A compendium on dimension reduction for slender elastic beams

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Abstract

Using the assumption of cylindrical invariance, we derive a one-dimensional beam model representing a prismatic elastic body. In this elementary presentation, we limit attention to *linear* elasticity and neglect any gradient effect (*i.e.*, we derive the leading-order contributions to the 1D energy). The approach works for arbitrary shapes of cross-section, arbitrary material symmetries and constitutive laws, and does not require the elastic properties to be homogeneous in the cross-section. It does not make any *ad hoc* kinematic assumption such as unshearability.

1 Formulation of the 3D elasticity problem

In reference configuration, the prismatic 3D body occupies the domain $\Omega \times \mathbb{R}$ where Ω is the reference cross-sectional domain. We denote by $\partial\Omega$ the boundary of the domain Ω in $\mathbb{R}^2$ and by $\boldsymbol{n}(S) \in \mathbb{R}^2$ the unit outer normal at any point of $\partial\Omega$.

The cross-sectional coordinate is denoted as $\boldsymbol{X} \in \mathbb{R}^2$ and the axial coordinate as $Z \in \mathbb{R}$.

The 3D displacement field is denoted as $\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z)$, where tildas are used to mark three-dimensional quantities (as opposed to two-dimensional quantities obtained by the forthcoming reduction):

$$\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^3.$$

The linearized strain $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{X}, Z)$ is given by

$$\tilde{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{u}}) = \frac{1}{2}\,(\tilde{\nabla}\tilde{\boldsymbol{u}} + \tilde{\nabla}\tilde{\boldsymbol{u}}^{\top}) \tag{1}$$

where $\tilde{\nabla}$ is the 3D gradient

$$\tilde{\nabla} = \left(\nabla, \frac{\partial}{\partial Z}\right) \tag{2}$$

and ∇ is the 2D gradient

$$\nabla = \left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}\right). \tag{3}$$

In the absence of distributed load, the potential energy takes the form

$$\tilde{\Phi}[\tilde{\boldsymbol{u}}] = \int_{\Omega \times \mathbb{R}} \tilde{W}(\tilde{\boldsymbol{\varepsilon}})\, \mathrm{d}^2\boldsymbol{X}\, \mathrm{d}Z - \tilde{\mathcal{L}}_{\mathrm{ts}}, \tag{4}$$

where $\tilde{W}(\tilde{\boldsymbol{\varepsilon}})$ is the quadratic strain energy density of the material, and $\tilde{\mathcal{L}}_{\mathrm{ts}}$ is the work of the loading applied on the remote terminal cross-sections, located at $Z = \pm\infty$. This remote loading is left unspecified.

Equilibrium solutions $\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z)$ are found by making the energy $\tilde{\Phi}[\tilde{\boldsymbol{u}}]$ in (4) stationary. We proceed to derive them variationally. Perturbing an equilibrium solution $\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z)$ with a test function $\tilde{\boldsymbol{v}}(\boldsymbol{X}, Z)$, we obtain the directional derivative of the total potential energy as

$$\begin{aligned} \mathrm{D}\tilde{\Phi}(\tilde{\boldsymbol{u}}; \tilde{\boldsymbol{v}}) &= \left.\frac{\mathrm{d}\tilde{\Phi}(\tilde{\boldsymbol{u}} + h\,\tilde{\boldsymbol{v}})}{\mathrm{d}h}\right|_{h=0} \\ &= \int_{\Omega \times \mathbb{R}} \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{v}})) : \tilde{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{v}})\, \mathrm{d}^2\boldsymbol{X}\, \mathrm{d}Z - \tilde{\mathcal{L}}[\tilde{\boldsymbol{v}}] \end{aligned} \tag{5}$$

where $\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}})$ is the elastic stress

$$\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}) := \frac{\partial \tilde{W}}{\partial \tilde{\boldsymbol{\varepsilon}}}(\tilde{\boldsymbol{\varepsilon}}). \tag{6}$$

