Determining the Acceleration Due to Gravity with a Simple Pendulum

(Your name)

Advanced Undergrad Lab, Department of Physics and Astronomy, University of Utah, Salt Lake City, 84112 Utah, USA

This is an example of a lab report associated with obtaining the acceleration due to gravity (*g*) and applying mathematical models. This example serves as a template to assist you in writing the lab reports. The bolded text represents wording or ideas that would be stated or included in an actual handout; in this exercise the bolded text is expected to form the core of your report. The unbolded text provides guidance, tips, and other information necessary to carry out this experiment. But use your own words when you write your lab reports. Remember that a paper (your lab report in this case) is a narrative, so everything (text, graphs, results, tables, etc.) has to be tied.

Abstract

Using a simple pendulum and the model that the square of the period of that pendulum is proportional to its length, we report that (a) the model is [not] supported by our data because the relationship between the two parameters is [not] linear based on a value of the correlation coefficient (*r*) = _____ and chi-squared χ^2 = ______, which we calculated from our least-squares fit and (b), the value of g we calculate based on measurements taken in Salt Lake City is ____±____. The precision of our measurement has been improved by corrections to our raw data: (a) an (in/de)crease of _________ m/sec^2 to account for the finite angular displacement of the bob; (b) an (in/de)crease of _________ m/sec^2 to account for the moment of inertia of the bob; (c) an (in/de)crease of _________ m/sec^2 to account for the viscous damping of the motion; (d) an (in/de)crease of _________ m/sec^2 to account for the elongation of the string as it accelerates in its motion. This measured value of *g* is within ______ standard deviations of our calculated value of _____m/sec^2, which includes corrections for the non-sphericity of the earth (+/-_____), the centripetal acceleration required to hold us to the earth (+/-_____) and the fact that in Salt Lake City we are 4800 feet above sea level (+/-_____).

This abstract is just an example. Write it using your own words including the following information: purpose/goal of your work; what technique(s), method(s) and model(s) you used; what results you obtained (values of *g* with corresponding uncertainties for each method); comparison with an accepted value for Salt Lake City; your model worked/didn't work; why?/why not?; what would you improve?

Introduction

Importance of this experiment (Motivation). History of the experiment.

The acceleration due to gravity on the Earth's surface (g) is a critical parameter in our everyday lives; of the four fundamental forces in nature, it is the one of which we are most frequently consciously aware.

Studying the motion of a simple pendulum achieves several goals for the physics undergraduate: (1) It allows us to actually determine the value of g, a common parameter often encountered in undergraduate physics problems, to greater precision than is available from textbooks; (2) it allows us to test a mathematical model of a physical system, namely that the square of the period of a pendulum is proportional to its length and (3) it gives a transparent introduction to sources of errors and their propagation through an experimental calculation and into the result of the experiment. This experiment has been described in great detail by Nelson and Olsson [1].

Theoretical background

When Newton's Law of Universal Gravitation is applied to an object on the surface of the earth, for all practical purposes the mass and radius of the earth may be considered constant. Hence the equation simplifies to:

$$\vec{F}_G = -G\frac{mM_E}{R_E^2}\hat{r} = -mg\hat{r} \qquad g = \frac{GM_E}{R_E^2} \tag{1}$$

where F_G is the gravitational force between a body of mass m and the earth. G is the Universal Gravitational Constant, M_E is the mass of the earth and R_E is the radius of the earth. "g" is referred to as the acceleration due to gravity on the surface of the earth.

Calculate the value of g from this equation; list the source(s) where you got the values of the individual terms in the References section.

One way to determine the value of this constant, g, is to use a simple pendulum. The equation of motion of a pendulum is derived below.

For the simple pendulum in the figure, show that the equation of motion is given by (your report must include the derivation)

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0 \tag{2}$$

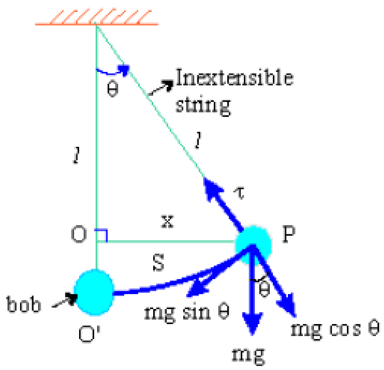

Figure 1. Pendulum free body diagram.

