Applied Mathematics III
Unit 4
Vector Differential Calculus

Solomon Amsalu (Asst.Prof.)

Mathematics Department
Wolkite University

solomon.amsalu@wku.edu.et

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Table of Content

Introduction

- Vector calculus is essential in physics and engineering, particularly in fluid dynamics, electromagnetism, and mechanics.
- We will explore fundamental vector operations such as dot and cross products.
- These concepts lead to gradient, divergence, and curl, which are crucial in vector fields.

Vectors in 2-Space and 3-Space

Definition

A vector is a quantity with both magnitude and direction.

- **Representation:**

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad \text{or} \quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

- **Vector operations:**
  - Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.
  - Scalar multiplication: $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$.

Example

Given $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 2 \rangle$, find $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a}$.

$$\mathbf{a} + \mathbf{b} = \langle 1 + (-1), 2 + 0, 3 + 2 \rangle = \langle 0, 2, 5 \rangle.$$
$$2\mathbf{a} = \langle 2(1), 2(2), 2(3) \rangle = \langle 2, 4, 6 \rangle.$$

Inner Product (Dot Product)

Definition

$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$

- Measures projection of one vector onto another.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta).$
- **Properties:**
  - Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$
  - Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$
  - $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$

Example

Compute $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u} = (2, 3, -1)$ and $\mathbf{v} = (4, -1, 5).$

Applications of Dot Product

- Angle between vectors: $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$.
- Projection of **a** onto **b**: $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$.
- Work done by a force: $W = \mathbf{F}\cdot\mathbf{d}$.

Vector Product (Cross Product)

- **Definition:** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.
- Result is a vector perpendicular to $\mathbf{u}$ and $\mathbf{v}$.
- Magnitude: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$.
- **Properties:**
  - Anti-commutative: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.
  - Distributive: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
  - $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

Example

Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (4, 5, 6)$.

In Applied Mathematics I, specifically in the linear algebra part, we have been discussing about constant vectors, but the most important applications of vectors involve also vector functions.

The simplest example is a position vector that depends on time. We can differentiate such a function with respect to time and the first derivative of such function is the velocity and its second derivative is the acceleration of the particle whose position is given by the position vector.

In this case, the coordinates of the tip of the position vector are functions of time. Therefore, it is worth to talk about such functions and in this course, specially in this unit we are going to address the calculus of vector fields (vector valued functions).

Vector Functions of One Variable in Space

Definition

A vector-valued function, or vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.

In this course, we are most interested in vector functions whose values are three-dimensional vectors. This means that for every number t in the domain there is a unique vector in $\mathbb{R}^3$ denoted by $r(t)$. If $f(t), g(t)$ and $h(t)$ are the components of the vector $r(t)$, then f, g and h are real-valued functions called the component functions of r and we can write

$$r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$$

Example

The function $r(t) = t^3 i + e^{-t} j + \sin tk$ is a vector valued function and the component functions of r are t^3, e^{-t} and $\sin t$.

Remark

The domain of a vector valued function r consists of all values of t for which the expression $r(t)$ is defined, that is the values of t for which all the component functions are defined.

For example, if $r(t) = \sqrt{t}i + \ln(t-2)j + 3tk$, then the domain of $r(t)$ is the set of points in $\mathbb{R}$, where $\sqrt{t}, \ln(t-2)$ and $3t$ are defined. That is, $t \geq 0$ and $t - 2 > 0$ and hence the domain of r is $(2, \infty)$.

For each t, where r is defined, draw $r(t)$ as as a vector from the origin to the point $(f(t), g(t), h(t))$. The end points of these vectors traces out a curve C as t varies.

Example

The function $r(t) = (1+t)i + tj + (3-t)k$ is a vector valued function of one variable. The curve that is traced out by the heads of the position vectors of this vector valued function is a line that passes through the point $(1, 0, 3)$ and with directional vector $(1, 1, -1)$.

Limit of a Vector Valued Function

Definition

A vector valued function $v(t)$ is said to have the limit l as t approaches t_0, if $v(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t \to t_0} \|v(t) - l\| = 0.$$

Then we write

$$\lim_{t \to t_0} v(t) = l.$$

A vector function $v(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} v(t) = v(t_0).$$

Theorem

If $r(t) = (f(t), g(t), h(t))$, *then*

$$\lim_{t \to t_0} r(t) = l$$

if and only if

$$(\lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t)) = l.$$

Example

Find $\lim_{t\to 0} r(t)$, if $r(t) = t^2 i + e^t j + \sin tk$.

Solution:

Remark

If $r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$ a vector valued function all t in the domain of r, then r is continuous at t_0 if and only if its (three) component functions f, g and h are continuous at t_0.

Derivative of a Vector Function

Definition

A vector function $V(t)$ is said to be differentiable at a point t in the domain of V if the limit

$$\lim_{h \to 0} \frac{V(t+h) - V(t)}{h}$$

exists and if the limit exists then it is denoted by $V'(t)$. That is,

$$V'(t) = \lim_{h \to 0} \frac{V(t+h) - V(t)}{h}.$$

Remark

If the function $V(t) = (V_1(t), V_2(t), V_3(t))$ is a vector field, then $V'(t) = (V_1'(t), V_2'(t), V_3'(t))$.

