Applied Mathematics III

Unit 5

Line and Surface Integrals

Solomon Amsalu (Asst.Prof.)

Mathematics Department
Wolkite University

solomon.amsalu@wku.edu.et

April 16, 2025

Table of Content

Line Integrals

Recall that, the integral $\int_a^b f(x)dx$ of a continuous function f represents the definite integral of f over a closed interval $[a, b] = \{x : a \leq x \leq b\}$. In the line integral we shall integrate a function F along a curve, say C.

Definition

The Line Integral of a vector valued function $F(r)$ over a curve C parameterized by $r(t) = x(t)i + y(t)j + z(t)k$ for $a \leq t \leq b$ is defined by

$$\int_C F(r).dr = \int_a^b F(r(t)).\frac{dr}{dt}dt, \quad \text{where} \;\; dr = (dx, dy, dz).$$

When we write it componentwise, i.e, if $F = (F_1, F_2, F_3)$, then it becomes:

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')dt.$$

The curve C is called path of integration.

Example

Find the line integral $\int_C (x^2y\,dx + y^2z\,dy + z^2x\,dz)$, where C is parameterized by $r(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.

Solution:

Example

Find the line integral $\int_C (x^2y\,dx + y^2z\,dy + z^2x\,dz)$, where C is parameterized by $r(t) = (t, t^2, t^3)$ for $0 \leq t \leq 1$.

Solution:

1. Compute $r'(t) = (1, 2t, 3t^2)$.
2. Evaluate $x = t$, $y = t^2$, $z = t^3$.
3. Substitute into the integral:

$$\int_0^1 (t^2(t^2) + t^7(2t) + t^7(3t^2))\,dt = \int_0^1 (t^4 + 2t^8 + 3t^9)\,dt.$$

4. Solve:

$$\int_0^1 t^4\,dt = \frac{1}{5}, \quad \int_0^1 2t^8\,dt = \frac{2}{9}, \quad \int_0^1 3t^9\,dt = \frac{3}{10}5.$$

5. Final answer: $\frac{1}{5} + \frac{2}{9} + \frac{3}{10} = \frac{13}{18}$.

Scalar Function Line Integrals

Scalar-Valued Function: If f is scalar-valued, the line integral becomes:

$$\int_C f(x, y, z) dS = \int_a^b f(x(t), y(t), z(t)) \sqrt{(r'(t))^2} dt$$

Thus

$$\int_C f(x, y, z) dS = \int_a^b f(x(t), y(t), z(t)) \sqrt{r'(t) \cdot r'(t)} dt.$$

Special Case: For closed curves where $r(a) = r(b)$:

$$\oint_C \text{ is used instead of } \int_C .$$

Example

Evaluate the scalar line integral $\int_C x^2\, ds$, where C is the helix parameterized by $r(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$.

Solution:

Example

Evaluate the scalar line integral $\int_C x^2 \, ds$, where C is the helix parameterized by $r(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$.

Solution:

1. Compute $r'(t) = (-\sin t, \cos t, 1)$.
2. Find $|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$.
3. Substitute $x = \cos t$ and integrate:

$$\int_0^{2\pi} (\cos t)^2 \sqrt{2} \, dt = \sqrt{2} \int_0^{2\pi} \frac{1 + \cos(2t)}{2} \, dt.$$

4. Solve:

$$\sqrt{2} \left[\frac{1}{2} \int_0^{2\pi} 1 \, dt + \frac{1}{2} \int_0^{2\pi} \cos(2t) \, dt \right] = \sqrt{2} \, [\pi + 0] = \sqrt{2}\pi$$

Example

Evaluate $\int_L (xyz\,dx - \cos(xy)\,dy + y\,dz)$ where L is the line segment from $(0,1,1)$ to $(2,1,-3)$.

Solution:

Example

Evaluate $\int_L (xyz\,dx - \cos(xy)\,dy + y\,dz)$ where L is the line segment from $(0,1,1)$ to $(2,1,-3)$.

Solution:

1. Parametrize the line segment:
$$r(t) = (2t, 1, 1-4t),\ 0 \le t \le 1.$$
2. Compute $r'(t) = (2, 0, -4)$ and substitute $x = 2t,\ y = 1,\ z = 1 - 4t$.
3. Substitute into the integral:
$$\int_0^1 [(2t)(1)(1-4t)(2) - \cos(2t \cdot 1)(0) + (1)(-4)]\,dt.$$
$$\int_0^1 [2t(1-4t)(2) - 4]\,dt = \int_0^1 [4t - 16t^2 - 4]\,dt.$$
4. Solve:
$$\int_0^1 (4t - 16t^2 - 4)\,dt = \left[2t^2 - \frac{16t^3}{3} - 4t\right]_0^1 = 2 - \frac{16}{3} - 4 = -\frac{22}{3}.$$

Example

Determine the mass of a wire shaped as $C : r(t) = (e^t, e^{-t}, t)$ with $\sigma = e^t$.

Solution:

Example

Determine the mass of a wire shaped as $C : r(t) = (e^t, e^{-t}, t)$ with $\sigma = e^t$.

Solution:

1. Compute $r'(t) = (e^t, -e^{-t}, 1)$.
2. Find $|r'(t)| = \sqrt{(e^t)^2 + (-e^{-t})^2 + 1^2} = \sqrt{e^{2t} + e^{-2t} + 1}$.
3. Set up the integral:

$$M = \int_C \sigma \, ds = \int_0^1 e^t \sqrt{e^{2t} + e^{-2t} + 1} \, dt.$$

4. This integral requires numerical computation for exact evaluation.

Final Answer: $M = \int_0^1 e^t \sqrt{e^{2t} + e^{-2t} + 1} \, dt$

Example

Evaluate $\int_C (x\,dy - y\,dx)$ over the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Solution:

Example

Evaluate $\int_C (x\,dy - y\,dx)$ over the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Solution:

1. Parametrize the circle: $x = 2\cos t$, $y = 2\sin t$, for $0 \leq t \leq 2\pi$.
2. Compute dx and dy:
$$dx = -2\sin t\,dt, \quad dy = 2\cos t\,dt.$$
3. Substitute into the integral:
$$\int_C (x\,dy - y\,dx) = \int_0^{2\pi} (2\cos t)(2\cos t)\,dt - (2\sin t)(-2\sin t)\,dt.$$
4. Simplify:
$$\int_C (x\,dy - y\,dx) = \int_0^{2\pi} 4\cos^2 t\,dt + 4\sin^2 t\,dt.$$

$$\int_C (x\,dy - y\,dx) = 4\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 4\int_0^{2\pi} 1\,dt = 4\cdot 2\pi = 8\pi.$$

Line Integrals Independent of Path

Consider the line integral

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz), \tag{1}$$

where C is the path of integration. The line integral (1) is said to be **independent of path of integration** in a domain D if for every pair of endpoints A and B in D the integral (1) has the same value for all paths in D that begin at A and end at B.

Theorem

Let F_1, F_2 and F_3 be continuous functions in a set D and let $F = (F_1, F_2, F_3)$. A line integral (1) is independent of path in D if and only if $F = (F_1, F_2, F_3)$ is the gradient of some potential function f in D, i.e., if there exists a function f in D such that $F = \nabla f$, which is equivalent to saying that

$$F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y}\ \text{ and }\ F_3 = \frac{\partial f}{\partial z}.$$

Let F be a conservative vector valued function with potential function f and let C be a curve with coordinate function $x = x(t), y = y(t), z = z(t)$ for $a \leq t \leq b$. Then

$$\begin{aligned}
\int_C F.dr &= \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \\
&= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} \left(f(x(t), y(t), z(t)) \right) dt = f(B) - f(A),
\end{aligned}$$

where $f(B) = f(x(b), y(b), z(b)), \;\; f(A) = f(x(a), y(a), z(a))$, the end points of the curve C. Therefore, we have the following theorem and some times it is called **Fundamental Theorem of Line Integrals**

Theorem

If the integral (1) is independent of path in D, then

$$\int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A), \quad \textit{where} \;\; F = \nabla f.$$

Example

Evaluate:

1. $\int_C (2xdx + 2ydy + 4zdz)$ if C is a curve with initial point $A = (0, 0, 0)$ and terminal point $B = (2, 2, 2)$.

