Functional Analysis

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Lecture 11
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Topics:

- §5.7: Spectra of bounded linear operators

A concrete question

Given $g \in \mathcal{C}([a,b],\mathbb{K})$ and $\lambda \in \mathbb{K}$ we want to solve

$$\int_a^b G(x,y)f(y)\,dy = \lambda f(x) + g(x)$$

Is this problem well-posed?

- does a solution exist?
- is the solution unique?
- does the solution depend continuously on g and λ?

An abstract reformulation

Given $T \in B(X)$

Find all $\lambda \in \mathbb{K}$ such that

$$Tx = \lambda x + y \quad \Rightarrow \quad x = (T - \lambda)^{-1} y$$

[Note: if $(T - \lambda)^{-1}$ is bounded, then this problem is well-posed]

Resolvent set and spectrum

Definition: for X Banach and $T \in B(X)$ we define the

resolvent set: $\rho(T) = \{\lambda \in \mathbb{K} : (T - \lambda)^{-1} \in B(X)\}$

resolvent operator: $R(\lambda) = (T - \lambda)^{-1} \qquad \lambda \in \rho(T)$

spectrum: $\sigma(T) = \mathbb{K} \setminus \rho(T)$

[X is assumed to be Banach to make the Open Mapping Theorem applicable]

Eigenvalues

Definition: if $T \in B(X)$, then

- $\lambda \in \mathbb{K}$ is called an eigenvalue of T if there exists $x \neq 0$ s.t.

$$(T - \lambda)x = 0$$

- $\ker(T - \lambda)$ is called the associated eigenspace
- nonzero elements of $\ker(T - \lambda)$ are called eigenvectors
- $\sigma_p(T) = \{\text{eigenvalues of } T\}$ is called the point spectrum of T

Important: $\sigma_p(T) \subset \sigma(T)$, but in general no equality!

Bounds on the spectrum

Lemma: assume X is Banach and $T \in B(X)$

If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$ and

$$R(\lambda) = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Corollary: if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$

Bounds on the spectrum

Proof: if $|\lambda| > \|T\|$, then

$$T - \lambda = -\lambda\left(I - \frac{T}{\lambda}\right) \qquad \text{and} \qquad \left\|\frac{T}{\lambda}\right\| < 1$$

Inversion by geometric series gives

$$\begin{aligned}(T-\lambda)^{-1} &= -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} \\ &= -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{T^n}{\lambda^n} = -\sum_{n=0}^{\infty}\frac{T^n}{\lambda^{n+1}}\end{aligned}$$

Spectra are closed

Lemma: assume X is Banach and $T \in B(X)$

If $\mu \in \rho(T)$ and $|\lambda - \mu| < 1/\|R(\mu)\|$, then

$$\lambda \in \rho(T) \quad \text{and} \quad R(\lambda) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\mu)^{n+1}$$

Corollary: $\rho(T)$ is open and thus $\sigma(T)$ is closed

Spectra are closed

Proof:

$$
\begin{aligned}
T-\lambda &= T-\mu-(\lambda-\mu)\\
&= [I-(\lambda-\mu)(T-\mu)^{-1}](T-\mu)\\
&= [I-(\lambda-\mu)R(\mu)](T-\mu) \qquad (*)
\end{aligned}
$$

$$
\begin{aligned}
|\lambda-\mu|\|R(\mu)\|<1 &\Rightarrow I-(\lambda-\mu)R(\mu) \text{ invertible}\\
&\Rightarrow (*) \text{ invertible}\\
&\Rightarrow T-\lambda \text{ invertible}\\
&\Rightarrow \lambda\in\rho(T)
\end{aligned}
$$

Spectra are closed

Proof (ctd):

$$T - \lambda = [I - (\lambda - \mu)R(\mu)](T - \mu)$$

$$\begin{aligned}(T-\lambda)^{-1} &= (T-\mu)^{-1}[I - (\lambda-\mu)R(\mu)]^{-1} \\ &= R(\mu)\sum_{n=0}^{\infty}(\lambda-\mu)^n R(\mu)^n \\ &= \sum_{n=0}^{\infty}(\lambda-\mu)^n R(\mu)^{n+1}\end{aligned}$$

