Functional Analysis

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Lecture 12
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Topics:

- §6.1: The dual space of a Hilbert space
- §6.2: Bounded operators in a Hilbert space
- §6.3: Special classes of operators

Adjoints of matrices

Definition: the adjoint (or conjugate transpose) of a $n \times n$ matrix A over $\mathbb{K}$ is defined as

$$A^* = (\overline{A})^\top \quad \text{i.e.} \quad (A^*)_{ij} = \overline{A_{ji}}$$

Fact: with the standard innerproduct on $\mathbb{K}^n$ given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$$

we have

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \text{for all} \quad x, y \in \mathbb{K}^n$$

Adjoints of matrices

Definition: an $n \times n$ matrix A over $\mathbb{K}$ is called selfadjoint if

$$A^* = A$$

Fact: selfadjoint matrices

- have real eigenvalues; eigenvectors corresponding to different eigenvalues are orthogonal
- are diagonalizable

Goal: generalize this to infinite-dimensional spaces

Dual spaces

Definition: let X be a NLS

The dual space of X is defined as

$$X' = B(X, \mathbb{K})$$

The norm on X' is given by

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

Dual spaces

Lemma: let X be a Hilbert space and $y \in X$

The map $f : X \to \mathbb{K}$ defined by $f(x) = \langle x, y \rangle$ belongs to X' and

$$\|f\| = \|y\|$$

Proof: w.l.o.g. we may assume $y \neq 0$ in which case

$$\frac{|f(x)|}{\|x\|} = \frac{|\langle x, y \rangle|}{\|x\|} \leq \frac{\|x\| \, \|y\|}{\|x\|} = \|y\| \qquad \forall\, x \neq 0$$

$$\frac{|f(y)|}{\|y\|} = \frac{|\langle y, y \rangle|}{\|y\|} = \frac{\|y\|^2}{\|y\|} = \|y\|$$

Riesz-Fréchet theorem

Theorem: assume is a X Hilbert space

For each $f \in X'$ there exist a unique $y \in X$ such that

$$f(x) = \langle x, y \rangle \quad \text{for all} \quad x \in X$$

Proof (uniqueness):

$$\begin{aligned} \langle x, y_1 \rangle = \langle x, y_2 \rangle \quad \forall\, x \in X \quad &\Rightarrow \quad \langle x, y_1 - y_2 \rangle = 0 \quad \forall\, x \in X \\ &\Rightarrow \quad \langle y_1 - y_2, y_1 - y_2 \rangle = 0 \\ &\Rightarrow \quad y_1 - y_2 = 0 \\ &\Rightarrow \quad y_1 = y_2 \end{aligned}$$

Riesz-Fréchet theorem

Proof (existence): if $f \neq 0$, then

$$\exists\, z \in (\ker f)^{\perp} \text{ with } f(z) = 1$$

For all $x \in X$ we have

$$\begin{aligned} x - f(x)z \in \ker f \quad &\Rightarrow \quad \langle x - f(x)z, z\rangle = 0 \\ &\Rightarrow \quad \langle x, z\rangle - f(x)\langle z, z\rangle = 0 \\ &\Rightarrow \quad f(x) = \langle x, y\rangle \quad \text{with} \quad y = \frac{z}{\|z\|^2} \end{aligned}$$

Existence of adjoints

Theorem: let X, Y be Hilbert spaces and $T \in B(X, Y)$

There exists a unique adjoint operator $T^* \in B(Y, X)$ such that

- $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x \in X$ and $y \in Y$
- $\|T^*\| \leq \|T\|$

Existence of adjoints

Proof: fix $y \in Y$ and define $f : X \to \mathbb{K}$ by $f(x) = \langle Tx, y \rangle$

For all $x \neq 0$ we have

$$\frac{|f(x)|}{\|x\|} = \frac{|\langle Tx, y \rangle|}{\|x\|} \leq \frac{\|Tx\| \, \|y\|}{\|x\|} \leq \|T\| \, \|y\|$$

Conclusion: $f \in X'$ and

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \|T\| \, \|y\|$$

Existence of adjoints

Proof (ctd): Riesz-Fréchet $\Rightarrow$ $\exists$ unique $u_y \in X$ s.t.

$$\langle Tx, y\rangle = \langle x, u_y\rangle \quad \forall\, x \in X \quad \text{and} \quad \|u_y\| = \|f\|$$

Define $T^* : Y \to X$ by setting $T^* y = u_y$, then $T^* \in L(Y, X)$
[Exercise: check that T^* is indeed linear]

Finally, for all $y \in Y$ we have

$$\|T^* y\| = \|u_y\| = \|f\| \leq \|T\|\,\|y\|$$

The shift operators

Example: on ℓ^2 consider

$$\begin{aligned} S_R(x_1, x_2, x_3, \dots) &= (0, x_1, x_2, \dots) \\ S_L(x_1, x_2, x_3, \dots) &= (x_2, x_3, x_4, \dots) \end{aligned}$$

