Functional Analysis

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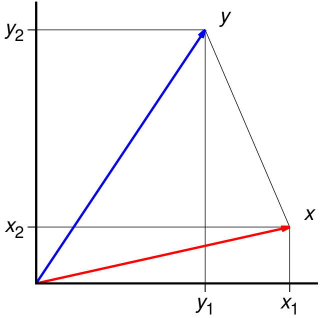
Lecture 4
Tuesday 13 February 2024

Topics:

- §2.4: Inner product spaces
- §2.5: Orthonormal systems and Gram-Schmidt

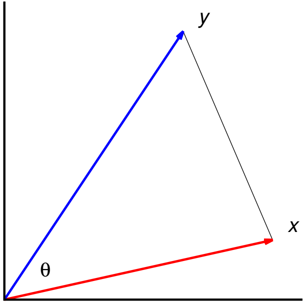
Inner product in $\mathbb{R}^2$



Euclidean distance between x and y:

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Inner product in $\mathbb{R}^2$



Alternative computation via the law of cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\|x\|\,\|y\|\cos(\theta)$$

Inner product in $\mathbb{R}^2$

$$\begin{aligned}
\|x-y\|^2 &= \|x\|^2+\|y\|^2-2\,\|x\|\,\|y\|\cos(\theta)\\
\|x\|\,\|y\|\cos\theta &= \tfrac{1}{2}\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right)\\
&= \tfrac{1}{2}\left(x_1^2+x_2^2+y_1^2+y_2^2-(x_1-y_1)^2-(x_2-y_2)^2\right)\\
&= x_1y_1+x_2y_2\\
&=: \langle x,y\rangle \qquad \text{“inner product of } x \text{ and } y\text{”}
\end{aligned}$$

Note: nonzero vectors x and y are orthogonal $\Leftrightarrow$ $\langle x,y\rangle=0$

Inner product spaces

Definition: let X be a linear space over $\mathbb{K}$

A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an inner product if

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
3. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle \qquad \lambda, \mu \in \mathbb{K}$
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

   [For $\mathbb{K} = \mathbb{R}$ we simply have $\langle x, y \rangle = \langle y, x \rangle$]

Inner product spaces

If $\mathbb{K} = \mathbb{R}$, then the IP is linear in the second component:

$$\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$$

If $\mathbb{K} = \mathbb{C}$, then the IP is conjugate-linear in the second component:

$$\langle x, \lambda y + \mu z \rangle = \overline{\lambda} \langle x, y \rangle + \overline{\mu} \langle x, z \rangle$$

[Exercise: prove these statements]

Inner product spaces

Examples:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i, \quad x, y \in \mathbb{K}^n$$

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i, \quad x, y \in \ell^2$$

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\, dt, \quad f, g \in \mathcal{C}([a, b], \mathbb{K})$$

[Exercise: for ℓ^2 show that the infinite sum converges absolutely using Hölder's ineq.]

Cauchy-Schwarz inequality

Lemma: if X is an IPS then

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \qquad \forall x, y \in X$$

Proof: for all $\lambda \in \mathbb{K}$

$$0 \leq \langle x - \lambda y, x - \lambda y\rangle = \langle x, x\rangle - \lambda\langle y, x\rangle - \bar{\lambda}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle$$

For $\lambda = t\langle x, y\rangle$ with $t \in \mathbb{R}$:

$$0 \leq \langle x, x\rangle - 2t|\langle x, y\rangle|^2 + t^2|\langle x, y\rangle|^2\langle y, y\rangle =: c + bt + at^2$$

Discriminant: $b^2 - 4ac \leq 0 \;\Rightarrow\;$ CS inequality

Cauchy-Schwarz inequality

Corollary: if X is an IPS, then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm

Proof of triangle inequality:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 \\
&\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \text{[by CS ineq.]} \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

[Exercise: verify the remaining properties of a norm]

Cauchy-Schwarz inequality

Corollary: if X is an IPS, then

$$x_n \to x, \quad y_n \to y \quad \Rightarrow \quad \langle x_n, y_n \rangle \to \langle x, y \rangle$$

Proof: with $M = \sup\{\|y_n\| \, : \, n \in \mathbb{N}\}$ we have

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
&\leq |\langle x_n, y_n \rangle - \langle x, y_n \rangle| + |\langle x, y_n \rangle - \langle x, y \rangle| \\
&= |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\
&\leq \|y_n\| \|x_n - x\| + \|x\| \|y - y_n\| \qquad \text{[by CS ineq.]} \\
&\leq M \|x_n - x\| + \|x\| \|y - y_n\| \to 0
\end{aligned}$$

Identities

$\|x\| = \sqrt{\langle x, x \rangle}$

Parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Polarization identity ($\mathbb{K} = \mathbb{R}$):

