Applied Mathematics III
Unit 6
Complex Analytic Functions

Solomon Amsalu (Asst.Prof.)

Mathematics Department
Wolkite University

solomon.amsalu@wku.edu.et

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Definition of Complex Numbers

A **complex number** z is expressed as:

$$z = x + iy$$

where:

- x is the **real part**, denoted by $\mathrm{Re}(z)$.
- y is the **imaginary part**, denoted by $\mathrm{Im}(z)$.
- i is the imaginary unit, defined by $i^2 = -1$.

Basic Operations

Let $a + bi$ and $c + di$ be two complex numbers. Then:

1. **Equality**: $a + bi = c + di$ if and only if $a = c$ and $b = d$.
2. **Addition**:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

3. **Multiplication**:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

4. **Division**: If $z = a + bi$ and $w = c + di$ $(w \neq 0)$, then:

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

$$\frac{w}{z} = w \cdot \frac{1}{z}$$

Example

Example

Express $\frac{3+i}{2-2i}$ in the form $a + bi$.

Solution:

Example

Example

Express $\frac{3+i}{2-2i}$ in the form $a+bi$.

Solution: Multiply numerator and denominator by the conjugate of $2-2i$, which is $2+2i$:

$$\frac{(3+i)(2+2i)}{(2-2i)(2+2i)}$$

Expanding both terms:

$$\frac{6+6i+2i+2i^2}{4+4i-4i-4i^2}$$

Since $i^2=-1$, simplify:

$$\frac{6+8i-2}{4+4}=\frac{4+8i}{8}=\frac{4}{8}+\frac{8i}{8}=\frac{1}{2}+i$$

Thus, $\frac{3+i}{2-2i}=\frac{1}{2}+i$.

Basic Properties

Remark

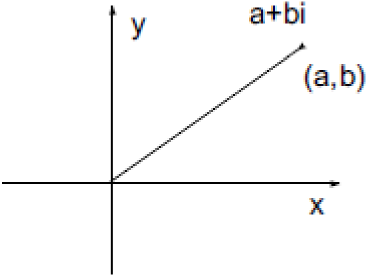
Some basic points about complex numbers:

1. The real and imaginary parts of any complex number are real numbers.
2. Any real number a can be considered as a complex number $a + 0i$. Therefore, the set of complex numbers is an extension of the set of real numbers.
3. The set of complex numbers is denoted by $\mathbb{C}$.
4. If x, y and z are complex numbers, then:
    - Addition is commutative: $x + y = y + x$
    - Multiplication is commutative: $xy = yx$
    - Associative law for addition: $x + (y + z) = (x + y) + z$
    - Associative law for multiplication: $x(yz) = (xy)z$
    - Distributive law: $x(y + z) = xy + xz$
    - 0 is the identity element for addition: $x + 0 = 0 + x = x$
    - 1 is the identity element for multiplication: $x \cdot 1 = 1 \cdot x = x$

Graphical Representation

A complex number $z = a + bi$ is represented in the Cartesian coordinate system as the point (a, b).

- The **Real Axis** is the horizontal axis.
- The **Imaginary Axis** is the vertical axis.



Magnitude and Argument

For a complex number $z = a + bi$:

1. The **Magnitude** (Modulus):

$$|z| = \sqrt{a^2 + b^2}$$

2. The **Argument**:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

3. The **Polar Form**:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Example

Example

Find the polar form of $z = 1 - i$.

Solution:

Example

Example

Find the polar form of $z = 1 - i$.

Solution:Calculate:

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arctan(-1) = \frac{3\pi}{4}$$

Thus:

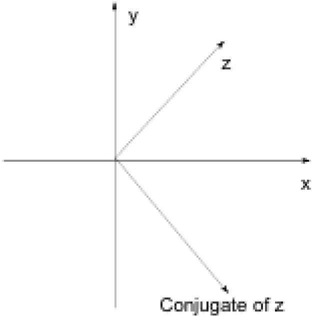
$$z = \sqrt{2}e^{i3\pi/4}$$

Definition of Complex Conjugates

For a complex number $z = a + bi$, the **conjugate** of z is given by:

$$\bar{z} = a - bi$$

On the complex plane, $\bar{z}$ is the reflection of z across the real axis.



