Functional Analysis

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Lecture 9
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Topics:

- §5.1: Baire's theorem
- §5.2: Zabreĭko's lemma

The interior of a set

Definition: let M be a subset of a metric space (X, d)

The interior of M is defined as

$$\mathrm{int}(M) = \bigcup_{O \subset M, O \text{ open}} O$$

[In other words: $\mathrm{int}(M)$ is the largest open set that is contained in M]

Examples: consider $X = \mathbb{R}$ with $d(x, y) = |x - y|$

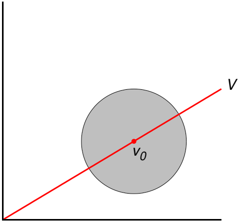
- $M = (0, 1) \quad \Rightarrow \quad \mathrm{int}(M) = (0, 1)$
- $M = [0, 1] \quad \Rightarrow \quad \mathrm{int}(M) = (0, 1)$
- $M = \{0, 1\} \quad \Rightarrow \quad \mathrm{int}(M) = \varnothing$

Nowhere dense sets

Definition: a subset M of a metric space (X, d) is called nowhere dense if $\operatorname{int}(\overline{M}) = \varnothing$

Example: let X be a NLS and $V \subset X$ a closed linear subspace

$V \neq X \;\Rightarrow\; V$ is nowhere dense



Nowhere dense sets

Claim: if $M \subset X$ is nowhere dense, then

$$B(x;\varepsilon) \cap (\overline{M})^c \neq \varnothing \qquad \forall\, \varepsilon > 0 \quad \forall\, x \in X$$

[In words: every ball intersects the complement of $\overline{M}$]

Proof: if $B(x;\varepsilon) \cap (\overline{M})^c = \varnothing$, then $B(x;\varepsilon) \subset \overline{M}$

Contradiction!

Meager sets

Definition: let (X, d) be a metric space

A subset $M \subset X$ is called meager if

$$M = \bigcup_{i=1}^{\infty} M_i \quad \text{where all } M_i \subset X \text{ are nowhere dense}$$

[In words: M is meager if it can be written as a countable union of nowhere dense sets]

Example: if $X = \mathbb{R}$ with $d(x, y) = |x - y|$, then

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \quad \text{is meager}$$

Baire's theorem

Theorem: if (X, d) is a complete metric space, then

$O \subset X$ nonempty and open $\Rightarrow$ O nonmeager

Proof: assume

- $O \subset X$ nonempty and open
- $O = \bigcup_{i=1}^{\infty} M_i$ all M_i nowhere dense

Strategy: show $\exists\, x \in O$ with $x \notin M_i \quad \forall\, i \in \mathbb{N}$

Baire's theorem

Proof (ctd): pick $x_0 \in O$

$\exists\, \varepsilon_0 > 0$ such that $B(x_0; \varepsilon_0) \subset O$

M_1 nowhere dense $\Rightarrow$ $\exists\, x_1 \in B(x_0; \varepsilon_0/2) \cap (\overline{M}_1)^c \neq \varnothing$

$\exists\, 0 < \varepsilon_1 < \varepsilon_0/2$ such that $B(x_1; \varepsilon_1) \subset B(x_0; \varepsilon_0/2) \cap (\overline{M}_1)^c$

Baire's theorem

Proof (ctd):

M_2 nowhere dense $\quad\Rightarrow\quad \exists\, x_2 \in B(x_1;\varepsilon_1/2) \cap (\overline{M}_2)^c \neq \varnothing$

$\exists\, 0 < \varepsilon_2 < \varepsilon_1/2$ such that $B(x_2;\varepsilon_2) \subset B(x_1;\varepsilon_1/2) \cap (\overline{M}_2)^c$

Baire's theorem

Proof (ctd): induction gives $x_i \in X$ and $\varepsilon_i > 0$ such that

$$B(x_i; \varepsilon_i) \subset B(x_{i-1}; \varepsilon_{i-1}/2) \cap (\overline{M}_i)^c \qquad \varepsilon_i < \varepsilon_{i-1}/2$$

$$B(x_i; \varepsilon_i) \subset B(x_{i-1}; \varepsilon_{i-1}/2) \subset B(x_{i-1}; \varepsilon_{i-1})$$

$$B(x_i; \varepsilon_i) \cap M_i = \varnothing$$

Baire's theorem

Proof (ctd):

