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HUMPTY DUMPTY HAD A GREAT FALL: MEASURING THE ACCELERATION DUE TO GRAVITY WITH A SIMPLE PENDULUM

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ABSTRACT

$g$man, g!

1. INTRODUCTION

Of the four fundamental forces of nature, gravity is the one that we have the most experience with in our everyday lives. It is such a constant in our lives that every movement we make factors in the force of gravity, even if we are not aware of it. We put objects down, expecting them to stay there because of gravity. A basketball player shoots a ball upwards, expecting the familiar parabolic arc due to gravity to bring it back down into the basket. Even the common phenomenon known as walking relies on gravity's pull to give our feet the friction needed to push us forward. Our experience with gravity is so intimate that we know exactly how things perform in earth's gravity field without having to measure and calculate (e.g. the basketball player knowing the right momentum to give the ball to score a basket without any calculation being performed).

g as a measure of force of gravity...

Importance of calculating g: study mantle, earthquakes?, earth's density (similar to recent lunar probes), clock timing (which use pendula as well)...

To calculate g, we will use a pendulum setup quite similar to that of the pendulum clocks mentioned above. Section 2 will introduce the physics behind this calculation more thoroughly. Here we state the intended goals of our study of the motion of a simple pendulum: (1) To be able to determine the value of g, which as discussed above is an important parameter for a variety of calculations; (2) to test a mathematical model (introduced in section 2) of a physical system and compare the correspondence between the two and (3) use the process as an introduction into the sources of errors and their propagation to gain greater skills as a physicist-in-training. The experiment we will be performing is discussed in great detail in ....

2. THEORETICAL BACKGROUND

Later gater...

Also talk about small angle approximation...

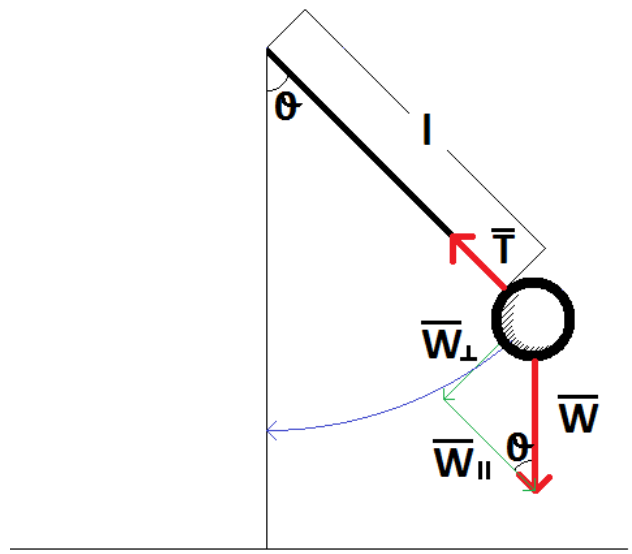

Figure 1. Diagram of a simple pendulum. The components listed are; θ is the angle between the string and the rest location of the pendulum, l is the length of the string, $\vec{W}$ is the weight force of the bob (i.e. W = mg), $\vec{W}_{\perp}$ is the component of the weight force perpendicular to l, $\vec{W}_{\parallel}$ is the component of the weight force parallel to l.

The motion of a simple pendulum can be derived by analysing the torques (or forces) that acts on the bob of the pendulum. Assume the string that the pendulum hangs from is massless, and its length l does not change. Friction on the string at the fulcrum as well as the air drag on the bob is also ignored. See Appendix A for the full derivation of the equation of motion for the simple harmonic oscillator.

From Figure 1; T is the tension force on the rope, l is the length of the rope (or r$\cdot\hat{r}$, where $\hat{r}$ is a unit vector), θ is the angle, and W is the weight given by m$\cdot$g.

Using the small angle approximation

$$\boxed{sin\theta \approx \theta} \tag{1}$$

for small angles, we derive the equation of motion

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \tag{2}$$

to model a simple pendulum.

3. EXPERIMENTAL PROCEDURE

3.1. *Setup*

The setup consisted of a metal frame attached to a wall, a string, and a metal ball.