Properties of Conjugates

Let z and w be complex numbers. Then:

1. $\bar{\bar{z}} = z$, and $\bar{z} = z$ if and only if z is real.
2. $\overline{z \pm w} = \bar{z} \pm \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$.
3. $\overline{|z|} = |\bar{z}|$, and $|z|^2 = z\bar{z}$.
4. $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$, and $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Definition of Complex Functions

Definition

A function w of a complex variable z is a rule that assigns a unique value $w(z)$ to each point z in some set D in the complex plane. If w is a complex function and $z = x + iy$, then we can always write

$$w(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions of x and y such that:

$$u(x, y) = \mathrm{Re}(f(z)), \quad v(x, y) = \mathrm{Im}(f(z))$$

Example

Example

Let w be a complex function defined by:

$$w(z) = z^2.$$

Find $\mathrm{Re}(w(z))$ and $\mathrm{Im}(w(z))$.

Solution:

Example

Example

Let w be a complex function defined by:

$$w(z) = z^2.$$

Find $\mathrm{Re}(w(z))$ and $\mathrm{Im}(w(z))$.

Solution: Since $z = x + yi$, then

$$w(z) = z^2 = (x + yi)^2 = (x^2 - y^2) + 2xyi$$

Hence:

$$\mathrm{Re}(w(z)) = u(x, y) = x^2 - y^2,$$

$$\mathrm{Im}(w(z)) = v(x, y) = 2xy$$

Definition of Limits

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function. Then clearly, f maps $\mathbb{R}^2$ into $\mathbb{R}^2$, and hence all concepts of limits and derivatives for vector functions of two variables also apply here, with notation adapted to complex numbers.

Definition

Let z_0 be an interior point in the domain of definition of a function $f : \mathbb{C} \to \mathbb{C}$. We say that the limit of $f(z)$ as z approaches z_0 is L, and we write:

$$\lim_{z \to z_0} f(z) = L$$

if for each $\epsilon > 0$ (no matter how small), there exists a $\delta > 0$ such that:

$$|f(z) - L| < \epsilon \quad \text{for all } z \text{ satisfying } 0 < |z - z_0| < \delta$$

In this definition, $z = x + yi$ and $f(z) = u(x, y) + iv(x, y)$. Moreover, $|z - z_0|$ represents the modulus of the complex number $z - z_0$, and the condition $|z - z_0| < \delta$ defines an open circle centered at z_0.

Properties of Limits

Remark

Let f and g be complex functions, and let z_0 and c be complex numbers such that:

$$\lim_{z \to z_0} f(z) = L, \quad \lim_{z \to z_0} g(z) = M$$

Then, the following properties hold:

1. $\lim_{z \to z_0} (f \pm g)(z) = L + M$
2. $\lim_{z \to z_0} (fg)(z) = LM$
3. $\lim_{z \to z_0} (f/g)(z) = L/M \quad \text{if } M \neq 0$
4. $\lim_{z \to z_0} (cf)(z) = cL$

Example

Example

Evaluate

$$\lim_{z \to 1+i} z^2$$

Solution:

Example

Example

Evaluate

$$\lim_{z \to 1+i} z^2$$

Solution:
Let $f(z) = z^2$. Substituting $z = x + yi$ (where x and y are real numbers), we have:

$$f(z) = (x + yi)^2 = x^2 - y^2 + 2xyi.$$

The limit as $z \to 1 + i$ means $x \to 1$ and $y \to 1$. Substituting $x = 1$ and $y = 1$ into $f(z)$, we calculate:

$$f(1 + i) = (1)^2 - (1)^2 + 2(1)(1)i = 2i.$$

$$\lim_{z \to 1+i} z^2 = 2i.$$

Continuity of Complex Functions

Definition

A function f is said to be **continuous** at z_0 if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A function is continuous in a set if it is continuous at each point of that set.

Example

Example

Prove that the function $f(z) = z^2 + 1$ is continuous at $z_0 = 2 + i$.

Solution:

Example

Example

Prove that the function $f(z) = z^2 + 1$ is continuous at $z_0 = 2 + i$.