The stationarity condition is that the first variation of energy in (5) is zero for any perturbation $\tilde{\boldsymbol{v}}$,

$$\forall \tilde{\boldsymbol{v}}, \qquad \mathrm{D}\tilde{\Phi}(\tilde{\boldsymbol{u}}; \tilde{\boldsymbol{v}}) = 0. \tag{7}$$

Integrating by parts, we obtain the equilibrium equations in the bulk in the form

$$\operatorname{div} \tilde{\boldsymbol{\sigma}}(\tilde{\varepsilon}) = \mathbf{0} \quad \text{in } \Omega \tag{8}$$

along with the boundary conditions

$$\tilde{\boldsymbol{\sigma}}(\tilde{\varepsilon}) \cdot \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{9}$$

The boundary conditions on the terminal cross-sections $Z = \pm\infty$ are not detailed.

2 Axially invariant solutions

We consider particular solutions such that the strain is independent of the axial coordinate Z,

$$\frac{\partial \tilde{\varepsilon}(\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z))}{\partial Z} = 0. \tag{10}$$

From the viewpoint of beam theory, this corresponds to *uniform* stretching, bending and twisting strains along the axis.

We denote as $e \in \mathbb{R}$ the constant axial strain, $\boldsymbol{k} = (k_1, k_2) \in \mathbb{R}^2$ the constant bending strain and $\tau \in \mathbb{R}$ the constant twisting strain:

- the axial strain e is dimensionless and measures the relative elongation of the centerline
- the curvature strain $\boldsymbol{k}$ is the curvature of the centerline, and is homogeneous to the inverse of a length
- the twisting strain $\boldsymbol{k}$ is the derivative of the twist angle of the cross-section with respect to the axial coordinate Z, and is homogeneous to the inverse of a length too.

The displacement $\boldsymbol{u} \in \mathbb{R}^3$ on a reference cross-section $Z = 0$ is denoted as

$$\boldsymbol{u}(\boldsymbol{X}) = \tilde{\boldsymbol{u}}(\boldsymbol{X}, 0) \in \mathbb{R}^3, \tag{11}$$

for $\boldsymbol{X} \in \Omega$. The displacement $\boldsymbol{u}(\boldsymbol{X})$ is a 3D-vector, defined on a 2D domain Ω,

$$\boldsymbol{u} : \Omega \to \mathbb{R}^3. \tag{12}$$

The projections of $\boldsymbol{u}$ onto the cross-section (included in the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$), and onto the axis (along $\boldsymbol{e}_3$) are denoted as $\boldsymbol{u}^{\parallel}(\boldsymbol{X}) \in \mathbb{R}^2$ and $u^{\perp}(\boldsymbol{X}) \in \mathbb{R}$, respectively:

$$\boldsymbol{u}(\boldsymbol{X}) = (\boldsymbol{u}^{\parallel}(\boldsymbol{X}), u^{\perp}(\boldsymbol{X})). \tag{13}$$

We postulate that the invariant solution $\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z)$ in the entire tube can be reconstructed from the displacement on the particular cross-section $Z = 0$ as

$$\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z) = e\, Z\, \boldsymbol{e}_3 - \frac{Z^2}{2} \boldsymbol{e}_3 \times \boldsymbol{k} + Z\, (\boldsymbol{k} + \tau \boldsymbol{e}_3) \times \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}), \tag{14}$$

where the successive terms represent

- a rigid-body translation of the cros-sections along the longitudinal vector $e\, Z\, \boldsymbol{e}_3$ representing uniform stretching,
- a rigid-body translation of the cros-sections along the transverse vector $-\frac{Z^2}{2} \boldsymbol{e}_3 \times \boldsymbol{k}$ capturing the bending of the centerline
- a rigid-body rotation of the cross-sections with (infinitesimal) rotation angle $Z\, (\boldsymbol{k} + \tau \boldsymbol{e}_3)$ caused by the combined action of bending and stretching,
- a *non-rigid* deformation of cross-sections associated with the displacement $\boldsymbol{u}(\boldsymbol{X})$, which by assumption is independent of the longitudinal coordinate Z.

