Applied Mathematics III
Unit 7
Complex Integral Calculus

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Complex Integration:

Integral of a Complex Valued Function of Real Variable

Definition

Let $f(t) = u(t) + iv(t)$ be a continuous complex function, then u and v are also continuous. Define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

If $U' = u, \quad V' = v$ and $F(t) = U(t) + iV(t)$, then by fundamental theorem of the complex integral calculus

$$\int_a^b f(t)dt = F(b) - F(a).$$

Example

1. $\displaystyle\int_0^1 (1+ti)^2 dt = \frac{2}{3} + i.$
2. $\displaystyle\int_{2i}^3 \sin z dz = -cos3 + \cos 2i = \cosh 2 - \cos 3$

Complex Integration

Integral of a Complex Valued Function of Real Variable

Definition

Let $f(t) = u(t) + iv(t)$ be a continuous complex-valued function, where $u(t)$ and $v(t)$ are the real and imaginary parts, respectively. Then both u and v are continuous. The integral is defined as:

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

If $U' = u$, $V' = v$, and $F(t) = U(t) + iV(t)$, then by the **Fundamental Theorem of Complex Integral Calculus**:

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example

Example

Compute $\int_0^1 (1+ti)^2 dt$.

Solution:

Example

Example

Compute $\int_0^1 (1+ti)^2 dt$.

Solution: Expand $(1+ti)^2$:

$$(1+ti)^2 = 1 + 2ti - t^2.$$

Separate into real and imaginary parts:

$$u(t) = 1 - t^2, \quad v(t) = 2t.$$

Compute the integrals:

$$\int_0^1 u(t)\,dt = \int_0^1 (1-t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3},$$

$$\int_0^1 v(t)\,dt = \int_0^1 2t\,dt = \left[t^2\right]_0^1 = 1.$$

Thus:

$$\int_0^1 (1+ti)^2 dt = \frac{2}{3} + i.$$

Example

Compute $\int_{2i}^{3} \sin z \, dz$.

Solution:

Example

Compute $\int_{2i}^{3} \sin z \, dz$.

Solution: Express $\sin z$ in terms of exponential functions:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The integral becomes:

$$\int_{2i}^{3} \sin z \, dz = \int_{2i}^{3} \frac{e^{iz} - e^{-iz}}{2i} \, dz.$$

This can be calculated by evaluating cos and cosh:

$$\int_{2i}^{3} \sin z \, dz = -\cos 3 + \cos 2i = \cosh 2 - \cos 3.$$

Contour Integral

Definition: A curve in complex analysis is a continuous function $\sigma(t) = x(t) + iy(t)$, where x and y are real-valued functions, and $t \in [a, b]$.

- A curve σ is called a **smooth curve** if σ is differentiable and σ' is continuous and nonzero for all t.
- A **contour** (or **piecewise smooth curve**) is obtained by joining finitely many smooth curves end to end.
- A curve σ is **simple** if it does not intersect itself except possibly at endpoints ($\sigma(t_1) \neq \sigma(t_2)$ when $a < t_1 < t_2 < b$).
- A curve σ is said to be a **closed curve** if $\sigma(a) = \sigma(b)$.
- If σ is a simple and closed curve, it is called a **simple closed curve** or **Jordan curve**.
- The **orientation** of σ is induced by its parametrization. If t moves from a to b in a counter-clockwise direction, the orientation is **positive**; otherwise, it is **negative**.

Length of a Curve

Definition: Let σ be a piecewise smooth curve defined on $[a, b]$. The length of σ is given by:

$$L(\sigma) = \int_a^b |\sigma'(t)|\, dt.$$

Line Integral or Contour Integral

Definition: Let C be a contour parametrically represented by $\sigma(t)$, $t \in [a, b]$, and f a complex-valued continuous function defined on C. The line integral (or contour integral) of f along C is defined as:

$$\int_C f(z)\,dz = \int_a^b f(\sigma(t))\sigma'(t)\,dt, \quad \text{where } \sigma'(t) = \frac{d\sigma}{dt}.$$

Examples

Example

Evaluate $\oint_C \frac{dz}{z}$, where C is the unit circle around the origin.

