Functional Analysis

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Lecture 8
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Topics:

- §4.4: Compact operators
- §4.5: Fredholm and Volterra operators
- Appendix A: Arzelà-Ascoli theorem

Bounded operators

Definition: $T \in L(X, Y)$ is bounded if there exists $c > 0$ such that

$$\|Tx\| \leq c\|x\| \qquad \forall x \in X$$

Exercise: show that T is bounded if and only if

$$V \text{ is a bounded set} \quad \Rightarrow \quad T(V) \text{ is a bounded set}$$

Compact operators

Definition: $T \in L(X, Y)$ is compact if

$$V \text{ is a bounded set} \quad \Rightarrow \quad T(V) \text{ is relatively compact}$$

$$\text{(i.e. } \overline{T(V)} \text{ is compact)}$$

Relation with boundedness

Lemma: T compact $\Rightarrow$ T bounded

Proof: take $V = \{x \in X \, : \, \|x\| = 1\}$

$T(V)$ is relatively compact and hence bounded

There exists $c > 0$ such that $\|Tx\| \leq c$ for all $x \in V$

If $z \neq 0$, then $x = z/\|z\| \in V$ so $\|Tx\| \leq c$ and thus $\|Tz\| \leq c\|z\|$

Sequential characterization

Lemma: the following statements are equivalent:

1. $T \in L(X, Y)$ is compact
2. (x_n) bounded seq. $\Rightarrow$ (Tx_n) has a convergent subseq.

Sequential characterization

Proof $(1 \Rightarrow 2)$**:**

If (x_n) is bounded, then $V = \{x_n : n \in \mathbb{N}\}$ is bounded

Since T is compact, $\overline{T(V)}$ is compact

The sequence (Tx_n) lies in $T(V)$ and thus in $\overline{T(V)}$

Conclusion: (Tx_n) has a convergent subsequence

Sequential characterization

Proof $(2 \Rightarrow 1)$**:** let $V \subset X$ be bounded

Take any sequence (y_n) in $T(V)$

Then $y_n = Tx_n$ for some sequence (x_n) in V

Since (x_n) is bounded, $(y_n) = (Tx_n)$ has a convergent subsequence

Conclusion: $T(V)$ is relatively compact

Finite-dimensional range

Lemma: if $T \in B(X, Y)$ and $\dim \operatorname{ran} T < \infty$, then T is compact

Proof: assume (x_n) is a bounded sequence in X

Since T is bounded, (Tx_n) is a bounded sequence in Y

Since $\dim \operatorname{ran} T < \infty$ we can apply Bolzano–Weierstrass!

So (Tx_n) has a convergent subsequence

Finite-dimensional range

Example: consider $\mathcal{C}([-\pi,\pi],\mathbb{K})$ with the sup-norm and

$$\begin{aligned}
Tf(x) &= \int_{-\pi}^{\pi} \sin(x-y) f(y)\,dy \\
&= \int_{-\pi}^{\pi} \big[\sin(x)\cos(y) - \cos(x)\sin(y)\big] f(y)\,dy \\
&= \sin(x)\int_{-\pi}^{\pi} \cos(y) f(y)\,dy - \cos(x)\int_{-\pi}^{\pi} \sin(y) f(y)\,dy
\end{aligned}$$

Note: T is bounded and $\dim \operatorname{ran} T = 2$

Conclusion: T is compact

The space of compact operators

Definition:

$$K(X, Y) = \left\{ T \in L(X, Y) \, : \, T \text{ is compact} \right\}$$

Lemma:

1. $K(X, Y)$ is a linear subspace of $B(X, Y)$
2. if $T \in B(X, Y)$ and $S \in B(Y, Z)$, then

$$T \text{ or } S \text{ compact} \;\Rightarrow\; ST \text{ compact}$$

Proof: see lecture notes

Closedness

Theorem: X NLS and Y Banach $\Rightarrow$ $K(X, Y)$ closed in $B(X, Y)$

Proof: assume $T \in \overline{K(X, Y)}$

Then $T_n \to T$ for some sequence (T_n) in $K(X, Y)$

Let (x_i) be a bounded sequence (so $\|x_i\| \leq C$ for all i)