Example

Find the derivatives the following functions.

1. $V(t) = (\sin t, \cos t)$.
2. $V(t) = (t^3, 3\cos t, 23)$.

Solution:

Differentiation Rules

Let $U(t)$ and $V(t)$ be a vector valued functions in space and c be any constant. Then

1. $(cV)' = cV'$
2. $(U+V)' = U' + V'$
3. $(U.V) = U'.V + U.V'$
4. $(U \times V)' = U' \times V + U \times V'$

Let $V(t)$ be a vector function of constant norm. i.e. $\|V(t)\| = c$ for a constant c or $V.V = c^2$.
Then $(V.V)' = (c^2)' = 0$ which implies $2V'.V = 0$. Then, either $V' = 0$ or $V' \perp V$.

Therefore, a nonzero vector field with constant norm is perpendicular to its derivative.

Vector and Scalar Fields

Definition

A function f whose value is a scalar (or a real number), say $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, is called a **scalar field**.
A function v whose value is a vector, say $v : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$, is called a **vector field**. That is a vector field is a vector valued function.

Example

1. If (x_0, y_0, z_0) is a point in $\mathbb{R}^3$, then the function $d : \mathbb{R}^3 \to \mathbb{R}$ given by $d(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ is a scalar field. (d is called the Euclidean Distance.)
2. The function $f : X \to \mathbb{R}^3$ given by $f(x, y) = (x^2 + y, \ln(x^2 + y^2), \sin(x + 3y))$, where $X = \mathbb{R}^2 \backslash \{(0, 0)\}$ is a vector field.

Definition

Let $v : \mathbb{R}^n \to \mathbb{R}^3,\ v = (v_1, v_2, v_3)$ where each v_i is a function of n variables, $t_1, t_2, \cdot, t_n$. Then the partial derivative of v with respect to t_i is denoted by $\dfrac{\partial v}{\partial t_i}$ and is defined as the vector function

$$\frac{\partial v}{\partial t_i} = \left(\frac{\partial v_1}{\partial t_i}, \frac{\partial v_2}{\partial t_i}, \frac{\partial v_3}{\partial t_i}\right)$$

Example

If $f(x, y) = ((x^2 + y^2), \ln(x + y), \sin(x + y))$, then

$$\frac{\partial f}{\partial x} = \left(2x, \frac{1}{x+y}, \cos(x+y)\right)$$
$$\frac{\partial f}{\partial y} = \left((2y, \frac{1}{x+y}, \cos(x+y))\right)$$

The Gradient Field

Definition

Let $F(x, y, z)$ be a real valued functions of three variables (i.e. F is a scalar field defined from $X \subset \mathbb{R}^3$ into $\mathbb{R}$.) The gradient of F , denoted by ∇F, is a vector field defined by

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$

and if P is a point in the domain of F, the gradient of F evaluated at P is denoted by $\nabla F(P)$ and also if f is a function of two variables, then the the gradient of f, denoted by ∇f, is defined by $\nabla f = \dfrac{\partial f}{\partial x} i + \dfrac{\partial f}{\partial y} j$.

Example

If $F(x, y, z) = 2x + xy - yz^2$, then find ∇F and $\nabla F(1, 2, 3)$.

Solution:

Definition

Let $P(x_0, y_0, z_0)$ be a point and $u = ai + bj + ck$ be a unit vector, i.e. $a^2 + b^2 + c^2 = 1$. Then the directional derivative of a scalar field F at the point P in the direction of u, denoted by $D_uF(p)$, is defined by

$$\begin{aligned} D_uF(P) &= a\frac{\partial F}{\partial x}(x_0, y_0, z_0) + b\frac{\partial F}{\partial y}(x_0, y_0, z_0) + c\frac{\partial F}{\partial z}(x_0, y_0, z_0) \\ &= \nabla F(x_0, y_0, z_0).u \end{aligned}$$

the scalar product of the vectors $F(x_0, y_0, z_0)$ and u.

Example

Given $F(x, y, z) = 2x + xy - yz^2$, find the directional derivative of F at the point $(1, 2, 2)$ in the direction of the unit vector $u = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Solution:

Remark

If F is a scalar field of three variables and v is any nonzero vector then the directional derivative of F at a point P in the direction of v is given by $D_uF(P)$, where $u = \frac{1}{\|v\|}v$.

Theorem

Let F be a scalar field and F and its partial derivatives be continuous in some sphere about a point P and suppose that $\nabla F(P) \neq 0$. Then

1. *At P, F has its maximum rate of change in the direction of $\nabla F(P)$ and this maximum rate of change is $\|\nabla F(P)\|$.*
2. *At P, F has its minimum rate of change in the direction of $-\nabla F(P)$ and this minimum rate of change is $-\|\nabla F(P)\|$.*

Proof: Use $D_uF(P) = \nabla F(P).u = \|\nabla F(P)\|\|u\| \cos\theta = \|\nabla F(P)\| \cos\theta$.