$$x_i \in B(x_{i-1}; \varepsilon_{i-1}/2) \quad \Rightarrow \quad d(x_i, x_{i-1}) < \varepsilon_{i-1}/2 < \varepsilon_0/2^i$$

$$\Rightarrow \quad (x_i) \text{ Cauchy}$$ [Exercise: show this]

$$\Rightarrow \quad \exists\, x \in X \text{ such that}$$

$$d(x, x_i) \to 0 \text{ as } i \to \infty$$

Baire's theorem

Proof (ctd): fix any $i \in \mathbb{N}$ and let $j > i$ then

$$B(x_j; \varepsilon_j) \subset B(x_i; \varepsilon_i/2)$$

$$d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon_i/2 + d(x_j, x)$$

$$d(x_i, x) \leq \varepsilon_i/2 \qquad (\text{let } j \to \infty)$$

$$x \in B(x_i; \varepsilon_i) \subset O$$

$$B(x_i; \varepsilon_i) \cap M_i = \varnothing \quad \Rightarrow \quad x \notin M_i$$

Application

Example: let $\|\cdot\|$ be any norm on

$$\mathcal{P} = \{p : \mathbb{K} \to \mathbb{K} \; : \; p \text{ is a polynomial}\}$$

Then $\mathcal{P}$ is NOT a Banach space

Idea: $\mathcal{P}$ is a countable union of nowhere dense sets

[See problems 1 and 2 of tutorial 9]

Semi-norms

Definition: a semi-norm on X is a map $p : X \to [0, \infty)$ s.t.

$$\begin{aligned} p(x+y) &\leq p(x) + p(y) \\ p(\lambda x) &= |\lambda|\, p(x) \end{aligned} \qquad \forall\, x, y \in X,\ \lambda \in \mathbb{K}$$

Example:

$$\left.\begin{aligned} &Y = \text{NLS} \\ &T \in L(X, Y) \end{aligned}\right\} \quad \Rightarrow \quad p(x) = \|Tx\| \quad \text{is a semi-norm on } X$$

Semi-norms

Definition: if X is a NLS, then a semi-norm p on X is called

bounded if there exists $c > 0$ s.t.

$$p(x) \leq c\|x\| \qquad \forall\, x \in X$$

Intended application: if $p(x) = \|Tx\|$, then

$$T \text{ is bounded} \quad \Leftrightarrow \quad p \text{ is bounded}$$

Semi-norms

Lemma: if a semi-norm $p : X \to [0, \infty)$ is bounded, then

$$|p(x) - p(y)| \le c\|x - y\| \qquad \forall\, x, y \in X$$

Hence:

$$x_n \to x \quad \Rightarrow \quad p(x_n) \to p(x)$$

Proof:

$$\begin{aligned} p(x) &= p(x - y + y) \\ &\le p(x-y) + p(y) \quad \Rightarrow \quad p(x) - p(y) \le c\|x - y\| \end{aligned}$$

Now swap x and y

Semi-norms

Lemma: bounded semi-norms are countably subadditive:

$$\sum_{j=1}^{\infty} x_j \text{ convergent} \quad \Rightarrow \quad p\left(\sum_{j=1}^{\infty} x_j\right) \leq \sum_{j=1}^{\infty} p(x_j)$$

Proof: for all $n \in \mathbb{N}$ we have

$$p\left(\sum_{j=1}^{n} x_j\right) \leq \sum_{j=1}^{n} p(x_j) \leq \sum_{j=1}^{\infty} p(x_j)$$

Now let $n \to \infty$

Zabreĭko's lemma

Lemma: assume

- X is a Banach space
- $p : X \to [0, \infty)$ is a semi-norm
- if the series $\sum_{j=1}^{\infty} x_j$ converges, then

$$p\left(\sum_{j=1}^{\infty} x_j\right) \leq \sum_{j=1}^{\infty} p(x_j) \in [0, \infty]$$

Then p is bounded

Zabreĭko's lemma

Proof: define $M_n = \{x \in X \,:\, p(x) \leq n\} \qquad n \in \mathbb{N}$

Note that:

$$x \in M_n \quad \Rightarrow \quad -x \in M_n \quad \text{since} \quad p(-x) = p(x)$$

$$\begin{aligned} x \in \overline{M}_n \quad &\Rightarrow \quad x_k \to x && (x_k) \text{ lies in } M_n \\ &\Rightarrow \quad -x_k \to -x && (-x_k) \text{ lies in } M_n \\ &\Rightarrow \quad -x \in \overline{M}_n \end{aligned}$$

$$x, y \in \overline{M}_n \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in \overline{M}_n \quad \forall\, \lambda \in [0,1]$$ [similar reasoning]

