Functional Analysis

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Lecture 10
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Topics:

- §5.3: Open mapping theorem
- §5.4: Closed graph theorem
- §5.6: Uniform boundedness principle

Zabreĭko's lemma

Definition: a semi-norm on X is a map $p : X \to [0, \infty)$ s.t.

$$\begin{aligned} p(x+y) &\leq p(x) + p(y) \\ p(\lambda x) &= |\lambda|\, p(x) \end{aligned} \qquad \forall\, x, y \in X,\ \lambda \in \mathbb{K}$$

If X is a NLS, then p is bounded if there exists $c > 0$ s.t.

$$p(x) \leq c\|x\| \qquad \forall\, x \in X$$

Lemma: X Banach & p countably subadditive $\Rightarrow$ p bounded

Open mapping theorem

Theorem: if X and Y are Banach spaces, then

$T \in B(X, Y)$ surjective $\Rightarrow$ T is an open map
[meaning: $O \subset X$ open $\Rightarrow$ $T(O) \subset Y$ open]

Proof: since T is surjective we can define

$$p : Y \to [0, \infty), \qquad p(y) = \inf\{\|x\| \, : \, x \in X \text{ s.t. } Tx = y\}$$

First aims:

- p is countably subadditive
- p is a semi-norm on Y

Open mapping theorem

Proof (ctd): we need to show that

$$\sum_{n=1}^{\infty} y_n \text{ convergent} \quad \Rightarrow \quad p\left(\sum_{n=1}^{\infty} y_n\right) \leq \sum_{n=1}^{\infty} p(y_n) \in [0, \infty]$$

If RHS $= \infty$, then nothing to show (since LHS $< \infty$)

Assume RHS $< \infty$, and let $\varepsilon > 0$ be arbitrary

For any $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$y_n = Tx_n \quad \text{and} \quad \|x_n\| < p(y_n) + \frac{\varepsilon}{2^n}$$

Open mapping theorem

Proof (ctd): we have

$$\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} p(y_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \sum_{n=1}^{\infty} p(y_n) + \varepsilon < \infty$$

Since X is a Banach space it follows that

$$\sum_{n=1}^{\infty} x_n \text{ converges}$$

Open mapping theorem

Proof (ctd): since T is bounded we have

$$T\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} y_n$$

Therefore,

$$p\left(\sum_{n=1}^{\infty} y_n\right) \leq \left\|\sum_{n=1}^{\infty} x_n\right\| \leq \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} p(y_n) + \varepsilon$$

$\varepsilon > 0$ arbitrary $\Rightarrow$ p countably subadditive

Open mapping theorem

Proof (ctd):

$$p(y_1 + y_2) \leq p(y_1) + p(y_2) \quad \text{[from countable subadditivity!]}$$

$$\begin{aligned}
\lambda \neq 0 \quad \Rightarrow \quad p(\lambda y) &= \inf\{\|x\| \,:\, x \in X,\ Tx = \lambda y\} \\
&= \inf\{\|x\| \,:\, x \in X,\ T(\lambda^{-1}x) = y\} \\
&= \inf\{\|\lambda x\| \,:\, x \in X,\ Tx = y\} \\
&= |\lambda|\, p(y)
\end{aligned}$$

So $p : Y \to [0, \infty)$ is a countably subadditive semi-norm on Y

Zabreĭko $\Rightarrow$ p is bounded [since Y is a Banach space]

Open mapping theorem

Proof (ctd):

$$\begin{aligned} y \in T(B(0;1)) &\Rightarrow y = Tx \text{ for some } x \text{ with } \|x\| < 1 \\ &\Rightarrow p(y) = \inf\{\|x\| \,:\, Tx = y\} < 1 \end{aligned}$$

$$\begin{aligned} y \notin T(B(0;1)) &\Rightarrow \|x\| \geq 1 \text{ for all } x \text{ with } Tx = y \\ &\Rightarrow p(y) \geq 1 \end{aligned}$$

$$T(B(0;1)) = \{y \in Y \,:\, p(y) < 1\}$$

The latter set is open by continuity of p

[Exercise: prove this last statement; it can be done in three ways]

Open mapping theorem

Proof (ctd): let $O \subset X$ be open and nonempty

$$y \in T(O) \quad \Rightarrow \quad y = Tx \text{ for some } x \in O$$

There exist $\delta, \varepsilon > 0$ s.t.

$$B(x;\delta) \subset O \quad \text{and} \quad B(0;\varepsilon) \subset T(B(0;1))$$

