Iterative Algorithms in Optimization, Variational Analysis and Fixed Point Theory

Unit 01: Introduction. Fundamentals of iterative algorithms.

university of groningen

About the course

Description

This course provides the theoretical bases for the numerical approximation of solutions to optimization problems, variational inequalities and fixed point problems by means of iterative algorithms.

Assessment

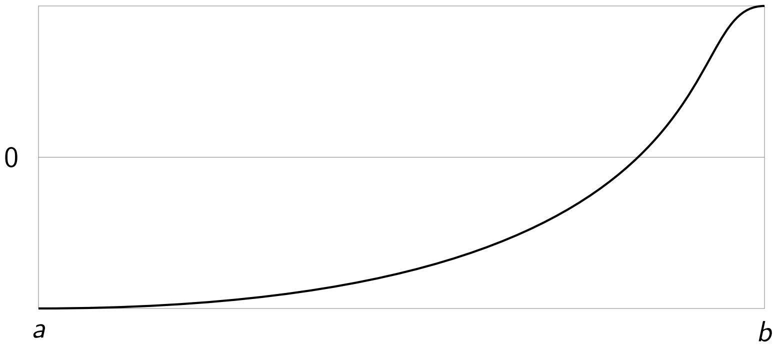
Computational exercises 20%, Homework assignments 50%, Exam 30%.

Dynamics and Schedule

16 sessions: Lectures, exercises. Mo, 9-11. Tu, 9-11. We, 17-19.

Introduction to iterative algorithms

Example: Bisection method to solve $g(x) = 0$



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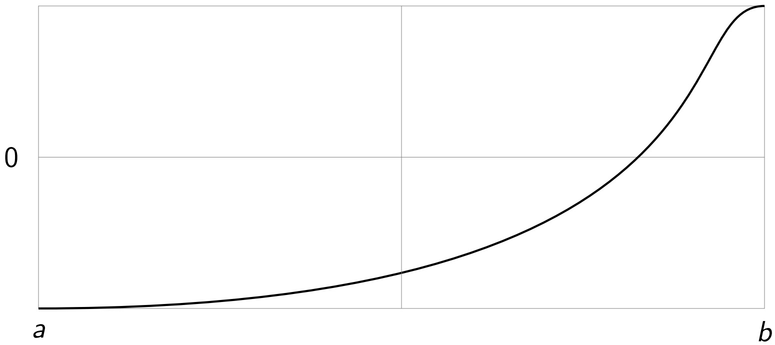

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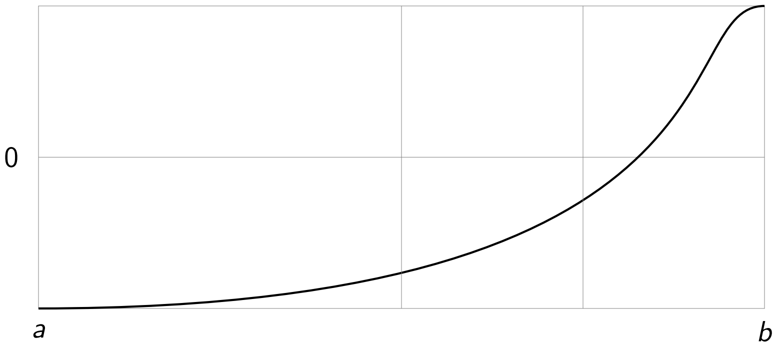

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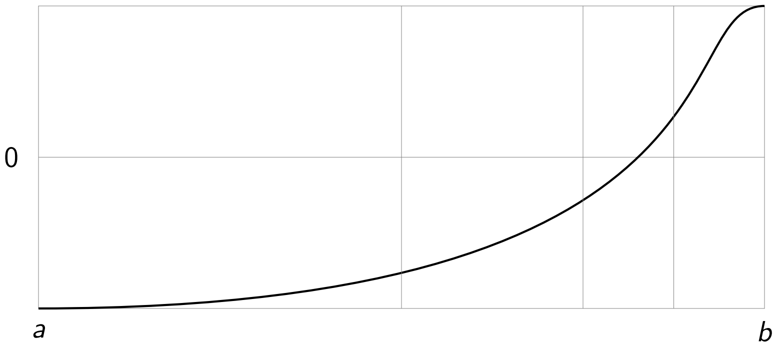