Solution:

Examples

Example

Evaluate $\oint_C \frac{dz}{z}$, where C is the unit circle around the origin.

Solution: Using the parametrization $\sigma(t) = e^{it}$, $t \in [0, 2\pi]$:

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

Example

Evaluate $\oint_C \overline{z}\, dz$, where $C : \sigma(t) = e^{it}$, $t \in [0, \pi]$.

Solution:

Examples

Example

Evaluate $\oint_C \frac{dz}{z}$, where C is the unit circle around the origin.

Solution: Using the parametrization $\sigma(t) = e^{it}$, $t \in [0, 2\pi]$:

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

Example

Evaluate $\oint_C \overline{z}\, dz$, where $C : \sigma(t) = e^{it}$, $t \in [0, \pi]$.

Solution: Using $z = e^{it}$ and $\overline{z} = e^{-it}$:

$$\oint_C \overline{z}\, dz = \int_0^{\pi} e^{-it} \cdot ie^{it}\, dt = \int_0^{\pi} i\, dt = i\pi.$$

Additional Examples

Example

Evaluate $I = \int_C z^2\, dz$, where C is the parabolic arc given by $x = 4 - y^2$, $-2 \leq y \leq 2$.

Solution: Parametrize C as $\sigma(t) = (4 - t^2) + it$, $-2 \leq t \leq 2$. Then:

$$\sigma'(t) = -2t + i.$$

The integral becomes:

$$I = \int_{-2}^{2} \big((4 - t^2) + it\big)^2 \big(-2t + i\big)\, dt.$$

Expand and evaluate each term.

$$I = \frac{16}{3} i$$

Example

Evaluate $\oint_C (z-a)^n \, dz$, where a is a complex number, n an integer, and C is a circle centered at a with radius r.

Solution:

Example

Evaluate $\oint_C (z-a)^n \, dz$, where a is a complex number, n an integer, and C is a circle centered at a with radius r.

Solution: The curve is parametrized by $z - a = re^{it}$ for $0 \leq t \leq 2\pi$. Thus

$$\oint_C (z-a)^n \, dz = \int_0^{2\pi} (re^{it})^n ire^{it\,dt}$$

If $n \neq -1$:

$$\oint_C (z-a)^n \, dz = 0.$$

If $n = -1$:

$$\oint_C \frac{1}{z-a} \, dz = 2\pi i.$$

Contour Integral: Additive Property

Definition: Let C be a piecewise smooth curve such that $C = C_1 \oplus C_2 \oplus \cdots \oplus C_n$, and let $f(z)$ be a continuous complex function on C. Then:

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz.$$

Example: Evaluating a Contour Integral

Example: Let C be a curve consisting of:

1. A portion of the parabola $y = x^2$ in the xy-plane from $(0, 0)$ to $(2, 4)$.
2. A horizontal line from $(2, 4)$ to $(4, 4)$.

If $f(z) = \text{Im}(z)$, evaluate:

$$I = \int_C f(z)\, dz.$$

Solution:

Example: Evaluating a Contour Integral

Example: Let C be a curve consisting of:

1. A portion of the parabola $y = x^2$ in the xy-plane from $(0,0)$ to $(2,4)$.
2. A horizontal line from $(2,4)$ to $(4,4)$.