For small displacements in the angle $\boldsymbol{\theta}$, we use a Taylor expansion to simplify the equation of motion. Later we will discuss the magnitude of the error introduced by using this approximation.

Show that in the "small oscillation" limit, this reduces to

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{3}$$

Explicitly write out the Taylor expansion; don't just state the result, justify it and describe the circumstances under which it is [not] true. [Hint: factor θ out of the Taylor expansion.]

We solve the above differential equation from which we obtain an expression for the period of oscillation, $\boldsymbol{T}$, as a function of the pendulum length and $\boldsymbol{g}$.

Show that the solution to Equation 3 is a simple harmonic oscillator (sine wave; note that "sine wave" here is a generic term, the actual expression will depend on boundary conditions). Include both the general and particular solutions to the differential equation and determine appropriate constants, specifically including the correlation between the period and length. Remember that this second order differential equation has three arbitrary constants. You must show how these are determined from appropriate initial (or other) conditions.

As mentioned above, we now discuss what error is produced in using the small angle approximation. The complete solution to the equation of motion of the pendulum without the small angle approximation is given by Nelson and Olsson [1]:

$$\boldsymbol{T(\theta_{max}) = T_0\left(1 + \frac{1}{4}sin^2\frac{\theta_{max}}{2} + \frac{9}{64}sin^4\frac{\theta_{max}}{2} + \cdots\right)} \tag{4}$$

From this equation we can determine the error induced by using the small angle approximation and subsequently see, as a function of the amplitude of the oscillation, $\boldsymbol{\theta_{max}}$, how that error will contribute to the period.

Determine the maximum pendulum amplitude, θ_{max}, for 1% and 0.1% systematic error in the measured period, T. Does it matter if θ_{max} is measured in degrees or radians? Note that $T(\theta_{max})$ is the period that you measure whereas T_0 is the value appropriate for the equation of motion resulting from the small angle approximation. Hence, your measured data must be corrected by this factor. Note also that this is a correction for a systematic error and does not get added in quadrature in determining your uncertainty in your measured value of g.

Experimental Procedure

Setup

Describe apparatus and setup as a narrative. Attach pictures and sketches. Be sure to explain the benefits as well as potential limitations of your setup.

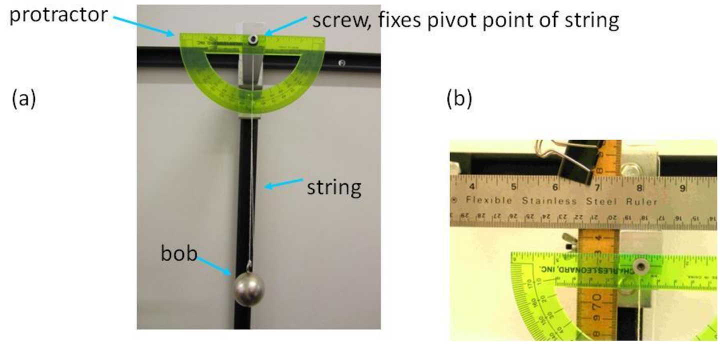

Figure 2. Experimental setup of pendulum. (a) Pendulum with protractor. (b) Close up of some measuring tools.

Describe timing method and equipment. You will want to reference the precision of your time interval measurements to an "atomic clock":

• Use the stopwatch to measure a 1 sec interval on the atomic clock.
• Repeat several times. Calculate the mean and standard deviation.
• Repeat for a 100 sec or longer time interval.
  - Which set of measurements gives you a better gauge of the accuracy of the stopwatch?
  - Which set gives a better gauge of your ability to make the measurement?
• Discuss the accuracy and precision of both sets of measurements in the "Discussion" section.

