Functional Analysis

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Lecture 3
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Topics:

- §2.3: Closed linear subspaces

Closure

Definition: let X be NLS and $V \subset X$ a subset

- distance between $x \in X$ and V:

$$d(x, V) := \inf\{\|x - v\| \,:\, v \in V\}$$

- closure of V:

$$\overline{V} := \{x \in X \,:\, d(x, V) = 0\}$$

- V is called closed when $\overline{V} = V$

[Exercise: show that $V \subset \overline{V}$ and that $\overline{\overline{V}} = \overline{V}$]

Open sets

Definition: let X be NLS and $O \subset X$ a subset

$O \subset X$ is called open when

$$\forall x \in O \quad \exists \varepsilon > 0 \quad \text{s.t.} \quad B(x;\varepsilon) := \{y \in X \ : \ \|x - y\| < \varepsilon\} \subset O$$

[Exercise: prove that $V \subset X$ is closed $\Leftrightarrow$ V^c is open]

[Exercise: if $O \subset X$ is a *linear subspace* and open, then $O = X$]

Characterization of closure

Lemma: if X is a NLS and $V \subset X$ is a subset, then

$$x \in \overline{V} \quad \Leftrightarrow \quad x_n \to x \quad \text{for some sequence } (x_n) \text{ in } V$$

Proof ($\Rightarrow$):

$$d(x, V) = 0 \quad \Rightarrow \quad \inf\{\|x - v\| \,:\, v \in V\} = 0$$

$$\Rightarrow \quad \forall n \in \mathbb{N} \quad \exists x_n \in V \text{ such that } \|x - x_n\| < \frac{1}{n}$$

$$\Rightarrow \quad x_n \to x$$

Characterization of closure

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Proof ($\Leftarrow$): if $x \notin \overline{V}$ then $d := d(x, V) > 0$

But $\|x_n - x\| < d$ for n sufficiently large

Contradiction, hence $x \in \overline{V}$

Closed sets

Example: $V = \{f \in \mathcal{C}([a,b],\mathbb{K}) : f(a) = 0\}$ closed in $\mathcal{C}([a,b],\mathbb{K})$

- $V \subset \overline{V}$ is trivial
- If $f \in \overline{V}$, then there exists (f_n) in V with $f_n \to f$, so

$$|f(a)| = |f(a) - f_n(a)| \leq \|f - f_n\|_\infty \to 0$$

Conclusion: $f(a) = 0$, so $f \in V$

Closed sets

Example:

$$\ell^\infty = \left\{ x = (x_1, x_2, x_3, \dots) \; : \; x_n \in \mathbb{K}, \quad \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}$$

$$s = \left\{ x = (x_1, x_2, x_3, \dots) \; : \; \exists N_x > 0 \text{ s.t. } n \geq N_x \;\Rightarrow\; x_n = 0 \right\}$$

Claim: s is NOT closed in ℓ^∞

Closed sets

Example (ctd):

$$x = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \dots\right) \in \ell^\infty$$

$$x^n = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, 0, \dots\right) \in s$$

$$\|x^n - x\|_\infty = \tfrac{1}{n+1} \to 0$$

So $x \in \overline{s}$ but $x \notin s$

Closed linear subspaces

Lemma: if X is a NLS and $V \subset X$ a linear subspace, then

$\dim V < \infty \ \Rightarrow\ V$ closed

Proof: let $V = \mathrm{span}\{e_1, \ldots, e_d\}$ and define the norm

$$v = \lambda_1 e_1 + \cdots + \lambda_d e_d, \qquad \|v\|_+ = \left(\sum_{i=1}^{d} |\lambda_i|^2\right)^{1/2}$$

Note: $\|\cdot\|$ and $\|\cdot\|_+$ are equivalent on V!