Solution:
To check the continuity of $f(z)$ at z_0, we verify that:

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Step 1: Compute $f(z_0)$:

$$f(2 + i) = (2 + i)^2 + 1.$$

Expanding $(2 + i)^2$:

$$(2 + i)^2 = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i.$$

Thus:

$$f(2 + i) = (3 + 4i) + 1 = 4 + 4i.$$

Step 2: Compute $\lim_{z \to z_0} f(z)$:
Let $z = x + yi$, then $f(z) = (x + yi)^2 + 1$.
Expand:

$$f(z) = (x^2 - y^2 + 2xyi) + 1 = (x^2 - y^2 + 1) + 2xyi.$$

As $z \to z_0 = 2 + i$, substitute $x \to 2$ and $y \to 1$:

$$f(z) \to (2^2 - 1^2 + 1) + 2(2)(1)i = (4 - 1 + 1) + 4i = 4 + 4i.$$

Since:

$$\lim_{z \to z_0} f(z) = f(z_0),$$

the function $f(z) = z^2 + 1$ is continuous at $z_0 = 2 + i$.

Definition of Derivatives

Definition

Let f be a complex function. The derivative of f at the point z_0, denoted by $f'(z_0)$, is defined as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}$$

if the limit exists and is a complex number.
The limit must be unique and independent of the path along which z approaches z_0.

Example

Example

Find the derivative of each of the following functions if it exists.

1. $f(z) = z^2$
2. $f(z) = \bar{z}$

Solution:

Example

Example

Find the derivative of each of the following functions if it exists.

1. $f(z) = z^2$
2. $f(z) = \bar{z}$

Solution:

1. For $f(z) = z^2$:

$$f'(z) = \lim_{\triangle z \to 0} \frac{(z + \triangle z)^2 - z^2}{\triangle z} = \lim_{\triangle z \to 0} \frac{z^2 + 2z\triangle z + (\triangle z)^2 - z^2}{\triangle z}$$

$$= \lim_{\triangle z \to 0} \frac{2z\triangle z + (\triangle z)^2}{\triangle z} = \lim_{\triangle z \to 0} (2z + \triangle z) = 2z$$

Thus, $f'(z) = 2z$.

2. For $f(z) = \bar{z}$:

$$f'(z) = \lim_{\triangle z \to 0} \frac{\overline{z + \triangle z} - \bar{z}}{\triangle z}$$

Since $\overline{z + \triangle z} = \bar{z} + \overline{\triangle z}$, this simplifies to:

$$= \lim_{\triangle z \to 0} \frac{\bar{z} + \overline{\triangle z} - \bar{z}}{\triangle z} = \lim_{\triangle z \to 0} \frac{\overline{\triangle z}}{\triangle z}$$

For general complex numbers, this limit does not always exist uniquely, meaning $f(z) = \bar{z}$ is not differentiable.

Rules of Differentiation for Complex Functions

Let f and g be complex functions, and let c be a complex number.

1. **Sum (Difference) Rule**: $(f \pm g)'(z) = f'(z) \pm g'(z)$
2. **Constant Multiple Rule**: $(cf)'(z) = cf'(z)$
3. **Product Rule**: $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
4. **Quotient Rule**:

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

5. **Chain Rule (Complex Version)**: $(f \circ g)'(z) = f'(g(z))g'(z)$

Theorem

If f is a differentiable complex function at z_0, then f is continuous at z_0.

Example

Example

Let $f(z) = \dfrac{1}{z}$. Find $f'(z)$.

Solution:

Example

Example

Let $f(z) = \frac{1}{z}$. Find $f'(z)$.

Solution:

Using the **quotient rule**, which states:

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

For $f(z) = 1$ and $g(z) = z$, we compute:

$$f'(z) = \frac{0 \cdot z - 1 \cdot 1}{z^2}$$

Simplifying:

$$f'(z) = \frac{-1}{z^2}$$

Definition

Since differentiability of a complex function (along with analyticity) plays a crucial role in the study of complex variables, we ask: **When is a complex function differentiable?**

Definition

Let f be a complex function. Then:

1. f is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points in D.
2. f is said to be analytic at a point $z_0 \in D$ if it is analytic in some neighborhood of z_0.
3. f is simply called an analytic function if it is analytic in some domain (open connected subset of $\mathbb{C}$).