Example

Let $F(x, y, z) = 2xz + yz^2$, then find the maximum rate of change of F at $(2, 1, 1)$.

Solution:

Level Surfaces, Tangent Planes and Normal Lines

Definition

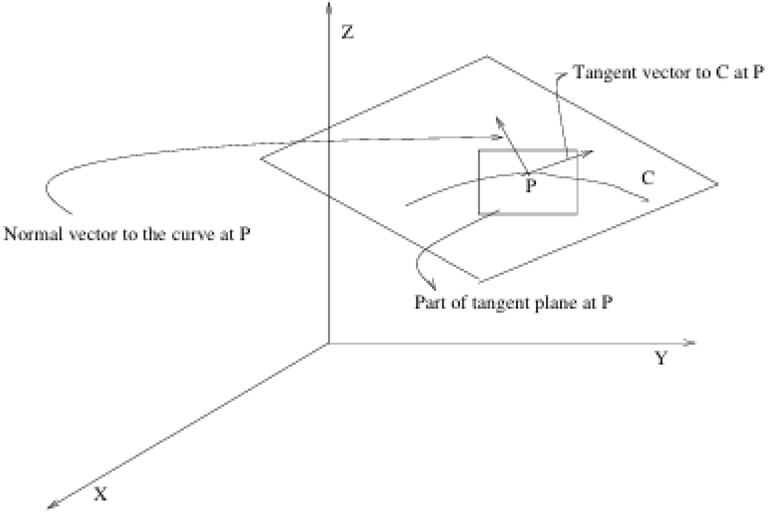
Let F be a function of three variables and c be a number. The set of points (x, y, z) such that $F(x, y, z) = c$ is called a level surface of F.

Example

Let $F(x, y, z) = x^2 + y^2 + z^2$ and $c = 9$. The level surface $F(x, y, z) = 9$ is a sphere $x^2 + y^2 + z^2 = 9$ with radius 3 and center at the origin.

Let F be a scalar function of three variables, c be a constant and S be the level surface given by $F(x, y, z) = c$. Let $P_0 = (x_0, y_0, z_0)$ be a point on S. Assume that there are smooth curves on the surface S passing through P_0. Then each such curve has a tangent vector at P_0.
The plane containing these tangent vectors is called the **tangent plane** to the surface S at P_0 and a vector orthogonal to this tangent plane at P_0 is called a **normal vector** to the surface S at P_0.

The line through P_0 in the direction of the normal vector is called a **normal line** to the surface S at the point P_0.



To determine equation of the tangent plane and normal line to a surface S at a given point P, we need to have a normal vector to the tangent plane and for this purpose we have the following theorem.

Theorem

Let F be a function of three variables and suppose that F and its first partial derivatives are continuous at a point P on the level surface S given by $F(x, y, z) = c$. Suppose that $\nabla F(P) \neq 0$. Then $\nabla F(P)$ is normal to the level surface S at the point P.

Example

Find the equation of the tangent plane and normal line to the surface $3x^4 + 3y^4 + 6z^4 = 12$ at the point $(1, 1, 1)$.

Solution:

Solution: Tangent Plane and Normal Line

Given Surface: $F(x,y,z) = 3x^4 + 3y^4 + 6z^4$ **Point of Interest:** $(1,1,1)$
Step 1: Compute Partial Derivatives

$$\frac{\partial F}{\partial x} = 12x^3, \quad \frac{\partial F}{\partial y} = 12y^3, \quad \frac{\partial F}{\partial z} = 24z^3$$

At $(1,1,1)$:

$$\frac{\partial F}{\partial x}(1,1,1) = 12, \quad \frac{\partial F}{\partial y}(1,1,1) = 12, \quad \frac{\partial F}{\partial z}(1,1,1) = 24.$$

Step 2: Gradient at $(1,1,1)$:

$$\nabla F(1,1,1) = 12\mathbf{i} + 12\mathbf{j} + 24\mathbf{k}.$$

Equation of the Tangent Plane

General Form:

$$\nabla F(1,1,1) \cdot (x-1, y-1, z-1) = 0.$$

Substituting values:

$$12(x-1) + 12(y-1) + 24(z-1) = 0.$$

Simplifying:

$$x + y + 2z = 4.$$

Thus, the equation of the tangent plane is $x + y + 2z = 4$.