Describe the length measuring method and equipment. Explicitly address the issue of how you measure the length of the pendulum, e.g. note that the pivot point, the top of the arm, is fixed by pinching the string with a screw. Which is more accurate: a tape or meter stick? What is the significance of the fact that the hook on the end of the measuring tape moves about 1 mm? What different ways can you use to minimize the relative error in measuring the length? How can you use statistical analysis to minimize the relative error the length; i.e. how many different ways can you think of to generate independent measurements of the length of the pendulum which can then be averaged to reduce error? Very important experimental design consideration, resulting from the fact that *g is extracted from the slope of your plot of your data using the least-squares method*: Will your experimental errors be less if you only take data with the maximum length of pendulum or should you include some data sets with shorter pendula? This is very important; it is a potential source of a huge error in your result; think it through carefully. Again, in the "Experimental Procedure" section, list the range of values of length over which you took data; in your "Discussion" section discuss why you made the choices you did and how they relate to your final accuracy and precision.

Procedure

The "Experimental Procedure" section of your report should be a <u>narrative</u> of what you did, addressing the experimental aspects of the issues raised above.

- Connect the bob to a string by means of the hook attached to the bob. (Alternatively there is one bob with no hook: the string is glued directly into the hole intended for attaching the hook).
- Pass the string through a hole in a stout aluminum bar. The wing screw and washer on top lock the string at a fixed length; the screw on front not only holds the protractor but also pins the string, fixing

a precise point of oscillation and length. Be sure that the pendulum can swing freely. Record all data in the provided lab notebook ("Bluebook") and in a software spreadsheet (e.g. Origin, Excel, etc.).

- With the stopwatch in one hand and bob in the other and a length of approximately 20 cm, time the first period as you release the pendulum. Use these measurements to calculate one value of g.
- Repeat, but time the second swing of the pendulum.
- Repeat, but average the measured period over 20 complete oscillations not including the first one. Repeat this procedure 5 times. Calculate the average of those 20 oscillation periods, standard deviation and standard deviation of the mean.
- Repeat after increasing the length of string and with an amplitude which is "quite small".
- Calculate g for each of the above cases and compare to the "accepted value".
- Think about how you can get statistical information on the length of the pendulum which allows you to use statistical methods to reduce the error in length measurements, similar to what you are doing with the time (period) measurements. This is not trivial. It is an important part of your effort to get maximum precision and accuracy in your measurement of g.
- How can you use a mirror to minimize parallax in the length measurement?
- For one length of pendulum, measure 1 set of 20 periods at $\theta_{max} = 3°, 5°, 10°, 20°, 40°$. Compare your results to the model $T(\theta_{max}) = T_0\left(1 + \frac{1}{4}sin^2\frac{\theta_{max}}{2} + \frac{9}{64}sin^4\frac{\theta_{max}}{2} + \cdots\right)$, keeping the third term, at least initially. What angle(s) would you use for your measurements to meet the approximations used in your "Theoretical Introduction" section? Note that here you are testing a non-linear model of the behavior of the system. Do your data support this model?
- For ten (or more) different lengths l of the pendulum (choose the range of lengths, based on your systematic error considerations and considerations discussed above) measure and record the time t_{20} for 20 complete oscillations by using a stopwatch.

Describe here any additional details of your technique which may be of importance. Did any "tricks" help you to get more repeatable measurements? Any pitfalls that you became aware of as you were taking data?

Data and Results

Your data may be attached as an appendix if you use a graph instead ("Data and Results" section), but all raw data must appear somewhere in your submission.

Plot your data for the pendulum period as a function of θ_{max}. Can this function be linearized?

By measuring T for a given measured l, g may be determined using the expression for the period of oscillation derived above. There are two ways to extract g from this equation: (1) simply measure T for a measured length l, and solve the equation for g. Repeat this procedure for those different lengths l, and obtain g as an average. (2) For a range of values of l measure a large number of values of the period, plot the square of the period as a function of the pendulum length (or vice versa depending on their relative errors), perform a least-squares fit to the slope and calculate g from that slope. We will use both approaches and compare the results.

For the first way:

Based on timing a single swing of a short pendulum beginning when the bob was released we determined g to be ____±____. Performing this measurement and calculation took ______ minutes.
For a single swing after the first, we determine g =__±__; this measurement and calculation took____ minutes.
For 20 periods of oscillation (not including the first swing) repeating this procedure 5 times, we determined g =__±__; this measurement and calculation took____ minutes.

Note that to calculate the uncertainty in these results you will have to estimate the errors in your measurements and propagate the estimates through the error propagation formula. Remember to take into account both systematic and random errors.

