MAP 4113 Probability, Random Processes and Applications

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Lecture Notes for Fall2022 semester.

Plan

1. Lecture 2

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What is probability?

- **As a limit of relative frequency:**

$$\lim_{\#\text{ of experiments}\to\infty} \frac{\#\text{ of occurring}}{\#\text{ of experiments}}.$$

- **As a person's belief:** Just a feeling base on your own experience.
- **A function defined on events.**
- Many other answers$\cdots$

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Sample Space and Events

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- The set of all elementary events will be called the **sample space**, denoted by Ω.
- An **event** is a subset of the sample space.
- **Example:** The experiment of flipping a coin twice. The sample space is

$$\Omega = \{HH, HT, TH, TT\}.$$

- Here are several examples of events:
  - At least one time is head :$\{HH, HT, TH\}$.
  - Exactly one head: $\{HT, TH\}$.

More Examples

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- Are they **mutually exclusive**? (Discussion)
- How to describe the relations among different events?

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Use the language of set to describe events

Definition

- The event $E \cup F$ is called the union of events E and F, which means at least one of the events occur.
- The event $E \cap F$ is called the intersection of E and F, conventionally denoted by EF, which means E and F occur at the same time.
- The event $\overline{E}$ stands for the event that E hasn't occurred.
- $E \subset F$ means the occurrence of E implies occurrence of F.
- $E - F$ means event E occurs but F doesn't occur.

Properties

Here are some fundamental properties from the set theory:

1. $E \cup F = F \cup E$ and $EF = FE$.
2. $(E \cup F) \cup G = E \cup (F \cup G)$ and $(EF)G = E(FG)$.
3. $(E \cup F)G = EG \cup FG$ and $EF \cup G = (E \cup G)(F \cup G)$.
4. (**DeMorgan's laws**) $\overline{E \cup F} = \overline{E} \cap \overline{F}$. [**Exercise**: Extend it to n events!]

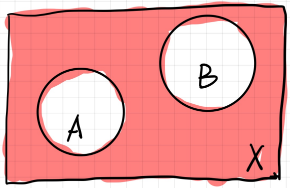
Let's see two exercises:

1. Find X such that

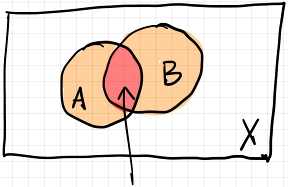
$$\overline{(X \cup A)} \cup \overline{(X \cup \overline{A})} = B.$$

2. Interpret the following relations involving events A,B and C: a). $AB = A$; b). $ABC = A$; c). $A \cup B \cup C = A$.

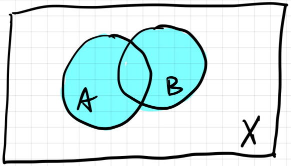
Venn Diagrams



A, B are mutually exclusive



$A \cap B$.



$A \cup B$

Proof of the DeMorgan's Laws

Proof.

Let $x \in \overline{E \cup F}$, then $x \notin E \cup F$, that is to say $x \notin E$ and $x \notin F$. In notation it means $x \in \overline{E}$ and $x \in \overline{F}$ which implies $x \in \overline{E} \cap \overline{F}$, the right hand side.

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On the other hand, one can see that the above argument is reversible and this completes the identity. □

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What is Probability again?

One **intuitive way** to define the probability of an event is as the limit of relative frequency:

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}, \tag{1}$$

where $n(E)$ denotes the time event E occurs after n trails. From this definition, we have

1. $0 \leq P(A) \leq 1$ for any event A. And $P(\Omega) = 1$.
2. Provided that A, B are mutually exclusive.

$$P(A \cup B) = P(A) + P(B)$$

3. Moreover, for mutually exclusive events $\{A_k\}_{k=1}^{n}$, we have

$$P(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k), \quad \text{(addition law for probabilities.)}$$

Definition of Probability

Definition

Let P be a function that assigns a real value to each events and satisfies the following three axioms: Let Ω be the sample space,

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. If events $\{A_k\}_{k\geq 1}$ mutually exclusive, we have $P(\cup_{k\geq 1} A_k) = \sum_{k\geq 1} P(A_k)$.

Such function P is called a probability. [Existence?]

Example

Flip coins once. Let $\Omega = \{H, T\}$. We define $P(\{H\}) = 2/3$ and $P(\{T\}) = 1/3$. Then such P is a probability.

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Example

Flip coins once. Let $\Omega = \{H, T\}$. We define $P(\{H\}) = 2/3$ and $P(\{T\}) = 1/3$. Then such P is a probability. Design a fair game using this coin.

Some properties of a probability

Theorem

For any events A, B, the following relations hold:

$$P(A - B) = P(A) - P(AB), \tag{2}$$

$$P(A \cup B) = P(A) + P(B) - P(AB), \tag{3}$$

$$P(A) \leq P(B) \quad \textit{if} \quad A \subset B. \tag{4}$$

Proof.

Use addition law for probabilities (**the third axiom of probability**). $\square$

An important theorem: The inclusion-exclusion identity

Theorem (The inclusion-exclusion identity)

Let $E_1, \ldots, E_n$ be n arbitrary events, then

$$P(\cup_{i=1}^{n} E_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} \cdots E_{i_r}).$$

Example

Flip a fair coin twice. Then $\Omega = \{HH, HT, TH, TT\}$. Let $E_1 = \{\text{at least one } H\}$ and $E_2 = \{\text{at least one } T\}$. Then
$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2) = 3/4 + 3/4 - 1/2 = 1.$
Note Here for a fair coin, $P(A) = |A|/|\Omega|$.