Equation of the Normal Line

General Form:

$$(x, y, z) = (x_0, y_0, z_0) + t\nabla F(x_0, y_0, z_0),\ t \in \mathbb{R}.$$

Substituting $(1, 1, 1)$ and $\nabla F(1, 1, 1) = (12, 12, 24)$:

$$(x, y, z) = (1, 1, 1) + t(1, 1, 2),\ t \in \mathbb{R}.$$

Parametric equations of the normal line:

$$x = 1 + t, \quad y = 1 + t, \quad z = 1 + 2t.$$

Curves and Arc length

Let $x = x(t), y = y(t)$ and $z = z(t)$ be continuous functions of a real parameter t over a closed interval $[a, b]$.
The points

$$r(t) = (x(t), y(t), z(t)),$$

for $a \leq t \leq b$ are said to constitute a curve C joining the endpoints $r(a)$ and $r(b)$ and

$$r(t) = (x(t), y(t), z(t))$$

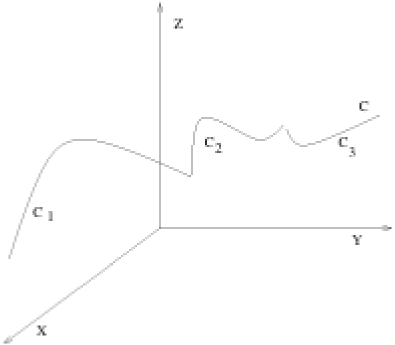
is called a **parametrization** of the curve.
We call the functions x, y and z, coordinate functions.

We call a curve C that is parameterized by $r(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$:

- continuous if each coordinate function is continuous;
- differentiable if each coordinate function is differentiable;
- closed if the initial and terminal points coincide, that is, $(x(a), y(a), z(a)) = (x(b), y(b), z(b))$ and if a curve is not closed it is called an arc;
- simple if $a < t_1 < t_2 < b$ implies that $(x(t_1), y(t_1), z(t_1)) \neq (x(t_2), y(t_2), z(t_2))$, in other words, if it does not intersect itself;
- smooth if the coordinate functions have continuous derivatives which are never all zero for the same value of t, that is, it possesses a tangent vector that varies continuously along the length of C.
- piecewise smooth if it has continuous tangent at all but finitely many points. Such a curve is a curve with a finite number of corner at which there is no tangent.

If C is a curve which is divided into smooth curves $C_1, C_2, \cdots, C_n$ such C begins with C_1, C_2 begins where C_1 ends and so on, but at the point where C_i and C_{i+1} join, there may be no tangent in the resulting curve , then C is piecewise smooth curve and we write such a curve as

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_n.$$



The following are examples of Curves.

1) **Stright Line:** A straight line L through a point P with position vector in the direction of a constant vector A can be represented as $r(t) = P + tA = (v_1 + ta_1, v_2 + ta_2, v_3 + ta_3)$, for all $t \in \mathbb{R}$, where $P = (v_1, v_2, v_3)$, $A = (a_1, a_2, a_3)$.
2) **Ellipse, circle:** The vector function: $r(t) = (a\cos t, b\sin t, 0)$ represents an ellipse and is a circle if $a = b$.
3) **Circular helix:** The twisted curve represented by the vector function: $r(t) = (a\cos t, a\sin t, ct)$, $c \neq 0$ is a circular helix.

Consider a curve C that is parameterized by $r(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$. If it exists, the derivative of $r(t_0)$ for $t_0 \in (a, b)$ is given by

$$r'(t_0) = x'(t_0)i + y'(t_0)j + z'(t_0)k.$$

The derivative of a function at a point is the slope of a tangent line to the graph of the function at the point. Hence the derivative $r'(t)$ (if it exists) of the curve is called the tangent vector to the curve at the point $r(t)$ and the equation of the tangent line to the curve C at point P is $g(s) = r + sr'$.

Example

Consider the curve $r(t) = (t^2, t^3, 2t)$. Find the tangent vector and the equation of the tangent line to the curve $r(t)$ at $t = 1$.

Solution:

1) **Tangent Vector:** The tangent vector $r'(t)$ is calculated by differentiating $r(t)$ component-wise:

$$r'(t) = \frac{d}{dt}(t^2)i + \frac{d}{dt}(t^3)j + \frac{d}{dt}(2t)k \implies r'(t) = 2ti + 3t^2j + 2k.$$

At $t = 1$, the tangent vector becomes:

$$r'(1) = 2(1)i + 3(1)^2j + 2k = 2i + 3j + 2k.$$

Hence, the tangent vector at $t = 1$ is $2i + 3j + 2k$.

2) **Tangent Line:** The equation of the tangent line at $t = 1$ is given by:

$$q(s) = r(t_0) + sr'(t_0),$$

where $t_0 = 1$, $r(t_0)$ is the position vector at $t = 1$, and $r'(t_0)$ is the tangent vector at $t = 1$.

First, calculate $r(1)$:

$$r(1) = ((1)^2, (1)^3, 2(1)) = (1, 1, 2) = i + j + 2k.$$

Now substitute $r(1)$ and $r'(1)$ into the equation:

$$q(s) = (i + j + 2k) + s(2i + 3j + 2k).$$

Thus, the equation of the tangent line is:

$$q(s) = (1 + 2s, 1 + 3s, 2 + 2s).$$

Definition

The length ℓ of the curve C which is given by the parametrization $r(t) = x(t)i + y(t)j + z(t)k$ on $[a, b]$ is defined by

$$\ell = \int_a^b \sqrt{r'(t).r'(t)}dt.$$

If we replace b (the fixed upper limit of integration) by a variable t, $a \leq t \leq b$, the integral becomes a function of t.

$$s(t) = \int_a^t \sqrt{r'.r'}d\tau, \quad \text{where} \quad r' = \frac{dr}{d\tau}$$

and is called the arc length function.

