# Functional Analysis

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Lecture 2
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Topics:

- §2.1: Linear spaces with a norm
- §2.2: Properties of norms

# Functional analysis > linear algebra!

Key word: topology

Using metrics induced by norms or inner products we can study:

- sequences, limits
- open, closed, compact sets
- continuity
- completeness

# Normed linear spaces

**Definition:** a norm on a linear space $X$ is a real-valued function

$$x \mapsto \|x\|$$

which satisfies:

1. $\|x\| \geq 0 \quad$ and $\quad \|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$

Note: $d(x, y) = \|x - y\|$ is a metric on $X$

# Normed linear spaces

**Example:** possible norms on $\mathbb{K}^n$ are:

$$\|x\|_p \quad = \quad \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad 1 \leq p < \infty$$

$$\|x\|_\infty \quad = \quad \max \left\{ |x_i| \, : \, i = 1, \ldots, n \right\}$$

Proof of triangle inequality nontrivial for $p > 1$!

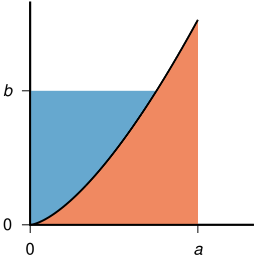
# Young's inequality

**Lemma:** if $1 < p < \infty$ and $a, b \geq 0$, then

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Proof:** if $f : [0, \infty) \to \mathbb{R}$ is strictly increasing and $f(0) = 0$, then

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$$




Apply with $f(x) = x^{p-1}$

[Exercise: show that $f^{-1}(y) = y^{q-1}$ using that $(p-1)(q-1) = 1$]

# Hölder's inequality

**Lemma:** let $1 < p < \infty$, then

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad \sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

**Proof:** apply Young's inequality:

$$\frac{|x_i|}{(\sum |x_i|^p)^{1/p}} \cdot \frac{|y_i|}{(\sum |y_i|^q)^{1/q}} \leq \frac{|x_i|^p}{p(\sum |x_i|^p)} + \frac{|y_i|^q}{q(\sum |y_i|^q)}$$

Sum over $i = 1, \ldots, n$:

$$\frac{\sum |x_i y_i|}{(\sum |x_i|^p)^{1/p}(\sum |y_i|^q)^{1/q}} \leq \frac{\sum |x_i|^p}{p(\sum |x_i|^p)} + \frac{\sum |y_i|^q}{q(\sum |y_i|^q)} = 1$$

# Minkowski's inequality

**Lemma:** let $1 \leq p < \infty$, then

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

**Proof ($p > 1$):**

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1} \leq (|x_i| + |y_i|)|x_i + y_i|^{p-1}$$

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}$$

# Minkowski's inequality

**Proof (ctd):** apply Hölder and note that $q(p-1) = p$

$$\sum |x_i||x_i + y_i|^{p-1} \leq \Big(\sum |x_i|^p\Big)^{1/p} \Big(\sum |x_i + y_i|^p\Big)^{1/q}$$

Hence

$$\sum |x_i+y_i|^p \leq \left[\Big(\sum |x_i|^p\Big)^{1/p} + \Big(\sum |y_i|^p\Big)^{1/p}\right] \Big(\sum |x_i+y_i|^p\Big)^{1/q}$$

# Normed linear spaces

**Examples:**

$$\ell^p = \left\{ x = (x_1, x_2, x_3, \dots) \, : \, x_i \in \mathbb{K}, \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}, \quad p \geq 1$$

$$\|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

$$\ell^\infty = \left\{ x = (x_1, x_2, x_3, \dots) \, : \, x_i \in \mathbb{K}, \quad \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$$

# Normed linear spaces

**Example:**

$$\mathcal{C}([a,b],\mathbb{K}) = \{f : [a,b] \to \mathbb{K} \,:\, f \text{ is continuous}\}$$

Possible norms:

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}$$

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$$

# Reverse triangle inequality

**Lemma:** if $X$ is a normed linear space, then

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \quad \text{for all} \quad x, y \in X$$

**Proof:**

$$\|x\| \;=\; \|x - y + y\| \;\leq\; \|x - y\| + \|y\|$$

$$\|x\| - \|y\| \;\leq\; \|x - y\|$$

$$\|y\| - \|x\| \;\leq\; \|y - x\| = \|x - y\| \qquad \text{(swap } x \text{ and } y\text{)}$$

Use that $|a| = \max\{a, -a\}$ for all $a \in \mathbb{R}$

# Convergence of sequences

Let $X$ be a linear space with a norm $\|\cdot\|$

**Definition:** a sequence $(x_n)$ in $X$ converges to $x \in X$ if

$$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty$$

Formally:

$$\forall\, \varepsilon > 0 \quad \exists\, N > 0 \quad \text{such that} \quad n \geq N \quad \Rightarrow \quad \|x_n - x\| \leq \varepsilon$$

**Notation:** $x_n \to x$ in $X$ (make sure w.r.t. which norm!)

# Convergence of sequences

**Lemma:** $x_n \to x$ in $X \quad \Rightarrow \quad \|x_n\| \to \|x\|$ in $\mathbb{R}$

**Proof:** by reverse triangle inequality

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| \to 0 \quad \text{as } n \to \infty$$