Cauchy-Riemann Equations: Theorem

Theorem: Let $f(z) = u(x, y) + iv(x, y)$ be a complex-valued function, where $z = x + yi$. If $f(z)$ is differentiable at $z = z_0$, then the real part $u(x, y)$ and the imaginary part $v(x, y)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

—

Proof: Start with the definition of the derivative:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Write $z = x + yi$, where $f(z) = u(x, y) + iv(x, y)$, and expand:

$$z - z_0 = (x - x_0) + i(y - y_0),$$

$$f(z) - f(z_0) = \big(u(x, y) - u(x_0, y_0)\big) + i\big(v(x, y) - v(x_0, y_0)\big).$$

Case 1: Limit Along the x-axis ($y = y_0$) If $z \to z_0$ along the x-axis:

$$f'(z_0) = \lim_{x \to x_0} \frac{\big(u(x, y_0) - u(x_0, y_0)\big) + i\big(v(x, y_0) - v(x_0, y_0)\big)}{x - x_0}.$$

Thus:

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Case 2: Limit Along the y-axis ($x = x_0$) If $z \to z_0$ along the y-axis:

$$f'(z_0) = \lim_{y \to y_0} \frac{\big(u(x_0, y) - u(x_0, y_0)\big) + i\big(v(x_0, y) - v(x_0, y_0)\big)}{i(y - y_0)}.$$

Simplify:

$$f'(z_0) = -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}.$$

Equating Real and Imaginary Parts: From the two cases:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, the Cauchy-Riemann equations are satisfied.

Test for Analyticity

If f is a complex function and $z = x + yi$, we can write:

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions of x and y.
For a function f defined in some domain D, if f is analytic in D (i.e., differentiable), then its partial derivatives exist, and at $z_0 = x_0 + y_0 i$, we have:

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

which leads to:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations are known as the **Cauchy-Riemann equations**, and they provide a necessary condition for analyticity.

Necessary and Sufficient Conditions for Analyticity

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be a function defined in some neighborhood of $z_0 = x_0 + iy_0$.

1. **Necessary Condition**: *If f is differentiable at z_0, then the Cauchy-Riemann equations are satisfied.*
2. **Sufficient Condition**: *If the Cauchy-Riemann equations hold at z_0 and u and v are continuously differentiable in some neighborhood of z_0, then f is analytic at z_0.*

Due to the Cauchy-Riemann equations, the derivative of $f(z)$ can be written in one of the following four equivalent forms:

$$f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

$$f'(z) = u_x(x,y) - iu_y(x,y) = v_y(x,y) + iv_x(x,y)$$

Definition of Neighborhood

The **neighborhood** of a point z_0 in the complex plane is a set of points surrounding z_0. It is formally defined as:

$$N(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

where:

- $r > 0$ is the radius of the neighborhood.
- $|z - z_0|$ represents the modulus (distance) between z and z_0.

Properties:

- $N(z_0, r)$ is an **open disk** centered at z_0 with radius r. It excludes the boundary $|z - z_0| = r$.
- A neighborhood is an **open set**, meaning it contains no boundary points.
- If a function is analytic in a neighborhood $N(z_0, r)$, it is analytic at all points within $N(z_0, r)$.

Example

Example

Show that $f(z) = |z|^2$ is differentiable only at $z = 0$ and is analytic nowhere.

Solution:

Example

Example

Show that $f(z) = |z|^2$ is differentiable only at $z = 0$ and is analytic nowhere.

Solution:
Since $f(z) = |z|^2 = x^2 + y^2$, we compute:

$$u_x = 2x, \quad u_y = 2y$$

$$v_x = 0, \quad v_y = 0$$

For analyticity, the Cauchy-Riemann equations must hold:

$$u_x = v_y, \quad v_x = -u_y$$

which simplifies to:

$$2x = 0, \quad 0 = -2y$$

The only solution is $x = 0$ and $y = 0$, meaning $f(z)$ is differentiable only at $z = 0$ and **not analytic elsewhere**.

Example

Let $f(z) = z^2 - 8z + 3$. If $z = x + yi$, show that f is differentiable for all z and find $f'(x + yi)$.

Solution:

Example

Example

Let $f(z) = z^2 - 8z + 3$. If $z = x + yi$, show that f is differentiable for all z and find $f'(x + yi)$.

