Functional Analysis

Alef Sterk
a.e.sterk@rug.nl

Lecture 6
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Topics:

- §3.2: Hilbert spaces
- §3.5: Orthonormal bases

Hilbert spaces

Definition: a Hilbert space is a Banach space of which the norm comes from an innerproduct

Examples:

$$\mathbb{K}^n \quad \text{with} \quad \langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i \quad \text{and} \quad \|x\| = \sqrt{\langle x, x\rangle}$$

$$\ell^2 \quad \text{with} \quad \langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \text{and} \quad \|x\| = \sqrt{\langle x, x\rangle}$$

Hilbert spaces

Example: $\mathcal{C}([a,b],\mathbb{K})$ is NOT a Hilbert space with

$$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

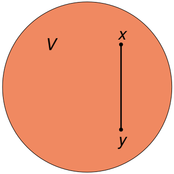
But $L^2(a,b)$ with the above inner product IS a Hilbert space!

Best approximations

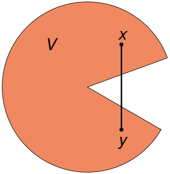
Definition: let X be a linear space

A subset $V \subset X$ is called convex if

$$x, y \in V \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in V \quad \text{for all} \quad \lambda \in [0,1]$$



V is convex



V is not convex

Best approximations

Theorem: assume that

- X is a Hilbert space
- $V \subset X$ is a nonempty, closed, and convex subset

Then $\forall\, x \in X \;\; \exists$ unique $v \in V$ s.t. $\|x - v\| = d(x, V)$

Proof:

$$d := d(x, V) = \inf\{\|x - v\| : \; v \in V\} \; \geq \; 0$$

$$\forall\, n \in \mathbb{N} \quad \exists\, v_n \in V \quad \text{s.t.} \quad d^2 \leq \|x - v_n\|^2 < d^2 + \frac{1}{n}$$

Best approximations

Proof (ctd): recall the parallellogram identity

$$\|y+z\|^2 + \|y-z\|^2 = 2\|y\|^2 + 2\|z\|^2$$

Taking $y = x - v_n$ and $z = x - v_m$ gives

$$\|2x - (v_n + v_m)\|^2 + \|v_n - v_m\|^2 < 4d^2 + \frac{2}{n} + \frac{2}{m}$$

On the other hand

$$\|2x - (v_n + v_m)\|^2 = 4\|x - \underbrace{\tfrac{1}{2}(v_n + v_m)}_{\in V}\|^2 \geq 4d^2$$

Best approximations

Proof (ctd): (v_n) is a Cauchy sequence since

$$\|v_n - v_m\|^2 < \frac{2}{n} + \frac{2}{m}$$

Since X is a Hilbert space there exists $v \in X$ such that $v_n \to v$

Since V is closed we even have $v \in V$

Recall that $d^2 \leq \|x - v_n\|^2 < d^2 + 1/n$ for all $n \in \mathbb{N}$

Taking $n \to \infty$ gives $\|x - v\| = d$

Best approximations

Proof (ctd): if $v, w \in V$ are best approximations, then

$$\|x - v\| = \|x - w\| = d$$

Again, the parallellogram identity gives

$$\|2x - (v + w)\|^2 + \|v - w\|^2 = 2\|x - v\|^2 + 2\|x - w\|^2 = 4d^2$$

On the other hand

$$\|2x - (v + w)\|^2 = 4\|x - \underbrace{\tfrac{1}{2}(v + w)}_{\in V}\|^2 \geq 4d^2$$

Hence $\|v - w\|^2 \leq 0$ so $v = w$

Orthogonal decompositions

Theorem: assume that

- X is a Hilbert space
- $V \subset X$ is a closed linear subspace

Then $\forall x \in X \;\; \exists$ unique $v \in V, w \in V^{\perp}$ such that

$$x = v + w$$

Orthogonal decompositions

Proof: for $x \in X$ there exists a unique $v \in V$ such that

$$\|x - v\| = d(x, V)$$

Define $w = x - v$ so that $x = v + w$

By characterization of best approximation in an IPS:

$$w \in V^{\perp}$$

Orthogonal decompositions

Proof (ctd): orthogonal decompositions are unique since

$$x = v_1 + w_1 = v_2 + w_2 \qquad v_1, v_2 \in V, \quad w_1, w_2 \in V^\perp$$

$$\Rightarrow \quad v_1 - v_2 = w_2 - w_1 \qquad v_1 - v_2 \in V, \quad w_2 - w_1 \in V^\perp$$

$$\Rightarrow \quad v_1 - v_2 = w_2 - w_1 = 0 \qquad \text{since} \qquad V \cap V^\perp = \{0\}$$

$$\Rightarrow \quad v_1 = v_2 \quad \text{and} \quad w_1 = w_2$$

[Exercise: show that $V \cap V^\perp = \{0\}$]

Bessel's inequality

Lemma: assume that

- X is an inner product space
- $\{e_i : i \in \mathbb{N}\}$ is an orthonormal set

Then for all $x \in X$ we have Bessel's inequality:

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

In particular: the series on the left converges

Bessel's inequality

Proof: for any $n \in \mathbb{N}$ we have

$$\begin{aligned} 0 &\leq \left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2 \\ &= \left\langle x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i, x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\rangle \\ &\vdots \\ &= \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \qquad \text{[Exercise: show this]} \end{aligned}$$

Rearrange and let $n \to \infty$

Convergence of series

Theorem: if X is a Hilbert space with ONS $\{e_i : i \in \mathbb{N}\}$ then

$$\sum_{i=1}^{\infty} \lambda_i e_i \quad \text{converges in } X \quad \Leftrightarrow \quad \sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$$

If either statement holds, then

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2$$

Convergence of series

Proof (⇒): observe that

$$\sum_{i=1}^{n} \lambda_i e_i \to x \quad \Rightarrow \quad \langle x, e_k \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i \langle e_i, e_k \rangle = \lambda_k$$

Bessel's inequality gives

$$\sum_{i=1}^{\infty} |\lambda_i|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2 < \infty$$

Convergence of series

Proof ($\Leftarrow$): for $s_n = \lambda_1 e_1 + \cdots + \lambda_n e_n$ we have

$$\|s_n - s_m\|^2 = \left\| \sum_{i=m+1}^{n} \lambda_i e_i \right\|^2 = \sum_{i=m+1}^{n} |\lambda_i|^2 \to 0 \quad \text{as} \quad n, m \to \infty$$

Since X is a Hilbert space (s_n) converges

If either statement is satisfied, then

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 = \lim_{n\to\infty} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2$$

Convergence of series

Corollary: if X is a Hilbert space with ONS $\{e_i : i \in \mathbb{N}\}$ then

$$\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \quad \text{converges} \quad \forall x \in X$$

Proof: for $\lambda_i := \langle x, e_i \rangle$ Bessel's inequality gives

$$\sum_{i=1}^{\infty} |\lambda_i|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2 < \infty$$

Now apply the previous theorem

Orthonormal bases

Definition: let X be a Hilbert space

The ONS $\{e_i \,:\, i \in \mathbb{N}\}$ is called an orthonormal basis for X if

$$\overline{\text{span}}\{e_i : \, i \in \mathbb{N}\} = X$$

Formally: for each $x \in X$, there exists (λ_i) in $\mathbb{K}$ such that

$$\sum_{i=1}^{n} \lambda_i e_i \to x \quad \text{as} \quad n \to \infty$$

[The next theorem implies that in fact $\lambda_i = \langle x, e_i \rangle$ for all $i \in \mathbb{N}$]

Characterization theorem

Theorem: let X be Hilbert with ONS $\{e_i \,:\, i \in \mathbb{N}\}$

The following statements are equivalent:

1. $\{e_i : i \in \mathbb{N}\}^\perp = \{0\}$
2. $\overline{\text{span}}\{e_i : i \in \mathbb{N}\} = X$ [the e_i form an ONB]
3. $\|x\|^2 = \sum_{i=1}^\infty |\langle x, e_i\rangle|^2 \quad \forall x \in X$
4. $x = \sum_{i=1}^\infty \langle x, e_i\rangle e_i \qquad \forall x \in X$