This implies that $T(O)$ is open:

$$\begin{aligned} B(y;\delta\varepsilon) &= Tx + \delta B(0;\varepsilon) \\ &\subset Tx + \delta T(B(0;1)) \\ &= T(B(x;\delta)) \\ &\subset T(O) \end{aligned}$$

Bounded inverse theorem

Corollary: if X, Y are Banach and $T \in B(X, Y)$ then

$$T \text{ bijective} \quad \Rightarrow \quad T^{-1} \in B(Y, X)$$

Proof: T is an open map

For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B(0; \delta) \subset T(B(0; \varepsilon))$$

Equivalently,

$$\|y\| < \delta \;\; \Rightarrow \;\; y = Tx \;\text{ with }\; \|x\| < \varepsilon \;\; \Rightarrow \;\; \|T^{-1}y\| < \varepsilon$$

Therefore T^{-1} is continuous at 0 and hence bounded

Closed range theorem

Theorem: assume X, Y are Banach and $T \in B(X, Y)$

The following statements are equivalent:

1. $\exists\, c > 0$ such that $\|Tx\| \geq c\|x\| \qquad \forall\, x \in X$
2. T injective & ran T closed

Proof $(1 \Rightarrow 2)$**:** see lecture notes

Closed range theorem

Proof $(2 \Rightarrow 1)$**:** note that $T \in B(X, \operatorname{ran} T)$ is bijective

$$\left.\begin{array}{l} Y \text{ Banach} \\ \operatorname{ran} T \text{ closed} \end{array}\right\} \quad \Rightarrow \quad \operatorname{ran} T \text{ Banach}$$

$$\Rightarrow \quad T^{-1} \in B(\operatorname{ran} T, X)$$

$$\Rightarrow \quad \exists\, c' > 0 \quad \text{s.t.} \quad \|T^{-1}y\| \leq c'\|y\| \quad \forall\, y \in \operatorname{ran} T$$

$$\Rightarrow \quad \|x\| \leq c'\|Tx\| \qquad (\text{set } y = Tx)$$

$$\Rightarrow \quad c\|x\| \leq \|Tx\| \qquad (\text{set } c = 1/c')$$

Closed operators

Definition: let X, Y be a NLS and $V \subset X$ a linear subspace

- The graph of $T \in L(V, Y)$ is defined as

$$G(T) = \{ (x, Tx) : x \in V \} \subset X \times Y$$

- The operator T is called closed if $G(T)$ is closed in $X \times Y$

  [Closed according to which norm?!]

Closed operators

Recall: on $X \times Y$ all the following norms are equivalent:

$$\begin{aligned} \|(x,y)\|_\infty &= \max\{\|x\|, \|y\|\} \\ \|(x,y)\|_p &= \left(\|x\|^p + \|y\|^p\right)^{1/p} \qquad 1 \le p < \infty \end{aligned}$$

[See problem 5 of tutorial 2]

Exercise: show that $(x,y) \in \overline{G(T)}$ if and only if

there exists a sequence (x_n) such that

$$x_n \to x \quad \text{and} \quad Tx_n \to y$$

Closed graph theorem

Lemma: if X, Y are NLS and $V \subset X$ a closed lin. subspace, then

$$T \in B(V, Y) \quad \Rightarrow \quad T \text{ is closed}$$

Proof: if $(x, y) \in \overline{G(T)}$, then there is a seq. (x_n) in V such that

$$x_n \to x \quad \text{and} \quad Tx_n \to y$$

Since V is closed we have $x \in V$

Since T is bounded we have $Tx_n \to Tx$

Limits are unique, so $y = Tx$ and thus $(x, y) \in G(T)$

Closed graph theorem

Theorem: if X, Y Banach and $V \subset X$ a closed lin. subspace, then

$$T \text{ is closed} \quad \Rightarrow \quad T \in B(V, Y)$$

Proof: define the semi-norm

$$p : V \to [0, \infty), \qquad p(x) = \| Tx \|$$

To show:

$$\sum_{n=1}^{\infty} x_n \text{ converges in } V \quad \Rightarrow \quad \left\| T\Big(\sum_{n=1}^{\infty} x_n \Big) \right\| \leq \sum_{n=1}^{\infty} \| Tx_n \|$$

Nothing to show when RHS $= \infty$

Closed graph theorem

Proof (ctd): if RHS $< \infty$, then since Y is Banach we have

$$\sum_{n=1}^{m} Tx_n \to \sum_{n=1}^{\infty} Tx_n \quad \text{as} \quad m \to \infty$$