**Note:** in this case $\|x_n\|$ is also bounded in $\mathbb{R}$

# Algebraic properties of limits

**Lemma:** $x_n \to x,\ y_n \to y$ in $X \quad \Rightarrow \quad x_n + y_n \to x + y$ in $X$

**Proof:** the triangle inequality gives

$$\begin{aligned}\|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\| \to 0\end{aligned}$$

# Algebraic properties of limits

**Lemma:** $x_n \to x$ in $X$ and $\lambda_n \to \lambda$ in $\mathbb{K}$ $\quad \Rightarrow \quad \lambda_n x_n \to \lambda x$ in $X$

**Proof:** taking $M = \sup \|x_n\|$ gives

$$\begin{aligned} \|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x\| \\ &\leq \|\lambda_n x_n - \lambda x_n\| + \|\lambda x_n - \lambda x\| \\ &= |\lambda_n - \lambda| \|x_n\| + |\lambda| \|x_n - x\| \\ &\leq |\lambda_n - \lambda| M + |\lambda| \|x_n - x\| \to 0 \end{aligned}$$

# Equivalent norms induce the same topology

**Definition:** two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $X$ are called equivalent

if there exist $m, M > 0$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \qquad \forall x \in X$$

**Important:** if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then

$$\|x_n - x\|_1 \to 0 \qquad \Longleftrightarrow \qquad \|x_n - x\|_2 \to 0$$

# Finite-dimensional spaces

**Theorem:** $\dim X < \infty \;\Rightarrow\;$ all norms on $X$ are equivalent

**Proof:** write $X = \operatorname{span}\{e_1, \ldots, e_n\}$ and define the norm

$$\|x\|_+ = \Big( \sum_{i=1}^{n} |\lambda_i|^2 \Big)^{1/2} \quad \text{where} \quad x = \lambda_1 e_1 + \cdots + \lambda_n e_n$$

For any norm $\|\cdot\|$ we have

$$\|x\| \le \sum_{i=1}^{n} |\lambda_i| \, \|e_i\| \le \Big( \sum_{i=1}^{n} |\lambda_i|^2 \Big)^{1/2} \Big( \sum_{i=1}^{n} \|e_i\|^2 \Big)^{1/2} =: M \|x\|_+$$

# Finite-dimensional spaces

**Proof (ctd):** the function

$$f : \mathbb{K}^n \to [0, \infty), \quad \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \|\lambda_1 e_1 + \cdots + \lambda_n e_n\|$$

is continuous since

$$\begin{aligned} \big| f(\lambda) - f(\mu) \big| &= \big| \|x\| - \|y\| \big| \\ &\leq \|x - y\| \\ &\leq M \|x - y\|_+ \\ &= M \Big( \sum_{i=1}^{n} |\lambda_i - \mu_i|^2 \Big)^{1/2} \end{aligned}$$

# Finite-dimensional spaces

**Proof (ctd):** the unit sphere $\mathbb{S}$ is compact in $\mathbb{K}^n$

[In finite-dimensional spaces: closed & bounded $\Rightarrow$ compact!]

Hence, $f$ attains a minimum on $\mathbb{S}$

$$\exists \mu \in \mathbb{S} \quad \text{such that} \quad 0 \leq m := f(\mu) \leq f(\lambda) \quad \forall \lambda \in \mathbb{S}$$

Note: $m > 0$ for if $m = 0$ then

$$f(\mu) = \|\mu_1 e_1 + \cdots + \mu_n e_n\| = 0 \quad \Rightarrow \quad \mu_1 e_1 + \cdots + \mu_n e_n = 0$$

but $|\mu_1|^2 + \cdots + |\mu_n|^2 = 1$. Contradicts that $\{e_1, \ldots, e_n\}$ is a basis!

# Finite-dimensional spaces

**Proof (ctd):**

$$\|x\|_+ = 1 \quad \Rightarrow \quad \|x\| = f(\lambda) \geq m > 0$$

Hence, for all $x \neq 0$ we have

$$\left\| \frac{x}{\|x\|_+} \right\|_+ = 1 \quad \Rightarrow \quad \left\| \frac{x}{\|x\|_+} \right\| \geq m \quad \Rightarrow \quad \|x\| \geq m\|x\|_+$$

# Infinite-dimensional spaces

**Theorem:** $\dim X < \infty \;\Rightarrow\;$ all norms on $X$ are equivalent

**Warning:** this is NOT TRUE in $\infty$-dimensional spaces!

# Infinite-dimensional spaces

**Example:**

$$\mathcal{C}([0,1],\mathbb{K}) = \{\text{all continuous functions } f : [0,1] \to \mathbb{K}\}$$

We have:

$$\|f\|_1 = \int_0^1 |f(x)|dx \leq \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty$$

There is no $m > 0$ such that $\|f\|_\infty \leq m\|f\|_1$ holds for all $f$:

$$f_n(x) = x^n \quad \Rightarrow \quad \|f_n\|_\infty = 1 \quad \text{but} \quad \|f_n\|_1 = \frac{1}{n+1}$$