Functional Analysis

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Lecture 5
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Topics:

- §3.1: Banach spaces
- §3.3: Completion of normed linear spaces

Cauchy sequences

Let X be a linear space with a norm $\|\cdot\|$

Definition: (x_n) is a Cauchy sequence in X if

$$\|x_n - x_m\| \to 0 \quad \text{as} \quad n, m \to \infty$$

Formally:

$$\forall\, \varepsilon > 0 \quad \exists\, N > 0 \quad \text{such that} \quad n, m \geq N \quad \Rightarrow \quad \|x_n - x_m\| \leq \varepsilon$$

Cauchy sequences

Claim: convergent $\Rightarrow$ Cauchy

Proof: if $x_n \to x$ then

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x + x - x_m\| \\ &\leq \|x_n - x\| + \|x - x_m\| \to 0 \quad \text{as} \quad n, m \to \infty \end{aligned}$$

Definition: a NLS is called a Banach space if every Cauchy sequence converges

Finite-dimensional spaces

Proposition: if X is a NLS with $\dim X < \infty$, then X is Banach

Proof: denote the given norm on X by $\|\cdot\|$

Let $X = \operatorname{span}\{e_1, \ldots, e_d\}$ and define another norm by

$$\|x\|_+ = \sum_{i=1}^{d} |\lambda_i| \qquad \text{where} \qquad x = \sum_{i=1}^{d} \lambda_i e_i$$

By norm equivalence there exist constants $a, b > 0$ such that

$$a\|x\| \leq \|x\|_+ \leq b\|x\| \quad \text{for all} \quad x \in X$$

Finite-dimensional spaces

Proof (ctd): let (x_n) be Cauchy in X w.r.t. $\|\cdot\|$

Write $x_n = \sum_{i=1}^d \lambda_{n,i} e_i$, then for fixed i we have

$$|\lambda_{n,i} - \lambda_{m,i}| \leq \|x_n - x_m\|_+ \leq b\|x_n - x_m\| \to 0 \quad \text{as} \quad m, n \to \infty$$

Since $\mathbb{K}$ is complete, $\lambda_i := \lim_{n\to\infty} \lambda_{n,i}$ exists

Define $x = \sum_{i=1}^d \lambda_i e_i$ and note that

$$\|x_n - x\| \leq \frac{1}{a}\|x_n - x\|_+ = \frac{1}{a}\sum_{i=1}^{d} |\lambda_{n,i} - \lambda_i| \to 0 \quad \text{as} \quad n \to \infty$$

The ℓ^p and ℓ^∞ spaces

Theorem: the following spaces are Banach spaces:

$$\begin{aligned}
\ell^p &= \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_i \in \mathbb{K}, \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \quad p \geq 1 \\
\|x\|_p &= \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
\ell^\infty &= \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_i \in \mathbb{K}, \quad \sup_{i \in \mathbb{N}} |x_i| < \infty \right\} \\
\|x\|_\infty &= \sup_{i \in \mathbb{N}} |x_i|
\end{aligned}$$

The ℓ^p and ℓ^∞ spaces

Proof: assume (x^n) is Cauchy in ℓ^∞ and write

$$x^n = (x_1^n, x_2^n, x_3^n, \dots)$$

For fixed $i \in \mathbb{N}$ we have

$$|x_i^n - x_i^m| \leq \|x^n - x^m\|_\infty \to 0 \quad \text{as} \quad m, n \to \infty$$

Completeness of $\mathbb{K}$: $x_i := \lim_{n\to\infty} x_i^n$ exists for all $i \in \mathbb{N}$

Claim: $x := (x_1, x_2, x_3, \dots) \in \ell^\infty$ and $\|x^n - x\|_\infty \to 0$