Characterization

Proposition: if X is Banach and $T \in B(X)$, then

$$\lambda \in \rho(T) \quad \Leftrightarrow \quad \begin{cases} \operatorname{ran}(T-\lambda) \text{ dense in } X & (1) \\ \quad \textcolor{red}{\text{AND}} & \\ \|(T-\lambda)x\| \geq c\|x\| \quad \forall x \in X & (2) \end{cases}$$

Proof ($\Leftarrow$):

$$\begin{aligned} (2) \quad &\Rightarrow \quad T-\lambda \text{ injective \& } \operatorname{ran}(T-\lambda) \text{ closed} \\ &\Rightarrow \quad T-\lambda \text{ bijective by (1)} \\ &\Rightarrow \quad (T-\lambda)^{-1} \in B(X) \text{ by open mapping thm.} \end{aligned}$$

Characterization

Corollary:

$$\lambda \in \sigma(T) \quad \Leftrightarrow \quad \begin{cases} \operatorname{ran}(T-\lambda) \text{ not dense in } X \\ \quad \text{OR} \\ \|(T-\lambda)x_n\| \to 0 \text{ for some seq. } (x_n) \\ \qquad\qquad\qquad \text{s.t. } \|x_n\| = 1 \quad \forall\, n \in \mathbb{N} \end{cases}$$

Definition: $\lambda \in \mathbb{K}$ is called an approximate eigenvalue of T if there exists a sequence (x_n) such that

$$\|x_n\| = 1 \quad \forall\, n \in \mathbb{N} \qquad \text{and} \qquad (T-\lambda)x_n \to 0$$

Spectral mapping theorem

Theorem: assume X is Banach over $\mathbb{K} = \mathbb{C}$ and $T \in B(X)$

For any polynomial $p : \mathbb{K} \to \mathbb{K}$ we have

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$$

Proof: see lecture notes

Component multiplication in ℓ^1

Example: consider

$$T : \ell^1 \to \ell^1, \qquad (x_1, x_2, x_3, \dots) \mapsto (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

If, $x^n = (0, 0, \dots, 0, 1, 0, 0, \dots)$, with 1 at the n-th position, then

$$Tx^n = \frac{1}{n}x^n \quad \text{and} \quad \| Tx^n \|_1 = \frac{1}{n} \to 0$$

Conclusions:

- $1/n$ is an eigenvalue for all $n \in \mathbb{N}$
- 0 is an approximate eigenvalue (but not an eigenvalue)

Component multiplication in ℓ^1

Example (ctd): if $|\lambda - \frac{1}{n}| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$, then

$$(T-\lambda)^{-1}x \;=\; \left(\frac{x_1}{1-\lambda}, \frac{x_2}{\frac{1}{2}-\lambda}, \frac{x_3}{\frac{1}{3}-\lambda}, \dots\right)$$

$$\|(T-\lambda)^{-1}x\|_1 \;=\; \sum_{n=1}^{\infty} \frac{|x_n|}{|\frac{1}{n}-\lambda|} \;\leq\; \frac{1}{\varepsilon}\|x\|_1$$

This means $(T-\lambda)^{-1} \in B(\ell^1)$ and thus $\lambda \in \rho(T)$

Hence $\sigma(T) = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

The left shift on ℓ^2

Example: consider $S_L : \ell^2 \to \ell^2$ given by

$$(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

If $|\lambda| < 1$, then $x = (1, \lambda, \lambda^2, \lambda^3, \dots) \in \ell^2$ and

$$S_L x = (\lambda, \lambda^2, \lambda^3, \dots) = \lambda x$$

In particular, S_L has uncountably many eigenvalues!

The left shift on ℓ^2

Example (ctd): so we have

- $|\lambda| < 1 \;\Rightarrow\; \lambda \in \sigma(S_L)$
- $|\lambda| = 1 \;\Rightarrow\; \lambda \in \sigma(S_L)$ since spectra are closed!
- $|\lambda| > 1 \;\Rightarrow\; \lambda \in \rho(S_L)$ since $\|S_L\| = 1$

So the spectrum is given by

$$\sigma(S_L) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$$

The right shift on ℓ^2

Example: consider $S_R : \ell^2 \to \ell^2$ given by

$$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$$

This operator has no eigenvalues:

$$\begin{aligned} S_R x = \lambda x \quad &\Rightarrow \quad (0, x_1, x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots) \\ &\Rightarrow \quad \lambda x_1 = 0 \quad \text{and} \quad \lambda x_{n+1} = x_n \quad \forall\, n \in \mathbb{N} \\ &\Rightarrow \quad \lambda = 0 \text{ or } x_1 = 0 \\ & \qquad\; x = 0 \quad \text{in both cases} \end{aligned}$$

