# Iterative Algorithms in Optimization, Variational Analysis and Fixed Point Theory

**Unit 07: Iterative algorithms in infinite dimension.**

university of groningen

# Some infinite-dimensional variational problems

- Calculus of variations
  - Braquistochrone, minimal surfaces, orbits of planets
- Optimal control
  - Linear-quadratic problem, automated vehicles, flight plans
- Partial differential equations
  - Poisson equation and harmonic functions, the obstacle problem
- Inverse problems
  - Tomography, signal processing, object detection

# Optimal Control and Calculus of Variations

# Calculus of variations

The classical problem of Calculus of Variations is

$$(CV) \qquad \min\{\ J[x]\ :\ x \in AC(0,T;\mathbb{R}^N),\ x(0)=x_0,\ x(T)=x_T\ \},$$

where the function $J$ is of the form

$$J[x] = \int_0^T \ell(t,\dot{x}(t),x(t))\,dt$$

for some function $\ell : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$.

# Least action principle

**Theorem (Euler-Lagrange Equation)**

*Let $x^*$ be a smooth solution of* (CV)*. Then, the function* $t \mapsto \nabla_3 L(t, x^*(t), \dot{x}^*(t))$ *is continuously differentiable and*

$$\frac{d}{dt}\left[\nabla_3 L(t, x^*(t), \dot{x}^*(t))\right] = \nabla_2 L(t, x^*(t), \dot{x}^*(t))$$

*for every* $t \in (0, T)$*.*

# Linear control systems

Consider the linear control system

$$(CS) \qquad \begin{cases} \dot{y}(t) &= A(t)y(t) + B(t)u(t) + c(t), \qquad t \in (0,T) \\ y(0) &= y_0. \end{cases}$$

Given $u \in L^p(0,T;\mathbb{R}^M)$ with $p \in [1,\infty]$, one can find $y_u \in \mathcal{C}(0,T;\mathbb{R}^N)$ such that the pair $(u, y_u)$ satisfies $(CS)$.

$$(1) \qquad y_u(t) = e^{tA}y_0 + e^{tA}\int_0^t e^{-sA}\left[B(s)u(s) + c(s)\right]\, ds.$$

The function $u \mapsto y_u$ is affine and continuous.

# Optimal control

Consider the functional $J$ defined by

$$J[u] = \int_0^T \ell(t, u(t), y_u(t))\, dt + h(y_u(T)) + \iota_{\mathcal{T}}(y_u(T))$$

with $\ell : \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$. The optimal control problem is

$$(OC) \qquad \min\{\, J[u] \;:\; u \in X \,\}.$$

**Theorem**

*Under suitable conditions, if $(OC)$ is feasible, it has a solution in $L^2(0, T; \mathbb{R}^M)$.*

# Linear-quadratic problem

$$J[u] = \frac{1}{2}\int_0^T u(t)^* U(t)u(t)\,dt + \frac{1}{2}\int_0^T y_u(t)^* W(t)y_u(t)\,dt + h(y_u(T))$$

## Theorem

*The pair $(\bar{u}, \bar{y}_u)$ is optimal if, and only if,*

$$\bar{u}(t) = U(t)^{-1}B(t)^* p(t) \quad \text{for a.e.} \quad t \in [0, T],$$

*where the adjoint state $p$ is the unique solution of the adjoint equation*

$$\dot{p}(t) = -A(t)^* p(t) + W(t)\bar{y}_u(t)$$

*with terminal condition $p(T) = -\nabla h(\bar{y}_u(T))$.*

# Some Elliptic Differential Equations

# Poisson Equation in $H^1$, with Dirichlet boundary conditions

Let $\mu \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ bounded, and $\bar{u} \in H^1(\Omega)$. Consider $v^* \in (H_0^1)^*$, and define $B : H_0^1 \times H_0^1 \to \mathbb{R}$ by

$$B(u, v) = \mu \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{(L^2)^N}.$$