For the second way:
In a second approach, we measure the periods and uncertainties for the respective lengths, as shown in Table 1.

Table 1. Data for pendulum length and period as well as corrections, calculation of *g* and uncertainties.

Generate your Table 1 (or Table A1 if you will include it in the Appendix A) and fill in with the data from the timing of 20 oscillations. Record all relevant calculations in your lab notebook. Feel free to take additional data at additional lengths.

Based on timing 20 periods of oscillation (not including the first swing) for each of______ different lengths of l, we calculated an average value of g to be ___±___. Performing these measurements and calculations took ______ minutes.
For each of the pairs of values of l and T^2 you measured, as recorded in your Table 1, calculate a value for g. Determine the mean, standard deviation, and standard deviation of the mean for these values of g. Propagate the uncertainties in l and T^2 to σ_g.

We perform a linear least squares fit to values of T^2 and l for the different measured lengths.
Plot your data, including the corresponding error bars, using a plotting and data analysis software (e.g. Origin, Kaleidagraph, etc.). Use the formalism of least-squares fit to determine the line that best fits your data ($y = A + Bx$, where A and B are the fit parameters) as well as the uncertainties in the fit parameters and the correlation coefficient (R or r) and chi-squared (χ^2) values.
Calculate r and χ^2 "by hand" using the expressions given in the corresponding lecture considering your experimental uncertainties. Compare these values with those obtained from the data analysis software you used.
From the slope of the data, we can extract a value for g. Report this value with its corresponding uncertainty. What information could you extract from the intercept?

Figure 3. Pendulum period squared as a function of pendulum length (*or vice versa depending on their relative errors*). The open circles (*or another kind of symbol, color, etc.*) represent the experimental data and the solid line (*or another kind of line, color, etc.*) represents the least-squares fit.

Comparison between methods

Compare the methods used in obtaining *g* with its corresponding uncertainty (for one particular length, as an average of at least 10 different lengths, and using the least-squares method). Compare these results with your corrected calculation g from G, R_E, and M_E.

Residual plot

A useful tool to check the goodness of your fit is the *residual graph* [2]. The residuals R_i are the differences between the T_i^2 values of your data points (or the *l* values depending on the relatives errors in *l* and T^2) and the "predictions" of your best fit line. Complete the table below:

Table 2. Residual calculations (considering the relative errors in T^2 are larger than the relative errors in *l*, but the relative errors in T^2 could be smaller than the relative errors in *l*, so you will have to calculate your residual values using *l* instead of T^2).

l_i	T_i^2	$T_{fit}^2 = A + Bl_i$	$R_i = T_i^2 - T_{fit}^2$

Make a plot of *R* versus *l*. Use the uncertainties in T_i^2 as your error bars for this graph. To have a good fit for a linear model your plot should show a random pattern and at least 2/3 of your data error bars should pass through zero.

Remember: the axes must have titles and units. The plot can be included in this section or in the "Discussion" section. Be sure to label it and reference it as if it were inserted right here.

Discussion

Here is where you get to address the issue: "What does it all mean?" Discuss the significance of your results. Your results were a bunch of numbers; here you get to analyze and explain the significance of these numbers, both the "theoretical" and "experimental" values of *g* that you have determined.

Discuss how (magnitude, increase or decrease, significance…) each of the corrections (finite θ_{max}, damping…) affects your calculated value of *g* (see Appendix of this handout).

Do your results for the measurement of period as a function of θ_{max} for a given length agree with the proposed model? Explain. Quantify.

Calculate a "theoretical" value of *g* from literature values of *G*, M_E and R_E (this handout equation (1)). Correct for the effects of elevation and latitude (there are three): centripetal acceleration, elevation and the non-sphericity of the earth. How does each of these factor in? (Hint: see http://www.ngs.noaa.gov/cgi-bin/grav_pdx.prl). In the "Results" section compare your calculated value to that from the NOAA website, for instance, and to your measurement. From the Nelson and Olsson sections on air resistance and "added mass", comment on the likely reason that pendula in traditional clocks are disc-shaped rather than spherical [1].

