占卜学: 作业 2

截止时间: 1919 年 8 月 10 日

计算机 1919 班
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题目 1

求一个合适的整数 c , 使得 $f(n) \le c \cdot g(n)$ 对 $n > 1$ 恒成立.

1. $f(n) = n^2 + n + 1$, $g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

解:
显然地, 我们可以通过占卜法来解 c 的值 c.

1.

$$\begin{aligned} n^2 + n + 1 &= \\ &\le n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\le c \cdot 2n^3 \end{aligned}$$

故 $c = 2$ 满足要求.

2.

$$\begin{aligned} n^2 + n\sqrt{n} &= \\ &= n^2 + n^{3/2} \\ &\le n^2 + n^{4/2} \\ &= n^2 + n^2 \\ &= 2n^2 \\ &\le c \cdot n^2 \end{aligned}$$

故 $c = 2$ 满足要求.

3.

$$\begin{aligned} n^2 - n + 1 &= \\ &\le n^2 \\ &\le c \cdot n^2/2 \end{aligned}$$

故 $c = 2$ 满足要求.

题目 2

令 $\Sigma = \{0, 1\}$. 建立一个有限状态自动机 A 来识别可以被 5 整除的二进制串.

令状态 q_k 表示 k 被 5 除的余数. 例如,7 对应 q_2 (因为 $7 \mod 5 = 2$).

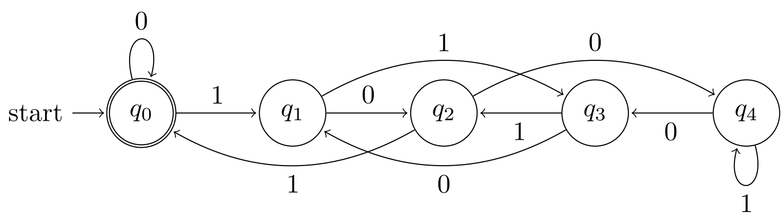

图 1: DFA A 真好看, 是不是?

解释

给出一个二进制表示的数, x. 由于我们的状态机只接受两种输入, x can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multipying by two and adding one.

通过这一点, 我们就可以建立出一个转移表:

	$x \mod 5 = 0$	$x \mod 5 = 1$	$x \mod 5 = 2$	$x \mod 5 = 3$	$x \mod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state q_0 or ($x \mod 5 = 0$), a transition line should go to state q_0 for the input 0 and a line should go to state q_1 for input 1. Continuing this gives us the Figure 1.

题目 3

写出快速排序算法的一部分 **Quick-Sort(**$list, start, end$**)**

function QUICK-SORT($list, start, end$)
  if $start \geq end$ **then**
    return
  end if
  $mid \leftarrow$ PARTITION($list, start, end$)
  QUICK-SORT($list, start, mid - 1$)
  QUICK-SORT($list, mid + 1, end$)
end function

算法 1: 快速排序的开始部分

题目 4

假设我们想要拟合一条过原点的直线, 如, $Y_i = \beta_1 x_i + e_i$ 其中 $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, $\mathrm{Var}[e_i] = \sigma_e^2$ 且 $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope β_1.

解:

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$
\begin{aligned}
RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\
&= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2
\end{aligned}
$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$
\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) = 0
$$

This gives us:

$$
\begin{aligned}
\sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\
&= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2
\end{aligned}
$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1:

$$
\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}
$$

题目 4 continued on next page...

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

解:

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta}_1]$:

$$\begin{aligned}
\mathrm{E}[\hat{\beta}_1] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i(\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}$$

Thus since our estimator's expected value is β_1, we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}_1] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}$$

题目 5

证明一个 k 阶的多项式, $a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$ 是 $\Theta(n^k)$ 的, 其中 $a_k \ldots a_0$ 都是非负常数.

证明 为了证明 $a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$, 只需证明:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

对于第一个不等式, 显然它是成立的. 因为无论常数是何值, $n^k \leq a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$ even if $c_1 = 1$ and $n_0 = 1$. 这是因为又 $n^k \leq c_1 \cdot a_kn^k$ 对任意的常数 c_1 和 a_k.

对于第二个不等式, 我们可以这样证明: 记 $A = \sum_{i=0}^{k} a_i$, 则有,

$$\begin{aligned}
a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0 = & \\
&\leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\
&= A \cdot n^k \\
&\leq c_2 \cdot n^k
\end{aligned}$$

其中 $n_0 = 1$ 和 $c_2 = A$. c_2 是常数. 这就证明了结论. $\square$

题目 18

比较 $\sum_{k=1}^{5} k^2$ 和 $\sum_{k=1}^{5} (k-1)^2$.

题目 19

求 $f(x) = x^4 + 3x^2 - 2$ 的导数.

题目 6

计算定积分 $\int_0^1 (1-x^2)\mathrm{d}x$ 和 $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.