Solution:

Expanding:

$$f(z) = (x + yi)^2 - 8(x + yi) + 3$$
$$= (x^2 - y^2 - 8x + 3) + i(2xy - 8y)$$

Setting $u = x^2 - y^2 - 8x + 3$ and $v = 2xy - 8y$, we compute derivatives:

$$u_x = 2x - 8, \quad u_y = -2y$$

$$v_x = 2y, \quad v_y = 2x - 8$$

Since $u_x = v_y$ and $v_x = -u_y$, the Cauchy-Riemann equations hold everywhere. Thus, f is differentiable for all z.

$$f'(z) = 2z - 8$$

Key Difference Between Differentiability and Analyticity:

- Differentiability is a *local property* at a point, whereas analyticity is a *global property* in a neighborhood.
- A function can be differentiable at a single point without being analytic (e.g., $f(z) = |z|^2$).
- If a function is analytic, it is automatically differentiable in the entire domain where it is analytic.

—

Example:

- The function $f(z) = z^2$ is analytic everywhere in the complex plane because it is differentiable in every neighborhood and satisfies the Cauchy-Riemann equations.
- The function $f(z) = |z|^2$ is differentiable at $z = 0$, but not analytic anywhere because it does not satisfy the Cauchy-Riemann equations.

Definition of Harmonic Functions

Definition

A real-valued function $u(x, y)$ satisfies **Laplace's equation**:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

If all first and second-order partial derivatives of u exist and are continuous, then u is called a **harmonic function**.

Theorem

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D, meaning they satisfy Laplace's equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad \nabla^2 v = v_{xx} + v_{yy} = 0$$

Since f is analytic, u and v are related by the Cauchy-Riemann equations. Such functions are called **conjugate harmonic functions**.

Example

Example

Show that $u = x^2 - y^2 - y$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.

Solution:

Example

Example

Show that $u = x^2 - y^2 - y$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.

Solution:

To check if u is harmonic, we compute its second-order partial derivatives:

$$u_x = \frac{\partial}{\partial x}(x^2 - y^2 - y) = 2x, \quad u_y = \frac{\partial}{\partial y}(x^2 - y^2 - y) = -2y - 1$$

Now, computing the second-order derivatives:

$$u_{xx} = \frac{\partial}{\partial x}(2x) = 2, \quad u_{yy} = \frac{\partial}{\partial y}(-2y - 1) = -2$$

Since:

$$\nabla^2 u = u_{xx} + u_{yy} = 2 + (-2) = 0$$

u satisfies the Laplace equation and is harmonic.

To find a conjugate harmonic function v, we use the Cauchy-Riemann equations:

$$v_x = -u_y = 2y + 1, \quad v_y = u_x = 2x$$

Integrating v_x with respect to x:

$$v = \int (2y + 1) dx = (2y + 1)x + C(y)$$

Differentiating with respect to y:

$$\frac{\partial}{\partial y}[(2y + 1)x + C(y)] = 2x$$

Setting $C'(y) = 0$, we conclude $C(y)$ must be a constant, thus:

$$v = (2y + 1)x$$

Therefore, $v = (2y + 1)x$ is the conjugate harmonic function of u.

Example

Example

Show that $u(x, y) = x^3 - 3xy^2 + 3x + 1$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.

Solution:

Example

Example

Show that $u(x, y) = x^3 - 3xy^2 + 3x + 1$ is harmonic in $\mathbb{C}$ and find a conjugate harmonic function v of u.

Solution:

First, compute the second-order partial derivatives:

$$u_x = \frac{\partial}{\partial x}(x^3 - 3xy^2 + 3x + 1) = 3x^2 - 3y^2 + 3$$

$$u_y = \frac{\partial}{\partial y}(x^3 - 3xy^2 + 3x + 1) = -6xy$$

Now, computing second-order derivatives:

$$u_{xx} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 3) = 6x$$

$$u_{yy} = \frac{\partial}{\partial y}(-6xy) = -6x$$

Since:

$$\nabla^2 u = u_{xx} + u_{yy} = 6x + (-6x) = 0$$

u satisfies the Laplace equation and is harmonic.
To find v, we apply the Cauchy-Riemann equations:

$$v_x = -u_y = 6xy, \quad v_y = u_x = 3x^2 - 3y^2 + 3$$

Integrating v_x with respect to x:

$$v = \int 6xy \, dx = 3x^2 y + C(y)$$

Differentiating v with respect to y:

$$\frac{\partial}{\partial y}(3x^2 y + C(y)) = 3x^2 - 3y^2 + 3$$

Setting $C'(y) = -3y^2 + 3$, integrating:

$$C(y) = -y^3 + 3y$$

Thus, the conjugate harmonic function is:

$$v = 3x^2 y - y^3 + 3y$$

Exponential Functions

For a complex number $z = x + yi$, the complex exponential function e^z is defined by:

$$e^z = e^{x+yi} = e^x(\cos y + i \sin y)$$

Euler's formula states:

$$e^{yi} = \cos y + i \sin y$$

Additionally:

$$|e^{yi}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

Thus:

$$|e^z| = |e^x e^{yi}| = e^x |e^{yi}| = e^x, \quad \text{for all } z = x + yi.$$

Example

Example

1. Compute $|e^{-2+4i}|$ and $|e^{3-5i}|$.
2. If $e^z = 2i$, then find z.

Solution:

Example

Example

1. Compute $|e^{-2+4i}|$ and $|e^{3-5i}|$.
2. If $e^z = 2i$, then find z.

Solution:

1. Since $|e^z| = e^x$, we have:

$$|e^{-2+4i}| = e^{-2}, \quad |e^{3-5i}| = e^3.$$

2. Given $e^z = 2i$, we express z as $x + yi$:

$$e^z = e^x e^{yi} = e^x(\cos y + i \sin y) = 2i$$

Comparing real and imaginary parts:

$$e^x \cos y = 0, \quad e^x \sin y = 2.$$

Since $\cos y = 0$, valid values for y are:

$$y = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

From $e^x \sin y = 2$, using $\sin y = \pm 1$, we get:

$$e^x = 2 \Rightarrow x = \ln 2.$$

Thus:

$$z = \ln 2 + i\left(\frac{\pi}{2} + k\pi\right), \quad k \in \mathbb{Z}.$$

Differentiability of Exponential Functions

Remark

For a complex number $z = x + yi$, we have $e^z \neq 0$ for all $z \in \mathbb{C}$, since $e^x \neq 0$ for all (finite) x and $\cos y$ and $\sin y$ do not vanish simultaneously.

Let $f(z) = e^z$. Then:

$$f(z) = e^x(\cos y + i \sin y)$$

where:

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y.$$

These satisfy the Cauchy-Riemann equations:

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x.$$

Since u and v are continuously differentiable, e^z is differentiable for all z, and:

$$f'(z) = e^z.$$

Trigonometric and Hyperbolic Functions

From Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding and subtracting:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

For any complex number z:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Similarly, hyperbolic functions are defined as:

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2}.$$

From these definitions, it follows:

$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z.$$

For a complex number z show that

i) $\sin(-z) = -\sin z$

ii) $\cos(-z) = \cos z$

iii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

iv) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

v) $\cos(z + 2\pi) = \cos z$ and $\sin(z + 2\pi) = \sin z$

vi) $\cos^2 z + \sin^2 z = 1$

vii) $\cosh^2 z - \sinh^2 z = 1.$

Example

Example

Let $z = x + yi$ and $f(z) = \sin z$. Show that $f(z)$ is differentiable and:

$$f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos z.$$

Solution:

Example

Example

Let $z = x + yi$ and $f(z) = \sin z$. Show that $f(z)$ is differentiable and:

$$f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos z.$$

Solution:

The complex sine function is defined as:

$$f(z) = \sin(x + yi) = \sin x \cosh y + i \cos x \sinh y.$$

From this, we identify:

$$u(x, y) = \sin x \cosh y, \quad v(x, y) = \cos x \sinh y.$$

Next, compute the partial derivatives:

$$u_x = \cos x \cosh y, \quad u_y = \sin x \sinh y,$$

$$v_x = -\sin x \sinh y, \quad v_y = \cos x \cosh y.$$

Since u, v, u_x, u_y, v_x, v_y are all continuous everywhere in $\mathbb{R}^2$, and the Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x$$

are satisfied, $f(z)$ is differentiable.
The derivative is given by:

$$f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y.$$

Now simplify:

$$\cos x \cosh y - i \sin x \sinh y = \cos x \cos(iy) - \sin x \sin(iy),$$

using the relations:

$$\cos(iy) = \cosh y, \quad \sin(iy) = i \sinh y.$$

This simplifies further to:

$$\cos(x + iy) = \cos z.$$

Thus:

$$f'(z) = \cos z.$$

Example

Example

Let $f(z) = z^2 e^{\cos z}$. Then find $f'(z)$.