Take subseq. (x_i^1) of (x_i) such that $(T_1 x_i^1)$ converges

Take subseq. (x_i^2) of (x_i^1) such that $(T_2 x_i^2)$ converges

Take subseq. (x_i^3) of (x_i^2) such that $(T_3 x_i^3)$ converges

$\vdots$

Closedness

Proof (ctd):

$$\begin{array}{lllll} x_1^1, & x_2^1, & x_3^1, & \ldots & \text{converges under } T_1 \\ x_1^2, & x_2^2, & x_3^2, & \ldots & \text{converges under } T_1, T_2 \\ x_1^3, & x_2^3, & x_3^3, & \ldots & \text{converges under } T_1, T_2, T_3 \\ \vdots & \vdots & \vdots & & \end{array}$$

$$z_i := x_i^i \quad \Rightarrow \quad \begin{cases} (z_i) \text{ is a subsequence of } (x_i) \\ (T_n z_i) \text{ converges for all fixed } n \in \mathbb{N} \end{cases}$$

Claim: (Tz_i) is Cauchy in Y

Closedness

Proof (ctd): for all $i, j, n \in \mathbb{N}$ we have

$$\begin{aligned}
\|Tz_i - Tz_j\| &= \|(T - T_n)(z_i - z_j) + T_n(z_i - z_j)\| \\
&\leq \|(T - T_n)(z_i - z_j)\| + \|T_n(z_i - z_j)\| \\
&\leq \|T - T_n\| \, \|z_i - z_j\| + \|T_n(z_i - z_j)\| \\
&\leq \|T - T_n\|(\|z_i\| + \|z_j\|) + \|T_n(z_i - z_j)\| \\
&\leq 2C\|T - T_n\| + \|T_n z_i - T_n z_j\|
\end{aligned}$$

Closedness

Proof (ctd): for all $i, j, n \in \mathbb{N}$

$$\|Tz_i - Tz_j\| \;\leq\; 2C\|T - T_n\| + \|T_n z_i - T_n z_j\|$$

Let $\varepsilon > 0$ and

1. fix $n_0 \in \mathbb{N}$ such that $\|T - T_{n_0}\| < \varepsilon/4C$
2. pick $N \in \mathbb{N}$ such that

$$\begin{aligned} i, j > N \quad &\Rightarrow \quad \|T_{n_0} z_i - T_{n_0} z_j\| < \varepsilon/2 \\ &\Rightarrow \quad \|Tz_i - Tz_j\| < \varepsilon \end{aligned}$$

Closedness

Example: consider the operators

$$T : \ell^1 \to \ell^1, \qquad (x_1, x_2, x_3, \dots) \mapsto (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

$$T_n : \ell^1 \to \ell^1, \qquad (x_1, x_2, x_3, \dots) \mapsto (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots, \tfrac{1}{n}x_n, 0, 0, \dots)$$

Note that:

- each T_n is compact [indeed: bounded and $\dim \operatorname{ran} T_n = n < \infty$]
- $\| T_n - T \| \leq \frac{1}{n+1} \to 0$ as $n \to \infty$

Since ℓ^1 is a Banach space we conclude that T is compact

Integral operators

Theorem: let $G : [a, b] \times [a, b] \to \mathbb{K}$ be continuous

Operators $T : \mathcal{C}([a, b], \mathbb{K}) \to \mathcal{C}([a, b], \mathbb{K})$ of the form

$$Tf(x) = \int_a^b G(x, y) f(y)\, dy \qquad \text{"Fredholm operator"}$$

or

$$Tf(x) = \int_a^x G(x, y) f(y)\, dy \qquad \text{"Volterra operator"}$$

are compact

Equicontinuity

Definition: a set $V \subset \mathcal{C}([a,b],\mathbb{K})$ is called equicontinuous if:

for all $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \qquad \forall\, x, y \in [a, b] \quad \forall\, f \in V$$

Moral:

- each $f \in V$ is uniformly continuous on $[a, b]$
- for a given ε the same δ works for all $f \in V$

Equicontinuity

Example: consider on $[0, 1]$ the set

$$V = \{ f(x) = c_0 + c_1 x + c_2 x^2 \, : \, |c_1| \leq 1, \; |c_2| \leq 1 \}$$

For all $f \in V$ we have

$$\begin{aligned} |f(x) - f(y)| &= |c_1(x - y) + c_2(x^2 - y^2)| \\ &= |c_1 + c_2(x + y)| \cdot |x - y| \\ &\leq (|c_1| + 2|c_2|)|x - y| \\ &\leq 3|x - y| \end{aligned}$$

If $\varepsilon > 0$ is given, then $\delta := \varepsilon/3$ works for all $f \in V$

