Applied Mathematics III
Unit 3
The Laplace Transform Method to Solve ODEs

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The Laplace Transform Method to Solve ODEs

In the previous sections, we have discussed how to solve differential equations of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{1}$$

by finding the general solutions and then evaluating the arbitrary constants in accordance with the given initial conditions. However, the solution methods mainly depend on the structure of the forcing function $f(x)$. Moreover, all the coefficients are assumed to be constants. To address problems with more general forcing function and some form of variable coefficients, we discuss the use of Laplace transform as a possible alternative.

Laplace Transform

Definition

The Laplace Transform of a function $f(t)$, if it exists, is denoted by $\mathcal{L}\{f(t)\}$ and is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \tag{2}$$

where s is a real number called a parameter of the transform. For short, we may write,

$$F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{3}$$

Example

Example

Find the Laplace Transform of the constant function $f(t) = 1$.

Solution

Example

Example

Find the Laplace Transform of the constant function $f(t) = 1$.

Solution

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{T\to\infty} \int_0^T e^{-st} dt = \lim_{T\to\infty} \left[-\frac{e^{-st}}{s} \right]_0^T$$

$$= \lim_{T\to\infty} \left(-\frac{e^{-sT}}{s} + \frac{1}{s} \right)$$

$$= \begin{cases} \frac{1}{s}, & if\ s > 0 \\ \infty, & otherwise \end{cases}$$

Therefore, $\mathcal{L}\{1\} = \dfrac{1}{s}$, *if* $s > 0$.

Basic Laplace Transforms

Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{1}{s},\ s > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}},\ s > 0$
e^{kt}	$\frac{1}{s-k},\ s > k$
$t^n e^{kt}$	$\frac{n!}{(s-k)^{n+1}},\ s > k$
$\sin(kt)$	$\frac{k}{s^2+k^2},\ s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2},\ s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2},\ s > \|k\|$
$\cosh(kt)$	$\frac{s}{s^2-k^2},\ s > \|k\|$

Table: Table of some basic Laplace Transforms

Inverse Laplace Transform

From the table above, we have

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{for } s > a.$$

Thus, the inverse operator applied on $\frac{1}{s-a}$ will give us back the function e^{at}

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \text{for } s > a.$$

In general, $\mathcal{L}^{-1}$, the inverse Laplace Operator, is given by

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds, \tag{4}$$

where γ is a positive real number, which is a complex improper integral.

Properties of the Laplace Transform

Here below we state some important properties of the transform in a series of theorems without proof.

Theorem (Linearity)

(a) *If $u(t)$ and $v(t)$ are functions and α, β are any constants, then*

$$\mathcal{L}\{\alpha u(t) + \beta v(t)\} = \alpha\mathcal{L}\{u(t)\} + \beta\mathcal{L}\{v(t)\}. \tag{5}$$

(b) *For any functions $U(s)$, $V(s)$ and any given scalars α, β, we have*

$$\mathcal{L}^{-1}\{\alpha U(s) + \beta V(s)\} = \alpha\mathcal{L}^{-1}\{U(s)\} + \beta\mathcal{L}^{-1}\{V(s)\}. \tag{6}$$

Examples

Example

Evaluate the following transform:

1. $\mathcal{L}\{3t + 5e^{-2t}\}$

Solution

Examples

Example

Evaluate the following transform:

1. $\mathcal{L}\{3t + 5e^{-2t}\}$

Solution

$$\begin{aligned}
\mathcal{L}\{3t + 5e^{-2t}\} &= 3\mathcal{L}\{t\} + 5\mathcal{L}\{e^{-2t}\} \\
&= 3 \cdot \frac{1}{s^2} + 5 \cdot \frac{1}{s+2} \\
&= \frac{3}{s^2} + \frac{5}{s+2}
\end{aligned}$$

Examples

Example

Evaluate the following transform:

2. $\mathcal{L}\{\cos(2\sqrt{3}t)\}$

Solution

Examples

Example

Evaluate the following transform:

2. $\mathcal{L}\{\cos(2\sqrt{3}t)\}$

Solution

$$\begin{aligned}\mathcal{L}\{\cos(2\sqrt{3}t)\} &= \frac{s}{s^2 + (2\sqrt{3})^2} \\ &= \frac{s}{s^2 + 12}\end{aligned}$$

Examples

Example

Evaluate the following transform:

3. $\mathcal{L}\{\cos^2(\sqrt{3}t)\}$

Solution

Examples

Example

Evaluate the following transform:

3. $\mathcal{L}\{\cos^2(\sqrt{3}t)\}$

Solution

Using the trigonometric identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$, *we have:*

$$\cos^2(\sqrt{3}t) = \frac{1+\cos(2\sqrt{3}t)}{2}$$

$$\begin{aligned}
\mathcal{L}\{\cos^2(\sqrt{3}t)\} &= \mathcal{L}\left\{\frac{1}{2} + \frac{\cos(2\sqrt{3}t)}{2}\right\} = \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cos(2\sqrt{3}t)\} \\
&= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+12} \\
&= \frac{1}{2s} + \frac{s}{2(s^2+12)}
\end{aligned}$$

Example

Evaluate the following transform:

4. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\}$

Solution

$$\begin{aligned}\frac{s^2}{(s+1)^3} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} \\ \Rightarrow s^2 &= A(s+1)^2 + B(s+1) + C \\ &= As^2 + 2As + A + Bs + B + C \\ &= As^2 + (2A+B)s + (A+B+C)\end{aligned}$$

By comparing coefficients, we get:

$$\begin{aligned} A &= 1 \\ 2A + B &= 0 \quad \Rightarrow B = -2 \\ A + B + C &= 0 \quad \Rightarrow C = 1\end{aligned}$$

Hence, we can rewrite the inverse transform and apply linearity to get:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} \\ &= e^{-t} - 2te^{-t} + \frac{t^2}{2}e^{-t} \\ &= (1 - 2t + \frac{t^2}{2})e^{-t}\end{aligned}$$

The other important property that leads us to use the Laplace transform in solving ordinary differential equation is how the transform performs on the derivative.

Transform of the Derivative

Theorem (Transform of the Derivative)

Let $f(t)$ be continuous and $f'(t)$ be piecewise continuous on some interval $[0, t_0]$ for every finite t_0, and let $|f(t)| < Ke^{ct}$ for some constants K, T, and c and for all $t > T$. Then the transform $\mathcal{L}\{f'(t)\}$ exists for all $s > c$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{7}$$

Example

Use the Laplace transform method to solve the initial-value problem:

$$y' + 2y = 0 \quad \text{with} \quad y(0) = 1.$$

Example

Solution

Applying the Laplace transform on both sides of the equation, we have:

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{0\}$$
$$\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = 0$$

Now, letting $\mathcal{L}\{y(t)\} := Y(s)$, *we get the algebraic equation:*

$$sY(s) - y(0) + 2Y(s) = 0 \quad \Rightarrow \quad Y(s) = \frac{1}{s+2}$$

Therefore, reading from the transform table, we get:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

i.e., $y(t) = e^{-2t}$ *is the solution for the differential equation.*

We can also use the Laplace method to solve higher order equations with constant coefficients. The following property of the transform, which is the continuouation of the above theorem, is required.

Theorem

Let $f(t)$ be continuous and $f^{(n)}(t)$ be piecewise continuous on some interval $[0, t_0]$ for every finite t_0, and let $|f(t)| < Ke^{ct}$ for some constants K, T, and c and for all $t > T$. Then we have

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \quad (8)$$

Theorem (First Shifting Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$ for $Re(s) > b$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $Re(s) > a + b$.

Example

Find the Laplace transform for the function $f(t) = e^{3t}\cos(4t)$.

Solution

Example

Find the Laplace transform for the function $f(t) = e^{3t}\cos(4t)$.