The ℓ^p and ℓ^∞ spaces

Proof (ctd): for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} m, n \geq N \quad &\Rightarrow \quad \|x^n - x^m\|_\infty \leq \varepsilon \\ &\Rightarrow \quad |x_i^n - x_i^m| \leq \varepsilon \qquad \forall\, i \in \mathbb{N} \end{aligned}$$

$$\begin{aligned} n \geq N \quad &\Rightarrow \quad |x_i^n - x_i| \leq \varepsilon \qquad \forall\, i \in \mathbb{N} \qquad \text{[take } m \to \infty\text{]} \\ &\Rightarrow \quad \sup_{i \in \mathbb{N}} |x_i^n - x_i| \leq \varepsilon \\ &\Rightarrow \quad \|x^n - x\|_\infty \leq \varepsilon \qquad \text{[in particular: } x^N - x \in \ell^\infty\text{]} \end{aligned}$$

Hence, $x = x^N - (x^N - x) \in \ell^\infty$ and $x^n \to x$

Relation with closedness

Proposition: if X is a NLS and $V \subset X$ a lin. subspace, then

1. X Banach and V closed $\Rightarrow$ V Banach
2. V Banach $\Rightarrow$ V closed in X

Proof (1): (v_n) Cauchy in V $\Rightarrow$ (v_n) Cauchy in X

$v_n \to x$ for some $x \in X$

Hence, $x \in \overline{V} = V$

Relation with closedness

Proposition: if X is a NLS and $V \subset X$ a lin. subspace, then

1. X Banach and V closed $\Rightarrow$ V Banach
2. V Banach $\Rightarrow$ V closed in X

Proof (2): if $x \in \overline{V}$, then $v_n \to x$ for some sequence (v_n) in V

(v_n) convergent $\Rightarrow$ (v_n) Cauchy in V

$v_n \to v$ for some $v \in V$

$x = v \in V$ since limits are unique

Relation with closedness

Exercise: for any nonempty set S the following space is Banach:

$$\mathcal{B}(S,\mathbb{K}) = \left\{ f : S \to \mathbb{K} \,:\, \sup_{s\in S} |f(s)| < \infty \right\}$$
$$\|f\|_\infty = \sup_{s\in S} |f(s)|$$

[Note that $\mathcal{B}(\mathbb{N},\mathbb{K})$ is isomorphic to ℓ^∞]

$\mathcal{C}([a,b],\mathbb{K})$ is closed in $\mathcal{B}([a,b],\mathbb{K})$ and thus a Banach space

Absolute convergence criterion

Theorem: if X is Banach, then

$$\sum_{i=1}^{\infty} \|x_i\| < \infty \quad \Rightarrow \quad \sum_{i=1}^{\infty} x_i \text{ converges}$$

Proof: writing $s_n = x_1 + x_2 + \cdots + x_n$ gives

$$\|s_n - s_m\| = \left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\| \to 0 \qquad n, m \to \infty$$

(s_n) Cauchy and X Banach $\Rightarrow$ (s_n) convergent

Absolute convergence criterion

Proposition: let X be a NLS such that for any series we have

$$\sum_{i=1}^{\infty} \|x_i\| < \infty \quad \Rightarrow \quad \sum_{i=1}^{\infty} x_i \quad \text{converges}$$

Then X is a Banach space

Absolute convergence criterion

Proof: assume (s_n) is Cauchy

For each $i \in \mathbb{N}$ there exists $N_i \in \mathbb{N}$ such that

$$n, m \geq N_i \quad \Rightarrow \quad \|s_n - s_m\| \leq \frac{1}{2^i}$$

Without loss of generality we may assume that

$$N_1 < N_2 < N_3 < \cdots$$

Absolute convergence criterion

Proof: note that for $k \geq 2$ we have

$$s_{N_k} = s_{N_1} + \sum_{i=1}^{k-1} (s_{N_{i+1}} - s_{N_i}) \quad \text{with} \quad \|s_{N_{i+1}} - s_{N_i}\| \leq \frac{1}{2^i}$$

By assumption the series $\sum_{i=1}^{\infty} (s_{N_{i+1}} - s_{N_i})$ converges

Thus the subsequence (s_{N_k}) converges

Hence, (s_n) itself converges

[Exercise: show that a Cauchy seq. with a convergent subseq. is convergent]

