3 Limit and Continuity

3.1 Definition of Limits

Until know we have been evaluating the limit of a function by using its intuitive definition. That is we have said that limit of $f(x)$ as x approaches to a is L and write

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ close enough to L by choosing x close enough to a but distinct from a. Although this intuitive definition is sufficient for solving limit problems it is not prices enough. In this section we see the formal definition of limit, which we call the $\varepsilon - \delta$ definition of limit.

Definition 3.1 (Formal definition of limit)

The limit of $f(x)$ as x approach a is L, written

$$\lim_{x \to a} f(x) = L$$

if every ε > 0, there exists a δ > 0 such that |f(*x*) - L| < ε whenever 0 < |*x* - *a*| < δ.

In Definition 3.1 above we should not that

I. The absolute value symbol is read as " the distance between" for instance **|*x* - *a*|** is the distance between x and *a*.

II. Notice that **|x - a| >0.** In other words *x* is not equal to *a*.

So with this in mind we can read the definition as:

" The distance between $f(x)$ and L can be made smaller than any positive number ε, whenever the distance between x and *a* is less than some number δ and x does not equal *a*." Fig 3.1 below represents this idea pictorially.

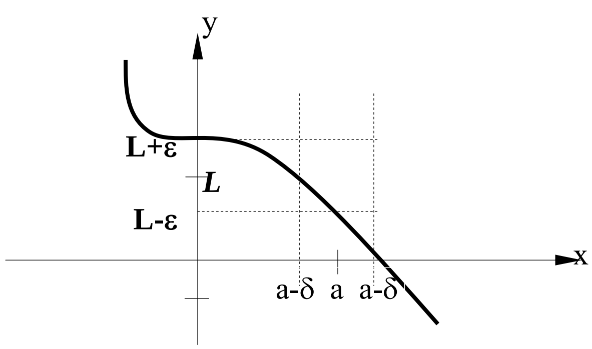


Fig 3.1

If we wish to use a form of Definition 3.1 that does not contain absolute value symbols we can have the following alternative definition of limit.

Definition 3.2 $\lim_{x \to a} f(x) = L$ if and only if for every ε>0, there is a δ>0 such that if x is in the open interval (a – δ, a +δ) and x ≠ a then f(x) is in the open interval (L –ε, L + ε).

Using either of the definitions of limit given above we can prove the following theorem.

Theorem 3.3 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$ then L=M.

The above theorem tells us that if a limit of a function f(x) at a exists then it must be *unique*.

3.2 Examples on limit

Even if it is very difficult to us the formal definition of limit to handle all limit problems let us see how we can use it for evaluating same important limits that may help us in developing rules by the way of which we can evaluate limits with out using the formal definition.

Example 1 Assume that $\lim_{x\to 2} 5x-7=3$. By using properties of inequalities, determine a δ> 0 such that

$$\text{if } 0<|x-2|<\delta \quad \textit{then} \quad |(5x-7)-3|<0.01.$$

Solution: By considering $|(5x-7)-3|<0.01$ we can see that

$$\begin{aligned} |(5x-7)-3|<0.01 &\Leftrightarrow |5x-10|<0.01 \\ &\Leftrightarrow 5|x-2|<0.01 \\ &\Leftrightarrow |x-2|<0.002 \end{aligned}$$

so now it is clear that if we choose δ = 0.002 statement holds but to check our result holds we proceed as follows:

$$\begin{aligned} \textit{if} \quad 0<|x-2|<\delta \quad \textit{then} \ |x-2|&<0.002 \\ \Rightarrow |x-2|&<0.01/5 \\ \Rightarrow 5|x-2|&<0.01 \\ \Rightarrow |5x-10|&<0.01 \\ \Rightarrow |(5x-7)-3|&<0.01 \end{aligned}$$

Thus we have shown that the choice of δ = 0.002 satisfies the statement

$$\text{if } 0<|x-2|<\delta \quad \textit{then} \quad |(5x-7)-3|<0.01.$$

This example is for the specific ε = 0.01. The general case can be seen as follows.

Example 2 Show that

$$\lim_{x\to 2} 5x-7=3$$

Solution:

We need to show that given **ε > 0** then there exists **δ > 0** such that

$$\text{if } 0<|x-2|<\delta \quad \textit{then} \quad |(5x-7)-3|<\varepsilon$$

To choose an appropriate δ we start with $|(5x-7)-3|<\varepsilon$ then we have

$$\begin{aligned} |5x-10|<\varepsilon &\Rightarrow 5|x-2|<\varepsilon \\ &\Rightarrow |x-2|<\frac{\varepsilon}{5} \end{aligned}$$

Hence, we let

$$\delta=\frac{\varepsilon}{5}$$

this proves that $\lim_{x\to 2} 5x-7=3$.

Example 3 Prove that $\lim_{x\to 5} 7 = 7$.

Solution:

Begin by letting ε > **0** be given. Find δ > **0** so that if , $0 < |x - 5| < \delta$ then $|f(x) - 7| < \epsilon$, i.e., $|7 - 7| < \epsilon$ i.e., $|0| < \epsilon$. But this trivial inequality is always true, no matter what value is chosen for δ. For example, $\delta = \frac{1}{2}$ will work. Thus, if $0 < |x - 5| < \delta$, then it follows that $|f(x) - 7| < \epsilon$. This completes the proof.

A similar proof as example 2 shows us that for any number *a* and k

$$\lim_{x \to a} k = k \qquad (1)$$

Example 4 Prove that $\lim_{x \to a} kx = ka$ for any real number k.

Solution: from (1) it is clear that if c = 0

$$\lim_{x \to a} kx = \lim_{x \to 0} 0 = 0 = 0.a = k.a.$$

If k≠0, letting ε>0 we must find a δ>0 so that

$$0 < |x - a| < \delta \;\Rightarrow |kx - ka| < \varepsilon$$

since

$$|kx - ka| < \varepsilon \Rightarrow |k||x - a| < \varepsilon$$

$$\Rightarrow |x - a| < \frac{\varepsilon}{|k|}$$

choose $\delta = \frac{\varepsilon}{|k|}$.

Example 5 Prove that $\lim_{x \to 1} (x^2 + 3) = 4$.

Solution: Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 1| < \delta$, then $|f(x) - 4| < \epsilon$. Begin with $|f(x) - 4| < \epsilon$ and "solve for" |*x*-1| .

Then,

$$|f(x) - 4| < \epsilon \text{ iff } |(x^2 + 3) - 4| < \epsilon$$

$$\text{iff } |x^2 - 1| < \epsilon$$

$$\text{iff } |(x - 1)(x + 1)| < \epsilon$$

$$\text{iff } |x - 1|\,|x + 1| < \epsilon.$$

We will now "replace" the term |*x*+1| with an appropriate constant and keep the term |*x*-1| , since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta \le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x-1| < \delta \le 1$ implies that -1 < x-1 < 1 and 0 < x < 2 so that 1 < |x+1| < 3 (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.)

$$|x-1|\,|x+1| < |x-1|\,(3) < \epsilon$$

iff $|x-1|\,(3) < \epsilon$

iff $|x-1| < \frac{\epsilon}{3}$.

Now choose $\delta = min\{1, \frac{\epsilon}{3}\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-1| < \delta$, it follows that $|f(x) - 4| < \epsilon$. This completes the proof.

Example 6 Prove that $\lim_{x \to 3} \frac{2}{x+3} = \frac{1}{3}$.

Solution: Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-3| < \delta$, then $\left|f(x) - \frac{1}{3}\right| < \epsilon$. Begin with $\left|f(x) - \frac{1}{3}\right| < \epsilon$ and "solve for" | x - 3 | . Then,

$\left|f(x) - \frac{1}{3}\right| < \epsilon$ iff $\left|\frac{2}{x+3} - \frac{1}{3}\right| < \epsilon$

iff $\left|\frac{3}{3}\,\frac{2}{x+3} - \frac{1}{3}\,\frac{x+3}{x+3}\right| < \epsilon$

iff $\left|\frac{6-(x+3)}{3(x+3)}\right| < \epsilon$

iff $\frac{|3-x|}{|3|\,|x+3|} < \epsilon$

iff $\frac{|(-1)(x-3)|}{|3|\,|x+3|} < \epsilon$

iff $\dfrac{|-1|\,|x-3)|}{|3|\,|x+3|} < \varepsilon$

iff $\dfrac{1}{3}\,\dfrac{|x-3|}{|x+3|} < \varepsilon$

iff $\dfrac{1}{3}\,|x-3|\,\dfrac{1}{|x+3|} < \varepsilon$.

We will now "replace" the term $|x+3|$ with an appropriate constant and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta \le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x-3| < \delta \le 1$ implies that $-1 < x-3 < 1$ and $2 < x < 4$ so that $5 < |x+3| < 7$ and $\dfrac{1}{7} < \dfrac{1}{|x+3|} < \dfrac{1}{5}$ (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ε. This guarantees the logic of the proof.)

$$\frac{1}{3}\,|x-3|\,\frac{1}{|x+3|} < \frac{1}{3}\,|x-3|\,\frac{1}{5} < \varepsilon$$

iff $\dfrac{1}{3}\,|x-3|\,\dfrac{1}{5} < \varepsilon$

iff $\dfrac{1}{15}\,|x-3| < \varepsilon$

iff $|x-3| < 15\varepsilon$.

Now choose $\delta = min\{1, 15\varepsilon\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-3| < \delta$, it follows that $\left|f(x) - \dfrac{1}{3}\right| < \varepsilon$. This completes the proof.

Example 7 Prove that $\lim\limits_{x \to 9} (2 + \sqrt{x}) = 5$.

Solution: Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 9| < \delta$, then $|f(x) - 5| < \epsilon$. Begin with $|f(x) - 5| < \epsilon$ and "solve for" $| x - 9 |$. Then,

$$|f(x) - 5| < \epsilon \text{ iff } |(2 + \sqrt{x}) - 5| < \epsilon$$

$$\text{iff } |\sqrt{x} - 3| < \epsilon$$

(At this point, we need to figure out a way to make $| x-9 |$ "appear" in our computations. Appropriate use of the conjugate will suffice.)

$$\text{iff } \left|(\sqrt{x} - 3)\ \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}\right| < \epsilon$$

(Recall that $(A - B)(A + B) = A^2 - B^2$.)

$$\text{iff } \left|\frac{x-9}{\sqrt{x}+3}\right| < \epsilon$$

$$\text{iff } \frac{|x-9|}{|\sqrt{x}+3|} < \epsilon .$$

$$\text{iff } |x-9|\ \frac{1}{|\sqrt{x}+3|} < \epsilon .$$

We will now "replace" the term $|\sqrt{x}+3|$ with an appropriate constant and keep the term $|x-9|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta \le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x-9| < \delta \le 1$ implies that $-1 < x-9 < 1$ and $8 < x < 10$ so that $\sqrt{8} + 3 < |\sqrt{x}+3| < \sqrt{10} + 3$ and $\frac{1}{\sqrt{10}+3} < \frac{1}{|\sqrt{x}+3|} < \frac{1}{\sqrt{8}+3}$ (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.)

$$|x-9|\ \frac{1}{|\sqrt{x}+3|} < |x-9|\ \frac{1}{\sqrt{8}+3} < \epsilon$$

$$\text{iff } |x-9|\,\frac{1}{\sqrt{8}+3} < \varepsilon$$

$$\text{iff } |x-9| < (\sqrt{8}+3)\varepsilon.$$

Now choose $\delta = min\{1, (\sqrt{8}+3)\varepsilon\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-9| < \delta$, it follows that $|f(x)-3| < \varepsilon$. This completes the proof.

Example 8:

$$\lim_{x\to 0} 3x\sin\frac{1}{x} = 0$$

Solution:

We need to show that given ε > **0** then there exists δ > **0** such that

$0 < |x-0| < \delta$ implies $\left|3x\sin\frac{1}{x} - 0\right| < \varepsilon$

Looking for δ:

$$\left|3x\sin\frac{1}{x}\right| < \varepsilon$$

$$3|x|\cdot\left|\sin\frac{1}{x}\right| < \varepsilon$$

$$3|x|\cdot 1 < \varepsilon$$

$$|x| < \frac{\varepsilon}{3}$$

Hence, we let

$$\delta = \frac{\varepsilon}{3}$$

Negation of the Existence of a Limit

Next we present an example of a function that does not have a limit at a certain point. For a function f not to have a limit at a means that for every real number L, the statement "L is the limit of f at a" is false. What does it mean for that statement to be false? By Definition 3.1, "L is the limit of f at a" means that

For every ε>0 there is a number δ> 0 such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$

for this statement to be false, there must be some $\varepsilon>0$ such that for ever $\delta>0$ it is false that

if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$ (2)

But to say that (2) is false is the same as to say that there must be a number x such that

$0<|x-a|<\delta$ and $|f(x)-L|\geq\varepsilon$

Thus to say that the statement $\lim_{x\to a} f(x)=L$ is false is the same as to say that there is some $\varepsilon>0$ such that for ever $\delta>0$ there is a number x satisfying

$0<|x-a|<\delta$ and $|f(x)-L|\geq\varepsilon$.

Example 8 Let *f* be defined by

$$f(x)=\begin{cases} x^2 & for\ \ x>0 \\ -1 & for\ \ x\leq 0 \end{cases}$$

Solution: Let L be any number. We will prove tht the statement "L is the limit of f at 0" is false by letting $\varepsilon=1/2$ and showing that for any $\delta>0$ there is an x satisfying

$0<|x-a|<\delta$ and $|f(x)-L|\geq\frac{1}{2}=\varepsilon$

Let δ be any positive number. If $L\leq -\frac{1}{2}$, then we let $x=\delta/2$ and note that $f(x)=x^2$ so that

$$|f(x)-L|=\left|\frac{\delta^2}{4}-L\right|\geq\left|\frac{\delta^2}{4}+\frac{1}{2}\right|>\frac{1}{2}=\varepsilon$$

If $L\geq -\frac{1}{2}$, then we let $x=-\delta/2$ and note that $f(x)=-1$, so that

$$|f(x)-L|=|-1-L|=|-1||1+L|\geq\left|1-\frac{1}{2}\right|=\frac{1}{2}=\varepsilon$$

In either case we have shown that for any $\delta>0$ there is an x satisfying

$0<|x-a|<\delta$ and $|f(x)-L|\geq\frac{1}{2}=\varepsilon$

There for f has no limit at 0.

Class work

Using the ε-δ definition of limit, prove that

1. $\lim_{x\to 1} 2x-1=1$ 2. $\lim_{x\to 2}\sqrt{x-1}=1$ 3. $\lim_{x\to 2} x^2=4$ 4. $\lim_{x\to 1} 2x-1\neq 3$

3.3 One-Sided Limits

The notion of limit discussed in the preceding sections can be extended to one-sided limit as we can see from the definition below.

Definition 3.4 a) A number L is the **right-hand limit of *f* at *a*** denoted by $\lim_{x\to a^+} f(x)=L$ if for every $\varepsilon>0$ there is a number $\delta>0$ such that

$$if\ \ 0<x-a<\delta,\ then\ \ |f(x)-L|<\varepsilon$$

b) A number L is the **left-hand limit of *f* at *a*** denoted by $\lim_{x\to a^-} f(x)=L$ if for every $\varepsilon>0$ there is a number $\delta>0$ such that

$$if\ \ -\delta<x-a<0,\ then\ \ |f(x)-L|<\varepsilon.$$

Example 9 Show that $\lim_{x\to 1^+}\sqrt{x-1}=0$

Solution: Let $\varepsilon>0$ be given we need to show there is a $\delta>0$ such that

$$if\ \ 0<x-1<\delta,\ then\ \ \left|\sqrt{x-1}-0\right|<\varepsilon$$

form $\left|\sqrt{x-1}\right| < \varepsilon$ sequaring both sides we get $0 < x-1 < \varepsilon^2$, hence choose $\delta = \varepsilon^2$.
Then

$$if \quad 0 < x-1 < \delta, \; then \quad \left|\sqrt{x-1} - 0\right| = \left|\sqrt{x-1}\right| < \sqrt{\delta} = \varepsilon.$$

Below we give a theorem that relates one sided limit with a general limit the student can see Robert Ellis and Denny Gulick for the proof of the theorem.

Theorem 3.5 $\lim_{x \to a} f(x)$ exists and is equal to L if and only if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and both are equal to L.

Example 2 Observe that in Example 1 even if the right hand side of *f* at 1 exists since the left side limit of f at 1 does not exist, as the function is not defined for *x* < 1 then $\lim_{x \to 1} \sqrt{x-1}$ does not exist.

3.4 Infinite Limits and Infinite Limits at infinity

According to Definition 3.1 if a function *f* has a limit L at *a* then L is a real number, so if the value of a function *f* becomes larger and larger in absolute value as *x* approach *a* from the right or from the left of *a* then f has no limit *a*. Now we introduce a definition that addresses such a case.

Infinite Limits

Definition 3.6 Let f be defined on some open interval (a, c).

a. If $\forall N, \exists \delta > 0$ such that

$$if \;\; 0 < x - a < \delta \;\; then \;\; f(x) > N$$

$then \; \lim_{x \to a^+} f(x) = \infty$

b. If $\forall N, \exists \delta > 0$ such that

$$if \;\; 0 < x - a < \delta \;\; then \;\; f(x) < N$$

$then \; \lim_{x \to a^+} f(x) = -\infty$

c. In either case (a) or (b) the vertical line *x* = *a* is called a **vertical asymptote** of the graph of f, and we say that *f* has an **infinite right-hand limit at *a***.

There are analogous definitions for the limits

$$\lim_{x \to a^-} f(x) = \infty \;\; and \;\; \lim_{x \to a^-} f(x) = -\infty$$

Note if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \infty$ then we right simply $\lim_{x \to a} f(x) = \infty$ for the common expression and say that the limit of f(x) as x approaches a is ∞ and that *f* has an infinit limit at a.

Example 10 Show that $\lim_{x \to a} 1/x^2 = \infty$. Show also that the line *x* = 0 is a vertical asymptote of the graph of 1/*x*.

Solution: Observe that for any N>0,

$$if \;\; 0 < x < \frac{1}{\sqrt{N}}, \; then \;\; \frac{1}{x^2} > N$$

Thus $\lim_{x \to a^+} 1/x^2 = \infty$, and thus the line *x* = 0 is a vertical asymptote of the gragh of $1/x^2$.

Once more for any N>0

$$if \ -\frac{1}{\sqrt{N}} < x < 0, \ then \ \frac{1}{x^2} > N$$

Thus $\lim_{x\to a^-} 1/x^2 = \infty$, again x = 0 is a vertical asymptote of the gragh of $1/x^2$.

Finaly since $\lim_{x\to a^-} 1/x^2 = \infty = \lim_{x\to a^+} 1/x^2$, $\lim_{x\to a} 1/x^2 = \infty$.

Limits at Infinity

Until now the limits we have seen have been limits of a function *f* at a number *a*. Now we consider the limit of f as x becomes larger and larger in absolute value.

Definition 3.7 a) $\lim_{x\to\infty} f(x) = L$ **if for every ε>0 there is a number M such that**

$$if \quad x > M, \ then \ \left|f(x) - L\right| < \varepsilon$$

b) $\lim_{x\to-\infty} f(x) = L$ if for every ε>0 there is a number M such that

$$if \quad x < M, \ then \ \left|f(x) - L\right| < \varepsilon$$

c) If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then we call the horizontal line y = L a horizontal asymptote of the graph of *f*.

Example 11 Show that $\lim_{x\to\infty} 1/x^2 = 0 \ and \ \lim_{x\to-\infty} 1/x^2 = 0$.

Solution: Let ε>0. To show that $\lim_{x\to\infty} 1/x^2 = 0$ **we must find an M such that**

$$if \quad x > M, \ then \ \left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \varepsilon$$

But then

$$\text{if } x > \frac{1}{\sqrt{\varepsilon}}, \ then \ \frac{1}{x^2} < \varepsilon$$

Therefore we let $M = 1/\sqrt{\varepsilon}$ and conclude that $\lim_{x\to\infty} 1/x^2 = 0$. To show that $\lim_{x\to-\infty} 1/x^2 = 0$
We simply choose $M = -1/\sqrt{\varepsilon}$ Then M<0, and thus

$$if \quad x < M, \ then \ \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \frac{1}{M^2} = \varepsilon$$

this proves that $\lim_{x\to-\infty} 1/x^2 = 0$.

Note here that y = 0 is the horizontal asymptote of the graph of $1/x^2$.

Infinite Limits at infinity

We now see the last possible formal definiton of limit that is not considered yet.

Defintion 3.8 $\lim_{x\to\infty} f(x) = \infty$ if for any real number N there is some number M such that

$$if \ \ x > M, \ \ then \ \ f(x) > N.$$

Note the definition of

$\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, *and* $\lim_{x\to-\infty} f(x) = -\infty$ are completely analogues.

Example 12 Show that $\lim_{x\to\infty} x^3 = \infty$.

Solution: We use the fact that $x^3 > x$ for $x > 1$. For any N, choose M so that M>1 and M > N. Then it follows that

$$\text{if } x > M, \text{ then } x^3 > x > M > N$$

therefore by Definition 3.8

$$\lim_{x\to\infty} x^3 = \infty.$$

Similarly, we conclude that for any positive integer n,

$$\lim_{x\to\infty} x^n = \infty.$$

3.5 Limit Theorems

Even if we have developed important techniques of solving limit problems by using the formal definition, I hope by now we have realized that it is not that easy to use this definition to solve each and every problem. Nevertheless the student had encountered in his or her earlier studies of calculus rather easy ways of evaluating limits by the help of different rules. Here we state and prove some of them by using Definition 3.1 and use them to evaluate more complex limit cases.

Theorem 3.9 Assume that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ and c is a constant then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
4. *if* $\lim_{x\to a} g(x) \neq 0$ *and* $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ exists then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Proof: Here we proof (1). Statement (2), (3), and (4) are left as exercise.

Let $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ we need to show for every ε>0 there is some δ>0 such that if $0 < |x-a| < \delta$, *then* $|f(x)+g(x)-(L+M)| < \varepsilon$. Observe that $\lim_{x\to a} f(x) = L$ iff for every ε/2 >0 there is a δ_1>0 such that *if* $0 < |x-a| < \delta_1$, *then* $|f(x)-L| < \varepsilon/2$.

Similarly $\lim_{x\to a} g(x) = M$ iff for every ε/2 >0 there is a δ_2>0 such that

$$\text{if } 0 < |x-a| < \delta_2, \text{ then } |g(x)-L| < \varepsilon/2.$$

Let δ = min {δ_1, δ_2} then we can see that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x)+g(x)-(L+M)| < |f(x)-L| + |g(x)-M| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M$.

In addition to these rules you have also seen that for instance if *f* is a polynomial or a rational function and *a* is in the domain of *f*, then

$\lim_{x\to a} f(x) = f(a)$ **etc**

Now let as quickly go through some important limit finding techniques that would require a little bit of caution before applying the rules in Theorem 3.9.

Example 13 Find $\lim_{x\to 2}\frac{x^2-4}{x-2}$

Solution: Direct substitution of 2 in $\frac{x^2-4}{x-2}$ implies that we have 0/0 which is indeterminate thus we cannot use Theorem 3.9 (4) but for x≠0 simplification of the rational expression would lead us to

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2$$

thus

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}(x+2)=4.$$

Example 14 Find $\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}$

Solution: Again here we cannot use Theorem 3.9 (4), as we get from direct substitution the indeterminate 0/0. But for x≠0 rationalizing the denominator we have:

$$\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}=\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$$

$$=\lim_{x\to 0}\frac{x^2\sqrt{x^2+1}+1}{(x^2+1)-1}=\lim_{x\to 0}\frac{x^2\sqrt{x^2+1}+1}{x^2}$$

$$=\lim_{x\to 0}\left(\sqrt{x^2+1}+1\right)=2.$$

Example 15 Find $\lim_{x\to 0}x|x|$

Solution: Observe that

$$x|x|=\begin{cases}x^2 & if\ \ x\geq 0\\ -x^2 & if\ \ x<0\end{cases}$$

Since $x|x|=x^2\ \ for\ \ x>0$**, we have**

$$\lim_{x\to 0^+}x|x|=\lim_{x\to 0^+}x^2=0$$

and also since $x|x|=-x^2\ \ for\ \ x<0,$ we have

$$\lim_{x\to 0^-}x|x|=\lim_{x\to 0^-}-x^2=0$$

therefore we conclude that

$$\lim_{x\to 0}x|x|=0.$$

Example 16 Prove that $\lim_{x\to -1}\frac{x+1}{|x+1|}$ **does not exist**

Solution:

$$\lim_{x\to -1^+}\frac{x+1}{|x+1|}=\lim_{x\to -1^+}\frac{x+1}{x+1}=\lim_{x\to -1^+}1=1\ \ and\ \ \lim_{x\to -1^-}\frac{x+1}{|x+1|}=\lim_{x\to -1^-}\frac{x+1}{-(x+1)}=\lim_{x\to -1^-}-1=-1$$

Consequently

$$\lim_{x\to -1^+}\frac{x+1}{|x+1|}\neq \lim_{x\to -1^-}\frac{x+1}{|x+1|}$$

Thus $\lim_{x\to -1}\frac{x+1}{|x+1|}$ does not exist.

