Linear elastic beam model (Bernoulli-Euler, Timoshenko)

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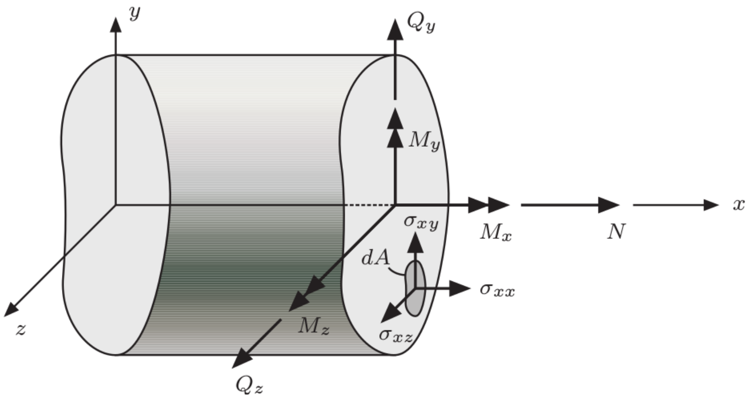
Section forces and stresses



Section forces and stresses

Section forces

$$N = \int_A \sigma_{xx} dA, \quad Q_y = \int_A \sigma_{xy} dA, \quad Q_z = \int_A \sigma_{xz} dA$$

Section moments

$$M_x = \int_A (y\sigma_{xz} - z\sigma_{xy}) dA,$$

$$M_y = \int_A z\sigma_{xx} dA,$$

$$M_z = -\int_A y\sigma_{xx} dA$$

Equilibrium equations

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0, \quad \frac{\mathrm{d}Q_y}{\mathrm{d}x} + q_y = 0, \quad \frac{\mathrm{d}Q_z}{\mathrm{d}x} + q_z = 0$$

$$\frac{\mathrm{d}M_x}{\mathrm{d}x} + m_x = 0, \quad \frac{\mathrm{d}M_y}{\mathrm{d}x} - Q_z + m_y = 0, \quad \frac{\mathrm{d}M_z}{\mathrm{d}x} + Q_y + m_z = 0$$

Distributed force load: $\mathbf{q} = [q_x, q_y, q_z]^T$
Distributed moment load: $\mathbf{m} = [m_x, m_y, m_z]^T$

Beam derormation

- A beam cross-section orthogonal to the x-axis at the coordinate x remains plane and keeps its shape during deformation (cross-section translates and rotates as a rigid body). This means that Poisson contractions in the transverse direction due to axial strains are ignored.
- Deformed position of the beam cross-section is defined by the displacement and rotation vectors of the beam mean line: $\mathbf{w} = [w_x, w_y, w_z]^T$, $\boldsymbol{\theta} = [\theta_x, \theta_y, \theta_z]^T$

Beam deformation

Displacement of a material point in the beam cross-section (assumed small displacements and rotations and symmetrical cross-section):

$$\begin{aligned} u_x &= w_x + z\theta_y - y\theta_z \\ u_y &= w_y - z\theta_x \\ u_z &= w_z - y\theta_x \end{aligned}$$

Strain components:

$$\begin{aligned} \epsilon_{xx} &= \frac{\mathrm{d}u_x}{\mathrm{d}x} = \frac{\mathrm{d}w_x}{dx} + z\frac{\mathrm{d}\theta_y}{\mathrm{d}x} - y\frac{\mathrm{d}\theta_z}{\mathrm{d}x} \\ \gamma_{xy} &= \frac{\mathrm{d}u_x}{\mathrm{d}y} + \frac{\mathrm{d}u_y}{\mathrm{d}x} = \frac{\mathrm{d}w_y}{\mathrm{d}x} - z\frac{\mathrm{d}\theta_x}{\mathrm{d}x} - \theta_z \\ \gamma_{xz} &= \frac{\mathrm{d}u_x}{\mathrm{d}z} + \frac{\mathrm{d}u_z}{\mathrm{d}x} = \frac{\mathrm{d}w_z}{\mathrm{d}x} + y\frac{\mathrm{d}\theta_x}{\mathrm{d}x} + \theta_y \end{aligned}$$