The homogeneous solutions introduced in (10) and (14) are meant to approximate solutions that vary *slowly* in the longitudinal direction. Homogeneous solutions are instrumental to dimensional reduction.

To check the consistency of (10) and (14), we calculate the strain associated with the displacement in (14) and the result is

$$\tilde{\varepsilon}(\tilde{\boldsymbol{u}}(\boldsymbol{X}, Z)) = \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{X})) - \boldsymbol{\varepsilon}_0(\boldsymbol{X}), \tag{15}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{X})$ is the strain arising from the cross-sectional displacement $\boldsymbol{u}(\boldsymbol{X})$,

$$\begin{aligned}\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \frac{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top}{2}\\ &= \boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel)+\nabla u^\perp\odot\boldsymbol{e}_3\\ &= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel)$}} & \frac{1}{2}\frac{\partial u^\perp}{\partial X_1}\\ & & \frac{1}{2}\frac{\partial u^\perp}{\partial X_2}\\ \hline \frac{1}{2}\frac{\partial u^\perp}{\partial X_1} & \frac{1}{2}\frac{\partial u^\perp}{\partial X_2} & 0\end{array}\right),\end{aligned} \tag{16}$$

the symbol $\odot$ denotes the symmetrized tensor product,

$$\boldsymbol{a}\odot\boldsymbol{b}=\frac{\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a}}{2}, \tag{17}$$

$\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel)$ is the apparent 2D strain,

$$\begin{aligned}\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) &= \frac{\nabla\boldsymbol{u}^\parallel+(\nabla\boldsymbol{u}^\parallel)^\top}{2}\\ &= \begin{pmatrix} \frac{\partial u_1^\parallel}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1^\parallel}{\partial X_2}+\frac{\partial u_2^\parallel}{\partial X_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_1^\parallel}{\partial X_2}+\frac{\partial u_2^\parallel}{\partial X_1}\right) & \frac{\partial u_1^\parallel}{\partial X_2}\end{pmatrix},\end{aligned}$$

and $\boldsymbol{\varepsilon}_0(\boldsymbol{X})$ is a pre-strain arising from the macroscopic strain $(e,\boldsymbol{k},\tau)$, defined by

$$\begin{aligned}\boldsymbol{\varepsilon}_0(\boldsymbol{X}) &= -\Big(e\,\boldsymbol{e}_3+(\boldsymbol{k}+\tau\boldsymbol{e}_3)\times\boldsymbol{X}\Big)\odot\boldsymbol{e}_3\\ &= -\Big(e\,\boldsymbol{e}_3+\boldsymbol{k}\times\boldsymbol{X}\Big)\odot\boldsymbol{e}_3-\tau\,(\boldsymbol{e}_3\times\boldsymbol{X})\odot\boldsymbol{e}_3\\ &= \left(\begin{array}{cc|c} 0 & 0 & \frac{\tau}{2}X_2\\ 0 & 0 & -\frac{\tau}{2}X_1\\ \hline \frac{\tau}{2}X_2 & -\frac{\tau}{2}X_1 & \varepsilon_{33}^0(\boldsymbol{X})\end{array}\right),\end{aligned} \tag{18}$$

whose longitudinal component is given by

$$\varepsilon_{33}^0(\boldsymbol{X})=-(e+k_1\,X_2-k_2\,X_1). \tag{19}$$

The effective 2D energy takes the form

$$\Phi[e,\boldsymbol{k},\tau;\boldsymbol{u}]=\int_\Omega W(\boldsymbol{\varepsilon}(\boldsymbol{u}),e,\boldsymbol{k},\tau)\,\mathrm{d}^2\boldsymbol{X}, \tag{20}$$

where $W(\boldsymbol{\varepsilon}(\boldsymbol{u}),e,\boldsymbol{k},\tau)$ is obtained by inserting (15–19) into the strain energy density $\tilde{W}(\tilde{\boldsymbol{\varepsilon}})$,

$$W(\boldsymbol{\varepsilon}(\boldsymbol{u}),e,\boldsymbol{k},\tau)=\tilde{W}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}_0). \tag{21}$$