Differentiating the arc length function gives us

$$\frac{ds}{dt} = \sqrt{r'.r'} = \|r'(t)\| = \|v(t)\|.$$

Example

Let $r(t) = (a\cos t, a\sin t, ct), c \neq 0$. represent circular helix. Then find the arc length function $s(t)$.

Solution:

Example

Let $r(t) = (a\cos t, a\sin t, ct), c \neq 0$. represent circular helix. Then find the arc length function $s(t)$.

Solution: To compute the magnitude of the tangent vector, consider the dot product:

$$r'(t) \cdot r'(t) = (-a\sin t, a\cos t, c) \cdot (-a\sin t, a\cos t, c) = a^2 + c^2.$$

Hence, the magnitude of the tangent vector is constant and equals $\sqrt{a^2 + c^2}$. The arc length of the circular helix from $t = 0$ to $t = T$ is calculated as:

$$s(T) = \int_0^T \sqrt{r'(t) \cdot r'(t)}\, dt = \int_0^T \sqrt{a^2 + c^2}\, dt.$$

Since $\sqrt{a^2 + c^2}$ is constant, the arc length simplifies to:

$$s(T) = T\sqrt{a^2 + c^2}.$$

Thus, the arc length of the circular helix is proportional to T.

Example

Consider the curve $r(t) = (3t, 4t)$, representing a straight line in the xy-plane. Calculate the arc length of the curve from $t = 0$ to $t = 1$.

Solution:

Example

Consider the curve $r(t) = (3t, 4t)$, representing a straight line in the xy-plane. Calculate the arc length of the curve from $t = 0$ to $t = 1$.

Solution:

1) **Find the derivative:** The derivative of $r(t)$ is:

$$r'(t) = \frac{d}{dt}(3t)i + \frac{d}{dt}(4t)j = 3i + 4j.$$

2) **Magnitude of the derivative:** The magnitude of $r'(t)$ is:

$$\|r'(t)\| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

3) **Set up the arc length integral:** The arc length s from $t = 0$ to $t = 1$ is:

$$s = \int_0^1 \|r'(t)\| \, dt = \int_0^1 5 \, dt.$$

4) **Simplify and compute:** Since 5 is constant, the integral simplifies to:

$$s = 5 \int_0^1 1 \, dt = 5[t]_0^1 = 5(1 - 0) = 5.$$

Example

Consider the curve $r(t) = (t, t^2)$, which represents a parabola in the xy-plane. Calculate the arc length of the curve from $t = 0$ to $t = 2$.

Solution:

Example

Consider the curve $r(t) = (t, t^2)$, which represents a parabola in the xy-plane. Calculate the arc length of the curve from $t = 0$ to $t = 2$.

Solution:

1) **Find the derivative:** The derivative of $r(t)$ is:

$$r'(t) = \frac{d}{dt}(t)i + \frac{d}{dt}(t^2)j = i + 2tj.$$

2) **Magnitude of the derivative:** The magnitude of $r'(t)$ is given by:

$$\|r'(t)\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

3) **Set up the arc length integral:** The arc length s from $t = 0$ to $t = 2$ is:

$$s = \int_0^2 \|r'(t)\|\, dt = \int_0^2 \sqrt{1 + 4t^2}\, dt.$$

4) **Simplify and compute:**Thus, the arc length of the parabola from $t = 0$ to $t = 2$ is given by the integral $\int_0^2 \sqrt{1 + 4t^2}\, dt$.

Tangent, Curvature, and Torsion

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the position vector of a curve C for $a \leq t \leq b$. Assume the functions x, y, z are twice continuously differentiable.

- **Velocity and Speed:** If a particle moves along C, the velocity is:

$$\mathbf{v}(t) = \mathbf{r}'(t),$$

  and the speed is:

$$v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|.$$

- **Acceleration:** The acceleration is:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

If $r'(t) \neq 0$, then the vector $r'(t)$ is tangent to the curve C. Let $T(t)$ be a unit vector in the direction of $r'(t)$, i.e.

$$T(t) = \frac{1}{\|r'(t)\|} r'(t)$$

Let C be a smooth curve with parametrization $r(t)$ such that $r(t)$ is differentiable. The norm of the rate of change of the unit vector $T(t)$ with respect to the arc length function S is called the **curvature** K of the curve C. That is,

$$K(S) = \left\| \frac{dT}{dS} \right\|.$$

Consider the relation

$$\frac{dT}{dS} = \frac{dT}{dt} . \frac{dt}{dS} = \frac{dT/dt}{dS/dt}.$$

But $dS/dt = \|r'(t)\|$ and hence we get

$$K(t) = \frac{1}{\|r'(t)\|} \|T'(t)\|$$

which is a function of t.