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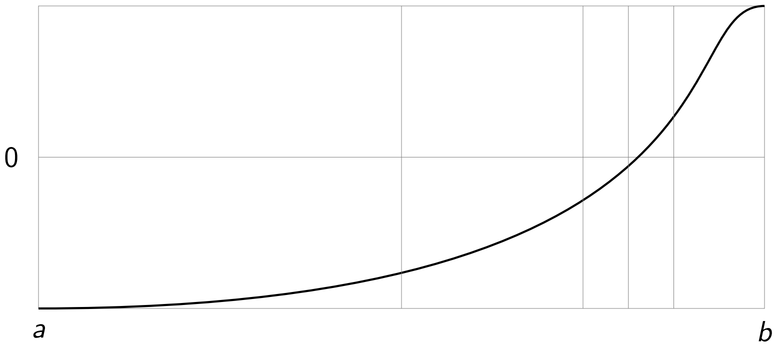

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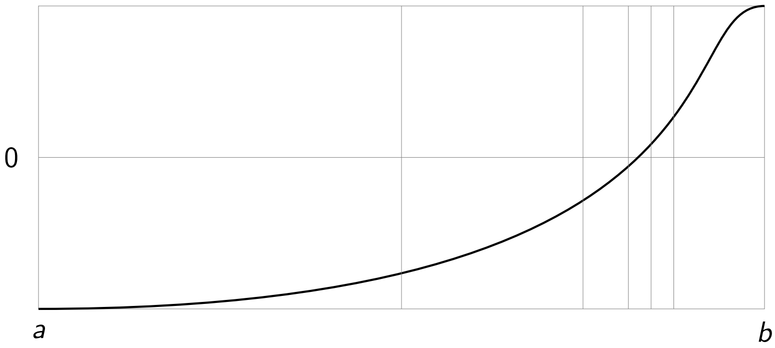

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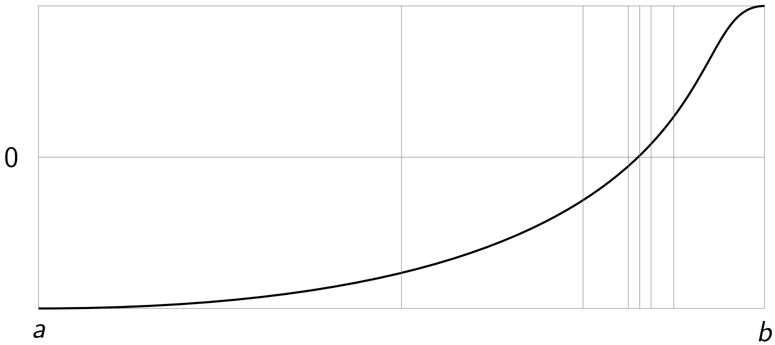

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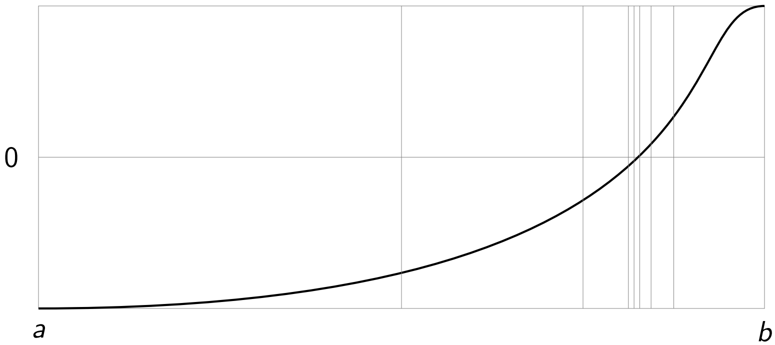

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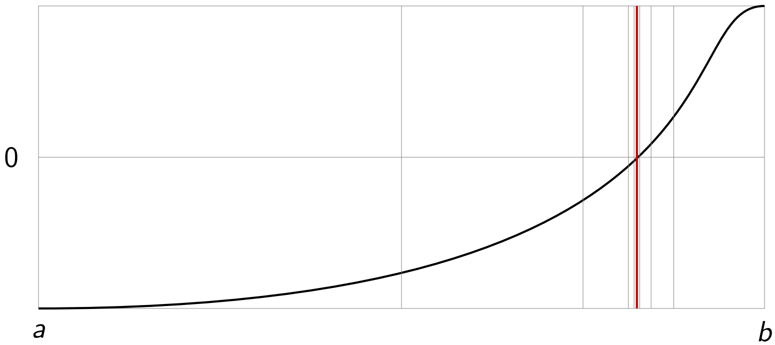