The right shift on ℓ^2

Example (ctd): if $|\lambda| < 1$, then

$$y = (1, \bar{\lambda}, \bar{\lambda}^2, \dots) \in \ell^2$$

For all $x \in \ell^2$ we have

$$y \perp (S_R - \lambda)x = (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \dots)$$

[Exercise: verify this statement]

Conclusion: $\operatorname{ran}(S_R - \lambda)$ is not dense in ℓ^2 and thus $\lambda \in \sigma(S_R)$

The right shift on ℓ^2

Example (ctd): so we have

- $|\lambda| < 1 \;\Rightarrow\; \lambda \in \sigma(S_R)$
- $|\lambda| = 1 \;\Rightarrow\; \lambda \in \sigma(S_R)$ since spectra are closed!
- $|\lambda| > 1 \;\Rightarrow\; \lambda \in \rho(S_R)$ since $\|S_R\| = 1$

So the spectrum is given by

$$\sigma(S_R) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$$

Spectral theorem for compact operators

Theorem: if X is Banach and $T \in K(X)$, then

1. for every $\varepsilon > 0$ the number of eigenvalues λ of T with $|\lambda| \geq \varepsilon$ is finite

   [Corollary: T only has *countably many* eigenvalues]

2. if $\lambda \neq 0$ is an eigenvalue of T, then $\dim \ker(T - \lambda) < \infty$

3. if $\dim X = \infty$, then $0 \in \sigma(T)$

Spectral theorem for compact operators

Proof (1): assume there exist distinct λ_n such that:

$$|\lambda_n| \geq \varepsilon \qquad Tx_n = \lambda_n x_n \qquad x_n \neq 0 \qquad n \in \mathbb{N}$$

For $M_n = \text{span}\{x_1, \dots, x_n\}$ we have

$$M_n \subset M_{n+1} \quad \text{and} \quad T(M_n) \subset M_n$$

By Riesz's lemma there exists $y_n \in M_n$ such that

$$\|y_n\| = 1 \quad \text{and} \quad \|y_n - x\| \geq 1/2 \quad \forall\, x \in M_{n-1}$$

Claim: $n \neq m \quad \Rightarrow \quad \|\, Ty_n - Ty_m\| \geq \varepsilon/2$

Spectral theorem for compact operators

Proof (1): we have that

$$y_n = \sum_{j=1}^{n} c_j x_j \in M_n = \mathrm{span}\{x_1, \ldots, x_n\}$$

$$(T - \lambda_n) y_n = \sum_{j=1}^{n} c_j (\lambda_j - \lambda_n) x_j \in M_{n-1}$$

[term for $j = n$ vanishes!]

$$T y_n - T y_m = \lambda_n y_n - \big(\underbrace{T y_m - (T - \lambda_n) y_n}_{=:\, u \in M_{n-1}}\big) \qquad \text{for } n > m$$

Spectral theorem for compact operators

Proof (1): if $n > m$, then

$$\begin{aligned} \|Ty_n - Ty_m\| &= \|\lambda_n y_n - u\| \qquad u \in M_{n-1} \\ &= |\lambda_n| \|y_n - u/\lambda_n\| \\ &\geq \tfrac{1}{2}|\lambda_n| \\ &\geq \tfrac{1}{2}\varepsilon \end{aligned}$$

Conlusion: (Ty_n) does not have a convergent subsequence

Contradiction, since T is compact

Spectral theorem for compact operators

Proof (2): assume $\lambda \neq 0$ and take (x_n) such that

$$x_n \in \ker(T - \lambda) \quad \text{and} \quad \|x_n\| = 1 \quad \forall\, n \in \mathbb{N}$$

T compact $\Rightarrow$ $x_n = (1/\lambda)Tx_n$ has convergent subsequence

Unit ball of $\ker(T - \lambda)$ compact $\Rightarrow$ $\dim \ker(T - \lambda) < \infty$

Spectral theorem for compact operators

Proof (3):

$$
\begin{aligned}
0 \in \rho(T) \quad &\Rightarrow \quad T^{-1} \in B(X) \\
&\Rightarrow \quad I = TT^{-1} \in K(X) \\
&\Rightarrow \quad \dim X < \infty
\end{aligned}
$$