Curvature

Curvature measures how sharply a curve bends at a point. It is defined as the norm of the rate of change of the unit tangent vector **T** with respect to arc length S.

$$K(S) = \left\| \frac{d\mathbf{T}}{dS} \right\|, \quad K(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Alternatively, using the cross product:

$$K(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Examples:

- For a straight line: $K(t) = 0$.
- For a circle of radius R: $K = \frac{1}{R}$.

Example: Curvature of a Line

Example

Let $\mathbf{r}(t) = (x_0 + ta)\mathbf{i} + (y_0 + tb)\mathbf{j} + (z_0 + tc)\mathbf{k}$. Find its curvature.

Solution:

Example: Curvature of a Line

Example

Let $\mathbf{r}(t) = (x_0 + ta)\mathbf{i} + (y_0 + tb)\mathbf{j} + (z_0 + tc)\mathbf{k}$. Find its curvature.

Solution:

$$\mathbf{r}'(t) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Since $\mathbf{T}'(t) = 0$, the curvature is:

$$K(t) = 0.$$

Curvature of a Circle

Example

For a circle $\mathbf{r}(t) = (a\cos t, a\sin t, 0)$, find its curvature.

Solution:

Curvature of a Circle

Example

For a circle $\mathbf{r}(t) = (a\cos t, a\sin t, 0)$, find its curvature.

Solution:

$$\mathbf{r}'(t) = (-a\sin t, a\cos t, 0), \quad \|\mathbf{r}'(t)\| = a.$$

The tangent vector is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

The curvature is constant:

$$K(t) = \frac{1}{a}.$$

Plane Curve: Curvature

Example

For $y = ax^2 + bx + c$, parameterize as $\mathbf{r}(t) = t\mathbf{i} + (at^2 + bt + c)\mathbf{j}$. Find the curvature.

Solution:

Plane Curve: Curvature

Example

For $y = ax^2 + bx + c$, parameterize as $\mathbf{r}(t) = t\mathbf{i} + (at^2 + bt + c)\mathbf{j}$. Find the curvature.

Solution:

- $\mathbf{r}'(t) = \mathbf{i} + (2at + b)\mathbf{j}, \quad \mathbf{r}''(t) = (2a)\mathbf{j}.$

The curvature:

$$K(t) = \frac{|2a|}{\left(1 + (2at + b)^2\right)^{3/2}}.$$

Given a curve C which is parameterized by the the position vector $r(t)$, we have a unit tangent vector T at a point where the coordinate functions are differentiable. Now we are looking to get a unit normal vector to the curve at a point where the coordinate functions are differentiable. Since T has constant length, $\frac{dT}{dS}$ is orthogonal to T. At a point where $K(S) \neq 0$, the vector

$$N = \frac{1}{K}.\frac{dT}{dS} = \frac{T'(t)}{\|T'(t)\|} = \frac{r''(S)}{\|r(S)\|}$$

is a unit vector parallel to $T'(t)$ and hence normal to the curve and it is called principal unit normal vector for a curve C.

Definition

The unit vector $B = T \times N$ is called the **binomial vector** of the curve C trace out by the vector field $r(t)$.
The vector $\frac{dB}{dS}$ is parallel to N. This implies that $\frac{dB}{dS} = -\tau N$ for some constant τ (the negative sign is traditional). Here the scalar τ is called the **torsion** along the curve and from $N.\frac{dB}{dS} = -\tau N.N = -\tau.1$ we have

$$\tau = -\frac{dB}{dS}.N.$$

Torsion

Torsion measures how much a curve twists out of the plane defined by its tangent and normal vectors. It is defined as:

$$\tau = -\frac{d\mathbf{B}}{dS} \cdot \mathbf{N}, \quad \tau = \frac{\mathbf{r}'(t) \cdot (\mathbf{r}''(t) \times \mathbf{r}'''(t))}{\|\mathbf{r}'(t)\|^2}$$

Key Concepts:

- **Binormal vector:** $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- For planar curves, $\tau = 0$.
- For a helix, τ is constant.

Curvature and Torsion Calculation

Example

For the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}$. Find the curvature and torsion.

Solution:

Curvature and Torsion Calculation

Example

For the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}$. Find the curvature and torsion.

Solution:

- $\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{a^2 + b^2}$.
- $\mathbf{r}''(t) = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j}$.
- $\mathbf{r}'(t) \times \mathbf{r}''(t) = (ab)\mathbf{k}$.

Curvature:

$$K(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{ab}{(a^2 + b^2)^{3/2}}.$$

Torsion:

$$\tau = \frac{\mathbf{r}'(t) \cdot (\mathbf{r}''(t) \times \mathbf{r}'''(t))}{\|\mathbf{r}'(t)\|^2} = \frac{b}{\sqrt{a^2 + b^2}}.$$

Example

Let $r(t) = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$. Find the curvature, principal unit vector, binomial vector, and torsion along the curve.

Solution:

Example

Let $r(t) = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$. Find the curvature, principal unit vector, binomial vector, and torsion along the curve.

