Functional Analysis

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Lecture 1
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Topics:

- §1.1: Linear spaces
- §1.2: Linear operators
- §1.3: Quotient spaces
- §1.4: Isomorphisms
- §1.5: Dual Spaces

Linear spaces

A linear space X over a field $\mathbb{K}$ is a set with two operations:

- addition: $x,\ y \in X \quad \Rightarrow \quad x + y \in X$
- scalar multiplication: $x \in X,\ \lambda \in \mathbb{K} \quad \Rightarrow \quad \lambda x \in X$
- 8 axioms

[In analysis: $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$]

Linear spaces

A familiar example:

$$\mathbb{K}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{K}\}$$

Infinite-dimensional examples:

$$\mathbb{K}^\infty = \{(x_1, x_2, x_3, \dots) : x_i \in \mathbb{K}\}$$

$$\mathcal{F}(S, \mathbb{K}) = \{f : S \to \mathbb{K}\}$$ [where S is an infinite set]

[The last two spaces are "too large" for analysis purposes]

Linear spaces

Important examples:

$$\ell^p = \left\{ (x_1, x_2, x_3, \dots) \,:\, x_i \in \mathbb{K}, \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \qquad (p \geq 1)$$

$$\ell^\infty = \left\{ (x_1, x_2, x_3, \dots) \,:\, x_i \in \mathbb{K}, \quad \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\mathcal{C}([a, b], \mathbb{K}) = \left\{ f : [a, b] \to \mathbb{K} \,:\, f \text{ is continuous} \right\}$$

Linear operators

Let X, Y be linear spaces over $\mathbb{K}$

Definition: a map $T : X \to Y$ is called linear if

- $\operatorname{dom} T$ is a linear subspace of X
- $T(x+y) = Tx + Ty, \quad x, y \in X$
- $T(\lambda x) = \lambda(Tx), \quad \lambda \in \mathbb{K}, \quad x \in X$

Notation:

$$L(X,Y) = \{T : X \to Y \ : \ T \text{ is linear and } \operatorname{dom} T = X\}$$

[If $X = Y$, then write $L(X, Y)$ as $L(X)$]

Linear operators

Definition: $P \in L(X)$ is called a projection if $P^2 = P$

Example:

$$P : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x_1, x_2) \mapsto (0, x_2)$$

Linear operators

Lemma: if $P \in L(X)$ is a projection, then

1. $I - P$ is a projection
2. $\operatorname{ran} P = \ker(I - P)$
3. $\ker P = \operatorname{ran}(I - P)$
4. $X = \ker P + \operatorname{ran} P$ is a direct sum [i.e. $\operatorname{ran} P \cap \ker P = \{0\}$]

Linear operators

Let $P \in L(X)$ be a projection

Claim: $I - P$ is a projection

Proof:

$$\begin{aligned}(I-P)^2 &= (I-P)(I-P) \\ &= I - P - P + P^2 \\ &= I - 2P + P \\ &= I - P\end{aligned}$$

Linear operators

Let $P \in L(X)$ be a projection

Claim: $\operatorname{ran} P = \ker(I - P)$ and $\ker P = \operatorname{ran}(I - P)$

Proof:

$$
\begin{aligned}
x \in \operatorname{ran} P \quad &\Leftrightarrow \quad x = Py \text{ for some } y \in X \\
&\Leftrightarrow \quad Px = P^2 y = Py = x \\
&\Leftrightarrow \quad (I - P)x = 0
\end{aligned}
$$

Linear operators

Let $P \in L(X)$ be a projection

Claim: $X = \ker P + \operatorname{ran} P$, direct sum

Proof:

"$\supset$": trivial

"$\subset$": $x = (I - P)x + Px$ for all $x \in X$

If $x \in \operatorname{ran} P \cap \ker P$, then

$$\left.\begin{array}{rcl} x & = & Py \\ Px & = & 0 \end{array}\right\} \;\Rightarrow\; x = Py = P^2 y = Px = 0$$