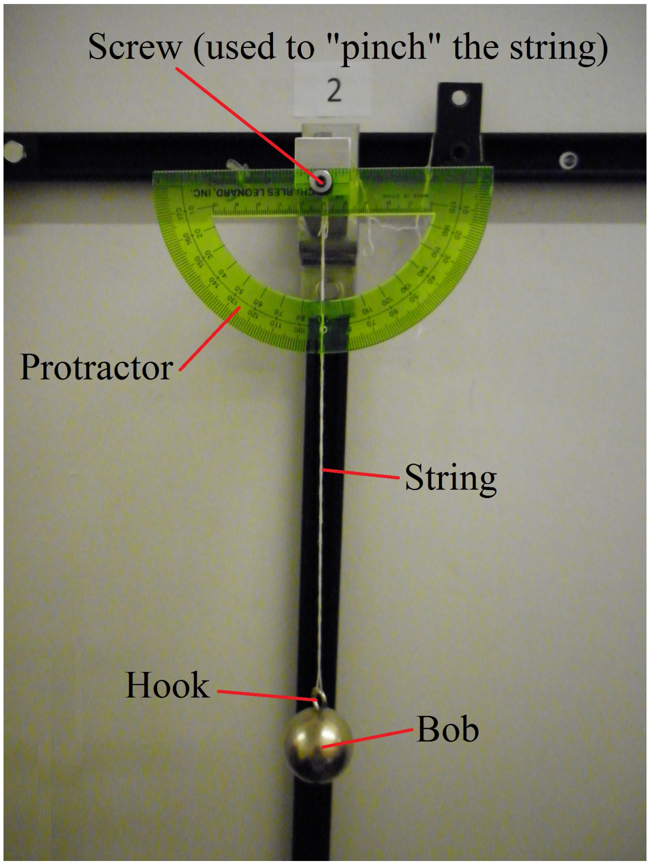

Figure 2. The bob is hung from the string by its hook. A screw pinches the string to act as the pendulum's fulcrum. We measured the angle of the initial period with the protractor.

3.2. *Procedure*

The first measurements we made were those of the ball diameter and the length of the hook on the top of the ball. These data are contained in table #####. These measurements were made using ???###a vernier caliper. Due to the size of the ball and the large uncertainty associated with measuring the full length from the pivot point to the ball center (the center of mass of the ball) we decided it would be best to measure the length of the string from the pivot point to the top of the hook of the ball and then add the measured values for the hook height and ball radius afterwards. The uncertainties from the caliper measurements are small enough compared to the uncertainty in using either a measuring stick or a tape measure in determining the length of the string that making three separate measurements like this do not create an unnecessarily large uncertainty in length. As stated above, a tape measure and/or measuring stick were used to measure the distance from the pivot point to the top of the hook. This was the distance that was measured when the pendulum length was varied.

The next step was to determine the uncertainty of the stopwatch we used. The stopwatches we used were brand #####. There were two sources of systematic uncertainty in the time measurements made by the stopwatch. The first was the gain/lag the stopwatch had over real time. The second was simply the finite gradations of time measured by the stopwatch ($.01s$). To account for the first source of systematic error (and also to determine which of the two authors were most accurate and precise in their time measurements) we timed ourselves against an atomic clock for 1s and 100s intervals. (On the second day, 20s and 50s intervals). These data are available in table ####????. JW was determined to have the best combination of accuracy and precision and was designated as the official timekeeper for the duration of the study. As seen in table ####????, the mean of the time measurements were not equal to the time interval attempted to be measured. This gain/lag were accounted for in our final results by finding an appropriate "weighting factor" by which to multiply all our time measurements to increase their accuracy. The weighting factor was calculated as follows:

$$W = 1 + \frac{t_{atomic} - t_{mean}}{t_{atomic}} \tag{3}$$

where t_{atomic} is the real time as measured by the atomic clock and t_{mean} is the mean of the times measured by the stopwatch. The fraction determines what percent of the real time was measured by the average of the stopwatch times, which is then added to 1 to provide a multiplicative factor to correct our time measurements. The standard deviation of the measured times is included in our final calculation of the error of T.

Measuring against the atomic clock provided a measure of both the stopwatch's own limitations and the user's reaction time. However, it is impossible to separate the two. In order to fully account for the user's reaction time the above error was assumed to be due entirely to the stopwatch's limitations and a separate calculation was made of the timekeeper's reaction time. The timekeeper recorded himself pushing the start and stop button on the stopwatch as fast as possible. The data from this exercise are recorded in table ####????. The average was $.17s$; this number was added to the overall uncertainty of T.