Example 17 Find $\lim_{x\to\infty}\frac{x-2x^2}{x^2-1}$ *and* $\lim_{x\to-\infty}\frac{x-2x^2}{x^2-1}$

Solution: Deveiding the numerator and the denominator of $\frac{x-2x^2}{x^2-1}$ by x^2 in the limit we have

$$\lim_{x\to\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to\infty}\frac{1/x-2}{1-1/x}=-2$$

similarly

$$\lim_{x\to-\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to-\infty}\frac{1/x-2}{1-1/x}=-2.$$

Observe here that y = -2 is the horizontal asymptote of the graph of $f(x)=\frac{x-2x^2}{x^2-1}$.

Example 18 Let $f(x)=\frac{x-2x^2}{x^2-1}$. Find all vertical asymptotes of the graph of *f*.

Solution: Since f is not defined at x =1 and x = –1 they are the possible vertical asymptotes but to confirm our claim we use limit:
Since

$$\lim_{x\to 1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to 1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=-\infty \quad \textit{and}$$

$$\lim_{x\to -1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to -1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=\infty$$

it follows that x =1 and x = –1 are the vertical asymptotes of the graph of *f*.

The next theorems give two additional properties of limits. For their proofs the student may refer any major calculus books.

Theorem 3.10 If $f(x)\le g(x)$ for all x in an open interval that contains a (except possibly at *a*) and the limits of *f* and g both exist as x approaches *a*, then

$$\lim_{x\to a} f(x)\le \lim_{x\to a} g(x).$$

Theorem 3.11 (The Squeezing Theorem) If $f(x)\le g(x)\le h(x)$ for all x in an a open interval that contains *a* (except a possibly at *a*) and

$$\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$$

then

$$\lim_{x\to a} g(x)=L.$$

I don't think the student is new for these theorems and for the special limit that is the consequence of especially the Squeezing Theorem. i.e.

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Example 19 Find $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

Solution: Since $-1 \le \sin\frac{1}{x} \le 1$, $\forall x \neq 0$, we have

$-x^2 \le x^2 \sin\frac{1}{x} \le x^2 \quad \forall \text{x} \neq 0$

Moreover $\lim_{x\to 0} - x^2 = 0$ *and* $\lim_{x\to 0} x^2 = 0$, **thus by the squeezing theorem we have**

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$

Example 20 Find $\lim_{x\to\infty}\frac{x^4-x^2}{x+1}$

Solution: Simplifying $\frac{x^4-x^2}{x+1}$ **we can evaluate the limit as below**

$$\lim_{x\to\infty}\frac{x^4-x^2}{x+1} = \lim_{x\to\infty}\frac{x^2(x^2-1)}{x+1} = \lim_{x\to\infty}\frac{x^2(x-1)(x+1)}{x+1}$$
$$= \lim_{x\to\infty} x^2(x-1) = \infty.$$

Class work

Evaluate each of the following limit as a real number, ∞, – ∞, if it exists.

1. $\lim_{x\to -1}\frac{x^2-x-2}{x+1}$
2. $\lim_{x\to 0}\frac{\sin 2x}{x}$
3. $\lim_{x\to 3^-}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
4. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$
5. $\lim_{x\to 1} f(x)$ *where* $f(x)=\begin{cases} x^3 & if\ \ x<1 \\ (x-2)^2 & if\ \ x>1 \end{cases}$

3.6 Continuity of a Function and the Intermediate Value Theorem

Definition 3.11 A function *f* is **continuous** at a number *a* in its domain if

$$\lim_{x\to a} f(x) = f(a)$$

f is said to be **discontinuous** at *a* if *f* is not continuous at *a*.

Notice that definition 3.11 implicitly requires three things if *f* is continuous at *a*:

1. *f*(*a*) is defined (that is, *a* is in the domain of *f*)
2. $\lim_{x\to a} f(x)$ exists (so *f* must be defined on an open interval that contains *a*).
3. $\lim_{x\to a} f(x) = f(a)$

Example 21 Let $f(x) = \lim_{x\to -1}\frac{x^2-x-2}{x+1}$. Determine the number at which f is not discontinuous.

Solution: Notice that f is a rational function. Since the denominator of f is 0 for x = -1, f is defined for all x except at –1. Thus f is discontinuous only at x = -1 else where it is continuous in its' domain.

Example 22 If we redefine the function *f* in Example 21 as:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x+1} & if \;\; x \neq -1 \\ -3 & if \;\; x = -1 \end{cases}$$

then since

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - x - 2}{x+1} = \lim_{x \to -1} \frac{(x-2)(x+1)}{x+1} = -3$$

and hence $\lim_{x \to -1} f(x) = -3 = f(-1)$

f is continuous.

Notice that we are able to make *f* in Example 21 to be continuous by redefining it at –1 as in Example 22. Such discontinuity points like –1 in our example are called **removable** discontinuities because we can remove the discontinuity of the function by redefining the function just at the discontinuity point.

Example 23 Let $f(x) = \dfrac{1}{x^2}$ *and* $g(x) = \begin{cases} 1 & if \quad x \leq 0 \\ 0 & if \quad x > 0 \end{cases}$ then we can see that, *f* is not defined at 0 and $\lim_{x \to 0} f(x) = \infty$, g is defined at 0 but $\lim_{x \to 0} g(x)$ does not exist as $\lim_{x \to 0^-} g(x) = 1$ and $\lim_{x \to 0^+} g(x) = 0$. Thus both functions are not continuous at 0. We say we have **infinite** discontinuity at 0 in case of *f* while we say we have **jump** discontinuity at 0 in case of *g*.

Clearly combinations of continuous functions follow immediately from the corresponding results for limits.

Theorem 3.12 If f and g are continuous at *a* and c is a constant, than the following functions are also continuous at *a*.

i. $f + g$ ii. $f - g$ iii. cf iv. fg v. f/g if $g(a) \neq 0$.

So using theorem 3.12 we can show that every polynomial function is continuous over **R** every rational function is continuous everywhere except at numbers where the denominator is 0.

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

Theorem 3.13 If *f* is continuous at b and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f(g(x)) = f(b) = f\left(\lim_{x \to a} g(x)\right)$$

The following theorem tells us that the composition of two continuous functions at a given number is continuous.

Theorem 3.14 If *g* is continuous at a and *f* is continuous at g(*a*), then $f \circ g(x) = f(g(x))$ is continuous at *a*.

Class work

Where are the following functions continuous

a) $f(x)=|x|$ b) $h(x)=\dfrac{1}{\sqrt{x^2+3}-2}$

One-Sided Continuity

Definition 3.15 A function f is continuous from the right at a point *a* in its domain if

$$\lim_{x\to a^+} f(x) = f(a)$$

A function f is continuous from the left at a point *a* in its domain if

$$\lim_{x\to a^-} f(x) = f(a).$$

Example 24 the step function $g(x)=\begin{cases}1 & if \quad x\le 0\\ 0 & if \quad x>0\end{cases}$ is continuous from the left at 0

Since $\lim_{x\to 0^-} g(x)=1=g(0)$ but it is not continuous from the right at 0 as $\lim_{x\to 0^+} g(x)\ne g(0)$ verify.

Continuity on interval

Definition 3.16 a) A function is continuous on (*a*,b), if it is continuous at every point in *(a,b).*

b. A function is continuous on [*a*, b] if it is continuous on (*a*,b) and is also continuous from the right at *a* and continuous from the left at b.

Class Work

Let $f(x)=\sqrt{1-x^2}$ .Show that f is continuous on [-1, 1].

An important property of continuous functions is expressed by the following theorem.

Theorem 3.17 (The Intermediate Value Theorem)

Suppose that f is continuous on the closed interval [a, b] and let N be any number strictly between *f*(*a*) and *f*(*b*). Then there exists a number *c* in (a, b) such that *f*(*c*) = N.

Example 24 Show that there is a root of the equation

$$4x^3-6x^2+3x-2=0$$

between 1 and 2.

Solution: Let $f(x)=4x^3-6x^2+3x-2$. We are looking for a solution of the given equation, that is, a number *c* between 1 and 2 such that *f*(c) = 0. Therefore we take a = 1, b = 2, and N = 0 in Theorem 3.17. We have

$$f(1)=4-6+3-2=-1<0 \quad and \qquad f(2)=32-24+6-2=12>0$$

Thus $f(1)<0<f(2)$, that is, N=0 is a number between *f*(1) and *f*(2). Now *f* is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number *c* between 1 and 2 such that $f(c)=0$. In other words, the equation $4x^3-6x^2+3x-2=0$ has a root c in the interval (1, 2).

Class Work

1. Find *A* that makes the function

$$f(x)=\begin{cases} x^2-2 & \text{if } x<1\\ Ax-4 & \text{if } 1\le x\end{cases}$$

continuous at *x*=1.

2. Demonstrate that the equation $\cos x + x = 0$ has at least one solution.

4 Derivatives

4.1 Definition and Properties of Derivative; the Chain Rule

In your previous calculus course you were introduced with the definition of the derivative of a function, properties of derivatives, the chain rule and important application of the derivative. Here our aim is to revise some of these concepts and introduce the derivatives of some more functions.

Definition 4.1 The **derivative** of a function f at a number ***a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (1)$$

if this limit exists.

If we write x = a + h, then x –a =h and x approaches a iff h approaches to 0. Therefore an equivalent way of stating the definition of the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \qquad (2)$$

This last definition is more convenient for finding the derivative of a function.

Example 1 Find the derivative of the function $f(x) = x^2 + 3x + 2$ at -1.

Solution: By definition

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}.$$

thus

$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{[(-1+h)^2 + 3(-1+h) + 2] - [(-1)^2 + 3(-1) + 2]}{h} \\ &= \lim_{h \to 0} \frac{1 - 2h + h^2 - 3 + 3h + 2}{h} \\ &= \lim_{h \to 0} \frac{h^2 + h}{h} = \lim_{h \to 0} (h+1) = 1. \end{aligned}$$

I hope the student remembers that the slope of the tangent line to the graph of the function f at a point (a, f(a)) is given by the derivative of f at a i.e $f'(a)$ consequently using the point-slope form of the equation of a line, we have the equation of the tangent line to the curve y = f(x) at a point (a, f(a)) is given by $y - f(a) = f'(a)(x - a)$. For instance the equation of the tangent line to the graph of $f(x) = x^2 + 3x + 2$ at (-1, 0) in our Example 1 is given by $y - f(-1) = f'(-1)(x - (-1))$ *or* $y - 0 = 1(x+1)$ or simply y = x+1.

Given a function f, we associate with it a new function f', called the **derivative** of f, defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, $f(x)$).

EXAMPLE 2

Find the derivative of $f(x) := x + \sqrt{x+1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The definition of derivative.

$$\lim_{h \to 0} \frac{\left[x + h + \sqrt{(x+h)+1}\right] - \left(x + \sqrt{x+1}\right)}{h}$$

Replace f(x+h) and f(x) by the corresponding expressions.

$$\lim_{h \to 0} \frac{h + \sqrt{(x+h)+1} - \sqrt{x+1}}{h}$$

Simplify.

$$\lim_{h \to 0} \frac{h + \left[\sqrt{(x+h)+1} - \sqrt{x+1}\right] \cdot \frac{\left[\sqrt{(x+h)+1} + \sqrt{x+1}\right]}{\left[\sqrt{(x+h)+1} + \sqrt{x+1}\right]}}{h}$$

Rationalize the radicals.

Some steps are omitted — see the **complete solution**

$$\lim_{h \to 0} \left[1 + \frac{1}{\sqrt{(x+h)+1} + \sqrt{x+1}}\right]$$

Divide both the numerator and denominator by h (h is nonzero).

$$1 + \frac{1}{2 \cdot \sqrt{x+1}}$$

Therefore, the derivative of $x + \sqrt{x+1}$ is $1 + \frac{1}{2 \cdot \sqrt{x+1}}$

Here, the domain of the function is $x \ge -1$, while the derivative is defined for all values of $x > -1$.

Definition 4.2 A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or (-∞, a) or (-∞, ∞)] if it is differentiable at every number in the interval.

Example 3 Show that $f(x) = |x|$ is not differentiable at 0.

Solution: Observe that

$$|x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$$

then for x >0 using (1) we have

$$\lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

and for x<0

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1.$$

which implies

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0} \text{ does not exist}$$

thus f is not differentiable at 0.

Theorem 4.3 If f is differentiable at a, then f is continuous at a.

Proof: To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$.

We do this by showing that the difference $f(x) - f(a)$ approaches 0.

For x ≠ a we can divide and multiply by x – a

We did this in order to involve the difference quotient. Thus we can use the Product Law of limits to write

$$\begin{aligned} \lim_{x \to a} \left[f(x) - f(a) \right] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} (x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + f(x) - f(a)] \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a). \end{aligned}$$

and so f is continuous at a.

Note: the converse of Theorem 4.3 is false: that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0).$$

But as we have seen in Example 3 that f is not differentiable at 0.

Let me remind u some of the differentiation rules that u have developed in your previous calculus course. I advice the student to check on these results using the definition of derivative.

The power rule: If $f(x) = x^n$ for any real number n is given by $f'(x) = nx^{n-1}$.

Derivatives of sine and cosine: $(\sin x)' = \cos x$ *and* $(\cos x) = -\sin x.$

Derivatives of exponential and logarithmic functions: $(e^x)' = e^x$ *and* $(\ln x) = \dfrac{1}{x}.$

etc.

We also need to revise the rules of finding the derivatives of combined functions as in the table below.

Let f and g be differentiable then

1. $(cf)' = cf'$ 2. $(f+g)' = f' + g'$
3. $(f-g) = f' - g'$ 4. $(fg)' = f'g + fg'$
5. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g + fg'}{g^2}$ 6. $(c)' = 0$

Class work

Find the derivative of each of the following functions

1. $f(x) = x^{25} + 5x^5 + 25$ 2. $f(x) = x^2 - \dfrac{1}{x^2}$
3. $f(x) = x^4 + \sqrt[4]{x}$ 4. $f(x) = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$
5. $f(x) = x\sin x$ 6. $f(x) = \sin x \cos x$
7. $f(x) = \tan x$ 8. $f(x) = \csc x$
9. $f(x) = \dfrac{\sec x}{1+\tan x}$ 10. $f(x) = \dfrac{x^2 \tan x}{\sec x}$

The Chain Rule

The rules that we have introduced till now are not enough to find composition of functions thus we need to develop an appropriate to handle these cases. The Chain Rule is such a rule.

Theorem 4.5 If the derivatives $g'(x)$ and $f'(g(x))$ both exist, then $(f \circ g)'(x) = f'(g(x))g'(x)$

Example 4 Find $h'(x)$ *if* $h(x) = \cos 2x$

Solution: Let $f(x) = \cos x$ *and* $g(x) = 2x$. Then $h = f \circ g$. Since

$$g'(x) = 2 \text{ and } f'(x) = -\sin x$$

we conclude that

$$h'(x) = f'(g(x))g'(x) = (-\sin 2x)(2) = -2\sin 2x.$$

Example 2 Find $h'(x)$ *if* $h(x) = \sqrt{1+x^2}$

Solution: Let $g(x) = 1 + x^2$ *and* $f(x) = \sqrt{x}$ **consequently** $h = f \circ g$.**Then**

$$g'(x) = 2x \text{ and } f'(x) = \frac{1}{2\sqrt{x}} \text{ for } x > 0.$$

Therefore

$$h'(x) = f'(g(x))g'(x) = \frac{1}{2\sqrt{1+x^2}}2x = \frac{x}{\sqrt{1+x^2}}.$$

Class Work

Find the derivative of the functions

1. $y = (x^5 + 2x^2 + 3)^{50}$ 2. $y = \frac{1}{\sqrt[3]{x^6 + 2x + 1}}$

3. $y = \cos(\sin(\tan x))$ 4. $y = \sqrt{\cos(\sin^2 x)}$

Find the equation of the tangent line to the curve at the given point

5. $(x^3 - x^2 + x - 1)^{10}$, $(1,0)$ 6. $y = \sqrt{x + 1/x}$, $(1, \sqrt{2})$

4.2 Inverse Functions and Their Derivatives

In pre-calculus mathematics courses we defined a function *f* as a relation in which no two elements of the relation have the same first coordinate. Also we have seen that for same of the functions the relation that is found by interchanging the entries of the ordered pairs can be again a function and we called such a function the *inverse* of the original function. In this section we discuss general properties of inverses and their derivatives.

4.2.1 Inverse Functions

In order to define the inverse of a function, it is essential that different numbers in the domain always give different values of *f*. Such functions are called one-to-one functions.

Definition 3.1 A function *f* with domain D and range R is **one-to-one function** if whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

Note from Definition 3.1 we see that every *strictly increasing* function is one-to-one, because if $a < b$, then $f(a) < f(b)$, and if $b < a$, then $f(b) < f(a)$ in short if $a \neq b$, then $f(a) \neq f(b)$. Similarly, every strictly decreasing function is one-to-one. We now give the definition of inverse functions in terms of one-to-one function.

Definition 3.2 Let *f* be a *one-to-one function* with domain D and range R. A function g with domain R and range D is the **inverse function of *f***, provided the following condition is true for every *x* in D and every *y* in R:

$$y = f(x) \text{ if and only if } x = g(y).$$

If a function *f* has an inverse function g, we often denote g by f^{-1}. Of course we must note here that almost always f^{-1} is different from 1/*f*.

If *f* is a one-to-one function with domain D and range R, then for each number y in R, there is exactly one number x in D such that $y = f(x)$. Since x is unique, we may define a function g from R to D by means of the rule $x = g(y)$. g reverses the correspondence given by *f*. We call g the inverse function of *f*. In summary, *a function f has an inverse if and only if it is one-to-one*. This conclusion is especially easy to apply to differentiable functions whose domains are intervals. We know that a function *f* is strictly increasing on I (and hence has an inverse) if $f'(x) > 0$ for all *x* in I or if $f'(x) \geq 0$ for all *x* in I and $f'(x) = 0$ for at most finitely many values of x. Similarly, f is strictly decreasing on I (and hence has an inverse) if $f'(x) < 0$ for all x in I or if $f'(x) \le 0$ for all x in I and $f'(x) = 0$ for at most finitely many values of x.

Example 1 Let $f(x) = 2x^7 + 3x^5 + 6x - 4$ then since $f'(x) = 17x^6 + 15x^2 + 6 > 0$ f is strictly increasing consequently it is invertible.

Properties of Inverses

From Definition 3.2 and the theories we developed above we can drive the following elementary relationships between f and f^{-1}.

I. Domain of f^{-1} = range of f and range of f^{-1} = domain of f.

II. $(f^{-1})^{-1} = f$

III. $f^{-1}(f(x)) = x$ for all x in the domain of f.

IV. $f(f^{-1}(y)) = y$ for all y in the range of f.

In some cases we can find the inverse of a one-to-one function by solving the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$. The following guidelines summarize this procedure.

Guidelines for finding f^{-1} is simple cases

1. Verify that f is a one-to-one function (or that f is increasing or is decreasing) throughout its domain.
2. Solve the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.

The success of this method depends on the nature of the equation $y = f(x)$, since we must be able to solve for x in terms of y.

Example 2 Let $f(x) = 2x + 3$. Find the inverse of f.

Solution: Following the guidelines, first since $f'(x) = 2 > 0$, f is increasing for all real number x and thus f^{-1} exists for all real number x.

Now as guideline 2, we consider the equation

$$y = 2x + 3$$

and solving for x in terms y, we obtain

$$x = \frac{y-3}{2}$$

we now let

$$f^{-1}(y) = \frac{y-3}{2}$$

Since we customarily use x as the independent variable, we replace y by x to obtain

$$f^{-1}(x) = \frac{x-3}{2}.$$

Example 3 Let $f(x) = x^2 - 3$ *for* $x \ge 0$. The inverse function of f.

Solution: The domain of f is $[0,\infty)$, and the range is $[-3,\infty)$. Since f is increasing, it is one-to-one and hence has an inverse function f^{-1} that has domain $[-3,\infty)$ and range $[0,\infty)$.

As in guideline 2, we consider the equation

$$y = x^2 - 3$$

and solve for *x*, obtaining

$$x = \pm\sqrt{y+3}.$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ **and let**

$f^{-1}(y) = \sqrt{y+3}$, or equivalently, $f^{-1}(x) = \sqrt{x+3}$.

Graphs of Inverse Functions

There is an interesting relationship between the graphs of a functions *f* and f^{-1}. We first not that b = *f*(*a*) is equivalent to *a* = f^{-1}(b). These equations imply that the point (a,b) is on the graph of *f* if and only if the point (b,a) is on the graph of f^{-1}. But (a,b) and (b,a) are symmetric with respect to the line *y* = *x*. Thus the graph of f^{-1} is obtained by simply reflecting the graph of f through the line *y* = *x*.

Example 4 For each function *f*, sketch the graph of *f* and f^{-1} on the same coordinate system.

a) $f(x) = 2x + 3$ c) $f(x) = x^2 - 3$ c) $f(x) = \sin x$

Solution: In each case the graph of f^{-1} is obtained by reflecting the graph of f through the line *y* = *x*. The graphs appear in fig 3.1 below.

Exercise 4.1

I Determine whether the given function has an inverse. If an inverse exists, give the domain and range of the inverse and graph the function and its inverse.

1. $f(x) = 4x + 3$
2. $f(x) = \sqrt{9 - x^2}$, $0 \le x \le 3$
3. $f(x) = x - \sin x$
4. $f(x) = \ln(3 - x)$
5. $f(x) = \dfrac{2x}{x-2}$
6. $f(x) = \sqrt[3]{x} + 1$

II Show *f* has an inverse if

7. $f(x) = \int_0^x \sqrt{1+t^4}\,dt \ \ for\ all\ x.$
8. $f(x) = \int_0^x \sin^4(t^2)\,dt \ \ for\ all\ x.$

4.2.2 Continuity and Differentiability of Inverse Functions

If f is continuous, then the graph of f has no breaks or holes, and hence the same is true for the (reflected) graph of f^{-1}. Thus we see intuitively that if f is continuous on [*a*,b], then f^{-1}

continuous on $[f(a), f(b)]$. We can also show that if f is increasing, then so is f^{-1}. These facts are stated in the next theorem that is given with out a proof.

Theorem 3.3 If f is continuous and increasing on [a,b], then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

We can also prove the analogous result obtained by replacing the word increasing in Theorem 3.3 by decreasing.

The next theorem provides us a method of finding of the derivative of an inverse function.

Theorem 3.4 Suppose that f has an inverse and is continuous on an open interval I containing a. Assume also that $f'(a)$ exists, $f'(a) \neq 0,$ and $f(a) = c.$ Then $(f^{-1})'(c)$ exists, and

$$(f^{-1})'(c) = \frac{1}{f'(a)} \tag{1}$$

Proof Using the fact that $f^{-1}(c) = a$ and definition of the derivative, we find that

$$(f^{-1})'(c) = \lim_{y \to c} \frac{f^{-1}(y) - f^{-1}(c)}{y - c} = \lim_{y \to c} \frac{f^{-1}(y) - a}{f(f^{-1}(y)) - f(a)} \tag{2}$$

provided that the latter limit exists. We will simultaneously show that it does exist and find its value. First notice that f^{-1} is continuous at c by theorem 3.3. Therefore

$$\lim_{y \to c} f^{-1}(y) = f^{-1}(c) = a$$

so that if $x = f^{-1}(y),$ then x approaches a as y approaches c. Moreover, the fact that f has an inverse and $f^{-1}(c) = a$ implies that $f^{-1}(y) \neq a$ for $y \neq c.$ Consequently (2) and the Substitution Theorem for Limits (with x substituting for $f^{-1}(y)$)imply that

$$(f^{-1})'(c) = \lim_{y \to c} \frac{f^{-1}(y) - a}{f(f^{-1}(y)) - f(a)} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)}$$

$$= \frac{1}{\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}} = \frac{1}{f'(a)}.$$

It is convenient to restate Theorem 3.4 as follows.

Corollary 3.5 If f^{-1} is the inverse function of a differentiable function f and if $f'(f^{-1}(x)) \neq 0,$ then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3}$$

Example 1 Let $f(x) = x^7 + 8x^3 + 4x - 2.$ Find $(f^{-1})'(-2).$

Solution: In order to use (1), we must first find the value of a for which f(a) = –2. But f(0) = –2, so a = 0. Since $f'(x) = 7x^6 + 24x^2 + 4,$ it follows that $f'(0) = 4.$ Thus we conclude from (1) that

$$(f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{4}$$

Example 2 If $f(x)=x^3+2x-1,$ prove that *f* has an inverse function f^{-1}, and find the slope of the tangent line to the graph of f^{-1} at the point P(2,1).