Solution:

Example

Evaluate:

1. $\int_C (2xdx + 2ydy + 4zdz)$ if C is a curve with initial point $A = (0, 0, 0)$ and terminal point $B = (2, 2, 2)$.

Solution:

1. Identify potential function $f(x, y, z) = x^2 + y^2 + 2z^2$. Then:

$$f(A) = f(0, 0, 0) = 0, \quad f(B) = f(2, 2, 2) = 2^2 + 2^2 + 2 \cdot 2^2 = 16.$$

Thus:

$$\int_C (2xdx + 2ydy + 4zdz) = f(B) - f(A) = 16 - 0 = 16.$$

Example

Evaluate:

1. $\int_C (e^z dx + 2y dy + xe^z dz)$ if C is a curve with initial point $A = (0, 1, 0)$ and terminal point $B = (-2, 1, 0)$.

Solution:

Example

Evaluate:

1. $\int_C (e^z dx + 2ydy + xe^z dz)$ if C is a curve with initial point $A = (0, 1, 0)$ and terminal point $B = (-2, 1, 0)$.

Solution:

1. Identify potential function $f(x, y, z) = xe^z + y^2$. Then:

$$f(A) = f(0, 1, 0) = (0 \cdot e^0) + 1^2 = 1,$$

$$f(B) = f(-2, 1, 0) = (-2 \cdot e^0) + 1^2 = -2 + 1 = -1.$$

Thus:

$$\int_C (e^z dx + 2ydy + xe^z dz) = f(B) - f(A) = -1 - 1 = -2.$$

Remark

1. The line integral (1) is independent of path in a domain D if and only if its value around every closed path in D is zero.
2. The line integral usually represents the work done by force F in the displacement of a body along path C. Hence if F has a potential function f, the line integral of F for displacement around any closed path is zero. In this case, the vector field F is called **conservative**, otherwise it is called **nonconservative**.

Definition

A domain D is said to be simply connected if every closed curve in D can be shrunk to any point in D.

Theorem

Suppose F_1, F_2 and F_3 are continuous and having continuous first order partial derivatives in a domain D and consider the line integral

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) \tag{2}$$

1. *If the line integral (2) is independent of path in D, then $CurlF = 0$. i.e.*
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \ \text{and} \ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$
2. *If $CurlF = 0$ in D and if D is simply connected then the line integral (2) is independent of path in D.*

Remark

In the plane, $\mathbb{R}^2$, the line integral $\int_C F(r).dr = \int_C (F_1 dx + F_2 dy)$ and $CurlF = 0$ means

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Example

Show that the following integrand is exact and evaluate the integral.

1. $\displaystyle\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy)$

Solution:

Remark

In the plane, $\mathbb{R}^2$, the line integral $\int_C F(r).dr = \int_C (F_1 dx + F_2 dy)$ and $CurlF = 0$ means

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Example

Show that the following integrand is exact and evaluate the integral.

1. $\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy)$

Solution:

1. Identify potential function $f(x, y) = e^x \cos y$. Then:

$$f(0, \pi) = e^0 \cos \pi = -1, \quad f(3, \frac{\pi}{2}) = e^3 \cos \frac{\pi}{2} = 0.$$

$$\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy) = f(3, \frac{\pi}{2}) - f(0, \pi) = 0 - (-1) = 1.$$

Example

Show that the following integrand is exact and evaluate the integral:

1. $\int_A^B \left(2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right)$, where $A = (0, 0, 1)$ and $B = (1, \frac{\pi}{4}, 2)$.

Solution:

Example

Show that the following integrand is exact and evaluate the integral:

1. $\int_A^B \left(2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right)$, where $A = (0,0,1)$ and $B = (1, \frac{\pi}{4}, 2)$.

Solution:

1. Identify potential function $f(x,y,z) = x^2yz^2 + \sin(yz)$. Then:

$$f(A) = f(0,0,1) = (0)^2(0)(1)^2 + \sin(0 \cdot 1) = 0,$$

$$f(B) = f(1, \frac{\pi}{4}, 2) = (1)^2\left(\frac{\pi}{4}\right)(2)^2 + \left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4} \cdot 2\right).$$

Evaluate:

$$f(B) = 4 \cdot \frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right) = \pi + 1$$

$$\int_A^B \left(2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right)$$

$=$

$$f(B) - f(A) = \pi + 1 - 0 = \pi + 1.$$

General Properties of Line integrals

Let F and G be continuous vector fields, C be a path joining points A and B. Furthermore, suppose that C is subdivided into two arcs C_1 and C_2 that have the same orientation as C. Then

1. $\int_C kF.dr = k \int_C F.dr$ for any constant k.
2. $\int_C (F + G).dr = \int_C F.dr + \int_C G.dr$
3. $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr$
4. If C' has an opposite orientation to that of C, then

$$\int_C F.dr = -\int_{C'} F.dr.$$

Example

Let C consist of the portion of the parabola $y = x^2$ in the xy-plane from $(0, 0)$ to $(2, 4)$ and a horizontal line from $(2, 4)$ to $(4, 4)$. Evaluate:

$$\int_C (y^2 dx + x^2 dy).$$

Solution:

Example

Let C consist of the portion of the parabola $y = x^2$ in the xy-plane from $(0, 0)$ to $(2, 4)$ and a horizontal line from $(2, 4)$ to $(4, 4)$. Evaluate:

$$\int_C (y^2 dx + x^2 dy).$$

Solution:

1. Divide C into two segments: C_1, the parabola $y = x^2$ for $0 \leq x \leq 2$, and C_2, the horizontal line $y = 4$ for $2 \leq x \leq 4$.
2. For C_1: Parametrize $x = t$, $y = t^2$, $dx = dt$, $dy = 2t\,dt$.

$$\int_{C_1} (y^2 dx + x^2 dy) = \int_0^2 (t^4 dt + t^2(2t)dt) = \int_0^2 (t^4 + 2t^3)dt.$$

Solve:

$$\int_{C_1} (y^2 dx + x^2 dy) = \left[\frac{t^5}{5} + \frac{t^4}{2}\right]_0^2 = \frac{32}{5} + 8 = \frac{72}{5}.$$

1. For C_2: Parametrize $x = t$, $y = 4$, $dx = dt$, $dy = 0$.

$$\int_{C_2} (y^2 dx + x^2 dy) = \int_2^4 (4^2 dt + 0) = \int_2^4 16 dt = 16(4-2) = 32.$$

2. Combine results:

$$\int_C (y^2 dx + x^2 dy) = \int_{C_1} (y^2 dx + x^2 dy) + \int_{C_2} (y^2 dx + x^2 dy)$$
$$= \frac{72}{5} + 32 = \frac{232}{5}.$$

Example

Evaluate the line integral $\int_C (3x^2y\,dx + (x^3 + z^2)\,dy + 2yz\,dz)$, where C consists of two segments:

- C_1: the line segment from $(0,0,0)$ to $(1,1,0)$,
- C_2: the curve $z = y^2$ in the plane $x = 1$, from $(1,1,0)$ to $(1,2,4)$.

Solution:

Example

Evaluate the line integral $\int_C (3x^2y\,dx + (x^3 + z^2)\,dy + 2yz\,dz)$, where C consists of two segments:

- C_1: the line segment from $(0, 0, 0)$ to $(1, 1, 0)$,
- C_2: the curve $z = y^2$ in the plane $x = 1$, from $(1, 1, 0)$ to $(1, 2, 4)$.