Solution:

- **Step 1: Compute the first and second derivatives.**

$$r'(t) = 2t\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad r''(t) = 2\mathbf{i}.$$

- **Step 2: Curvature.** Compute the cross product:

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & -2 & 1 \\ 2 & 0 & 0 \end{vmatrix} = (-2\mathbf{i} - 2\mathbf{j} - 4t\mathbf{k}).$$

Find the norm:

$$\|r'(t) \times r''(t)\| = \sqrt{(-2)^2 + (-2)^2 + (-4t)^2} = \sqrt{8 + 16t^2}.$$

The curvature is:

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.$$

Compute the norm of $r'(t)$:

$$\|r'(t)\| = \sqrt{(2t)^2 + (-2)^2 + (1)^2} = \sqrt{4t^2 + 5}.$$

Substituting:

$$K(t) = \frac{\sqrt{8 + 16t^2}}{(4t^2 + 5)^{3/2}}.$$

- **Step 3: Principal Unit Normal Vector.** The principal unit normal vector is:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

where $\mathbf{T}(t) = \frac{r'(t)}{\|r'(t)\|}$.

- **Step 4: Binomial Vector.** The binomial vector is:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

- **Step 5: Torsion.** The torsion is:

$$\tau = \frac{r'(t) \cdot (r''(t) \times r'''(t))}{\|r'(t)\|^2}.$$

Frenet-Serret Formulas

- **Tangent Vector:** $\mathbf{T}'(t) = K(t)\mathbf{N}(t)$
- **Normal Vector:** $\mathbf{N}'(t) = -K(t)\mathbf{T}(t) + \tau\mathbf{B}(t)$
- **Binormal Vector:** $\mathbf{B}'(t) = -\tau(t)\mathbf{N}(t)$

Here, K is the curvature and τ is the torsion.

Applications of Curvature and Torsion

Curvature and torsion have wide-ranging applications in various fields:

- **Physics:** Modeling particle trajectories, bending and twisting mechanics.
- **Engineering:** Design of roads, bridges, and railways; structural optimization.
- **Biology:** Studying shapes of DNA and proteins.

Divergence and Curl

In this section we will discuss two important vector operations. One produces a scalar field from a vector field and the other produces a vector field from a vector field.

Definition

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field given by

$$V(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k.$$

1) The **divergence** of V, denoted by $divV$, is the scalar field defined by

$$divV = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

2) The **curl** of V, denoted by $curlV$, is the vector field defined by

$$curlV = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) k.$$

Example

Let $V(x, y, z) = 3xy\mathbf{i} - 2yz\mathbf{j} + x^2\mathbf{k}$. Find div$V$ and curl$V$.

Solution:

Example

Let $V(x, y, z) = 3xy\mathbf{i} - 2yz\mathbf{j} + x^2\mathbf{k}$. Find $\mathrm{div} V$ and $\mathrm{curl} V$.

Solution:

- $\mathrm{div} V = \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(-2yz) + \frac{\partial}{\partial z}(x^2) = 3y - 2z + 0 = 3y - 2z.$
- $\mathrm{curl} V = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & -2yz & x^2 \end{vmatrix} =$
  $(-2y - 0)\mathbf{i} - (2x - 0)\mathbf{j} + (-2z - 3x)\mathbf{k} = -2y\mathbf{i} - 2x\mathbf{j} - (2z + 3x)\mathbf{k}.$

Let ∇ be the operator defined from the set of scalar fields of three variables into the set of vectors in $\mathbb{R}^3$ by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$

If F is a scalar field of three variables, then the products $\frac{\partial}{\partial x}(F), \frac{\partial}{\partial y}(F)$ and $\frac{\partial}{\partial z}(F)$ are defined to be $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ respectively.

Remark (The ∇ operator and gradient, divergence and curl.)

1. The product of ∇ and a scalar field F in the given order is the gradient of F, that is,

$$\nabla F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)F = \frac{\partial F}{\partial x}i + \frac{\partial F}{\partial y}j + \frac{\partial F}{\partial z}k = gradF.$$

2. The product of ∇ and a vector field V in the given order is the divergence of V , that is, if $V = fi + gj + hk$, then

$$\nabla .V = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(fi+gj+hk) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = divV.$$

3. The cross product of ∇ and a vector field V is the curl of V, that is, if $V = fi + gj + hk$, then

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k = curlV$$

Exercise 1

Exercise: Find the divergence and curl of $V(x, y, z) = (xz, -yz, x^2y)$.

Exercise 1

Exercise: Find the divergence and curl of $V(x, y, z) = (xz, -yz, x^2 y)$.
Solution:

- **Divergence:**
  - $\operatorname{div} V = \frac{\partial(xz)}{\partial x} + \frac{\partial(-yz)}{\partial y} + \frac{\partial(x^2 y)}{\partial z}$
  - $\operatorname{div} V = z - z + 0 = 0$
- **Curl:**
  - $\operatorname{curl} V = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -yz & x^2 y \end{vmatrix}$
  - $\operatorname{curl} V = (x^2 + y)\mathbf{i} - (2xy - x)\mathbf{j}$

Potential Function

Recall that, if a scalar field f is differentiable at every point D of its domain, then $V(P) = \nabla f(P)$ defines a vector field V on D.