Have your efforts tested a model? What model? Do your results support or refute the model? Quantify your support. How do your values for g in compare to the “correct value”. If your values are not within the error of your calculated “correct value”, try to explain unexplored sources of error, starting with inadequate estimates of the errors in your measurements. Was your method of testing the accuracy of the stopwatch adequate; the ability of your reflexes to start and stop the watch? Meter stick and tape measure, both their inherent accuracy and your ability to read them? If your value of *g* differs from the “correct value” by more than two standard deviations of the mean, try to think of plausible explanations, a.k.a. “models”. How would you test your models?

Conclusion

Write a one-paragraph conclusion to your paper. Your conclusion should recap your major numerical result(s), and your main inferences. Begin with a statement of the model you are testing. Then address the question: Do the data support the model you are testing, or not? Use a discrepancy criterion. How certain are you of this claim? Certainty in physics is established by the number of significant digits you can justify. Was the experiment you performed a good way of testing the model? How could you improve your method/experiment?

*The Discussion and Conclusion sections are the most important parts of your report: they are where your intellectual input goes; they are where you convince your audience that you actually know what you are talking about.

References

Cite all references you consulted in the writing of this report. References must be formatted using the conventions of the AID style manual:
http://www.aip.org/pubservs/style/4thed/toc.html

Appendix A

Include here the tables with your data and the corresponding uncertainties.

Appendix B

Include here all sources of errors describing if they are systematic or random (if they weren’t include them in your previous sections), and the formulas with the final mathematical expressions you have obtained and used to propagate the uncertainties involved.

References (cited on this handout)

[1] "The Pendulum – Rich Physics from a Simple System", R. A. Nelson and M. G. Olsson, *Am. J. Phys.* **54**, 112-121 (1986).

[2] http://stattrek.com/statistics/dictionary.aspx?definition=residual_plot.

Appendix (for this handout)

Other considerations you may take into account to explain your results (points extra credit)

Up to now, we have neglected several other factors that will affect our calculated value of g. We address some of those now. You can find a more detailed description in Nelson and Olsson's work [1].

Extensibility of the String-

Consider the simple pendulum shown in the Figure 2. Take the pendulum to be a point-like bob of mass m, attached to a string that doesn't stretch, which has length *l*. How will you determine the extent (quantify, of course) to which the string stretches as the tension in it changes during the course of a period? How can you determine the extent to which the string stretches? How can you arrange other parameters to minimize the effects of a finite spring constant in the string? How does the stretchiness of the string factor into your measurements?

***Damping Contribution* – Air resistance will damp the amplitude of the oscillation over time, slowing the bob and increasing the period of the oscillations. The actual period for a damped oscillator is given by [1]:**

$$T_d = \frac{T_0}{\sqrt{1-\xi^2}} \tag{5}$$

where $\xi = T_0/2\pi\tau$, T_0 is the period in a vacuum and τ is the time for the oscillations to decrease in magnitude by a factor of $1/e$.

Using your setup, estimate τ. Determine the relative systematic error of the measured period T_d compared to T_0. Does damping contribute significantly to the error in your experiment? Note that, as was with the case with $T(\theta_{max})$, T_d gives a correction to be applied to your measurements.

***Rotational Inertia Contribution* – A key difference between a "real world" pendulum and the simple pendulum is the fact that any real pendulum bob will not be point-like. The total energy of the system must be modified to include a rotational inertia term:**

Point-like bob: $$E_{tot} = \frac{1}{2}m(l\dot{\theta})^2 + mgl(1-cos\theta) \quad (KE+PE) \tag{6}$$

Physical bob: $$E_{tot} = \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2 + mgl(1-cos\theta) \tag{7}$$

where I is the moment of inertia of the bob around its center of mass.

(a) From these equations, show that, for a spherical bob, the effect of the rotational term is to modify the period of oscillation by a factor

$$T_{physical} = T_{simple} \times \sqrt{1 + \frac{2a^2}{5l^2}} \tag{8}$$

where a is the radius of the bob. As with the others, this is another correction to your measurement of the period.

(b) Measure the mass and radius of the bob used in our experiment. For what values of the length of the string, l, will this effect produce a 1% change in the period? 0.1%? What is the contribution of the moment of inertia of the bob to your measurements?