Solution:

1. Divide the curve into two segments C_1 and C_2 and compute the integral for each.
2. For C_1: Parametrize C_1 as $r(t) = (t, t, 0)$, $0 \leq t \leq 1$, so that $x = t$, $y = t$, $z = 0$, and $dx = dt$, $dy = dt$, $dz = 0$. Substitute into the integral:

$$\int_{C_1} (3x^2y\,dx + (x^3 + z^2)\,dy + 2yz\,dz) = \int_0^1 \left(3t^2t \cdot dt + (t^3 + 0^2) \cdot dt + 0\right)$$

Simplify:

$$= \int_0^1 (3t^3 + t^3)\,dt = \int_0^1 4t^3\,dt = \left[t^4\right]_0^1 = 1 - 0 = 1.$$

1. For C_2: Parametrize C_2 as $r(t) = (1, t, t^2)$, $1 \leq t \leq 2$, so that $x = 1$, $y = t$, $z = t^2$, and $dx = 0$, $dy = dt$, $dz = 2t\,dt$. Substitute into the integral:

$$\int_{C_2} (3x^2 y\,dx + (x^3 + z^2)\,dy + 2yz\,dz) = \int_1^2 \left(3 \cdot 1^2 \cdot t \cdot 0 + (1^3 + t^4) \cdot dt +\right.$$

Simplify:

$$\int_{C_2} = \int_1^2 \left(0 + (1 + t^4) + 4t^4\right)\,dt = \int_1^2 (1 + 5t^4)\,dt.$$

Solve:

$$\int_{C_2} = \left[t + \frac{5t^5}{5}\right]_1^2 = \left[t + t^5\right]_1^2 = (2 + 32) - (1 + 1) = 34 - 2 = 32.$$

2. Combine results:

$$\int_C = \int_{C_1} + \int_{C_2} = 1 + 32 = 33.$$

Green's Theorem

Over a plane region, double integrals can be transformed into line integrals over the boundary of the regions and conversely. This can be done using Green's Theorem which is stated stated below.

Theorem (Green's Theorem)

Let R be a closed bounded region in the $xy-$plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives every where in some domain containing R, (i.e.$(F_1)_y$ and $(F_2)_x$ are continuous in R.) Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy), \tag{3}$$

where C is the boundary of R.

Example

1. Use Green's Theorem to evaluate $\int_C (x^2ydx + ydy)$ over the triangular path $(0,0)$ to $(1,0)$, $(1,0)$ to $(1,2)$ and $(1,2)$ back to $(0,0)$.
2. Find the work done by the force field $F(x,y) = (e^x - y^3)i + (\cos y + x^3)j$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction.

Solution:

Example

1. Use Green's Theorem to evaluate $\int_C (x^2 ydx + ydy)$ over the triangular path $(0,0)$ to $(1,0)$, $(1,0)$ to $(1,2)$ and $(1,2)$ back to $(0,0)$.
2. Find the work done by the force field $F(x,y) = (e^x - y^3)i + (\cos y + x^3)j$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction.

Solution:

1. Apply Green's Theorem:

$$\iint_R \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2 y) \right) dxdy.$$

Compute partial derivatives:

$$\frac{\partial}{\partial x}(y) = 0, \quad \frac{\partial}{\partial y}(x^2 y) = x^2.$$

The double integral becomes:

$$\iint_R (0 - x^2)\, dxdy = -\iint_R x^2\, dxdy,$$

where R is the triangular region. The limits are $0 \leq x \leq 1$ and $0 \leq y \leq 2x$. Evaluate:

$$-\int_0^1 \int_0^{2x} x^2\, dy\, dx = -\int_0^1 x^2(2x)\, dx = -\int_0^1 2x^3\, dx = -\left[\frac{x^4}{2}\right]_0^1 = -\frac{1}{2}.$$

Final Answer: $\int_C \left(x^2y\, dx + y\, dy\right) = -\frac{1}{2}$.

1. For the work done, apply Green's Theorem:

$$\iint_R \left(\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3) \right) dxdy.$$

Compute partial derivatives:

$$\frac{\partial}{\partial x}(\cos y + x^3) = 3x^2, \quad \frac{\partial}{\partial y}(e^x - y^3) = -3y^2.$$

The double integral becomes:

$$\iint_R (3x^2 + 3y^2)\, dxdy,$$

where R is the unit circle. Change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,drd\theta$, and $r \in [0, 1]$, $\theta \in [0, 2\pi]$. Evaluate:

$$\iint_R (3x^2 + 3y^2)\, dxdy = \int_0^{2\pi} \int_0^1 3r^2\, r\, drd\theta = \int_0^{2\pi} \int_0^1 3r^3\, drd\theta = \frac{3\pi}{2}$$

Application of Green's Theorem to Area

Green's Theorem can be used to find areas of a plane region.
Let R be a plane region with boundary C.

1. Area of the region R in cartesian coordinates. First choose $F_1 = 0$ and $F_2 = x$. Then, as in the above,

$$\iint_R dxdy = \oint_C xdy$$

and then choose $F_1 = -y$ and $F_2 = 0$ to get

$$\iint_R dxdy = -\oint_C ydx$$

By adding up the two we get

$$2\iint_R dxdy = \oint_C xdy - \oint_C ydx = \oint_C (xdy - ydx)$$

Therefore, the area $A(R)$ of the region bounded by the curve C is given by:

$$A(R) = \iint_R dxdy = \frac{1}{2}\oint_C (xdy - ydx)$$

Example

Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution:

Example

Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution:

- Parametrize the ellipse: $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$.
- Compute $dx = -a\sin\theta\, d\theta$ and $dy = b\cos\theta\, d\theta$.
- Substitute into the formula for area:

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} [a\cos\theta(b\cos\theta) - b\sin\theta(-a\sin\theta)]\, d\theta.$$

- Simplify:

$$A = \frac{1}{2}\int_0^{2\pi} \left[ab\cos^2\theta + ab\sin^2\theta\right] d\theta = \frac{1}{2}\int_0^{2\pi} ab\, d\theta = \frac{1}{2}ab(2\pi) = \pi ab.$$

2. Area of a plane region in polar coordinates.
Let $x = r\cos\theta$ and $y = r\sin\theta$, where (r, θ) is the polar coordinate of point (x, y). Then
$dx = \cos\theta dr - r\sin\theta d\theta, \quad dy = \sin\theta dr + r\cos\theta d\theta$. Hence (4) becomes

$$A(R) = \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\oint_C r^2 d\theta.$$

Example

find the area of the region bounded by the cardioid $r = a(1 - \cos\theta)$, where $0 \leq \theta \leq 2\pi$ and a is a positive constant.