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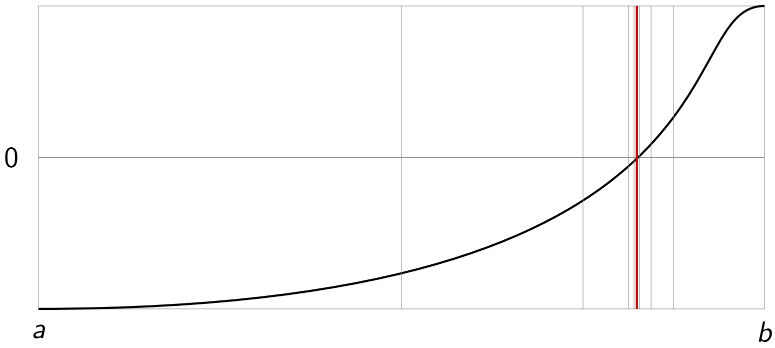

After k iterations, the distance to a solution is $|x_k - \hat{x}| \leq \dfrac{b-a}{2^k}$.

Problems to be solved

We shall approximate points in a set S of solutions in different contexts:

- $S = \text{Argmin}(f)$. Minimizers of an extended real-valued function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$, especially in the convex case.

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- $S = \mathrm{Fix}(T)$. Fixed points of a function $T : \mathbb{R}^N \to \mathbb{R}^N$, especially in the nonexpansive case.

Quality of approximate solutions

A merit function $\Phi : \mathbb{R}^N \to \mathbb{R}_+$ for a problem with solution set S satisfies:

- $\Phi(z) = 0$ if $z \in S$, and $\Phi(z) > 0$ otherwise.
- If $\Phi(z) < \Phi(w)$, then z is a better approximate solution than w.

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- $\Phi(z) = \mathrm{dist}(z, S)$ in all cases ($\Phi(z) = \|z - \hat{x}\|$ if $S = \{\hat{x}\}$).

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Important questions: convergence and complexity.

Convergence

A sequence (x_k) in $\mathbb{R}^N$ converges to a limit $\hat{x} \in \mathbb{R}^N$ as $k \to \infty$ if, for every $\varepsilon > 0$, there is $K \in \mathbb{N}$ such that

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for every $k \geq K$.

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Proposition

The following statements about a sequence (x_k) in $\mathbb{R}^N$ are equivalent:

i) *it is convergent;*

i) *it has Cauchy's property; and*

i) *it is bounded and has at most one limit point.*

Keynote examples

Theorem (Banach-Picard Iterations and the Fixed Point Theorem)

Let $q \in (0,1)$, and let $T : \mathbb{R}^N \to \mathbb{R}^N$ be $q-$Lipschitz. Then, T has a unique fixed point $\hat{x}$. Moreover, for each $x_0 \in \mathbb{R}^N$, the sequence (x_k), defined by iterating $x_{k+1} = Tx_k$, converges to $\hat{x}$.

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Theorem (Krasnoselskii-Mann Iterations)

Let $\gamma \in (0,1)$ *and let* $T : \mathbb{R}^N \to \mathbb{R}^N$ *be nonexpansive (*$1-$*Lipschitz), with* $\mathrm{Fix}(T) \neq \emptyset$. *For each* $x_0 \in \mathbb{R}^N$, *the sequence* (x_k), *defined by iterating* $x_{k+1} = \gamma Tx_k + (1-\gamma)x_k$, *converges to a fixed point of* T.

Useful shortcuts

Lemma (Opial)

Let Ω be a nonempty subset of $\mathbb{R}^N$, and let (x_k) be a sequence in $\mathbb{R}^N$. Suppose that:

i) *For every $y \in \Omega$, $\lim\limits_{k\to\infty} \|x_k - y\|$ exists; and*

ii) *Every limit point of (x_k) belongs to Ω.*

Then, (x_k) converges, as $k \to \infty$, to a point in Ω.

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Remark (Fejér)

If $\|x_{k+1} - y\| \leq \|x_k - y\|$ for all $k \geq 0$ and $y \in \Omega$, then $\lim_{k\to\infty} \|x_k - y\|$ exists for every $y \in \Omega$.

