Applied Mathematics III
Unit 4
Line and Surface Integrals

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Line Integrals

Recall that, the integral $\int_a^b f(x)dx$ of a continuous function f represents the definite integral of f over a closed interval $[a,b] = \{x : a \leq x \leq b\}$. In the line integral we shall integrate a function F along a curve, say C.

Definition

The Line Integral of a vector valued function $F(r)$ over a curve C parameterized by $r(t) = x(t)i + y(t)j + z(t)k$ for $a \leq t \leq b$ is defined by

$$\int_C F(r).dr = \int_a^b F(r(t)).\frac{dr}{dt}dt, \quad \text{where} \quad dr = (dx, dy, dz).$$

When we write it componentwise, i.e, if $F = (F_1, F_2, F_3)$, then it becomes:

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')dt.$$

The curve C is called path of integration.

Example

1) Let $F(r) = (y, x, z)$ and C be the helix $r(t) = (\cos tt, \sin t, 3t)$ for $0 \leq t \leq 2\pi$. Then find $\int_C F(r).dr$.

Solution:

2) Evaluate $\int_L (xyzdx - \cos(xy)dy + ydz)$, where L is the line segment from $(0, 1, 1)$ to $(2, 1, -3)$.

Solution:

Let C be a curve with parametrization r on the closed interval $[a, b]$ and F be a vector valued function defined on C.

1. The integrand in the line integral is a scalar not a vector, because we take a dot (scalar) product of two vectors, $F(r(t)).r'(t)$.
2. If the integrand function F is a scalar valued function, the line integral will take the following form.

$$\int_C f(x,y,z)dS = \int_a^b f(x(t),y(t),z(t))\sqrt{(r'(t))^2}.dt = \int_a^b f(x(t),y(t),$$

3. If the path of integration C is a closed curve, that is, if $r(a) = r(b)$, then the line integral will be denoted by

$$\oint_C \quad \text{instead of} \quad \int_C .$$

Example

Determine the mass M of a wire that is in the shape of a helical curve $C : r(t) = (a\cos t, a\sin t, bt)\ 0 \leq t \leq 2n\pi, n \in \mathbb{N}$ and that has a mass density $\sigma = ct$ that varies along C.

Line Integrals Independent of Path

Consider the line integral

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz), \tag{1}$$

where C is the path of integration. The line integral (1) is said to be **independent of path of integration** in a domain D if for every pair of endpoints A and B in D the integral (1) has the same value for all paths in D that begin at A and end at B.

Theorem

Let F_1, F_2 and F_3 be continuous functions in a set D and let $F = (F_1, F_2, F_3)$. A line integral (1) is independent of path in D if and only if $F = (F_1, F_2, F_3)$ is the gradient of some potential function f in D, i.e., if there exists a function f in D such that $F = \nabla f$, which is equivalent to saying that

$$F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y} \ \text{ and } \ F_3 = \frac{\partial f}{\partial z}.$$

Let F be a conservative vector valued function with potential function f and let C be a curve with coordinate function $x = x(t), y = y(t), z = z(t)$ for $a \leq t \leq b$. Then

$$\begin{aligned}
\int_C F.dr &= \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \\
&= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} \left(f(x(t), y(t), z(t)) \right) dt = f(B) - f(A),
\end{aligned}$$

where $f(B) = f(x(b), y(b), z(b)), \;\; f(A) = f(x(a), y(a), z(a))$, the end points of the curve C. Therefore, we have the following theorem and some times it is called **Fundamental Theorem of Line Integrals**

Theorem

If the integral (1) is independent of path in D, then

$$\int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A), \quad \textit{where} \;\; F = \nabla f.$$

Example

Evaluate:

1. $\displaystyle\int_C (2xdx + 2ydy + 4zdz)$ if C is a curve with initial point $A = (0, 0, 0)$ and terminal point $B = (2, 2, 2)$.
2. $\displaystyle\int_C (e^z dx + 2ydy + xe^z dz)$ if C is a curve with initial point $A = (0, 1, 0)$ and terminal point $B = (-2, 1, 0)$.

Solution:

Remark

1. The line integral (1) is independent of path in a domain D if and only if its value around every closed path in D is zero.
2. The line integral usually represents the work done by force F in the displacement of a body along path C. Hence if F has a potential function f, the line integral of F for displacement around any closed path is zero. In this case, the vector field F is called **conservative**, otherwise it is called **nonconservative**.