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2$$

Polarization identity ($\mathbb{K} = \mathbb{C}$):

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

Orthogonality

Notation: $x \perp y$ if $\langle x, y \rangle = 0$ (x and y are called orthogonal)

Pythagorean theorem:

$$x \perp y \quad \Rightarrow \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned}$$

Orthogonality

Lemma: if X is an IPS and $V \subset X$ a subset, then the orthogonal complement of V defined by

$$V^{\perp} = \{x \in X \,:\, \langle x, v\rangle = 0 \text{ for all } v \in V\}$$

is a closed linear subspace

Proof: if $x, y \in V^{\perp}$ and $\lambda, \mu \in \mathbb{K}$, then

$$\langle \lambda x + \mu y, v\rangle = \lambda\langle x, v\rangle + \mu\langle y, v\rangle = 0 \quad \text{for all} \quad v \in V$$

If (x_n) in $V^{\perp}$ and $x_n \to x$, then

$$\langle x, v\rangle = \lim_{n\to\infty}\langle x_n, v\rangle = 0 \quad \text{for all} \quad v \in V$$

Best approximations

Definition: let X be a NLS and $V \subset X$ a subset

$v_0 \in V$ is called a best approximation of $x \in X$ if

$$\|x - v_0\| = d(x, V) := \inf\{\|x - v\| \, : \, v \in V\}$$

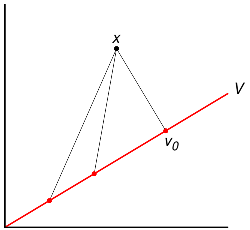
Remark: in an arbitrary NLS the existence and uniqueness of best approximations is a delicate matter!

Characterization in an IPS

Lemma: let X be an IPS and $V \subset X$ a linear subspace

If $x \in X$ and $v_0 \in V$ then

$$\|x - v_0\| = d(x, V) \iff x - v_0 \in V^{\perp}$$



[Exercise: where do we use that V is a linear subspace in the proof?]

Characterization in an IPS

Claim: $\|x - v_0\| = d(x, V) \quad \Leftrightarrow \quad x - v_0 \in V^\perp$

Proof ($\Rightarrow$): for all $v \in V$ and $\lambda \in \mathbb{K}$:

$$\begin{aligned} \|x - v_0\|^2 &\leq \|x - v_0 - \lambda v\|^2 \\ &= \|x - v_0\|^2 - \bar{\lambda}\langle x - v_0, v\rangle - \lambda\langle v, x - v_0\rangle + |\lambda|^2\|v\|^2 \end{aligned}$$

Let $\lambda = t\langle x - v_0, v\rangle$ with $t > 0$:

$$2|\langle x - v_0, v\rangle|^2 \leq t|\langle x - v_0, v\rangle|^2\|v\|^2$$

$t \to 0 \ \Rightarrow \ x - v_0 \in V^\perp$

Characterization in an IPS

Claim: $\|x - v_0\| = d(x, V) \quad \Leftrightarrow \quad x - v_0 \in V^\perp \qquad (v_0 \in V)$

Proof ($\Leftarrow$): for all $v \in V$ we have

$$\begin{aligned} \|x - v\|^2 &= \|x - v_0 + v_0 - v\|^2 \\ &= \|x - v_0\|^2 + \|v_0 - v\|^2 \qquad (x - v_0 \perp v_0 - v) \\ &\geq \|x - v_0\|^2 \end{aligned}$$

Taking the infimum over all $v \in V$ gives $d(x, V) \geq \|x - v_0\|$

[Recall: inf = *greatest* lower bound]

We also have $d(x, V) \leq \|x - v_0\|$

[Recall: inf is a lower bound]

Existence and uniqueness in an IPS

Lemma: let X be IPS and $V \subset X$ a linear subspace

$\dim V < \infty \;\Rightarrow\; \forall\, x \in X \;\; \exists$ a unique best approximation $v_0 \in V$

Existence and uniqueness in an IPS

Proof: let $V = \operatorname{span}\{e_1, \ldots, e_n\}$

Writing $v_0 = c_1 e_1 + \cdots + c_n e_n$ gives

$$\begin{aligned} x - v_0 \in V^\perp &\Leftrightarrow \langle x - v_0, v\rangle = 0 \qquad \forall\, v \in V \\ &\Leftrightarrow \langle x - v_0, e_k\rangle = 0 \qquad \forall\, k = 1, \ldots, n \\ &\Leftrightarrow \langle x, e_k\rangle = \langle v_0, e_k\rangle \qquad \forall\, k = 1, \ldots, n \\ &\Leftrightarrow \langle x, e_k\rangle = \sum_{j=1}^{n} c_j \langle e_j, e_k\rangle \qquad \forall\, k = 1, \ldots, n \end{aligned}$$