If $f(z) = \operatorname{Im}(z)$, evaluate:

$$I = \int_C f(z)\,dz.$$

Solution: Step 1: Parametrize the Parabolic Segment For the parabola $y = x^2$, set:

$$\sigma_1(t) = t + it^2, \quad 0 \leq t \leq 2.$$

Compute the derivative:

$$\sigma_1'(t) = 1 + 2it.$$

Thus, the integral over C_1 is:

$$I_1 = \int_0^2 t^2(1+2it)\,dt.$$

Evaluating separately:

$$\int_0^2 t^2\,dt = \frac{8}{3}, \quad \int_0^2 2it^3\,dt = \frac{16i}{2} = 8i.$$

$$I_1 = \frac{8}{3} + 8i.$$

Step 2: Parametrize the Horizontal Segment
For the horizontal line at $y = 4$:

$$\sigma_2(t) = t + 4i, \quad 2 \leq t \leq 4.$$

Derivative:

$$\sigma_2'(t) = 1.$$

The integral over C_2 is:

$$I_2 = \int_2^4 4(1)\,dt = 4(4-2) = 8.$$

Final Computation: Summing both integrals:

$$I = I_1 + I_2 = \left(\frac{8}{3} + 8i\right) + 8 = \frac{32}{3} + 8i.$$

Cauchy's Integral Theorem.

Definition

1. A domain D is called **simply connected** if every simple closed contour (within it) encloses points of D only.
2. A domain D is called **multiply connected** if it is not simply connected. For example $\mathbb{C}' = \mathbb{C}/\{0\}$ and the annulus $A(a,b) = \{z \in \mathbb{C} : a < |z| < b\}$.

Theorem (Cauchy's Theorem)

If a function f is analytic on a simply connected domain D and C is a simple closed contour lying in D then

$$\oint_C f(z)dz = 0.$$

Proof Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $C : \sigma(t) = x(t) + iy(t)$; $a \leq t \leq b$ is the curve C. Then

$$\oint_C f(z)dz = \int_a^b f(\sigma(t))\sigma'(t)dt$$

$$
\begin{aligned}
&= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]dt \\
&= \int_a^b (ux' + vy')dt + i\int_a^b (vx' + uy')dt \\
&= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \\
&= \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy, \quad \text{(by Greens Theorem)} \\
&= 0 \quad \text{(by CR equations} \quad u_x = v_y \;\; and \;\; u_y = -v_x).
\end{aligned}
$$

Examples

Example 1: Evaluating $\oint_C f(z)dz$ for analytic functions

We are given a unit circle C parametrized by:

$$\sigma(t) = e^{it}, \quad -\pi \leq t \leq \pi.$$

By *Cauchy's Theorem*, if $f(z)$ is analytic inside and on C, then:

$$\oint_C f(z)dz = 0.$$

Solution: Since $f(z) = e^{z^n}$, $f(z) = \cos z$, and $f(z) = \sin z$ are analytic everywhere in $\mathbb{C}$, applying *Cauchy's Theorem*, we conclude:

$$\oint_C e^{z^n} dz = 0, \quad \oint_C \cos z \, dz = 0, \quad \oint_C \sin z \, dz = 0.$$

Example 2: Evaluating $\oint_C f(z)dz$ when $f(z)$ is not analytic at $z = 0$ We are given:

$$f(z) = \frac{1}{z^2}, \quad f(z) = \csc^2 z.$$

Since both functions have singularities at $z = 0$, *Cauchy's Theorem* cannot be applied directly. Instead, we use the *Fundamental Theorem of Contour Integration*, which states:

$$\oint_C \frac{d}{dz} g(z) dz = g(B) - g(A) = 0.$$

where A and B are the endpoints of C.
Solution: Using differentiation:

$$\frac{d}{dz}\left(-\frac{1}{z}\right) = \frac{1}{z^2}, \quad \frac{d}{dz}(-\cot z) = \csc^2 z.$$

Applying the fundamental theorem:

$$\oint_C \frac{1}{z^2} dz = 0, \quad \oint_C \csc^2 z \, dz = 0.$$

Since the integral of a derivative over a closed contour is always zero, this confirms our results.

Additional Examples

Let C be a unit circle given by $\sigma(t) = e^{it}, -\pi \leq t \leq \pi$.

Example 1: Evaluate $\oint_C \frac{e^{(iz)^2}}{z^2+4} dz$.

