector $F'(t) = 2i - 2tj + 4k$ is tangent to the curve at the point $(2t, -t^2, 4t)$.

Definition

The length l of the curve C which is given by the parametrization $r(t) = x(t)i + y(t)j + z(t)k$ on $[a, b]$ is defined by

$$l = \int_a^b \sqrt{r'(t).r'(t)}dt.$$

If we replace b (the fixed upper limit of integration) by a variable t, $a \leq t \leq b$, the integral becomes a function of t.

$$s(t) = \int_a^t \sqrt{r'.r'}d\tau, \quad \text{where} \quad r' = \frac{dr}{d\tau}$$

and is called the arc length function.

Differentiating the arc length function gives us

$$\frac{ds}{dt} = \sqrt{r'.r'} = \|r'(t)\| = \|v(t)\|.$$

Example

Let $r(t) = (a\cos t, a\sin t, ct), c \neq 0$. represent circular helix. Then find the arc length function $s(t)$.

Solution:

Tangent, Curvature and Torsion

Let $F(t) = x(t)i + y(t)j + z(t)k$ be the position vector of a curve C for $a \leq t \leq b$. Assume that the coordinate functions x, y and z are twice continuously differentiable.

If a particle is moving along the curve C with a position vector $F(t) = x(t)i + y(t)j + z(t)k$, then the the velocity $v(t)$ of the particle at time t is:

$$v(t) = F'(t)$$

and the speed $v(t)$ of the particle is the norm of the velocity, i.e

$$v(t) = \|v(t)\| = \|F'(t)\|.$$

which is the rate of change of the distance covered by the particle along the curve with respect to the time and the acceleration $a(t)$ of the moving particle is the rate of change of the velocity with respect to time, i.e.

$$a(t) = v'(t) = F''(t).$$

If $F'(t) \neq 0$, then the vector $F'(t)$ is tangent to the curve C. Let $T(t)$ be a unit vector in the direction of $F'(t)$, i.e.

$$T(t) = \frac{1}{\|F'(t)\|} F'(t)$$

Let C be a smooth curve with parametrization $F(t)$ such that $F(t)$ is differentiable. The norm of the rate of change of the unit vector $T(t)$ with respect to the arc length function S is called the **curvature** K of the curve C. That is,

$$K(S) = \left\| \frac{dT}{dS} \right\|.$$

Consider the relation

$$\frac{dT}{dS} = \frac{dT}{dt} . \frac{dt}{dS} = \frac{dT/dt}{dS/dt}.$$

But $dS/dt = \|F'(t)\|$ and hence we get

$$K(t) = \frac{1}{\|F'(t)\|} \|T'(t)\|$$

which is a function of t.

Example

Curvature of a line at any point is zero.
To see this, let l be a line that passes through a point $P(x_0, y_0, z_0)$ with directional vector $A = (a, b, c)$. Then the parametric equation of l is given by

$$F(t) = (x_0 + ta)i + (y_0 + tb)j + (z_0 + tc)k, \ \ t \in \mathbb{R}.$$

Then $F'(t) = ai + bj + ck$ and

$$T(t) = \frac{1}{\|F'(t)\|}F'(t) = \frac{1}{\sqrt{a^2 + b^2 + c^2}}(ai + bj + ck).$$

This implies that, $T'(t) = 0$ for all t and hence $K(t) = 0$ for all t. This is clear from the fact that a particle moving on a straight line does not change its direction.

Example

(Curvatures of ellipses and circles). Recall that the vector function $r(t) = (a\cos t, b\sin t, 0), \quad t \in \mathbb{R}$ represents an ellipse and it represents a circle if $a = b$ in space. Then $r'(t) = (-a\sin t, b\cos t, 0)$ and $\|r'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$. Thus

$$T(t) = \frac{1}{\|r'(t)\|} r'(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, b\cos t, 0)$$

If $a = b$, then $T(t) = (-\sin t, \cos t, 0)$. This implies $T'(t) = (-\cos t, -\sin t, 0)$ and hence the curvature of the circle is $K(t) = \frac{1}{a}$.

Remark

If C is a curve that is traced-out by a vector field $r(t)$, then the curvature K of the curve C is given by

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.$$

Example

Find the curvature of the helix $r(t) = (a\cos t)i + (a\sin t)j + btk$, where $a, b \geq 0$ and $a^2 + b^2 \neq 0$.

Solution:

In the case of plane curves, that is, graph of functions of the form $y = f(x)$ can be considered as curves traced out by a vector field $r(t) = ti + f(t)j$. Here the k^{th} component is considered to be zero. Therefore $r'(t) = i + f'(t)j$ and $r''(t) = f''(t)j$ and then the curvature of this curve is given by

$$K(t) = \frac{|f''(t)|}{(1 + (f'(t))^2)^{\frac{3}{2}}}, \quad \text{since} \quad r'(t) \times r''(t) = f''(t)k.$$

Example

Find the curvature of the parabola $y = ax^2 + bx + c$, where $a \neq 0$.