Proof of the inclusion-exclusion identity

We will prove it by math induction. $n = 1$ is trivial and $n = 2$ is proved in the previous theorem. Suppose the statement is true for n events, let's show it is also true for $n + 1$ events. In fact,

$$
\begin{aligned}
P(\cup_{i=1}^{n+1} E_i) &= P(\cup_{i=1}^{n} E_i \cup E_{n+1}) \\
&= P(\cup_{i=1}^{n} E_i) + P(E_{n+1}) - P((\cup_{i=1}^{n} E_i) E_{n+1}) \\
&= \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} P(E_{i_1} \cdots E_{i_r}) + P(E_{n+1}) \\
&\quad - \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} P(E_{i_1} \cdots E_{i_r} E_{n+1}).
\end{aligned}
$$

Proof of the inclusion-exclusion identity

Since for any $r \leq n$, we have

$$(-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} P(E_{i_1} \cdots E_{i_r}) - (-1)^r \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n} P(E_{i_1} \cdots E_{i_r} E_{n+1})$$

$$= (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n+1} P(E_{i_1} \cdots E_{i_r}).$$

Thus, we have

$$\begin{aligned} P(\cup_{i=1}^{n+1} E_i) &= \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n+1} P(E_{i_1} \cdots E_{i_r}) \\ &+ (-1)^{n+2} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq n} P(E_{i_1} \cdots E_{i_r} E_{n+1}) \\ &= \sum_{r=1}^{n+1} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n+1} P(E_{i_1} \cdots E_{i_r}). \end{aligned}$$

A remark on the inclusion-exclusion identity

For simplicity, one can denote

$$P_k = \sum_{1 \le i_1 < ... < i_k \le n} P(E_{i_1} ... E_{i_k}).$$

Then one can write the inclusion-exclusion identity as follows:

$$P(\bigcup_{i=1}^{n} E_i) = P_1 - P_2 + P_3 - ... + (-1)^{n+1} P_n.$$

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Practice Problems

Example (Coincidences)

Suppose n students put their ID cards inside a box, then draw randomly from the box, what is the probability that at least one student get its own ID card?

Solution: Apply inclusion-exclusion identity. Denote A_k the event that k-th student gets his own ID. We need to compute

$$P(\bigcup_{k=1}^{n} A_k).$$

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Let's first compute P_1. In fact, since $P(A_k) = \frac{(n-1)!}{n!}$, we obtain

$$P_1 = \binom{n}{1}\frac{(n-1)!}{n!}.$$

Cont.

Similarly, one can show that

$$P_m = \binom{n}{m} \frac{(n-m)!}{n!} = \frac{1}{m!}.$$

Thus, from the inclusion-exclusion identity, we have

$$P(\bigcup_{k=1}^{n} A_k) = \sum_{m=1}^{n} (-1)^{m+1} \frac{1}{m!}.$$

What happens if $n \to \infty$?

Limiting probability

If we let $n \to \infty$, we will get

$$P(\bigcup_{k=1}^{\infty} A_k) = 1 - \frac{1}{e} \approx 0.63212055882. \tag{5}$$

Using Taylor's theorem, we have

$$P(\bigcup_{k=1}^{n} A_k) = 1 - \frac{1}{e} + R_n,$$

where the error term is bounded as follows

$$|R_n| \leq \frac{c}{(n+1)!}.$$

More Practice

If n people are in a room, what is the probability that no two of them were born at the same date of the year (ignoring Feb 29th)?

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Proof.

The size of the sample space is 365^n. The event can also be easily computed: that is why?

$$\frac{365!}{(365-n)!}$$

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Proof.

The size of the sample space is 365^n. The event can also be easily computed: that is why?

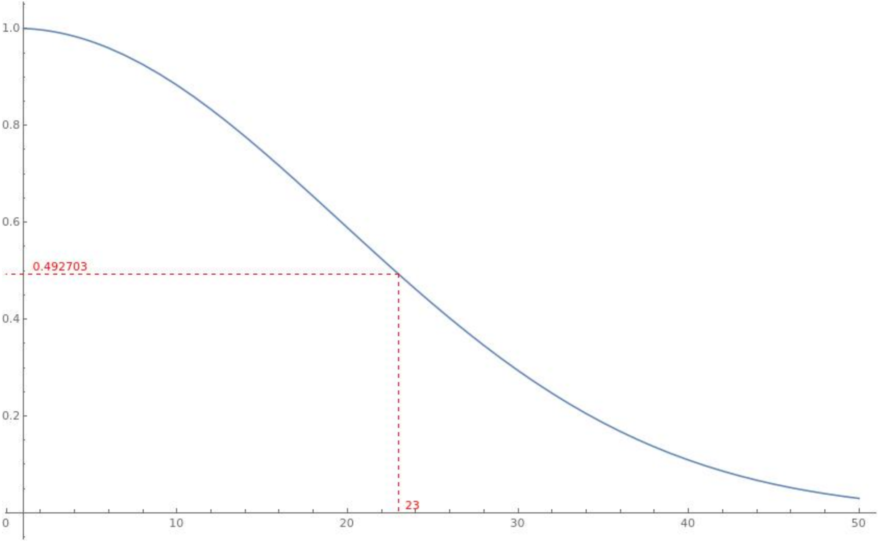
$$\frac{365!}{(365-n)!}$$

Thus, suppose each outcome is equally likely, the probability is

$$\frac{365!}{(365-n)!(365)^n}$$

□

Cont.



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Homework

- ★ Ross Chapter 2, Problem: 2, 4, 15, 27, 41
- ★ Ross Chapter 2, Theoretical Exercises: 4, 10, 8, 16, 19,