Solution: Since $f'(x)=3x^2+2>0$ for every x, f is increasing and hence is one-to-one. Thus, f has an inverse function f^{-1}. Since *f*(1) = 2, it follows that $f^{-1}(2)=1,$ and consequently the point P(2,1) is on the graph of f^{-1}. It would be difficult to find f^{-1} using Guidelines, because we would have to solve the equation $y=x^3+2x-1,$ for x in terms of y. However, even if we cannot find f^{-1} explicitly, we can find the slope $f^{-1}(2)$ of the tangent line to the graph of g at P(2,1). Thus, by Theorem 3.4

$$f^{-1}(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(1)}=\frac{1}{5}.$$

An easy way to remember Corollary 3.5 is to let $y=f(x).$ If f^{-1} is the inverse function of f, then $f^{-1}(y)=f^{-1}(f(x))=x.$ Then

$$(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}=\frac{1}{f'(x)}$$

or, in differential notation,

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Example 3 Let *f* be the function in example 2 then let $y=x^3+2x-1$ *and* $x=f^{-1}(y)$.

Then $$\frac{dx}{dy}=\frac{1}{dy/dx}=\frac{1}{3x^2+2};$$

That is $$(f^{-1})'(y)=\frac{1}{3x^2+2}=\frac{1}{3(f^{-1}(y))^2+2}.$$

Or using x

$$(f^{-1})'(x)=\frac{1}{3(f^{-1}(x))^2+2}.$$

Consequently, to find $(f^{-1})'(x)$ it is necessary to know $f^{-1}(x),$ just as in corollary 3.5.

Exercise 4.2

I Find $(f^{-1})'(c).$

1. $f(x)=x^3+7;\ c=6$
2. $f(x)=x+\sin x;\ c=0$
3. $f(x)=x+\sqrt{x};\ c=2$
4. $f(x)=x\ln x;\ c=2e^2$

II a) Use f' to prove that *f* has an inverse function. b) Find the slope of the tangent line at the point P on the graph of f^{-1}.

5. $f(x)=x^5+3x^2+2x-1;\quad P(5,1)$
6. $f(x)=4x^5-(1/x^3);x>0;\quad P(3,1)$

III Find dx/dy

7. $f(x)=4-x^2,\quad x\geq 0$
8. $f(x)=\ln(x^3+1)$

4.2.3 Inverse Trigonometric Functions

Since the trigonometric functions are not one-to-one, they do not have inverse functions. By restricting their domains, however, we may obtain one-to-one functions that have the same values as the trigonometric functions and that do have inverse over these restricted domains.

The Arcsine Function

If we restrict the domain of the sine function to $[-\pi/2, \pi/2]$, then the resulting function is strictly increasing (because its derivative is positive except $-\pi/2$ *and* $\pi/2$.) Hence the restricted function which is called **arcsine function** has domain [–1,1], and range $[-\pi/2, \pi/2]$. Its value at x is usually written $\arcsin x$ or $\sin^{-1} x$. As a consequence,

$$\arcsin x = y \ \ if \ \ and \ \ only \ \ if \ \ \sin y = x$$

$$for \ -1 \le x \le 1 \ \ and \ -\pi/2 \le y \le \pi/2$$

We also see from the property of inverse functions that

$i.$ $\arcsin(\sin x) = x \ \ for \ -\pi/2 \le x \le \pi/2$ $\qquad ii.$ $\sin(\arcsin x) = x \ \ for \ -1 \le x \le 1.$

Example 1 Evaluate

a) $\sin\left(\arcsin\dfrac{1}{2}\right)$ b) $\arcsin\left(\sin\dfrac{\pi}{4}\right)$ c) $\arcsin\left(\sin\dfrac{5\pi}{6}\right)$

Solution:

a) $\sin\left(\arcsin\dfrac{1}{2}\right) = \dfrac{1}{2}$ since $-1 < \dfrac{1}{2} < 1$

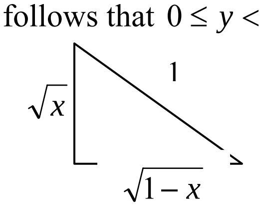
b) $\arcsin\left(\sin\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$

c) $\arcsin\left(\sin\dfrac{5\pi}{6}\right) = \arcsin\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}.$

In Example 1c) $5\pi/6$ is not between $-\pi/2$ *and* $\pi/2$, and hence we cannot use ii. Instead we use properties of special angles to first evaluate $\sin(5\pi/6)$ and then find $\arcsin(1/2)$.

Example 2 Simplify the expression $\sec(\arcsin\sqrt{x})$

Solution: We will evaluate $\sec(\arcsin\sqrt{x})$ by evaluating $\sec y$ for the value of y in $(-\pi/2, \pi/2)$ such that $\arcsin\sqrt{x} = y$, that is, $\sin y = \sqrt{x}$. Since $\sin y = \sqrt{x} \ge 0$, it follows that $0 \le y < \pi/2$. Applying the Pythagorean Theorem to the triangle in Fig (3.2)



We find $\sec y = \dfrac{1}{\sqrt{1-x}}$. Therefore

$$\sec(\arcsin\sqrt{x}) = \sec y = \frac{1}{\sqrt{1-x}}.$$

The Arccosine Function

If the domain of the cosine

one continuous decreasing function that has a continuous decreasing inverse function. We call the inverse function of cosine **arccosine function**. The domain of the $\arccos ine$ is [–

1,1], and its range is $[0,\pi]$. Its value at x is usually written $\arccos x$ *or* $\cos^{-1} x$. As a consequence,

$$\arccos x = y \;\; if \;\; and \;\; only \;\; if \;\; \cos y = x$$
$$for \; -1 \le x \le 1 \;\; and \;\; 0 \le y \le \pi$$

Since cos and arccos are inverse functions of each other, we obtain the following properties.

$$i.\ \arccos(\cos x) = x \;\; for \;\; 0 \le x \le \pi \qquad ii.\ \cos(\arccos x) = x \;\; for \;\; -1 \le x \le 1.$$

Example 2 Evaluate

a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$ b) $\arccos\left(\cos\frac{2\pi}{3}\right)$ c) $\arccos\left[\cos\left(-\frac{1}{2}\right)\right]$

Solution:

a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$ since $-1 < -\frac{1}{2} < 1$

b) $\arccos\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ since $0 < \frac{2\pi}{3} < \pi$

c) $\arccos\left[\cos\left(-\frac{\pi}{4}\right)\right] = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Note that in the c) part of the preceding Example 2, $-\pi/4$ is not between 0 and π, and hence we cannot use property ii. above. Instead, we first evaluate $\cos(-\pi/4)$ and then find $\cos^{-1}(\sqrt{2}/2)$.

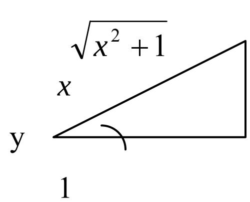
Example 3 Simplify the expression $\cos(\arctan x)$.

Solution: Let $y = \arctan x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$
$$\sec y = \sqrt{1+x^2} \qquad (as \;\; \sec y > 0 \;\; for \; -\pi/2 < y < \pi/2)$$

Thus $$\cos(\arctan x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}.$$

Note instead of using trigonometric identities as in the solution above, it is easy to use a triangular diagram. If we let $y = \arctan x$ then $\tan y = x$, and using the right triangle below we can read from the fig that



$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}.$$

The Arctangent Function

To find an inverse for the tangent function, we restrict the tangent function to $(-\pi/2, \pi/2)$. The resulting inverse function is called the **arctangent function**. Its domain is $(-\infty,\infty)$, and its range is $(-\pi/2, \pi/2)$. We usually write its value at x as $\arctan x$ or $\tan^{-1} x$. As a consequence,

$$\arctan x = y \textit{ if and only if } \tan y = x$$
$$\textit{for any } x \textit{ and for } -\pi/2 < y < \pi/2$$

Thus for any x, $\arctan x$ is the number y between $-\pi/2$ *and* $\pi/2$ whose tangent is x.

As with arcsin and arccox, we have the following properties of arctan

$i.\ \arctan(\tan x) = x \textit{ for } -\pi/2 \le x \le \pi/2 \qquad ii.\ \tan(\arctan x) = x \textit{ for every } x.$

Example 4

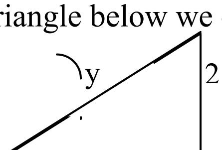
a) $\tan(\arctan 99) = 99$

b) $\arctan(\tan\frac{\pi}{4}) = \frac{\pi}{4}$

c) $\arctan(\tan\pi) = \arctan 0 = 0$

Example 5 Evaluate $\sec(\arctan\frac{2}{3})$

Solution: If we let $y = \arctan\frac{2}{3}$, then $\tan y = \frac{2}{3}$. We wish to find $\sec y$. Since $-\pi/2 < \arctan x < \pi/2$ for every x and $\tan y > 0$, it follows that $0 < y < \pi/2$ and from the triangle below we obtain that



$$\sec\left(\arctan\frac{2}{3}\right) = \sec y = \frac{\sqrt{13}}{3}.$$

The remaining trigonometric funct tly and are summarized here as below:

$$y = \csc^{-1} x(|x| \ge 1) \iff \csc y = x \quad \textit{and} \quad y \in (0,\pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1} x(|x| \ge 1) \iff \sec y = x \quad \textit{and} \quad y \in (0,\pi/2] \cup (\pi, 3\pi/2]$$

$$y = \cot^{-1} x(x \in \Re) \iff \cot y = x \quad \textit{and} \quad y \in (0,\pi)$$

Of these functions only the arcsecant function appears with any frequency in the sequel.

Exercise 4.3

I Find the exact value of the expression, whenever it is defined.

1. $\arcsin(-\sqrt{2}/2)$
2. $\arccos(-1/2)$
3. $\arctan(-\sqrt{3})$
4. $\sin(\arcsin 2/3)$
5. $\arcsin(\sin 5\pi/4)$
6. $\arccos(\cos 5\pi/4)$
7. $\cos[\arctan(-3/4) - \arcsin(4/5)]$
8. $\tan[\arctan(3/4) + \arccos(8/17)]$

II Rewrite as an algebraic expression in x for $x > 0$.

9. $\sec(\arcsin(x/3))$
10. $\tan(arc\csc(x/2))$
11. $\cos(2\arcsin x)$
12. $\sin(2\arcsin x)$

Derivatives and Integrals

We know see the derivatives and integrals of the inverse trigonometric functions in the following two theorems.

Theorem 3.1

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1}x)=-\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$$

Proof

To proof $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$, put $y=\sin^{-1}x$ so that $\sin y = x$ whenever $-1<x<1$ and $-\pi/2<y<\pi/2$. Then differentiating $\sin y = x$ implicitly, we have

$$\cos y\frac{dy}{dx}=1$$

and hence
$$\frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\cos y}$$

Since $-\pi/2<y<\pi/2$, $\cos y$ is positive and, therefore,

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}.$$

Thus,
$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

For $|x|<1$. Observe that the inverse sine function is not differentiable at ±1.

Since the inverse tangent function is differentiable at every real number, let us consider the equivalent equation

$$y=\arctan x \quad and \quad \tan y = x$$

for $-\pi/2\le y\le\pi/2$. Differentiating $\tan y$ and trigonometric identities we have

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{\frac{d\tan y}{dy}}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

In other words, $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$.

The rest of the formulas can be obtained in similar fashion.

Example 1 Find

a) $\frac{d}{dx}\arcsin 3x^2$ b) $\frac{d}{dx}\arccos(\ln x)$ c) $\frac{d}{dx}\arctan e^{2x}$ d) $\frac{d}{dx}arc\sec 3x^2$

Solution: Using Theorem 3.1 along the Chain Rule, we have

a) $$\frac{d}{dx}(\arcsin 3x^2)=\frac{1}{\sqrt{1-(3x^2)^2}}\frac{d}{dx}(3x^2)=\frac{6x}{\sqrt{1-9x^4}}$$

b) $$\frac{d}{dx}\arccos(\ln x)=-\frac{1}{\sqrt{1-(\ln x)^2}}\frac{d}{dx}(\ln x)=-\frac{1}{x\sqrt{1-(\ln x)^2}}$$

c) $$\frac{d}{dx}\arctan e^{2x}=\frac{1}{1+(e^{2x})^2}\frac{d}{dx}(e^{2x})=\frac{2e^{2x}}{1+(e^{2x})^2}$$

d) $\dfrac{d}{dx} arc\sec 3x^2 = \dfrac{1}{3x^2\sqrt{(3x^2)^2 - 1}} \dfrac{d}{dx}(3x^2) = \dfrac{2}{x\sqrt{9x^4 - 1}}$

Each of the formulas in Theorem 3.1 gives rise to an integration formula. The three most useful of these are given in the following theorem.

Theorem 3.2

i) $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin\frac{u}{a} + C$

ii) $\displaystyle\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\frac{u}{a} + C$

iii) $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}} du = \frac{1}{a} arc\sec\frac{u}{a} + C$

The proof of the above example is left as exercise.

Example 2 Evaluate $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$

Solution: If we let $u = e^{2x}$ so that $du = 2e^{2x}dx$, the integral may be written as in Theorem (i) as below.

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx = \int \frac{1}{\sqrt{1-u^2}} \frac{1}{2} du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin e^{2x} + C.$$

Example 3 Evaluate $\displaystyle\int \frac{x^2}{4+x^6} dx.$

Solution: The integral may be written as in the second formula of Theorem (3.2) by letting $a^2 = 4$ and using the substitution

$$u = x^3, \qquad du = 3x^2 dx$$

and proceed as follows:

$$\int \frac{x^2}{4+x^6} dx = \int \frac{1}{4+u^2}\left(\frac{du}{3}\right) = \frac{1}{3}\int \frac{1}{2^2+u^2} du$$

$$= \frac{1}{3}\cdot\frac{1}{2}\arctan\frac{u}{2} + C$$

$$= \frac{1}{6}\arctan\frac{u}{2} + C.$$

Example 4 Evaluate $\displaystyle\int \frac{1}{x\sqrt{x^4 - 9}} dx$

Solution: The integral may be written as in Theorem 3.2(iii) by letting $a^2 = 9$ and using the substitution

$$u = x^2, \qquad du = 2xdx,$$

we introduce $2x$ is the integrand by multiplying numerator and denominator by $2x$ and then proceed as follows:

$$\int \frac{1}{x\sqrt{x^4-9}}dx = \int \frac{1}{2x\cdot x\sqrt{(x^2)^2-3^2}}2xdx$$
$$= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-3^2}}du$$
$$= \frac{1}{2}\cdot\frac{1}{3} arc\sec\frac{u}{3} + C$$
$$= \frac{1}{6} arc\sec\frac{x^2}{3} + C.$$

Exercise 3.3

I Find the derivative of the function. Simplify where possible.

1. $f(x) = \sin^{-1}(2x-1)$
2. $f(x) = (1+x^2)\arctan x$
3. $f(x) = \tan^{-1}(x-\sqrt{1+x^2})$
4. $f(x) = \cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1} x$
5. $f(x) = (\tan x)^{\arctan x}$
6. $f(x) = (\tan^{-1} 4x)e^{\arctan 4x}$

II Evaluate the integral

7. $\int_0^4 \frac{1}{x^2+16}$
8. $\int \frac{\cos x}{\sqrt{9-\sin^2 x}}dx$
9. $\int \frac{1}{\sqrt{e^{2x}-25}}dx$
10. $\int \frac{1}{\sqrt{x}(1+x)}dx$

4.2.4 Hyperbolic Functions

The exponential expressions

$$\frac{e^x - e^{-x}}{2} \text{ and } \frac{e^x + e^{-x}}{2}$$

occur in advanced applications of calculus. Their properties are similar in many ways to those of $\sin x$ and $\cos x$, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine** and **hyperbolic cosine**. We also define the rest of the hyperbolic functions in terms of these functions.

Definition 3.3

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \csc hx = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sec hx = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

The hyperbolic functions satisfy a number of identities that are analogues of well-known trigonometric identities. We least some of the as below

Hyperbolic Identities

$\sinh(-x) = -\sinh x$ $\qquad$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $\qquad$ $\cosh^2 x + \sinh^2 x = \cosh 2x$

$\sinh 2x = 2\sinh x \cosh x$ $\qquad$ $1 - \tanh^2 x = \sec h^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

The proof the above identities are left as exercise.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as below. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric

Theorem 3.4

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\csc hx = -\csc hx \coth x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sec hx = -\sec hx \tanh x$$

$$\frac{d}{dx}\tanh x = \sec h^2 x \qquad \frac{d}{dx}\coth x = -\csc h^2 x$$

Example 1 If $f(x) = \cosh(e^{2x} + x)$, find $f'(x)$.

Solution: Applying Theorem 3.4, with the chain rule, we obtain

$$f'(x) = [\sinh(e^{2x} + x)] \cdot [(2e^{2x} + 1)] = (2e^{2x} + 1)\sinh(e^{2x} + x)$$

The integration formulas that correspond to the derivative formulas in theorem 3.4 are as follows.

Theorem 3.5

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \sec h^2 x dx = \tanh x + C \qquad \int \csc h^2 x dx = -\coth x + C$$

$$\int \sec hx \tanh x = -\sec h + C \qquad \int \csc hx \coth x = -\csc h + C$$

Example 2 Evaluate $\int x^2 \sinh x^3 dx$.

Solution: If we let $u = x^3$, then $du = 3x^2 dx$ and

$$\begin{aligned} \int x^2 \sinh x^3 dx &= \int \sinh u (\tfrac{1}{3} du) \\ &= \frac{1}{3}\cosh u + C \\ &= \frac{1}{3}\cosh x^3 + C. \end{aligned}$$

Exercise 3.4

I Find $f'(x)$ if $f(x)$ it the given expression.

1. $e^x \sinh x$
2. $\cosh(x^4)$
3. $\cos(\sinh x)$
4. $e^{\tanh x}\cosh(\cosh x)$

II Evaluate the integral

5. $\int \tanh 3x \sec h3x dx$
6. $\int \sinh x \sec h^2 dx$
7. $\int \sec hx dx$
8. $\int \tanh x dx$

III. Verify the identity.

9. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
10. $\sinh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{2}$

4.2.5 Inverse Hyperbolic Functions

The hyperbolic sine function is continuous and increasing for every x and hence, has a continuous, increasing inverse, function, denoted by $\sinh^{-1}$. Since $\sinh x$ is defined in terms of e^x, we might expect that $\sinh^{-1}$ can be expressed in terms of the inverse, ln, of the natural exponential function. The first formula of the next theorem shows that this is the case.

Theorem 3.6

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2+1}),\ x \ge 1$
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x},\ |x| < 1$
4. $\sec h^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x},\ 0 < x \le 1$

Proof: To prove (1), let $y = \sinh^{-1} x$. **Then**

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

then $$e^y - 2x - e^{-y} = 0.$$

Multiplying by e^y, we have

$$e^{2y} - 2xe^y - 1 = 0$$

which is a quadratic equation in e^y:

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $x - \sqrt{x^2+1} < 0$ **and** $e^y > 0$**, we must have**

$$e^y = x + \sqrt{x^2+1}.$$

The equivalent logarithmic form is

$$y = \ln(x + \sqrt{x^2+1})$$

that is, $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

The proofs of the formulas 2-4 in theorem 3.6 are left as exercise.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in theorem 3.7 below can be proved by the method for inverse functions or by differentiating the formulas in theorem 3.6.

Theorem 3.7

1. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
2. $\frac{d}{dx}(\operatorname{csc}h^{-1} x) = \frac{1}{|x|\sqrt{x^2+1}}$
3. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$
4. $\frac{d}{dx}(\operatorname{sec}h^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$
5. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$
6. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Proof: To proof (1) let $y = \sinh^{-1} x.$ Then $\sinh y = x$ and $\frac{dx}{dy} = \cosh y$. Since $\cosh y \geq 0$ and $\cosh^2 y - \sinh^2 y = 1,$ we have $\cosh y = \sqrt{1+\sinh^2 y}$. Then applying the method for inverse functions, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

Observe that we could have done the proof (1) by using formula (1) of Theorem 3.6 as below.

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1} x) &= \frac{d}{dx}\ln(x+\sqrt{x^2+1}) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.\end{aligned}$$

Example 1 Find $\frac{d}{dx}\sinh^{-1}(\tan x).$

Solution: Using Theorem 3.7 and the Chain rule, we have

$$\begin{aligned}\frac{d}{dx}\sinh^{-1}(\tan x) &= \frac{1}{\sqrt{\tan^2 x+1}}\frac{d}{dx}\tan x = \frac{1}{\sqrt{\sec^2 x}}\sec^2 x \\ &= \frac{1}{|\sec^2 x|}\sec^2 x = |\sec x|.\end{aligned}$$

Example 2 Evaluate $\int_0^{1/2} \frac{1}{1-x^2}dx$.

Solution: Referring to Theorem 3.7 we can see that the antiderivative of $1/1-x^2$ is $\tanh^{-1} x.$ Therefore

$$\int_0^{1/2} \frac{1}{1-x^2}dx = \tanh^{-1} x\Big|_0^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\Bigg|_0^{1/2}$$

$$= \frac{1}{2}\ln 3$$

Exercise 3.5

I Find $f'(x)$ *if* $f(x)$ is the given expression.

1. $\sinh^{-1} 5x$
2. $\sqrt{\cosh^{-1} x}$
3. $x\tanh^{-1} x + \ln\sqrt{1-x^2}$
4. $\sec h^{-1}\sqrt{1-x^2}, x > 0$

II Evaluate the integral

5. $\int \sinh 2x dx$
6. $\int \frac{\sinh x}{1+\cosh x}dx$
7. $\int \frac{e^x}{\sqrt{e^x - 16}}$
8. $\int \frac{\sin x}{\sqrt{1+\cos^2 x}}dx$

III Prove that the formulas 2,3, and 4 in Theorem 3.6.

4.2.6 L'Hôpital's Rule

While we study limits in the previous course of calculus we considered limits of quotients such as

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \quad \text{and} \quad \lim_{x\to 0}\frac{\sin x}{x}$$

and calculated the limits by using algebraic, geometric, and trigonometric methods even if the limits have the undefined form 0/0. In this section we develop another technique that employs the derivatives of the numerator and denominator of the quotient. This new technique is called L'Hôpital's rule. For the proof of this rule we need the following generalization of the Mean Value Theorem.

Theorem 4.1 (Cauchy's formula)

If *f* and *g* are continuous on [a,b] and differentiable on (a,b) and if $g'(x) \neq 0$ for every *x* in (a,b), then there is a number c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{f'(c)}$$

Proof: We first note that $g(b)-g(a) \neq 0,$ because otherwise g(a)=g(b) and, by Rolle's Theorem, there is a number c is (a,b) such that $g'(c) = 0,$ contrary to our assumption about g'.

Let us introduce a new function h as follows:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

for ever x in [a,b]. It follows that h is continuous on [a,b] and differentiable on (a,b) and that h(a) = h(b). By Rolle's Theorem, there is a number c in (a,b) such that $h'(c) = 0$; that is,

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0.$$

This is equivalent to Cauchy's formula.

The Indeterminate Form 0/0

If $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$, then we say that $\lim_{x\to a^+} f(x)/g(x)$ has the **indeterminate form 0/0**. The same notion applies if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}, \lim_{x\to c}, \lim_{x\to\infty}, or \lim_{x\to -\infty}$. The limits

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \quad and \quad \lim_{x\to 0}\frac{\sin x}{x}$$

therefore have the indeterminate form 0/0. Our first version of L'Hôpital's rule is concerning the indeterminate form 0/0.

Theorem 4.2 (L'Hôpital's rule)

Let L be a real number or $\infty \; or \; -\infty$.

a. Suppose *f* and g are differentiable on (a,b) and $g'(x) \neq 0 \; for \; a < x < b$. If

$$\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \qquad and \qquad \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c}$, where c is any number in (a,b). In the letter case *f* and g need not be differentiable at c.

b. Suppose f and g are differentiable on (a,∞) and $g'(x) \neq 0$ for $x > a$. If

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \qquad and \qquad \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to\infty} \; is \; replaced \; by \lim_{x\to -\infty}$.

Proof: We establish the formula involving the right-hand limits in (a). Define F and G on [a,b) by

$$F(x) = \begin{cases} f(x) & for \quad a < x < b \\ 0 & for \quad x = b \end{cases}$$

$$G(x) = \begin{cases} g(x) \; for \; a < x < b \\ 0 \quad for \quad x = a \end{cases}$$

Then

$$\lim_{x\to a^+} F(x) = \lim_{x\to a^+} f(x) = 0 = F(a)$$

so that F is continuous [a,b). The same is true of G. Moreover, F and G are differentiable on (a,b), since they agree with f and g, respectively, on (a,b). Consequently if x is any number in (a,b), the F and G are continuous on [a,x] and differentiable on (a,x). By the Generalized Mean Value Theorem, this means that there is a number c(x) in (a,x) such that

$$\frac{F(x)}{G(x)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F'(c(x))}{G'(c(x))}.$$

Because F = f and G =g on (a,b), this means that

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))}.$$

Since $a < c(x) < x,$ we know that

$$\lim_{x\to a^+} c(x) = a$$

so we can use the Substitution Theorem with y = c(x) to conclude that

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(c(x))}{g'(c(x))} = \lim_{y\to a^+}\frac{f'(y)}{g'(y)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L.$$

This proves the equation involving right-hand limit in (a). The results involving left-hand and two-sided limits are proved analogously. Part (b) is more difficult to prove, and we omit its proof.