Next, we minimize the effective 2D potential $\Phi[\boldsymbol{u};e,\boldsymbol{k},\tau]$ with respect to the cross-sectional displacement, and denote as $\boldsymbol{u}^\star$ the optimum,

$$\forall\boldsymbol{v},\quad \mathrm{D}\Phi[e,\boldsymbol{k},\tau;\boldsymbol{u}^\star;\boldsymbol{v}]=\boldsymbol{0}.$$

The optimal cross-sectional displacement is $\boldsymbol{u}^\star$ typically unique up to rigid-body translation, and rigid-body rotation about the axis $\boldsymbol{e}_3$.

Remark 1. The minimization problem above is *not* invariant by rigid-body rotations about the cross-sectional vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, which correspond to shear modes of the 3D beam by (14).

By linearity, the relaxed strain $\tilde{\boldsymbol{\varepsilon}}^\star$ can be written as

$$\begin{aligned}\tilde{\boldsymbol{\varepsilon}}^\star(\boldsymbol{X},Z) &= \boldsymbol{\varepsilon}(\boldsymbol{u}^\star(\boldsymbol{X}))-\boldsymbol{\varepsilon}_0(\boldsymbol{X})\\ &= \begin{pmatrix}\tilde{\boldsymbol{\varepsilon}}_e^\star(\boldsymbol{X})\\ \tilde{\boldsymbol{\varepsilon}}_{k_1}^\star(\boldsymbol{X})\\ \tilde{\boldsymbol{\varepsilon}}_{k_2}^\star(\boldsymbol{X})\\ \tilde{\boldsymbol{\varepsilon}}_\tau^\star(\boldsymbol{X})\end{pmatrix}\cdot\begin{pmatrix}e\\ k_1\\ k_2\\ \tau\end{pmatrix}\end{aligned} \tag{22}$$

where $\tilde{\boldsymbol{\varepsilon}}_i(\boldsymbol{X})$ is the cross-sectional distribution of strain arising from the 1D strain $i\in\{e,k_1,k_2,\tau\}$.

The equivalent beam model is then defined by the strain energy

$$\Phi_{\mathrm{beam}} = \int_{-\infty}^{+\infty} \frac{1}{2} \begin{pmatrix} e(Z) \\ k_1(Z) \\ k_2(Z) \\ \tau(Z) \end{pmatrix} \cdot \boldsymbol{K} \cdot \begin{pmatrix} e(Z) \\ k_1(Z) \\ k_2(Z) \\ \tau(Z) \end{pmatrix} \mathrm{d}Z \tag{23}$$

where the entries in the 4×4 matrix of 1D moduli $\boldsymbol{K}$ is given by

$$K_{ij} = \int_\Omega \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}_i^\star(\boldsymbol{X})) : \tilde{\boldsymbol{\varepsilon}}_j^\star(\boldsymbol{X})\, \mathrm{d}^2\boldsymbol{X}. \tag{24}$$

The 1D constitutive law yield the normal stress N, the bending moments M_1 and M_2 and the twisting moment M_3 as

$$\begin{pmatrix} N \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \boldsymbol{K} \cdot \begin{pmatrix} e \\ k_1 \\ k_2 \\ \tau \end{pmatrix}. \tag{25}$$

3 Analytical solution for a Hookean material having uniform properties

For a linear, isotropic elastic material, the strain energy density $\tilde{W}(\tilde{\boldsymbol{\varepsilon}})$ can be expressed in terms of the material constants K (bulk modulus) and μ (shear modulus) as

$$\tilde{W}(\tilde{\boldsymbol{\varepsilon}}) = \frac{K}{2} \operatorname{tr}^2 \tilde{\boldsymbol{\varepsilon}} + \mu\, \|\tilde{\boldsymbol{\varepsilon}}^D\|^2, \tag{26}$$

where $\boldsymbol{t}^D$ denotes the deviatoric part of a symmetric rank-2 tensor,

$$\boldsymbol{t}^D := \boldsymbol{t} - \frac{1}{3} \boldsymbol{I}_3 \operatorname{tr} \boldsymbol{t}, \tag{27}$$