Example

If $f(x,y) = 3x^2 + xy + y^3$, then $\nabla f(x,y) = V(x,y) = (6x + y, x + 3y^2)$ is a vector field. Here the function f is called a potential of the vector V. However, not every vector field has a potential, that is, not every vector field is a gradient of some scalar field.

How do we check whether a vector field has a potential or not? The following proposition will answer this question.

Proposition (Test for Existence of a Potential Function)

1) Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by $V(p) = (V_1(p), V_2(p))$. Then V has a potential function if and only if

$$\frac{\partial V_1}{\partial y}(p) = \frac{\partial V_2}{\partial x}(p) \text{ for all } p \text{ in the Domain of } V.$$

2) Let $V : \mathbb{R}^n \to \mathbb{R}^3,\ n = 2, 3$ be given by $V(p) = (V_1(p), V_2(p), V_3(p))$. Then V has a potential function if and only if $curlV = \nabla \times V = 0 = (0, 0, 0)$.

Example

1. Let $V(x, y, z) = (2 + y, x - z^2, -2yz)$. Then show that V has a potential function.
2. Let $V(x, y) = (-cy, cx), c \in \mathbb{R}\backslash 0$. Then show that V has no potential function.

Proof:

To find the potential function f consider the following example.
Let $V(x,y) = (6x + y, x + 3y^2)$ be given. Then if there is a potential function f for V it must satisfy

$$f_x(x,y) = 6x + y \quad \text{and} \quad f_y(x,y) = x + 3y^2.$$

This implies

$$f(x,y) = \int (6x + y)dx = 3x^2 + xy + A(y),$$

where $A(y)$ is constant with respect to x (or, it is a function of y only).
Then from $f_y(x,y) = x + 3y^2$, we get $f_y(x,y) = x + A'(y) = x + 3y^2$, which implies that $A'(y) = 3y^2$ and hence

$$A(y) = 3y^2 dy = y^3 + C, \quad \text{where } C \text{ is a constant}$$

Therefore the scalar field $f(x,y) = 3x^2 + xy + y^3 + C$ is the potential of the vector field

$$V(x,y) = (6x + y, x + 3y^2).$$

Example

Let $V(x,y,z) = (2+y, x-z^2, -2yz)$. Find a potential function f for V.

Example

Let $V(x, y, z) = (2 + y, x - z^2, -2yz)$. Find a potential function f for V.

Solution

- $f_x = 2 + y \implies f = 2x + xy + A(y, z)$
- $f_y = x - z^2 \implies x + A_y = x - z^2 \implies A_y = -z^2 \implies A = -yz^2 + B(z)$
- $f_z = -2yz \implies -2yz + B'(z) = -2yz \implies B'(z) = 0 \implies B(z) = C$
- $f(x, y, z) = 2x + xy - yz^2 + C$

Exercise 1

Exercise 1: Determine if $W(x, y) = (2xy, x^2 + 1)$ has a potential function, and if so, find it.

Solution:

Exercise 1

Exercise 1: Determine if $W(x, y) = (2xy, x^2 + 1)$ has a potential function, and if so, find it.

Solution:

- **Test for potential:**
  - $\frac{\partial(2xy)}{\partial y} = 2x$
  - $\frac{\partial(x^2+1)}{\partial x} = 2x$
  - Since $\frac{\partial W_1}{\partial y} = \frac{\partial W_2}{\partial x}$, W has a potential function.
- **Finding the potential function:**
  - $f_x = 2xy \implies f = x^2y + A(y)$
  - $f_y = x^2 + 1 \implies A'(y) = 1 \implies A(y) = y + C$
  - $f(x, y) = x^2y + y + C$

Exercise 2

Exercise 2: Show that $V(x, y, z) = (y^2z^3, 2xyz^3, 3xy^2z^2)$ has a potential function and find it.
Solution:

Exercise 2

Exercise 2: Show that $V(x, y, z) = (y^2z^3, 2xyz^3, 3xy^2z^2)$ has a potential function and find it.

Solution:

- **Test for potential:**
  - $\mathrm{curl}V = \mathbf{0}$
  - Thus, V has a potential function.
- **Finding the potential function:**
  - $f_x = y^2z^3 \implies f = xy^2z^3 + A(y, z)$
  - $f_y = 2xyz^3 \implies A_y(y, z) = 0 \implies A(y, z) = B(z)$
  - $f = xy^2z^3 + B(z)$
  - $f_z = 3xy^2z^2 \implies B'(z) = 0 \implies B(z) = C$
  - $f(x, y, z) = xy^2z^3 + C$

Conclusion

- Divergence and curl are powerful tools for analyzing vector fields.
- Potential functions provide a scalar representation of conservative vector fields.
- These concepts are fundamental in various areas of applied mathematics and physics.