Solution:

Example

Example

Let $f(z) = z^2 e^{\cos z}$. Then find $f'(z)$.

Solution:

We start by differentiating $f(z) = z^2 e^{\cos z}$ using the product rule:

$$f'(z) = \frac{d}{dz}\left(z^2\right) \cdot e^{\cos z} + z^2 \cdot \frac{d}{dz}\left(e^{\cos z}\right).$$

The derivative of z^2 is $2z$, so the first term becomes:

$$\frac{d}{dz}\left(z^2\right) \cdot e^{\cos z} = 2z e^{\cos z}.$$

For the second term, use the chain rule to differentiate $e^{\cos z}$:

$$\frac{d}{dz}\left(e^{\cos z}\right) = e^{\cos z} \cdot \frac{d}{dz}(\cos z) = -\sin z \cdot e^{\cos z}.$$

Now substitute back into the product rule:

$$f'(z) = 2z e^{\cos z} + z^2 \cdot (-\sin z \cdot e^{\cos z}) = e^{\cos z}\left(2z - z^2 \sin z\right).$$

Polar Form and Multi-Valuedness

The Polar form of a complex number z is:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg z$. The angle θ can be determined only within an arbitrary integer multiple of 2π, thus:

$$\theta = \arg z + 2k\pi, \quad k \in \mathbb{Z}.$$

If the exponent k is rational (e.g., $\frac{m}{n}$), then $f(z) = z^k$ is n-valued (since there are exactly n nth roots of a complex number z).

Example

Example

Let $z = 1 + i$, then find $z^{\frac{1}{3}}$.

Solution:

Example

Example

Let $z = 1 + i$, then find $z^{\frac{1}{3}}$.

Solution:

Expressing z in polar form:

$$z = \sqrt{2}e^{i\frac{\pi}{4}}$$

Taking the cube root:

$$z^{\frac{1}{3}} = \sqrt[3]{\sqrt{2}}e^{i\frac{\pi}{12}+i\frac{2k\pi}{3}}, \quad k = 0, 1, 2.$$

$$z_0 = 2^{\frac{1}{6}}e^{i\frac{\pi}{12}}, \quad z_1 = 2^{\frac{1}{6}}e^{i\frac{9\pi}{12}}, \quad z_2 = 2^{\frac{1}{6}}e^{i\frac{17\pi}{12}}.$$

Example

Example

Let $z = -1 + i\sqrt{3}$. Find $z^{\frac{1}{4}}$.

Solution:

Example

Example

Let $z = -1 + i\sqrt{3}$. Find $z^{\frac{1}{4}}$.

Solution:

Expressing z in polar form:

To find the modulus $|z|$:

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$

To determine the argument θ:

$$\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Thus:

$$z = 2e^{i\frac{2\pi}{3}}$$

Now compute the fourth root:

$$z^{\frac{1}{4}} = \sqrt[4]{2}e^{i\frac{\frac{2\pi}{3}}{4}+i\frac{2k\pi}{4}}, \quad k = 0, 1, 2, 3.$$

This simplifies:

$$z^{\frac{1}{4}} = 2^{\frac{1}{4}}e^{i\frac{\pi}{6}+i\frac{k\pi}{2}}, \quad k = 0, 1, 2, 3.$$

The four distinct roots are:

$$z_0 = 2^{\frac{1}{4}}e^{i\frac{\pi}{6}}, \quad z_1 = 2^{\frac{1}{4}}e^{i\frac{2\pi}{3}},$$

$$z_2 = 2^{\frac{1}{4}}e^{i\frac{7\pi}{6}}, \quad z_3 = 2^{\frac{1}{4}}e^{i\frac{11\pi}{6}}.$$

Thus, the fourth roots of z are evenly spaced on the complex plane.