Example 1 Evaluate $\lim\limits_{x\to 0}\dfrac{1-3^x}{x}.$

Solution: Both the numerator and the denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form 0/0 at x = 0. By L'Hôpital's rule

$$\lim_{x\to 0}\frac{1-3^x}{x} = \lim_{x\to 0}\frac{-3^x \ln 3}{1} = -\ln 3$$

Example 2 Evaluate $\lim\limits_{x\to 0}\dfrac{\ln(1-x^2)}{\ln\cos 2x}$

Solution: Observe that

$$\lim_{x\to 0}\ln(1-x^2) = 0 = \lim_{x\to 0}\ln\cos 2x$$

thus by applying L'Hôpital's rule we get

$$\lim_{x\to 0}\frac{\ln(1-x^2)}{\ln\cos 2x} = \frac{\frac{-2x}{1-x^2}}{-2\tan 2x} = \lim_{x\to 0}\left(\frac{1}{1-x^2}\cdot\frac{-2x}{-2\tan 2x}\right)$$

$$= \lim_{x\to 0}\frac{x}{\tan 2x}, \quad \sin ce \lim_{x\to 0}\frac{1}{1-x^2} = 1$$

$$= \lim_{x\to 0}\left[\frac{x}{\sin 2x}\cdot(\cos 2x)\right]$$

$$= \lim_{x\to 0}\frac{x}{\sin 2x}\cdot\lim_{x\to 0}(\cos x) = \frac{1}{2}$$

Example 3 Evaluate $\lim\limits_{x\to\infty}\dfrac{(\pi/2)-\arctan x}{1/x}.$

Solution: Since $\lim_{x\to\infty} \arctan x = \pi/2,$ we have

$$\lim_{x\to\infty}\left(\frac{\pi}{2} - \arctan x\right) = 0 = \lim_{x\to\infty}\frac{1}{x}$$

Hence by L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{(\pi/2) - \arctan x}{1/x} = \lim_{x\to\infty}\frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{x^2}{1+x^2} = 1 .$$

In same limits we need to apply L'Hôpital's rule several times in succession. The next example is one.

Example 4 Evaluate $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{2x - \sin 2x}$

Solution: The given quotient has the indeterminate form 0/0. By L'Hôpital's rule we have

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{2x - \sin 2x} = \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{2 - 2\cos 2x}$$

provided the second limit exists. Because the last quotient has the indeterminate form 0/0, we apply L'Hôpital's rule again, to obtain

$$\lim_{x\to 0}\frac{e^x + e^{-x} - 2}{2 - 2\cos 2x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin 2x}$$

still the last quotient has the indeterminate form 0/0, hence applying the L'Hôpital's rule for third time we get

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin 2x} = \lim_{x\to 0}\frac{e^x + e^{-x}}{8\cos 2x} = \frac{2}{8} = \frac{1}{4}.$$

The Indeterminate Form ∞/∞

Our second version of L'Hôpital's rule involves limits with indeterminate form ∞/∞. We give it now, with out proof.

Theorem 4.3 (L'Hôpital's rule)

Let L be a real number or ∞ *or* $-\infty$.

a. Suppose *f* and g are differentiable on (a,b) and $g'(x) \neq 0$ *for* $a < x < b$. If

$$\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty,\ \lim_{x\to a^+} g(x) = \infty \text{ or } -\infty,\ \text{and } \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c},$ where c is any number in (a,b). In the letter case *f* and g need not be differentiable at c.

b. Suppose f and g are differentiable on (a,∞) and $g'(x) \neq 0$ for $x > a$. If

$$\lim_{x\to\infty} f(x) = \infty \text{ or } -\infty,\ \lim_{x\to\infty} g(x) = \infty \text{ or } -\infty,\ \text{and } \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to\infty}$ *is replaced by* $\lim_{x\to-\infty}$.

Evaluate 5 Evaluate $\lim_{x\to(\pi/2)^-}\frac{4\tan x}{1+\sec x}$.

Solution: Observe that the limit has the indeterminate form ∞/∞. Then by L'Hôpital's rule we have

$$\lim_{x\to(\pi/2)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-}\frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to(\pi/2)^-}\frac{4\sec x}{\tan x}.$$

The last quotient again has the indeterminate form ∞/∞ at $x=\pi/2$; however, additional applications of L'Hôpital's rule always produce the form ∞/∞. In this case the limit may be found by using trigonometric identities to change the quotient as follows:

$$\frac{4\sec x}{\tan x} = \frac{4/\cos x}{\sin x/\cos x} = \frac{4}{\sin x}$$

Consequently

$$\lim_{x\to(\pi/2)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-}\frac{4}{\sin x} = \frac{4}{1} = 4.$$

Example 6 Evaluate $\lim_{x\to\infty}\frac{e^{2x}}{x^2}$.

Solution: Since the limit has the indeterminate form ∞/∞ by applying L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{e^{2x}}{x}.$$

The last quotient has the indeterminate form ∞/∞, so we apply L'Hôpital's rule for a second time, to obtain

$$\lim_{x\to\infty}\frac{e^{2x}}{x} = \lim_{x\to\infty}\frac{2e^{2x}}{1} = \infty.$$

Particularly in a similar fashion we can show that

$$\lim_{x\to\infty}\frac{e^x}{x^n} = \infty \quad \textit{for every real number } n.$$

Other Indeterminate Forms

Various indeterminate forms, such as $0\cdot\infty, 0^0, 1^\infty, \infty^0$, *and* $\infty-\infty$, can usually be converted into the indeterminate form 0/0 or ∞/∞ and then evaluated by one of the versions of L'Hôpital's rule given in Theorem 4.2 and 4.3.

Example 7 Find $\lim_{x\to0^+} x^2\ln x$

Solution: Since $\lim_{x\to0^+} x^2 = 0$ *and* $\lim_{x\to0^+}\ln x = -\infty$ the given limit is of the form $0\cdot\infty$ (more precisely, $0\cdot(-\infty)$). However, we can transform it into the indeterminate form ∞/∞ by writing it as

$$\lim_{x\to0^+}\frac{\ln x}{1/x^2}$$

and apply the L'Hôpital's rule we get

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3}.$$

The last quotient has the indeterminate form ∞/∞; however, further application of L'Hôpital's rule would again lead to ∞/∞. In this case we simplify the quotient algebraically and find the limit as follows:

$$\lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{x^3}{-2x} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Example 8 Find $\lim\limits_{x\to 0^+} x^x$.

Solution: The limit evidently has the indeterminate form 0^0. But then since $x^x = e^{x\ln x}$ and consequently

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}.$$

Since the exponential function is continuous, it follows that

$$\lim_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+}(x\ln x)}.$$

it the limit on the right side exists. But since

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+}(-x) = 0$$

by L'Hôpital's rule,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$$

Example 9 Show that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

Solution: Observe that the limit has the indeterminate form 1^∞. As in example 8, since $\left(1+\dfrac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x}$ first let us evaluate $\lim\limits_{x\to\infty}\ln\left(1+\dfrac{1}{x}\right)^x$. But in doing so we have

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x}.$$

This expression is now prepared for L'Hôpital's rule as the limit has 0/0 form. As a result

$$\lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty}\frac{\dfrac{1}{(1+1/x)}\left(-\dfrac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{1}{1+1/x} = 1.$$

Thus

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^0 = 1.$$

Example 10 Find $\lim\limits_{x\to 0^+}\left(\dfrac{1}{e^x-1}-\dfrac{1}{x}\right)$.

Solution: The limit has the indeterminate form $\infty-\infty$; however; if the difference is written as a single fraction, then

$$\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}.$$

This gives us the indeterminate form 0/0. It is necessary to apply L'Hôpital's rule twice, since the first application leads to the indeterminate form 0/0. Thus,

$$\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}=\lim_{x\to 0^+}\frac{1-e^x}{xe^x+e^x-1}$$

$$=\lim_{x\to 0^+}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.$$

Exercise

I Find the limit

1. $\lim_{x\to -1}\frac{x^6-1}{x^4-1}$
2. $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$
3. $\lim_{x\to 0}\frac{\sin x-x}{x^3}$
4. $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$
5. $\lim_{x\to \infty}\frac{\tan 1/x}{1/x}$
6. $\lim_{x\to 0}\frac{\tan^{-1}(2x)}{3x}$
7. $\lim_{x\to 0}\frac{e^x-1-x-(x^2/2)}{x^3}$
8. $\lim_{x\to -\infty} xe^x$
9. $\lim_{x\to \infty}(x-\sqrt{x^2-1})$
10. $\lim_{x\to \infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)$
11. $\lim_{x\to \infty}\left(1-\frac{1}{x}\right)^x$
12. $\lim_{x\to \infty}x^2\left(1-x\sin\frac{1}{3}\right)$
13. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$
14. $\lim_{x\to \pi/4}(\tan x)^{\tan 2x}$
15. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

16. Why is the following "application" of L'Hôpital's rule invalid?

$$\frac{1}{\pi/2}=\lim_{x\to \pi/2}\frac{\sin x}{x}=\lim_{x\to \pi/2}\frac{\cos x}{1}=0$$

17. Evaluate $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \sin(t^2)dt.$

4.3 Implicit Differentiation Problems

The following problems require the use of implicit differentiation. Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions y written EXPLICITLY as functions of x . For example, if

$$y=3x^2-\sin(7x+5)$$

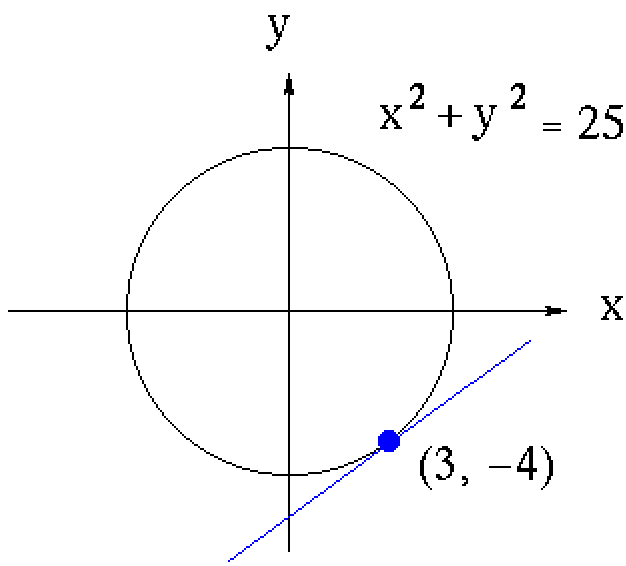
then the derivative of y is

$$y'=6x-7\cos(7x+5)$$

However, some functions y are written IMPLICITLY as functions of x . A familiar example of this is the equation

$$x^2+y^2=25\ ,$$

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point (3, -4) .



How could we find the derivative of y in this instance ? One way is to first write y explicitly as a function of x . Thus,

$$x^2 + y^2 = 25 ,$$
$$y^2 = 25 - x^2 ,$$

and

$$y = \pm\sqrt{25 - x^2}$$

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point (3, -4) lies on the bottom semi-circle given by

$$y = -\sqrt{25 - x^2} ,$$

the derivative of y is

$$y' = -(1/2)(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}} ,$$

i.e.,

$$y' = \frac{x}{\sqrt{25 - x^2}} .$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{(3)}{\sqrt{25 - (3)^2}} = \frac{3}{4} .$$

Unfortunately, not every equation involving x and y can be solved explicitly for y . For the sake of illustration we will find the derivative of y WITHOUT writing y explicitly as a function of x . Recall that the derivative (D) of a function of x squared, $(f(x))^2$, can be found using the chain rule :

$$D\{(f(x))^2\} = 2f(x)\ D\{f(x)\} = 2f(x)f'(x) .$$

Since y symbolically represents a function of x, the derivative of y^2 can be found in the same fashion :

$$D\{y^2\} = 2y\, D\{y\} = 2yy'$$
.

Now begin with $x^2 + y^2 = 25$. Differentiate both sides of the equation, getting

$$D(x^2 + y^2) = D(25),$$
$$D(x^2) + D(y^2) = D(25),$$

and

$$2x + 2\,y\,y' = 0,$$

so that

$$2\,y\,y' = -2x,$$

and

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$
,

i.e.,

$$y' = \frac{-x}{y}$$
.

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{-(3)}{(-4)} = \frac{3}{4}$$
.

This second method illustrates the process of implicit differentiation. It is important to note that the derivative expression for explicit differentiation involves *x* only, while the derivative expression for implicit differentiation may involve BOTH *x* AND *y* .

The following problems range in difficulty from average to challenging.

Example 1 Assume that *y* is a function of *x* . Find $y' = dy/dx$ for $x^3 + y^3 = 4$.

SOLUTION: Begin with $x^3 + y^3 = 4$. Differentiate both sides of the equation, getting

$$D(x^3 + y^3) = D(4),$$
$$D(x^3) + D(y^3) = D(4),$$

(Remember to use the chain rule on $D(y^3)$.)

$$3x^2 + 3y^2\, y' = 0,$$

so that (Now solve for y' .)

$$3y^2\, y' = -3x^2,$$

and

$$y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

Exercise 2 Assume that *y* is a function of *x* . Find $y' = dy/dx$ for $(x-y)^2 = x + y - 1$.

SOLUTION: Begin with $(x-y)^2 = x + y - 1$. Differentiate both sides of the equation, getting

$$D(x-y)^2 = D(x + y - 1),$$
$$D(x-y)^2 = D(x) + D(y) - D(1),$$

(Remember to use the chain rule on $D(x-y)^2$.)

$$2(x-y)\, D(x-y) = 1 + y' - 0$$
,

$$2(x-y)(1 - y') = 1 + y',$$

so that (Now solve for y' .)

$$2(x-y) - 2(x-y)\,y' = 1 + y',$$
$$-2(x-y)\,y' - y' = 1 - 2(x-y),$$

(Factor out y' .)

$$y' [- 2 (x-y) - 1] = 1 - 2 (x-y) ,$$

and

$$y' = \frac{1 - 2(x-y)}{-2(x-y) - 1} = \frac{2y - 2x + 1}{2y - 2x - 1} .$$

Example 3 Assume that y is a function of x . Find $y' = dy/dx$ for $y = \sin(3x + 4y)$.

SOLUTION: Begin with $y = \sin(3x + 4y)$. Differentiate both sides of the equation, getting

$$D(y) = D(\sin(3x + 4y)) ,$$

(Remember to use the chain rule on $D(\sin(3x + 4y))$.)

$$y' = \cos(3x + 4y)\ D(3x + 4y) ,$$

$$y' = \cos(3x + 4y)(3 + 4y') ,$$

so that (Now solve for y' .)

$$y' = 3\cos(3x + 4y) + 4y' \cos(3x + 4y) ,$$

$$y' - 4y' \cos(3x + 4y) = 3\cos(3x + 4y) ,$$

(Factor out y' .)

$$y'[1 - 4\cos(3x + 4y)] = 3\cos(3x + 4y) ,$$

and

$$y' = \frac{3\cos(3x + 4y)}{1 - 4\cos(3x + 4y)}$$

Example 4 Assume that y is a function of x . Find $y' = dy/dx$ for $y = x^2 y^3 + x^3 y^2$.
SOLUTION: Begin with $y = x^2 y^3 + x^3 y^2$. Differentiate both sides of the equation, getting

$$D(y) = D (x^2 y^3 + x^3 y^2) ,$$

$$D(y) = D (x^2 y^3) + D (x^3 y^2) ,$$

(Use the product rule twice.)

$$y' = \{x^2 D(y^3) + D(x^2) y^3\} + \{x^3 D(y^2) + D(x^3) y^2\} ,$$

(Remember to use the chain rule on $D (y^3)$ and $D (y^2)$.)

$$y' = \{x^2 (3y^2 y') + (2x) y^3\} + \{x^3 (2yy') + (3x^2) y^2\} ,$$

$$y' = 3x^2 y^2 y' + 2x y^3 + 2x^3 y y' + 3x^2 y^2 ,$$

so that (Now solve for y' .)

$$y' - 3x^2 y^2 y' - 2x^3 y y' = 2x y^3 + 3x^2 y^2 ,$$

(Factor out y' .)

$$y' [1 - 3x^2 y^2 - 2x^3 y] = 2x y^3 + 3x^2 y^2 ,$$

and

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$$

.

PROBLEM 5 Assume that y is a function of x . Find $y' = dy/dx$ for $e^{xy} = e^{4x} + e^{5y}$.
SOLUTION: Begin with $e^{xy} = e^{4x} - e^{5y}$. Differentiate both sides of the equation, getting

$$D(e^{xy}) = D(e^{4x} + e^{5y}),$$
$$D(e^{xy}) = D(e^{4x}) + D(e^{5y}),$$
$$e^{xy} D(xy) = e^{4x} D(4x) + e^{5y} D(5y),$$
$$e^{xy}(xy' + (1)y) = e^{4x}(4) + e^{5y}(5y'),$$

so that (Now solve for y' .)

$$xe^{xy} y' + y e^{xy} = 4 e^{4x} + 5e^{5y} y',$$
$$xe^{xy} y' - 5e^{5y} y' = 4 e^{4x} - y e^{xy},$$

(Factor out y' .)

$$y' [xe^{xy} - 5e^{5y}] = 4 e^{4x} - y e^{xy},$$

and

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} - 5e^{5y}}$$

Example 6 Assume that y is a function of x . Find $y' = dy/dx$ for $\cos^2 x + \cos^2 y = \cos(2x + 2y)$

.

SOLUTION: Begin with $\cos^2 x + \cos^2 y = \cos(2x + 2y)$. Differentiate both sides of the equation, getting

$$D(\cos^2 x + \cos^2 y) = D(\cos(2x + 2y)),$$

$$D(\cos^2 x) + D(\cos^2 y) = D(\cos(2x + 2y)),$$

$$(2\cos x)D(\cos x) + (2\cos y)D(\cos y) = -\sin(2x + 2y)D(2x + 2y),$$

$$2\cos x(-\sin x) + 2\cos y(-\sin y)(y') = -\sin(2x + 2y)(2 + 2y'),$$

so that (Now solve for y' .)

$$-2\cos x \sin x - 2y' \cos y \sin y = -2\sin(2x + 2y) - 2y' \sin(2x + 2y),$$

$$2y' \sin(2x + 2y) - 2y' \cos y \sin y = -2\sin(2x + 2y) + 2\cos x \sin x,$$

(Factor out y' .)

$$y'[2\sin(2x + 2y) - 2\cos y \sin y] = 2\cos x \sin x - 2\sin(2x + 2y),$$

$$y' = \frac{2\cos x \sin x - 2\sin(2x + 2y)}{2\sin(2x + 2y) - 2\cos y \sin y},$$

$$y' = \frac{2[\cos x \sin x - \sin(2x + 2y)]}{2[\sin(2x + 2y) - \cos y \sin y]},$$

and

$$y' = \frac{\cos x \sin x - \sin(2x+2y)}{\sin(2x+2y) - \cos y \sin y}$$

Example 7 Assume that y is a function of x . Find $y' = dy/dx$ for $x = \sqrt{x^2 + y^2}$.

SOLUTION: Begin with $x = \sqrt{x^2 + y^2}$. Differentiate both sides of the equation, getting

$$D(x) = D(\sqrt{x^2 + y^2})$$

,

$$1 = (1/2)(x^2 + y^2)^{-1/2} D(x^2 + y^2) ,$$

$$1 = (1/2)(x^2 + y^2)^{-1/2} (2x + 2y\, y') ,$$

so that (Now solve for y' .)

$$1 = \frac{(1/2)(2)(x + yy')}{\sqrt{x^2 + y^2}} ,$$

$$1 = \frac{x + yy'}{\sqrt{x^2 + y^2}} ,$$

$$\sqrt{x^2 + y^2} = x + yy' ,$$

$$\sqrt{x^2 + y^2} - x = yy' ,$$

and

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y} .$$

Exercise 8: Assume that y is a function of x . Find $y' = dy/dx$ for $\frac{x - y^3}{y + x^2} = x + 2$.

SOLUTION: Begin with $\frac{x - y^3}{y + x^2} = x + 2$. Clear the fraction by multiplying both sides of the equation by $y + x^2$, getting

$$\frac{x - y^3}{y + x^2}(y + x^2) = x + 2(y + x^2)$$

or

$$x - y^3 = xy + 2y + x^3 + 2x^2 .$$

Now differentiate both sides of the equation, getting

$$D(x - y^3) = D(xy + 2y + x^3 + 2x^2) ,$$

$$D(x) - D(y^3) = D(xy) + D(2y) + D(x^3) + D(2x^2) ,$$

(Remember to use the chain rule on $D(y^3)$.)

$$1 - 3y^2 y' = (xy' + (1)y) + 2y' + 3x^2 + 4x ,$$

so that (Now solve for y' .)

$$1 - y - 3x^2 - 4x = 3y^2 y' + xy' + 2y' ,$$

(Factor out y' .)

$$1 - y - 3x^2 - 4x = (3y^2 + x + 2) y' ,$$

and

$$y' = \frac{1 - y - 3x^2 - 4x}{3y^2 + x + 2}$$
.

Class Work

PROBLEM 9 : Assume that *y* is a function of *x* . Find $y' = dy/dx$ for $\frac{y}{x^3} + \frac{x}{y^3} = x^2 y^4$.

Example 10 Find an equation of the line tangent to the graph of $(x^2+y^2)^3 = 8x^2y^2$ at the point (-1, 1) .

SOLUTION Begin with $(x^2+y^2)^3 = 8x^2y^2$. Now differentiate both sides of the equation, getting

$$D\,(x^2+y^2)^3 = D\,(\,8x^2y^2\,)\,,$$
$$3\,(x^2+y^2)^2\,D\,(x^2+y^2) = 8x^2\,D\,(y^2\,) + D\,(\,8x^2\,)\,y^2\,,$$

(Remember to use the chain rule on $D\,(y^2\,)$.)

$$3\,(x^2+y^2)^2\,(\,2x + 2\,y\,y'\,) = 8x^2\,(2\,y\,y'\,) + (\,16\,x\,)\,y^2\,,$$

so that (Now solve for y' .)

$$6x\,(x^2+y^2)^2 + 6\,y\,(x^2+y^2)^2\,y' = 16\,x^2\,y\,y' + 16\,x\,y^2\,,$$
$$6\,y\,(x^2+y^2)^2\,y' - 16\,x^2\,y\,y' = 16\,x\,y^2 - 6x\,(x^2+y^2)^2\,,$$

(Factor out y' .)

$$y'\,[\,6\,y\,(x^2+y^2)^2 - 16\,x^2\,y\,] = 16\,x\,y^2 - 6x\,(x^2+y^2)^2\,,$$

and

$$y' = \frac{16xy^2 - 6x(x^2 + y^2)^2}{6y(x^2 + y^2)^2 - 16x^2y}$$
.

Thus, the slope of the line tangent to the graph at the point (-1, 1) is

$$m = y' = \frac{16(-1)(1)^2 - 6(-1)((-1)^2 + (1)^2)^2}{6(1)((-1)^2 + (1)^2)^2 - 16(-1)^2(1)} = \frac{8}{8} = 1$$
,

and the equation of the tangent line is

$$y - (\,1\,) = (1)\,(\,x - (\,-1\,)\,)$$

or

$$y = x + 2$$

Example 11 Find an equation of the line tangent to the graph of $x^2 + (y-x)^3 = 9$ at x=1.

SOLUTION: Begin with $x^2 + (y-x)^3 = 9$. If x=1 , then

$$(1)^2 + (\,y-1\,)^3 = 9$$

so that

$$(\,y-1\,)^3 = 8\,,$$
$$y-1 = 2\,,$$
$$y = 3\,,$$

and the tangent line passes through the point (1, 3) . Now differentiate both sides of the original equation, getting

$$D\,(\,x^2 + (y-x)^3\,) = D\,(\,9\,)\,,$$
$$D\,(\,x^2\,) + D\,(y-x)^3 = D\,(\,9\,)\,,$$
$$2x + 3\,(y-x)^2\,D\,(y-x) = 0\,,$$
$$2x + 3\,(y-x)^2\,(y'-1) = 0\,,$$

so that (Now solve for y' .)

$$2x + 3\,(y-x)^2\,y' - 3\,(y-x)^2 = 0\,,$$
$$3\,(y-x)^2\,y' = 3\,(y-x)^2 - 2x\,,$$

and

$$y' = \frac{3(y-x)^2 - 2x}{3(y-x)^2}$$

.

Thus, the slope of the line tangent to the graph at (1, 3) is

$$m = y' = \frac{3(3-1)^2 - 2(1)}{3(3-1)^2} = \frac{10}{12} = \frac{5}{6}$$

and the equation of the tangent line is

$$y - (3) = (5/6) (x - (1)) ,$$

or

$$y = (7/6) x + (13/6) .$$

Finally let us see how to find the second derivative of a function that is defined implicitly.