$\boldsymbol{I}_n$ denotes the identity matrix in dimension n, $\|\boldsymbol{t}\|$ denotes the Euclidean norm of a tensor,

$$\|\boldsymbol{t}\|^2 := \boldsymbol{t} : \boldsymbol{t}. \tag{28}$$

In the Hookean case, the stress is given by (6) as

$$\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}) = K \operatorname{tr} \tilde{\boldsymbol{\varepsilon}}\, \boldsymbol{I}_3 + 2\,\mu\, \tilde{\boldsymbol{\varepsilon}}^D. \tag{29}$$

In view of Equations (15–21), the energy density takes the form

$$\begin{aligned}
W(\boldsymbol{\varepsilon}(\boldsymbol{u}), e, \boldsymbol{k}, \tau) \;&:=\; \frac{K}{2} \operatorname{tr}^2 (\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \boldsymbol{\varepsilon}_0) + \mu\, \|(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \boldsymbol{\varepsilon}_0)^D + \nabla u^\perp \odot \boldsymbol{e}_3\|^2 \\
&=\; \frac{K}{2} \operatorname{tr}^2 \left(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\right) + \mu \left\| \begin{matrix} (\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)^D \\ + (\nabla u^\perp + \tau\, \boldsymbol{e}_3 \times \boldsymbol{X}) \odot \boldsymbol{e}_3 \end{matrix} \right\|^2 \\
&=\; \frac{K}{2} \operatorname{tr}^2 \left(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\right) + \mu\, \|(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)^D\|^2 \\
&\qquad + \mu\, \|(\nabla u^\perp + \tau\, \boldsymbol{e}_3 \times \boldsymbol{X}) \odot \boldsymbol{e}_3\|^2 \\
&=\; \frac{K}{2} \operatorname{tr}^2 \left(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\right) + \mu\, \|(\boldsymbol{\varepsilon}^\parallel(\boldsymbol{u}^\parallel) - \varepsilon_{33}^0\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)^D\|^2 \\
&\qquad + \frac{\mu}{2} \|\nabla u^\perp + \tau\, \boldsymbol{e}_3 \times \boldsymbol{X}\|^2,
\end{aligned} \tag{30}$$

where the norm appearing on the last line is the norm of a *vector.*

The first two terms concerns to the combined effect of stretching and bending, while the last term concerns twisting.

We first deal with the bending and stretching problems, which are coupled. Minimizing the first two terms in (30) with respect to the apparent 2D strain $\boldsymbol{\varepsilon}(\boldsymbol{X})$ yields the classical solution of Hookean elasticity in simple traction,

$$\boldsymbol{\varepsilon}_{\mathrm{bd+st}}^\parallel(\boldsymbol{X}) \;=\; \nu\, \varepsilon_{33}^0(\boldsymbol{X})\, \boldsymbol{I}_2 \tag{31}$$

that gives rise to a 3D strain and stress,

$$\begin{aligned}
\tilde{\boldsymbol{\varepsilon}}_{\mathrm{bd+st}}(\boldsymbol{X}) \;&=\; \begin{pmatrix} \nu\, \boldsymbol{I}_2 & \\ & -1 \end{pmatrix} \varepsilon_{33}^0(\boldsymbol{X}) \\
\tilde{\boldsymbol{\sigma}}_{\mathrm{bd+st}}(\boldsymbol{X}) \;&=\; -Y \varepsilon_{33}^0(\boldsymbol{X})\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3.
\end{aligned}$$

Here, ν is Poisson's coefficient and Y is the Young modulus,

$$\begin{aligned} \nu &= \frac{3\,K - 2\,\mu}{2\,(3\,K + \mu)} \\ Y &= \frac{9\,K\,\mu}{3\,K + \mu}. \end{aligned} \tag{32}$$

We have minimized the energy at every point over the cross-sectional strain $\boldsymbol{\varepsilon}$: to confirm that this approach is valid, we need to check that the 3D strain $\tilde{\boldsymbol{\varepsilon}}$ found in this way is geometrically compatible. This verification is left to the reader.