Arzelà–Ascoli theorem

Theorem: if $V \subset \mathcal{C}([a, b], \mathbb{K})$, then

$$V \text{ relatively compact } \Leftrightarrow V \text{ bounded and equicontinuous}$$

Proof ($\Leftarrow$): let (f_n) be a sequence in V

To show: (f_n) has a convergent subsequence

There exists $C > 0$ such that

$$|f_n(x)| \leq \|f_n\|_\infty \leq C \qquad \forall\, x \in [a, b] \quad \forall\, n \in \mathbb{N}$$

Let $E = \{x_1, x_2, x_3, \dots\}$ be countable and dense in $[a, b]$

Arzelà–Ascoli theorem

Proof (ctd):

Note: $(f_n(x))$ is a bounded sequence in $\mathbb{K}$ for each fixed $x \in E$

Repeated application of Bolzano–Weierstrass:

- take subsequence (f_n^1) of (f_n) such that $(f_n^1(x_1))$ converges
- take subsequence (f_n^2) of (f_n^1) such that $(f_n^2(x_2))$ converges
- take subsequence (f_n^3) of (f_n^2) such that $(f_n^3(x_3))$ converges
- $\ldots$

Arzelà–Ascoli theorem

Proof (ctd): we have that

$$\begin{array}{llllll} f_1^1, & f_2^1, & f_3^1, & \dots & \text{converges at } x_1 \\ f_1^2, & f_2^2, & f_3^2, & \dots & \text{converges at } x_1, x_2 \\ f_1^3, & f_2^3, & f_3^3, & \dots & \text{converges at } x_1, x_2, x_3 \\ \vdots & \vdots & \vdots & & \end{array}$$

Diagonalization trick:

$$g_n := f_n^n \quad \Rightarrow \quad \begin{cases} (g_n) \text{ is a subsequence of } (f_n) \\ g_n(x_i) \text{ converges for all fixed } i \in \mathbb{N} \end{cases}$$

Claim: (g_n) is a Cauchy sequence in $\mathcal{C}([a, b], \mathbb{K})$

Arzelà–Ascoli theorem

Proof (ctd): let $\varepsilon > 0$ be arbitrary

Since V is equicontinuous there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |h(x) - h(y)| < \varepsilon/3 \quad \text{for all} \quad h \in V$$

$$E = \{x_1, x_2, \dots\} \text{ dense in } [a, b] \;\Rightarrow\; [a, b] \subset \bigcup_{i=1}^{\infty} (x_i - \delta, x_i + \delta)$$

$$[a, b] \text{ compact } \;\Rightarrow\; [a, b] \subset \bigcup_{i=1}^{r} (x_i - \delta, x_i + \delta) \text{ for some } r \in \mathbb{N}$$

Arzelà–Ascoli theorem

Proof (ctd): there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad |g_n(x_i) - g_m(x_i)| < \varepsilon/3 \quad \text{for all} \quad i = 1, \ldots, r$$

$$x \in [a, b] \quad \Rightarrow \quad |x - x_i| < \delta \quad \text{for some} \quad i = 1, \ldots, r$$

$$m, n \geq N \quad \Rightarrow \quad \begin{cases} |g_n(x) - g_n(x_i)| < \varepsilon/3 \\ |g_n(x_i) - g_m(x_i)| < \varepsilon/3 \\ |g_m(x) - g_m(x_i)| < \varepsilon/3 \end{cases}$$

$$\Rightarrow \quad |g_n(x) - g_m(x)| < \varepsilon \quad \forall\, x \in [a, b]$$

$$\Rightarrow \quad \|g_n - g_m\|_\infty \leq \varepsilon$$

Integral operators

Consider the Fredholm operator

$$Tf(x) = \int_a^b G(x, y) f(y)\, dy$$

Claim: if $V \subset \mathcal{C}([a, b], \mathbb{K})$ is bounded, then $T(V)$ is

1. bounded [see lecture notes]
2. equicontinuous [see next slide]

Therefore $T(V)$ is relatively compact and so T is compact

Integral operators

Proof: for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sqrt{(x-x')^2+(y-y')^2} < \delta \quad \Rightarrow \quad |G(x,y) - G(x',y')| < \varepsilon$$

If $|x - x'| < \delta$, then

$$\begin{aligned} |Tf(x) - Tf(x')| &\leq \int_a^b |G(x,y) - G(x',y)|\,|f(y)|\,dy \\ &< \varepsilon(b-a)\|f\|_\infty \\ &\leq \varepsilon(b-a)C \qquad\qquad \forall f \in V \end{aligned}$$