Example 12 Find y'' *if* $x^4 + y^4 = 25$

Solution: Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3 y' = 0$$

solving for y'gives

$$y' = -\frac{x^3}{y^3} \qquad (1)$$

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3 D(x^3) - x^3 D(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 .3x^2 - x^3(3y^2 y')}{y^6}$$

If we now substitute Equation 1 into this expression we get

$$y'' = \frac{3x^2y^3 - 3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6}$$

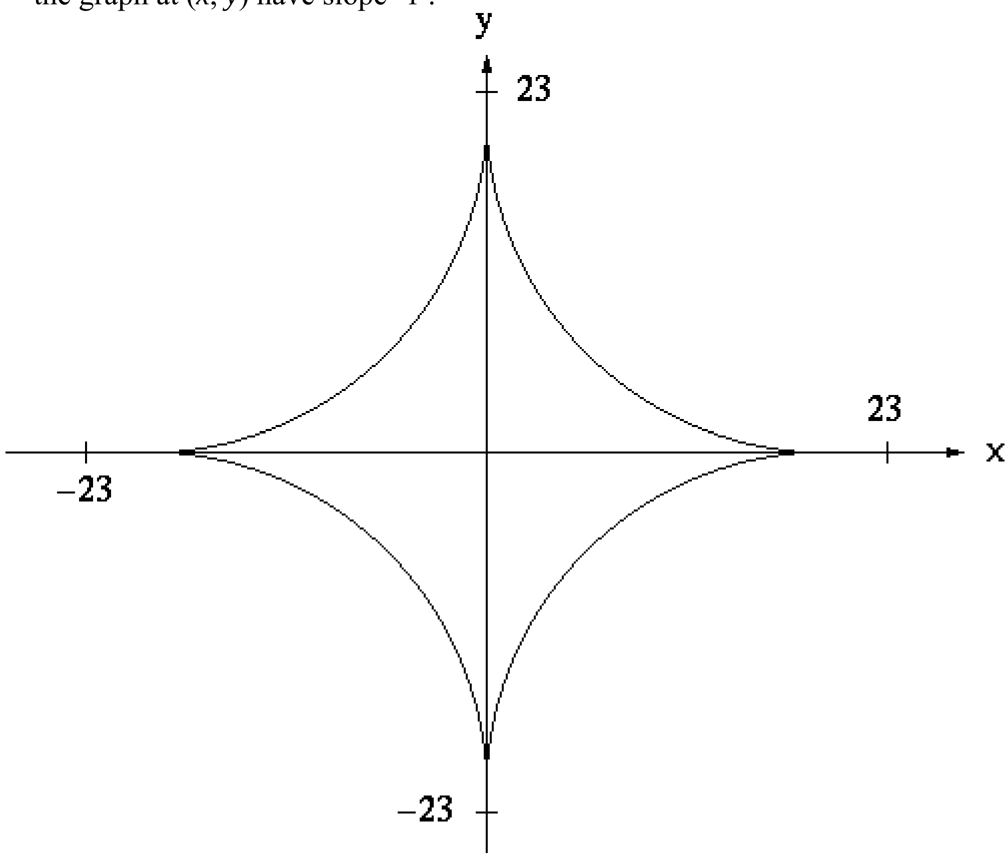
$$= -\frac{3(x^2y + x^6)}{y^7} = -\frac{3x(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 25$. So that answer simplifies to $y'' = -\frac{3x^2(25)}{y^7} = -75\frac{x^2}{y^7}$

Class Work

1. Find the slope and concavity of the graph of $x^2y + y^4 = 4 + 2x$ at the point (-1, 1) .
2. Consider the equation $x^2 + xy + y^2 = 1$. Find equations for y' and y" in terms of x and y
3. Find all points (x, y) on the graph of $x^{2/3} + y^{2/3} = 8$ (See diagram.) where lines tangent to

the graph at (x, y) have slope -1 .



4.Find y'' by implicit differentiation.

a. $x^3 + y^3 = 1$

b. $x^2 + 6xy + y^2 = 8$

c. $\sqrt{x} + \sqrt{y} = 1.$

4.4 Application of the derivative

4.4.1 Extrema of a function

Definition 1 A function f has an **absolute maximum** at ***c*** if f©≥ f(x) for all x in D, where D is the domain of f. The number f© is called the **maximum value of f** on D. Similarly, f has an **absolute minimum** at c if f© ≤ f(x) for all x in D and the number f© is called the **minimum value of f** on D. The maximum and minimum values of are called the **extreme** values of f.

Definition 2 A function f has a **local maximum** (or relative maximum) at c if there is an open interval I containing c such that f©≥ f(x) for all x in I. Similarly, f has a **local minimum** at c if there is an open interval I containing c such that f© ≤ f(x) for all x in I.

Example 1 If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x. therefore f(0) = 0 is the absolute (and local) minimum value of f. this corresponds to the fact that the origin is that lowest point on the parabola $y = x^2$. However, there is no highest point on the parabola and so this function has no maximum value.

Example 2 From the graph of the function f(x) = x^3 we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either.

Theorem 3 If f has a relative (local) extremum (that is, maximum or minimum) at c, and that $f'(c)$ exists, then $f'(c) = 0$.

Definition 4 A number c in the domain of a function f is a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 3 Find the critical numbers of f $f(x) = 4x^{3/5} - x^{8/5}$.

Solution The derivative of f is given by

$$f'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5} = \frac{12-8x}{5x^{2/5}}$$

Therefore $f'(c) = 0$ if 12-8x = 0, that is, x = 3/2 and $f'(x)$ does not exist when x =0. Thus the critical numbers are 3/2 and 0.//

To find the absolute extreme value of a function on a closed interval a similar theorem to Theorem 3 is given bellow.

Maximum-Minimum Theorem

Theorem 5 Let f be continuous on a closed interval [a, b]. Then f has a maximum and a minimum value on [a, b].

Note that according to Maximum-Minimum Theorem an extreme value can be taken on more than once.

The following Theorem will simplify our effort of searching for an extreme value on a closed interval.

Theorem 6 Let f be defined on [a, b]. If an *absolute* extreme value of f on [a, b] occurs at a number c in (a, b) at which f has a derivative, then $f'(c) = 0$.

In using theorem 5 to find the extreme value we followthe three-step procedure bellow.

1. Find the values of f at the critical numbers of f in (a,b)
2. Find the values of f(a) and f(b).
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 2 \quad \tfrac{1}{2} \le x \le 3$$

Solution: Since f is continuous on [-½,3], we can use the procedure outlined above:
Since

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, x = 0 or x = 2. Notice that each of these critical numbers lies in the interval [-½,3]. The values of f at these critical numbers are

f(0) = 2 f(2) = -3

The values of f at the endpoints of the interval are

$$f(-\tfrac{1}{2}) = \tfrac{1}{8} \qquad f(3) = -2$$

Comparing these four numbers, we see that the absolute maximum value is f(0)=f(3)=2 and the absolute minimum value is f(2) = –2.

Class Work

1. Find the critical numbers of each function a) $f(x) = x^3 - 6x + 1$ b) $f(x)=|x|$ c) $\cos\sqrt{x}$ d) $f(x) = \frac{1}{\sqrt{x^2+1}}$
2. Find all extreme values (if any) of the given function on the given interval. Determine at which numbers in the interval these values occur. a) $f(x)=x^2 - 2x + 2$, [0,3] b) $f(x)=x^2 + 2/x$, [1/2, 2] c) $f(x)=x^{2/3}$, [-8,8].
3. Show that 0 is a critical number of the function $f(x) = x^5$ but f does not have a local extremum at 0.
4. Prove that the function $f(x) = x^{51} + x^{21}+x+1$ has neither a local maximum nor a local minimum.

4.4.2 The Mean Value Theorem

Theorem 6 Let f be continuous on [a, b] and differentiable on (a, b). Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b-a).$$

Example 5 Let $f(x) = x^3 - 8x + 5$. Find a number c in (0,3) that satisfies the Mean Value Theorem.

Solution Since f is continuous on [0,3] and $f'(c)$ should satisfy the condition

$$f'(c) = \frac{f(3) - f(0)}{3-0} = \frac{8-5}{3} = 1$$

we seek a number c in (0,3) such that $f'(c) = 1$. But

$$f'(x) = 3x^2 - 8$$

so that c must satisfy

$$3c^2 - 8 = 1$$

$$c = \pm\sqrt{3}$$

Since $-\sqrt{3} \notin (0,3)$, the value of c that satisfies the mean value theorem in the interval (0,3) is $\sqrt{3}$.

Class Work

1. Verify that the function bellow satisfies the hypothesis of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   a) $f(x)=1 - x^2$, [0, 3] b) $f(x) = 3\left(x + \frac{1}{x}\right), [\frac{1}{3}, 3]$ c) $f(x) = \sqrt{x}, [1,4]$
2. Let f(x)=|x-1|. Show that there is no value of c such that f(3)-f(0)=f'©(3-0). Why does this not contradict the Mean Value Theorem?
3. Show that the equation $x^5+10x+3=0$ has exactly one real root.
4. Show that the equation $x^4+4x+c=0$ has at most two real roots.

4.4.3 First and Second Derivative Tests; Curve sketching

I hope you remember that a function that is increasing or decreasing on an interval I is called **monotonic** on I and we used the test stated in the theorem bellow to identify whether a function is monotonic or not on a given interval.

Theorem 7 Suppose f is continuous on [a,b] and differentiable on (a,b).

a) If $f'(x) > 0$ for all x in (a,b), then f is increasing on [a,b].
b) If $f'(x) < 0$ for all x in (a,b), then f is decreasing on [a,b].

Theorem 7 lays the bases for the proof of the first derivative test stated as follows.

Theorem 8 (The first derivative test)

Suppose that c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.
b) If f' changes from negative to positive at c, then f has a local minimum at c.
c) If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 6 Find the local extrema of $x^{1/3}(8-x)$ and sketch its graph.

Solution By the product rule we have

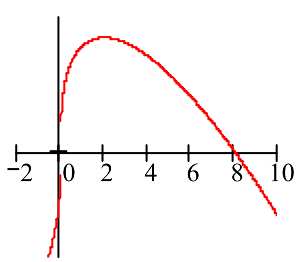
$$f'(x) = \tfrac{1}{3}x^{-2/3}(8-x) - x^{1/3}$$
$$= \frac{8-x-3x}{3x^{2/3}} = \frac{4(2-x)}{3x^{2/3}}$$

The derivative $f'(x) = 0$ when x =2 more over f'(x) does not exist when x = 0. So the critical numbers are 0 and 2.

Bellow we give the sign chart for f'(x).

	0	2	
4(2-x) $3x^{2/3}$ f'(x) f	+ + + f is increasing	+ + + f is increasing	– + – f is decreasing

Then the function does not have an extreme value at 0. Since f' does not change sign at 0. But f has a local maximum at 2 since f' changes sign from positive to negative and the local maximum value is given by f(2)=$2^{1/3}$(8 –2)=$6\sqrt[3]{2}$. Then using the sign chart and the extreme value we sketch the graph as bellow.



Class Work

If $f(x) = x^{2/3}(x^2 - 8)$, find the local extrema, and sketch the graph of *f*.

As the first derivative is useful to sketch the graph of a function the second derivative gives also additional information that enables us to sketch a better picture of the graph. The tests that we give below involve second derivative the student can consult advanced books for there proofs.

Theorem 9 (The Test For Concavity) Suppose f is twice differentiable on an interval I.

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 10 A point (a,b) on a curve is called a **point of inflection** if the curve changes from concave upward to concave downward or from concave downward to concave upward at (a,b).

Example 7 Determine where the curve $y = x^3 - 3x + 1$ is concave upward and where it is concave downward. Find the inflection points and sketch the curve.

Solution If $f(x) = x^3 - 3x + 1$**, then**

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Since $f'(x) = 0$ when $x^2 = 1$, the critical numbers are ±1. Also

$$f'(x) < 0 \quad \Leftrightarrow \quad x^2 - 1 < 0 \quad \Leftrightarrow x^2 < 1 \quad \Leftrightarrow \quad |x| < 1$$

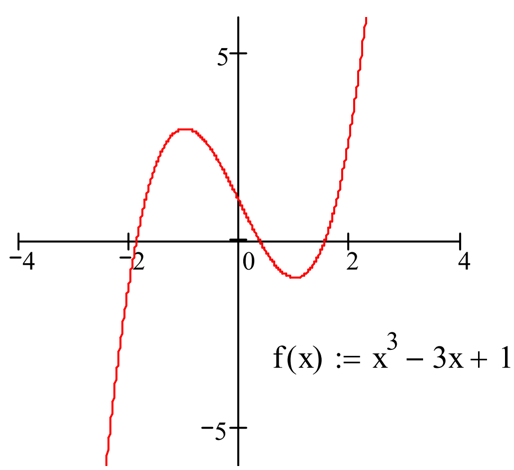
$$f'(x) < 0 \quad \Leftrightarrow \quad x^2 < 1 \quad \Leftrightarrow \quad x > 1 \quad or \quad x < -1$$

Therefore f is increasing on the interval (-∞, -1] and [1,∞) and is decreasing on [-1,1]. By the first derivative test, f(-1) = 3 is local maximum value and f(1) = -1 is a local minimum value.

To determine the concavity we compute the second derivative:

$$f''(x) = 6x$$

Thus $f''(x) > 0$ when x>0 and $f''(x) < 0$ when x<0. The Test for concavity then tells us that the curve is concave downward on (-∞, 0) and concave upward on (0, ∞). Since the curve changes from concave downward to concave upward when x = 0, the point (0, 1) is a point of inflection. We use this information to sketch the curve in Fig below.



Another Application of the second derivative is in finding maximum and minimum values of a function.

Theorem 11 Suppose f'' is continuous on an open interval of a function.

a) If $f'(x)=0$ *and* $f''(c)>0$, then f has a local minimum at c.

b) If $f'(x)=0$ *and* $f''(c)<0$, then f has a local maximum at c.

EXAMPLE

Use the Second Derivative Test to find relative extrema of

$f(x) = 3\cdot x^4 + 8\cdot x^3 + 4$.

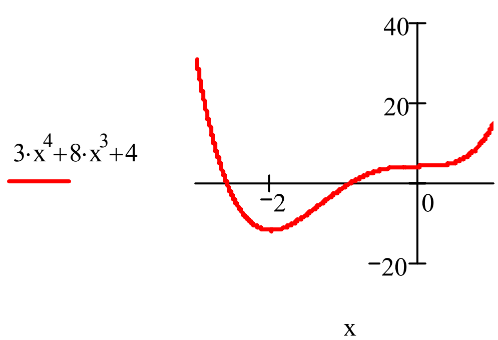
Solution

$f'(x) = 12\cdot x^3 + 24\cdot x^2$ $= 12\cdot x^2\cdot(x+2)$ Critical numbers: x = 0, x = -2	**Find critical numbers of f.** **(Note that the Second Derivative Test can only be applied at critical numbers where f '=0.)**
$f''(x) = 36\cdot x^2 + 48\cdot x$	**Find f ".** **Evaluate f " at the critical numbers where f ' = 0.**
$f''(-2) = 48 > 0$	**Relative minimum**
$f''(0) = 0$	**The Second Derivative Test fails in this case.**

In the last case, x = 0 could still be a relative maximum, relative minimur or neither; but the Second Derivative Test fails to produce any useful information.

If you used the First Derivative Test, you would find out that x = 0 is not relative extremum (there is an inflection point there instead).

The graph on the right illustrates these findings.



Class Work

Find (a) the intervals of increasing or decreasing, b) the local maximum and minimum values of the points of inflection. Then use this information to sketch the graph.

a) $f(x)=x^3 - x$ b) $f(x) = x\sqrt{x+1}$ c) $f(x) = x^{1/3}(x+3)^{2/3}$

4.4.3 Curve Sketching

Now we apply the knowledge that we have developed in this chapter for sketching the graphs of different functions. The table below lists the items that are most important in graphing a function f.

Property	Test
f has y intercept c	*f*(0)=c
f has x intercept c	*f*©=c
symmetric with respect to the $\begin{cases} y\ axis \\ origin \end{cases}$	$f(-x) = f(x)$ $f(-x) = -f(x)$
f has a relative maximum value at c	$\begin{cases} f'(c) = 0\ and\ \ f'changes\ from\ v+\ to\,v- \\ f'(c) = 0\ and\ f''(c) < 0 \end{cases}$
f has a relative minimum value at c	$\begin{cases} f'(c) = 0\ and\ \ f'changes\ from\ v-\ to\,v+ \\ f'(c) = 0\ and\ f''(c) > 0 \end{cases}$
f is strictly increasing on an open interval I	f'>0 for all except finitely many x in I
f is strictly decreasing on an open interval I	f'<0 for all except finitely many x in I
Graph of f is concave upward on I	$f''(x) < 0 \quad \forall x \in I$
Graph of f is concave downward on I	$f''(x) > 0 \quad \forall x \in I$
(c, f©) is an inflection point	f'' changes sign at c and usually $f''(c) = 0$
f has a vertical asymptote $x = c$	$\lim\limits_{x\to c^+} f(x) = \pm\infty\ \ or\ \lim\limits_{x\to c^-} f(x) = \pm\infty$
f has a horizontal asymptote $x = d$	$\lim\limits_{x\to\infty} f(x) = d\ \ or\ \lim\limits_{x\to-\infty} f(x) = d$

Example If $g(x) = \dfrac{x^2}{1-x^2}$, discuss and sketch the graph of *g*.

Solution:

1. **Analyze the first derivative.**

$$g'(x) := \frac{2\cdot x}{\left(1 - x^2\right)^2}$$

This has a root at x = 0. Possible local maximum or minimum here.

Notice that neither g(x) nor its derivative are defined at x = 1 and x = –1. The derivative is negative for x < 0, except at x = –1, where it is not defined. It is positive for x > 0, except at x = 1, where it is not defined.

2. **Analyze the second derivative.**

$$g''(x) := \frac{2 + 6\cdot x^2}{\left(1 - x^2\right)^3}$$

There are no values of x where the second derivative equals zero, so the graph of g has no inflection points.

$g''(0) = 2$

At x = 0, a critical number, the second derivative is positive, so the graph is concave up at this point, and has a local minimum.

3. Find horizontal asymptotes.

$$\lim_{x \to \infty} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad -1$$

$$\lim_{x \to -\infty} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad -1$$

$h(x) := -1$ is a horizontal asymptote.

4. Find vertical asymptotes.

Since g is undefined at 1 and –1, examine the limits of g as x approaches these values.

$$\lim_{x \to 1^+} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad -\infty$$

$$\lim_{x \to 1^-} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad \infty$$

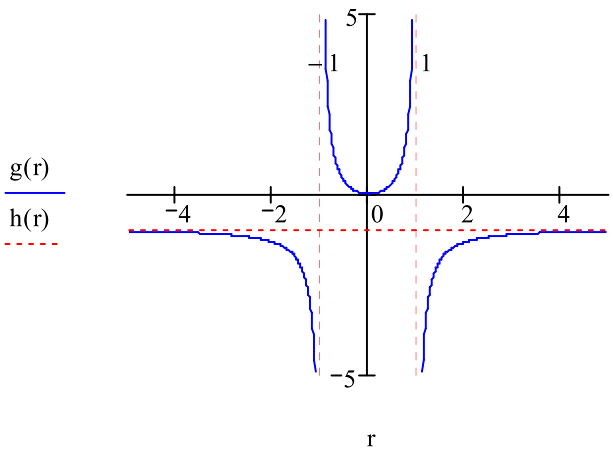
$$\lim_{x \to -1^+} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad \infty$$

$$\lim_{x \to -1^-} \frac{x^2}{1 - x^2} \quad \text{simplifies to} \quad -\infty$$

g has vertical asymptotes at x = 1 and x = –1.

5. Put it all together.

$r := -5, -4.99 .. 5$ **Range for graphing**



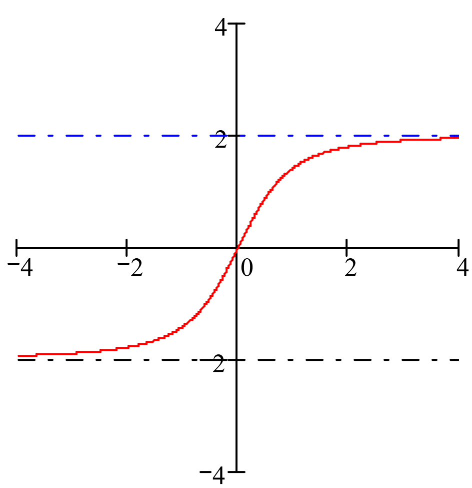
Notice that all the aspects of the graph you found in your analysis are present: a local minimum at x = 0, vertical asymptotes at x = 1 and x = –1, a horizontal asymptote at y = –1, downward sloping when x < 0, upward sloping when x > 0.

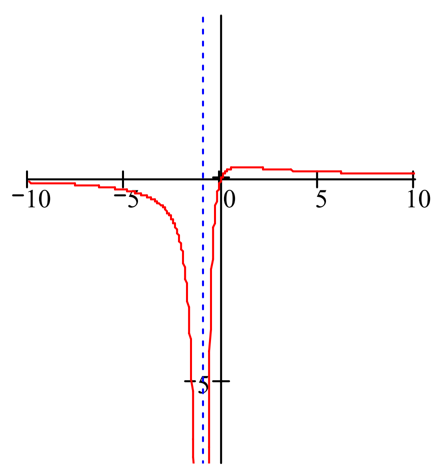
Class Work

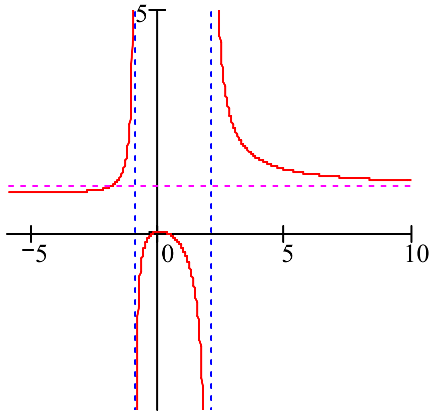
Discuss and sketch the graph of *f* if

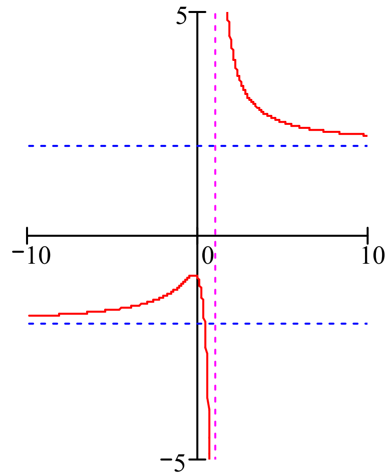
a) $f(x)=\dfrac{x^2}{x^2-x-2}$ b) $f(x)=\dfrac{x}{(x+1)^2}$ c) $f(x)=\dfrac{2x}{\sqrt{x^2+1}}$ d) $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Note your sketch should look like one of the graphs bellow.









5 Review of Techniques of Integration

5.0 Introduction

Before we see techniques of integration let us revise the integrals of important functions in the following table.

Derivative	Indefinite integral
$D_x(x) = 1$	1. $\int 1dx = \int dx = x + c$
$D_x\left(\dfrac{x^{r+1}}{r+1}\right) = x^r \ (r \neq -1)$	2. $\int x^r dx = \dfrac{x^{r+1}}{r+1} + c \ \ (r \neq -1)$
$D_x(\sin x) = \cos x$	3. $\int \cos xdx = \sin x + c$
$D_x(-\cos x) = \sin x$	4. $\int \sin xdx = -\cos x + c$
$D_x(\tan x) = \sec^2 x$	5. $\int \sec^2 xdx = \tan x + c$
$D_x(-\cot x) = \csc^2 x$	6. $\int \csc^2 xdx = -\cot x + c$
$D_x(\sec x) = \sec x \tan x$	7. $\int \sec x \tan xdx = \sec x + c$
$D_x(-\csc x) = \csc x \cot x$	8. $\int \csc x \cot xdx = -\csc x + c$
$D_x(e^x) = e^x$	9. $\int e^x dx = e^x + c$
$D_x\left(\dfrac{a^x}{\ln a}\right) = a^x$	10. $\int a^x dx = \dfrac{a^x}{\ln a} + c$
$D_x(\ln\lvert x\rvert) = \dfrac{1}{x}$	11. $\int \dfrac{1}{x} dx = \ln\lvert x\rvert + c$

$D_x(\sin^{-1}\frac{x}{a}) = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} D_x(\frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$	12. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$
$D_x(\cos^{-1}\frac{x}{a}) = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} D_x(\frac{x}{a}) = -\frac{1}{\sqrt{a^2-x^2}}$	13. $\int \frac{1}{\sqrt{a^2-x^2}}dx = -\cos^{-1}\frac{x}{a} + c$
$D_x(\frac{1}{a}\tan^{-1}\frac{x}{a}) = (\frac{1}{a})\frac{1}{1+(\frac{x}{a})^2} D_x(\frac{x}{a}) = \frac{1}{a^2+x^2}$	14. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
$D_x\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2-a^2}}$	15. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

Table 1.0

I hope the student does not forget how to evaluate the definite integral by using the following fundamental theorem of calculus:

Theorem 1.0 (Fundamental theorem of calculus)
Suppose f is continuous on a closed interval [a,b].
Part I If the function G is defined by

$$G(x) = \int_a^x f(t)dt$$

for every x in [a,b], then G is an antiderivative of f on [a,b].
Part II If F is any antiderivative of f on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 1.0 Evaluate $\int_{-2}^{3}(6x^2-5)dx.$

Solution: An antiderivative of $6x^2-5$ is $F(x) = 2x^3-5x.$ Then

$$\int_{-2}^{3}(6x^2-5)dx = 2x^3-5x\Big|_{-2}^{3}$$

$$= [2(3)^3-5(3)] - [2(-2)^3-5(-2)] = 45.$$

5.1 Integration by Substitution

The formulas for indefinite integrals in Table (1.0) are limited in scope, because we cannot use them directly to evaluate such as

$$\int \sqrt{2x-5}dx \;\; or \;\; \int \sin 3x dx$$

In this section we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (1.0).