Existence and uniqueness in an IPS

Proof (ctd):

$$x - v_0 \in V^\perp \Leftrightarrow \underbrace{\begin{bmatrix} \langle e_1, e_1 \rangle & \dots & \langle e_n, e_1 \rangle \\ \vdots & & \vdots \\ \langle e_1, e_n \rangle & \dots & \langle e_n, e_n \rangle \end{bmatrix}}_{G} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle x, e_1 \rangle \\ \vdots \\ \langle x, e_n \rangle \end{bmatrix}$$

Exercise: $\det G = 0 \;\Leftrightarrow\; \{e_1, \dots, e_n\}$ linearly dependent

$\{e_1, \dots, e_n\}$ linearly indep. $\;\Rightarrow\; c_1, \dots, c_n$ uniquely determined

Orthonormal sets

Definition: if X is an IPS, then $\{e_i : i \in I\} \subset X$ is called an orthonormal set if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Remark: for any finite set $F \subset I$ we have

$$\left\| \sum_{i \in F} \lambda_i e_i \right\|^2 = \left\langle \sum_{i \in F} \lambda_i e_i, \sum_{j \in F} \lambda_j e_j \right\rangle = \sum_{i \in F} |\lambda_i|^2$$

In particular, orthonormal vectors are linearly independent

Gram-Schmidt procedure

Theorem: Let X be an IPS and $f_1, \dots, f_n$ be linearly independent

There exist orthonormal vectors $e_1, \dots, e_n$ such that

$$\operatorname{span}\{e_1, \dots, e_k\} = \operatorname{span}\{f_1, \dots, f_k\} \quad \forall\, k = 1, \dots, n$$

Gram-Schmidt procedure

Proof:

$$e_1 = \frac{f_1}{\|f_1\|} \quad \Rightarrow \quad \|e_1\| = 1 \qquad \text{span}\{e_1\} = \text{span}\{f_1\}$$

$$\widetilde{e}_2 = f_2 - \langle f_2, e_1 \rangle e_1 \quad \Rightarrow \quad \langle \widetilde{e}_2, e_1 \rangle = 0 \quad \widetilde{e}_2 \neq 0$$

$$e_2 = \frac{\widetilde{e}_2}{\|\widetilde{e}_2\|} \quad \Rightarrow \quad \langle e_2, e_1 \rangle = 0 \quad \|e_2\| = 1$$

$$\text{span}\{e_1, e_2\} = \text{span}\{f_1, f_2\}$$

Gram-Schmidt procedure

Proof (ctd): assume $\{e_1, \ldots, e_k\}$ are orthonormal and

$$\begin{aligned}
\operatorname{span}\{e_1, \ldots, e_k\} &= \operatorname{span}\{f_1, \ldots, f_k\} \\
\widetilde{e}_{k+1} &= f_{k+1} - \sum_{i=1}^{k} \langle f_{k+1}, e_i \rangle e_i \quad \Rightarrow \quad \widetilde{e}_{k+1} \neq 0 \\
\langle \widetilde{e}_{k+1}, e_j \rangle &= 0 \quad j = 1, \ldots, k \\
e_{k+1} &= \widetilde{e}_{k+1} / \|\widetilde{e}_{k+1}\|
\end{aligned}$$

Then $\{e_1, \ldots, e_{k+1}\}$ are orthonormal and

$$\operatorname{span}\{e_1, \ldots, e_{k+1}\} = \operatorname{span}\{f_1, \ldots, f_{k+1}\}$$

Best approximations revisited

Lemma: let X be IPS and $V \subset X$ a linear subspace

$\dim V < \infty \;\Rightarrow\; \forall\, x \in X \;\; \exists$ a unique best approximation $v_0 \in V$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V then

$$v_0 = \sum_{j=1}^{n} \langle x, e_j \rangle e_j$$

Best approximations revisited

Proof: let $c_j = \langle x, e_j \rangle$ then

$$\begin{aligned}\left\| x - \sum_{j=1}^{n} \lambda_j e_j \right\|^2 &= \|x\|^2 - \sum_{j=1}^{n} \bar{\lambda}_j c_j - \sum_{j=1}^{n} \lambda_j \bar{c}_j + \sum_{j=1}^{n} |\lambda_j|^2 \\ &= \|x\|^2 + \sum_{j=1}^{n} |\lambda_j - c_j|^2 - \sum_{j=1}^{n} |c_j|^2\end{aligned}$$

Minimum attained $\Leftrightarrow \lambda_j = c_j$ for all j

[Exercise: verify the equalities above]