Solution:

Additional Examples

Let C be a unit circle given by $\sigma(t) = e^{it}, -\pi \leq t \leq \pi$.

Example 1: Evaluate $\oint_C \frac{e^{(iz)^2}}{z^2+4} dz$.

Solution: We are given the contour C, which is the unit circle parametrized as:

$$\sigma(t) = e^{it}, \quad -\pi \leq t \leq \pi.$$

By **Cauchy's Integral Theorem**, if $f(z)$ is analytic inside and on C, then:

$$\oint_C f(z) dz = 0.$$

The integrand $f(z) = \frac{e^{(iz)^2}}{z^2+4}$ is not analytic at $z = 2i$, but these points are outside C. Hence, applying **Cauchy's Theorem**:

$$\oint_C \frac{e^{(iz)^2}}{z^2+4} dz = 0.$$

Example 2: Evaluate $\oint_C f(z)dz$ **where** $f(z) = (\mathbf{Im}z)^2$.

Solution:

Example 2: Evaluate $\oint_C f(z)dz$ **where** $f(z) = (\mathbf{Im} z)^2$.

Solution: We are given the unit circle C parametrized as:

$$\sigma(t) = e^{it}, \quad -\pi \leq t \leq \pi.$$

Since $f(z) = (\text{Im} z)^2$, we express $z = e^{it}$, giving:

$$\text{Im}(z) = \sin t \quad \text{so} \quad f(z) = \sin^2 t.$$

Using the contour integral definition:

$$\oint_C f(z)dz = \oint_C \sin^2(t)dz.$$

Since $(\text{Im} z)^2$ is not analytic anywhere in C, **Cauchy's Theorem does not apply directly**, but direct integration over a symmetric contour confirms:

$$\oint_C (\text{Im} z)^2 dz = 0.$$

The Deformation Theorem

Theorem (The Deformation Theorem)

Let C_1 and C_2 be closed paths in the complex plane, with C_2 inside C_1. Suppose f is analytic in an open set containing both paths and the region between them. Then:

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Remark on Path Independence

Remark

If f is analytic in a simply connected domain D, then the integral:

$$\int_C f(z)dz$$

is independent of the path in D. That is, if C_1 and C_2 are open curves with the same initial and terminal points, then:

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Hence, we can deform C_1 into C_2 without changing the value of the integral. However, if f is not analytic in D, then **Cauchy's Theorem does not hold in general**.

Key Contour Integral Example

Consider:

$$\oint_C \frac{dz}{z-a},$$

where C is a piecewise smooth simple closed curve, oriented counterclockwise and enclosing a. Since $f(z) = \frac{1}{z-a}$ is analytic everywhere **except at** $z = a$, we conclude:
By deformation, we assume C_1 is a circular path centered at a with radius r:

$$\oint_C \frac{dz}{z-a} = \oint_{C_1} \frac{dz}{z-a}.$$

Setting $z - a = re^{i\theta}$, we get $dz = rie^{i\theta} d\theta$, so:

$$\oint_{C_1} \frac{rie^{i\theta}}{re^{i\theta}} d\theta = i \oint_{C_1} d\theta.$$

Since the path completes one full cycle:

$$i \int_0^{2\pi} d\theta = 2\pi i.$$

Thus:

$$\oint_C \frac{dz}{z-a} = 2\pi i.$$

Cauchy's Integral Formula

Definition

A Complex function g is said to be singular at a point, say $z = z_0$, if it is not analytic at that point.

Theorem (Cauchy Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then

$$\oint_C \frac{f(z)}{z-a}dz = 2i\pi f(a)$$

for all a in D. This implies

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz.$$

Examples

Example 1: Evaluate $\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz$, where C is the unit circle centered at $z = 1$.

Solution:

Examples

Example 1: Evaluate $\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz$, where C is the unit circle centered at $z = 1$.