Method of Substitution

If the integral to be evaluated is of the form

$$\int f(g(x))g'(x)dx$$

we substitute $u = g(x)$ and $du = g'(x)dx,$ then the integral becomes $\int f(u)du.$

Example 1 Evaluate $\int \sqrt{2x-5}dx.$

Solution: We let u = 2x – 5 and calculate du:

u = 2x – 5 , du = 2dx

Since du contains the factor 2, the integral is not in the proper form $\int f(u)du$ required in the method of substitution given above. However, we can introduce the factor 2 into the integrand, provided we also multiply by $\frac{1}{2}$. Doing this and property of integral we have

$$\begin{aligned}\int \sqrt{2x-5}dx &= \int \sqrt{2x-5}\,\tfrac{1}{2}2dx \\ &= \tfrac{1}{2}\int \sqrt{2x-5}\,2dx\end{aligned}$$

We now substitute and use the power rule for integration:

$$\begin{aligned}\int \sqrt{2x-5}dx &= \tfrac{1}{2}\int \sqrt{u}du \\ &= \tfrac{1}{2}\int u^{\frac{1}{2}}du \\ &= \tfrac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c \\ &= \tfrac{1}{3}u^{\frac{3}{2}}+c \\ &= \tfrac{1}{3}(2x-5)^{3/2}+c\end{aligned}$$

Example 2 Evaluate $\int \sin 2x dx.$

Solution: We make the substitution

$u = 2x,\ du = 2dx$.

Since du contains the factor 2, we adjust the integrand by multiplying by 2 and compensate by multiplying the integral by $\frac{1}{2}$ before substituting:

$$\begin{aligned}\int \sin 2x dx &= \tfrac{1}{2}\int (\sin 2x)2dx \\ &= \tfrac{1}{2}\int \sin u\, du \\ &= -\tfrac{1}{2}\cos u + c \\ &= -\tfrac{1}{2}\cos 2x + c\end{aligned}$$

It is not always easy to decide what substitution u = g(x) is needed to transform an indefinite integral into a form that can be readily evaluated. It may be necessary to try several different possibilities before finding a suitable substitution. In most cases no substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for changing variables in indefinite integrals

1. Decide on a reasonable substitution $u = g(x)$.
2. Calculate $du = g'(x)dx.$
3. Using 1 and 2, try to transform the integral into a form that involves only the variable u. If necessary, introduce a constant factor k into the integrand and compensate by 1/k. If any part of the resulting integrand contains the variable x, use a different substitution in 1.
4. Evaluate the integral obtained in 3, obtaining an antiderivative involving u.

5. Replace u in the antiderivative obtained in guideline 4 by $g(x)$. The final result should contain only the variable x.

The following examples illustrate the use of the guidelines.

Example 3 Evaluate $\int x^2(3x^3+2)^{10}\,dx$.

Solution: If an integrand involves an expression raised to a power, such as $(3x^3+2)^{10}$, we often substitute u for the expression. Thus, we let

$$u = 3x^3+2, \quad du = 9x^2dx \Leftrightarrow \tfrac{1}{9}du = x^2dx.$$

Comparing $du = 9xdx$ with x^2dx in the integral suggests that we introduce the factor 9 into the integrand. Doing this and compensating by multiplying the integral by 1/9, we obtain the following:

$$\begin{aligned}\int x^2(3x^3+2)^{10}\,dx &= \int u^{10}\,\tfrac{1}{9}du\\ &= \tfrac{1}{9}\int u^{10}du\\ &= \frac{1}{9}\left(\frac{u^{11}}{11}\right)+c\\ &= \tfrac{1}{99}(3x^2+2)^{11}+c.\end{aligned}$$

Example 4 Evaluate $\int x\sqrt{3x-1}dx.$

Solution: To simplify the expression $\sqrt{3x-1}$, we let

$$u = 3x-1, \ \textit{so that} \ \ du = 3dx.$$

Then

$$\int x\sqrt{3x-1}dx = \int x\overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du}$$

Thus we still need to find x in terms of u. From the equation u=3x –1 we deduce that

$$x = \frac{1}{3}(u+1).$$

Therefore

$$\begin{aligned}\int x\sqrt{3x-1}dx &= \int \overbrace{x}^{\frac{1}{3}(u+1)}\ \overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du} = \int \tfrac{1}{3}(u+1)\sqrt{u}\,\tfrac{1}{3}du\\ &= \tfrac{1}{9}\int\left(u^{3/2}+u^{1/2}\right)du\\ &= \frac{1}{9}\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+c\\ &= \frac{2}{45}(3x-1)^{5/2}+\frac{2}{27}(3x-1)^{3/2}+c.\end{aligned}$$

Sometimes there is more than one substitution that will work. For instance, in Example 4 we could have let $u=\sqrt{3x-1}$, then $u^2 = 3x-1$ *or* $x = \frac{1}{3}(u^2+1)$ and

$$2udu = 3dx \ \textit{so} \ \tfrac{2}{3}udu = dx,$$

As a result,

$$\int x\sqrt{3x-1}dx = \int \overbrace{x}^{\frac{1}{3}(u^2+1)} \overbrace{\sqrt{3x-1}}^{u} \tfrac{2}{3}udu = \int \tfrac{1}{3}(u^2+1)u\tfrac{2}{3}udu$$
$$= \tfrac{2}{9}\int (u^2+1)u^2du$$
$$= \tfrac{2}{9}\int (u^4+u^2)du = \tfrac{2}{9}(\tfrac{1}{5}u^5+\tfrac{1}{3}u^3)+c$$
$$= \frac{2}{45}(3x-1)^{5/2} + \frac{2}{27}(3x-1)^{3/2} + c.$$

Even though we used a different substitution, the final answer remains the same. **Example 5** Evaluate $\int xe^{x^2}dx$

Solution: We let

$$u = x^2, \text{ so that } du = 2xdx.$$

Then

$$\int xe^{x^2}dx = \int \overbrace{e^{x^2}}^{e^u} \overbrace{xdx}^{\frac{1}{2}du} = \int e^u \tfrac{1}{2}du$$
$$= \frac{1}{2}e^u + c$$
$$= \frac{1}{2}e^{x^2} + c.$$

Example 6 Evaluate $\int \frac{1}{x}(1+\ln x)^4 dx.$

Solution: We let

$$u = 1+\ln x, \text{ so than } du = \frac{1}{x}dx.$$

Then

$$\int \frac{1}{x}(1+\ln x)^4 dx = \int u^4du = \frac{1}{5}(1+\ln x)^5 + c.$$

Example 7 Evaluate $\int \tan xdx.$

Solution: First write the integral in the following form

$$\int \tan xdx = \int \frac{\sin x}{\cos x}dx.$$

Now let $u = \cos x$, *so that* $du = -\sin xdx.$

Then $\int \tan xdx = \int \frac{\sin x}{\cos x}dx = \int \frac{1}{u}(-du) = -\ln|u| + c = -\ln|\cos x| + c.$

Example 8 Evaluate $\int \sec xdx.$

Solution: We first put the integral in the form

$$\int \sec xdx = \int \sec x\frac{\sec x+\tan x}{\sec x+\tan x}dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$$

If we now let $u = \sec x + \tan x$, *so that* $du = (\sec x\tan x + \sec^2 x)dx$, then

$$\int \sec x dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
$$= \int \frac{1}{u} du = \ln|u| + c$$
$$= \ln|\sec x + \tan x| + c.$$

Example 9 Evaluate $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx.$

Solution: The integral may be written as in the first formula 12 of table 1.0 by letting a=1 and using the substitution

$$u = e^{2x}, \qquad du = 2e^{2x} dx.$$

Then

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2}\sin^{-1} u + c = \frac{1}{2}\sin^{-1} e^{2x} + c.$$

Example 10 Evaluate $\int_0^{\pi/2} \sin x \cos^4 x dx$

Solution: Let $u = \cos x$, *then* $du = -\sin x dx$ hence

$$\int_0^{\pi/2} \sin x \cos^4 x dx = \int u^4 (-du) = -\frac{1}{5}u^5 = -\frac{1}{5}\cos^5 x\Big|_0^{\frac{\pi}{2}}$$
$$= -\frac{1}{5}\cos^5 \frac{\pi}{2} + \frac{1}{5}\cos^5 0 = \frac{1}{5}.$$

Exercise 1.2 Evaluate the following integrals.

1. $\int \sin^2 x dx$
2. $\int \csc x dx$
3. $\int \frac{x}{(x^2+5)^3} dx$
4. $\int \frac{x}{\sqrt[3]{1-2x^2}} dx$
5. $\int x^5 \sqrt{x^2 - 1} dx$
6. $\int_{-1}^{2} \frac{t^2}{\sqrt{t+2}} dt$
7. $\int_1^2 \frac{e^{3/x}}{x^2} dx$
8. $\int \frac{1}{x(\ln x)^2} dx$
9. $\int \frac{3\sin x}{1+2\cos x} dx$
10. $\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}} dx$

5.2 Integration by parts

If we try to evaluate integrals of the type

$$\int xe^x dx, \textit{ and } \int \ln x dx$$

by using the method of substitution we obviously fail. But don’t worry the next formula will enable us to evaluate not only these, but also many other types of integrals.

Integration by parts formula
If $u = f(x)$ *and* $v = g(x)$ and if f' *and* g' are continuous, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \text{ or using } u \text{ and } v$$

$$\int udv = uv - \int vdu.$$

Proof By the product rule,

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

or equivalently, $f(x)g'(x) = [f(x)g(x)]' - g(x)f'(x).$
Integrating both sides of the last equation gives us

$$\int f(x)g'(x)dx = \int [f(x)g(x)]'dx - \int g(x)f'(x)dx.$$

The first integral on the right side equals f(x)g(x)+c. Since another constant of integration is obtained from the second integral, we may omit c in the formula; that is

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \quad (1)$$

Since $dv = g'(x)dx$ and $du = f'(x)dx$, we may write the preceding formula as

$$\int udv = uv - \int vdu.$$

Since applying (1) involves splitting the integrand into two parts, the use of (1) is referred to as **integrating by parts**. A proper choice for *dv* is crucial. We usually let *dv* equal the most complicated part of the integrand that can be readily integrated. The following examples illustrate this method of integration.

Example 1 Evaluate $\int xe^x dx.$

Solution: The integrand xe^x can be split into two parts x and e^x. We let

$$u = x \quad and \quad dv = e^x dx$$

Then $$du = x \quad and \quad v = \int e^x dx = e^x$$

Consequently integration by parts yields

$$\int \overbrace{x}^{u}\overbrace{e^x dx}^{dv} = \overbrace{x}^{u}\overbrace{e^x}^{v} - \int \overbrace{e^x}^{v}\overbrace{dx}^{du} = xe^x - e^x + c.$$

Example 2 Evaluate

a) $\int x\sec^2 x dx$ b) $\int_0^{\pi/3} x\sec^2 x dx$

Solution: a) We let here

$$u = x \quad and \quad dv = \sec^2 x dx$$

then $$du = dx \quad and \quad v = \tan x.$$

Hence integration by parts yields

$$\int x\sec^2 x dx = x\tan x - \int \tan x dx = x\tan x - (-\ln|\cos x|) + c$$

$$= x\tan x + \ln|\cos x| + c.$$

b) The indefinite integral obtained in part (a) is an antiderivative of $x\sec^2 x.$ Using the fundamental theorem of calculus (and dropping the constant of integration c), we obtain

$$\int_0^{\pi/3} x\sec^2 x dx = \left[x\tan x + \ln|\cos x|\right]_0^{\pi/3}$$

$$= \left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - (0 + \ln 1)$$

$$= \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right) - (0 + 0)$$

$$= \frac{\pi}{3}\sqrt{3} - \ln 2.$$

Example 3 Evaluate $\int \ln xdx.$

Solution: Let $u = \ln x$ *and* $dv = dx$

Then $du = \frac{1}{x}dx$ *and* $v = x$

and integrating by parts yields:

$$\int \ln xdx = x\ln x - \int x(\frac{1}{x}dx) = x\ln x - \int dx = x\ln x - x + c.$$

Sometimes it is necessary to usse integration by parts more than once in the same problem. This is illustrated in the next example.

Example 4 Evaluate $\int_0^{\pi/2} x^2 \sin 2xdx.$

Solution: Let

$u = x^2$ *and* $dv = \sin 2xdx$

Then $du = 2xdx$ *and* $v = -\frac{1}{2}\cos 2x$.

Thus using integration by parts we have;

$$\int_0^{\pi/2} x^2 \sin 2xdx = \left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} - \int_0^{\pi/2} 2x\left(-\frac{1}{2}\cos 2x\right)dx$$

$$= \left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} + \int_0^{\pi/2} x\cos 2xdx$$

but then since

$$\left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} = -\frac{1}{2}\left(\frac{\pi}{2}\right)^2\cos 2(\tfrac{\pi}{2}) - 0 = \frac{\pi^2}{8}$$

and

$$\int_0^{\pi/2} x\cos 2x dx = \left[x\frac{\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2x}{2}dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\sin 2(\frac{\pi}{2}) - 0\right] - \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\cos 2(\frac{\pi}{2}) - \cos 0\right] = \frac{1}{4}[-1-1] = -\frac{1}{2}$$

Hence,

$$\int_0^{\pi/2} x^2 \sin 2x dx = \frac{\pi^2}{8} - \frac{1}{2}.$$

The following example illustrates another device for evaluating an integral by means of two applications of the integration by parts formula.

Example 5 Evaluate $\int e^x \cos x dx.$

Solution: We could either let $dv = \cos x dx$ or let $dv = e^x dx,$ since each of these expression is readily integrable. Let us choose

$$u = e^x \quad and \quad dv = \cos x dx$$

so that $\qquad du = e^x dx \quad and \quad v = \sin x$

Then by integrating by parts we have;

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \qquad (1)$$

We next apply integration by parts to the integral of the right side of equation (1). Since we chose a trigonometric form for *dv* in the first integration by parts, we shall also choose a trigonometric form for the second. Letting

$$u = e^x \quad and \quad dv = \sin x dx \quad so\ that$$

$$du = e^x dx \quad and \quad v = -\cos x$$

integrating by parts, we have

$$\int e^x \sin x dx = e^x(-\cos x) - \int(-\cos x)e^x dx$$

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx. \qquad (2)$$

If we now use equation (2) to substitute on the right side of equation (1), we obtain

$$\int e^x \cos x dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x dx\right]$$

or $\qquad \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$.

Adding $\int e^x \cos x dx$ to both sides of the last equation gives us

$$2\int e^x \cos x dx = e^x(\sin x + \cos x).$$

Finally, dividing both sides by 2 and adding the constant of integration yiels

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + c.$$

We could have evaluated the given integral by using $dv = e^x dx$ for both the first and second applications of the integration by parts formula.

In conclusion we remark that integration by parts is effective with integrals involving a polynomial and either an exponential, a logarithmic, or a trigonometric function. More specifically, integration by parts is especially well adapted to integrals of the form

$$\int (polynomial)\sin axdx, \qquad \int (polynomial)\cos axdx,$$

$$\int (polynomial)e^{ax}dx, \qquad \int (polynomial)\ln xdx.$$

In all except $\int (polynomial)\ln xdx$, the most effective choice of u is the polynomial, since the derivatives of a polynomial are simpler than the polynomial itself, while the choice $u = \ln x$ is effective for $\int (polynomial)\ln xdx$.

Example 6 Evaluate $\int \sin^{-1} xdx.$

Solution: Let

$u = \sin^{-1} x$ *and* $dv = dx$ so that $du = \dfrac{1}{\sqrt{1-x^2}}dx$ *and* $v = x$.

Then

$$\int \sin^{-1} xdx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

Now we use substitution to solve the integral to the right. That is let

$$w = \sqrt{1-x^2} \;\; or \;\; w^2 = 1-x^2 \;\; so\ that \;\; 2wdw = -2xdx$$

we then have

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{wdw}{w} = -\int dw = -w + c = -\sqrt{1-x^2} + c$$

Consequently

$$\int \sin^{-1} xdx = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

Integration by parts may sometimes be employed to obtain **reduction formulas** for integrals. We now find reduction formulas of $\int \sin^n xdx$ *and* $\int \cos^n xdx$ with the help of integration by parts.

Example 7 Find a reduction formula for $\int \sin^n xdx.$

Solution: First write $\int \sin^n xdx = \int \sin^{n-1} x\sin xdx$ and let

$u = \sin^{n-1} x$ *and* $dv = \sin xdx$ so that

$$du = (n-1)\sin^{n-2} x\cos xdx \;\; and \;\; v = -\cos x$$

then using integration by parts we have:

$$\int \sin^n xdx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 xdx$$

since $\cos^2 x = 1 - \sin^2 x,$ we may write

$$\int \sin^n xdx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx.$$

Consequently,

$$\int \sin^n x dx + (n-1)\int \sin^n x dx. = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx.$$

The left side of the last equation reduces to $n\int \sin^n x dx$. Dividing both sides by n, we obtain

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

In a similar fashion we can show the reduction formula for $\int \cos^n x dx$ is given by:

$$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Example 8 Evaluate $\int \sin^5 x dx.$

Solution: Using the reduction formula for sine with n = 5 gives us

$$\int \sin^5 x dx = -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x dx$$

A second application of the reduction formula, to $\int \sin^5 x dx$, yields

$$\begin{aligned}\int \sin^3 x dx &= -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x dx \\ &= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C_1\end{aligned}$$

Consequently

$$\begin{aligned}\int \sin^5 x dx &= -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\left(-\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C_1\right) \\ &= -\frac{1}{5}\cos x \sin^4 x - \frac{4}{15}\cos x \sin^2 x - \frac{8}{15}\cos x + C.\end{aligned}$$

Exercise 1.2

Evaluate the integral

1. $\int xe^{-x} dx$
2. $\int x \ln x dx$
3. $\int \sec^3 x dx$
4. $\int x2^x dx$
5. $\int x \tan x \sec x dx$
6. $\int_0^{\pi/2} 2t \sin 2t dt$
7. $\int (x+1)^{10}(x+2)dx$
8. $\int \sin(\ln x)dx$ (Hint: Let $u = \sin(\ln x)$)
9. $\int \tan^{-1} x dx$
10. $\int \cos\sqrt{x} dx$

Evaluate the integral with the help of the reduction formulas

11. $\int_0^{\pi/2} \cos^3 \frac{x}{2} dx$
12. $\int \cos^5 x dx$

5.3 Integration by Partial Fractions

An expression for rational function is called a ***proper fraction*** if the degree of the numerator is strictly less than the degree of the denominator; otherwise it is called an ***improper fraction***. In case of improper fraction we actually divide the numerator by the denominator and the improper fraction is expressed in terms of a polynomial and a proper fraction. For example,

$$\frac{2x+1}{x-3}=2+\frac{7}{x-3} \quad \text{and} \quad \frac{4x^3-3x^2+2x-1}{x^2+9}=4x-3-\frac{34x-26}{x^2+9}$$

Let us consider a proper fraction $\frac{P(x)}{Q(x)}$ where P and Q are polynomials in *x*, then it can be proved that

$$\frac{P(x)}{Q(x)}=F_1+F_2+\cdots+F_r$$

Such that each term F_k of the sum has one of the forms

$$\frac{A}{(ax+b)^n} \quad or \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

for real numbers A and B and a nonnegative integer n, where ax^2+bx+c is **irreducible** in the sense that this quadratic polynomial has no real zeros (that is, $b^2-4ac<0$). In this case, ax^2+bx+c cannot be expressed as a product of two first-degree polynomials with real coefficients.

The sum $F_1+F_2+\cdots+F_r$ is the partial **fraction decomposition** of P(*x*)/Q(*x*), and each F_k is a **partial fraction**. We state guidelines for obtaining this decomposition.

Guidelines for partial fraction decompositions of P(*x*)/Q(*x*)

1. If the degree of *P*(*x*) is not lower than the degree of Q(*x*), use long division to obtain the proper form.
2. Express Q(x) as a product of linear factors a*x* + b or irreducible quadratic factors ax^2+bx+c, and collect repeated factors so that Q(*x*) is a product of different factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ for a nonnegative integer n.
3. Apply the following rules.

   Rule a For each factor $(ax+b)^n$ with $n\geq 1$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$

where each numerator A_k is a real number.

Rule b For each factor $(ax^2+bx+c)^n$ with $n\geq 1$, and with ax^2+bx+c irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n},$$

where each A_k and B_k is a real number.

Example 1 Evaluate $\int\frac{4x^2+13x-9}{x^3+2x^2-3x}dx.$

Solution: We may factor the denominator of the integrand as follows:

$$x^3+2x^2-3x = x(x^2+2x+3) = x(x+3)(x-1)$$

Each factor has the form stated in Rule (a) of the guideline, with n = 1. Therefore the partial fraction decomposition has the form

$$\frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}.$$

Multiplying by the LCM of the denominators gives us

$$4x^2+13x-9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3). \qquad (*)$$

In a case such as this, in which the factors are linear and nonrepeated, the values of A, B and C can be found by substituting values for *x* that make the various factors zero. If we let x = 0 in ($*$), then

$$-9 = -3A, \quad or \quad A = 3.$$

Letting x = 1 in ($*$) gives us

$$8 = 4C, \quad or \quad C = 2.$$

Finally, if x = -3 in ($*$), we have

$$-12 = 12B, \quad or \quad B = -1.$$

The partial fraction decomposition is, therefore,

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}.$$

Integrating and letting C denote the sum of the constants of integration we have

$$\begin{aligned}\int \frac{4x^2+13x-9}{x(x+3)(x-1)}dx &= \int \frac{3}{x}dx + \int \frac{-1}{x+3}dx + \int \frac{2}{x-1}dx.\\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C\\ &= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + C\\ &= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + C.\end{aligned}$$

Another technique for finding A, B, and C is to expand the right-hand side of ($*$) and collect like powers of x as follows:

$$4x^2+13x-9 = (A+B+C)x^2 + (2A-B+3C)x - 3A$$

We now use the fact that if two polynomials are equal, then coefficients of like powers of *x* are the same. It is convenient to arrange our work in the following way, which we call **comparing coefficients of *x***.

Coefficients of x^2:	$A + B + C = 4$
Coefficients of x:	$2A - B + 3C = 13$
Constant terms:	$-3A = -9$

We may show the solution of this system of equations is A = 3, B = -1, and C = 2.

Example 2 Evaluate $\int \frac{13-7x}{(x+2)(x-1)^3}dx$

Solution: By Rule (a) of the Guidelines the partial fraction of the integrand has the form

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+2}$$

Multiplying both sides by $(x+2)(x-1)^3$ gives us

$$13-7x=A(x-1)^2(x+2)+B(x-1)(x+2)+C(x+2)+D(x-1)^3 \qquad (*)$$

Two of the unknown constants may be determined easily as follows.
Let x = 1 in ($*$) the we obtain $13-7=C(1+2)$ or C=2.

Similarly, letting x = -2 in ($*$) yields $13+14=D(-2-1)^3$ *or* $D=-1$.

The remaining constants may be found by comparing coefficients. So comparing the coefficients of x^3 on both sides of ($*$), gives

0=A+D or A = - D = 1.