Constitutive relations for elastic beam

Stress components:

$$
\begin{aligned}
\sigma_{xx} &= E\epsilon_{xx} = E\left(\frac{\mathrm{d}w_x}{dx} + z\frac{\mathrm{d}\theta_y}{\mathrm{d}x} - y\frac{\mathrm{d}\theta_z}{\mathrm{d}x}\right) \\
\sigma_{xy} &= G\gamma_{xy} = G\left(\frac{\mathrm{d}w_y}{\mathrm{d}x} - z\frac{\mathrm{d}\theta_x}{\mathrm{d}x} - \theta_z\right) \\
\sigma_{xz} &= G\gamma_{xz} = G\left(\frac{\mathrm{d}w_z}{\mathrm{d}x} + y\frac{\mathrm{d}\theta_x}{\mathrm{d}x} + \theta_y\right)
\end{aligned}
$$

Integrated forces:

$$
\begin{aligned}
N &= E\left(A\frac{\mathrm{d}w_x}{dx} + S_y\frac{\mathrm{d}\theta_y}{\mathrm{d}x} - S_z\frac{\mathrm{d}\theta_z}{\mathrm{d}x}\right) \\
Q_y &= G\left[A\left(\frac{\mathrm{d}w_y}{\mathrm{d}x} - \theta_z\right) - S_y\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\right] \\
Q_z &= G\left[A\left(\frac{\mathrm{d}w_z}{\mathrm{d}x} + \theta_y\right) + S_z\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\right]
\end{aligned}
$$

Constitutive relations for elastic beam

Integrated moments:

$$
\begin{aligned}
M_x &= G\left[S_z\left(\frac{\mathrm{d}w_z}{\mathrm{d}x}+\theta_y\right)-S_y\left(\frac{\mathrm{d}w_y}{\mathrm{d}x}-\theta_z\right)+(I_{yy}+I_{zz})\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\right]\\
M_y &= E\left(S_y\frac{\mathrm{d}w_x}{dx}+I_{yy}\frac{\mathrm{d}\theta_y}{\mathrm{d}x}-I_{yz}\frac{\mathrm{d}\theta_z}{\mathrm{d}x}\right)\\
M_z &= E\left(-S_z\frac{\mathrm{d}w_x}{dx}-I_{yz}\frac{\mathrm{d}\theta_y}{\mathrm{d}x}+I_{zz}\frac{\mathrm{d}\theta_z}{\mathrm{d}x}\right)
\end{aligned}
$$

Constants of the beam cross-section:

$$
A=\int_A \mathrm{d}A, \quad S_y=\int_A z\,\mathrm{d}A, \quad S_z=\int_A y\,\mathrm{d}A,
$$

$$
I_{yy}=\int_A z^2\,\mathrm{d}A, \quad I_{zz}=\int_A y^2\,\mathrm{d}A, \quad I_{yz}=\int_A yz\,\mathrm{d}A,
$$

Note: For symmetrical beam cross-section (circular cross-section) some of the sectional constants vanish: $S_y=0,\ S_z=0,\ I_{yz}=0$

Equilibrium equations for Timoshenko beam

$$\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} = -\frac{q_x}{EA}$$

$$\frac{\mathrm{d}^2 \theta_x}{\mathrm{d}x^2} = -\frac{m_x}{2GI}$$

$$\frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2} = \frac{\mathrm{d}\theta_z}{\mathrm{d}x} - \frac{q_y}{GA}$$

$$\frac{\mathrm{d}^2 \theta_z}{\mathrm{d}x^2} = -\frac{GA}{EI}\left(\frac{\mathrm{d}w_y}{\mathrm{d}x} - \theta_z\right) - \frac{m_z}{EI}$$

$$\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} = -\frac{\mathrm{d}\theta_y}{\mathrm{d}x} - \frac{q_z}{GA}$$

$$\frac{\mathrm{d}^2 \theta_y}{\mathrm{d}x^2} = \frac{GA}{EI}\left(\frac{\mathrm{d}w_z}{\mathrm{d}x} + \theta_y\right) - \frac{m_y}{EI}$$

Equilibrium equations for Timoshenko beam

Matrix form of the equilibrium equations

$$\frac{\mathrm{d}^2\mathbf{w}}{\mathrm{d}x^2} = -\mathbf{C}_q \cdot \mathbf{q} + \mathbf{C}'_\theta \cdot \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}x^2} = -\mathbf{C}_m \cdot \mathbf{m} + \mathbf{C}'_w \cdot \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}x} + \mathbf{C}_\theta \cdot \boldsymbol{\theta}$$

Note: Second order system of PDEs which can be naturally discretized using second order finite volume method (FVM).

