

For a predicate $\varphi(x)$, I define $\hat{x}\varphi(x)$ to be the x that satisfies $\varphi(x)$; it is undefined if there's more than one such x . Then,

$$\begin{aligned} A_0 &= \{\emptyset\} \\ A_{n+1} &= \{(\alpha_i, \beta_i)_{i \in k} \mid k \in \omega \setminus 1 \wedge \alpha_i \in \text{Ord} \setminus 1 \wedge \beta_i \in A_n \wedge |\{\beta_i\}_{i \in k}| = k\} \\ A &= \bigcup_{n \in \omega} (A_n) \\ A' &= A \setminus \{\emptyset\} \\ B_0 &= \{\emptyset\} \\ B_{n+1} &= \{(\alpha_i, \beta_i)_{i \in k} \mid k \in \omega \setminus 1 \wedge \alpha_i \in \text{Ord} \wedge \beta_i \in B_n \wedge |\{\beta_i\}_{i \in k}| = k\} \\ B &= \bigcup_{n \in \omega} (A_n) \end{aligned}$$

$$\begin{aligned} a &: A \rightarrow \bigcup_{n \in \omega} (\{\{\alpha_i\}_{i \in n, \alpha_i \in \text{Ord}}\}) \\ a(X) &: X \mapsto \{X' \mid \exists X'' \in (\text{Ord} \setminus 1)[(X'', X') \in X]\} \end{aligned}$$

$$\begin{aligned} b &: A' \rightarrow A \\ b(X) &: X \mapsto \hat{X}'(\exists X'' \in a(X)[X' < X'']) \end{aligned}$$

$$\begin{aligned} \bar{b} &: A' \rightarrow A \\ \bar{b}(X) &: X \mapsto \hat{X}'(\exists X'' \in a(X)[X'' < X']) \end{aligned}$$

$$\begin{aligned} c_X(Y) &: A^2 \rightarrow \text{Ord} \\ c_X(Y) &: (X, Y) \mapsto \hat{X}'((X', Y) \in X) \iff Y \in a(X) \\ & \quad (X, Y) \mapsto 0 \iff Y \notin a(X) \end{aligned}$$

$$X < Y \iff \begin{cases} \perp & [Y = \emptyset] \\ \top & [X = \emptyset \wedge Y \neq \emptyset] \\ b(X) < b(Y) & [X \neq \emptyset \wedge Y \neq \emptyset \wedge b(X) \neq b(Y)] \\ c_X(b(X)) < c_Y(b(Y)) & [X \neq \emptyset \wedge Y \neq \emptyset \wedge b(X) = b(Y) \wedge c_X(b(X)) \neq c_Y(b(Y))] \\ X \setminus \{(c_X(b(X)), b(X))\} < Y \setminus \{(c_Y(b(Y)), b(Y))\} & [X \neq \emptyset \wedge Y \neq \emptyset \wedge b(X) = b(Y) \wedge c_X(b(X)) = c_Y(b(Y))] \end{cases}$$

$$\begin{aligned} d &: B \rightarrow A \\ d(X) &: X \mapsto \{(X', X'') \mid (X', X'') \in X \wedge X' > 0\} \end{aligned}$$

$$\begin{aligned} e &: A \rightarrow A \\ e(X) &: X \mapsto \begin{cases} X & [c_X(\emptyset) \in \text{Lim} \cup 1] \\ d(X \setminus \{(c_X(\emptyset), \emptyset)\} \cup \{(\hat{\alpha}(\alpha + 1 = c_X(\emptyset)), \emptyset)\}) & [c_X(\emptyset) \in \text{Succ}] \end{cases} \end{aligned}$$

$$\begin{aligned} f &: A \rightarrow \text{Ord} \cup \{\text{Ord}\} \\ f(X) &: X \mapsto \begin{cases} 0 & [X = \emptyset] \\ 1 & [e(X) \neq X] \\ c_X(\bar{b}(X)) & [c_X(\bar{b}(X)) \in \text{Lim}] \\ \text{Ord} & [e(\bar{b}(X)) \neq \bar{b}(X)] \\ f(\bar{b}(X)) & [e(\bar{b}(X)) = \bar{b}(X)] \end{cases} \end{aligned}$$

$$\begin{aligned} g &: A \times \text{Ord} \rightarrow A \\ g(X, \alpha) &: (X, \alpha) \mapsto \begin{cases} \emptyset & [X = \emptyset] \\ e(X) & [e(X) \neq X] \\ d(X \setminus \{(c_X(\bar{b}(X)), \bar{b}(X))\} \cup \{(\hat{\beta}(\beta + 1 = c_X(\bar{b}(X))), \bar{b}(X)), (\alpha, e(\bar{b}(X)))\}) & [e(\bar{b}(X)) \neq \bar{b}(X) \wedge c_X(\bar{b}(X)) \in \text{Succ}] \\ d(X \setminus \{(c_X(\bar{b}(X)), \bar{b}(X))\} \cup \{(\hat{\beta}(\beta + 1 = c_X(\bar{b}(X))), \bar{b}(X)), (1, g(\bar{b}(X), \alpha))\}) & [e(\bar{b}(X)) = \bar{b}(X) \wedge c_X(\bar{b}(X)) \in \text{Succ}] \\ d(X \setminus \{(c_X(\bar{b}(X)), \bar{b}(X))\} \cup \{(\alpha, \bar{b}(X))\}) & [c_X(\bar{b}(X)) \in \text{Lim}] \end{cases} \end{aligned}$$

$$\begin{aligned} h &: A \rightarrow A \\ h(X) &: X \mapsto \begin{cases} \emptyset & [X = \emptyset] \\ X \setminus \{(c_X(\emptyset), \emptyset)\} & [\emptyset \in a(X)] \\ X & [\emptyset \notin a(X)] \end{cases} \end{aligned}$$

$$\begin{aligned} \varphi &: A \rightarrow \text{Ord} \\ \varphi X &: (X, \alpha) \mapsto \begin{cases} 1 & [X = \emptyset] \\ \omega^{c_X(\emptyset)} & [a(X) = \{\emptyset\}] \\ \text{enum}(\{\alpha \mid \forall \beta < f(h(X))[\alpha = \varphi(g(h(X), \beta) \cup \{(\alpha, \emptyset)\}]\})(c_X(\emptyset)) & [f(h(X)) \in \text{Ord}] \\ \text{enum}(\{\alpha \mid \alpha = g(h(X), \alpha)\})(c_X(\emptyset)) & [f(h(X)) = \text{Ord}] \end{cases} \end{aligned}$$

$$\begin{aligned} \{(\alpha_i, i)_{i \in n, n \in \omega}\} &= (\alpha_i)_{i \in n, n \in \omega} \\ \{(\alpha_i, X_i)_{i \in n, n \in \omega}\} &= \left(\alpha_i \right)_{i \in n, n \in \omega} \\ & \quad X_i \end{aligned}$$