Definition

A domain D is said to be simply connected if every closed curve in D can be shrunk to any point in D.

Theorem

Suppose F_1, F_2 and F_3 are continuous and having continuous first order partial derivatives in a domain D and consider the line integral

$$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz) \tag{2}$$

1. *If the line integral (2) is independent of path in D, then $CurlF = 0$. i.e.*
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \ \text{ and } \ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$
2. *If $CurlF = 0$ in D and if D is simply connected then the line integral (2) is independent of path in D.*

Remark

In the plane, $\mathbb{R}^2$, the line integral $\int_C F(r).dr = \int_C (F_1 dx + F_2 dy)$ and $CurlF = 0$ means

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Example

Show that the following integrands are exact and evaluate the integrals.

1. $\int_{(0,\pi)}^{(3,\frac{\pi}{2})} e^x(\cos y dx - \sin y dy)$
2. $\int_A^B \left(2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right)$, where $A = (0,0,1)$ and $B = (1, \frac{\pi}{4}, 2)$.

Solution:

General Properties of Line integrals

Let F and G be continuous vector fields, C be a path joining points A and B. Furthermore, suppose that C is subdivided into two arcs C_1 and C_2 that have the same orientation as C. Then

1. $\int_C kF.dr = k\int_C F.dr$ for any constant k.
2. $\int_C (F+G).dr = \int_C F.dr + \int_C G.dr$
3. $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr$
4. If C' has an opposite orientation to that of C, then
$$\int_C F.dr = -\int_{C'} F.dr.$$

Example

Let C be a curve consisting of portion of a parabola $y = x^2$ in the $xy-$plane from $(0,0)$ to $(2,4)$ and a horizontal line from $(2,4)$ to $(4,4)$. Evaluate $\int_C (y^2dx + x^2dy)$.

Green's Theorem

Over a plane region, double integrals can be transformed into line integrals over the boundary of the regions and conversely. This can be done using Green's Theorem which is stated stated below.

Theorem (Green's Theorem)

Let R be a closed bounded region in the $xy-$plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives every where in some domain containing R, (i.e.$(F_1)_y$ and $(F_2)_x$ are continuous in R.) Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy), \tag{3}$$

where C is the boundary of R.

Example

1. Use Green's Theorem to evaluate $\int_C (x^2 y dx + y dy)$ over the triangular path $(0,0)$ to $(1,0)$, $(1,0)$ to $(1,2)$ and $(1,2)$ back to $(0,0)$.
2. Find the work done by the force field $F(x,y) = (e^x - y^3)i + (\cos y + x^3)j$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction.

Solution:

Green's Theorem can be used to find areas of a plane region.
Let R be a plane region with boundary C.

1. Area of the region R in cartesian coordinates. First choose $F_1 = 0$ and $F_2 = x$. Then, as in the above,

$$\iint_R dxdy = \oint_C xdy$$

and then choose $F_1 = -y$ and $F_2 = 0$ to get

$$\iint_R dxdy = -\oint_C ydx$$

By adding up the two we get

$$2\iint_R dxdy = \oint_C xdy - \oint_C ydx = \oint_C (xdy - ydx)$$

Therefore, the area $A(R)$ of the region bounded by the curve C is given by:

$$A(R) = \iint_R dxdy = \frac{1}{2}\oint_C (xdy - ydx)$$

Example

Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution:

2. Area of a plane region in polar coordinates.
   Let $x = r\cos\theta$ and $y = r\sin\theta$, where (r, θ) is the polar coordinate of point (x, y). Then
   $dx = \cos\theta dr - r\sin\theta d\theta, \quad dy = \sin\theta dr + r\cos\theta d\theta$. Hence (4) becomes
$$A(R) = \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\oint_C r^2 d\theta.$$

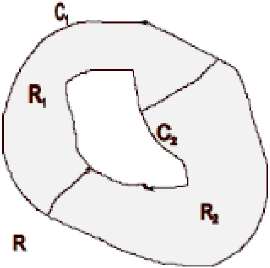
Example

find the area of the region bounded by the cardioid $r = a(1 - \cos\theta)$, where $0 \le \theta \le 2\pi$ and a is a positive constant.