Proof: see lecture notes

Existence theorem

Theorem: if X is an ∞-dimensional Hilbert space, then

$$X \text{ has an orthonormal basis } \Leftrightarrow X \text{ is separable}$$

Proof ($\Leftarrow$): assume $E = \{x_n : n \in \mathbb{N}\}$ dense in X

Construct $F \subset E$ as follows:

- pick smallest n_1 such that $x_{n_1} \neq 0$
- pick smallest $n_2 > n_1$ such that $\{x_{n_1}, x_{n_2}\}$ is lin. indep.
- continue inductively

Apply Gram-Schmidt to F and check that $\overline{\text{span}}(F) = \overline{\text{span}}(E) = X$

Orthonormal bases

Exericse: Let X be Hilbert with ONB $\{e_i \,:\, i \in \mathbb{N}\}$, then

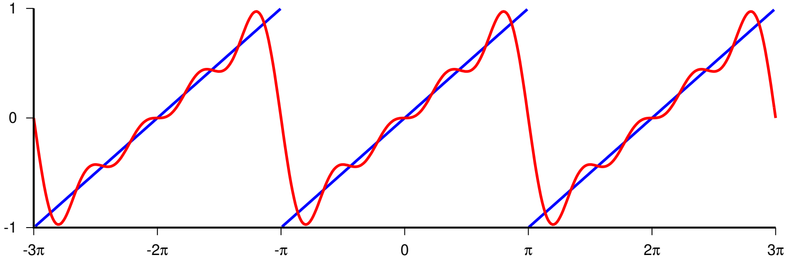
$$T : X \to \ell^2, \quad Tx = (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \langle x, e_3 \rangle, \dots)$$

is an isometric isomorphism

Corollary: all separable, ∞-dimensional Hilbert spaces

are isomorphic with ℓ^2

Motivation



Question: can we write any 2π-periodic function f as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

What do convergence and "=" mean here?

An orthonormal basis for $L^2(-\pi, \pi)$

Consider the functions

$$e_n : [-\pi, \pi] \to \mathbb{C}, \quad e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}$$

In $L^2(-\pi, \pi)$ these functions form an orthonormal set:

$$\langle e_n, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-ikx} \, dx = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

[Exercise: verify this computation]

An orthonormal basis for $L^2(-\pi, \pi)$

Let $f \in L^2(-\pi, \pi)$ and $\varepsilon > 0$ be arbitrary

There exists $g \in \mathcal{C}([-\pi, \pi], \mathbb{C})$ such that

$$\|f - g\|_2 = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2 \, dx \right)^{1/2} < \frac{\varepsilon}{2}$$

We may even assume $g(-\pi) = g(\pi)$

Exercise: argue that the last statement is true

Using the L^2 norm is essential here; it is not true with the sup norm!

An orthonormal basis for $L^2(-\pi, \pi)$

Weierstrass Aproximation Theorem: for any $g \in \mathcal{C}([-\pi, \pi], \mathbb{K})$ with $g(-\pi) = g(\pi)$ and $\varepsilon > 0$ there exists $p \in \operatorname{span}\{e_n \,:\, n \in \mathbb{Z}\}$ such that

$$\|g - p\|_\infty = \sup_{x \in [-\pi, \pi]} |g(x) - p(x)| < \frac{\varepsilon}{2\sqrt{2\pi}}$$

Hence $\operatorname{span}\{e_n \,:\, n \in \mathbb{Z}\}$ is dense in $L^2(-\pi, \pi)$:

$$\begin{aligned} \|f - p\|_2 &\leq \|f - g\|_2 + \|g - p\|_2 \\ &\leq \|f - g\|_2 + \sqrt{2\pi}\|g - p\|_\infty < \varepsilon \end{aligned}$$

Convergence of Fourier series

The best approximation of $f \in L^2(-\pi,\pi)$ in $\operatorname{span}\{e_{-n},\dots,e_n\}$ is

$$s_n(x) = \sum_{k=-n}^{n} c_k \frac{e^{ikx}}{\sqrt{2\pi}} \quad \text{where} \quad c_k = \int_{-\pi}^{\pi} f(x)\frac{e^{-ikx}}{\sqrt{2\pi}}\,dx$$

We have convergence in L^2 and Parseval's equality:

$$\|f - s_n\|_2 = \left(\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2\,dx\right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty$$

$$\|f\|_2^2 = \int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2$$