Equivalence relations

Definition: $\sim$ is an equivalence relation on a set X if

1. $x \sim x$ for all $x \in X$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y$ and $y \sim z \Rightarrow x \sim z$

Equivalence class of x: $[x] := \{y \in X \ : \ x \sim y\}$

Set of equivalence classes: $X/\sim := \{[x] \ : \ x \in X\}$

Natural map: $\pi : X \to X/\sim, \quad x \mapsto [x]$

Equivalence relations

Example: $X = \{$ books with a single author $\}$

$$x \sim y \quad \Leftrightarrow \quad x \text{ and } y \text{ have the same author}$$

is an equivalence relation on X

"The Hobbit" $\sim$ "The Lord of the Rings"

"The Hobbit" $\nsim$ "Harry Potter"

[Does this example still work with books that have *multiple* authors?]

Equivalence relations

Example: on $X = \mathbb{Z}$ define the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad x - y \text{ is even}$$

Equivalence classes:

$$[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

$$[1] = \{\dots, -5, -3, -1, 1, 3, \dots\}$$

$$\mathbb{Z}/\sim \, = \{[0], [1]\}$$

Quotient spaces

Lemma: let X be a linear space and $V \subset X$ a linear subspace

$$x \sim y \quad \Leftrightarrow \quad x - y \in V$$

is an equivalence relation on X

Proof: since V is a linear subspace we have

1. $x - x = 0 \in V$
2. $x - y \in V \Rightarrow y - x = (-1) \cdot (x - y) \in V$
3. $x - y \in V$ and $y - z \in V \Rightarrow x - z = (x - y) + (y - z) \in V$

Quotient spaces

Recall: $x \sim y \quad \Leftrightarrow \quad x - y \in V$

Equivalence classes:

$$\begin{aligned}
[x] &= \{y \in X : x \sim y\} \\
&= \{y \in X : x - y \in V\} \\
&= \{x - v \in X : v \in V\} \\
&= \{x + v \in X : v \in V\} \\
&= x + V
\end{aligned}$$

Note: $[x + v] = [x]$ for all $v \in V$

Quotient spaces

Recall: $x \sim y \quad \Leftrightarrow \quad x - y \in V$

Proposition: $X/V := X/\sim$ becomes a linear space with

$$(x + V) + (y + V) := (x + y) + V$$

$$\lambda(x + V) := (\lambda x) + V$$

where $x, y \in X$ and $\lambda \in \mathbb{K}$

Note: need to show that these operations are well defined!

Isomorphisms

Theorem: if X, Y are linear spaces, $T \in L(X, Y)$, and $V \subset \ker T$ a linear subspace, then

$$\widehat{T} : X/V \to Y, \quad [x] = x + V \mapsto T(x)$$

is well defined and linear

Proof: $\widehat{T}$ is well defined since

$$\begin{aligned} [x] = [y] \quad &\Rightarrow \quad x - y \in V \subset \ker T \\ &\Rightarrow \quad T(x - y) = 0 \\ &\Rightarrow \quad T(x) = T(y) \end{aligned}$$

Isomorphisms

Corollary: let X, Y be linear spaces and $T \in L(X, Y)$, then

$$\widehat{T} : X/\ker T \to \operatorname{ran} T, \quad x + \ker T \mapsto T(x)$$

is an isomorphism, so $X/\ker T$ and $\operatorname{ran} T$ are isomorphic

Proof: $\widehat{T}$ is injective since

$$\begin{aligned} \widehat{T}(x + \ker T) = 0 \quad &\Rightarrow \quad T(x) = 0 \\ &\Rightarrow \quad x \in \ker T \\ &\Rightarrow \quad x + \ker T = 0 + \ker T \end{aligned}$$