Solution:

Solution:

1. The area of the region is:

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta.$$

2. Substitute $r = a(1 - \cos\theta)$:

$$A = \frac{1}{2}\int_0^{2\pi} \left[a^2(1 - \cos\theta)^2\right] \, d\theta = \frac{1}{2}\int_0^{2\pi} \left[a^2(1 - 2\cos\theta + \cos^2\theta)\right] \, d\theta.$$

3. Split the integral:

$$A = \frac{a^2}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) \, d\theta.$$

4. Use the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$:

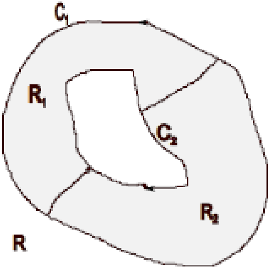
$$A = \frac{a^2}{2}\left[\int_0^{2\pi} 1 \, d\theta - 2\int_0^{2\pi} \cos\theta \, d\theta + \frac{1}{2}\int_0^{2\pi} (1 + \cos 2\theta) \, d\theta\right].$$

5. Simplify:

$$A = \frac{a^2}{2}\left[2\pi + \frac{1}{2}(2\pi)\right] = \frac{a^2}{2}(2\pi + \pi) = \frac{a^2}{2} \cdot 3\pi = \frac{3\pi a^2}{2}.$$

Green's Theorem for Multiply Connected Regions

Recall that, a region in $\mathbb{R}^2$ is called **simply connected** if it is connected and has no holes, and is called **multiply connected** if it is connected but it has finitely many holes.
Now consider the vector field $F = (F_1, F_2)$ which is continuously differentiable over the plane region R, which is multiply connected as shown in Figure below



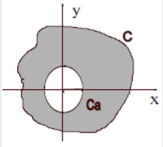
First divide R into simply connected regions R_1 and R_2. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$
$$= \oint_{C_1} (F_1 dx + F_2 dy) + \oint_{C_2} (F_1 dx + F_2 dy),$$

where the curves C_1 and C_2 are the boundaries of the regions R_1 and R_2 respectively. The orientation of the curves should be in such a way that when traveling along the curves the region should be to the left.

Example

Evaluate the integral $\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2}\right)$, if C is a piecewise smooth simply closed curve oriented counterclockwise such that C incloses the origin. Consider the Figure below.



Solution:

Solution:

1. By Green's Theorem, let R be the region bounded by the curves C_a and C, and compute:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2}.$$

Thus:

$$\iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) dA = 0.$$

This implies:

$$\oint_C \left(\frac{-y\,dx + x\,dy}{x^2 + y^2} \right) + \oint_{-C_a} \left(\frac{-y\,dx + x\,dy}{x^2 + y^2} \right) = 0.$$

2. Hence:

$$\oint_C \left(\frac{-y\,dx + x\,dy}{x^2+y^2} \right) = \oint_{C_a} \left(\frac{-y\,dx + x\,dy}{x^2+y^2} \right).$$

3. Parametrize C_a as $x = a\cos t$, $y = a\sin t$, $0 \leq t \leq 2\pi$. Then:

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt.$$

4. Substitute into the integral:

$$\oint_{C_a} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)\,dt + (a\cos t)(a\cos t)\,dt}{(a\cos t)^2 + (a\sin t)^2}.$$

5. Simplify:

$$\oint_{C_a} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^{2\pi} \frac{a^2 \sin^2 t + a^2 \cos^2 t}{a^2}\,dt = \int_0^{2\pi} 1\,dt.$$

6. Evaluate:

$$\int_0^{2\pi} 1\,dt = 2\pi.$$

7. Therefore:

$$\oint_C \left(\frac{-y\,dx + x\,dy}{x^2 + y^2} \right) = 2\pi.$$

Final Answer: 2π.

Example

Evaluate:

$$\oint_C \left(\frac{y\,dx - x\,dy}{x^2 + y^2} \right),$$

where C is the boundary of a multiply connected region consisting of an outer circle C_{outer} with radius 2, enclosing an inner circle C_{inner} with radius 1, both centered at the origin and oriented counterclockwise.

Solution:

Example

Evaluate:

$$\oint_C \left(\frac{y\,dx - x\,dy}{x^2 + y^2} \right),$$

where C is the boundary of a multiply connected region consisting of an outer circle C_{outer} with radius 2, enclosing an inner circle C_{inner} with radius 1, both centered at the origin and oriented counterclockwise.

Solution:

1. Divide the region R into two parts: R_1, the area between the inner and outer circles, and R_2, the area enclosed by the inner circle.
2. Apply Green's Theorem to R:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_{C_{outer}} (F_1\,dx + F_2\,dy) - \oint_{C_{inner}} (F_1\,dx + F_2\,dy$$

where $F_1 = y/(x^2 + y^2)$ and $F_2 = -x/(x^2 + y^2)$.

3. Compute the curl of F:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right), \quad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right).$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0,$$

because F is divergence-free everywhere in R. Thus, the double integral over R vanishes:

$$\iint_R 0\, dxdy = 0.$$

4. Parametrize C_{outer} and C_{inner}. For C_{outer}, let $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$:

$$dx = -2\sin t\, dt, \quad dy = 2\cos t\, dt.$$

Substitute into the integral:

$$\oint_{C_{outer}} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{(2\sin t)(-2\sin t) - (2\cos t)(2\cos t)}{4}\, dt.$$

Simplify:

$$\oint_{C_{outer}} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-4\sin^2 t - 4\cos^2 t}{4}\,dt = \int_0^{2\pi} -1\,dt = -2\pi.$$

5. For C_{inner}, let $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$:

$$dx = -\sin t\,dt, \quad dy = \cos t\,dt.$$

Substitute into the integral:

$$\oint_{C_{inner}} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{(\sin t)(-\sin t) - (\cos t)(\cos t)}{1}\,dt.$$

Simplify:

$$\oint_{C_{inner}} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} -1\,dt = -2\pi.$$

6. Combine results:

$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_{C_{outer}} \frac{y\,dx - x\,dy}{x^2+y^2} - \oint_{C_{inner}} \frac{y\,dx - x\,dy}{x^2+y^2}.$$

$$\oint_C = -2\pi - (-2\pi) = 0.$$

Surface Integrals

In the previous sections, we worked on integrals of vector fields over curves. Now, we focus on integrals of vector fields over surfaces. Let us begin with some basic concepts about surfaces.

Surface Parametrization: While a curve in $\mathbb{R}^3$ is represented parametrically as:

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad \text{for } a \leq t \leq b,$$

a surface is defined by parametric functions of two variables:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

where (u, v) are parameters in a set within the uv-plane.
For a surface, the position vector is:

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k.$$

Here, $r(u, v)$ can be considered as a vector in $\mathbb{R}^3$ originating at the origin and terminating at $(x(u, v), y(u, v), z(u, v))$, a point on the surface.

Parametrization of Some Surfaces

Examples of Surface Parametrization:

1. **Cylinder:** For $x^2 + y^2 = a^2$, $-1 \leq z \leq 1$, the parametric representation is:

$$r(u, v) = a\cos u\, i + a\sin u\, j + v\, k.$$

2. **Sphere:** For $x^2 + y^2 + z^2 = a^2$, the parametric representation is:

$$r(u, v) = a\cos v\cos u\, i + a\cos v\sin u\, j + a\sin v\, k.$$

3. **Cone:** For $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq T$, the parametric representation is:

$$r(u, v) = u\cos v\, i + u\sin v\, j + u\, k,$$

where $0 \leq u \leq T$ and $0 \leq v \leq 2\pi$.

A surface parametrization $r(u, v)$ is called **simple** if it does not fold over or intersect itself. This means $r(u_1, v_1) = r(u_2, v_2)$ occurs only when $u_1 = u_2$ and $v_1 = v_2$.