Solution: Factor the denominator:

$$z^2 - 1 = (z-1)(z+1).$$

Singularities occur at $z = \pm 1$. The contour C encloses $z = 1$ but not $z = -1$, meaning we apply **Cauchy's Integral Formula**:

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Define $f(z)$ as:

$$f(z) = \frac{z^2+1}{z+1}.$$

Since $z = 1$ is enclosed by C, we evaluate:

$$f(1) = \frac{1^2+1}{1+1} = \frac{2}{2} = 1.$$

Applying **Cauchy's Integral Formula**:

$$\oint_C \frac{z^2+1}{z^2-1} dz = 2\pi i \times 1 = 2\pi i.$$

Example 2: Evaluate $\oint_C \left(\frac{z^3 - 6}{2z - i}\right) dz$, where C encloses $a = \frac{i}{2}$.

Solution:

Example 2: Evaluate $\oint_C \left(\frac{z^3-6}{2z-i}\right) dz$, where C encloses $a = \frac{i}{2}$.

Solution: Since $f(z) = \frac{z^3-6}{2z-i}$ has a singularity at $z = \frac{i}{2}$, apply Cauchy's Integral Formula:

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Substituting $a = \frac{i}{2}$:

$$f\left(\frac{i}{2}\right) = \frac{1}{2}\left(\left(\frac{i}{2}\right)^3 - 6\right) = -\frac{i}{16} - 3.$$

Thus:

$$\oint_C \frac{z^3-6}{2z-i} dz = 2\pi i\left(-\frac{i}{16} - 3\right) = \frac{\pi}{8} - 6\pi i.$$

Example 3: Show that:

a) $\oint_C \frac{\cos z}{z} dz = 2\pi i$, where C is the circle $|z-4| = 5$.

b) $\oint_C \frac{z^2}{z^2+1} dz = -\pi$, where C is the circle $|z-i| = 1$.

c) $\oint_C \frac{e^z}{z(z-1)} dz = 2\pi i(e-1)$, where C is the circle centered at $z = 0$ with radius 2.

Solution:

Example 3: Show that:

a) $\oint_C \frac{\cos z}{z} dz = 2\pi i$, where C is the circle $|z-4| = 5$.

b) $\oint_C \frac{z^2}{z^2+1} dz = -\pi$, where C is the circle $|z-i| = 1$.

c) $\oint_C \frac{e^z}{z(z-1)} dz = 2\pi i(e-1)$, where C is the circle centered at $z = 0$ with radius 2.

Solution: For part **(a)**: The singularity $z = 0$ is inside C. By Cauchy's Integral Formula:

$$\oint_C \frac{\cos z}{z} dz = 2\pi i \cos(0) = 2\pi i.$$

For part **(b)**: The singularity at $z = i$ is enclosed by C. Using Cauchy's Integral Formula:

$$\oint_C \frac{f(z)}{z-i} dz = 2\pi i f(i).$$

Define $f(z) = \frac{z^2}{z+i}$, so:

$$f(i) = \frac{i^2}{i+i} = \frac{-1}{2i}.$$

Applying the formula:

$$\oint_C \frac{z^2}{z^2+1} dz = 2\pi i \times \frac{-1}{2i} = -\pi.$$

For part **(c)**: The singularities of the integrand are at $z = 0$ and $z = 1$.
Applying Cauchy's Integral Formula:

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a),$$

where $f(z) = e^z$.
Since both singularities are enclosed by C, we evaluate $f(z)$ at $z = 0$ and $z = 1$:

$$f(0) = e^0 = 1, \quad f(1) = e^1 = e.$$

Applying the formula to both singularities:

$$\oint_C \frac{e^z}{z(z-1)}dz = 2\pi i f(1) + 2\pi i f(0).$$

Substituting the values:

$$\oint_C \frac{e^z}{z(z-1)}dz = 2\pi i e + 2\pi i(1) = 2\pi i(e-1).$$

Cauchy Integral Formula for Higher Derivatives

Theorem (Cauchy Integral Formula for Higher Derivatives)

Let $f(z)$ be analytic in a simply connected domain D and let C be a piecewise smooth simple closed curve in D oriented counterclockwise. Then for all a in D

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

for any nonnegative integer n.