We have that $S_R^* = S_L$ since

$$\langle S_R x, y \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_{n+1} = \langle x, S_L y \rangle \quad \forall\, x, y \in \ell^2$$

Fredholm and Volterra operators

Exercise: on $L^2(a,b)$ it follows from Fubini that

$$Tf(x) = \int_a^b G(x,y)f(y)\,dy \quad \Rightarrow \quad T^*f(x) = \int_a^b \overline{G(y,x)}f(y)\,dy$$

$$Tf(x) = \int_a^x G(x,y)f(y)\,dy \quad \Rightarrow \quad T^*f(x) = \int_x^b \overline{G(y,x)}f(y)\,dy$$

Properties of adjoints

Lemma: $(T^*)^* = T$

Proof: for all $x \in X$ and $y \in Y$ we have

$$\begin{aligned}
\langle (T^*)^* x, y \rangle &= \overline{\langle y, (T^*)^* x \rangle} \\
&= \overline{\langle T^* y, x \rangle} \\
&= \langle x, T^* y \rangle \\
&= \langle Tx, y \rangle
\end{aligned}$$

Subtracting and taking $y = (T^*)^* x - Tx$ gives

$$\|(T^*)^* x - Tx\|^2 = 0 \qquad \forall\, x \in X$$

Properties of adjoints

Lemma: $\|T^*\| = \|T\|$

Proof:

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \leq \|T\|$$

Properties of adjoints

Lemma: $\|T^*T\| = \|T\|^2$

Proof: on the one hand

$$\|T^*T\| \leq \|T^*\|\,\|T\| = \|T\|^2$$

On the other hand we have for all $x \in X$ that

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2$$

This implies $\|T\|^2 \leq \|T^*T\|$

Properties of adjoints

Lemma: if X, Y, and Z are Hilbert spaces, then

1. $T, S \in B(X, Y) \quad \Rightarrow \quad (\lambda T + \mu S)^* = \bar{\lambda} T^* + \bar{\mu} S^*$
2. $T \in B(X, Y)$ and $S \in B(Y, Z) \quad \Rightarrow \quad (ST)^* = T^* S^*$
3. $T \in K(X, Y) \quad \Rightarrow \quad T^* \in K(Y, X)$

Properties of adjoints

Proof (3):

$$\begin{aligned} T \in K(X,Y) &\Rightarrow TT^* \in K(Y) \\ \|y_n\| \le c \quad \forall\, n \in \mathbb{N} &\Rightarrow TT^* y_{n_k} \text{ converges for some subsequence} \end{aligned}$$

$$\begin{aligned} \|T^*(y_n - y_m)\|^2 &= \langle T^*(y_n - y_m), T^*(y_n - y_m)\rangle \\ &= \langle TT^*(y_n - y_m), y_n - y_m\rangle \\ &\le \|TT^*(y_n - y_m)\|\,\|y_n - y_m\| \\ &\le 2c\|TT^*(y_n - y_m)\| \end{aligned}$$

$$T^* y_{n_k} \text{ Cauchy} \Rightarrow T^* y_{n_k} \text{ convergent}$$

The inverse of an adjoint

Lemma: if $T \in B(X)$ is invertible, then so is T^* and

$$(T^*)^{-1} = (T^{-1})^*$$

Proof: since T is invertible we have $T^{-1} \in B(X)$ and

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T = I$$

Taking adjoints gives

$$\left.\begin{aligned} I = I^* = (TT^{-1})^* = (T^{-1})^* T^* \\ I = I^* = (T^{-1}T)^* = T^*(T^{-1})^* \end{aligned}\right\} \quad \Rightarrow \quad (T^*)^{-1} = (T^{-1})^*$$

The spectrum of the adjoint

Lemma: if $T \in B(X)$, then

$$\rho(T^*) = \overline{\rho(T)} \quad \text{and} \quad \sigma(T^*) = \overline{\sigma(T)}$$

Proof:

$$\begin{aligned} \lambda \in \rho(T) &\Leftrightarrow (T - \lambda)^{-1} \in B(X) \\ &\Leftrightarrow ((T - \lambda)^{-1})^* \in B(X) \\ &\Leftrightarrow (T^* - \bar{\lambda})^{-1} \in B(X) \\ &\Leftrightarrow \bar{\lambda} \in \rho(T^*) \end{aligned}$$

Range/kernel orthogonality

Lemma: for $T \in B(X)$ and $\lambda \in \mathbb{K}$ we have

$$(\mathrm{ran}(T-\lambda))^{\perp} = \ker(T^* - \bar{\lambda})$$

$$(\mathrm{ran}(T^*-\bar{\lambda}))^{\perp} = \ker(T - \lambda)$$

[Note: for $T \in B(X, Y)$ we only have $(\mathrm{ran}\ T)^{\perp} = \ker T^*$ and $(\mathrm{ran}\ T^*)^{\perp} = \ker T$]