The next thing we did was determine the best way to measure the period of the pendulum's swing, i.e., a method that would allow us to both be accurate (allowing us to obtain a true value for g) and precise (thus reducing random error). We first compared whether it was better to start timing a period from when the ball was released or to wait a period and start timing with the second period. We wanted to see if there was a difference in accuracy and precision between the simultaneous double hand movement of releasing the ball while starting the stopwatch and just timing the period by eye. For this we timed a single swing of the pendulum eight times for both cases (i.e. starting timing with the swing and starting timing with the second swing) this was done for two different lengths: one around $26cm$ and the other around ####????cm. These data are shown in table ####????. Unfortunately the method with the lowest mean (timing from release) had the largest spread in times measured and vice-versa (for timing the second period). We chose to go with the method with the least uncertainty (i.e. lowest spread, or timing the second swing) over greatest accuracy because we figured that the method we used to determine the stopwatch gain/lag was closest to the method that timed the second swing; for both methods, the start and stop time were determined by eye. If we had pushed a button on the atomic clock to start its timing at the same moment we had started the stopwatch, then this method would have been more similar to the method that used the release time of the pendulum to measure T. Therefore, the weighting factor determined by the timing against the atomic clock method factors in the inaccuracy of the second method more than the first method (in addition to the stopwatch's own gain/lag). The inaccuracy of the second method being thus accounted for we are free to choose

4. DATA AND RESULTS

Table 1
Atomic Clock Measurements

Time Lapse	Stop Watch Time (s)
50s	49.85
	49.99
	50.05
	50.09
	49.98
$\overline{At}$	49.99
$\sigma_{\overline{At}}$	0.04

Table 2
Reaction Time

	Stop Watch Reaction Time (s)
	0.15
	0.13
	0.15
	0.14
	0.17
$\overline{Rt}$	0.15
$\sigma_{\overline{Rt}}$	0.01

Table 3
Period Measurement for Varying Lengths

Length (m)			T_{20} (s)			$\overline{T^2}$ (s^2)
0.280	22.68	22.64	22.64	22.58	22.56	6.19
0.415	27.03	27.05	27.07	27.03	27.06	5.78
0.546	30.61	30.74	30.57	30.65	30.58	5.37
0.638	32.97	33.04	33.05	33.03	32.96	4.75
0.800	36.70	36.65	36.73	36.74	36.71	4.13
0.989	40.65	40.73	40.62	40.62	40.72	3.37
1.143	43.60	43.61	43.59	43.56	43.53	2.72
1.293	46.33	46.46	46.40	46.26	46.29	2.34
1.395	48.01	48.06	48.12	48.17	48.09	1.83
1.498	49.77	49.82	49.81	49.72	49.70	1.28

Table 4
Dimensions of the bob

Quantity	Uncertainty in Measurment	Measured Value
Mass (g)	± 0.1	226.2
Diameter (mm)	± 0.03	38.02
Radius (m)	± 0.00002	0.01901
Hook Length (mm)	± 0.03	11.95
Hook Length (m)	± 0.00003	0.01195

5. DISCUSSION

6. CONCLUSION

APPENDIX

APPENDIX A: DERIVING THE EQUATION OF MOTION FOR THE SIMPLE PENDULUM

The motion of a simple pendulum can be derived by analysing the torques (or forces) that acts on the bob of the pendulum. Assume the string that the pendulum hangs from is massless, and its length l does not change. Friction on the string at the fulcrum as well as the air drag on the bob is also ignored. The definition of torque is given by;

$$\boxed{\tau = \vec{r} \times \vec{F} \equiv |\vec{r}||\vec{F}|\hat{n}sin\theta} \tag{A.1}$$

From Figure 1; T is the tension force on the rope, l is the length of the rope (or r$\cdot\hat{r}$, where $\hat{r}$ is a unit vector), θ is the angle, and W is the weight given by m$\cdot$g. Using the definition of the cross product;

$$\begin{aligned} \tau_T &= \vec{l} \times \vec{T} = 0 && (\vec{l} \text{ is } || \text{ to } \vec{T}) \\ \tau_W &= \vec{l} \times \vec{W} \\ &= |\vec{l}||\vec{W}|\hat{n}sin\theta \\ &= -l \cdot W \cdot sin\theta && (\hat{n} \text{ is "-"}) \\ &= -l \cdot mg \cdot sin\theta && (\text{W = m}\cdot\text{g}) \end{aligned}$$

$$\boxed{\tau_W = -lmg sin\theta} \tag{A.2}$$

Newton's second law applied to rotational motion is given by,

$$\tau = I \cdot \alpha \tag{A.3}$$

where I is the moment of inertia. The simple pendulum will be treated as a point particle at a distance l from the origin. Equating the two we get;

$$\begin{aligned} \tau_W = -lmg sin\theta &= I \cdot \alpha \\ &= (ml^2)\alpha && (\text{I} = ml^2 \text{ for a point particle}) \\ &= (ml^2)\ddot{\theta} && (\alpha = \ddot{\theta}) \\ \Rightarrow -g sin\theta &= l\ddot{\theta} \\ \Rightarrow l\ddot{\theta} + g sin\theta &= 0 \end{aligned}$$