Summary of Cauchy's Integral Formula for Higher Derivatives

Let C be a simple closed curve contained in a simply connected domain D, and let f be analytic on D. Then:

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \begin{cases} 2i\pi f(a), & \text{if } n = 0 \text{ and } a \text{ is enclosed by } C. \\ \frac{2i\pi}{n!} f^{(n)}(a), & \text{if } n \geq 1 \text{ and } a \text{ is enclosed by } C. \\ 0, & a \text{ lies outside of the region enclosed by } C. \end{cases}$$

Examples

Example 1: Evaluate $\oint_C \frac{\sin z}{(z-\pi i)^2} dz$, where C encloses πi and is oriented counterclockwise.

Solution:

Examples

Example 1: Evaluate $\oint_C \frac{\sin z}{(z - \pi i)^2} dz$, where C encloses πi and is oriented counterclockwise.

Solution: Using **Cauchy's Integral Formula for Higher Derivatives**:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} dz.$$

For $n = 1$, we substitute $f(z) = \sin z$ and $a = \pi i$:

$$\oint_C \frac{\sin z}{(z - \pi i)^2} dz = 2\pi i f'(\pi i).$$

Compute the derivative:

$$f'(z) = \cos z.$$

Evaluating at $z = \pi i$:

$$f'(\pi i) = \cos(\pi i) = \frac{e^{\pi i} + e^{-\pi i}}{2} = \frac{-1 + (-1)}{2} = -1.$$

Thus:

$$\oint_C \frac{\sin z}{(z - \pi i)^2} dz = 2\pi i(-1) = -2\pi i.$$

Example 2: Show that:

a) $\oint_C e^z z^{-3} dz = i\pi$, where C is the circle $|z| = 1$.

b) $\oint_C \frac{1}{(z-4)(z+1)^4} dz = \frac{-2i\pi}{81}$, where C is the circle $|z-1| = \frac{5}{2}$.

Solution:

Example 2: Show that:

a) $\oint_C e^z z^{-3} dz = i\pi$, where C is the circle $|z| = 1$.

b) $\oint_C \frac{1}{(z-4)(z+1)^4} dz = \frac{-2i\pi}{81}$, where C is the circle $|z-1| = \frac{5}{2}$.

Solution:

For part **(a)**: The singularity at $z = 0$ is inside C. Using the formula for higher derivatives:

$$\oint_C e^z z^{-3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(e^z)|_{z=0}.$$

Since $e^0 = 1$, we get:

$$\oint_C e^z z^{-3} dz = \frac{2\pi i}{2} = i\pi.$$

For part **(b)**: Singularities occur at $z = 4$ and $z = -1$. Since C encloses only $z = -1$, we focus on that term:

$$\oint_C \frac{1}{(z-4)(z+1)^4} dz.$$

Using Cauchy's Integral Formula:

$$\oint_C \frac{f(z)}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1).$$

Computing the function value:

$$f(z) = \frac{1}{z-4} \implies f'''(-1) = \frac{6}{-5} = -\frac{6}{5}.$$

Applying the formula:

$$\oint_C \frac{1}{(z-4)(z+1)^4} dz = \frac{2\pi i}{6} \times \frac{-6}{5} = \frac{-2i\pi}{5}.$$

Thus:

$$\oint_C \frac{1}{(z-4)(z+1)^4} dz = \frac{-2i\pi}{5}.$$

Cauchy's Theorem for Multiply Connected Domains

Theorem

Let C be a closed path and $C_1, C_2, \cdots, C_n$ be closed paths enclosed by C. Assume that any two of $C, C_1, C_2, \cdots, C_n$ intersect and no interior point to any C_i is interior to any other C_k. Let f be analytic on an open set containing C and each C_i and all the points that are both interior to C an exterior to each C_i. Then

$$\oint_C f(z)dz = \sum_{i=1}^{n} \oint_{C_i} f(z)dz.$$

Examples

Example: Evaluate $\oint_C \frac{dz}{z(z-1)}$, where C is the circle $|z| = 3$ oriented counterclockwise.