Solution:

Green's Theorem for Multiply Connected Regions

Recall that, a region in $\mathbb{R}^2$ is called **simply connected** if it is connected and has no holes, and is called **multiply connected** if it is connected but it has finitely many holes.
Now consider the vector field $F = (F_1, F_2)$ which is continuously differentiable over the plane region R, which is multiply connected as shown in Figure below



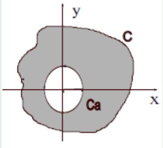
First divide R into simply connected regions R_1 and R_2. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$
$$= \oint_{C_1} (F_1 dx + F_2 dy) + \oint_{C_2} (F_1 dx + F_2 dy),$$

where the curves C_1 and C_2 are the boundaries of the regions R_1 and R_2 respectively. The orientation of the curves should be in such a way that when traveling along the curves the region should be to the left.

Example

Evaluate the integral $\oint_C \left(\frac{-ydx + xdy}{x^2 + y^2}\right)$, if C is a piecewise smooth simply closed curve oriented counterclockwise such that C incloses the origin. Consider the Figure below.



Surface Integrals

In the previous sections we have been working on integral of vector fields over curves. Now we are going to consider integrals of vector fields over surfaces. Let us start by discussing some facts about surfaces.
In the case of line integrals we represented a curve in $\mathbb{R}^3$ perimetrically as

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

for $a \leq t \leq b$. That is, a curve is given by coordinate functions of one variable, where as, a surface is defined by parametric functions of two variables

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

for (u, v) in some set in the $uv-$plane and the variables u and v are called parameters.

Parametrization of some surfaces

1. The parametric representation of a cylinder

$$x^2 + y^2 = a^2, \quad -1 \le z \le 1 \quad \text{is} \quad r(u, v) = a\cos ui + a\sin uj + vk.$$

2. The parametric representation of a sphere

$$x^2 + y^2 + z^2 = a^2 \text{is} r(u, v) = a\cos v\cos ui + a\cos v\sin uj + a\sin vk.$$

3. The parametric representation of a cone

$$z = \sqrt{x^2 + y^2}, 0 \le z \le T \text{is} r(u, v) = u\cos vi + u\sin vj + uk$$

where $0 \le u \le T$ and $0 \le v \le 2\pi$.

For a surface we write a position vector as

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$$

and $r(u, v)$ can be considered as a vector in $\mathbb{R}^3$ with initial point the origin and terminal point $(x(u, v), y(u, v), z(u, v))$ which is on the surface.
A surface with parametrization r is simple if it does not fold over and intersect itself. This means $r(u_1, v_1) = r(u_2, v_2)$ can occur only when $u_1 = u_2$ and $v_1 = v_2$.

Normal Vector and Tangent plane to a Surface

If C is a curve with coordinate functions $x(t), y(t), z(t)$, then

$$T = x'(t_0)i + y'(t_0)j + z'(t_0)k$$

is a vector that is tangent to the curve at a point $p_0 = (x(t_0), y(t_0), z(t_0))$.
Let Ω be a surface in $\mathbb{R}^3$ with coordinate functions $x(u, v), y(u, v), z(u, v)$ and let P_0 be the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ on the surface Ω.
We want to find a normal vector N to the surface at P_0.
Let Ω_u be the curve with coordinate functions $x(u, v_0), y(u, v_0), z(u, v_0)$.
Then the tangent vector

$$T_{U_0} = \frac{\partial x}{\partial u}(u, v_0)i + \frac{\partial y}{\partial u}(u, v_0)j + \frac{\partial z}{\partial u}(u, v_0)k$$

is a tangent vector to the curve Ω_u at P_0. Similarly, the vector

$$T_{v_0} = \frac{\partial x}{\partial u}(u_0, v)i + \frac{\partial y}{\partial u}(u_0, v)j + \frac{\partial z}{\partial u}(u_0, v)k$$

is a tangent vector to the curve Ω_v with coordinate functions $x(u_0, v), y(u_0, v), z(u_0, v)$.

