# Wolkite University
## Mathematics Department
### Applied Mathematics III
### *Worksheet 6*

1. Evaluate the following integrals
   (a) $\displaystyle\int_1^4 \left(\frac{1}{t} - i\right)^2 \, dt$
   (b) $\displaystyle\int_0^\infty e^{-zt} \, dt \quad (\Re z > 0)$
   (c) $\displaystyle\int_0^{4+2i} \bar{z} \, dz$ along the curve $z = t^2 + ti$.
   (d) $\displaystyle\int_C \frac{z^2}{z-3} \, dz$ and $C$ is the circle $|z| = 1$.
   (e) $\displaystyle\int_C \frac{1}{z^2+2z+2} \, dz$ and $C$ is the circle $|z| = 1$.
   (f) $\displaystyle\int_C (z-l) \, dz$ and $C$ is the arc from $z = 0$ to $z = 2$ consisting of
      1. the semicircle $z = 1 + e^{ti} \quad (\pi < t < 2\pi)$;
      2. the segment $0 \leq x \leq 2$ of the real axis.
   (g) $\displaystyle\int_C \frac{z^2+1}{z^2-1} dz$, where $C$ is the circle $|z+1| = 1$.
   (h) $\displaystyle\int_C \frac{2z+5}{z^2-2z} dz$, where $C$ is the circle $|z-3| = 2$.

2. $f(z)$ is defined by

$$f(z) = \begin{cases} 1, & \text{if } \ y < 0 \\ 4y, & \text{if } \ y > 0 \end{cases}$$

   and $C$ is the arc from $z = -1-i$ to $z = 1+i$ along the curve $y = x^3$. Find $\displaystyle\int_C f(z) \, dz$.

3. Show that if $m$ and $n$ are integers,

$$\int_0^{2\pi} e^{mti} e^{-nti} \, dt = \begin{cases} 0, & \text{if } \ m \neq n \\ 2\pi, & \text{if } \ m = n \end{cases}$$

4. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t) \ \ (a < t < b)$. Show that if $w(t) = f(z(t))$, then

$$w'(t) = f'(z(t))z'(t)$$

   when $t = t_0$.

5. Show that if $C$ is the boundary of the triangle with vertices at the points $0, 3i$ and $-4$, oriented in the counterclockwise direction, then

$$\left| \int_C (e^z - \overline{z})\, dz \right| \leq 60.$$

6. Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of the following integrals:
   (a) $\displaystyle\int_C \frac{z}{2z+1} dz;$
   (b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)} dz;$
   (c) $\displaystyle\int_C \frac{\cosh z}{z^4} dz;$
   (d) $\displaystyle\int_C \frac{e^{-z}}{z - i\frac{\pi}{2}}$

7. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when
   (a) $g(z) = \dfrac{1}{z^2+4}$
   (b) $g(z) = \dfrac{1}{(z^2+4)^2}$
   (c) $g(z) = \dfrac{1}{(z^2+4)^3}$

8. Show that if $f$ is analytic within and on a simple closed contour $C$ and $z_0$ is not on $C$, then

$$\int_C \frac{f'(z)}{z - z_0} dz = \int_C \frac{f(z)}{(z - z_0)^2} dz.$$

9. Evaluate $\displaystyle\oint_C f(z)\, dz$ where $C$ is a closed curve oriented positively.
   (a) $f(z) = \dfrac{\cos(2z)}{z^5}$; $C$ is the circle $|z| = 1$.
   (b) $f(z) = \dfrac{e^z}{z-4}$; $C$ is the circle $|z - 6| = 3$.
   (c) $f(z) = \dfrac{\sin(2z)}{z^2 + (\frac{\pi}{2})^2}$; $C$ is the circle $|z - 1| = 1$.
   (d) $f(z) = \dfrac{e^{z^3}}{(z-i)^3}$; $C$ is the circle $|z - 2i = 3$.
   (e) $f(z) = \dfrac{\sin(z^2)}{z-4}$; $C$ is the circle $|z| = 3$.
   (f) $f(z) = \dfrac{\sin^2 z}{(z - \frac{\pi}{2})^3}$; $C$ is the circle $|z| = \pi$.