Solution: Since C encloses both singularities at $z = 0$ and $z = 1$, we apply **Cauchy's Theorem for Multiply Connected Domains**:

$$\oint_C \frac{dz}{z(z-1)} = \oint_{C_1} \frac{dz}{z(z-1)} + \oint_{C_2} \frac{dz}{z(z-1)}.$$

Step 1: Integral over C_1, enclosing $z = 0$.

For C_1, define $f(z) = \frac{1}{z-1}$, which is analytic inside C_1. By Cauchy's Integral Formula:

$$\oint_{C_1} \frac{dz}{z(z-1)} = 2\pi i f(0) = 2\pi i \times \frac{1}{-1} = -2\pi i.$$

Step 2: Integral over C_2, enclosing $z = 1$.

For C_2, define $f(z) = \frac{1}{z}$, which is analytic inside C_2. By Cauchy's Integral Formula:

$$\oint_{C_2} \frac{dz}{z(z-1)} = 2\pi i f(1) = 2\pi i \times \frac{1}{1} = 2\pi i.$$

Step 3: Compute Final Integral Summing the contributions:

$$\oint \frac{dz}{z(z-1)} = (-2\pi i) + (2\pi i) = 0.$$

Example 2: Evaluate $\oint_C \frac{z+1}{z(z-2)(z-4)^3} dz$, where C is the circle $|z-3| = 2$ oriented counterclockwise.

Solution:

Example 2: Evaluate $\oint_C \frac{z+1}{z(z-2)(z-4)^3}dz$, where C is the circle $|z-3|=2$ oriented counterclockwise.

Solution: Since C encloses the singularities $z=2$ and $z=4$ but not $z=0$, we apply **Cauchy's Theorem for Multiply Connected Domains**:

$$\oint_C \frac{z+1}{z(z-2)(z-4)^3}dz = \oint_{C_1} \frac{z+1}{z(z-2)(z-4)^3}dz + \oint_{C_2} \frac{z+1}{z(z-2)(z-4)^3}dz.$$

Step 1: Integral over C_1, enclosing $z=2$.

For C_1, define $f(z) = \frac{z+1}{z(z-4)^3}$, which is analytic inside C_1. By Cauchy's Integral Formula:

$$\oint_{C_1} \frac{f(z)}{z-2}dz = 2\pi i f(2).$$

Evaluating $f(2)$:

$$f(2) = \frac{2+1}{2(2-4)^3} = \frac{3}{2(-8)} = -\frac{3}{16}.$$

Thus:

$$\oint_{C_1} \frac{z+1}{z(z-2)(z-4)^3}dz = 2\pi i \times -\frac{3}{16} = -\frac{3\pi i}{8}.$$

Step 2: Integral over C_2, enclosing $z = 4$.

For C_2, define $f(z) = \dfrac{z+1}{z(z-2)}$, which is analytic inside C_2. By Cauchy's Integral Formula:

$$\oint_{C_2} \frac{dz}{(z-4)^3} = \frac{2\pi i}{2!} f''(4).$$

Computing $f''(4)$:

$$f''(4) = \frac{d^2}{dz^2}\left(\frac{z+1}{z(z-2)}\right)\Big|_{z=4}.$$

After computation, we find:

$$f''(4) = \frac{23}{64}.$$

Thus:

$$\oint_{C_2} \frac{z+1}{z(z-2)(z-4)^3} dz = \frac{2\pi i}{2} \times \frac{23}{64} = \frac{23\pi i}{64}.$$

Step 3: Compute Final Integral Summing the contributions:

$$\oint_{C} \frac{z+1}{z(z-2)(z-4)^3} dz = -\frac{3\pi i}{8} + \frac{23\pi i}{64} = -\frac{\pi i}{64}.$$