Normal Vector and Tangent Plane to a Surface

Tangent Vector: For a curve C with coordinate functions $x(t), y(t), z(t)$, the tangent vector is:

$$T = x'(t_0)i + y'(t_0)j + z'(t_0)k,$$

which is tangent to the curve at point $P_0 = (x(t_0), y(t_0), z(t_0))$.
Normal Vector: For a surface Ω in $\mathbb{R}^3$ with coordinate functions $x(u, v), y(u, v), z(u, v)$, the normal vector at a point $P_0 = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ can be found using the tangent vectors:

$$T_{u_0} = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k,$$

$$T_{v_0} = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k.$$

Assuming these vectors are non-zero, they lie in the tangent plane at P_0, and the normal vector N is given by:

Normal Vector Expression: The normal vector at (u_0, v_0) can be expressed as:

$$N(P_0) = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} i - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} j + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} k,$$

where the partial derivatives are evaluated at (u_0, v_0).
For any point (u, v) on the surface, the normal line to the tangent plane is given by $N = r_u \times r_v$, and the corresponding unit normal vector n is:

$$n = \frac{N}{\|N\|} = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$

Example

Find the normal vector and unit normal vector for the parametrized sphere:

$$r(u, v) = a \cos u \sin v\, i + a \sin v \sin u\, j + a \sin v\, k.$$

Solution:

Example

Find the normal vector and unit normal vector for the parametrized sphere:

$$r(u,v) = a\cos u \sin v\, i + a\sin v \sin u\, j + a\sin v\, k.$$

Solution:

1. Compute r_u and r_v:

$$r_u = \frac{\partial r}{\partial u} = -a\sin u \sin v\, i + a\cos u \sin v\, j + 0\, k,$$
$$r_v = \frac{\partial r}{\partial v} = a\cos u \cos v\, i + a\sin u \cos v\, j - a\sin v\, k.$$

2. Compute $N = r_u \times r_v$:

$$N = \begin{vmatrix} i & j & k \\ -a\sin v \sin u & a\sin v \cos u & 0 \\ -a\cos v \cos u & -a\cos v \sin u & -a\sin v \end{vmatrix}.$$

Expand the determinant:

$$N = -a^2 \sin^2 v \cos u\, i - a^2 \sin^2 v \sin u\, j - a^2 \sin v \cos v\, k.$$

3. Compute the magnitude of N:

$$\|N\| = a^2\sqrt{\sin^4 v \cos^2 u + \sin^4 v \sin^2 u + \sin^2 v \cos^2 v}.$$

Using $\cos^2 u + \sin^2 u = 1$, simplify:

$$\|N\| = a^2\sqrt{\sin^4 v + \sin^2 v \cos^2 v} = a^2 \sin v\sqrt{\sin^2 v + \cos^2 v}.$$

Since $\sin^2 v + \cos^2 v = 1$, we have:

$$\|N\| = a^2 \sin v.$$

4. Compute the unit normal vector:

$$n = \frac{N}{\|N\|} = -\sin v \cos u\, i - \sin v \sin u\, j - \cos v\, k.$$

Final Answer: The normal vector is:

$$N = -a^2 \sin^2 v \cos u\, i - a^2 \sin^2 v \sin u\, j - a^2 \sin v \cos v\, k,$$

and the unit normal vector is:

$$n = -\sin v \cos u\, i - \sin v \sin u\, j - \cos v\, k.$$

Remark

If the surface Ω is a surface represented by the equation $g(x, y, z) = 0$, then the unit normal vector is given by

$$n = \frac{1}{\|\nabla g\|} \nabla g.$$

where $\nabla g = \frac{\partial g}{\partial x}\, i + \frac{\partial g}{\partial y}\, j + \frac{\partial g}{\partial z}\, k$.

Example

1. Find the equation of the tangent plane to the surface given by $r(u, v) = ui + (u + v)j + (u + v^2)k$ at a point$(2, 4, 6)$.
2. If Ω is the sphere $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$ and $a \neq 0$, then find n.

Solution:

Example

1. Find the equation of the tangent plane to the surface given by $r(u,v) = ui + (u+v)j + (u+v^2)k$ at a point$(2,4,6)$.
2. If Ω is the sphere $f(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$ and $a \neq 0$, then find n.

Solution: The surface is parametrized as:

$$r(u,v) = u\,i + (u+v)\,j + (u+v^2)\,k.$$

Compute the partial derivatives:

$$r_u = \frac{\partial r}{\partial u} = 1\,i + 1\,j + 1\,k,$$
$$r_v = \frac{\partial r}{\partial v} = 0\,i + 1\,j + 2v\,k.$$

At the point (u,v), the tangent vectors are:

$$r_u = i + j + k, \quad r_v = j + 2v\,k.$$

At the given point $(2,4,6)$, solve for u and v.

From the parametrization:

$$x = u, \quad y = u + v, \quad z = u + v^2.$$

Substituting $x = 2, y = 4, z = 6$:

$$u = 2, \quad 2 + v = 4 \implies v = 2, \quad 2 + 2^2 = 6.$$

Thus, at $(u, v) = (2, 2)$, the tangent plane has normal vector:

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}.$$

Expand the determinant:

$$N = i(1 \cdot 2 - 1 \cdot 1) - j(1 \cdot 2 - 1 \cdot 0) + k(1 \cdot 1 - 1 \cdot 0),$$
$$N = i(2 - 1) - j(2 - 0) + k(1 - 0),$$
$$N = i - 2j + k.$$

The equation of the tangent plane at $(2, 4, 6)$ is:

$$N \cdot (r - r_0) = 0,$$

where $r_0 = 2i + 4j + 6k$.

Substitute:

$$(i - 2j + k) \cdot (x - 2, y - 4, z - 6) = 0.$$

Expand:

$$(x - 2) - 2(y - 4) + (z - 6) = 0.$$

Simplify:

$$x - 2y + z = 2.$$

The tangent plane is:

$$x - 2y + z = 2.$$

Find the unit normal vector for the sphere: The sphere is given by:

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.$$

The gradient of $f(x, y, z)$ is:

$$\nabla f = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k = 2x\, i + 2y\, j + 2z\, k.$$

At a point (x, y, z) on the sphere, the unit normal vector is:

$$n = \frac{\nabla f}{\|\nabla f\|}.$$

Compute $\|\nabla f\|$:

$$\|\nabla f\| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2} = 2\sqrt{x^2 + y^2 + z^2}.$$

Since $x^2 + y^2 + z^2 = a^2$ on the sphere:

$$\|\nabla f\| = 2a.$$

Thus, the unit normal vector is:

$$n = \frac{2x\, i + 2y\, j + 2z\, k}{2a} = \frac{x}{a}\, i + \frac{y}{a}\, j + \frac{z}{a}\, k.$$

Suppose Ω represents a surface in $\mathbb{R}^3$ with equation $z = g(x, y)$ and let R be its projection on the $xy-$plane. If g has continuous first partial derivatives on R, then the surface area of Ω is

$$\text{Area of } \Omega = \iint_R \sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)} dA.$$

But the integrant is the norm of the normal vector $N(x, y)$ to the surface, that is,

$$\text{Area of } \Omega = \iint_R \|N(x, y)\| dA.$$

Definition

A surface S is called a **smooth** surface if the unit normal vector n is continuous on S and surface S is called **piecewise smooth** if it consists of finitely many smooth portions.

Example

A sphere is smooth but a cube is picewise smooth.

Definition

Suppose S is a smooth surface parameterized by $r(u, v)$ with normal vector $N(u, v) = r_u \times r_v$. Let F be a continuous function on S. Then the surface integral of F over S is denoted by $\iint_S F(x, y, z)d\sigma$ and is defined by

$$\iint_S F(x, y, z)d\sigma = \iint_R F(r(u, v))\|N(u, v)\|dudv$$

and if F is a vector field then the surface integral of F over S $\iint_S F(x, y, z)d\sigma$ is defined by

$$\iint_S F(x, y, z)d\sigma = \iint_R F(r(u, v)).N(u, v)dudv.$$

Example: Surface Integral of a Scalar Field

Example

Evaluate $\iint_S (x+y)\sigma$ where S is the portion of the cylinder $x^2+y^2=3$ between the planes $z=0$ and $z=6$.

Solution:

Example: Surface Integral of a Scalar Field

Example

Evaluate $\iint_S (x+y)\sigma$ where S is the portion of the cylinder $x^2+y^2=3$ between the planes $z=0$ and $z=6$.