Surjectivity of $\widehat{T}$ is trivial

Isomorphisms

Theorem: if X is a linear space and $V \subset X$ a linear subspace, then

$$\dim X < \infty \quad \Rightarrow \quad \dim X/V = \dim X - \dim V$$

Proof: extend a basis of V to a basis of X:

$$\begin{aligned} V &= \operatorname{span}\{e_1, \ldots, e_k\} \\ X &= \operatorname{span}\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\} \end{aligned}$$

Define the linear map $P : X \to X$ by

$$P(c_1 e_1 + \cdots + c_n e_n) = c_{k+1} e_{k+1} + \cdots + c_n e_n$$

Since X/V and $\operatorname{ran} P$ are isomorphic we have

$$\dim(X/V) = \dim(\operatorname{ran} P) = n - k = \dim X - \dim V$$

Isomorphisms

Theorem: if X is a linear space and $V \subset X$ a linear subspace, then

$$\dim X < \infty \quad \Rightarrow \quad \dim X/V = \dim X - \dim V$$

Corollary: if $\dim X < \infty$ and $T \in L(X, Y)$, then

$$\dim X = \dim(\ker T) + \dim(\operatorname{ran} T)$$

Dual spaces

Definition: let X be a linear space over $\mathbb{K}$, then

$$X' = L(X, \mathbb{K}) = \{f : X \to \mathbb{K} \ : \ f \text{ is linear } \}$$

is called the dual space of X

Elements in X' are called functionals on X

Dual spaces

Lemma: $\dim X = n < \infty \;\Rightarrow\; \dim X' = n$

Proof: let $X = \operatorname{span}\{e_1, \ldots, e_n\}$, and define

$$f_i : X \to \mathbb{K}, \quad c_1 e_1 + \cdots + c_n e_n \mapsto c_i$$

Claim: $X' = \operatorname{span}\{f_1, \ldots, f_n\}$

- $\sum_{j=1}^n a_j f_j(x) = 0 \;\; \forall x \in X \quad \Rightarrow \quad a_i = \sum_{j=1}^n a_j f_j(e_i) = 0 \;\; \forall i$
- $f \in X' \quad \Rightarrow \quad f(x) = \sum_{j=1}^n f(e_j) f_j(x)$

  [true for $x = e_i$ and hence for all x by linearity]

Dual spaces

Definition: let X be a linear space over $\mathbb{K}$, then

$$X'' = L(X', \mathbb{K}) = \{\varphi : X' \to \mathbb{K} \, : \, \varphi \text{ is linear } \}$$

is called the second dual space of X

We define the natural map as

$$J : X \to X'', \quad J(x)(f) = f(x), \quad x \in X, \quad f \in X'$$

Corollary: $\dim X < \infty \;\Rightarrow\; J : X \to X''$ is bijective

[If $\dim X = \infty$, then J need not be surjective!]

Dual spaces

Proof (J injective): let $X = \operatorname{span}\{e_1, \ldots, e_n\}$ and

$$f_i : X \to \mathbb{K}, \quad x = c_1 e_1 + \cdots + c_n e_n \mapsto c_i$$

Then $f_i \in X'$ and

$$\begin{aligned}
J(x) = 0 \quad &\Rightarrow \quad J(x)(f) = 0 && \forall f \in X' \\
&\Rightarrow \quad J(x)(f_i) = 0 && \forall i = 1, \ldots, n \\
&\Rightarrow \quad f_i(x) = 0 && \forall i = 1, \ldots, n \\
&\Rightarrow \quad c_i = 0 && \forall i = 1, \ldots, n \\
&\Rightarrow \quad x = 0
\end{aligned}$$

Dual spaces

Proof (J surjective): recall that

$$\dim X = \dim(\ker J) + \dim(\operatorname{ran} J) = \dim(\operatorname{ran} J)$$

The previous lemma implies

$$\dim X'' = \dim X' = \dim X = \dim(\operatorname{ran} J)$$

Functional analysis > linear algebra!

Key word: topology

Using metrics induced by norms or inner products we can study:

- sequences, limits
- open, closed, compact sets
- continuity
- completeness