Quotient spaces

Proposition: Let $(X, \|\cdot\|)$ be Banach

$V \subset X$ closed linear subspace $\Rightarrow$ X/V is Banach

Proof: the strategy is to show that

$$\sum_{i=1}^{\infty} \|x_i + V\| < \infty \;\Rightarrow\; \sum_{i=1}^{\infty} (x_i + V) \quad \text{converges in } X/V$$

Quotient spaces

Proof (ctd): recall that

$$\|x_i + V\| = d(x_i, V) = \inf\{\|x_i - v\| \,:\, v \in V\}$$

There exists $v_i \in V$ such that $\|x_i - v_i\| < \|x_i + V\| + 1/2^i$

Note that

$$\sum_{i=1}^{\infty} \|x_i + V\| < \infty \;\; \Rightarrow \;\; \sum_{i=1}^{\infty} \|x_i - v_i\| < \infty$$

Since X is Banach there exists $x \in X$ such that $\sum_{i=1}^{n}(x_i - v_i) \to x$

Quotient spaces

Proof (ctd): $x_i + V = (x_i - v_i) + V$ since $v_i \in V$ so

$$\begin{aligned}
\left\|(x+V) - \sum_{i=1}^{n}(x_i + V)\right\| &= \left\|(x+V) - \sum_{i=1}^{n}(x_i - v_i + V)\right\| \\
&= \left\|\left(x - \sum_{i=1}^{n}(x_i - v_i)\right) + V\right\| \\
&\leq \left\|x - \sum_{i=1}^{n}(x_i - v_i)\right\| \to 0
\end{aligned}$$

The completion theorem

Theorem: Let X be a NLS

There exists a Banach space $\widetilde{X}$ and a lin. map $\iota : X \to \widetilde{X}$ s.t.

1. X and $\iota(X)$ are isometrically isomorphic

2. $\iota(X)$ dense in $\widetilde{X}$

Moral: every NLS can be completed

Proof: step 1

Consider the following sets:

$$\mathcal{X} = \{\text{all Cauchy sequences } \mathbf{x} = (x_i)_{i=1}^{\infty} \text{ in } X\}$$

$$\mathcal{V} = \{\text{all sequences } \mathbf{x} = (x_i)_{i=1}^{\infty} \text{ in } X \text{ such that } x_i \to 0\}$$

Note: $\mathcal{X}$ is a linear space and $\mathcal{V} \subset \mathcal{X}$ is a linear subspace

Turn $\widetilde{X} := \mathcal{X}/\mathcal{V}$ into a normed linear space via

$$\|\mathbf{x} + \mathcal{V}\| = \lim_{i\to\infty} \|x_i\|$$

[Exercise: show that this indeed gives a norm on $\widetilde{X}$]

Proof: step 2

Define the linear map

$$\iota : X \to \widetilde{X}, \qquad \iota(x) = (x, x, x, \dots) + \mathcal{V}$$

Note that ι is an isometry: for any $x \in X$ we have

$$\|\iota(x)\| = \|(x, x, x, \dots) + \mathcal{V}\| = \lim_{i \to \infty} \|x\| = \|x\|$$

In particular:

- $\iota : X \to \iota(X)$ is linear, bijective, and isometric
- X and $\iota(X)$ are isometrically isomorphic

Proof: step 3

Let $\mathbf{x} + \mathcal{V} \in \widetilde{X}$ and $\varepsilon > 0$ be arbitrary

Since $\mathbf{x} = (x_i)_{i=1}^{\infty}$ is Cauchy in X, there exists $N \in \mathbb{N}$ such that

$$\|x_i - x_j\| \leq \varepsilon \qquad \forall\, i, j \geq N$$

Fixing $j = N$ gives

$$\|x_i - x_N\| \leq \varepsilon \qquad \forall\, i \geq N$$

This implies

$$\|(\mathbf{x} + \mathcal{V}) - \iota(x_N)\| = \lim_{i \to \infty} \|x_i - x_N\| \leq \varepsilon$$

