On the phenomenon of series LCR resonance

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1 Introduction

Capacitors and inductors form the basis for all electronics - both in the analog and the digital realm. These reactive components have enabled engineers to work with all sorts of time-varying signals which has many applications in signal processing, radio communication and even power electronics through devices such like buck-boost converters. In this report we will be looking into a specific characteristic of a series circuit with a capacitor, inductor and resistor (otherwise known as a LCR circuit) called resonance. Resonance enables engineers to build band-pass and band-stop filters which are very useful for extracting a narrow frequency band of signals into a circuit. The aims of this experiment are to,

1. Characterize the transfer function of LCR circuits
2. Determine how the series resistance affects the frequency response of the circuit
3. Determine how the phase difference of the waveform is altered by the circuit

2 Theory

Resonance is a phenomenon that occurs in circuits with both capacitive and inductive components. This effect arises due to the fact that capacitive impedance decreases as the frequency increases, as compared to inductive impedance which increases as the frequency increases. Additionally, capacitors have a phase shift of -90° while inductors have a phase shift of -90°. The equations for the impedances of capacitors and inductors are Z_C and Z_I correspondingly, and are shown below [McHutchon, 2013].

$$Z_C = \frac{-j}{\omega C},\ Z_I = j\omega C.$$

Together, the combined effect of a series LCR circuit is that, at some frequency ω_0, the impedances from the capacitor and inductor will cancel each other out, which results in a purely real impedance (from the resistor with resistance R) and no phase shift. This frequency is known as the resonance frequency of the circuit [McHutchon, 2013].

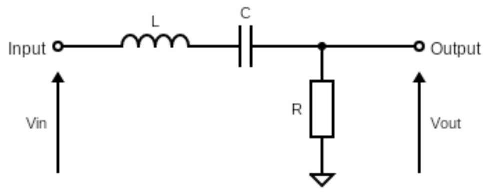

Figure 1: Simple LCR Circuit Diagram

2.1 Deriving the resonant frequency

Analyzing the simple LCR circuit as shown in Figure 1, the transfer function can be obtained by treating the circuit as a normal resistive voltage divider, except the resistances are replaced by the equivalent impedances of the reactive components.

$$\frac{V_out}{V_in} = \frac{R}{R + j\omega L - \frac{j}{\omega C}} \tag{1}$$

From which we can see that at the resonance frequency ω_0,

$$j\omega_0 L = \frac{j}{\omega_0 C} \ and \ \left|\frac{V_out}{V_in}\right| = 1$$

Which means that the resonance frequency ω_0 is,

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2}$$

2.2 Deriving the 3dB bandwidth

The 3dB bandwidth is obtained from the range of values for which $\left|\frac{V_out}{V_in}\right| = \frac{1}{\sqrt{2}}$. The 3dB bandwidth is an important characteristic as it shows us the range of frequencies for which the signal not attenuated more than half-power. To derive the 3dB bandwidth, we start with Equation (1)

$$\left|\frac{V_out}{V_in}\right| = \frac{|R|}{|R + j\omega L - \frac{j}{\omega C}|} = \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_1 = \frac{-R + \sqrt{R^2 + 4L/C}}{2L} \ and \ \omega_2 = \frac{R + \sqrt{R^2 + 4L/C}}{2L} \tag{3}$$

The 3dB bandwidth is then given by,

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L} \tag{4}$$

2.3 Deriving the Q-factor

The Q-factor of a LCR filter characterizes the sharpness of the filter, with high Q-factor corresponding to a sharp and narrow filter response and low Q-factor corresponding to a broad filter response. The Q-factor is defined to be the ratio between the resonant frequency ω_0 and the 3dB bandwidth $\Delta\omega$.

From Equation (2) and (4),

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{5}$$

3 Procedure

Through this experiment, a basic LCR circuit is observed and the frequency response of the circuit is plotted. The circuit that is being investigated has $L = 2.2mH$ and $C = 100nF$, where L is the inductance of the series inductor and C is the capacitance of the series capacitor. The frequency response is also observed for 2 values of R, 51Ω and 100Ω.

3.1 Required Equipment

- Function Generator (capable of 10Hz-1Mhz range)
- Oscilloscope (at least 2 channels)
- Oscilloscope Probes (x2)
- Multimeter with resistance measurement functionality
- Multimeter probes
- Coaxial Cables
- BNC splitter

3.2 Circuit Construction

The test circuit can be constructed as shown in Figure 2. An SPDT switch is used so that the circuit can be switched between the 51Ω and 100Ω resistors. The output voltage straight from the signal generator is sampled on CH1 of the oscilloscope and the voltage across the resistor is sampled on CH2. Sampling both of these signals at the same time, enables comparison between the waveforms to understand the behaviour of the resonance circuit.

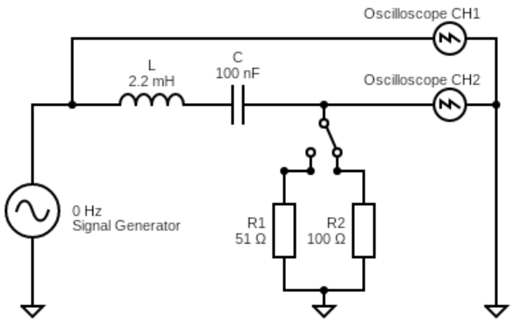

Figure 2: Test circuit diagram

3.3 Measured parameters

First, measure the actual resistances of the resistors $R1$ and $R2$. With these values, it is possible to avoid errors due to the tolerances of the resistors.

Next, configure the oscilloscope such that the following parameters are displayed on the screen.

- Amplitude (CH1) measuring V_{in}
- Amplitude (CH2) measuring V_{out}
- Phase Difference between CH1 and CH2 measuring $\Delta\phi$

Then configure the signal generator to pass in a $5V_{pp}$ input sinusoidal signal corresponding to V_{in}. Consider different values of ω to measure, and for every value of ω considered, record V_{in}, V_{out} and $\Delta\phi$ in a table.

3.4 Choosing sample values for ω

It is important to efficiently choose sample values for ω such that more samples are collected around points of interest, namely the resonance frequency. For the provided values of L and C, the predicted resonant frequency can be calculated according to Equation (2) which is $\omega_0 = 67420rad/s \implies f = 10730Hz$.

Outside of this resonant frequency, beyond the 3dB cutoff, a constant rolloff is expected, and therefore not many measurements need to be taken in that region.

4 Results and Discussion

First, the measured resistances of the resistors is given in Table **??**.

The collected data also has to be transformed such that the ratio $\frac{V_{out}}{V_{in}}$ is expressed in dB using Equation (6).

Table 1: Measured Resistances

Resistor	**Expected**	**Measured**
R1	51Ω	51.862Ω
R2	100Ω	113.347Ω

$$dB = 20 \log \frac{V_{out}}{V_{in}} \tag{6}$$

Then, plotting the obtained values for the 51Ω and 100Ω against frequency (on a logarithmic scale), yields the transfer function for both resistors as shown Figure 3.

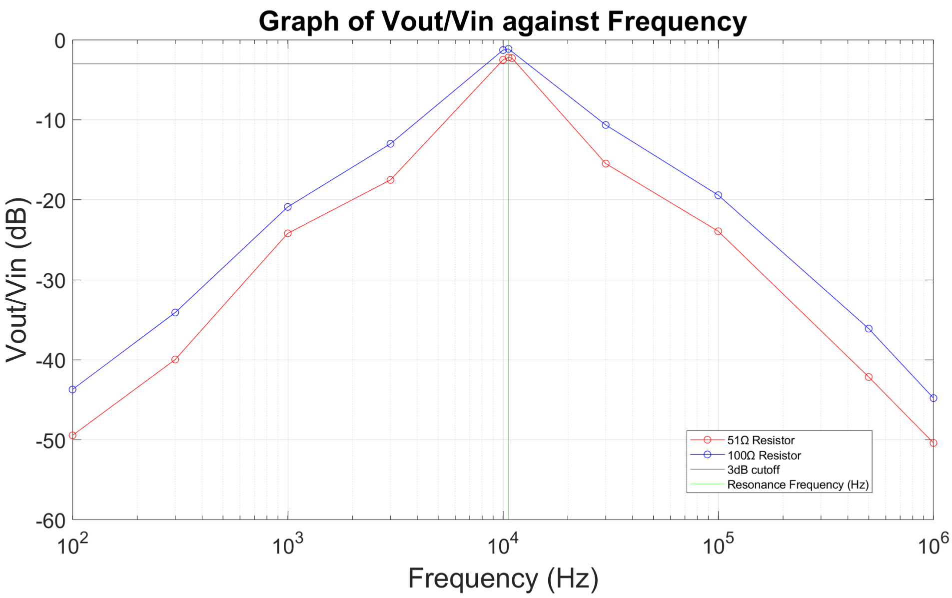

Figure 3: Transfer Function of LCR circuit for $R = 51\Omega$ and $R = 100\Omega$

4.1 Resonant frequency analysis

The first observation that is made is the noticeable peak at $f \approx 10700Hz$. That corresponds to our theoretical resonance frequency of $10730Hz$. It is also clear from the graph that the change in resistor value between 51Ω and 100Ω does not impact the resonant frequency as predicted by the resonance frequency equation in Equation (2). The transfer function depicted shows the characteristics of a bandpass filter, where only a certain range of frequencies (between the 3dB cutoff frequencies) are not extremely attenuated.

4.2 Q-factor and 3dB cutoff analysis

The second observation is that the graph for $R = 51\Omega$ is noticeably narrower than the graph for $R = 100\Omega$. This is a confirmation of the prediction by Equation (5), as the Q-factor is lower when the series resistance in the LCR circuit is greater.

The 3dB cutoff point is labelled by the grey horizontal line in the graph. As predicted by Equation (4), the bandwidth with the 100Ω resistor is wider as shown by the fact that more of the blue graph is above the line compared to the red graph.

Reading from Figure 4, we can estimate the following values for the bandwidth and Q-factors of the 100Ω and 51Ω resistors using Equation (4) and Equation (5).

$$\Delta\omega_{51\Omega} = 11.6kHz - 9.64kHz = 1.96kHz \implies Q_{51\Omega} = 10.7kHz/1.96kHz = 5.45$$
$$\Delta\omega_{100\Omega} = 12.98kHz - 8.44kHz = 4.54kHz \implies Q_{100\Omega} = 10.7kHz/4.54kHz = 2.37$$

Reading from Figure (3), it is also visible that the slope of the graph well beyond the 3dB cutoff point is $\approx 25dB/decade$.

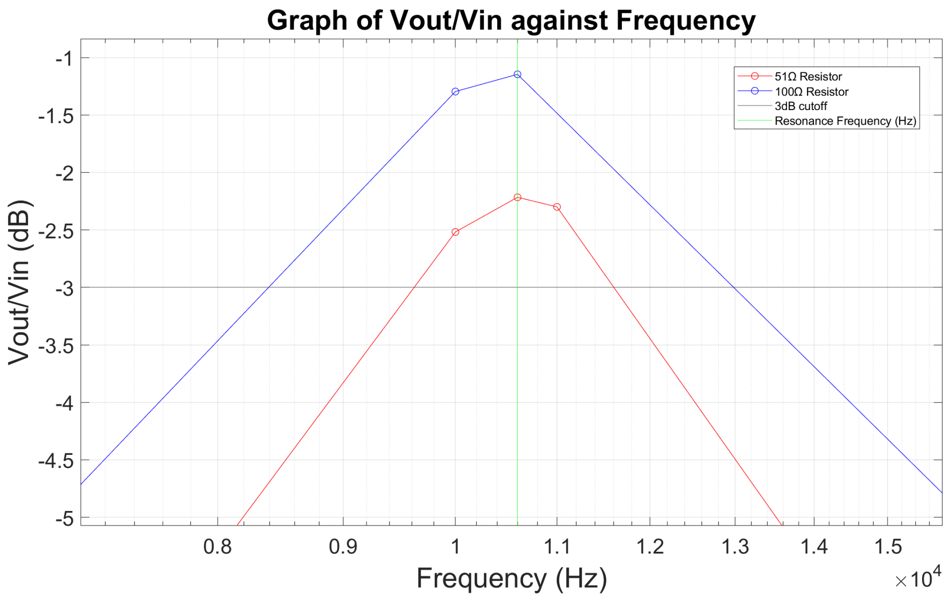

Figure 4: Transfer function zoomed in around the resonance frequency (ω_0)

4.3 Change in V_{in} with frequency

An interesting observation would be that V_{in} fluctuates along with the frequency.

In fact, the V_in is observed to drop significantly in the region around the resonance frequency of 10kHz. A proposed explanation for this is that, as the frequency approaches resonance, the impedance from the inductor and capacitor cancel out, leaving a low series

Table 2: V_{in} changing with Frequency (51Ω Resistor)

Frequency	V_{in}
1kHz	5.18V
10kHz	3.00V
100kHz	5.12V

resistance of just 51Ω. This low resistance, could be loading the output of the signal generator, causing the generated signal voltage to sag.

4.4 Investigating phase around resonance

One consequence of the inductive and capacitive properties of the circuit, is that the circuit imposes a phase shift of $\Delta\phi$ on the waveform passing through it. However, the reactive components of the circuit cancel out during resonance, thus the phase shift must approach zero as the frequency approaches the resonant frequency.

The phase shift around resonance is plotted using linear regression in Figure 5. The R^2 value from the linear regression is 0.9921, which is high and indicates that the regression model fits the data well. As expected from the theory, as the frequency increases towards the resonance frequency, the phase difference drops from a positive value towards zero. Near the resonance frequency, $\Delta\phi = 0$. Similarly, as the frequency increases away from the resonance frequency, the phase difference continues to decrease and becomes negative.

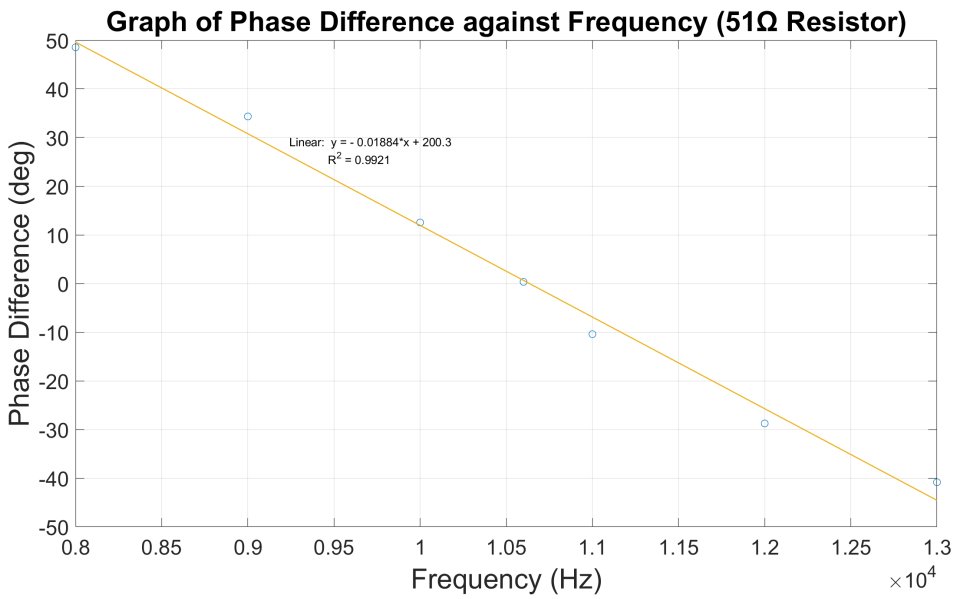

Figure 5: Phase difference ($\Delta\phi$) around the resonance frequency (ω_0)

5 Conclusion

As the aims of the experiment highlighted, the transfer function of the LCR circuit has been characterized in this report. As predicted by the theory, a peak is observed at the resonant frequency and the frequencies which do not lie between the 3dB cutoff frequencies are significantly attenuated by the filter. The effect of the series resistance on the filter response has also been shown, as a higher resistance leads to a larger bandwidth and a broader curve seen in Figure 4. Lastly, the change in phase difference around the resonant frequency was analyzed in Figure 5 and it was demonstrated that at resonance, the phase difference is zero.

The results of this experiment enable the creation of filters which can be used in various signal processing applications, especially in radio and audio equipment. For example, AM radios use bandpass filters to select radio stations, and band stop filters could be used to remove frequencies from audio tracks to enhance the listening experience.

Further analysis can be done around other interesting configurations of LC circuits such as low-pass, high-pass and band-stop filters.

References

Andrew McHutchon. *RLC Resonant Circuits*. Cambridge University, 2013. URL `http://mlg.eng.cam.ac.uk/mchutchon/ResonantCircuits.pdf`.

6 Appendix A

6.1 Raw Data (51Ω)

6.2 Raw Data (100Ω)

6.3 Matlab Code Reference