The solution of the twisting problem is found by minimizing the last term in (30). The solution takes the form

$$u^{\perp}(\boldsymbol{X}) = \tau\,\psi(\boldsymbol{X}), \tag{33}$$

where $\psi(\boldsymbol{X})$ is the warping function. It is the solution of the minimization problem

$$\min_{\psi} \frac{1}{2} \int_{\Omega} \|\nabla\psi + \boldsymbol{e}_3 \times \boldsymbol{X}\|^2 \,\mathrm{d}^2\boldsymbol{X}. \tag{34}$$

The warping function $\psi(\boldsymbol{X})$ is defined up to a constant representing a longitudinal rigid-body translation.The minimization problem for ψ leads to the elliptic boundary-value problem in the cross-section

$$\begin{aligned} \Delta\psi &= 0 \qquad \text{in } \Omega \\ \nabla\psi \cdot \boldsymbol{n}_j + (\boldsymbol{X} \times \boldsymbol{n}_j) \cdot \boldsymbol{e}_3 &= 0 \qquad \text{on } \partial\Omega_j. \end{aligned} \tag{35}$$

In the particular case of an elliptical cross-section,

$$\Omega = \Omega_{\mathrm{ell}} := \left\{ (X_1, X_2) \in \mathbb{R}^2 \quad \text{such that } \left(\frac{X_1}{a}\right)^2 + \left(\frac{X_2}{b}\right)^2 \leqslant 1 \right\}, \tag{36}$$

where a and b are the half-length of the principal axes, the solution is given by

$$\psi_{\mathrm{ell}}(\boldsymbol{X}) = -\frac{a^2 - b^2}{a^2 + b^2}\,X_1\,X_2. \tag{37}$$

Returning to the general case and collecting the above results, we obtain the four contributions to the relaxed strain and stress listed in (22) in the form

$$\begin{aligned} \tilde{\boldsymbol{\varepsilon}}_e^{\star} &= \begin{pmatrix} -\nu\,\boldsymbol{I}_2 & \\ & 1 \end{pmatrix} & \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}_e^{\star}) &= Y \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\ \tilde{\boldsymbol{\varepsilon}}_{k_1}^{\star} &= \begin{pmatrix} -\nu\,\boldsymbol{I}_2 & \\ & 1 \end{pmatrix} X_2 & \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}_{k_1}^{\star}) &= Y X_2\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\ \tilde{\boldsymbol{\varepsilon}}_{k_2}^{\star} &= \begin{pmatrix} -\nu\,\boldsymbol{I}_2 & \\ & 1 \end{pmatrix} (-X_1) & \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}_{k_2}^{\star}) &= -Y X_1\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\ \tilde{\boldsymbol{\varepsilon}}_{\tau}^{\star} &= (\nabla\psi(\boldsymbol{X}) + \boldsymbol{e}_3 \times \boldsymbol{X}) \odot \boldsymbol{e}_3 & \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\varepsilon}}_{\tau}^{\star}) &= 2\,\mu\,(\nabla\psi(\boldsymbol{X}) + \boldsymbol{e}_3 \times \boldsymbol{X}) \odot \boldsymbol{e}_3. \end{aligned} \tag{38}$$

The explicit expression of the microscopic stress is

$$\tilde{\boldsymbol{\sigma}} = Y \begin{pmatrix} 0 & 0 & \mu\,\tau\,(\psi_{,1} - X_2) \\ 0 & 0 & \mu\,\tau\,(\psi_{,2} + X_1) \\ \mu\,\tau\,(\psi_{,1} - X_2) & \mu\,\tau\,(\psi_{,2} + X_1) & e + k_1\,X_2 - k_2\,X_1 \end{pmatrix}. \tag{39}$$