Assume that these two vectors are not zero. Then these two vectors lie in a plane tangent to the surface Ω at the point P_0 and hence the vectors

$$N(P_0) = T_{u_0} \times T_{v_0}$$

is a normal vector to the tangent plane and hence the surface to at the point P_0.
Then the normal vector

$$N(P_0) = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}i + \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}j + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}k,$$

where the partial derivatives are evaluated at (u_0, v_0). For an arbitrary point (u, v) on the surface, the normal line to the tangent plane is given by $N = r_u \times r_v$ and we denote the corresponding unit vector in the direction of N by n and it is given by

$$n = \frac{1}{\|N\|}N = \frac{1}{\|r_u \times r_v\|}(r_u \times r_v).$$

Remark

If the surface Ω is a surface represented by the equation $g(x, y, z) = 0$, then the unit normal vector is given by

$$n = \frac{1}{\|\nabla g\|} \nabla g.$$

Example

1. Find the equation of the tangent plane to the surface given by $r(u, v) = ui + (u + v)j + (u + v^2)k$ at a point$(2, 4, 6)$.
2. If Ω is the sphere $f(x, y, z = x^2 + y^2 + z^2 - a^2$ and $a \neq 0$, then find n.

Solution:

Suppose Ω represents a surface in $\mathbb{R}^3$ with equation $z = g(x, y)$ and let R be its projection on the $xy-$plane. If g has continuous first partial derivatives on R, then the surface area of Ω is

$$\text{Area of } \Omega = \iint_R \sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)} dA.$$

But the integrant is the norm of the normal vector $N(x, y)$ to the surface, that is,

$$\text{Area of } \Omega = \iint_R \|N(x, y)\| dA.$$

Definition

A surface S is called a **smooth** surface if the unit normal vector n is continuous on S and surface S is called **piecewise smooth** if it consists of finitely many smooth portions.

Example

A sphere is smooth but a cube is picewise smooth.

Definition

Suppose S is a smooth surface parameterized by $r(u, v)$ with normal vector $N(u, v) = r_u \times r_v$. Let F be a continuous function on S. Then the surface integral of F over S is denoted by $\iint_S F(x, y, z)d\sigma$ and is defined by

$$\iint_S F(x, y, z)d\sigma = \iint_R F(r(u, v))\|N(u, v)\|dudv$$

and if F is a vector field then the surface integral of F over S $\iint_S F(x, y, z)d\sigma$ is defined by

$$\iint_S F(x, y, z)d\sigma = \iint_R F(r(u, v)).N(u, v)dudv.$$

Example

Evaluate $\iint_S (x + y)\sigma$ where S is the portion of the cylinder $x^2 + y^2 = 3$ between the planes $z = 0$ and $z = 6$.

Applications of Surface Integrals

Flux of A fluid Across a Surface

Suppose a fluid moves in some region of the space with velocity. The volume of fluid crossing a certain surface S per unit time is known as the **flux** across the surface S and the surface integral of a vector function F over a surface S describes the flux across S, when $F = \rho v, \quad \rho$ the density of fluid, v velocity of the flow.

Hence the above surface integral is known as the **flux integral** and if $F = (F_1, F_2, F_3)$ and $N = (N_1, N_2, N_3)$, then

$$\iint_S F.ndA = \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3) dudv.$$

Similarly, for surface S in surface integrals we parameterize the surfaces. But since surfaces are two dimensional; S can be represented as

$$Y(u,v) = x(u,v)i + y(u,v)j + z(u,v)k, \quad (u,v) \in R,$$

where R is some region in $uv-$plane.

A normal vector N of a surface S whose parametric form is

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$$

at the point P is $N = r_u \times r_v$.
We denote the corresponding unit vector in the direction of N by n,

$$n = \frac{1}{\|N\|}N = \frac{1}{\|r_u \times r_v\|}(r_u \times r_v).$$

If a surface S is represented by a the equation $g(x, y, z) = 0$, then

$$n = \frac{1}{\|\nabla g\|}\nabla g.$$

Example

Let S be the portion of the surface $z = 1 - x^2 - y^2$ that lie above the $xy-$plane, and suppose that S is oriented upward (i.e. n is in the upward direction at all points of S). Find the flux ϕ of the flow field $F(x, y, z) = (x, y, z)$ across S.

Surface Area

If Ω is a piecewise smooth surface, then the area of the surface Ω is given by

$$\text{Area of } \Omega = \iint_{\Omega} dA.$$

But $\|N\| = \|r_u \times r_v\|$ represents the area of a parallelogram with adjacent side vectors r_u and r_v. Therefore, we can write dA as $dA = \|r_u \times r_v\| dudv$. Hence

$$\text{Area of } \Omega = \iint_{\Omega} dA = \iint_{R} \|r_u \times r_v\| dudv,$$

where R is the projection on the $uv-$plane of the surface Ω.