And comparing the constant terms on both sides of ($*$), gives

$$13=2A-2B+2C-D \quad or \quad B=\frac{1}{2}(2+4+1-13)=-3.$$

Therefore

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{1}{x-1}+\frac{-3}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{-1}{x+2}.$$

Thus

$$\int\frac{13-7x}{(x+2)(x-1)^3}dx=\int\frac{1}{x-1}dx-\int\frac{3}{(x-1)^2}dx+\int\frac{2}{(x-1)^3}dx-\int\frac{1}{x+2}dx$$

$$=\ln|x-1|+\frac{3}{x-1}-\frac{1}{(x-1)^2}-\ln|x+2|+C$$

$$=\ln\left|\frac{x-1}{x+2}\right|+\frac{3}{x-1}-\frac{1}{(x-1)^2}+C.$$

Example 3 Evaluate $\int\frac{x^2+2x+7}{x^3+x^2-2}dx.$

Solution: The denominator of the integrand may be factored as follows:

$$x^3+x^2-2=(x-1)(x^2+2x+2)$$

Applying Rule (b) of the Guidelines to the irreducible quadratic factor x^2+2x+2 we have

$$\frac{x^2+2x+7}{x^3+x^2-2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+2}$$

This leads to

$$x^2+2x+7=A(x^2+2x+2)+(Bx+C)(x-1) \qquad (*)$$

As in previous examples, substituting $x=1$ in ($*$) gives us

$10=A(5)$ or $A=2$

The remaining constants may be found by combining like powers of x:

$$x^2+2x+7=(2+B)x^2+(4+C-B)x+(4-C) \qquad (**)$$

and comparing coefficients in ($**$).

Coefficients of x^2: $1=2+B$ or $B=-1$

Constant terms: 7=4-C or $C=-3$

Thus the partial fraction decomposition of the integrand is

$$\frac{x^2+2x+7}{x^3+x^2-2}=\frac{2}{x-1}+\frac{-x-3}{x^2+2x+2}.$$

Consequently

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx-\int\frac{x+3}{x^2+2x+2}dx$$

To evaluate the right-hand integral, we first complete the square in the denominator to obtain $x^2+2x+2=(x+1)^2+1$

and substitute $u=x+1$, so that du = dx and x +3 = u + 2

Therefore

$$\int\frac{x+3}{x^2+2x+2}dx=\int\frac{x+3}{(x+1)^2+1}dx=\int\frac{u+2}{u^2+1}du$$

$$=\int\frac{u}{u^2+1}du+2\int\frac{1}{u^2+1}du$$

$$=\frac{1}{2}\int\frac{2u}{u^2+1}du+2\int\frac{1}{u^2+1}du$$

$$=\frac{1}{2}\ln(u^2+1)+2\arctan u+C$$

$$=\frac{1}{2}\ln((x+1)^2+1)+2\arctan(x+1)+C.$$

Hence

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx-\int\frac{x+3}{x^2+2x+2}dx$$

$$=2\ln|x-1|-\tfrac{1}{2}\ln((x+1)^2+1)-2\arctan(x+1)+C$$

Example 4 Evaluate $\int\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx.$

Solution: Applying Rule b) of the Guidelines, with n = 2, yields

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}.$$

Multiplying by both sides of the equation by $(x^2+1)^2$ gives

$$5x^3-3x^2+7x-3=(Ax+B)(x^2+1)+Cx+D$$

$$5x^3-3x^2+7x-3=Ax^3+Bx^2+(A+C)x+(B+D)$$

We next compare coefficients as follows:

coefficients of x^3: 5 = A
coefficients of x^2: -3 = B
coefficients of x : 7 = A + C or C = 2
constant terms : -3 = B + D or D = 0

Therefore

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}=\frac{5x-3}{x^2+1}+\frac{2x}{(x^2+1)^2}$$

so that

$$\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx = \int \frac{5x}{x^2+1}dx - \int \frac{3}{x^2+1}dx + \int \frac{2x}{(x^2+1)^2}dx$$
$$= \frac{5}{2}\ln(x^2+1) - 3\arctan(x^2+1) - \frac{1}{x^2+1} + C.$$

Example 5 Evaluate $\int \frac{dx}{\sin x(2+\cos^2 x)}$.

Solution: Since
$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin x^2(2+\cos^2 x)}$$
substituting

$u = \cos x$ *and* $du = -\sin x dx$, we get
$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin x^2(2+\cos^2 x)} = \int \frac{-du}{(1-u^2)(2+u^2)} = \int \frac{du}{(u^2-1)(u^2+2)}$$
But then the partial fraction representation for the integrand of the last integral has the form
$$\frac{1}{(u^2-1)(u^2+2)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+2}$$
Then by similar procedure as the above examples we have
$$1 = A(u+1)(u^2+2) + B(u-1)(u^2+2) + (Cu+D)(u-1)(u+1) \qquad (*)$$
Then putting u=1 gives us 1=6A or A=1/6
Putting u=-1 gives us 1=-6B or B=-1/6
We now compare coefficients to find the remaining two constants

Coefficients of x^3: 0=A+B+C or C=0

Constant terms: 1=2A- 2B-D or D=-1/3 Therefore
$$\frac{1}{(u^2-1)(u^2+2)} = \frac{1/6}{u-1} + \frac{-1/6}{u+1} + \frac{-1/3}{u^2+2}$$
so that
$$\int \frac{du}{(u^2-1)(u^2+2)} = \frac{1}{6}\int \frac{1}{u-1}du - \frac{1}{6}\int \frac{1}{u+1}du + \frac{1}{3}\int \frac{1}{u^2+2}du$$
$$= \frac{1}{6}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C$$
Consequently resubstituting cos*x* for *u* we have
$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \frac{1}{6}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{\cos x}{\sqrt{2}} + C.$$

Exercise 1.3

1. $\int \frac{x^2}{x^2-1}dx$
2. $\int \frac{2x^2-12x+4}{x^3-4x^2}dx$
3. $\int_{-1}^{0} \frac{x^2+x+1}{x^2+1}dx$
4. $\int \frac{-x^3+x^2+x+3}{(x+1)(x^2+1)^2}dx$

5. $\int \frac{x^2 - 1}{x^3 + 3x + 4} dx$

6. $\int \frac{\sqrt{x} + 1}{x + 1} dx$; ($H$ int : *Substitute* $u = \sqrt{x}$)

7. $\int \frac{\sin^2 x \cos x}{\sin^2 x + 1} dx$

8. $\int_0^{\pi/4} \tan^3 x dx$; ($H$ int : *Substitute* $u = \tan x$)

9. $\int \frac{e^x}{1 - e^{3x}} dx$

10. $\int \frac{dx}{1 + 3e^x + 2e^{2x}}$

5.4 Trigonometric Integrals

Integrals such as

$$\int \sin^5 x \cos^3 x dx, \qquad \int \tan^2 x \sec^3 x dx, \qquad and \quad \int \sin 3x \cos 4x dx$$

are called **trigonometric integrals** because their integrands are combinations of trigonometric functions. This section is devoted to trigonometric integrals especially those in which the integrands are composed of the basic trigonometric functions.

Guidelines for evaluating integrals of the form $\int \sin^m x \cos^n x dx$

1. **If m is an odd integer**: Write the integrals as $\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x \sin x dx$ and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x.$ Make the substitution

$$u = \cos x, \quad du = -\sin x dx$$

and evaluate the resulting integral.

2. **If n is an odd integer**: write the integral as

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x \cos x dx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1 - \sin^2 x.$ Make the substitution

$$u = \sin x, \qquad du = \cos x dx$$

and evaluate the resulting integral.

3. **If m and n are even**: Use half-angle formulas for $\sin^2 x = \frac{1 - \cos 2x}{2}$ *and* $\cos^2 x = \frac{1 + \cos 2x}{2}$ and the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

to reduce the exponents by one-half.

Example 1 Evaluate $\int \sin^3 x \cos^2 x dx.$

Solution: By guideline 1

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^2 x \sin x dx.$$

If we let $u = \cos x,$ then $du = -\sin x dx,$ and the integral may be written

$$\int \sin^3 x\cos^2 xdx = \int (1-u^2)u^2(-du) = \int (u^4 - u^2)du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C$$
$$= \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C.$$

Example 2 Evaluate $\int \sin^2 x\cos^4 xdx.$

Solution: By guideline 3 we have

$$\int \sin^2 x\cos^4 xdx = \int (\sin^2 x\cos^2 x)\cos^2 xdx$$
$$= \int (\sin x\cos x)^2 \cos^2 xdx$$
$$= \int \left(\frac{1}{2}\sin 2x\right)^2 \left(\frac{1+\cos 2x}{2}\right)dx$$
$$= \frac{1}{8}\int \sin^2 2xdx + \frac{1}{8}\int \sin^2 2x\cos 2xdx$$

Putting $\sin^2 2x = \dfrac{1-\cos 4x}{2}$ and $u = \sin 2x$ *so that* $du = 2\cos 2xdx$ in the first and second integrals of the right of the last equation we get:

$$= \frac{1}{8}\int \frac{1-\cos 4x}{2}dx + \frac{1}{8}\int \frac{1}{2}u^2du$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}u^3 + C$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

An alternative way to evaluate $\int \sin^m x\cos^n xdx$ when m and n are even is to use the identity $\sin^2 x + \cos^2 x = 1,$ but this time we transform the integral into integrals of the form $\int \sin^k xdx$ or of the form $\int \cos^k xdx$, which can be evaluated by the reduction formulas.

Guidelines for evaluating integrals of the form $\int \tan^m x\sec^n xdx$

1. **If m is an odd integer**: Write the integrals as
$\int \tan^m x\sec^n xdx = \int \tan^{m-1} x\sec^{n-1} x\sec x\tan xdx$ and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 x = \sec^2 x - 1.$ Make the substitution
$$u = \sec x, \quad du = \sec x\tan xdx$$
and evaluate the resulting integral.

2. **If n is an even integer**: write the integral as
$$\int \tan^m x\sec^n xdx = \int \tan^m x\sec^{n-2} x\sec^2 xdx$$
and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x.$ Make the substitution
$$u = \tan x, \qquad du = \sec^2 xdx$$

and evaluate the resulting integral.

3. **If m is even and n is odd**: Reduce to powers of $\sec x$ alone by using the identity $\tan^2 x = \sec^2 x - 1$.

Example 3 Evaluate $\int \tan^3 x \sec^5 x dx.$

Solution: By guideline 1 above

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x(\sec x \tan x)dx$$
$$= \int (\sec^2 x - 1)\sec^4 x(\sec x \tan x)dx.$$

Substituting $u = \sec x$ **and** $du = \sec x \tan x dx,$ **we obtain**

$$\int \tan^3 x \sec^5 x dx = \int (u^2 - 1)u^4 du$$
$$= \int (u^6 - u^4)du.$$
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C$$

Example 4 Evaluate $\int \tan^3 x \sec^4 x dx.$

Solution: By guideline 2 above

$$\int \tan^3 x \sec^4 x dx = \int \tan^3 x \sec^2 x \sec^2 x dx$$
$$= \int \tan^3 x(1 + \tan^2 x)\sec^2 x dx$$

If we let $u = \tan x$, *then* $du = \sec^2 x dx$, and

$$\int \tan^3 x \sec^4 x dx = \int u^3(1 + u^2)du$$
$$= \int (u^5 + u^3)du$$
$$= \tfrac{1}{6}u^6 + \tfrac{1}{4}u^4 + C$$
$$= \tfrac{1}{6}\tan^6 x + \tfrac{1}{4}\tan^4 x + C.$$

Integrals of the form $\int \cot^m x \csc^n x dx$ may be evaluated in similar fashion.

Finally, the evaluation of integrals of the form $\int \sin ax \cos bx dx$ depends on the trigonometric identity

$$\sin x \cos y = \frac{1}{2}\sin(x - y) + \frac{1}{2}\sin(x + y)$$

With the appropriate replacements, this identity becomes

$$\sin ax \cos bx = \frac{1}{2}\sin(a - b)x + \frac{1}{2}\sin(a + b)x \qquad (*)$$

Notice that $\tfrac{1}{2}\sin(a - b)x$ *and* $\tfrac{1}{2}\sin(a + b)x$ are easy to integrate by substitution.

Example 5 Evaluate $\int \sin 4x \cos 2x dx.$

Solution: Using ($*$) with $a = 4$ and b = 2, we find that

$$\int \sin 4x \cos 2x dx = \int \left(\frac{1}{2} \sin 2x + \frac{1}{2} \sin 6x \right) dx$$
$$= -\frac{1}{4} \cos 2x - \frac{1}{12} \sin 6x + C.$$

Note that integrals of the form

$$\int \sin ax \sin bx dx \qquad and \qquad \int \cos ax \cos bx dx$$

can be found by similar techniques.

Exercise 1.4 Evaluate the following integrals.

1. $\int \sin^3 x \cos^4 x dx$
2. $\int_0^{\pi/2} \sin^2 x \cos^5 x dx$
3. $\int \sqrt{\sin x} \cos^3 x dx$
4. $\int (\tan x + \cot x)^2 dx$
5. $\int \tan^3 x \csc^4 x dx$
6. $\int \cot^3 x \csc^3 x dx$
7. $\int \sin 5x \sin 3x dx$
8. $\int_0^{\pi/4} \cos x \cos 5x dx$
9. $\int_0^{\pi/3} \tan x \sec^{3/2} x dx$
10. $\int \tan^6 x dx$

5.5 Trigonometric Substitutions

Observe that the trigonometric substitution $x = a \sin \theta$ simplifies the expression $\sqrt{a^2 - x^2}$, with a>0, into a trigonometric expression without radical i.e

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a\sqrt{1 - \sin^2 \theta} = a \cos \theta .$$

We can use a similar procedure for $\sqrt{a^2 + x^2}$,and $\sqrt{x^2 - a^2}$. This technique is useful for eliminating radicals from these types of integrands. The substitutions are listed in the table 1.1.

When making a trigonometric substitution we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution $x = a \sin \theta$, we have $-\pi/2 \le \theta \le \pi/2,$ In this case, $\cos \theta \ge 0$.

Trigonometric Substitutions

Expressions in integrand	Trigonometric substitution	Interval(s)
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$-\pi/2 \le \theta \le \pi/2,$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$-\pi/2 < \theta < \pi/2,$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$0 \le \theta < \pi/2,\ or\ \pi \le \theta < \frac{3\pi}{2}$

Table 1.1

Example 1 Evaluate $\int \frac{1}{x^2\sqrt{16 - x^2}} dx.$

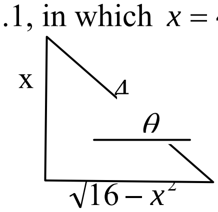
Solution: Since $\sqrt{16 - x^2} = \sqrt{4^2 - x^2}$, we substitute

$x = 4\sin\theta,$ so that $dx = 4\cos\theta\, d\theta$ for $-\pi/2 < \theta < \pi/2.$

Then

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \int \frac{1}{16\sin^2\theta 4\sqrt{1-\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta$$
$$= -\frac{1}{16}\cot\theta + C.$$

In order to write the answer in terms of the original variable *x*, we draw the triangle as fig1.1, in which $x = 4\sin\theta.$



$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Fig 1.1

Thus

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\cot\theta + C = -\frac{\sqrt{16-x^2}}{16x} + C.$$

Example 2 Evaluate $\int_{-5/2}^{5/2} \sqrt{25-4x^2}\,dx$

Solution: Because $\sqrt{25-4x^2} = \sqrt{5^2-(2x)^2}$, we are led to substitute

$$2x = 5\sin\theta, \text{ so that } x = \frac{5}{2}\sin\theta, \text{ and thus } dx = \frac{5}{2}\cos\theta d\theta$$

For the limits of integration we notice that

if $x = -\frac{5}{2}$ *then* $\theta = -\frac{\pi}{2},$ *and if* $x = \frac{5}{2}$ *then* $\theta = \frac{\pi}{2}.$

Therefore

$$\int_{-5/2}^{5/2}\sqrt{25-4x^2}\,dx = \int_{-5/2}^{5/2}\sqrt{5^2-(2x)^2}\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\sqrt{5^2-5^2\sin^2\theta}\left(\frac{5}{2}\cos\theta\right)d\theta$$

$$= \frac{25}{2}\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\theta}(\cos\theta)d\theta$$

$$= \frac{25}{2}\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta$$

$$= \frac{25}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{25}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right) = \frac{25}{4}\pi$$

Example 3 Evaluate $\int \frac{1}{x^2\sqrt{16+x^2}}dx.$

Solution: The denominator of the integrand has an expression of the form $\sqrt{a^2+x^2}$ with $a=4.$ Hence, using table 1.1, we make the substitution

$$x = 4\tan\theta,\ dx = 4\sec^2\theta d\theta.$$

Consequently

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sqrt{1+\tan^2\theta} = 4\sqrt{\sec^2\theta} = 4\sec\theta$$

and

$$\int \frac{1}{x^2\sqrt{16+x^2}}dx = \int \frac{1}{16\tan^2\theta(4\sec\theta)}4\sec^2\theta d\theta$$

$$= \frac{1}{16}\int\frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{16}\int\frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= -\frac{1}{16\sin\theta}$$

To give the answer in terms of *x*, we use the triangle in Fig 1.2, with $x = 4\tan\theta$. We then find that

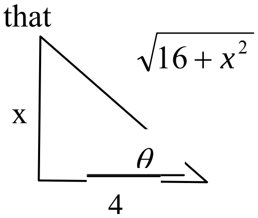


Fig 1.1

$$\sin\theta = \frac{x}{\sqrt{16+x^2}} \quad \textit{and}$$

$$\int\frac{1}{x^2\sqrt{16+x^2}}dx = -\frac{1}{16\sin\theta} = \frac{\sqrt{16+x^2}}{16x}+C.$$

Example 4 Evaluate $\int_{-6}^{-3}\frac{\sqrt{x^2-9}}{x}dx.$

Solution: The domain of the integrand consists of $(-\infty,-3]$ and $[3,\infty)$, but since the interval over which we must integrate is [-6,-3], we seek an antiderivative whose domain is contained in $(-\infty,-3]$. Since $\sqrt{x^2-9}=\sqrt{x^2-3^2}$, we let

$$x = 3\sec\theta,\ \textit{so that}\ dx = 3\sec\theta\tan\theta d\theta$$

and notice that $\sqrt{x^2-9}=\sqrt{9\sec^2\theta-9}=3\tan\theta.$ For the limits of integration we observe that

$$\textit{if}\ \ x=-6\ \textit{then}\ \theta=\sec^{-1}(-2)=\frac{4\pi}{3},\ \ \textit{and}\ \ \textit{if}\ \ x=-3\ \ \textit{then}\ \ \theta=\pi.$$

Therefore

$$\int_{-6}^{-3}\frac{\sqrt{x^2-9}}{x}dx=\int_{4\pi/3}^{\pi}\frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}(3\sec\theta\tan\theta)d\theta$$

$$=\int_{4\pi/3}^{\pi}\frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)d\theta=3\int_{4\pi/3}^{\pi}\tan^2\theta d\theta$$

$$=3\int_{4\pi/3}^{\pi}(\sec^2\theta-1)d\theta=3(\tan\theta-\theta)\Big|_{4\pi/3}^{\pi}$$

$$=\pi-3\sqrt{3}.$$

Integrals containing $\sqrt{bx^2+cx+d}$

By completing the square in bx^2+cx+d we can express $\sqrt{bx^2+cx+d}$ in terms of $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, *or* $\sqrt{x^2-a^2}$ for suitable $a>0$. Then a trigonometric substitution eliminates the square root as before.

Example 5 Evaluate $\displaystyle\int\frac{1}{\sqrt{x^2+8x+25}}\,dx.$

Solution: We complete the square for the quadratic expression as follows:

$$x^2+8x+25=(x^2+8x\qquad)+25$$

$$=(x^2+8x+16)+25-16$$

$$=(x+4)^2+9$$

Thus,

$$\int\frac{1}{\sqrt{x^2+8x+25}}\,dx=\int\frac{1}{\sqrt{(x+4)^2+9}}dx.$$

If we make the trigonometric substitution

$$x+4=3\tan\theta,\quad dx=3\sec^2\theta\,d\theta$$

then

$$\sqrt{(x+4)^2+9}=\sqrt{9\tan^2\theta+9}=3\sqrt{\tan^2\theta+1}=3\sec\theta$$

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \int \frac{1}{3\sec\theta}3\sec^2\theta\, d\theta$$

and $$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C.$$

Using our formulas for $\tan\theta$ *and* $\sec\theta$, we conclude that

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C.$$

Exercise 1.4 In Exercises1-10 evaluate the integral.

1. $\int \frac{1}{x^2\sqrt{9-x^2}}dx$
2. $\int \frac{(1-x^2)^{3/2}}{x^6}dx.$
3. $\int_0^1 \frac{1}{(3x^2+2)^{5/2}}dx$
4. $\int_1^{\sqrt{2}} \frac{1}{\sqrt{2x^2-1}}dx$
5. $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}dx$
6. $\int_{\sqrt{2}}^{2} arc\sec dx$
7. $\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}dx$
8. $\int \frac{1}{\sqrt{4x-x^2}}dx$
9. $\int \frac{1}{x^2-2x+2}dx$
10. $\int \frac{x+5}{9x^2+6x+17}dx$

5.6 Improper integrals

The definite integral $\int_a^b f(x)dx$ has meaning only when *f* is continuous on [a,b] consequently bounded on [a,b]. We say *f* is bounded on an interval **I** if there is a constant M such that $|f(x)| \le M$ for all x in **I**. In this section, we shall extend the definition of the definite integral when either the integrand or the interval of integration is unbounded. Such integrals are called **improper integrals**.

1.6.1 Integrals Over Unbounded Intervals

If *f* is continuous on $[a,\infty)$, then the improper integral $\int_a^\infty f(x)dx$ **converges** if

$\lim_{t\to\infty}\int_a^t f(x)dx$ exists. In that case

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \qquad (1)$$

If the limit does not exist, the improper integral **diverges**.
Again if *f* is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_{t}^{a} f(x)dx \qquad (2)$$

provided the limit exists.

Example 1 Determine whether the integral converges or diverges, and if it converges, find its value.

(a) $\int_{0}^{\infty}\frac{1}{(x+1)^2}dx$ (b) $\int_{0}^{\infty}\frac{1}{x+1}dx$

Solution: (a) Following the discussion above and equation (1) we have

$$\int_{0}^{\infty}\frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\int_{0}^{t}\frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\left[\frac{-1}{x+1}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{t+1}+\frac{1}{0+1}\right] = 0+1 = 1$$

Thus, the improper integral converges and has the value 1.

(b) Using equation (2)

$$\int_{0}^{\infty}\frac{1}{x+1}dx = \lim_{t\to\infty}\int_{0}^{t}\frac{1}{x+1}dx$$

$$= \lim_{t\to\infty}\left[\ln(x+1)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln(t+1)-\ln(0+1)\right]$$

$$= \lim_{t\to\infty}\left[\ln(t-1)\right] = \infty.$$

Since the limit does not exist, the improper integral diverges.

Example 2 Determine whether the integral $\int_{-\infty}^{1} e^x dx$ converges or diverges, and if it converges, find its value.

Solution: As in Example 1;

$$\int_{-\infty}^{1} e^x dx = \lim_{t\to-\infty}\int_{t}^{1} e^x dx = \lim_{t\to-\infty}\left[e^x\right]_t^1$$

$$= \lim_{t\to-\infty}\left[e^1 - e^t\right] = e.$$

Thus, the integral converges and has the value e.

Finally, for integrals over the range $(-\infty,\infty)$, we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \qquad (3)$$

provided both of the improper integrals on the right converge.

If either of the integrals on the right in (3) diverges, then $\int_{-\infty}^{\infty} f(x)dx$ is said to **diverge**. It can be shown that (3) does not depend on the choice of the real number a.

Example 3 Determine whether $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ diverges.

Solution: Using (3), with $a = 0$, we have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx.$$

Next, applying (2)

$$\int_{-\infty}^{0} \frac{1}{1+x^2}dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}dx = \lim_{t\to-\infty}\left[\arctan x\right]_t^0$$

$$= \lim_{t\to-\infty}\left[\arctan 0 - \arctan t\right] = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}.$$

Similarly, we may show, by using (1) that

$$\int_{0}^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

Consequently the given improper integral converges and has the value $\frac{\pi}{2} + \frac{\pi}{2} = \pi.$

1.6.2 Integrals with Unbounded Integrands

We now consider a function *f* that is continuous at every point in $(a,b]$ and unbounded near a. By assumption *f* is continuous on the interval $[t,b]$ for any *t* in (a,b), so that $\int_{t}^{b} f(x)dx$ is defined for such *t*. If the one-sided limit

$$\lim_{t\to a^+}\int_{t}^{b} f(x)dx$$

exists, then we define $\int_{a}^{b} f(x)dx$ to be the limit. This idea leads us to the following definitions:

(i) If *f* is continuous on $[a,b)$ and discontinuous at b, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to b^-}\int_{a}^{t} f(x)dx, \qquad (4)$$ provided the limit exists.

(ii) If *f* is continuous on $(a,b]$ and discontinuous at a, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to a^+}\int_{t}^{b} f(x)dx, \qquad (5)$$

provided the limit exists.

As in the preceding section, the integrals defined in (4) and (5) are referred to as improper integrals and they converge if the limits exist. The limits are called the values of the improper integrals. If the limits do not exist, the improper integrals diverge.

Example 4 Evaluate $\int_{1}^{2} \frac{1}{\sqrt{2-x}}dx.$

Solution: Since the integrand has an infinite discontinuity at $x = 2$, we apply (4) and have

$$\int_1^2 \frac{1}{\sqrt{2-x}}dx = \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt{2-x}}dx$$
$$= \lim_{t\to 2^-}[-2\sqrt{2-x}]_1^t$$
$$= \lim_{t\to 2^-}[-2\sqrt{2-t} - (-2\sqrt{2-1}] = 2.$$

Example 5 Determine whether the improper integral $\int_0^1 \frac{1}{x}dx$ converges or diverges.

Solution: The integrand is unbounded near 0. Applying (5) gives us

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+}[\ln x]_t^1 = \lim_{t\to 0^+}[\ln 1 - \ln t] = \infty.$$

Consequently the improper integral diverges, since the limit does not exist.

We give the definition of another improper integral as follows.

If f has a discontinuity at a number c in the open interval (a,b) but continuous elsewhere on $[a,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad (6)$$

provided both of the improper integrals on the right converge. If both converge, then the value of the improper integral $\int_a^b f(x)dx$ is the sum of the two values.

Example 6 Determine whether the improper integral $\int_0^4 \frac{1}{(x-3)^2}dx$ converges or diverges.

Solution: The integrand is undefined at $x = 3$. Since this number is in the interval (0,4), we use (6), with c = 3:

$$\int_0^4 \frac{1}{(x-3)^2}dx = \int_0^3 \frac{1}{(x-3)^2}dx + \int_3^4 \frac{1}{(x-3)^2}dx$$

For the integral on the left to converge, both integrals on the right must converge. However, since

$$\int_0^3 \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}dx$$
$$= \lim_{t\to 3^-}\left[\frac{-1}{x-3}\right]_0^t$$
$$= \lim_{t\to 3^-}\left[\frac{-1}{t-3} - \frac{1}{3}\right] = \infty$$

the given improper integral diverges.

The other kind of improper integral is found if f is continuous in $(a,b)$and is unbounded near both a and b. We say that $\int_a^b f(x)dx$ **converges** if for some point c in (a,b) both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. Otherwise we say that the integral is divergent.

Example 7 Determine whether $\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}dx$ diverges.

Solution: The integrand is unbounded near both the endpoints 0 and 1 and is continuous on (0,1). Consequently the integral is of the type under consideration. If we let $c=\frac{3}{4}$,then we need to analyze the convergence of

$$\int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx \quad \text{and} \quad \int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx$$

For $0<t<\frac{3}{4}$ we have

$$\int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx = \lim_{t\to 0^+}\int_t^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx = \lim_{t\to 0^+}[2\sqrt{x-x^2}\,]\Big|_t^{3/4}$$

$$= \lim_{t\to 0^+}[2(\sqrt{\tfrac{3}{16}}-\sqrt{t-t^2})] = \tfrac{\sqrt{3}}{2}$$

A similar computation shows that the second improper integral also converges and that

$$\int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx = -\frac{\sqrt{3}}{2}.$$

Therefore the original integral converges, and

$$\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}dx = \int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx + \int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx = \frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=0.$$

Exercise 1.6

Determine whether the integral converges or diverges, and if it converges, find its value.

1. $\int_0^\infty \frac{x}{1+x^2}dx$
2. $\int_{-\infty}^0 \frac{1}{(x+3)^2}dx$
3. $\int_1^\infty \frac{1}{\sqrt{x^2-1}}dx$
4. $\int_{-\infty}^\infty xe^{-x^2}dx$
5. $\int_0^9 \frac{1}{\sqrt{x}}dx$
6. $\int_0^{\pi/2} \sec^2 x dx$
7. $\int_{-2}^0 \frac{1}{\sqrt{4-x^2}}dx$
8. $\int_0^\pi \sec x dx$
9. $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}dx$
10. $\int_0^1 \frac{3x^2-1}{x^3-x}dx$

5.7 Application of the Integral

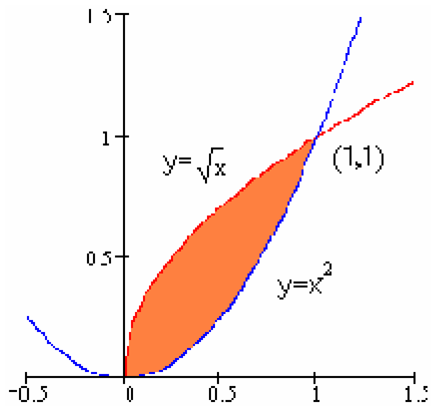
Area (Review)

Definition: Let f and g be continuous on [a,b], with $f(x) \geq g(x)$ *for* $a \leq x \leq b$. The area **A** of the region between the graphs of f and g on [a,b] is given by

$$\mathbf{A} = \int_a^b [f(x) - g(x)]dx$$

Example 1 Find the area of the region bounded by the graphs of the equations $y = x^2$ *and* $y = \sqrt{x}$.

Solution: First sketch the graphs on the same plane. And find the intesection of the two graphs by putting $x^2 = \sqrt{x}$. Observe that



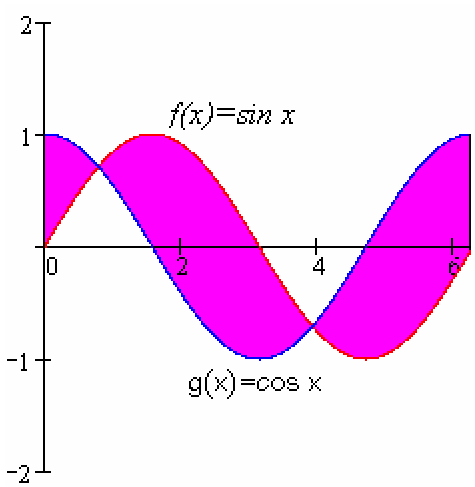
$$x^2 = \sqrt{x} \Rightarrow x^4 = x \Leftrightarrow x^4 - x = 0$$

Hence x=0 or $\Leftrightarrow x(x-1)(x^2+x+1)$ x=1 since x^2+x+1>0 for every real x the two graphs intersect at (0,0) and (1,1). Moreover $x^2 \leq \sqrt{x}$ on [0,1]. Thus the area A of the region bounded by the graphs is given by

$$\mathbf{A} = \int_0^1 (\sqrt{x} - x^2) = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3\bigg|_0^1 = \frac{1}{3}$$

Example 2 Let $f(x) = \sin x$ *and* $g(x) = \cos x$. Find the area A of the region between the graphs of *f* and *g* on [0,2π].

Solution: $\sin x = \cos x$, on [0,2π] implies that $\tan x = 1$ on [0,2π]. And $\tan x = 1$ on [0,2π] for $x = \frac{\pi}{4}$ *and* $x\frac{5\pi}{4}$. Thus the two graphs intersect at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ *and* $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$ and the region bounded by the two graphs on [0,2π] is as below.



Observe that $\sin x \geq \cos x$ *on* $[0, \pi/4]$,
$\sin x \geq \cos x$ *on* $[\pi/4, 5\pi/4]$ *and* $\sin x \geq \cos x$

[5π/4,2π] and it follows that

$$\mathbf{A} = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx$$

$$+ \int_{5\pi/4}^{2\pi} (\cos x - \sin x)dx$$

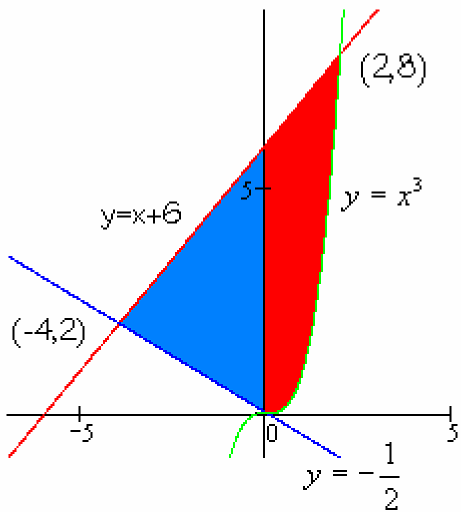
$$= (\sin x + \cos x)_0^{\pi/4} + (-\cos x - \sin x)_{\pi/4}^{5\pi/4}$$

$$(\sin x + \cos x)_{5\pi/4}^{2\pi}$$

$$= (\sqrt{2} - 1) + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2}$$

Example 3 Find the area of the region bounded by the graph of y-x = 6, y-x^3=0 and 2y+x=0.

Solution: First we graph the region as follows. We divid the region in two two regions R_1 and R_2 as in the plote to the right



$$\mathbf{A}_1 = \int_{-4}^{0}[x+6+\tfrac{1}{2}x]dx = \tfrac{3}{4}x^2 + 6x\Big|_{-4}^{0} = 12$$

and

$$\mathbf{A}_2 = \int_{0}^{2}(x+6-x^3)dx = 10$$

Thus the area **A** of the entire region **R** is

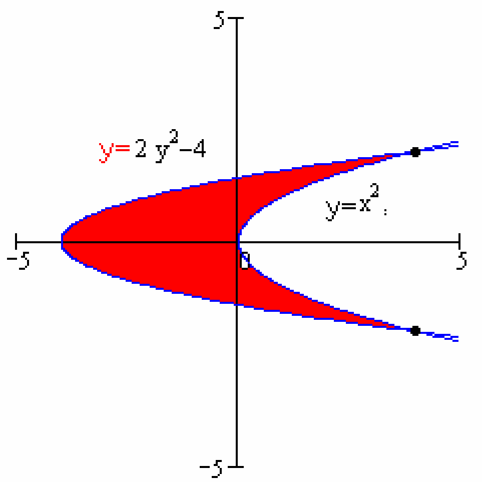
A=A_1+A_2=22.

Reversing the roles of x and y

Instead of considering a region **R** that is bounded between the graphs of two functions of x, it is sometimes convenient to consider R as the region between the graphs of two functions of y. Then the area is computed by integrating along the y-axis, instead of along the x-axis.

Example 4 Find the area of the region bounded by the graphs of the equations 2y^2=x+4 and y^2=x.

Solution: First we sketch the region as below



'e can see that

points (4,-2)

y^2=x+4 lays

ence the area

the two graphs

$$\mathbf{A} = \int_{-2}^{2}[y^2-(2y^2-4)]dy = 4y - \frac{1}{3}y^3\Big|_{-2}^{2}$$

$$= \frac{32}{3}.$$

Class Work

Find the area A of the regions bounded between the graphs of the equations bellow.

a) y=x^2+1 and y=2x+9 b) x=y^2-y and x=y-y^2

Volume

The cross –section method

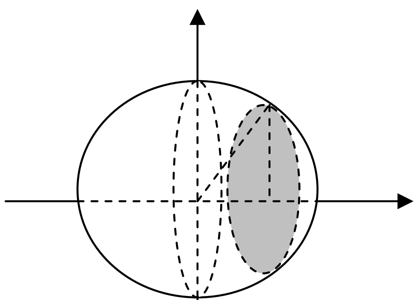
If a solid region D has cross-sectional area **A**(x) for $a \le x \le b$, and if **A** is continuous on [a,b] , then we define the volume V of D by the fromula

$$V=\int_a^b \mathbf{A}(x)dx.$$

Example 1 Show that the volume of a sphere of radius r is

$$V=\tfrac{4}{3}\pi r^3$$

Solution: If we place the sphere so that its center is at the origin then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean theorem) is



$y=\sqrt{r^2-x^2}$. So that cross-section area is

$$\mathbf{A}(x)=\pi y^2=\pi(r^2-x^2)$$

Using the formula with $a=-r$ *and* $b=r$, we have

$$V=\int_{-r}^{r} A(x)dx=\int_{-r}^{r}\pi(r^2-x^2)dx$$

$$=\pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}$$

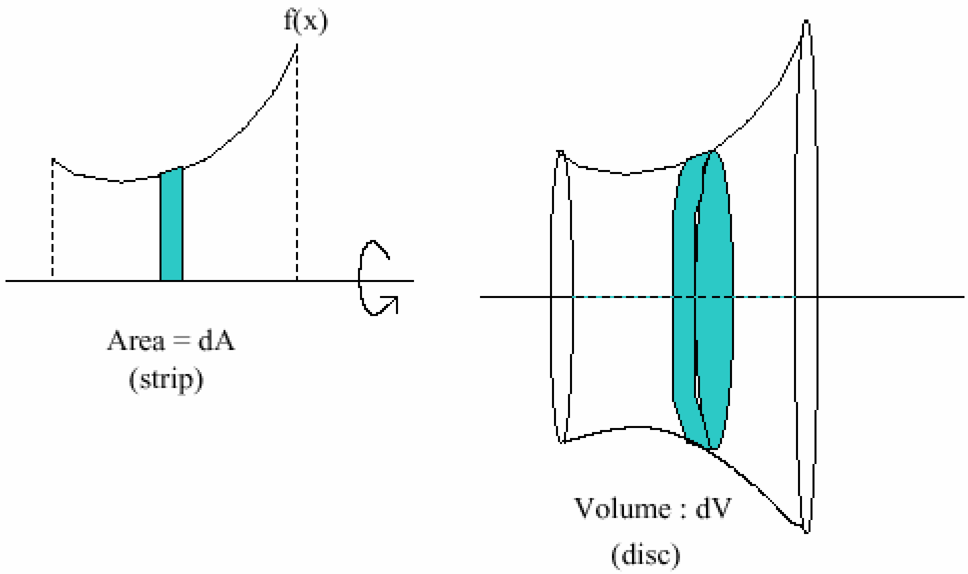
$$=\tfrac{4}{3}\pi r^3$$

Class work

Suppose a pyramid is 4 units tall and has a squere base 3 units on a side. Find the volume V of the pyramid.

The Disc Method

We now move on to yet another application of definite integrals: **volumes of revolution**. Volumes of revolution are solids whose shapes can be generated by revolving some curve(s) about some axis in three-space. If we can set things up so that a solid of revolution is generated by revolving the region between the graph of a continuous function $f(x), a \leq x \leq b$ and the x axis, and the axis of rotation is the x axis (see diagram below), we can then calculate the volume in the following way:

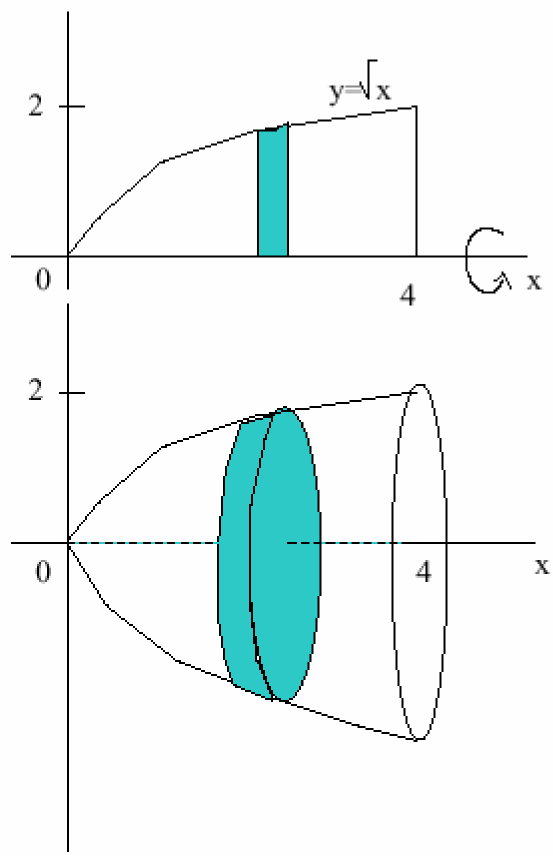


The steps to follow are very familliar:

(1) sketch the region to be revolved
(2) Draw a small strip perpendicular to the axis of revolution ,then revolve it about the axis of rotation and calculate the volume that it generates, say dV (see Fig 6)
(3) Integrate dV to find the entire volume

Example 2 Find the volume generated by revolving the region bounded by $y = \sqrt{x}, y = 0$, and $x = 4$ about the x-axis.

Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):



Rotating the strip about the x-axis we see that we get something of the form:

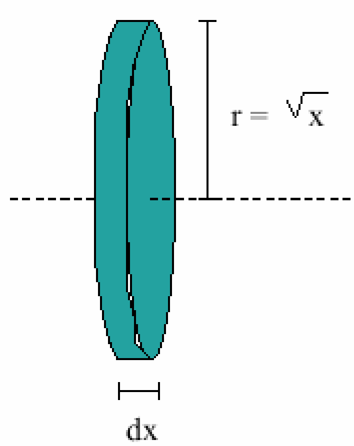


Figure 8: $volume = \pi r^2 h$

This is clearly a cylindrical shape and so has volume given by the classical formula: $v = \pi r^2 h$, where r is the radius of the cylinder, and h is the height. Looking at the specific solid generated by the strip here, we see that $h = dx$ and r=the length of the strip =the y-value of the curve $=\sqrt{x}$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi r^2 h \\ &= \pi(\sqrt{x})^2 dx \\ &= \pi x dx \end{aligned}$$

We also see from the sketch that x varies from 0 to 4 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^4 \pi x dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 \\ &= \pi(8-0) \\ &= 8\pi \end{aligned}$$

The Washer Method

The next examples illustrate the above process which is sometimes called the **method of washers**, for a soon obvious reason (the strip generates a solid resembling a washer).

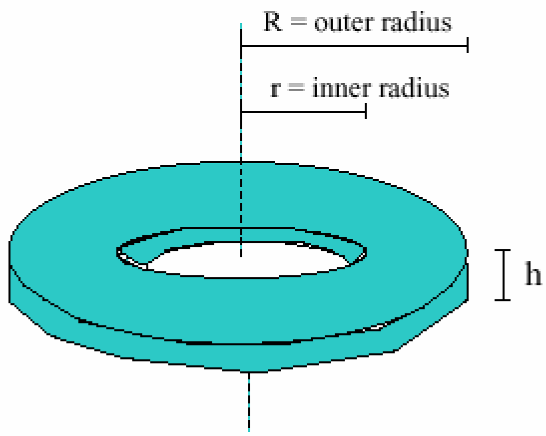


Figure 9: washer volume $= \pi(R^2 - r^2)h$

To find the volume dV, of such an animal, we simply find the volume of the large disc as if it were solid $(\pi R^2 h)$and then subtract the volume of the hole $(\pi r^2 h)$. This gives us the formula:

$$\boxed{dV = \pi(R^2 - r^2)h}$$

The use of the above formula is better illustrated through some examples:

Example 7. Find the volume generated by revolving the region bounded by $y = x^2 + 2$, $y = 1$, $x = 0$ and $x = 2$ about the x-axis.

Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):

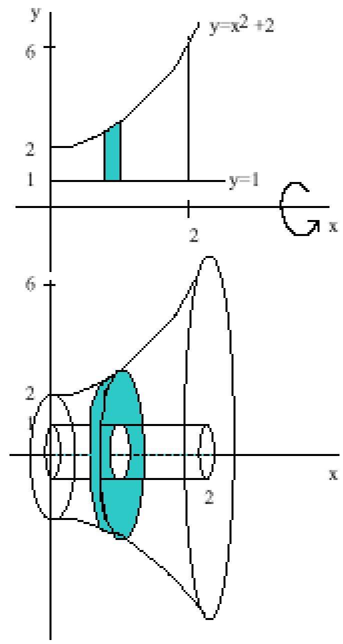
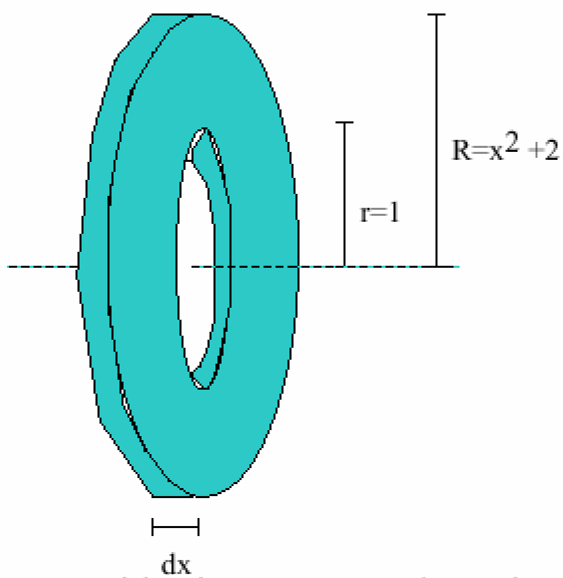


Figure 10:

Rotating the strip about the x-axis we see that we get something resembling figure 11.



The volume generated by the strip is one of a washer with $R =$ (the distance from the x-axis to the outer edge of the strip) $= x^2 + 2$, $r =$ (the distance from the x-axis to the inner edge of the strip) $= 1$, and $h = dx$. So the volume generated by the strip is given by:

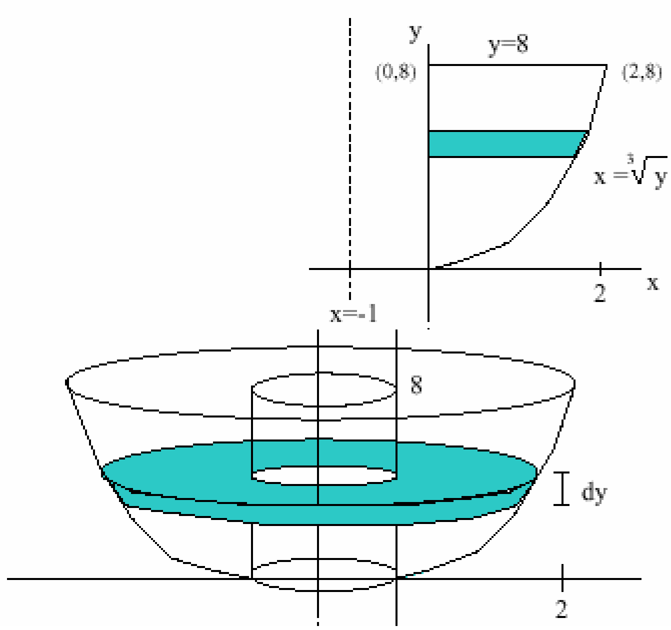
$$
\begin{aligned}
dV &= \pi(R^2 - r^2)h \\
&= \pi[(x^2+2)^2 - (1^2)]dx \\
&= \pi(x^4 + 4x^2 + 3)dx
\end{aligned}
$$

We also see from the sketch that x varies from 0 to 2 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$
\begin{aligned}
V &= \int_0^2 \pi[(x^2+2)^2 - (1^2)]dx \\
&= \pi \int_0^2 (x^4 + 4x^2 + 3)dx \\
&= \pi \left[\tfrac{1}{5}x^5 + \tfrac{4}{3}x^3 + 3x\right]_0^2 \\
&= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6) - 0\right] \\
&= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6)\right] \approx 72.5
\end{aligned}
$$

Just as with areas, we sometimes use horizontal strips for finding volumes. This comes about since the method we learned above requires the strips to be perpendicular to the axis of rotation, so if we revolve a region about, say, the y-axis then our strips must be horizontal. All other mechanics of such a problem are business as usual as we shall see:

Example 8. Find the volume generated by revolving the region bounded by $y = x^3$, $y = 8$,and $x = 0$ about the line $x = -1$.
Solution: We first sketch the region in question, and draw our small strip (with width dy) perpendicular to the axis of rotation :



Rotating the strip about the axis of rotation we see that we get something resembling figure 13.

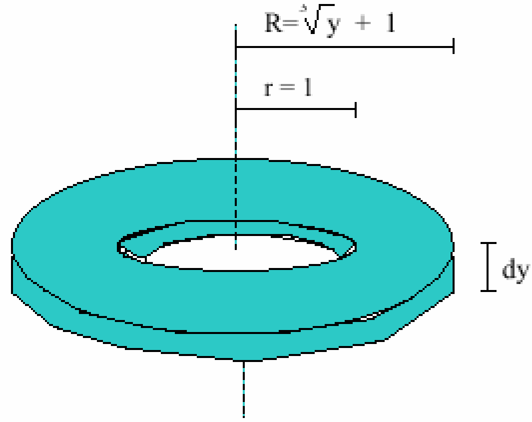


Figure 13:

The volume generated by the strip is one of a washer with $R =$ (the distance from the axis of rotation to the outer edge of the strip)=(1+the x value of the outer curve) $= 1+y^{\frac{1}{3}}$, $r =$ (the distance from the axis of rotation to the inner edge of the strip) $= 1$, and $h = dy$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi(R^2 - r^2)h \\ &= \pi[(1+y^{\frac{1}{3}})^2 - (1^2)]dy \\ &= \pi(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \end{aligned}$$

We also see from the sketch that y varies from 0 to 8 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^8 \pi[(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi \int_0^8 (y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi\left[\tfrac{3}{5}y^{\frac{5}{3}} + \tfrac{3}{2}y^{\frac{4}{3}}\right]_0^8 \\ &= \pi\left[\tfrac{3}{5}(8)^{\frac{5}{3}} + \tfrac{3}{2}(8)^{\frac{4}{3}}\right] \\ &= \pi\left[(\tfrac{96}{5} + 24\right] \approx 135.7 \end{aligned}$$

Class Work

Let f(x)=5x and g(x)=x^2 and let R be the region between the graphs of f and g on [0,3]. Find the volume of the solid obtained by revolving R about the x-axis.