The 4×4 matrix of 1D elastic moduli appearing in (23) can the be written using (24) as

$$\boldsymbol{K} = \begin{pmatrix} Y \begin{pmatrix} \overbrace{\int_{\Omega} 1^2 \,\mathrm{d}X_1\,\mathrm{d}X_2}^{A} & \int_{\Omega} 1 \cdot X_2 \,\mathrm{d}X_1\,\mathrm{d}X_2 & \int_{\Omega} 1 \cdot (-X_1) \,\mathrm{d}X_1\,\mathrm{d}X_2 \\ \text{sym.} & \int_{\Omega} (X_2)^2 \,\mathrm{d}X_1\,\mathrm{d}X_2 & \int_{\Omega} (X_2)\,(-X_1) \,\mathrm{d}X_1\,\mathrm{d}X_2 \\ \text{sym.} & \text{sym.} & \int_{\Omega} (-X_1)^2 \,\mathrm{d}X_1\,\mathrm{d}X_2 \end{pmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \mu\,J \end{pmatrix}, \tag{40}$$

where A is the cross-sectional area and J the torsional constant,

$$\begin{aligned} A &= \int_{\Omega} 1 \,\mathrm{d}X_1\,\mathrm{d}X_2 \\ J &= \int_{\Omega} \|\nabla\psi(\boldsymbol{X}) + \boldsymbol{e}_3 \times \boldsymbol{X}\|^2 \,\mathrm{d}X_1\,\mathrm{d}X_2 \end{aligned}. \tag{41}$$

If the origin of coordinates is taken at the center of mass of the cross-section, the stretching and bending decouple and we have

$$\boldsymbol{K} = \begin{pmatrix} Y A & \boldsymbol{0} & 0 \\ \boldsymbol{0} & Y \boldsymbol{I} & \boldsymbol{0} \\ 0 & \boldsymbol{0} & \mu\, J \end{pmatrix},$$

where $\boldsymbol{I}$ is the matrix of geometrical moments,

$$\boldsymbol{I} = Y \begin{pmatrix} \int_\Omega (X_2)^2\, \mathrm{d}X_1\, \mathrm{d}X_2 & \int_\Omega (X_2)\, (-X_1)\, \mathrm{d}X_1\, \mathrm{d}X_2 \\ \text{sym.} & \int_\Omega (-X_1)^2\, \mathrm{d}X_1\, \mathrm{d}X_2 \end{pmatrix}. \tag{42}$$

If in addition the coordinates coincide with the principal bending axes, the matrix of geometrical moments becomes diagonal,

$$\boldsymbol{I} = Y \begin{pmatrix} \overbrace{\int_\Omega (X_2)^2\, \mathrm{d}X_1\, \mathrm{d}X_2}^{I_1} & 0 \\ 0 & \underbrace{\int_\Omega (-X_1)^2\, \mathrm{d}X_1\, \mathrm{d}X_2}_{I_2} \end{pmatrix},$$

and we recover the standard expressions of the 1D moduli,

$$\boldsymbol{K} = \begin{pmatrix} Y A & & & \\ & Y I_1 & & \\ & & Y I_2 & \\ & & & \mu\, J \end{pmatrix}. \tag{43}$$

The energy of the equivalent beam in (23) then takes the form

$$\Phi_{\text{beam}} = \int_{-\infty}^{+\infty} \frac{1}{2}\, (Y A\, e^2 + Y I_1\, k_1^2 + Y I_2\, k_2^2 + \mu\, J\, \tau^2)\, \mathrm{d}Z, \tag{44}$$

and the constitutive laws in (25) are written as

$$\begin{aligned} N &= Y A\, e \\ M_1 &= Y I_1\, k_1 \\ M_2 &= Y I_2\, k_2 \\ M_3 &= \mu\, J\, \tau. \end{aligned} \tag{45}$$

In the particular case of an elliptical cross-section, see Equation (36), the 1D moduli are given by

$$\begin{aligned} A_{\text{ell}} &= \pi\, a\, b \\ B_{1,\text{ell}} &= \frac{\pi}{4}\, a\, b^3 \\ B_{2,\text{ell}} &= \frac{\pi}{4}\, a^3\, b \\ J_{\text{ell}} &= \pi\, \frac{a^3\, b^3}{a^2 + b^2}. \end{aligned} \tag{46}$$