Mass and Center of Mass of a Shell

Consider a shell of negligible thickness in the shape of piecewise smooth surface Ω. Let $\delta(x, y, z)$ be the density of the material of the shell at point (x, y, z).

Let $x(u, v), \quad y(u, v)$ and $z(u, v)$ be the coordinate functions of Ω for $(u, v) \in R$, where R is the projection of the surface in the $xy-$plane. Then the mass of Ω is given by

$$\text{Mass of } \Omega = \iint_{\Omega} \delta(x, y, z) d\sigma$$

and the center of mass of the shell is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_{\Omega} x\delta(x, y, z) d\sigma, \ \bar{y} = \frac{1}{m} \iint_{\Omega} y\delta(x, y, z) d\sigma \ \& \ \bar{z} = \frac{1}{m} \iint_{\Omega} z\delta(x, y, z$$

where m is the mass of the shell.

If the surface is given by $z = f(x, y)$ for $(x, y) \in R$, then the mass is given by

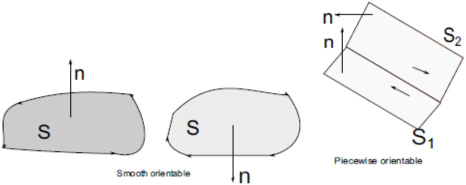
$$m = \iint_{\Omega} \delta(x, y, z) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dydx.$$

Example

Find the center of mass of the sphere Ω, $x^2 + y^2 + z^2 = a^2$, in the first octant, if it has constant density μ_0.

Solution:

If a surface S is smooth and P is any point in S we can choose a unit normal vector n of S at P. Then we can take the direction of n as the positive normal direction of S at P(two possibilities).
A smooth surface is said to be **orientable** if the positive normal direction, given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface.
A smooth surface is said to be **piecewise orientable** if we can orient each smooth piece of the surface S in such a manner that along each curve C^* which is a common boundary of two pieces S_1 and S_2 the positive direction of C^* relative to S_1 is opposite to the positive direction of V^* relative to S_2.



There are also non-orientable surfaces. Mobius strip [no inward and no outward directions once in once out word.]
Consider a boundary surface of a solid region D in 3-space. Such surfaces are called **closed**. If a closed surface is orientable or piecewise orientable, then there are only two possible orientations: inward (to ward the solid) and outward (away from the solid).

Theorem (Divergence Theorem of Gauss)

Let D be a solid in $\mathbb{R}^3$ with surface S oriented outward. If $F = F_1 i + F_2 j + F_3 k$, where F_1, F_2 and F_3 have continuous first and second partial derivatives on some open set containing D, then

$$\iint_S F.ndA = \iiint_D divFdv,$$

that is,

$$\iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy).$$

Example

1. Let S be the sphere $x^2 + y^2 + z^2 = a^2$ oriented outward. Find the flux of the vector function $F(x, y, z) = zk$ across S.
2. Let S be the surface of the solid enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$, oriented outward. use the Divergence theorem to find the flux ϕ of the vector field $F(x, y, z) = x^3 i + y^3 j + z^2 k$ across S.

Solution:

Theorem (Stoke's Theorem)

Let S be a piecewise smooth orientable surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the components of $F = (F_1, F_2, F_3)$ are continuous and have continuous first partial derivatives on some open set containing S, and if T is the unit tangent vector of C, then.

$$\oint_C F.TdS = \iint_S (curlF).ndA.$$

Recall that, $T = \frac{dr}{dS}$ which implies $dr = TdS$. Hence the above formula takes the following form

$$\oint_C F.dr = \iint_S (curlF).ndA.$$

Example

Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$, and let the vector field $F(x, y, z) = 2zi + 3xj + 5yk$ is defined on S. Verify Stoke's Theorem , if S is oriented upward.

Solution:

Remark

1. If S_1 and S_2 have the same boundary C which is oriented positively, then for any vector function F that satisfy the hypotheses in Stoke's Theorem, we have:

$$\iint_{S_1} (curlF).ndA = \iint_{S_2} (curlF).ndA.$$

2. If $F = (F_1, F_2)$ is a vector function that is continuously differentiable in a domain in the $xy-$plane containing a simply connected domain S whose boundary C is a piecewise smooth simple closed curve, then

$$(curlF).n = (curlF).k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Hence from Stoke's Theorem we have:

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C (F_1 dx + F_2 dy),$$

which is the result of Green's Theorem.