The Logarithmic Function

Let $z = re^{i\theta}$ be a nonzero complex number expressed in its polar form, where $r = |z|$ and $\theta = \arg z$.
The logarithm of z is defined as:

$$\log z = \ln r + i(\theta + 2k\pi), \quad k \in \mathbb{Z}.$$

Key Points:

- The logarithmic function for $z \neq 0$ is **infinitely valued** because $\arg z$, the angle of z, can take infinitely many values due to its periodicity $\theta + 2k\pi$, where k is any integer.
- This definition is derived from expressing z as e^w, where $w = \ln r + i(\theta + 2k\pi)$ satisfies the equation $e^{i(b-\theta)} = 1 = e^{2k\pi i}$, allowing $b = \theta + 2k\pi$.

Properties:

- The real part of $\log z$ is $\ln r = \ln |z|$, corresponding to the modulus of z.
- The imaginary part is $\arg z + 2k\pi$, describing the infinitely many arguments of z.

Example

Example

Compute $\log(1+i)$.

Solution:

Example

Example

Compute $\log(1+i)$.

Solution:

Express $z = 1 + i$ in polar form:

$$z = \sqrt{2}e^{i\frac{\pi}{4}}$$

Applying the logarithm definition:

$$\log(1+i) = \ln|z| + i(\arg z + 2k\pi), \quad k \in \mathbb{Z}$$

Since $|z| = \sqrt{2}$, we get:

$$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

$$= \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

Thus, $\log(1+i)$ is infinitely valued.

Example

Example

Compute $\log(-\sqrt{3} - i)$.

Solution:

Example

Example

Compute $\log(-\sqrt{3} - i)$.

Solution:
Express $z = -\sqrt{3} - i$ in polar form:
To find the modulus $|z|$:

$$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2.$$

To determine the argument θ:

$$\theta = \arctan\left(\frac{-1}{-\sqrt{3}}\right).$$

Since both real and imaginary parts are negative, z lies in the third quadrant:

$$\theta = \pi + \arctan\left(\frac{1}{\sqrt{3}}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}.$$

Thus, z in polar form is:

$$z = 2e^{i\frac{7\pi}{6}}.$$

Applying the logarithm definition:

$$\log(-\sqrt{3} - i) = \ln|z| + i(\arg z + 2k\pi), \quad k \in \mathbb{Z}.$$

Substituting $|z| = 2$ and $\arg z = \frac{7\pi}{6}$:

$$\log(-\sqrt{3} - i) = \ln 2 + i\left(\frac{7\pi}{6} + 2k\pi\right), \quad k \in \mathbb{Z}.$$

Thus:

$$\log(-\sqrt{3} - i) = \ln 2 + i\left(\frac{7\pi}{6} + 2k\pi\right), \quad k \in \mathbb{Z}.$$

Principal Argument

The argument $\arg z$ of a complex number $z = x + yi$ represents the angle made by the line joining z to the origin with the positive real axis.

Principal Argument: The **Principal Argument** of z, denoted as $\mathrm{Arg}(z)$, is the unique value of $\arg z$ restricted to the interval:

$$-\pi < \mathrm{Arg}(z) \leq \pi.$$

For any complex number $z \neq 0$, the argument can take infinitely many values given by:

$$\arg z = \mathrm{Arg}(z) + 2k\pi, \quad k \in \mathbb{Z}.$$

The principal argument simplifies the representation of $\arg z$ by selecting the canonical angle within $(-\pi, \pi]$.

Example

Example

Find the Principal Argument $\mathrm{Arg}(-\sqrt{3} - i)$.

Solution:

Example

Example

Find the Principal Argument $\text{Arg}(-\sqrt{3} - i)$.

Solution:

The complex number $z = -\sqrt{3} - i$ lies in the third quadrant.

1. **Compute** $\arg z$**:**

$$\theta = \arctan\left(\frac{\text{Imaginary part}}{\text{Real part}}\right) = \arctan\left(\frac{-1}{-\sqrt{3}}\right).$$

Since both real and imaginary parts are negative, the angle needs to be adjusted to the third quadrant:

$$\theta = \pi + \arctan\left(\frac{1}{\sqrt{3}}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}.$$

1. **Check for the principal argument restriction:**

$$-\pi < \text{Arg}(-\sqrt{3} - i) \leq \pi.$$

The calculated angle $\frac{7\pi}{6}$ already lies within $(-\pi, \pi]$. Hence:

$$\text{Arg}(-\sqrt{3} - i) = \frac{7\pi}{6}.$$