Equilibrium equations for Timoshenko beam

Matrix form of the equilibrium equations

$$\mathbf{C}_q = \begin{bmatrix} \frac{1}{EA} & 0 & 0 \\ 0 & \frac{1}{GA} & 0 \\ 0 & 0 & \frac{1}{GA} \end{bmatrix}, \quad \mathbf{C}'_\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{C}_m = \begin{bmatrix} \frac{1}{2GI} & 0 & 0 \\ 0 & \frac{1}{EI} & 0 \\ 0 & 0 & \frac{1}{EI} \end{bmatrix}, \quad \mathbf{C}'_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{GA}{EI} \\ 0 & -\frac{GA}{EI} & 0 \end{bmatrix},$$

$$\mathbf{C}_\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{GA}{EI} & 0 \\ 0 & 0 & \frac{GA}{EI} \end{bmatrix}$$

Equilibrium equations for Timoshenko beam

Neumann boundary conditions

$$\frac{\mathrm{d}w_x}{\mathrm{d}x} = \frac{N}{EA}, \quad \frac{\mathrm{d}\theta_x}{\mathrm{d}x} = \frac{M_x}{2GI},$$

$$\frac{\mathrm{d}w_y}{\mathrm{d}x} = \frac{Q_y}{GA} + \theta_z, \quad \frac{\mathrm{d}\theta_z}{\mathrm{d}x} = \frac{M_z}{EI}$$

$$\frac{\mathrm{d}w_z}{\mathrm{d}x} = \frac{Q_z}{GA} - \theta_y, \quad \frac{\mathrm{d}\theta_y}{\mathrm{d}x} = \frac{M_y}{EI}$$

Bernoulli-Euler beam deformation

Additional constraints following form the Bernoulli-Euler beam model assumption (rotated cross-section is always orthogonal to the deformed beam axis):

$$
\begin{aligned}
\theta_y &= -\frac{\mathrm{d}w_z}{\mathrm{d}x} \\
\theta_z &= \frac{\mathrm{d}w_y}{\mathrm{d}x} \\
\kappa_y &= \frac{\mathrm{d}\theta_y}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} \\
\kappa_z &= \frac{\mathrm{d}\theta_z}{\mathrm{d}x} = \frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2}
\end{aligned}
$$

Constitutive equation for Bernouli-Euler beam

$$
\begin{aligned}
N &= E\left(A\frac{dw_x}{dx} - S_y\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} - S_z\frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2}\right)\\
Q_y &= -GS_y\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\\
Q_z &= GS_z\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\\
M_x &= G(I_{yy}+I_{zz})\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\\
M_y &= E\left(S_y\frac{dw_x}{dx} - I_{yy}\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} - I_{yz}\frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2}\right)\\
M_z &= E\left(-S_z\frac{dw_x}{dx} + I_{yz}\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} + I_{zz}\frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2}\right)
\end{aligned}
$$

Equilibrium equations for Bernoulli-Euler beam

Modified equilibrium equations for M_x, M_y and M_z (original equations are derived with respect to x):

$$
\begin{aligned}
\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{\mathrm{d}m_x}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} - \frac{\mathrm{d}Q_z}{\mathrm{d}x} + \frac{\mathrm{d}m_y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2 M_z}{\mathrm{d}x^2} + \frac{\mathrm{d}Q_y}{\mathrm{d}x} + \frac{\mathrm{d}m_z}{\mathrm{d}x} &= 0
\end{aligned}
$$