By assumption we also have

$$\sum_{n=1}^{m} x_n \to \sum_{n=1}^{\infty} x_n \quad \text{as} \quad m \to \infty$$

Since T is closed it follows that

$$\left(\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} Tx_n \right) \in \overline{G(T)} = G(T) \quad \text{so} \quad T\left(\sum_{n=1}^{\infty} x_n \right) = \sum_{n=1}^{\infty} Tx_n$$

Closed graph theorem

Proof (ctd): therefore

$$\left\| T\Big(\sum_{n=1}^{\infty} x_n\Big)\right\| = \left\|\sum_{n=1}^{\infty} Tx_n\right\| \leq \sum_{n=1}^{\infty} \|Tx_n\|$$

Since X is Banach and V is a closed lin. subspace, V is Banach

Zabreĭko implies that $p(x) = \|Tx\|$ is bounded

Hence T is bounded

Bounded projections

Example: assume that

- X is Banach
- $V, W \subset X$ are closed linear subspaces
- $X = V + W$ and $V \cap W = \{0\}$

Then $\forall\, x \in X \quad \exists$ unique $v \in V, w \in W$ with $x = v + w$

Define the projection onto V as

$$P : X \to X, \quad Px = v$$

Claim: P is closed [and thus bounded by the closed graph theorem!]

Bounded projections

Example (ctd): if $(x, y) \in \overline{G(P)}$, then there exists (x_n) s.t.

$$x_n \to x \quad \text{and} \quad Px_n \to y$$

Writing $x_n = v_n + w_n$ with $v_n \in V$ and $w_n \in W$ gives

$$\begin{aligned} v_n &= Px_n \to y && \Rightarrow \quad y \in V \quad \text{since } V \text{ is closed} \\ w_n &= x_n - v_n \to x - y && \Rightarrow \quad x - y \in W \quad \text{since } W \text{ is closed} \end{aligned}$$

This gives $P(x - y) = 0$ so $Px = Py = y$

Therefore $(x, y) \in G(P)$ and thus P is closed

Uniform boundedness

Theorem: assume X is Banach and Y is a NLS

For any set $F \subset B(X, Y)$ we have

$$\sup_{T \in F} \|Tx\| < \infty \quad \forall x \in X \quad \Rightarrow \quad \sup_{T \in F} \|T\| < \infty$$

Proof: define the following semi-norm on X:

$$p : X \to [0, \infty), \qquad p(x) = \sup_{T \in F} \|Tx\|$$

a.k.a. Banach–Steinhaus theorem

Uniform boundedness

Proof (ctd): assume that $\sum_{n=1}^{\infty} x_n$ converges

For all $T \in F$ we have

$$\left\| T\left(\sum_{n=1}^{\infty} x_n \right) \right\| = \left\| \sum_{n=1}^{\infty} Tx_n \right\| \leq \sum_{n=1}^{\infty} \| Tx_n \| \leq \sum_{n=1}^{\infty} p(x_n)$$

Taking the supremum over all $T \in F$ gives

$$p\left(\sum_{n=1}^{\infty} x_n \right) \leq \sum_{n=1}^{\infty} p(x_n)$$

Uniform boundedness

Proof (ctd): by Zabreĭko's lemma there exists $c > 0$ such that

$$p(x) \le c\|x\| \quad \forall\, x \in X$$

For any $T \in F$ we have

$$\|Tx\| \le \sup_{T \in F} \|Tx\| = p(x) \le c\|x\| \quad \forall\, x \in X$$

Therefore

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \le c$$

Conclusion: $\|T\| \le c$ for all $T \in F$

Pointwise limits

Corollary: let X be a Banach space and Y be a NLS

Let (T_n) be a sequence in $B(X, Y)$ such that

$$(T_n x) \text{ converges for all } x \in X$$

If $T \in L(X, Y)$ is defined pointwise by

$$Tx := \lim_{n \to \infty} T_n x$$

then $T \in B(X, Y)$

Pointwise limits

Proof: since convergent sequences are bounded we have

$$\sup_n \| T_n x\| < \infty \qquad \forall x \in X$$

The uniform boundedness principle implies

$$c := \sup_n \| T_n\| < \infty$$

For all $x \in X$ and $n \in \mathbb{N}$ we have

$$\| T_n x\| \leq \| T_n\| \, \|x\| \leq c\|x\|$$

Finally, we have

$$\| Tx\| = \lim_{n\to\infty} \| T_n x\| \leq c\|x\|$$