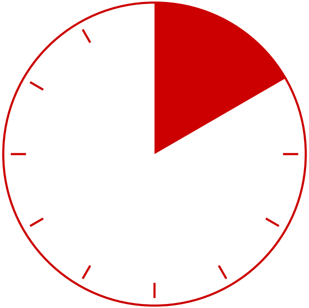
The following are equivalent:

1. $\bar{u}$ is the unique weak solution $\bar{u}$ for the problem

$$\begin{cases} -\Delta u + \mu\, u & = & v^* & \text{in } \Omega \\ u & = & 0 & \text{in } \partial\Omega. \end{cases}$$

2. $\mu \langle \bar{u}, v \rangle_{L^2} + \langle \nabla \bar{u}, \nabla v \rangle_{(L^2)^N} = \langle v^*, v \rangle_{(H_0^1)^*, H_0^1}$ for all $v \in H_0^1$.
3. $\bar{u}$ minimizes $\varphi(x) = B(x, x)$ on $H_0^1$.

# Break



# The reduction process

## From infinite to finite dimension

Let $J : X \to \mathbb{R}$ be bounded from below.

### Discretize, then optimize

- Approximate the problem in $X_N = \text{span}\{e_1, \ldots, e_N\} \subset X$
- Minimize $J_N(x_1, \ldots, x_N) = J(\sum x_i e_i)$ by applying an algorithm

### Optimize, then discretize

- Apply an algorithm or use optimality conditions
- Discretize to implement

# The way back

At the $n$-th iteration, given a point $x_{n-1} \in X_{n-1}$ and a tolerance $\varepsilon_n \geq 0$, find $x_n$ such that

$$x_n \in \varepsilon_n - \mathrm{Argmin}\{\ J(x) : x \in X_n\ \} \quad \text{and} \quad J(x_n) \leq J(x_{n-1}).$$

This is always possible since $J$ is bounded from below and $X_{n-1} \subset X_n$.

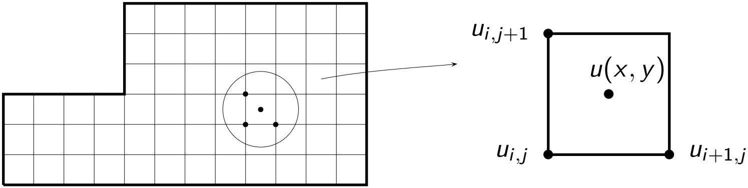
If $\varepsilon_n = 0$, the condition $J(x_n) \leq J(x_{n-1})$ holds automatically.

**Ritz's Theorem**

If $J$ is upper-semicontinuous and $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then $\lim\limits_{n\to\infty} J(x_n) = \inf\limits_{x\in X} J(x)$.
If, moreover, $J$ is continuous, then every limit point of $(x_n)$ minimizes $J$.

# Building the finite-dimensional approximations
## Finite differences

Draw a grid in the domain, partitioning it into small $N$-dimensional blocks.



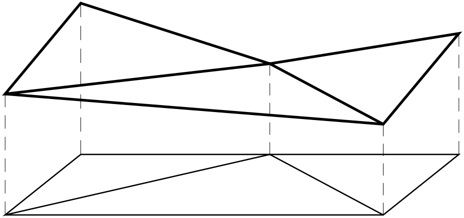

Piecewise constant approximations.

# Building the finite-dimensional approximations

## Finite elements

Domain partitioned into small polyhedral units, where the function is represented by a polynomial. For example:

1. $N = 1$: Trapezoids, Simpson's Rule, Runge Kutta, etc.
2. $N = 2$: triangles, with function values assigned to the vertices



# Building the finite-dimensional approximations

## Fourier Series: Approximation by trigonometric polynomials

Pros:

+ Infinitely differentiable
+ Easy to compute derivatives and integrals
+ Basis generated by combining translations and contractions

Cons

– Lack of localization. A function with a small support may need many Fourier coefficients.

– Poor approximation around discontinuities

Main advantages:

- *Localization:* Elements with arbitrarily small (essential) support, to identify peaks, cliffs, or sparsity
- *Simple representation:* Basis constructed from one particular element, by translations and contractions
- *Customization:* One wavelet basis for every need (e.g. smoothness, identification of main frequencies)