[So there exist $m, M > 0$ such that $m\|v\| \le \|v\|_+ \le M\|v\|$ for all $v \in V$]

Closed linear subspaces

Proof (ctd): assume

$$v_n = \lambda_{n,1} e_1 + \cdots + \lambda_{n,d} e_d \in V \quad \text{and} \quad \|v_n - x\| \to 0$$

[Since (v_n) is in V we have $x \in \overline{V}$. Next, we prove that in fact $x \in V$]

Then for all $i = 1, \ldots, d$:

$$\begin{aligned} |\lambda_{n,i} - \lambda_{m,i}| &\leq \|v_n - v_m\|_+ \\ &\leq M\|v_n - v_m\| \\ &= M\|v_n - x + x - v_m\| \\ &\leq M(\|v_n - x\| + \|x - v_m\|) \to 0 \quad n, m \to \infty \end{aligned}$$

Closed linear subspaces

Proof (ctd): $(\lambda_{n,i})$ is Cauchy in $\mathbb{K}$ for all $i = 1, \ldots, d$ so define

$$u = \lambda_1 e_1 + \cdots + \lambda_d e_d, \quad \lambda_i = \lim_{n\to\infty} \lambda_{n,i}$$

Then

$$\|v_n - u\| \leq \frac{1}{m}\|v_n - u\|_+ = \frac{1}{m}\left(\sum_{i=1}^{d} |\lambda_{n,i} - \lambda_i|^2\right)^{1/2} \to 0 \quad n \to \infty$$

Since $\|v_n - x\| \to 0$ as well it follows that $x = u \in V$

[Here we use uniqueness of limits]

Closed linear subspaces

Lemma: let X be a NLS and $V \subset X$ a linear subspace

Then $\overline{V}$ is closed and a linear subspace

Proof (addition): assume $x, y \in \overline{V}$

There are sequences (x_n) and (y_n) in V such that

$$x_n \to x \quad \text{and} \quad y_n \to y$$

We have $x + y \in \overline{V}$ since $(x_n + y_n)$ is a sequence in V and

$$x_n + y_n \to x + y$$

[Exercise: show that $x \in \overline{V}$ and $\lambda \in \mathbb{K}$ implies that $\lambda x \in \overline{V}$]

Norms on quotient spaces

Proposition: if X is a NLS and $V \subset X$ a linear subspace, then

1. $\|x + V\| := d(x, V)$ is a semi-norm on X/V
2. $\|x + V\|$ is a norm $\Leftrightarrow$ V is closed
3. $\|x + V\| \leq \|x\|$ for all $x \in X$

Proof (3):

$$\|x + V\| = d(x, V) = \inf\{\|x - v\| \,:\, v \in V\} \leq \|x - 0\| = \|x\|$$

Norms on quotient spaces

Proof (1): if $\lambda \neq 0$ then

$$\begin{aligned}
\|\lambda(x+V)\| &= \|\lambda x + V\| \\
&= d(\lambda x, V) \\
&= \inf\{\|\lambda x - v\| \,:\, v \in V\} \\
&= |\lambda| \inf\{\|x - v/\lambda\| \,:\, v \in V\} \\
&= |\lambda| \inf\{\|x - v\| \,:\, v \in V\} \\
&= |\lambda| \|x + V\|
\end{aligned}$$

If $\lambda = 0$ then $d(\lambda x, V) = d(0, V) = 0$ since $0 \in V$

Norms on quotient spaces

Proof (1):

$$\begin{aligned}
\|(x+V)+(y+V)\| &= \|(x+y)+V\| \\
&= d(x+y, V) \\
&= \inf\{\|x+y-z\| \,:\, z \in V\} \\
&= \inf\{\|x+y-(u+v)\| \,:\, u, v \in V\} \\
&\leq \inf\{\|x-u\| + \|y-v\| \,:\, u, v \in V\} \\
&= d(x, V) + d(y, V)
\end{aligned}$$

Hence, $\|x+V\|$ is a semi-norm

Norms on quotient spaces

Proof (2):

$$
\text{If } V \text{ is closed:} \quad \|x+V\| = 0 \;\Rightarrow\; d(x,V)=0
$$
$$
\Rightarrow\; x \in \overline{V} = V
$$
$$
\Rightarrow\; x+V = 0+V
$$
$$
\Rightarrow\; \|x+V\| \text{ is a norm}
$$

$$
\text{If } \|x+V\| \text{ is a norm:} \quad x \in \overline{V} \;\Rightarrow\; d(x,V)=0
$$
$$
\Rightarrow\; \|x+V\| = 0
$$
$$
\Rightarrow\; x+V = 0+V
$$
$$
\Rightarrow\; x \in V
$$

Separable spaces

Definition: let X be a metric space

1. a subset $E \subset X$ is called dense when $\overline{E} = X$
2. X is called separable if it contains a countable dense subset

Separable spaces

Examples:

- $\mathbb{R}$ is separable since $\mathbb{Q}$ is dense and countable
- $\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is dense and countable
- $\mathbb{C}$ is separable since $\mathbb{Q} + i\mathbb{Q}$ is dense and countable
- $\mathbb{C}^n$ is separable since $(\mathbb{Q} + i\mathbb{Q})^n$ is dense and countable

Separable spaces

Example: ℓ^p is separable for all $1 \le p < \infty$

$$E = \bigcup_{n=1}^{\infty} \left\{ (r_1, r_2, r_3, \ldots, r_n, 0, 0, 0, \ldots) \,:\, r_i \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is a countable set

Exercise: show that E is dense in ℓ^p

Separable spaces

Example: ℓ^∞ is NOT separable

Assume that $E = \{e^1, e^2, e^3, \dots\} \subset \ell^\infty$ is countable

Define $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$ by

$$x_n = \begin{cases} e_n^n + 1 & \text{if } |e_n^n| \leq 1 \\ 0 & \text{if } |e_n^n| > 1 \end{cases}$$

Then $\|x - e^n\|_\infty \geq 1$ for all $n \in \mathbb{N}$ so E cannot be dense

Riesz's lemma

Lemma (Riesz): assume

- X is NLS
- $V \subset X$ is a closed linear subspace with $V \neq X$

Then for all $0 < \lambda < 1$ there exists $x_\lambda \in X$ such that

$$\|x_\lambda\| = 1 \quad \text{and} \quad \|x_\lambda - v\| > \lambda \quad \forall v \in V$$

Riesz's lemma

Proof: there exists $x \in X \setminus V$ such that

$$0 < d(x, V) = \inf\{\|x - v\| \,:\, v \in V\}$$

Since $0 < \lambda < 1$ there is $w \in V$ such that

$$\|x - w\| < \frac{d(x, V)}{\lambda} \quad \Rightarrow \quad \frac{1}{\|x - w\|} > \frac{\lambda}{d(x, V)}$$

Define $x_\lambda = \dfrac{x - w}{\|x - w\|}$ so that $\|x_\lambda\| = 1$

Riesz's lemma

Proof (ctd): for all $v \in V$

$$\begin{aligned}
\|x_\lambda - v\| &= \left\| \frac{x-w}{\|x-w\|} - v \right\| \\
&= \frac{1}{\|x-w\|} \, \|x - (\underbrace{w + \|x-w\| v}_{\in V})\| \\
&\geq \frac{1}{\|x-w\|} \, d(x, V) \\
&> \frac{\lambda}{d(x, V)} \, d(x, V) = \lambda
\end{aligned}$$

Compactness versus dimension

Theorem: let X be NLS, then

$$B = \{x \in X \ : \ \|x\| \leq 1\} \quad \text{compact} \quad \Rightarrow \quad \dim X < \infty$$

Proof: assume $\dim X = \infty$

Idea: construct a sequence in B without a convergent subsequence

Compactness versus dimension

Proof: choose any $e_1 \in X$ with $\|e_1\| = 1$

Apply Riesz with $V_1 = \text{span}\{e_1\}$ and $\lambda = \frac{1}{2}$:

$$\exists e_2 \in X \quad \text{s.t.} \quad \|e_2\| = 1 \quad \text{and} \quad d(e_2, V_1) \geq \frac{1}{2}$$

Apply Riesz with $V_2 = \text{span}\{e_1, e_2\}$ and $\lambda = \frac{1}{2}$:

$$\exists e_3 \in X \quad \text{s.t.} \quad \|e_3\| = 1 \quad \text{and} \quad d(e_3, V_2) \geq \frac{1}{2}$$

Proceed by induction...

Compactness versus dimension

Proof (ctd): we now have a sequence (e_i) in X such that

$$\|e_i\| = 1, \qquad d(e_i, \mathrm{span}\{e_1, \ldots, e_{i-1}\}) \geq \frac{1}{2}$$

This implies

$$\|e_i - e_j\| \geq \frac{1}{2} \qquad \forall i \neq j$$

Hence (e_i) has no convergent subsequence, so B is NOT compact

Compactness versus dimension

Warning: the following statements are NOT equivalent in ∞-dimensional linear spaces!

1. V compact
2. V closed and bounded

Only (1) $\Rightarrow$ (2) remains true