Convergence rates

Let (ϕ_k) be nonnegative, and let (ρ_k) be positive, with $\lim_{k\to\infty} \rho_k = 0$

We shall use the following notation:

- $\phi_k = \mathcal{O}(\rho_k)$ means that $\phi_k \to 0$ at least as fast as ρ_k:

$$C := \sup_{k\geq 0} \left[\frac{\phi_k}{\rho_k}\right] < +\infty,$$

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Clearly, $\phi_k = o(\rho_k)$ implies $\phi_k = \mathcal{O}(\rho_k)$, but the constant C is important!

- If $\phi_k = \mathcal{O}(q^k)$ for some $q \in (0, 1)$, we say ϕ_k converges linearly to zero.

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- If $\phi_k \neq \mathcal{O}(q^k)$ for every $q \in (0,1)$, ϕ_k converges sublinearly.
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- Finally, if $\phi_k = o(q^k)$ for all $q \in (0,1)$, then ϕ_k converges superlinearly.

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- Finally, if $\phi_k = o(q^k)$ for all $q \in (0,1)$, then ϕ_k converges superlinearly.
  - It is the case for Newton's method, under favorable conditions.

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$$\Psi(m, b) = \frac{1}{2} \sum_{k=1}^{K} (v_k - mu_k - b)^2.$$

We look for the values of m and b that minimize Ψ.

Practical aspects

Linear regression

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If we suspect that $\Phi(x_k)$ converges linearly to zero, we can use this with $u_k = k$, and $v_k = \ln\left(\Phi(x_k)\right)$.

Complexity

Consider a problem with merit function Φ and a sequence (x_k) produced by an algorithm

Suppose that for every $\varepsilon > 0$, which represents a tolerance or precision level, there is a number k_ε such that

$$\Phi(x_{k_\varepsilon}) < \varepsilon.$$

A complexity bound is either a map $\varepsilon \mapsto k_\varepsilon$, or an inequality that maps ε to a set of valid k_ε's.

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- From $\Phi(x_k) \leq C/k^p$, we get $k > \sqrt[p]{C/\varepsilon}$.

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- From $\Phi(x_k) \leq C/k^p$, we get $k > \sqrt[p]{C/\varepsilon}$.
- From $\Phi(x_k) \leq Ae^{-Bk}$, we get $k > \ln\left(\sqrt[B]{A/\varepsilon}\right)$.

Examples

Banach-Picard Iterations

We have

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Krasnoselskii-Mann Iterations

We can prove that

$$\|Tx_k - x_k\|^2 \leq \frac{\mathrm{dist}(x_0, S)^2}{\gamma(1-\gamma)k} = \mathcal{O}\left(\frac{1}{k}\right) \quad \text{and} \quad \|Tx_k - x_k\|^2 = o\left(\frac{1}{k}\right).$$

Exercises

1. Let (a_k), (b_k) and (c_k) be nonnegative sequences, and suppose
$$a_{k+1} + b_k \leq a_k + c_k$$
for all $k \geq 0$. Assume $C := \sum c_k < +\infty$. Show that
$$\lim_{k\to\infty} a_k \text{ exists,} \qquad \text{and} \qquad \sum b_k \leq a_0 + C < +\infty.$$

2. Let (d_k) be nonnegative, with $D := \sum d_k < +\infty$, and write $\delta_k = \min\{d_0, \ldots, d_k\}$. Prove that
$$(k+1)\delta_k \leq D \qquad \text{and} \qquad \lim_{k\to\infty} (k+1)\delta_k = 0.$$

Exercises, continued

3. Let T be nonexpansive, with $\mathrm{Fix}(T) \neq \emptyset$. Apply Krasnoselskii-Mann iterations
$$x_{k+1} = \gamma T x_k + (1-\gamma)x_k.$$
   1. Show that $\|x_{k+1} - \hat{x}\|^2 + \gamma(1-\gamma)\|Tx_k - x_k\|^2 \leq \|x_k - \hat{x}\|^2$ for every $\hat{x} \in \mathrm{Fix}(T)$. Conclude that (x_k) converges to a fixed point of T.
   2. Verify that $k \mapsto \|Tx_k - x_k\|^2$ is nonincreasing.
   3. Prove that the following convergence rates hold:
$$\|Tx_k - x_k\|^2 \leq \frac{\mathrm{dist}(x_0, S)^2}{\gamma(1-\gamma)k} = \mathcal{O}\left(\frac{1}{k}\right) \quad \text{and} \quad \|Tx_k - x_k\|^2 = o\left(\frac{1}{k}\right).$$