Solution:

Given a curve C which is parameterized by the the position vector $r(t)$, we have a unit tangent vector T at a point where the coordinate functions are differentiable. Now we are looking to get a unit normal vector to the curve at a point where the coordinate functions are differentiable. Since T has constant length, $\frac{dT}{dS}$ is orthogonal to T. At a point where $K(S) \neq 0$, the vector

$$N = \frac{1}{K}.\frac{dT}{dS} = \frac{T'(t)}{\|T'(t)\|} = \frac{r''(S)}{\|r(S)\|}$$

is a unit vector parallel to $T'(t)$ and hence normal to the curve and it is called principal unit normal vector for a curve C.

Definition

The unit vector $B = T \times N$ is called the **binomial vector** of the curve C trace out by the vector field $r(t)$.
The vector $\frac{dB}{dS}$ is parallel to N. This implies that $\frac{dB}{dS} = -\tau N$ for some constant τ (the negative sign is traditional). Here the scalar τ is called the **torsion** along the curve and from $N.\frac{dB}{dS} = -\tau N.N = -\tau.1$ we have

$$\tau = -\frac{dB}{dS}.N.$$

Unlike K, which is always positive, τ can be positive, negative or zero. Since B, T and N are mutually orthogonal, they are linearly independent. Hence any vector in $\mathbb{R}^3$ can be represented as a linear combination of these vectors.
If B', T' and N' exist, then we get the following:

$$T' = KN$$
$$N' = -KT + \tau B$$
$$B = -\tau N$$

and this formula is called **Frenet** formula.

Example

Let $F(t) = t^2 i - 2tj + tk$. Find the curvature, principal unit vector, binomial vector of the curve C with position vector F and the torsion along the curve.

Solution:

Divergence and Curl

In this section we will discuss two important vector operations. One produces a scalar field from a vector field and the other produces a vector field from a vector field.

Definition

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field given by

$$F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k.$$

1) The **divergence** of F, denoted by $divF$, is the scalar field defined by

$$divF = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

2) The **curl** of F, denoted by $curlF$, is the vector field defined by

$$curlF = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) k.$$

Example

Let $F(x, y, z) = 3xyi - 2yzj + x^2k$. Then find $divF$ and $curlF$.

Solution:

Let ∇ be the operator defined from the set of scalar fields of three variables into the set of vectors in $\mathbb{R}^3$ by

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k.$$

If F is a scalar field of three variables, then the products $\frac{\partial}{\partial x}(F)$, $\frac{\partial}{\partial y}(F)$ and $\frac{\partial}{\partial z}(F)$ are defined to be $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ respectively.

Remark (The ∇ operator and gradient, divergence and curl.)

1. The product of ∇ and a scalar field F in the given order is the gradient of F, that is,

$$\nabla F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = gradF.$$

2. The product of ∇ and a vector field F in the given order is the divergence of F , that is, if $F = fi + gj + hk$, then

$$\nabla .F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(fi+gj+hk) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = divF.$$

3. The cross product of ∇ and a vector field F is the curl of F, that is, if $F = fi + gj + hk$, then

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k = curlF$$

Potential Function

Recall that, if a scalar field f is differentiable at every point D of its domain, then $V(P) = \nabla f(P)$ defines a vector field V on D.

Example

If $f(x, y) = 3x^2 + xy + y^3$, then $\nabla f(x, y) = V(x, y) = (6x + y, x + 3y^2)$ is a vector field. Here the function f is called a potential of the vector V. However, not every vector field has a potential, that is, not every vector field ia a gradient of some scalar field.

How do we check whether a vector field has a potential or not? The following proposition will answer this question.

Proposition (Test for Existence of a Potential Function)

1) Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by $V(p) = (V_1(p), V_2(p)$.
Then V has a potential function if and only if

$$\frac{\partial V_1}{\partial y}(p) = \frac{\partial V_2}{\partial x}(p) \text{ for all } p \text{ in the Domain of } V.$$

2) Let $V : \mathbb{R}^n \to \mathbb{R}^3$, $n = 2, 3$ be given by $V(p) = (V_1(p), V_2(p), V_3(p))$. Then V has a potential function if and only if $curlV = \nabla \times V = 0 = (0, 0, 0)$.

Example

1. Let $V(x, y, z) = (2 + y, x - z^2, -2yz)$. Then show that V has a potential function.
2. Let $V(x, y) = (-cy, cx), c \in \mathbb{R} \backslash 0$. Then show that V has no potential function.

To find the potential function f consider the following example.

Let $V(x, y) = (6x + y, x + 3y^2)$ be given. Then if there is a potential function f for V it must satisfy

$$f_x(x, y) = 6x + y \quad \text{and} \quad f_y(x, y) = x + 3y^2.$$

This implies

$$f(x, y) = \int (6x + y)dx = 3x^2 + xy + A(y),$$

where $A(y)$ is constant with respect to x (or, it is a function of y only).

Then from $f_y(x, y) = x + 3y^2$, we get $f_y(x, y) = x + A'(y) = x + 3y^2$, which implies that $A'(y) = 3y^2$ and hence

$$A(y) = 3y^2 dy = y^3 + C, \quad \text{where } C \text{ is a constant}$$

Therefore the scalar field $f(x, y) = 3x^2 + xy + y^3 + C$ is the potential of the vector field

$$V(x, y) = (6x + y, x + 3y^2).$$

Example

Let V be a vector field given by $V(x, y, z) = (2 + y, x - z^2, -2yz)$. Then find a potential function f for V.

Solution: