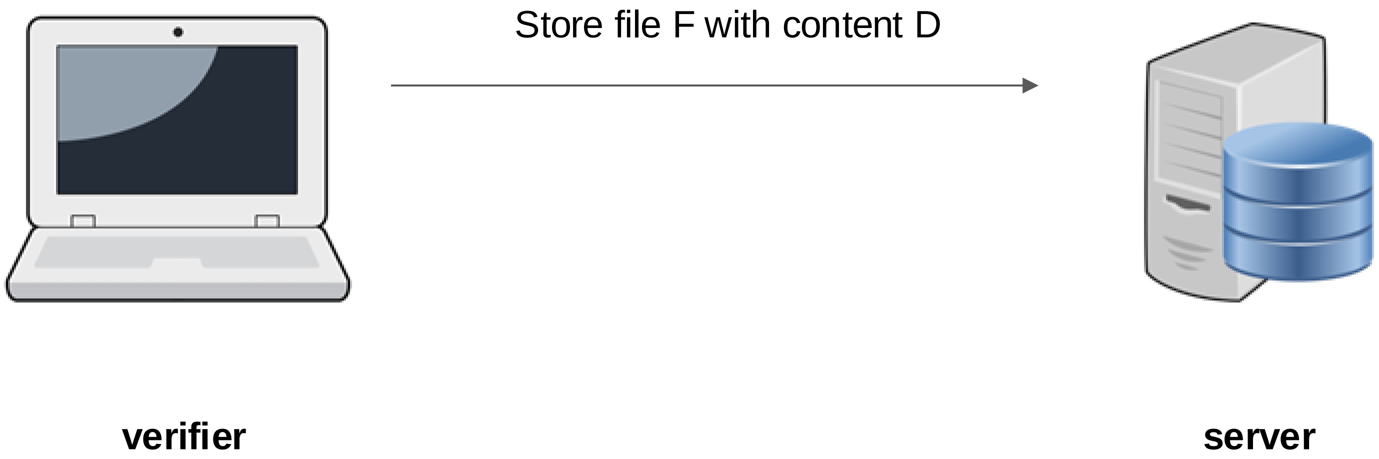
Blockchains & Distributed Ledgers

Lecture 02

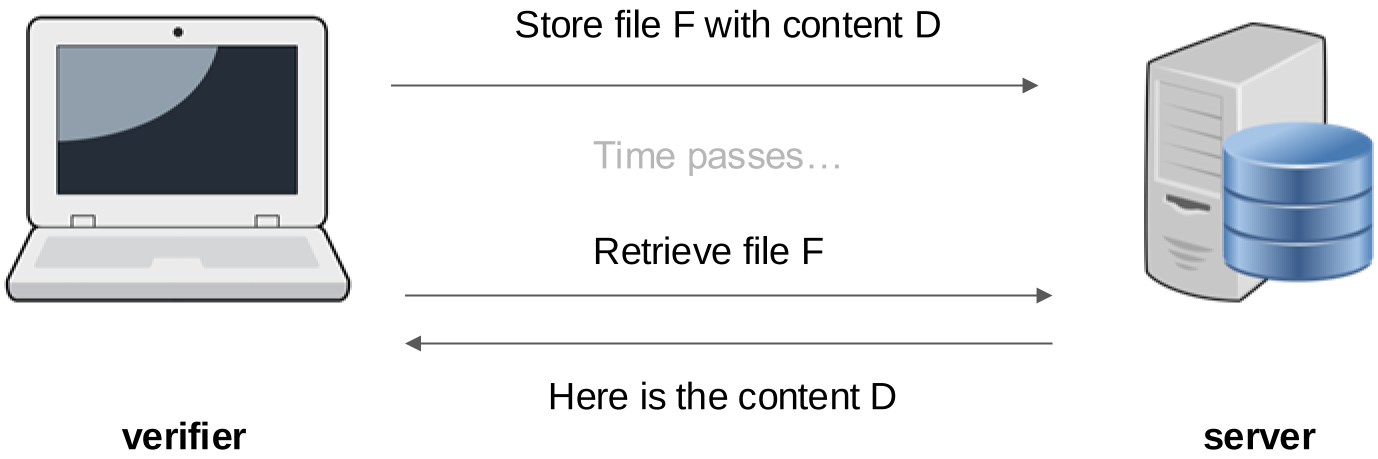
Aggelos Kiayias

Slide credits: AK, Dimitris Karakostas, Michele Ciampi, Dionysis Zindros, Christos Nasikas

The authenticated file storage problem



The authenticated file storage problem



The authenticated file storage problem

The problem

- Client wants to store a file, with identifier F and content D, on a server
- Clients wants to retrieve D later in time

Usecases

- Save storage space (e.g., cloud)
- Redundancy (e.g., backup)

File storage: Basic protocol

- Client sends file F with content D to server
- Server stores (F, D)
- Client deletes D
- Client requests F from server
- Server returns D
- Client has recovered D

File storage: Basic protocol

- Client sends file F with content D to server
- Server stores (F, D)
- Client deletes D
- Client requests F from server
- Server returns D
- Client has recovered D

*What if **server is corrupted** and returns D’ != D?*

File storage: Protocol against adversaries

Trivial solution:

- Client does not delete D
- When server returns D’, client compares D and D’

...what if client doesn’t have enough memory to store D for a long time?

Authenticated Data Structures

- Like regular data structures, but cryptographically authenticated
- A **verifier** can store/retrieve/operate on data held by an **untrusted prover**
  - Client wants to store a file, with identifier F and content D, on a server
  - Client wants to delete D
  - Clients wants to retrieve D later in time
  - Prover is *not trusted* - it has to *prove* that the returned data is the correct/original D
- How can this problem be solved using:
  a. A hash function H
  b. A signature scheme Σ = <KeyGen, Sign, Verify>

File storage: Authenticated protocols

Hash-based

- Client sends file *F* with data *D* to server
- Server stores *(F, D)*
- Client computes and stores *H(D)*, deletes *D*

Time passes…

- Client requests *F* from server
- Server returns *D'*
- Client compares *H(D') = H(D)*

File storage: Authenticated protocols

Digital signature-based

- Client creates and stores key pair (sk, vk)
- Client computes $\sigma = Sign(sk, \langle F, D \rangle)$
- Client sends (F, D, σ) to server, deletes D, σ
- Server stores (F, D, σ)

Time passes…

- Client requests F from server
- Server returns (D', σ')
- Client checks if $Verify(vk, \langle F, D' \rangle, \sigma') = True$

File storage: Authenticated protocols

Hash-based

- Client sends file *F* with data *D* to server
- Server stores *(F, D)*
- Client computes and stores *H(D)*, deletes *D*

Time passes…

- Client requests *F* from server
- Server returns *D’*
- Client compares *H(D’) = H(D)*

Digital signature-based

- Client creates and stores key pair *(sk, vk)*
- Client computes *σ = Sign(sk, <F, D>)*
- Client sends *(F, D, σ)* to server, deletes *D, σ*
- Server stores *(F, D, σ)*

Time passes…

- Client requests *F* from server
- Server returns *(D’, σ’)*
- Client checks if *Verify(vk, <F, D’>, σ’) = True*

What if client needs only one byte of the file?

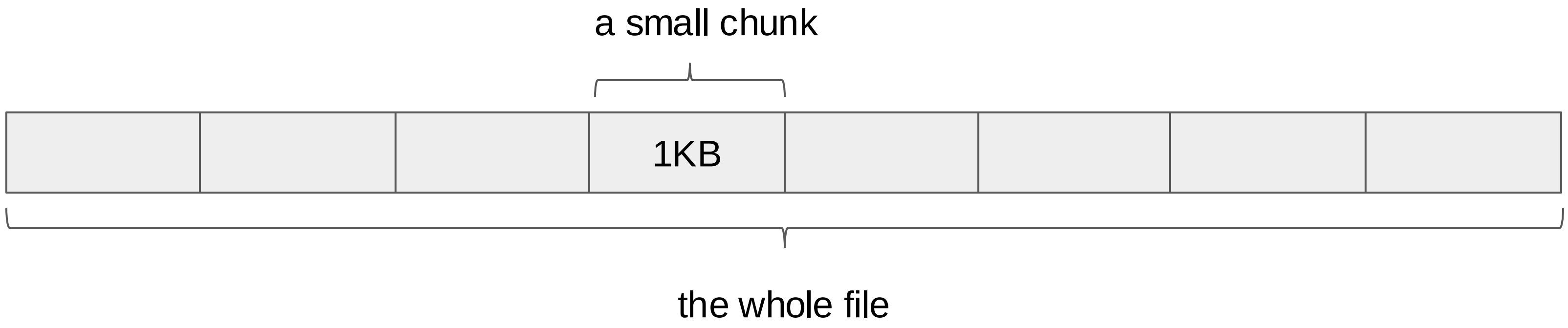
Merkle Trees

Tree definitions

- **Binary**: every node has at most 2 children
- Binary **full**: every node has either 0 or 2 children
- Binary **complete**: every node in every level, except possibly the second-to-last, has exactly 2 children, and all nodes in the last level are as far left as possible
- **Merkle tree**: an *authenticated* binary tree

Merkle Tree

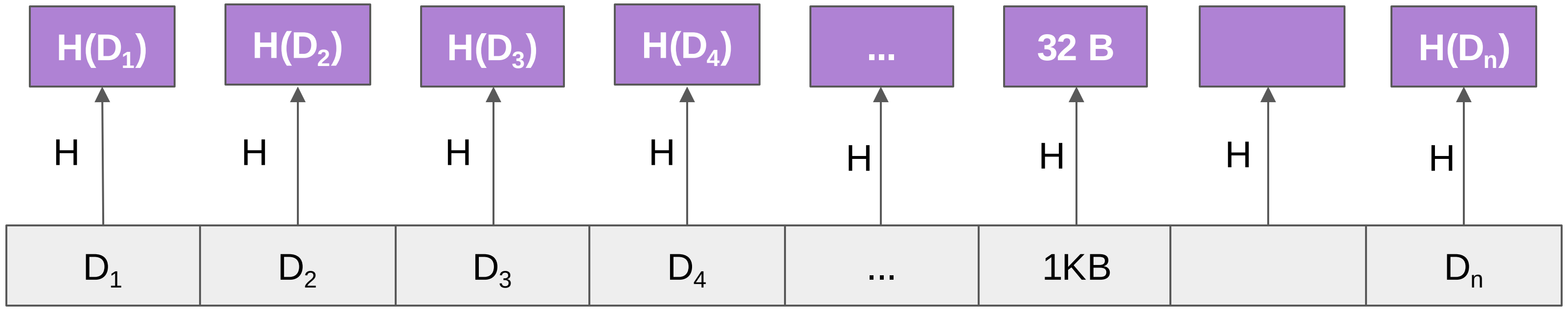
- Split file into *small* **chunks** (e.g., 1KB)



Merkle Tree

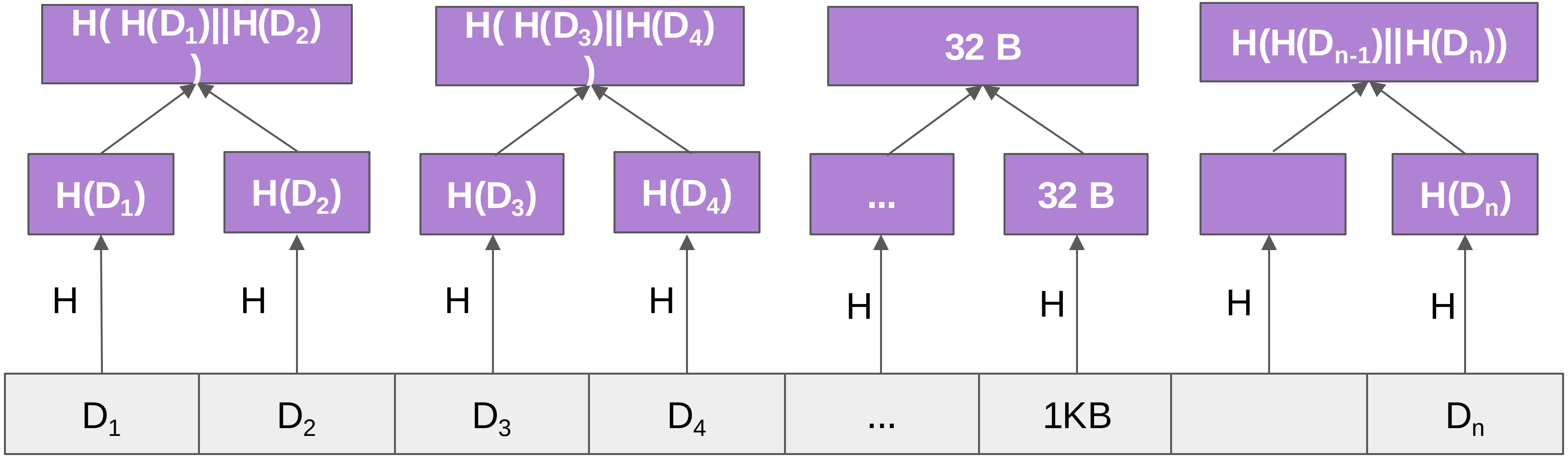
- **Hash** each chunk using a cryptographic hash function (e.g., SHA256)

*Arrows show direction of hash function application

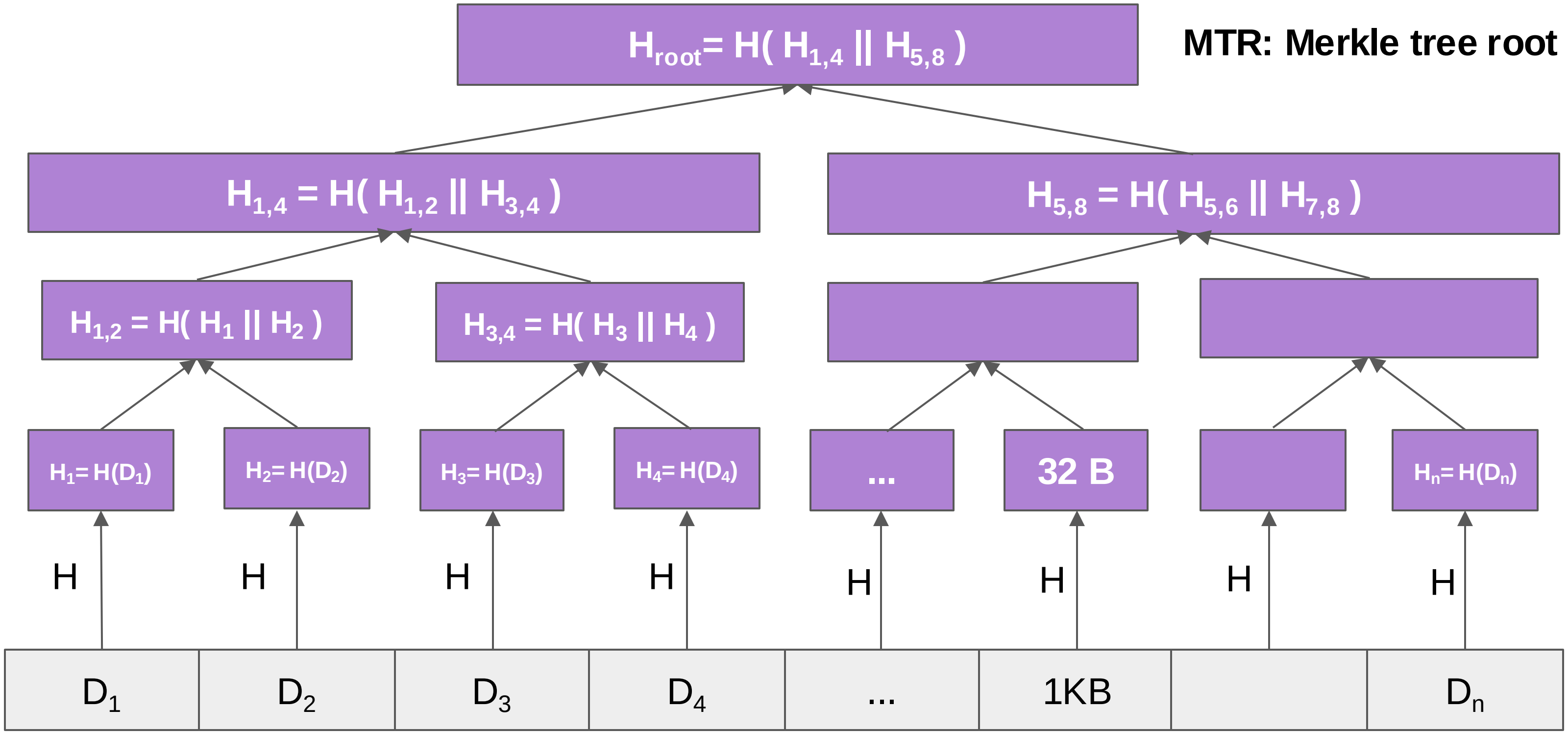


Merkle Tree

- **Combine** them by two to create a binary tree
- Each node stores the **hash** of the **concat** of its children



Merkle Tree



$H_{root}= H(H_{1,4} || H_{5,8})$
MTR: Merkle tree root
$H_{1,4} = H(H_{1,2} || H_{3,4})$
$H_{5,8} = H(H_{5,6} || H_{7,8})$
$H_{1,2} = H(H_1 || H_2)$
$H_{3,4} = H(H_3 || H_4)$
$H_1= H(D_1)$
$H_2= H(D_2)$
$H_3= H(D_3)$
$H_4= H(D_4)$
...
32 B
$H_n= H(D_n)$
H
H
H
H
H
H
H
H
D_1
D_2
D_3
D_4
...
1KB
D_n

File storage: Merkle tree-based protocol

- Client sends file data D to server
- Client creates Merkle Tree root **MTR** from initial file data D
- Client deletes data D, but stores MTR (32 bytes)

File storage: Merkle tree-based protocol

- Client sends file data D to server
- Client creates Merkle Tree root **MTR** from initial file data D
- Client deletes data D, but stores MTR (32 bytes)

Time passes…

- Client requests chunk x from server
- Server returns chunk x and *short* proof-of-inclusion π
- Client checks whether proof π of chunk x is correct w.r.t. stored MTR

Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$
Prover: a, b, c, d, e, f, g, h

Merkle tree: proof of inclusion

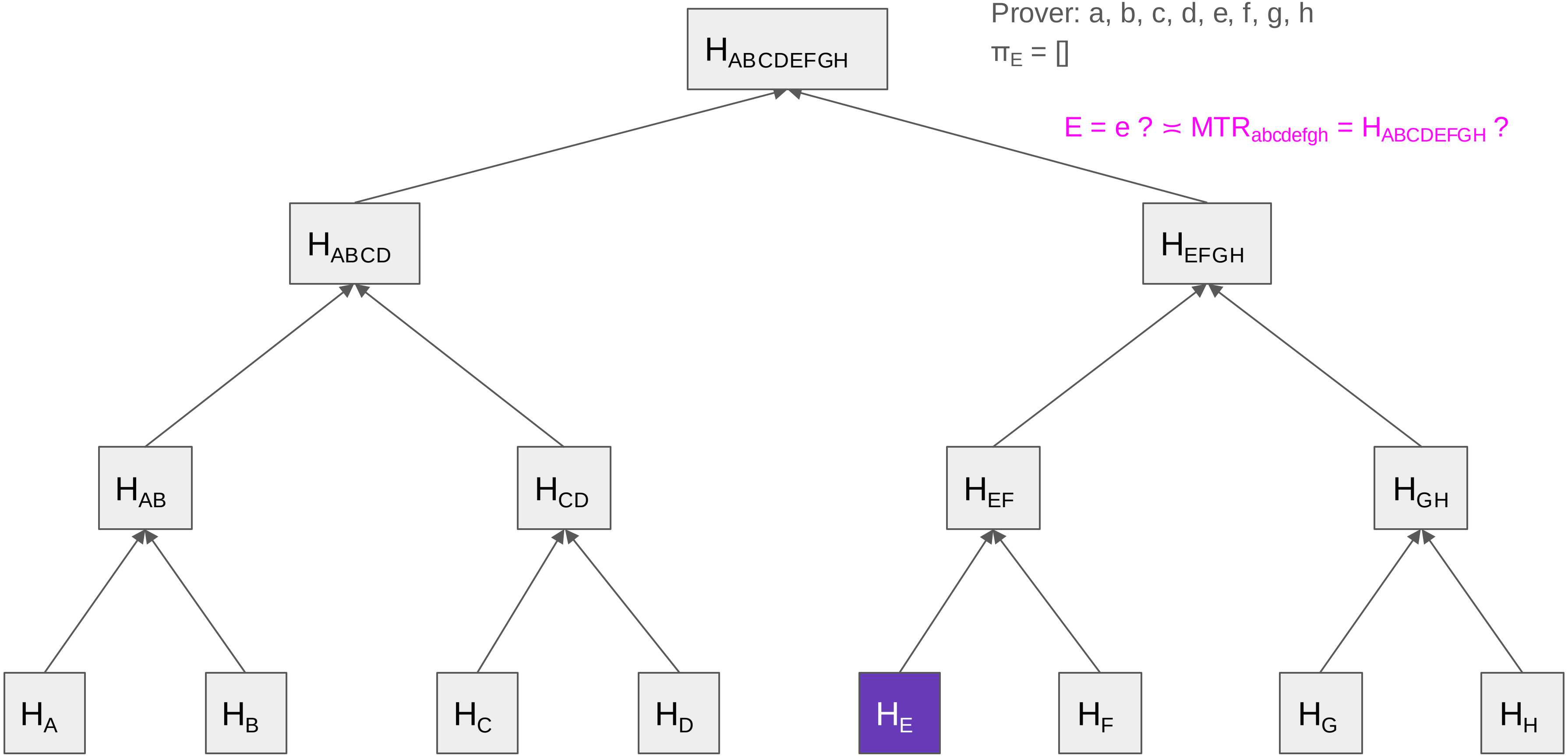
Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h

E = e ?

Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
π_E = []

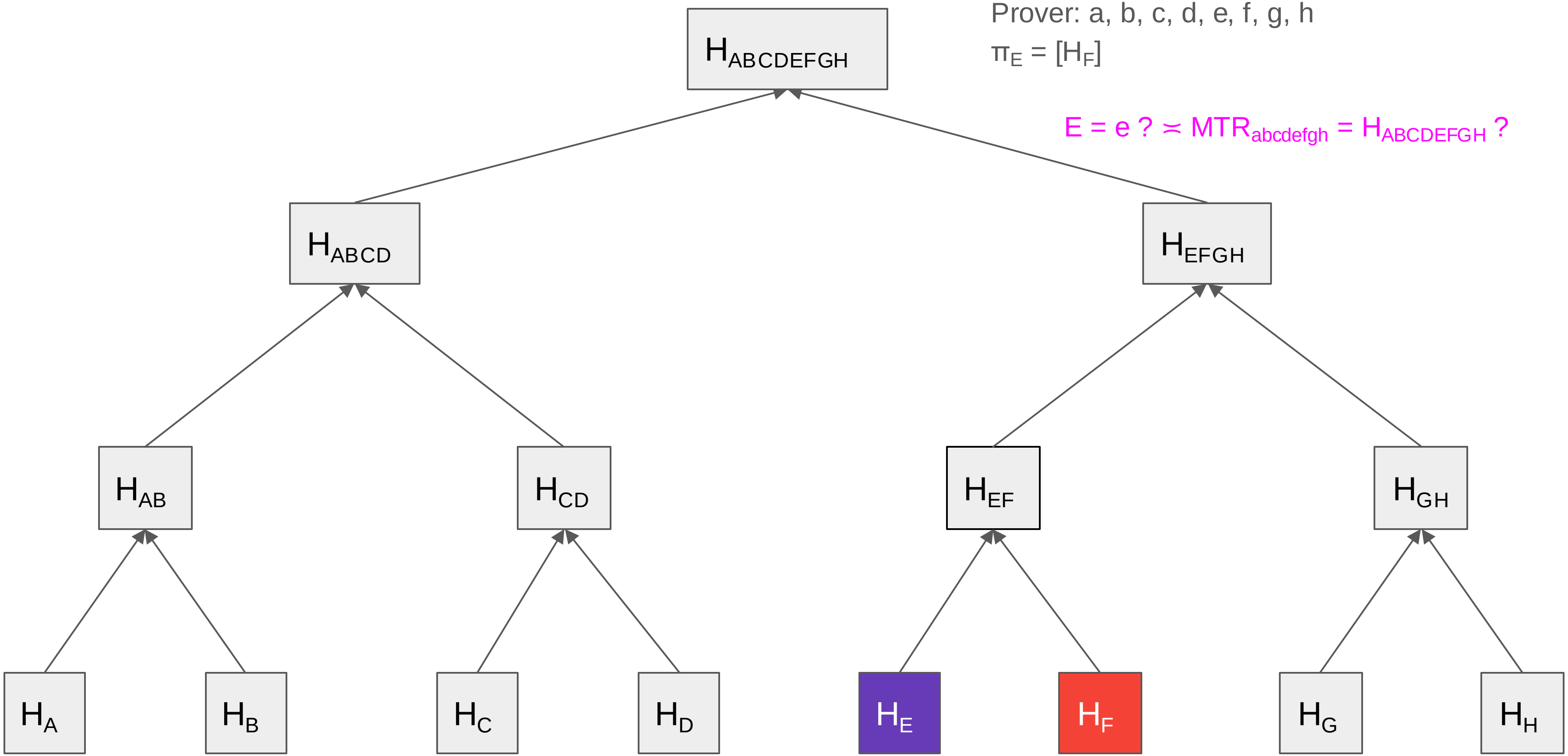
E = e ? ≍ $MTR_{abcdefgh} = H_{ABCDEFGH}$?



Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
$\pi_E = [H_F]$

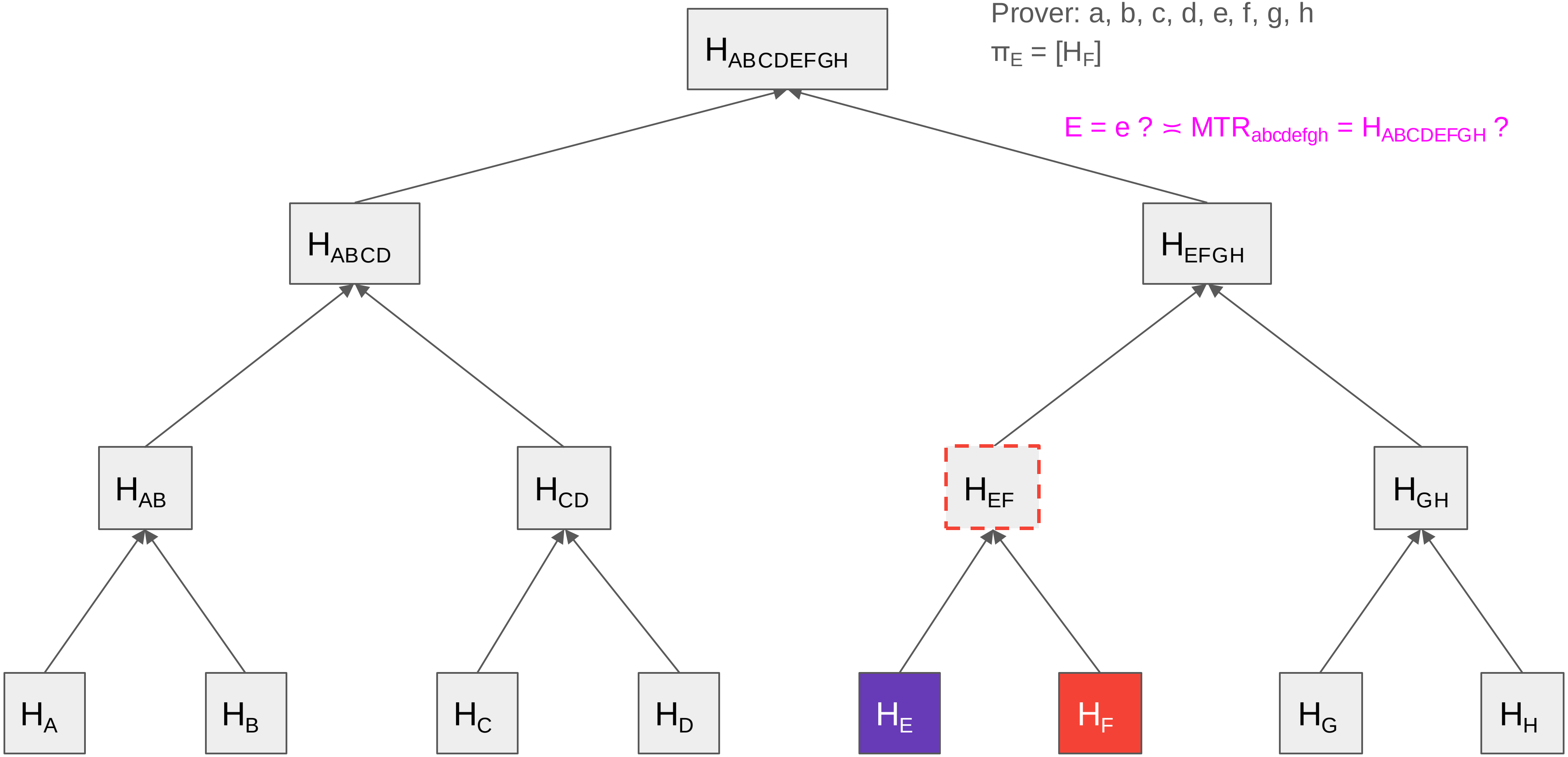
$E = e\ ? \asymp MTR_{abcdefgh} = H_{ABCDEFGH}\ ?$



Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
$\pi_E = [H_F]$

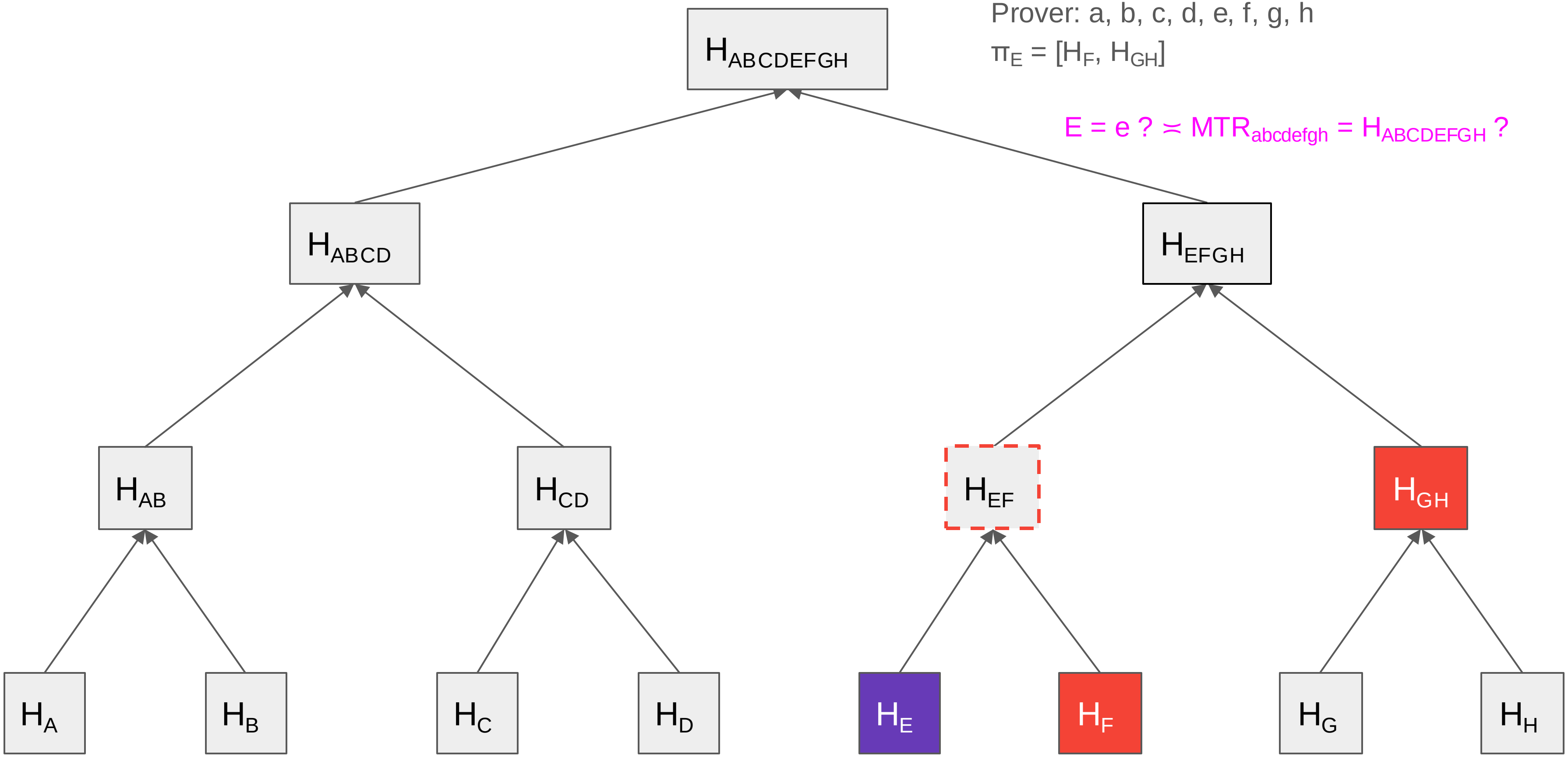
$E = e\ ? \asymp MTR_{abcdefgh} = H_{ABCDEFGH}\ ?$



Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
$\pi_E = [H_F, H_{GH}]$

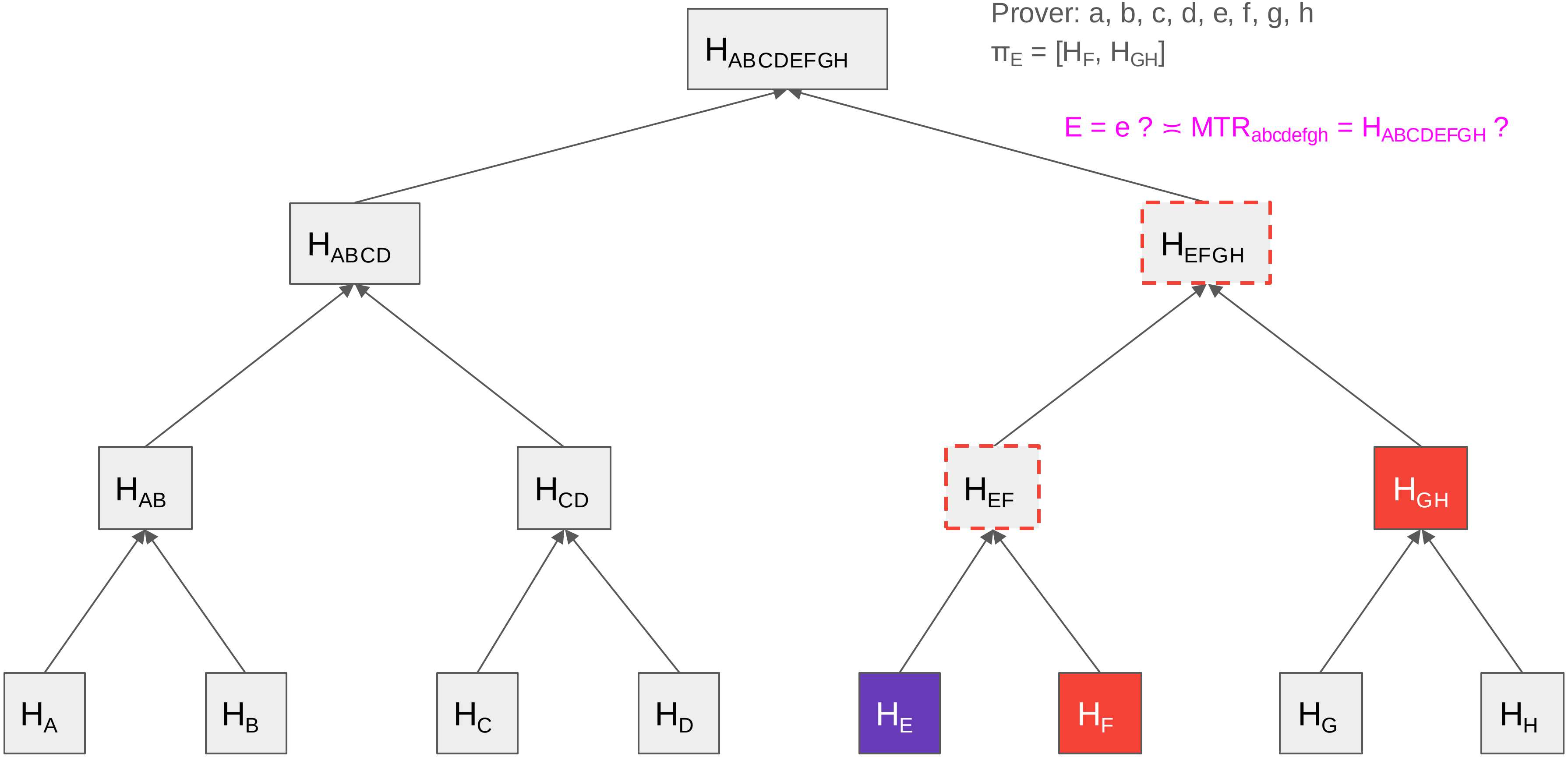
$E = e \; ? \asymp MTR_{abcdefgh} = H_{ABCDEFGH} \; ?$



Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
$\pi_E = [H_F, H_{GH}]$

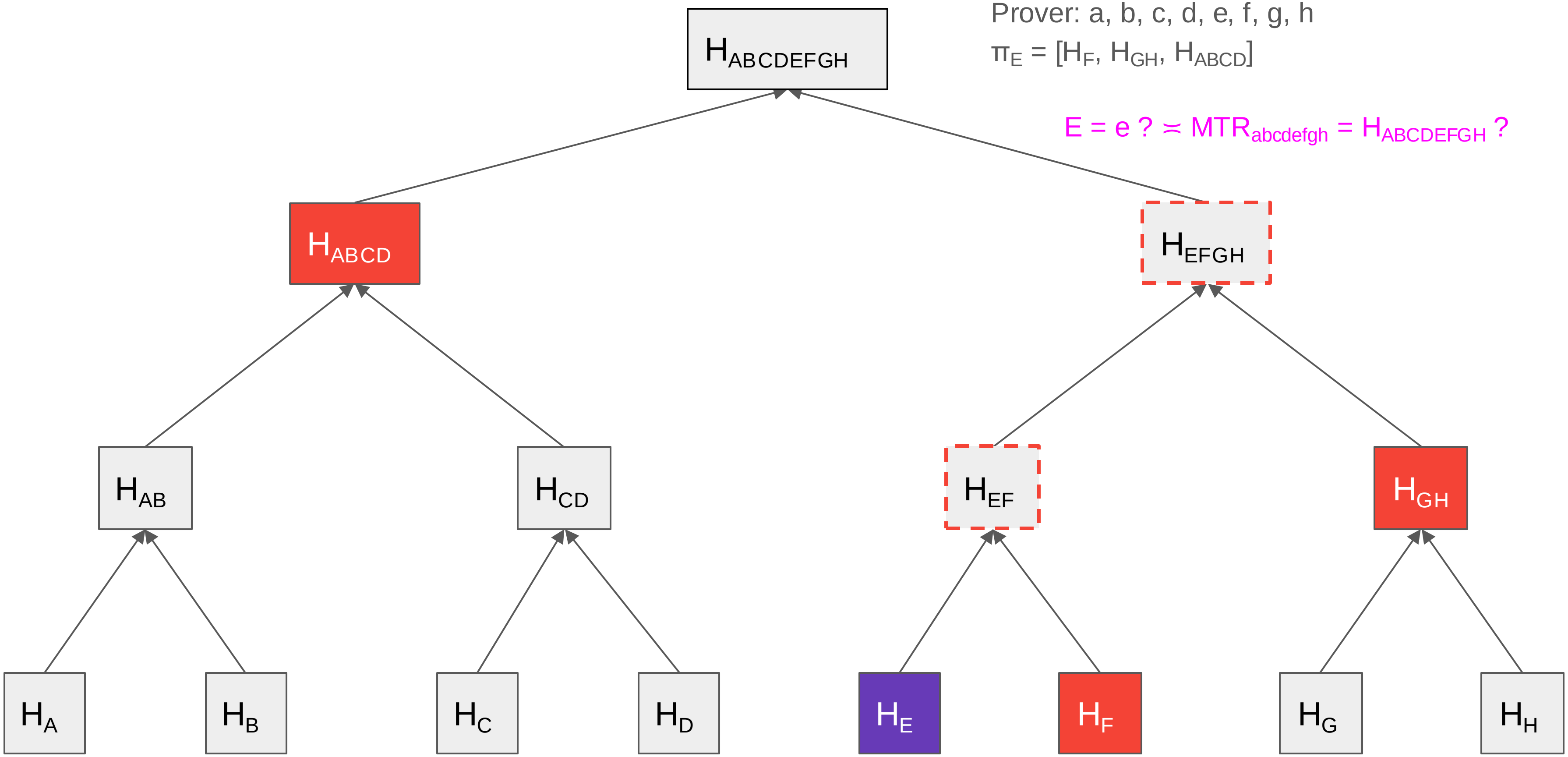
E = e ? ≍ $MTR_{abcdefgh} = H_{ABCDEFGH}$?



Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E
Prover: a, b, c, d, e, f, g, h
$\pi_E = [H_F, H_{GH}, H_{ABCD}]$

$E = e\ ?\ \asymp MTR_{abcdefgh} = H_{ABCDEFGH}\ ?$



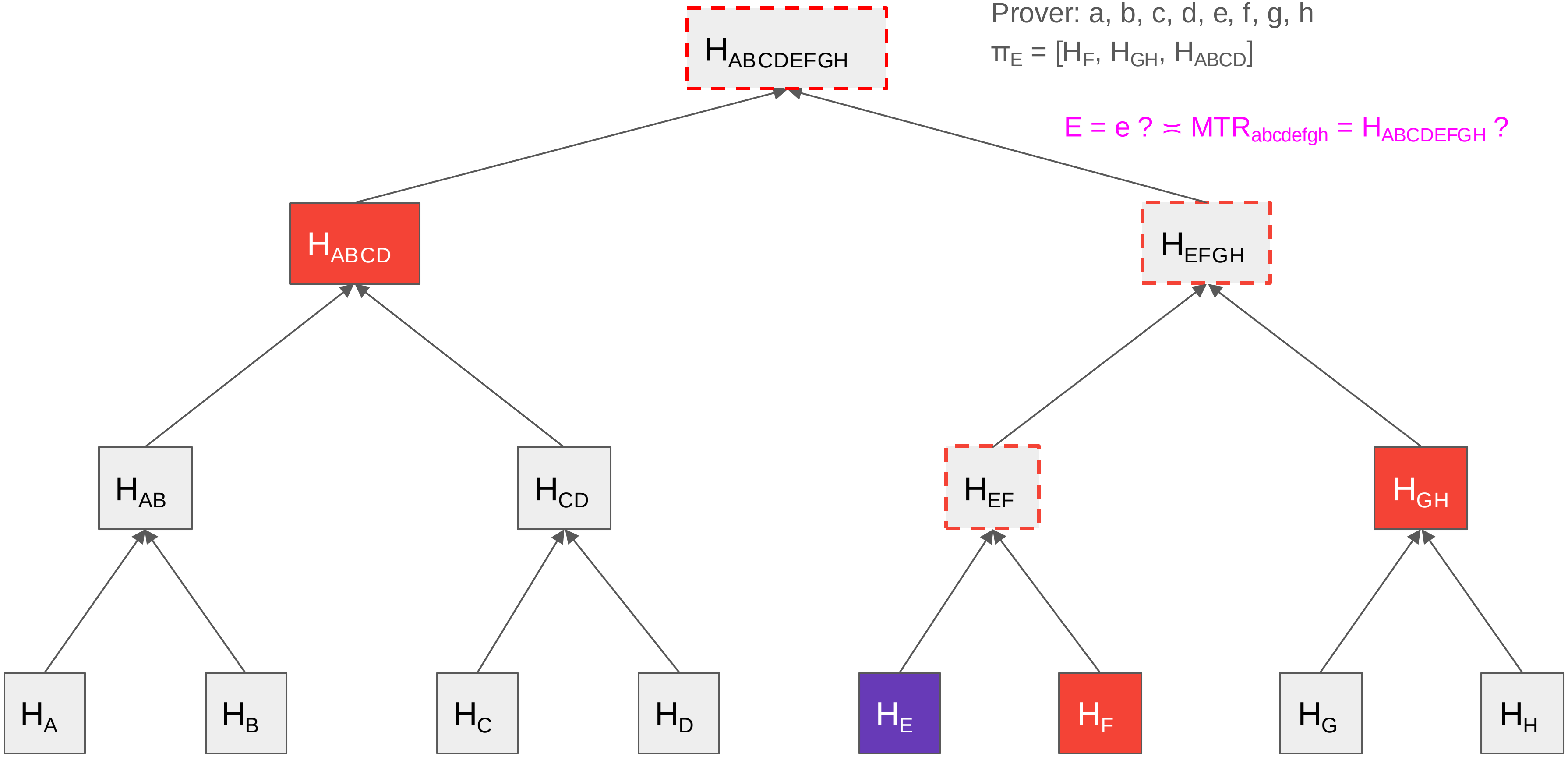
Merkle tree: proof of inclusion

Verifier: $MTR_{abcdefgh}$, E, π_E

Prover: a, b, c, d, e, f, g, h

$\pi_E = [H_F, H_{GH}, H_{ABCD}]$

$E = e\ ?\ \asymp\ MTR_{abcdefgh} = H_{ABCDEFGH}\ ?$



Merkle Tree proof-of-inclusion

- Prover sends chunk
- Prover sends **siblings** along path connecting leaf to MTR
- Verifier computes hashes along the path connecting leaf to MTR
- Verifier checks that computed root is equal to MTR
- How big is proof-of-inclusion?

Merkle Tree proof-of-inclusion

- Prover sends chunk
- Prover sends **siblings** along path connecting leaf to MTR
- Verifier computes hashes along the path connecting leaf to MTR
- Verifier checks that computed root is equal to MTR
- How big is proof-of-inclusion?

$$|\pi| \in \Theta(\log_2 |D|)$$

Merkle tree applications

- BitTorrent uses Merkle trees to verify exchanged files
- Bitcoin uses Merkle trees to store transactions
- Ethereum uses Merkle-Patricia tries for storage and transactions

Storing *sets* instead of files/lists

- Merkle trees can be used to store *sets* of keys instead of lists
- Verifier asks prover to store a set
- Verifier deletes set
- Verifier later asks prover if key belongs to set
- Prover provides proof-of-inclusion or proof-of-non-inclusion
- Prover can be adversarial

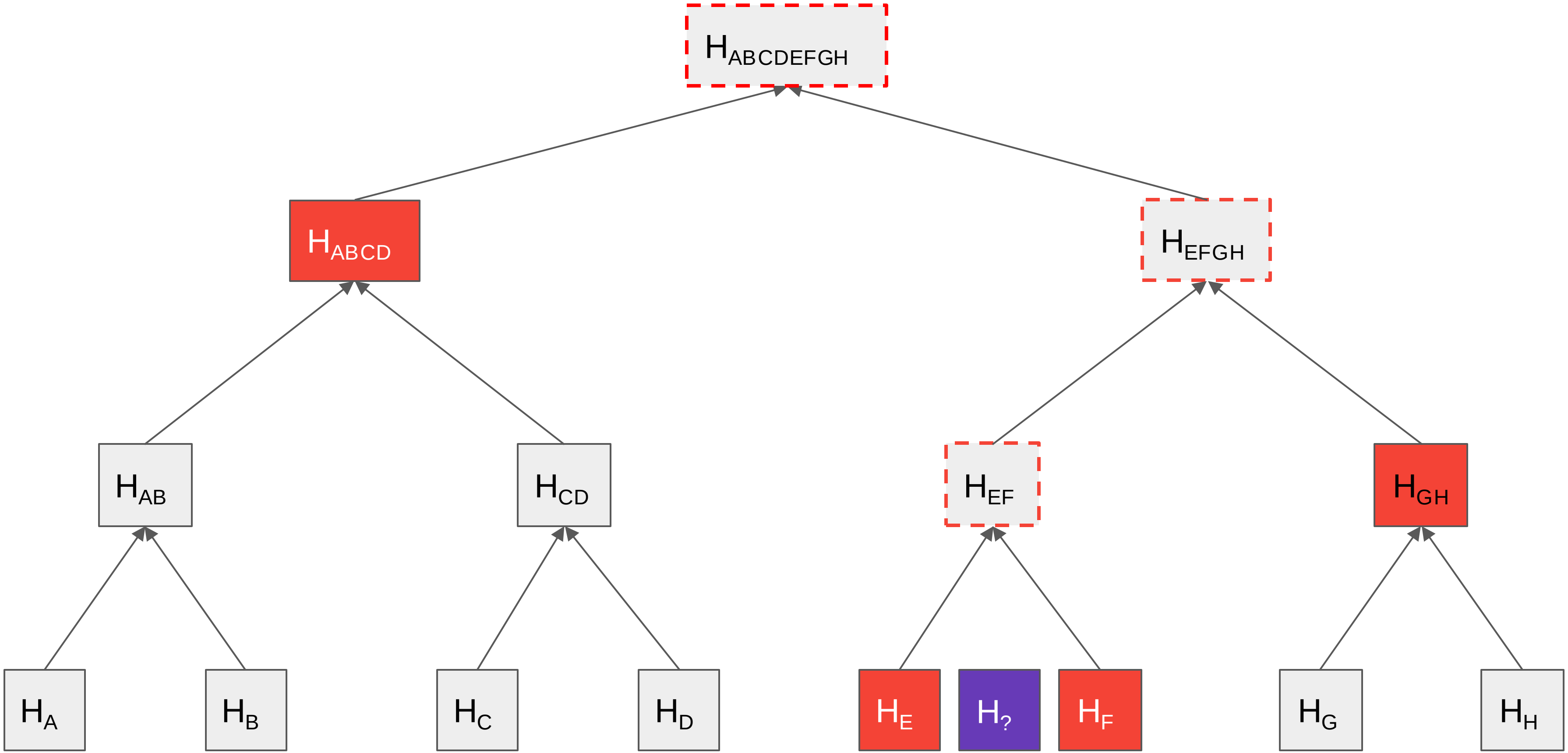
Merkle trees for set storage

- Verifier sorts set elements
- Creates MTR on sorted set
- Proof-of-inclusion as before

Merkle trees for set storage

- Verifier sorts set elements
- Creates MTR on sorted set
- Proof-of-inclusion as before
- Proof-of-non-inclusion for x
    - Show proof-of-inclusion for previous $H_<$ and next $H_>$ element in set
    - Verifier checks that $H_<$, $H_>$ proofs-of-inclusion are correct
    - Verifier checks that $H_<$, $H_>$ are adjacent in tree
    - Verifier checks that $H_< < x$ and $H_> > x$
    - Question: How to compress the two proofs-of-inclusion into one?

Merkle tree: proof of inclusion / non-inclusion



Tries

Tries

- Also called radix or prefix tree
- Search tree: ordered data structure
- Used to store an associative array (key/value store)
    - <key, value>, <key, value> …
- Keys are usually strings

Tries

- **Initialize**: Start with empty root
- Supports two operations: **add** and **query**
- **add** adds a <key,value> pair to the set
- **query** checks if a key is in the set and returns its value

Tries / Patricia tries as key/value store

- Marking can contain arbitrary value
- This allows to map keys to values
- **add(key, value)**
- **query(key) → value**

Tries: add(<key,value>)

- Start at root
- Split key string into characters
- For every character, follow an edge labelled by that character
- If edge does not exist, create it
- Mark the node you arrive by value

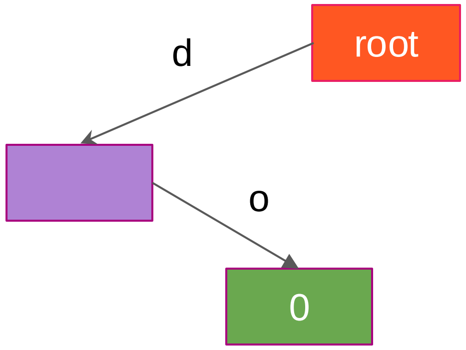
Tries: query(key)

- Start at root
- Split key into characters
- For every character, follow an edge labelled by that character
- If edge does not exist, return **false**
- When you arrive at a node and your string is consumed, check if node is marked
    - If it is marked, return marked **value**
    - Otherwise, return **false**

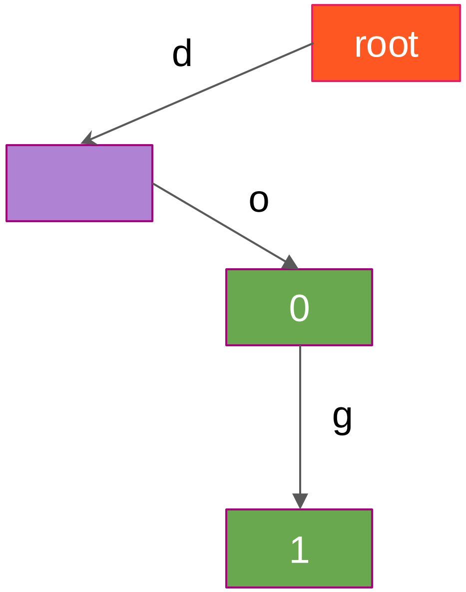
{ }

root

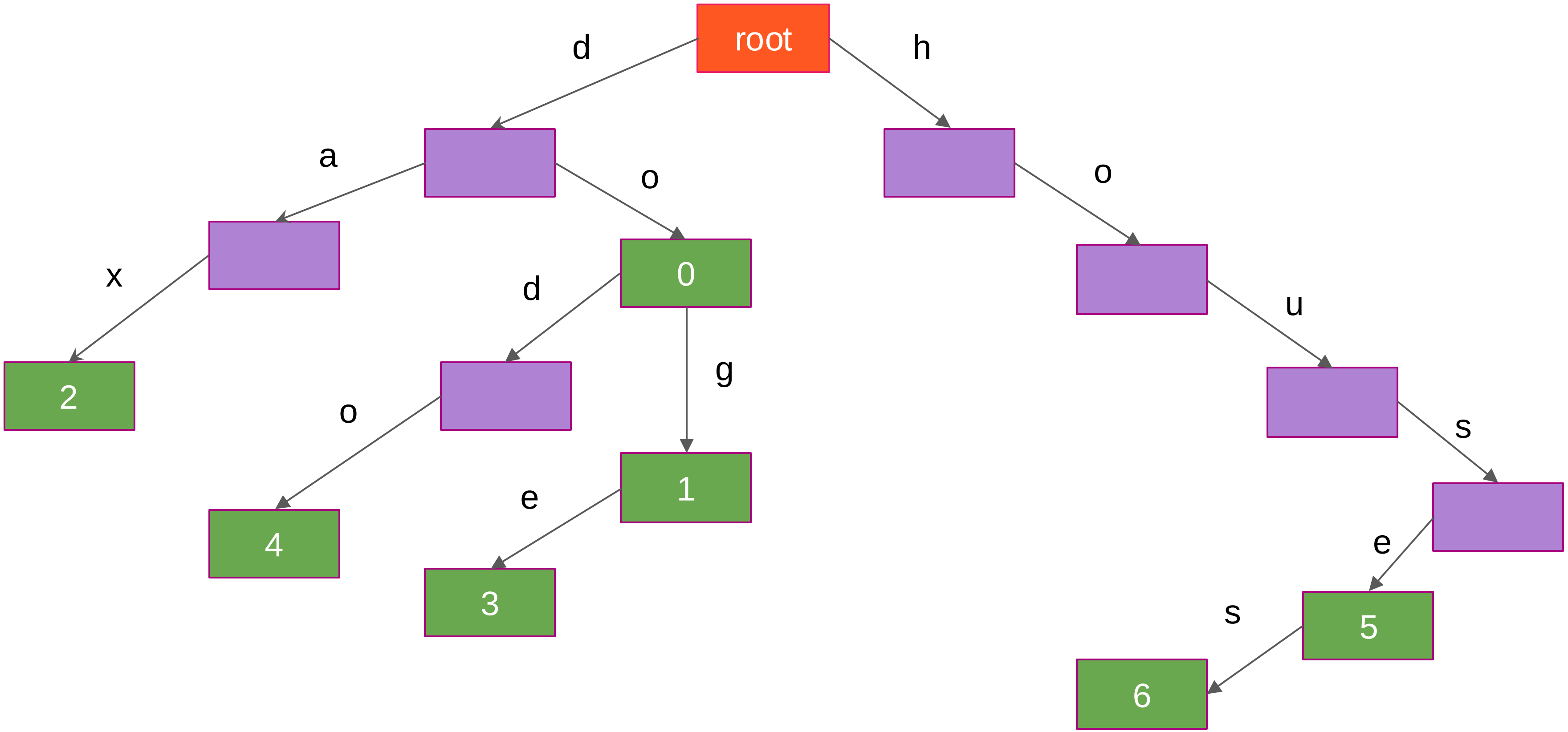
{ **do**: 0 }



{ **do**: 0, **dog**: 1 }



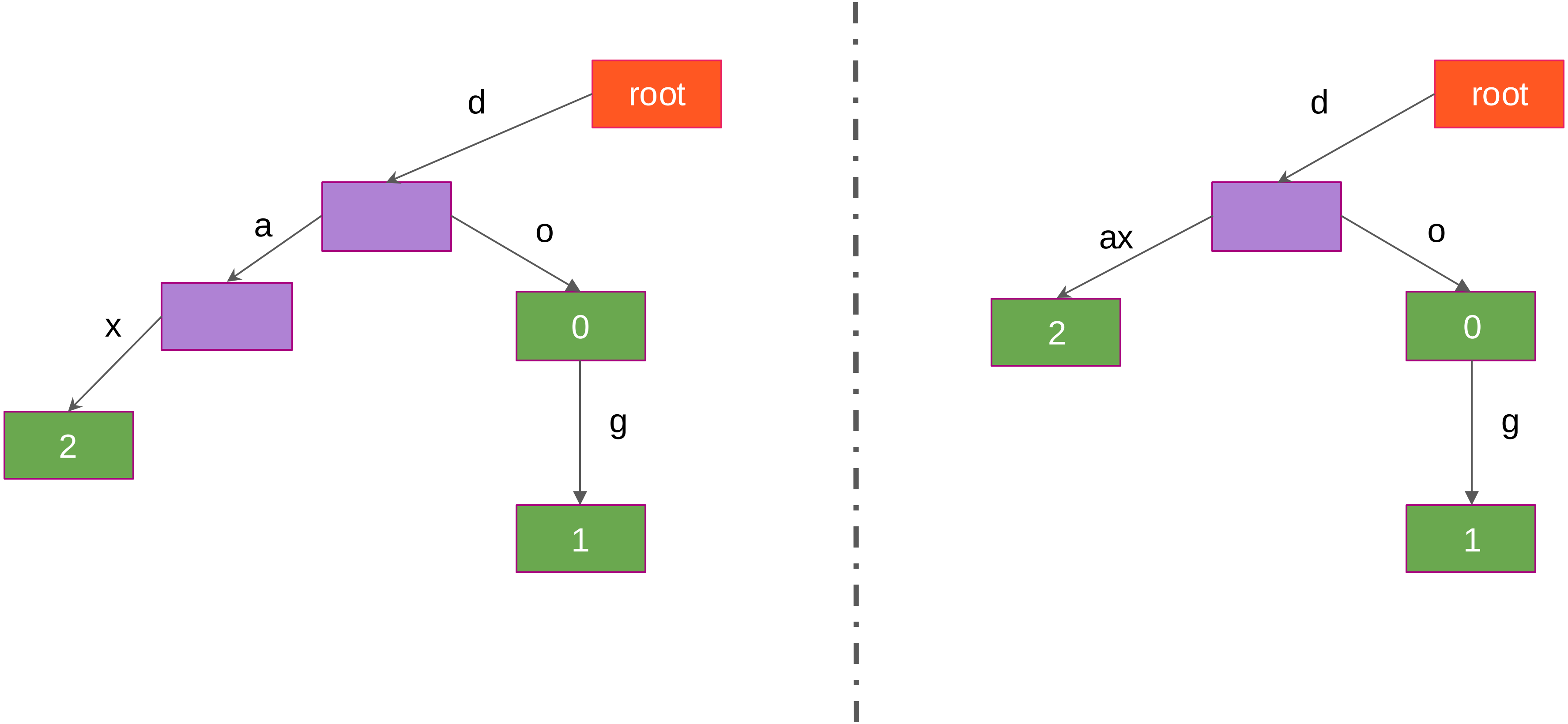
{ **do**: 0, **dog**: 1, **dax**: 2, **doge**: 3, **dodo**: 4, **house**: 5, **houses**: 6 }



Patricia (or radix) tree

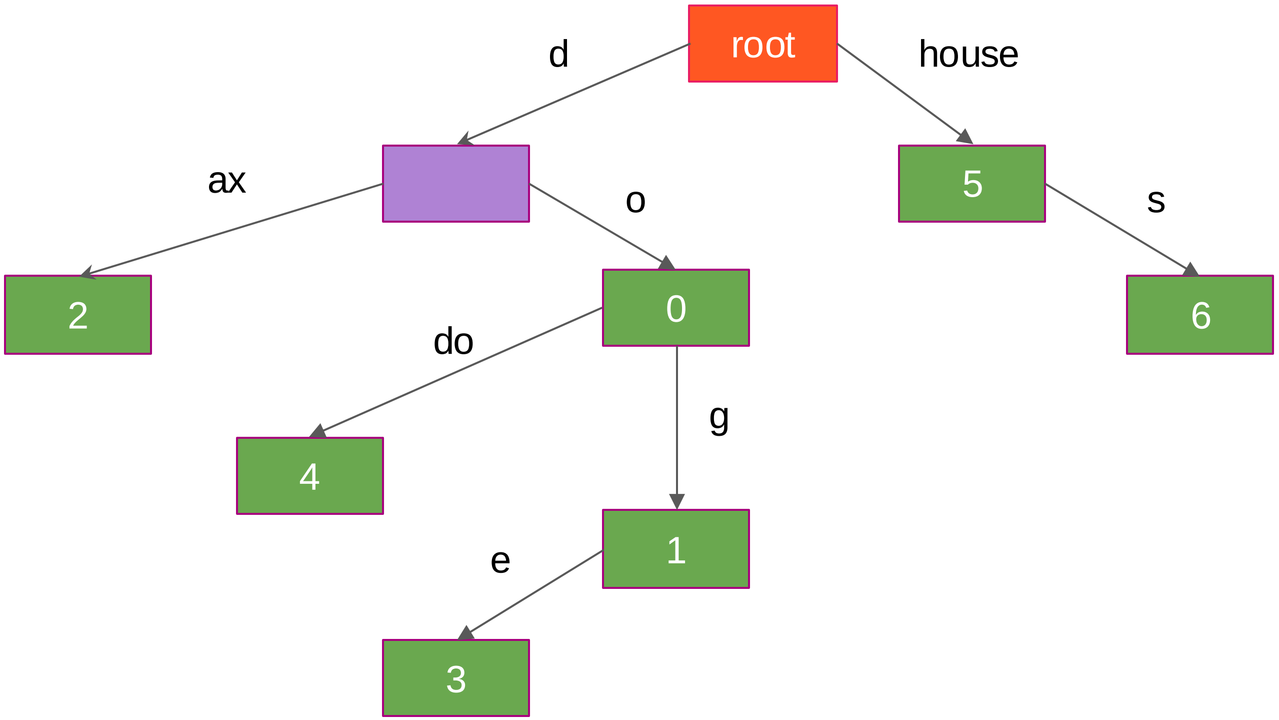
- Space-optimized trie
- An isolated path, with *unmarked* nodes which are *only children*, is merged into single edge
- The label of the merged edge is the concatenation of the labels of merged nodes

Trie vs. Patricia trie



Patricia trie

{ **do**: 0, **dog**: 1, **dax**: 2, **doge**: 3, **dodo**: 4, **house**: 5, **houses**: 6 }

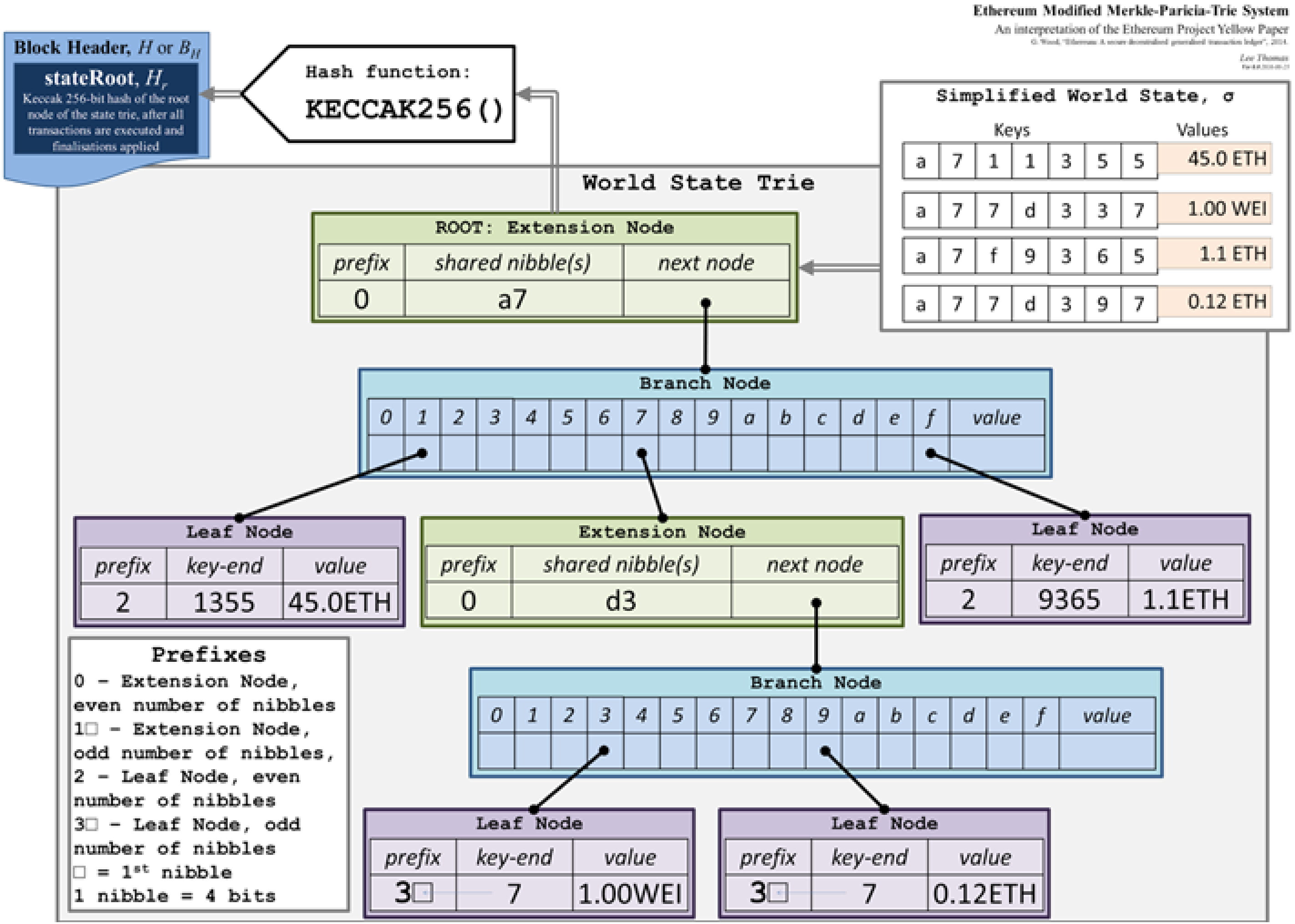


Merkle Patricia trie

- Authenticated Patricia trie
- First implemented in Ethereum
- Allows proof of inclusion (of key, with particular value)
- Allows proof of non-inclusion (by showing key does not exist in trie)

Merkle Patricia trie

- Split nodes into three types:
  - **Leaf**: Stores edge string leading to it, and **value**
  - **Extension**: Stores **string** of a single edge, **pointer** to next node, and **value** if node marked
  - **Branch**: Stores one pointer to another node per alphabet symbol, and **value** if node marked
- Encode keys as hex, so alphabet size is 16
- Encode all child edges in every node with some encoding (e.g., JSON)
- Pointers are by hash application (authenticated inclusion)
- Arguments for correctness and security are same as for Merkle Trees

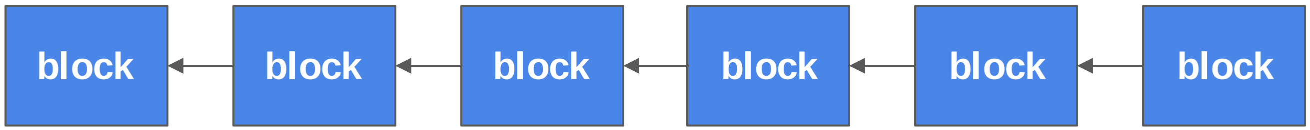


Ethereum Modified Merkle-Paricia-Trie System
An interpretation of the Ethereum Project Yellow Paper
Lee Thomas
Block Header, H or B_H
stateRoot, H_r
Keccak 256-bit hash of the root node of the state trie, after all transactions are executed and finalisations applied
Hash function:
KECCAK256()
Simplified World State, σ
Keys
Values
a 7 1 1 3 5 5
45.0 ETH
a 7 7 d 3 3 7
1.00 WEI
a 7 f 9 3 6 5
1.1 ETH
a 7 7 d 3 9 7
0.12 ETH
World State Trie
ROOT: Extension Node
prefix
shared nibble(s)
next node
0
a7
Branch Node
0 1 2 3 4 5 6 7 8 9 a b c d e f
value
Leaf Node
prefix
key-end
value
2
1355
45.0ETH
Extension Node
prefix
shared nibble(s)
next node
0
d3
Leaf Node
prefix
key-end
value
2
9365
1.1ETH
Prefixes
0 – Extension Node, even number of nibbles
1□ – Extension Node, odd number of nibbles,
2 – Leaf Node, even number of nibbles
3□ – Leaf Node, odd number of nibbles
□ = 1st nibble
1 nibble = 4 bits
Branch Node
0 1 2 3 4 5 6 7 8 9 a b c d e f
value
Leaf Node
prefix
key-end
value
3□
7
1.00WEI
Leaf Node
prefix
key-end
value
3□
7
0.12ETH

Authenticated data in blockchains

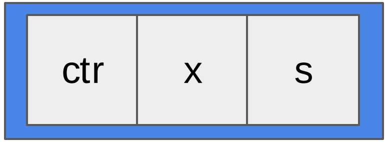
Blockchain

- Each block references a **previous** block
- This reference is by **hash** to its **previous** block
- This linked list is called the **blockchain**
- Blocks contain list of **transactions** (more on this later)



*Convention: Arrows show authenticated inclusion

Blocks



- Data structure with three parts:
  - nonce (ctr), data (**x**), reference (s)
  - Typically called the **block header**
- data (**x**) is application-dependent
  - In Bitcoin it stores financial data (“UTXO”-based)
  - In Ethereum it stores contract data (account-based)
- Block validity:
  - Data must be valid (application-defined validity)
- *s*: pointer to the previous block by hash

Proof-of-work in blocks

- Blocks must satisfy proof-of-work equation

$$H(ctr \,||\, \mathbf{x} \,||\, s) <= T$$

  for some (protocol-parameter) T

- ctr is the nonce used to solve Proof-of-work
- The value H(ctr || x || s) is known as the **blockid**

Bitcoin at a high level

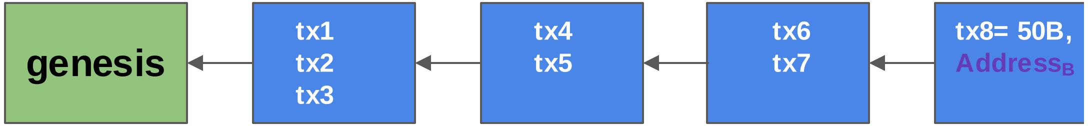
1. New transactions are broadcast to all nodes.
2. Each node collects new transactions into a block.
3. Each node works on finding a difficult proof-of-work for its block.
4. When a node finds a proof-of-work, it broadcasts the block to all nodes.
5. Nodes accept the block only if all transactions in it are valid and not already spent.
6. Nodes express their acceptance of the block by working on creating the next block in the chain, using the hash of the accepted block as the previous hash.

Digital Signature Scheme

- Three algorithms: **KeyGen, Sign, Verify**
- **KeyGen**
  - Input: *security parameter* (bits of security)
  - Output: a pair of keys <sk, vk> (sk: signing/private key, vk: verification/ public – key)
- **Sign**
  - Input: <sk, m> (m: message)
  - Output: σ (σ: signature)
- **Verify**
  - Input: <vk, m, σ>
  - Output: {True, False}

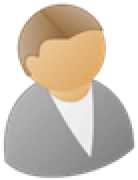
Blockchain

- The **first** block of a blockchain is called the Genesis Block



High level idea (more details later)

PK_B, SK_B



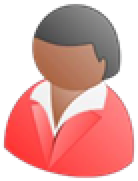
$H(PK_B)=Address_B$

m=I want to give 50 bitcoin to Alice $Address_A$

s_B**=Sign(**SK_B**,**m**)**

$tx11=(m,s_B)$

PK_A, SK_A



$H(PK_A)=Address_A$

Transactions

A simple transaction for financial data

- Input: contains a proof of spending an existing UTxO*
- Output: contains a verification procedure and a value

*UTxO = “Unspent Transaction Output”

Field	Description
In-counter	positive integer
list of inputs	the first input of the first transaction is also called "coinbase"
Out-counter	positive integer
list of outputs	the outputs of the first transaction spend the mined bitcoins for the block

Transactions

Input

Field	Description
Outpoint hash	The previous transaction that contains the spendable output
Outpoint index	The index within the previous transaction's output array to identify the spendable output
Script signature (ScriptSig)	Information required to spend the output (see below for details)

Output

Field	Description
Value	The monetary value of the output in satoshis
Script (ScriptPubKey)	A calculation which future transactions need to satisfy in order to spend it

Transaction Verification

scriptSig (input): <**sig**> <**pubKey**>

scriptPubKey (output): OP_DUP OP_HASH160 <**pubKeyHash**> OP_EQUALVERIFY OP_CHECKSIG

Transaction Verification

scriptSig (input): <**sig**> <**pubKey**>

scriptPubKey (output): OP_DUP OP_HASH160 <**pubKeyHash**> OP_EQUALVERIFY OP_CHECKSIG

0th output in the previous transaction

ata

```
Input:
Previous tx: f5d8ee39a430901c91a5917b9f2dc19d6d1a0e9cea205b009ca73dd04470b9a6
Index: 0
scriptSig: 304502206e21798a42fae0e854281abd38bacd1aeed3ee3738d9e1446618c4571d10
90db022100e2ac980643b0b82c0e88ffdfec6b64e3e6ba35e7ba5fdd7d5d6cc8d25c6b241501

Output:
Value: 5000000000
scriptPubKey: OP_DUP OP_HASH160 404371705fa9bd789a2fcd52d2c580b65d35549d
OP_EQUALVERIFY OP_CHECKSIG
```

Hash of the recipient's public key

The input in this transaction imports 50 BTC from output #0 in transaction f5d8... Then the output sends 50 BTC to a Bitcoin address. When the recipient wants to spend this money, he will reference output #0 of this transaction in an input of his own transaction.

Transaction Verification

tx10

Input:

…

Output:

Value: 5000000000

scriptPubKey: OP_DUP OP_HASH160 **Address$_B$**

OP_EQUALVERIFY OP_CHECKSIG

tx11

Input:

Previous tx: **tx10**

Index: **0**

scriptSig: **s_B** **PK_B**

Output:

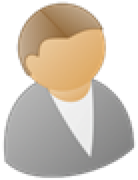
Value: 4000000000

scriptPubKey: OP_DUP OP_HASH160 **Address$_A$**

OP_EQUALVERIFY OP_CHECKSIG

PK_B, SK_B

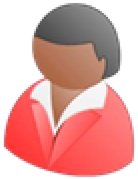
m=H(output* from **tx10**)



s_B=Sign(SK_B,m)

H(PK_B)=Address$_B$

PK_A, SK_A



H(PK_A)=Address$_A$

Data and Transactions

- Financial data is encoded in the form of *transactions*
- Each block organizes transactions in an authenticated data structure
  - Bitcoin: Merkle Tree
  - Ethereum: Merkle Patricia Trie
- Every transaction is sent on the network to everyone via a gossip protocol

- Question: Is it necessary to download the entire block (header + transactions) to verify whether a transaction is included in it?

The Bitcoin network

The bitcoin network

- All bitcoin nodes connect to a common p2p network
- Each node runs (code that implements) the Bitcoin protocol
- Open source code
- Each node connects to its (network) neighbours
- They continuously exchange data
- Each node can **freely** enter the network – no permission needed!
    - A “permissionless network”
- **The adversarial assumption:**
  There is no trust placed on any specific node or participant, anyone individually may lie

Peer discovery

- Each node stores a list of peers (by IP address)
- When Alice connects to Bob, Bob sends Alice his own known peers
- That way, Alice can learn about new peers

Bootstrapping the p2p network

- Peer-to-peer nodes come “pre-installed” with some peers by IP / host
- When running a node, you can specify extra “known peers”

The *gossip* protocol

- **Alice** generates some new data
- Alice **broadcasts** data to its peers
- Each peer multicasts this data to *its* peers
- If a peer has seen this data before, it ignores it
- If this data is new, it multicasts it to its peers
- That way, the data spreads like an epidemic, until the whole network learns it
- This process is called peer to peer **diffusion**

Eclipse attacks

- Isolate some honest nodes in the network, effectively causing a “network split” in two partitions A and B
- If peers in A and peers in B are disjoint and don’t know about each other, the networks will remain isolated
    - Highlight: “liveness favoring operation”

- The connectivity assumption:
    - There is a path between two nodes on the network
    - **If a node broadcasts a message, every other node *will* learn it**

Summary: what we learned

- Hash functions and signatures: useful primitives, and building blocks for more complex protocols
- Authenticated data structures
    - Merkle trees
    - Tries / Patricia Merkle Trees
- Bitcoin
    - Blockchain Data Structure
    - Transactions
    - Payments

Thanks

Additional material

- A primer in cryptographic proofs of security

Merkle Tree proof-of-inclusion security

- If adversary can present proof-of-inclusion for incorrect leaf, then we can break the hash function
- Proof is by computational reduction

Merkle Tree protocol

`MT-construct(D)`

- Construct a Merkle Tree with given data D
- Returns the Merkle Tree root
- If |D| <= chunk size, then: `MT-construct(D) = H(D)`
- Otherwise:
  $\texttt{MT-construct(D) = H(MT-construct(}D_1\texttt{) || MT-construct(}D_2\texttt{))}$
  where
  - $D = D_1 \,||\, D_2$ and $|D_1|$ has length a multiple of chunk size that is no less than |D|/2

Merkle Tree protocol

`MT-prove(D, x)`

- Given data D and element x in D, construct proof-of-inclusion
- Returns the proof-of-inclusion π to be used with `MT-construct(D)`
- Proof contains:
  - Siblings on path connecting x to root
  - A bit for each sibling indicating whether the path we are taking is left or right

Merkle Tree protocol

`MT-verify(r, π, x)`

- Given Merkle Tree root r, element x, and proof-of-inclusion π
- Outputs true/false based on whether verification was successful

Correctness

For all D, x:

`MT-verify(MT-construct(D), MT-prove(D, x), x) = True`

(Proof by direct application of hashes on path)

Proof-of-inclusion security

- Assume the hash function is **collision-resistant**
- Collision resistance formal definition:
  - $\forall$ PPT A: $\exists$ negl: $\Pr[\text{coll-find}_{A,H}(\lambda)] \leq \text{negl}(\lambda)$
- Where coll-find is the collision finding game:

```
def coll-find_{A,H}(λ):
    x_1, x_2 ← A(1^λ)
    if x_1 ≠ x_2 ∧ H(x_1) = H(x_2):
        return 1
    return 0
```

Threat modelling with bad events

- When defining a security property precisely, specify what **bad event** we are trying to avoid
- In this case, the construction of a proof about a non-existent element
- It is important to allow the adversary to **choose** which Merkle Tree to attack
- It is possible that the vast majority of trees are not attackable…
- Hence, we define a **game** where the adversary chooses a data set D to construct the tree from, an element x, and a proof of π
- The adversary can construct these arbitrarily. π does not need to be produced out of a tree!

The Merkle Tree forgery game

```
def MT-forgery_{A,Π(H)}(λ):
    (D, x, π) ← A(1^λ)
    if MT-verify(MT-construct(D), π, x) ∧ x ∉ D:
        return 1
    return 0
```

The Merkle Tree security

$\forall$ PPT A: $\exists$ negl: $\Pr[\text{MT-forgery}_{A,\Pi(H)}(\lambda)] \leq \text{negl}(\lambda)$

The theorem: Assumption → Desirable

Theorem: If H is collision-resistant, then the MT constructed from H is secure:

$$\forall \text{ PPT } A: \exists \text{ negl}: \Pr[\text{coll-find}_{A,H}(\lambda)] \leq \text{negl}(\lambda)$$

$$\rightarrow$$

$$\forall \text{ PPT } A: \exists \text{ negl}: \Pr[\text{MT-forgery}_{A,\Pi(H)}(\lambda)] \leq \text{negl}(\lambda)$$

Proof strategy: Contraposition

By reductio ad absurdum using contraposition:

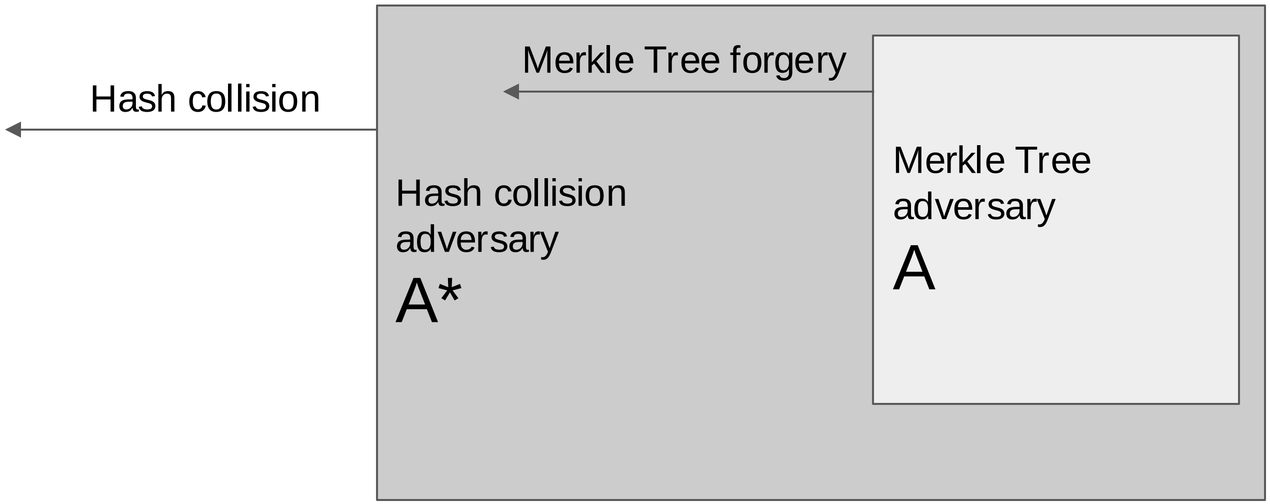
- Suppose for contradiction that
  not $\forall$ `PPT A:` $\exists$ `negl: Pr[MT-forgery`$_{A,\Pi(H)}$`(λ)] ≤ negl(λ)`
  i.e., the Merkle Tree construction is **not** secure
- It suffices to show that
  not $\forall$ `PPT A:` $\exists$ `negl: Pr[coll-find`$_{A,H}$`(λ)] ≤ negl(λ)`
  i.e., the hash function is **not** collision-resistant

Proof strategy: Contraposition

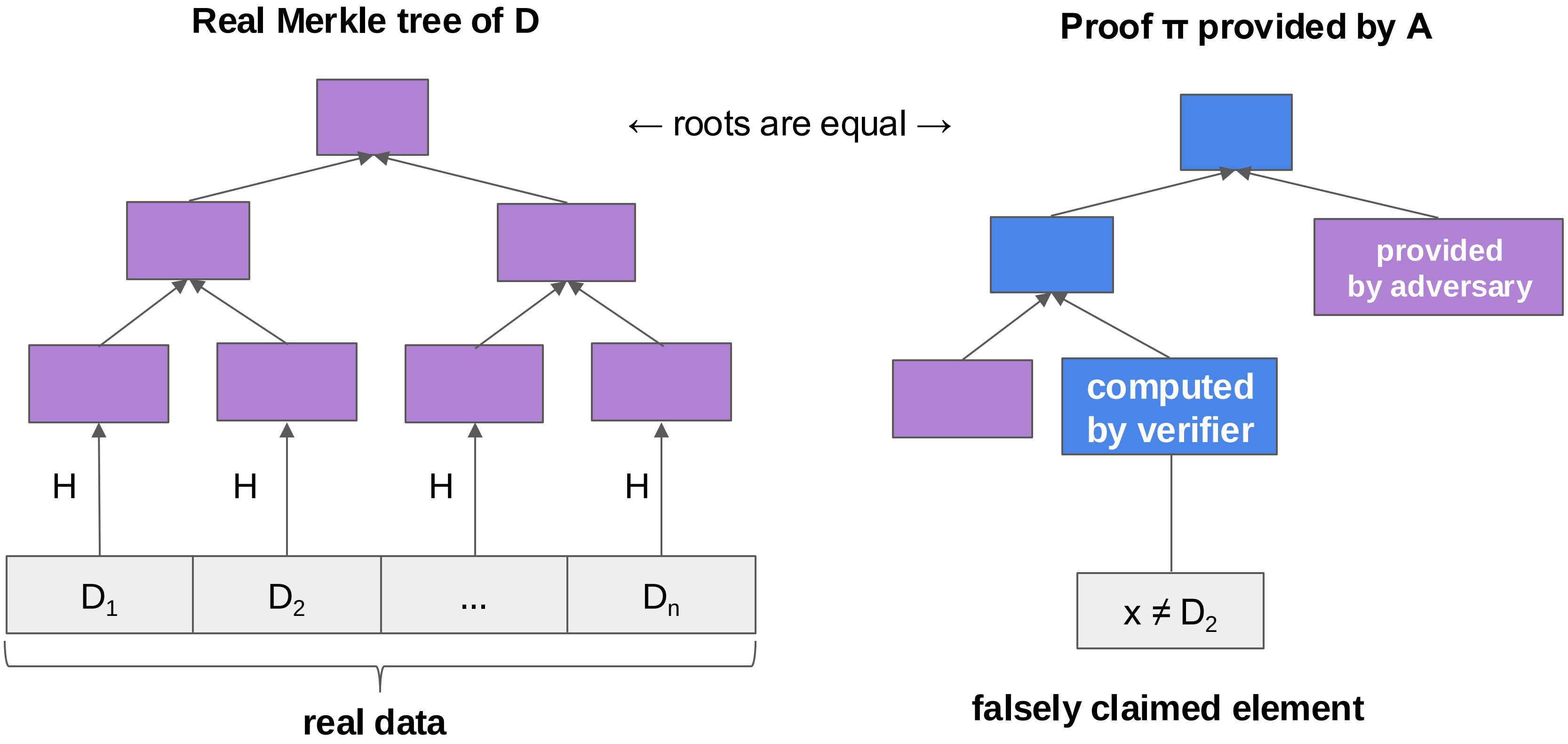
- Suppose for contradiction that
  $\exists$ PPT A: $\Pr[\text{MT-forgery}_{A,\Pi(H)}(\lambda)]$ is non-negl
- It suffices to show that
  $\exists$ PPT A*: $\Pr[\text{coll-find}_{A,H}(\lambda)]$ is non-negl

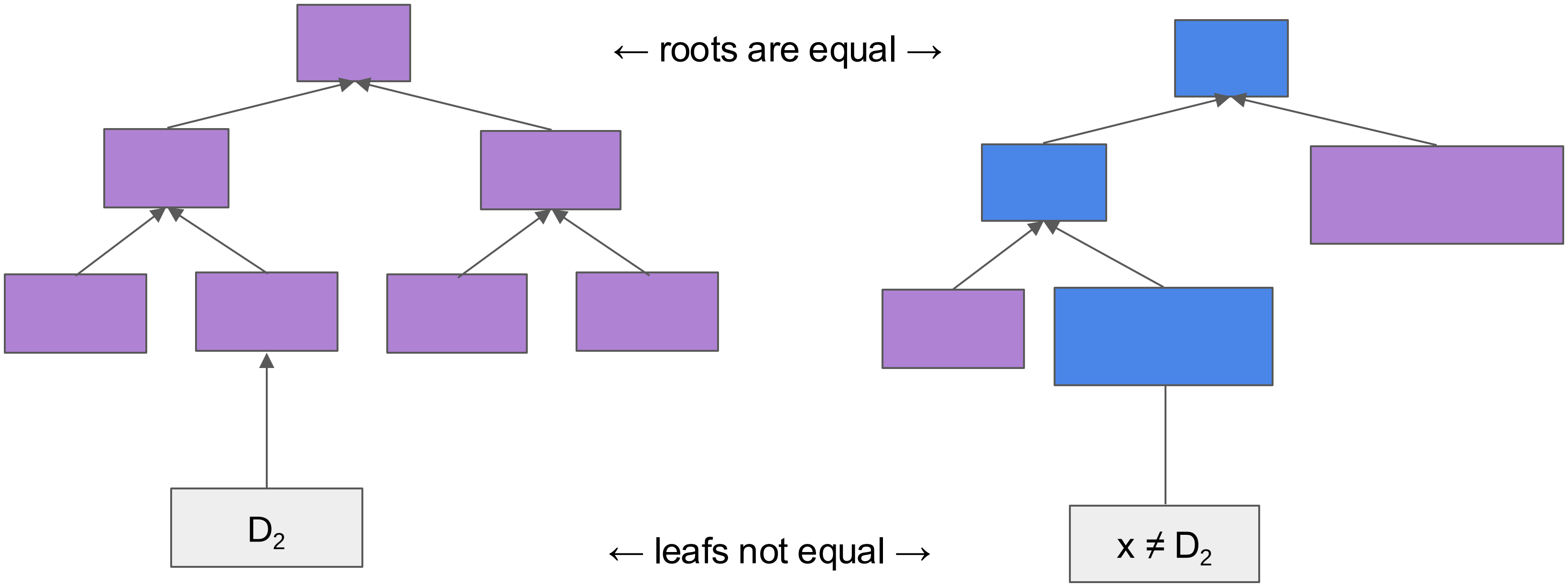
The PPT A is arbitrary, so we must use it as black box.
We show the existence of A* by construction.
Since A is a machine, we can have A* call A in its code.

Proof strategy: Computational reduction

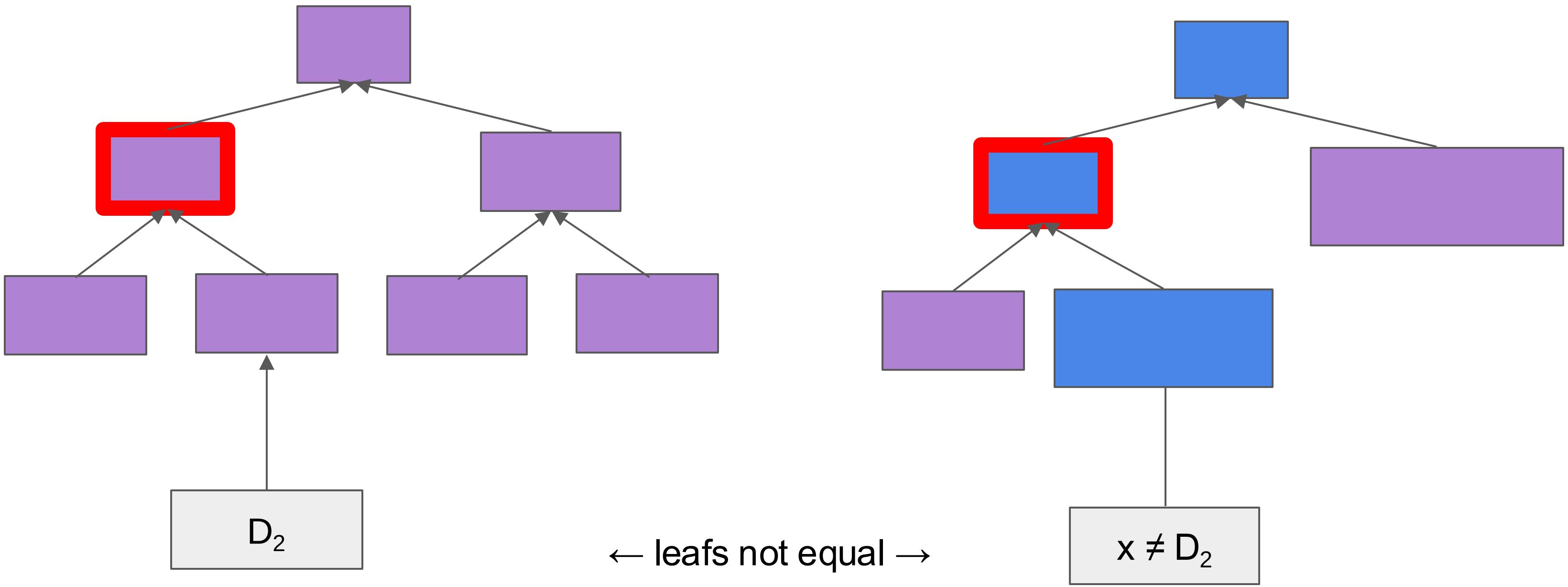


Situation if adversary A wins



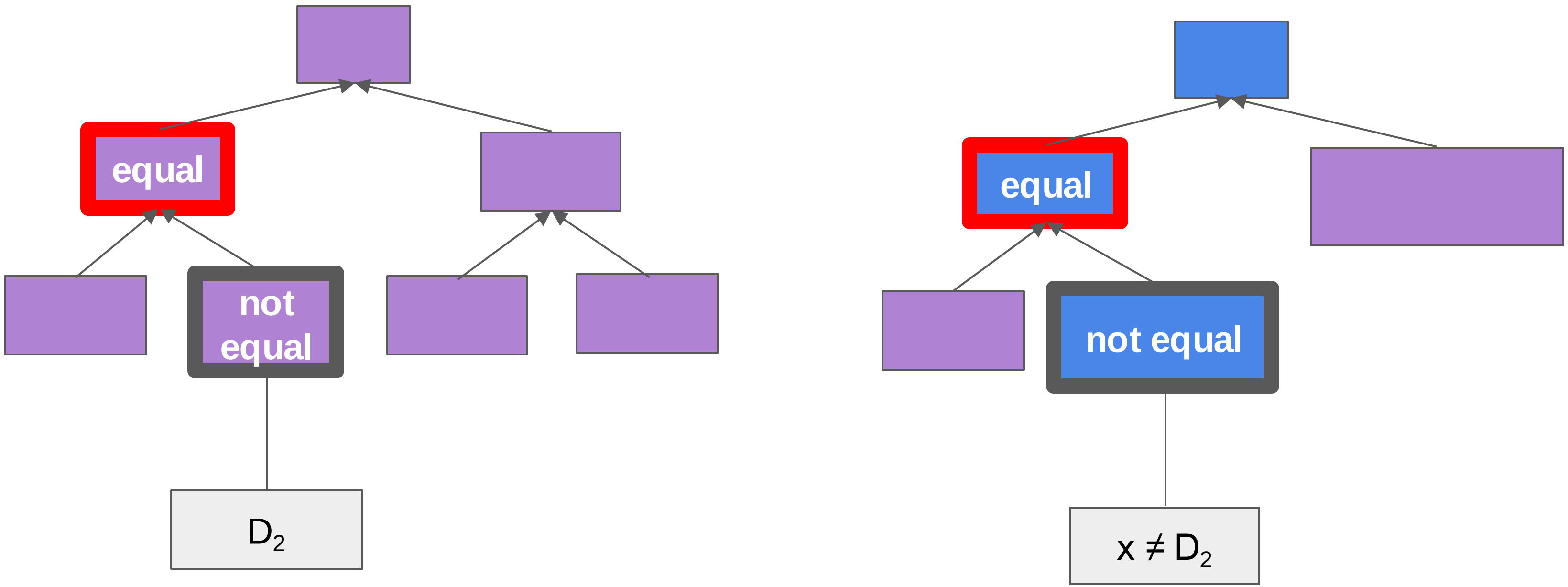

← roots are equal →
D2
← leafs not equal →
x ≠ D2

Induction:
Take **lowest** tree level where nodes are **equal**
That level must exist, as roots are equal
That level cannot be a leaf, as leafs are not equal



$H(L^a \mid\mid R^a) = H(L^b \mid\mid R^b)$ but $R^a \neq R^b$

We can extract a hash collision!



Proof conclusion

A* works as follows:

- Checks if A has found forgery
- If not, aborts
- If yes, finds minimum level where hashes are equal
- This gives a hash collision

If A finds a MT forgery, then A* finds a hash collision

$$\underbrace{\mathrm{Pr}[\text{MT-forgery}_{A,\Pi(H)}(\lambda)]}_{\text{non-negligible by for-contradiction assumption}} \overbrace{=}^{\text{equal by computational reduction}} \underbrace{\mathrm{Pr}[\text{coll-find}_{A*,H}(\lambda)]}_{\text{non-negligible, therefore contradiction}}$$

Proof of security is nuanced

Don’t roll your own crypto.

Use standard code by others which you know is secure.