Solution

Using the First Shifting Theorem, we have:

$$
\begin{aligned}
f(t) &= e^{3t}\cos(4t) \\
\mathcal{L}\{e^{3t}\cos(4t)\} &= \mathcal{L}\{\cos(4t)\} \quad \textit{(shift by } 3) \\
&= \frac{s}{s^2+16} \quad \textit{(shift by } 3) \\
&= \frac{s-3}{(s-3)^2+16}
\end{aligned}
$$

Example

Find the inverse Laplace transform for the function $\mathcal{F}(s) = \frac{s}{s^2+s+1}$.

Solution

Example

Find the inverse Laplace transform for the function $\mathcal{F}(s) = \frac{s}{s^2+s+1}$.

Solution

First, let us rewrite the function $\mathcal{F}(s)$ as:

$$\mathcal{F}(s) = \frac{s}{s^2+s+1} = \frac{s}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

and hence,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} - \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}.$$

Then, using the first shifting theorem, we have:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right).$$

Application of the Shifting Theorem

Example

1. Solve the initial-value problem:

$$y'' + 4y' + 4y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution

Application of the Shifting Theorem

Example

1. Solve the initial-value problem:

$$y'' + 4y' + 4y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution

Taking the Laplace transform of both sides:

$$s^2 Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{1}{s+1}.$$

Substituting initial conditions:

$$s^2 Y(s) - 1 + 4sY(s) + 4Y(s) = \frac{1}{s+1}.$$

$$(s^2 + 4s + 4)Y(s) = \frac{1}{s+1} + 1.$$

$$Y(s) = \frac{1}{(s+2)^2(s+1)} + \frac{1}{(s+2)^2}.$$

Using partial fractions on the first term:

$$\frac{1}{(s+2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}.$$

Solving gives $A = 1, B = -1, C = 1$. Thus:

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{(s+2)^2} + \frac{1}{(s+2)^2}.$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{2}{(s+2)^2}.$$

Taking the inverse Laplace transform:

$$y(t) = e^{-t} - e^{-2t} + 2te^{-2t}.$$

Notice the te^{-2t} term, which comes from the $\frac{1}{(s+2)^2}$, illustrating the shifting theorem.

Example

2. Solve the initial-value problem:

$$y'' - 6y' + 9y = e^{4t}, \quad y(0) = 1, \quad y'(0) = 5.$$

Solution

Example

2. Solve the initial-value problem:

$$y'' - 6y' + 9y = e^{4t}, \quad y(0) = 1, \quad y'(0) = 5.$$

Solution

Taking the Laplace transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{1}{s-4}.$$

Substituting initial conditions:

$$s^2Y(s) - s - 5 - 6sY(s) + 6 + 9Y(s) = \frac{1}{s-4}.$$

$$(s^2 - 6s + 9)Y(s) = \frac{1}{s-4} + s - 1.$$

$$(s-3)^2Y(s) = \frac{1}{s-4} + s - 1.$$

$$Y(s) = \frac{1}{(s-3)^2(s-4)} + \frac{s-1}{(s-3)^2}.$$

Using partial fractions on the first term:

$$\frac{1}{(s-3)^2(s-4)} = \frac{A}{s-4} + \frac{B}{s-3} + \frac{C}{(s-3)^2}.$$

Solving gives $A = 1, B = -1, C = 1$. Thus:

$$Y(s) = \frac{1}{s-4} - \frac{1}{s-3} - \frac{1}{(s-3)^2} + \frac{s-1}{(s-3)^2}.$$

$$Y(s) = \frac{1}{s-4} + \frac{(s-3)+1}{(s-3)^2} = \frac{1}{s-4} + \frac{1}{s-3} + \frac{1}{(s-3)^2}$$

Taking the inverse Laplace transform:

$$y(t) = e^{4t} + te^{3t}.$$

Derivative of the Transform

Consider the general Laplace transform formula:

$$\mathcal{F}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Taking the derivative with respect to s on both sides, we get:

$$\mathcal{F}'(s) = \int_0^\infty (-t) e^{-st} f(t) dt = \mathcal{L}\{-tf(t)\}.$$

By further differentiating the above equation with respect to s, we get:

$$\mathcal{F}''(s) = \mathcal{L}\{t^2 f(t)\}.$$

In general, we have:

Theorem (Derivative of the Transform)

For a piecewise continuous function $f(t)$ and for any positive integer n, it holds that:

$$\mathcal{L}\{(-1)^n t^n f(t)\} = \mathcal{F}^{(n)}(s).$$

The formula in this theorem can be used to find transforms of functions of the form $x^n f(x)$ when the Laplace transform of $f(t)$ is known.