The Taylor series of $\sin(\theta)$ is given by

$$\boxed{sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - ...} \tag{A.4}$$

For small angles we get

$$\boxed{sin\theta \approx \theta} \tag{A.5}$$

which is the small angle approximation. Note that the small angle approximation is valid to about 15°.
Applying the small angle approximation to our equation of motion we get

$$\boxed{l\ddot{\theta} + g\theta = 0} \tag{A.6}$$

Solving the the differential equation;

$$\begin{aligned} l\ddot{\theta} + g\theta &= 0 \\ \Rightarrow \ddot{\theta} + \frac{g}{l}\theta &= 0 \end{aligned}$$

The characteristic equation for this Differential Equation is

$$\begin{aligned} x^2 + \frac{g}{l} &= 0 \\ \Rightarrow x &= \pm i\sqrt{\frac{g}{l}} \end{aligned}$$

Which has the general equation

$$\begin{aligned} \theta(t) &= C_1 e^{i\sqrt{(g/l)}t} + C_2 e^{-i\sqrt{(g/l)}t} \\ &= C_1[cos(\sqrt{\frac{g}{l}}t) + isin(\sqrt{\frac{g}{l}}t)]+ \\ &\quad + C_2[cos(\sqrt{\frac{g}{l}}t) - isin(\sqrt{\frac{g}{l}}t)] \\ &= (C_1 + C_2)cos(\sqrt{\frac{g}{l}}t) + i(C_1 - C_2)sin(\sqrt{\frac{g}{l}}t) \\ &= Acos(\sqrt{\frac{g}{l}}t) + Bsin(\sqrt{\frac{g}{l}}t) \end{aligned}$$

If we create a right triangle with the constants A and B, to get a hypotenuse constant C and angle ϕ. That triangle can be describe with the terms;

$$\begin{aligned} sin\phi &= \frac{B}{C} \\ cos\phi &= \frac{A}{C} \end{aligned}$$

Then if we multiply the general solution by C/C;

$$\begin{aligned}\theta(t) &= C[\frac{A}{C}cos(\sqrt{\frac{g}{l}}t) + \frac{B}{C}sin(\sqrt{\frac{g}{l}}t)] \\ &= C[cos(\phi)cos(\sqrt{\frac{g}{l}}t) + sin(\phi)cos(\sqrt{\frac{g}{l}}t)] \\ &= C[cos(\phi)cos(\sqrt{\frac{g}{l}}t) - sin(-\phi)cos(\sqrt{\frac{g}{l}}t)] \\ &= C[cos(\sqrt{\frac{g}{l}}t - \phi)]\end{aligned}$$

ϕ is the phase shift. For the simple pendulum the phase shift is zero because the initial angle is θ_0. This gives

$$\theta(t) = Ccos(\sqrt{\frac{g}{l}}t)$$

Then for t = 0

$$\begin{aligned}\theta(0) &= C[cos(0)] \\ &= C \\ &\equiv \theta_0\end{aligned}$$

Then the solution to the equation of motion is

$$\boxed{\theta(t) = \theta_0[cos(\sqrt{\frac{g}{l}}t)]} \tag{A.7}$$

We know that the derivative of $\theta(t)$ is ω (angular velocity)

$$\begin{aligned}\dot{\theta}(t) &= \omega(t) \\ &= -\sqrt{\frac{g}{l}}\theta_0[sin(\sqrt{\frac{g}{l}}t)]\end{aligned}$$

Angular velocity is zero every time it reaches the max height during an oscillation. We can than see for for what times this occurs by setting $\omega \equiv 0$.

$$\begin{aligned}\omega &\equiv 0 \\ -\sqrt{\frac{g}{l}}\theta_0[sin(\sqrt{\frac{g}{l}}t_n)] &= 0 \\ \Rightarrow \sqrt{\frac{g}{l}}t_n &= sin^{-1}(0) \\ &= 2\pi n \\ \Rightarrow t_n &= 2\pi n\sqrt{\frac{l}{g}}\end{aligned}$$

This periodicity is what we call the Period;

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \tag{A.8}$$

We thank ....

REFERENCES