Equilibrium equations for Bernoulli-Euler beam

Final form of the equilibrium equations follows after elimination of shear forces Q_y and Q_z using corresponding equilibrium equation:

$$
\begin{aligned}
\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{\mathrm{d}m_y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} - q_z + \frac{\mathrm{d}m_y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2 M_z}{\mathrm{d}x^2} + q_y + \frac{\mathrm{d}m_z}{\mathrm{d}x} &= 0
\end{aligned}
$$

Equilibrium equations for Bernoulli-Euler beam

Final equilibrium equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}w_x}{dx} - ES_y\frac{\mathrm{d}^2w_z}{\mathrm{d}x^2} - ES_z\frac{\mathrm{d}^2w_y}{\mathrm{d}x^2}\right) + q_x &= 0\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[G(I_{yy}+I_{zz})\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\right] + \frac{\mathrm{d}m_x}{\mathrm{d}x} &= 0\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(ES_y\frac{\mathrm{d}w_x}{dx} - EI_{yy}\frac{\mathrm{d}^2w_z}{\mathrm{d}x^2} - EI_{yz}\frac{\mathrm{d}^2w_y}{\mathrm{d}x^2}\right) - q_z + \frac{\mathrm{d}m_y}{\mathrm{d}x} &= 0\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(-ES_z\frac{\mathrm{d}w_x}{dx} + EI_{yz}\frac{\mathrm{d}^2w_z}{\mathrm{d}x^2} + EI_{zz}\frac{\mathrm{d}^2w_y}{\mathrm{d}x^2}\right) + q_y + \frac{\mathrm{d}m_z}{\mathrm{d}x} &= 0
\end{aligned}
$$

Equilibrium equations for Bernoulli-Euler beam

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}w_x}{dx}\right) + q_x &= 0 \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[G(I_{yy}+I_{zz})\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\right] + \frac{\mathrm{d}m_x}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{yy}\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2}\right) + q_z - \frac{\mathrm{d}m_y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{zz}\frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2}\right) + q_y + \frac{\mathrm{d}m_z}{\mathrm{d}x} &= 0
\end{aligned}
$$

Equilibrium equations for Bernoulli-Euler beam

Equilibrium equation expressed in terms of mean line displacement and curvature (circular cross-section)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}w_x}{dx}\right) = -q_x$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[G(I_{yy}+I_{zz})\kappa_x\right] = -\frac{\mathrm{d}m_y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{yy}\kappa_y\right) = -q_z + \frac{\mathrm{d}m_y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{zz}\kappa_z\right) = -q_y - \frac{\mathrm{d}m_z}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} = -\kappa_y, \qquad \frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2} = \kappa_z$$

Equilibrium equations for Bernoulli-Euler beam

Circular cross-section ($I_{yy} = I_{zz} = I$)

$$\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} = -\frac{1}{EA} q_x$$

$$\frac{\mathrm{d}^2 \kappa_x}{\mathrm{d}x^2} = -\frac{1}{2GI} \frac{\mathrm{d}m_y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2 \kappa_y}{\mathrm{d}x^2} = -\frac{1}{EI} \left(q_z - \frac{\mathrm{d}m_y}{\mathrm{d}x} \right)$$

$$\frac{\mathrm{d}^2 \kappa_z}{\mathrm{d}x^2} = -\frac{1}{EI} \left(q_y + \frac{\mathrm{d}m_z}{\mathrm{d}x} \right)$$

$$\frac{\mathrm{d}^2 w_z}{\mathrm{d}x^2} = -\kappa_y, \quad \frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2} = \kappa_z$$

Mathematical model consists of six Poisson equations for unknown variables: w_x, w_y, w_z, κ_x, κ_y and κ_z. Equations are not mutually coupled and can be solve using segregated solution procedure. By using curvature (κ_y and κ_z), the system is reduced from fourth to second order.

Equilibrium equations for Bernoulli-Euler beam

Boundary conditions at the fixed moment end (M_z, $Q_y = 0$):

$$\kappa_z = \frac{\mathrm{d}^2 w_y}{\mathrm{d}x^2} = \frac{M_z}{EI}$$
$$\frac{\mathrm{d}^3 w_y}{\mathrm{d}x^3} = 0$$

Note: It will be problem to implement this using second-order FVM. Higher order FVM would be required.