Solution:

1. **Parametrize the surface:**
$$r(u,v) = \sqrt{3}\cos u\, i + \sqrt{3}\sin u\, j + v\, k.$$
2. **Compute partial derivatives:**
$$r_u = -\sqrt{3}\sin u\, i + \sqrt{3}\cos u\, j + 0\, k,$$
$$r_v = 0\, i + 0\, j + 1\, k.$$
3. **Find the normal vector:**
$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ -\sqrt{3}\sin u & \sqrt{3}\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Expanding:

$$N = \sqrt{3}\cos u\, i + \sqrt{3}\sin u\, j.$$

4. Compute $\|N\|$:

$$\|N\| = \sqrt{3}.$$

5. Compute the integral:

$$\iint_S (x+y)\, d\sigma = 3\int_0^{2\pi}\int_0^{6} (\cos u + \sin u)\, du\, dv.$$

$$\iint_S (x+y)\, d\sigma = 3\int_0^{2\pi} (\cos u + \sin u)\, du \int_0^{6} 1\, dv.$$

Since $\int_0^{2\pi} \cos u\, du = 0$ and $\int_0^{2\pi} \sin u\, du = 0$, the integral evaluates to:

$$\iint_S (x+y)\, d\sigma = 0.$$

Example: Surface Integral of a Vector Field

Example

Evaluate:

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the upper hemisphere of the sphere $x^2 + y^2 + z^2 = a^2$, and the vector field is given by:

$$\mathbf{F} = x\,i + y\,j + z\,k.$$

Solution:

Example: Surface Integral of a Vector Field

Example

Evaluate:

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the upper hemisphere of the sphere $x^2 + y^2 + z^2 = a^2$, and the vector field is given by:

$$\mathbf{F} = x\,i + y\,j + z\,k.$$

Solution:

1. **Parametrize the surface:** The hemisphere can be parametrized using spherical coordinates:

$$r(u, v) = a \cos u \sin v\, i + a \sin u \sin v\, j + a \cos v\, k,$$

where $0 \leq u \leq 2\pi$, $0 \leq v \leq \frac{\pi}{2}$.

2. **Compute the tangent vectors:**

$$r_u = \frac{\partial r}{\partial u} = -a \sin u \sin v \, i + a \cos u \sin v \, j + 0 \, k,$$

$$r_v = \frac{\partial r}{\partial v} = a \cos u \cos v \, i + a \sin u \cos v \, j - a \sin v \, k.$$

3. **Find the normal vector:** Compute $N = r_u \times r_v$:

$$N = \begin{vmatrix} i & j & k \\ -a \sin u \sin v & a \cos u \sin v & 0 \\ a \cos u \cos v & a \sin u \cos v & -a \sin v \end{vmatrix}.$$

Expanding:

$$N = a^2 \sin v (\cos u \, i + \sin u \, j + \cos v \, k).$$

4. **Compute the surface integral:** The surface integral of the vector field is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(r(u, v)) \cdot N \, dudv.$$

Substitute $\mathbf{F}(r)$:

$$\mathbf{F}(r) = a \cos u \sin v \, i + a \sin u \sin v \, j + a \cos v \, k.$$

Compute $\mathbf{F} \cdot N$:

$(a \cos u \sin v \; i + a \sin u \sin v \; j + a \cos v \; k) \cdot (a^2 \sin v(\cos u \; i + \sin u \; j + \cos v \; k))$.

Expand:

$$a^3 \sin^2 v \cos^2 u + a^3 \sin^2 v \sin^2 u + a^3 \sin v \cos v.$$

Factor $a^3 \sin v$:

$$a^3 \sin v(\sin v + \cos v).$$

Integrate over $0 \leq u \leq 2\pi$, $0 \leq v \leq \frac{\pi}{2}$:

$$\int_0^{2\pi} du \int_0^{\pi/2} a^3 \sin v(\sin v + \cos v) dv.$$

Solve:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi a^3 \int_0^{\pi/2} \sin v(\sin v + \cos v) dv.$$

Evaluate:

$$\int_0^{\pi/2} (\sin^2 v + \sin v \cos v) dv.$$

Using $\frac{1}{2}(1 - \cos 2v)$ for $\sin^2 v$, and integrating:

$$\frac{1}{2}v - \frac{1}{4}\sin 2v + \frac{1}{2}\sin^2 v \Big|_0^{\pi/2}.$$

$$\frac{1}{2}\frac{\pi}{2} - 0 + \frac{1}{2}(1-0).$$

$$\frac{\pi}{4} + \frac{1}{2} = \frac{\pi}{4} + \frac{2}{4} = \frac{\pi+2}{4}.$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi a^3 \frac{\pi+2}{4}.$$

$$\boxed{\frac{2\pi a^3(\pi+2)}{4}.}$$

Flux of a Fluid Across a Surface

Suppose a fluid moves in a region of space with velocity v. The volume of fluid crossing a surface S per unit time is known as the **flux** across S. The surface integral of a vector function F over S represents the flux when:

$$F = \rho v,$$

where ρ is the fluid density and v is the velocity of the flow. Hence, the above surface integral is known as the **flux integral**. If $F = (F_1, F_2, F_3)$ and $N = (N_1, N_2, N_3)$, then:

$$\iint_S F \cdot n \, dA = \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3) \, du \, dv.$$

Parametrization of a Surface

For surface integrals, we parameterize surfaces as they are two-dimensional. A surface S can be represented as:

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k, \quad (u,v) \in R,$$

where R is some region in the uv-plane.

Normal Vector to a Surface: A normal vector N to a surface S given in parametric form is:

$$N = r_u \times r_v.$$

The unit normal vector n is defined as:

$$n = \frac{N}{\|N\|} = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$

If a surface S is represented by an implicit equation $g(x,y,z) = 0$, then:

$$n = \frac{\nabla g}{\|\nabla g\|}, \quad \text{where } \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right).$$

Example: Flux Across a Surface

Example

Let S be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane, with orientation in the upward direction. Find the flux Φ of the vector field:

$$F(x, y, z) = (x, y, z)$$

across S.

Solution:

Example: Flux Across a Surface

Example

Let S be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane, with orientation in the upward direction. Find the flux Φ of the vector field:

$$F(x, y, z) = (x, y, z)$$

across S.

Solution:

1. **Parametrize the surface:** The surface is given as $z = 1 - x^2 - y^2$, so we set:

$$r(x, y) = x\,i + y\,j + (1 - x^2 - y^2)\,k.$$

The projection of S onto the xy-plane is the disk:

$$x^2 + y^2 \leq 1.$$

2. **Compute the normal vector:** The normal vector is given by:

$$N = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{vmatrix}.$$

Compute derivatives:

$$r_x = (1, 0, -2x), \quad r_y = (0, 1, -2y).$$

$$N = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}.$$

Expanding:

$$N = (2x\, i + 2y\, j + k).$$

3. **Compute the flux integral:** The flux integral is:

$$\iint_S F \cdot n \, dA = \iint_R (x, y, z) \cdot (2x, 2y, 1) \, dA.$$

$$= \iint_R (2x^2 + 2y^2 + (1 - x^2 - y^2)) \, dA.$$

   Simplify:

$$\iint_R (x^2 + y^2 + 1) \, dA.$$

4. **Convert to polar coordinates:** Using $x = r\cos\theta, y = r\sin\theta, dA = r \, dr \, d\theta$, we transform the integral:

$$\int_0^{2\pi} \int_0^1 (r^2 + 1) r \, dr \, d\theta.$$

Evaluating the r-integral:

$$\int_0^1 (r^3 + r)\, dr = \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Evaluating the θ-integral:

$$\int_0^{2\pi} 1\, d\theta = 2\pi.$$

Final Answer:

$$\Phi = 2\pi \times \frac{3}{4} = \frac{3\pi}{2}.$$

Surface Area

If Ω is a piecewise smooth surface, then its area is given by:

$$\text{Area of } \Omega = \iint_{\Omega} dA.$$

Since $\|N\| = \|r_u \times r_v\|$ represents the area of a parallelogram with adjacent side vectors r_u and r_v, we express dA as:

$$dA = \|r_u \times r_v\| dudv.$$

Thus, the surface area is:

$$\text{Area of } \Omega = \iint_{\Omega} dA = \iint_{R} \|r_u \times r_v\| dudv,$$

where R is the projection of Ω onto the uv-plane.