Zabreĭko's lemma

Proof (ctd):

$$M_n = \{x \in X \,:\, p(x) \le n\} \quad \Rightarrow \quad X = \bigcup_{n=1}^{\infty} M_n$$

$$X \text{ nonmeager} \quad \Rightarrow \quad \exists\, n \in \mathbb{N} \text{ s.t. } \operatorname{int}(\overline{M}_n) \ne \varnothing$$

$$\Rightarrow \quad \exists\, x_0 \in X, \varepsilon > 0 \text{ s.t.}$$

$$B(x_0; \varepsilon) \subset \overline{M}_n$$

Zabreĭko's lemma

Proof (ctd): note that

$$x \in B(0;\varepsilon) \quad \Rightarrow \quad \begin{cases} x + x_0 \in B(x_0;\varepsilon) \subset \overline{M}_n \\ x - x_0 \in B(-x_0;\varepsilon) = -B(x_0;\varepsilon) \subset \overline{M}_n \end{cases}$$

$$\Rightarrow \quad x = \tfrac{1}{2}(x - x_0) + \tfrac{1}{2}(x + x_0) \in \overline{M}_n$$

Conclusion: $B(0;\varepsilon) \subset \overline{M}_n$

Next goal: show that $B(0;\varepsilon) \subset M_{2n}$

Zabreĭko's lemma

Proof (ctd): since $B(0;\varepsilon) \subset \overline{M}_n$ it follows that

$$\begin{aligned}
x \in B(0;\varepsilon) \quad &\Rightarrow \quad \exists\, x_1 \in M_n \quad \text{s.t.} \quad \|x - x_1\| < \tfrac{1}{2}\varepsilon \\
&\Rightarrow \quad 2(x - x_1) \in B(0;\varepsilon) \\
&\Rightarrow \quad \exists\, x_2 \in M_n \quad \text{s.t.} \quad \|2(x - x_1) - x_2\| < \tfrac{1}{2}\varepsilon \\
&\qquad\qquad\qquad\qquad\qquad\;\; \|x - x_1 - \tfrac{1}{2}x_2\| < \tfrac{1}{4}\varepsilon \\
&\Rightarrow \quad 4(x - x_1 - \tfrac{1}{2}x_2) \in B(0;\varepsilon) \\
&\Rightarrow \quad \exists\, x_3 \in M_n \quad \text{s.t.} \quad \|4(x - x_1 - \tfrac{1}{2}x_2) - x_3\| < \tfrac{1}{2}\varepsilon \\
&\qquad\qquad\qquad\qquad\qquad\;\; \|x - x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{4}x_3\| < \tfrac{1}{8}\varepsilon
\end{aligned}$$

Zabreĭko's lemma

Proof (ctd): by induction there is a sequence (x_j) in M_n s.t.

$$\left\| x - \sum_{j=1}^{k} \frac{x_j}{2^{j-1}} \right\| < \frac{\varepsilon}{2^k} \qquad \forall\, k \in \mathbb{N}$$

By taking $k \to \infty$ we find

$$x = \sum_{j=1}^{\infty} \frac{x_j}{2^{j-1}}$$

Zabreĭko's lemma

Proof (ctd): we have

$$x \in B(0;\varepsilon) \quad \Rightarrow \quad x = \sum_{j=1}^{\infty} \frac{x_j}{2^{j-1}}, \qquad x_j \in M_n \quad \forall\, j \in \mathbb{N}$$

By countable subadditivity:

$$p(x) \leq \sum_{j=1}^{\infty} \frac{p(x_j)}{2^{j-1}} \leq n \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 2n$$

Conclusion: $B(0;\varepsilon) \subset M_{2n}$

Zabreĭko's lemma

Proof (ctd):

$$x \neq 0 \quad \Rightarrow \quad y := \tfrac{1}{2}\varepsilon x/\|x\| \in B(0;\varepsilon) \subset M_{2n}$$

$$\Rightarrow \quad p(y) \leq 2n$$

$$\Rightarrow \quad p(x) \leq \frac{4n}{\varepsilon}\|x\|$$

Hence, p is bounded

Application

Zabreĭko's lemma gives unified proofs for:

- open mapping theorem
- closed graph theorem
- uniform boundedness principle

Key theorems which guarantee boundedness of operators!