Proof: follows from

$$\langle (T-\lambda)x, y\rangle = \langle x, (T^*-\bar{\lambda})y\rangle \qquad \forall\, x, y \in X$$

Range/kernel orthogonality

Corollary: let $T \in B(X)$ and $\lambda \in \mathbb{K}$

We have the following orthogonal decompositions:

$$X = \ker(T^* - \bar{\lambda}) \oplus \overline{\mathrm{ran}}(T - \lambda)$$

$$X = \ker(T - \lambda) \oplus \overline{\mathrm{ran}}(T^* - \bar{\lambda})$$

Notation: $\oplus$ = direct sum of orthogonal closed linear subspaces

Special classes of operators

Definition:

$T \in B(X)$ is normal if: $TT^* = T^*T$

$T \in B(X)$ is selfadjoint if: $T = T^*$

$T \in B(X, Y)$ is unitary if: $T^*T = I_X$ and $TT^* = I_Y$

Normal operators

Lemma: if $T \in B(X)$ is normal, then

$$\|Tx\| = \|T^*x\| \qquad \forall\, x \in X$$

Proof: for all $x \in X$ we have

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \\
&= \langle x, T^*Tx\rangle \\
&= \langle x, TT^*x\rangle \\
&= \langle T^*x, T^*x\rangle = \|T^*x\|^2
\end{aligned}$$

Normal operators

Corollary: if $T \in B(X)$ is normal, then

$$\ker(T - \lambda) = \ker(T^* - \bar{\lambda}) \qquad \forall\, \lambda \in \mathbb{K}$$

Proof: $T - \lambda$ is also normal and thus

$$\|(T - \lambda)x\| = \|(T^* - \bar{\lambda})x\| \qquad \forall\, x \in X$$

Normal operators

Lemma: if $T \in B(X)$ is normal, then

$$\rho(T) = \{\lambda \in \mathbb{K} : \exists\, c > 0 \text{ s.t. } \|(T-\lambda)x\| \geq c\|x\| \ \forall x \in X\}$$

Proof: we have that

$$\lambda \in \rho(T) \quad \Leftrightarrow \quad \begin{cases} \operatorname{ran}(T-\lambda) \text{ dense in } X & (1) \\ \quad \text{AND} & \\ \|(T-\lambda)x\| \geq c\|x\| \quad \forall x \in X & (2) \end{cases}$$

Statement (2) implies statement (1) since

$$\left.\begin{aligned} \ker(T^* - \bar{\lambda}) = \ker(T-\lambda) = \{0\} \\ X = \ker(T^* - \bar{\lambda}) \oplus \overline{\operatorname{ran}}(T-\lambda) \end{aligned}\right\} \quad \Rightarrow \quad \overline{\operatorname{ran}}(T-\lambda) = X$$

Normal operators

Lemma: if $T \in B(X)$ is normal, then

$$\rho(T) = \big\{\lambda \in \mathbb{K} \,:\, \exists\, c > 0 \ \text{ s.t. } \ \|(T-\lambda)x\| \geq c\|x\| \ \ \forall\, x \in X\big\}$$

Corollary:

$$\sigma(T) = \big\{\lambda \in \mathbb{K} \,:\, \exists\, (x_n) \ \text{ s.t. } \ \|x_n\| = 1 \ \text{ and } \ (T-\lambda)x_n \to 0\big\}$$

Normal operators

Lemma: if $T \in B(X)$ is normal, then

$$\left.\begin{array}{l} Tx = \lambda x \quad \text{and} \quad Ty = \mu y \\ \lambda \neq \mu \end{array}\right\} \quad \Rightarrow \quad \langle x, y \rangle = 0$$

Proof:

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle = \langle x, \bar{\mu} y \rangle = \mu \langle x, y \rangle$$

$$\Rightarrow \quad (\lambda - \mu) \langle x, y \rangle = 0$$

$$\Rightarrow \quad \langle x, y \rangle = 0$$

Orthogonal projections

Definition: X Hilbert

$P \in B(X)$ is called an orthogonal projection if:

1. $P^2 = P$
2. $\ker P \perp \operatorname{ran} P$

Orthogonal projections

Lemma: if X is Hilbert and $P \in B(X)$ is a projection, then

P is an orthogonal projection $\Leftrightarrow$ $P = P^*$

Proof (⇒): for all $x, y \in X$ we have

$$\begin{aligned} \langle Px, y\rangle &= \langle Px, Py + (I-P)y\rangle \\ &= \langle Px, Py\rangle \\ &= \langle Px + (I-P)x, Py\rangle \\ &= \langle x, Py\rangle \end{aligned}$$

Orthogonal projections

Proof ($\Leftarrow$): since $P = P^*$ we have

$$\langle Px, (I-P)y\rangle = \langle x, P(I-P)y\rangle = 0 \qquad \forall\, x, y \in X$$

This shows that $\operatorname{ran} P \perp \operatorname{ran}(I-P)$

But $\operatorname{ran}(I-P) = \ker P$