Conclusion: $\iota(X)$ is dense in $\widetilde{X}$

Proof: step 4

Let $\mathbf{x}^n + \mathcal{V}$ be a Cauchy sequence in $\widetilde{X}$

To show: there exists $\mathbf{z} = (z_i)_{i=1}^{\infty}$ in $\mathcal{X}$ such that

$$\|(\mathbf{x}^n + \mathcal{V}) - (\mathbf{z} + \mathcal{V})\| \to 0 \quad \text{as} \quad n \to \infty$$

$\iota(X)$ is dense in $\widetilde{X}$: for each $n \in \mathbb{N}$ there exists $z_n \in X$ such that

$$\|(\mathbf{x}^n + \mathcal{V}) - \iota(z_n)\| \leq \frac{1}{n}$$

Proof: step 4 (ctd)

For each $\varepsilon > 0$ there exists $N > 0$ such that

$$n, m \geq N \quad \Rightarrow \quad \begin{cases} \|\iota(z_n) - (\mathbf{x}^n + \mathcal{V})\| \leq \frac{1}{6}\varepsilon \\ \|(\mathbf{x}^n + \mathcal{V}) - (\mathbf{x}^m + \mathcal{V})\| \leq \frac{1}{6}\varepsilon \\ \|(\mathbf{x}^m + \mathcal{V}) - \iota(z_m)\| \leq \frac{1}{6}\varepsilon \end{cases}$$

$$\Rightarrow \quad \|\iota(z_n) - \iota(z_m)\| \leq \tfrac{1}{6}\varepsilon + \tfrac{1}{6}\varepsilon + \tfrac{1}{6}\varepsilon = \tfrac{1}{2}\varepsilon$$

$$\Rightarrow \quad \|z_n - z_m\| \leq \tfrac{1}{2}\varepsilon$$

$$\Rightarrow \quad \mathbf{z} = (z_i)_{i=1}^{\infty} \text{ Cauchy in } X \text{ so } \mathbf{z} \in \mathcal{X}$$

Proof: step 4 (ctd)

If $n \geq N$, then

$$\begin{aligned} \|(\mathbf{x}^n + \mathcal{V}) - (\mathbf{z} + \mathcal{V})\| &\leq \|(\mathbf{x}^n + \mathcal{V}) - \iota(z_n)\| + \|\iota(z_n) - (\mathbf{z} + \mathcal{V})\| \\ &\leq \tfrac{1}{6}\varepsilon + \|\iota(z_n) - (\mathbf{z} + \mathcal{V})\| \\ &= \tfrac{1}{6}\varepsilon + \lim_{i\to\infty} \|z_n - z_i\| \\ &\leq \tfrac{1}{6}\varepsilon + \tfrac{1}{2}\varepsilon \; < \; \varepsilon \end{aligned}$$

Conclusion: $\mathbf{x}^n + \mathcal{V} \to \mathbf{z} + \mathcal{V}$ in $\widetilde{X}$

The L^p spaces

Example: $\mathcal{C}([a,b],\mathbb{K})$ is NOT Banach w.r.t. the norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} \qquad (1 \leq p < \infty)$$

For a suitable discontinuous function $f_0 : [a,b] \to \mathbb{K}$ define

$$X = \{f + \lambda f_0 \,:\, f \in \mathcal{C}([a,b],\mathbb{K}), \lambda \in \mathbb{K}\}$$

Now show that $\mathcal{C}([a,b],\mathbb{K})$ is NOT closed in X w.r.t. $\|\cdot\|_p$

The L^p spaces

Definition: $L^p(a,b)$ is the completion of $\mathcal{C}([a,b],\mathbb{K})$ w.r.t.

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \qquad (1 \leq p < \infty)$$

Alternative: obtain $L^p(a,b)$ via Lebesgue measure and integral

[See §3.4 in the lecture notes]