Example: Surface Area of a Paraboloid

Example

Find the surface area of the portion of the paraboloid

$$z = x^2 + y^2$$

that lies above the disk $x^2 + y^2 \leq a^2$.

Solution:

Step 1: Parameterization of the Surface

The paraboloid can be expressed in terms of parameters (x, y):

$$r(x, y) = x\,i + y\,j + (x^2 + y^2)\,k.$$

The projection onto the xy-plane is the disk $x^2 + y^2 \leq a^2$. Switching to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r^2.$$

Thus, the surface is parametrized as:

$$r(r, \theta) = r\cos\theta\,i + r\sin\theta\,j + r^2\,k,$$

where $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$.

Step 2: Compute Normal Vectors

Compute the partial derivatives:

$$r_r = \frac{\partial r}{\partial r} = \cos\theta\, i + \sin\theta\, j + 2r\, k.$$

$$r_\theta = \frac{\partial r}{\partial \theta} = -r\sin\theta\, i + r\cos\theta\, j + 0\, k.$$

Compute the cross product:

$$N = r_r \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}.$$

Expanding:

$$N = (-2r^2\cos\theta)\, i + (-2r^2\sin\theta)\, j + (r)\, k.$$

Compute the magnitude:

$$\|N\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}.$$

Step 3: Compute the Surface Area Integral

The surface area is:

$$\text{Area} = \iint_{\Omega} \|N\| \, dA.$$

Substituting $dA = r\,dr\,d\theta$, we get:

$$\text{Area} = \int_0^{2\pi} \int_0^a r\sqrt{4r^2+1}\,dr\,d\theta.$$

Step 5: Evaluate the Integral

Let $u = 4r^2 + 1$, then $du = 8r\,dr$. Rewriting:

$$\int_0^a r\sqrt{4r^2+1}\,dr = \frac{1}{8}\int_1^{4a^2+1} \sqrt{u}\,du.$$

Evaluating:

$$\frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_1^{4a^2+1}.$$

$$= \frac{1}{12}\left((4a^2+1)^{3/2} - 1^{3/2}\right).$$

Computing the θ-integral:

$$\int_0^{2\pi} d\theta = 2\pi.$$

Thus, the total surface area is:

$$\text{Area} = 2\pi \times \frac{1}{12}\left((4a^2+1)^{3/2} - 1\right).$$

$$= \frac{\pi}{6}\left((4a^2+1)^{3/2} - 1\right).$$

Mass and Center of Mass of a Shell

Consider a shell with negligible thickness, shaped as a piecewise smooth surface Ω. Let $\delta(x, y, z)$ be the density of the material at point (x, y, z). If the coordinates of Ω are parameterized as $x(u, v), y(u, v), z(u, v)$ for $(u, v) \in R$, where R is the projection of Ω onto the xy-plane, then the mass of Ω is:

$$\text{Mass of } \Omega = \iint_\Omega \delta(x, y, z) d\sigma.$$

The center of mass $(\bar{x}, \bar{y}, \bar{z})$ is computed as:

$$\bar{x} = \frac{1}{m} \iint_\Omega x\delta(x, y, z) d\sigma, \quad \bar{y} = \frac{1}{m} \iint_\Omega y\delta(x, y, z) d\sigma, \quad \bar{z} = \frac{1}{m} \iint_\Omega z\delta(x, y,$$

where m is the mass.

If the surface is represented by $z = f(x, y)$ over R, then the mass is:

$$m = \iint_\Omega \delta(x, y, z) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy.$$

Example: Center of Mass of a Sphere

Example

Find the center of mass of the sphere Ω, given by:

$$x^2 + y^2 + z^2 = a^2,$$

in the first octant, assuming a constant density μ_0.

Solution:

Example: Center of Mass of a Sphere

Example

Find the center of mass of the sphere Ω, given by:

$$x^2 + y^2 + z^2 = a^2,$$

in the first octant, assuming a constant density μ_0.

Solution:
Step 1: Convert to Spherical Coordinates Using spherical coordinates:

$$x = a\cos\theta\sin\phi, \quad y = a\sin\theta\sin\phi, \quad z = a\cos\phi.$$

The first octant corresponds to:

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Thus, the parameterization of the sphere in the first octant is:

$$r(\theta,\phi) = a\cos\theta\sin\phi\, i + a\sin\theta\sin\phi\, j + a\cos\phi\, k.$$

Step 2: Compute the Normal Vector

Compute the partial derivatives:

$$r_\theta = -a\sin\theta\sin\phi\, i + a\cos\theta\sin\phi\, j,$$

$$r_\phi = a\cos\theta\cos\phi\, i + a\sin\theta\cos\phi\, j - a\sin\phi\, k.$$

Compute the cross product:

$$N = r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \end{vmatrix}.$$

Expanding:

$$N = -a^2\sin^2\phi\cos\theta\, i - a^2\sin^2\phi\sin\theta\, j - a^2\sin\phi\cos\phi\, k.$$

Compute the magnitude:

$$\|N\| = a^2\sin\phi.$$

Step 3: Compute the Mass

The mass integral is:

$$m = \iint_{\Omega} \mu_0 \|N\| \, d\theta \, d\phi.$$

Substituting $\|N\| = a^2 \sin\phi$:

$$m = \mu_0 a^2 \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi \, d\theta \, d\phi.$$

Evaluating:

$$m = \mu_0 a^2 \frac{\pi}{2}.$$

Step 4: Compute the Center of Mass

Compute the integral for $\bar{x}$:

$$\bar{x} = \frac{1}{m} \iint_{\Omega} x \mu_0 \, d\sigma.$$

Substituting $x = a \cos\theta \sin^2\phi$:

$$\bar{x} = \frac{1}{m} \int_0^{\pi/2} \int_0^{\pi/2} \mu_0 a^3 \cos\theta \sin^2\phi \, d\theta \, d\phi.$$

Evaluating:

$$\bar{x} = \frac{a}{2}.$$

Similarly:

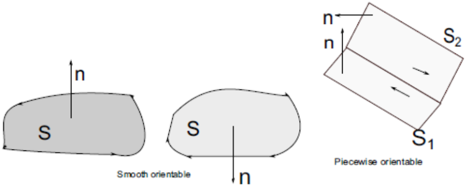
$$\bar{y} = \frac{a}{2}, \quad \bar{z} = \frac{a}{2}.$$

The center of mass of the portion of the sphere in the first octant is:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right).$$

Divergence and Stock's Theorems

If a surface S is smooth and P is any point in S we can choose a unit normal vector n of S at P. Then we can take the direction of n as the positive normal direction of S at P(two possibilities).
A smooth surface is said to be **orientable** if the positive normal direction, given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface.
A smooth surface is said to be **piecewise orientable** if the surface consists of multiple smooth sections, each oriented so that along any common boundary curve C^*, the positive orientation on one section is opposite to that on the other.



There are also non-orientable surfaces. Mobius strip [no inward and no outward directions once in once out word.]
Consider a boundary surface of a solid region D in 3-space. Such surfaces are called **closed**. If a closed surface is orientable or piecewise orientable, then there are only two possible orientations: inward (to ward the solid) and outward (away from the solid).