Application of the Derivative of the Transform

Example

1. Solve the initial-value problem:

$$y'' + 2y' + y = te^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution

Application of the Derivative of the Transform

Example

1. Solve the initial-value problem:

$$y'' + 2y' + y = te^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution

Taking the Laplace transform of both sides:

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \mathcal{L}\{te^{-t}\}.$$

Substituting initial conditions:

$$s^2 Y(s) - 1 + 2sY(s) + Y(s) = \mathcal{L}\{te^{-t}\}.$$

$$(s^2 + 2s + 1)Y(s) = 1 + \mathcal{L}\{te^{-t}\}.$$

Using the derivative of the transform theorem,
$\mathcal{L}\{te^{-t}\} = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$.

$$(s+1)^2 Y(s) = 1 + \frac{1}{(s+1)^2}.$$

$$Y(s) = \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}.$$

Taking the inverse Laplace transform:

$$y(t) = te^{-t} + \frac{t^3 e^{-t}}{6}.$$

Notice the $\frac{t^3e^{-t}}{6}$ term, which comes from $\frac{1}{(s+1)^4}$, illustrating the derivative of the transform effect.

Example

2. Solve the initial-value problem:

$$y'' + 4y' + 4y = t\cos(t)e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution

Example

2. Solve the initial-value problem:

$$y'' + 4y' + 4y = t\cos(t)e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution

Taking the Laplace transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \mathcal{L}\{t\cos(t)e^{-2t}\}.$$

Substituting initial conditions:

$$(s^2 + 4s + 4)Y(s) = \mathcal{L}\{t\cos(t)e^{-2t}\}.$$

$$(s+2)^2Y(s) = \mathcal{L}\{t\cos(t)e^{-2t}\}.$$

Using the derivative of the transform theorem,

$$\mathcal{L}\{t\cos(t)e^{-2t}\} = -\frac{d}{ds}\left(\frac{s+2}{(s+2)^2+1}\right).$$

$$\frac{d}{ds}\left(\frac{s+2}{(s+2)^2+1}\right) = \frac{((s+2)^2+1)-(s+2)(2(s+2))}{((s+2)^2+1)^2} = \frac{1-(s+2)^2}{((s+2)^2+1)^2}$$

Solving for Y(s):

$$Y(s) = \frac{1-(s+2)^2}{((s+2)^2+1)^2(s+2)^2}$$

Taking the inverse Laplace transform:

$$y(t) = \frac{1}{2}te^{-2t}\sin(t)$$

Notice how using the derivative of the transform theorem allowed us to solve for the laplace transform of $t\cos(t)e^{-2t}$.

Exercise

Use the Laplace transform method to solve the following ODE:

1. $y'' - 4y' + 4y = e^{3t}, \quad y(0) = 1, \quad y'(0) = 2.$
2. $y'' + 2y' + y = te^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$
3. $y'' + 6y' + 9y = te^{-3t}, \quad y(0) = 1, \quad y'(0) = -2.$
4. $y'' + 2y' + 5y = e^{-t}\sin(2t), \quad y(0) = 0, \quad y'(0) = 1.$
5. $y'' - 2y' + y = te^{t}, \quad y(0) = 1, \quad y'(0) = 2.$
6. $xy'' + (2x+3)y' + (x+3)y = 3e^{-x}; \quad y(0) = 0, \quad y'(0) = 1.$

Remark

The main idea in using the Laplace transform in solving ODEs is that it transforms the differential equation into an algebraic equation. Once the transformation is completed, we seek for a solution to $\mathcal{L}\{y(t)\}$ algebraically. Then the final step will be to get back the value of $y(t)$ using the inverse Laplace transform.