Theorem (Divergence Theorem of Gauss)

Let D be a solid in $\mathbb{R}^3$ with surface S oriented outward. If $F = F_1 i + F_2 j + F_3 k$, where F_1, F_2 and F_3 have continuous first and second partial derivatives on some open set containing D, then

$$\iint_S F \cdot n \, dA = \iiint_D \nabla \cdot F \, dv.$$

that is,

$$\iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy).$$

Divergence Theorem converts a surface integral into a volume integral, making flux calculations much easier.

Example 1: Flux Across a Sphere

Example

Let S be the sphere $x^2 + y^2 + z^2 = a^2$, oriented outward. Find the flux of the vector function:

$$F(x, y, z) = zk$$

across S.

Solution:

Step 1: Compute the Divergence

By the **Divergence Theorem**, the flux across S is given by:

$$\iint_S F \cdot n \, dA = \iiint_D \nabla \cdot F \, dv.$$

Compute the divergence of $F(x, y, z) = zk$:

$$\nabla \cdot F = \frac{\partial(0)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(z)}{\partial z} = 1.$$

Note: Since the divergence is constant, integration over D simplifies to multiplying by the volume.

Step 2: Compute the Volume of the Sphere

The volume of the sphere $x^2 + y^2 + z^2 = a^2$ is:

$$V = \frac{4}{3}\pi a^3.$$

Since $\nabla \cdot F = 1$, the flux is:

$$\Phi = \iiint_D 1\, dv = \frac{4}{3}\pi a^3.$$

Final Answer:

$$\Phi = \frac{4}{3}\pi a^3.$$

Example 2: Flux Across a Cylinder

Example

Let S be the surface of the solid enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$, oriented outward. Use the **Divergence Theorem** to find the flux Φ of the vector field:

$$F(x, y, z) = x^3 i + y^3 j + z^2 k$$

across S.

Solution:

Step 1: Compute the Divergence

Compute the divergence of $F(x, y, z) = x^3 i + y^3 j + z^2 k$:

$$\nabla \cdot F = \frac{\partial(x^3)}{\partial x} + \frac{\partial(y^3)}{\partial y} + \frac{\partial(z^2)}{\partial z}.$$

$$= 3x^2 + 3y^2 + 2z.$$

Note: The divergence function gives a measure of how much the vector field expands or contracts at each point.

Step 2: Convert to Cylindrical Coordinates

Switching to cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2.$$

Since $x^2 + y^2 = 9$, the volume element is:

$$dv = r\,dr\,d\theta\,dz.$$

Set the bounds:

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2.$$

Note: Converting to cylindrical coordinates makes solving the integral much easier, as it accounts for the radial symmetry of the problem.

Step 3: Evaluate the Flux Integral

Evaluate:

$$\Phi = \int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z) r \, dz \, dr \, d\theta.$$

Final result:

$$\Phi = 279\pi.$$

Final Answer:

$$\Phi = 279\pi.$$

Introduction to Stokes' Theorem

Concept of Stokes' Theorem: Stokes' Theorem provides a relationship between a surface integral and a line integral over the boundary of that surface. It states that the circulation of a vector field along a closed curve is equal to the surface integral of the curl of the field.

Theorem (Stoke's Theorem)

If S is a piecewise smooth orientable surface bounded by a simple, closed curve C, and the vector field $F = (F_1, F_2, F_3)$ is continuously differentiable on an open set containing S and if T is the unit tangent vector of C, then:

$$\oint_C F \cdot T \, dS = \iint_S (\nabla \times F) \cdot n \, dA.$$

Key Intuition:

- The line integral represents the circulation of F along C.
- The surface integral represents how much F "rotates" over S, as measured by its curl.

Example 1: Verifying Stokes' Theorem

Example

Let S be the portion of the paraboloid:

$$z = 4 - x^2 - y^2, \quad \text{for } z \geq 0,$$

and let the vector field be:

$$F(x, y, z) = 2zi + 3xj + 5yk.$$

Verify **Stokes' Theorem**, assuming S is oriented upward.

Solution:

Step 1: Compute the Curl of F

By Stokes' Theorem:

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dA.$$

First, compute $\nabla \times F$:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix}.$$

Expanding:

$$\nabla \times F = \left(\frac{\partial(5y)}{\partial y} - \frac{\partial(3x)}{\partial z} \right) i + \left(\frac{\partial(2z)}{\partial z} - \frac{\partial(5y)}{\partial x} \right) j + \left(\frac{\partial(3x)}{\partial x} - \frac{\partial(2z)}{\partial y} \right) k.$$

$$= (5-0)i + (2-0)j + (3-0)k.$$

Final result:

$$\nabla \times F = 5i + 2j + 3k.$$

Step 2: Parametrize the Surface

The boundary curve of S is given by setting $z = 0$, which yields:

$$x^2 + y^2 = 4.$$

Parametrize in cylindrical coordinates:

$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = 0.$$

where $0 \leq \theta \leq 2\pi$.
The tangent vector to C is:

$$dr = \frac{dx}{d\theta}d\theta = (-2\sin\theta i + 2\cos\theta j)d\theta.$$

Note: The parameterization simplifies the line integral computation over the boundary.

Step 3: Compute the Line Integral

Evaluate:

$$\oint_C F \cdot dr.$$

Substituting $F(x, y, z)$ into the integral:

$$F(2\cos\theta, 2\sin\theta, 0) = 2(0)i + 3(2\cos\theta)j + 5(2\sin\theta)k.$$
$$= 0i + 6\cos\theta j + 10\sin\theta k.$$

Now, dot F with dr:

$$(6\cos\theta j + 10\sin\theta k)\cdot(-2\sin\theta i + 2\cos\theta j) = (6\cos\theta \cdot 2\cos\theta) + 0 = 12\cos^2\theta.$$

Integrating:

$$\oint_C F \cdot dr = \int_0^{2\pi} 12\cos^2\theta \, d\theta.$$

Using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, evaluate:

$$\oint_C F \cdot dr = 12 \times \frac{2\pi}{2} = 12\pi.$$

Step 4: Compute the Surface Integral

Using Stokes' Theorem:

$$\iint_S (\nabla \times F) \cdot n \, dA.$$

Since S is oriented upward, the normal is:

$$n = \frac{\nabla g}{\|\nabla g\|}.$$

Compute:

$$\frac{\partial g}{\partial x} = -2x, \quad \frac{\partial g}{\partial y} = -2y, \quad \frac{\partial g}{\partial z} = 1.$$

Thus:

$$n = \frac{-2xi - 2yj + 1k}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Now, compute:

$$(\nabla \times F) \cdot n = (5, 2, 3) \cdot \left(-\frac{2x}{\sqrt{4(x^2 + y^2) + 1}}, -\frac{2y}{\sqrt{4(x^2 + y^2) + 1}}, \frac{1}{\sqrt{4(x^2 + y^2)}} \right.$$

Solving:

$$\iint_S (\nabla \times F) \cdot n \, dA = 12\pi.$$

Final Verification

Since:

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dA = 12\pi.$$

Stokes' Theorem holds!

Key Takeaways:

- The line integral was computed using parameterization.
- The surface integral used the curl of F.
- Both integrals gave the same result, proving Stokes' Theorem.

This verification confirms the fundamental relationship between circulation and curl in vector calculus!

Remark

1. If S_1 and S_2 have the same boundary C which is oriented positively, then for any vector function F that satisfy the hypotheses in Stoke's Theorem, we have:

$$\iint_{S_1} (curlF).ndA = \iint_{S_2} (curlF).ndA.$$

2. If $F = (F_1, F_2)$ is a vector function that is continuously differentiable in a domain in the $xy-$plane containing a simply connected domain S whose boundary C is a piecewise smooth simple closed curve, then

$$(curlF).n = (curlF).k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Hence from Stoke's Theorem we have:

$$\iint_{S} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_{C} (F_1 dx + F_2 dy),$$

which is the result of Green's Theorem.