Functional Analysis

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Lecture 14
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Topics:

- §1.6: Hahn-Banach for linear spaces
- §7.1: Hahn-Banach for normed spaces

Are dual spaces always nontrivial?

Definition: if X is a NLS, its dual space is defined as

$$X' = B(X, \mathbb{K}) = \{f : X \to \mathbb{K} \,:\, f \text{ is linear and bounded } \}$$

Examples: the following maps belong to the dual of $\mathcal{C}([a,b], \mathbb{K})$:

$$f(\varphi) = \int_a^b \varphi(t)\, dt \quad \text{and} \quad g(\varphi) = \varphi(a)$$

Question: can we have $X' = \{0\}$ for some NLS?

[Spoiler alert: no!]

Can we find functionals with specific properties?

Aim: find $f \in X'$ that satisfies a property "P"

Approach:

- choose a suitable linear subspace $V \subset X$
- construct $f \in V'$ that satisfies "P"
- extend $f : V \to \mathbb{K}$ to all of X

Question: can the extension $f : X \to \mathbb{K}$ chosen to be bounded?

[Spoiler alert: yes!]

Partial orders

Assume X is a nonempty set

Definition: $\preceq$ is called a partial order on X if:

1. $x \preceq x$ for all $x \in X$
2. $x \preceq y$ and $y \preceq x \;\Rightarrow\; x = y$
3. $x \preceq y$ and $y \preceq z \;\Rightarrow\; x \preceq z$

$\preceq$ is called a total order if for all $x, y \in X$ we have

$$x \preceq y \quad \text{or} \quad y \preceq x$$

Partial orders

Example: we can define a partial order on $\mathbb{N}$ by

$$x \preceq y \quad \Leftrightarrow \quad x \leq y$$

This is also a total order

Similar for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$

Partial orders

Example: let $X = \{A : A \subset \mathbb{N}\}$

We can define a partial order on X by

$$A \preceq B \quad \Leftrightarrow \quad A \subset B$$

This is NOT a total order:

$$A = \{1, 2\} \quad B = \{2, 3\} \quad \Rightarrow \quad \text{neither } A \preceq B \quad \text{nor } B \preceq A$$

Partial orders

Example: we can define a partial order on $\mathcal{F}([0,1],\mathbb{R})$ by

$$f \preceq g \quad \Leftrightarrow \quad f(x) \leq g(x) \qquad \forall\, x \in [0,1]$$

This is NOT a total order:

$$\left.\begin{aligned} f(x) &= x \\ g(x) &= 1 - x \end{aligned}\right\} \quad \Rightarrow \quad \text{neither } f \preceq g \quad \text{nor } g \preceq f$$

Zorn's lemma

Definition: if $\preceq$ is a partial order on X and $V \subset X$, then $y \in X$ is called:

- upper bound for V if: $\quad x \preceq y \quad \forall x \in V$
- maximal element of X if: $\quad y \preceq x \;\Rightarrow\; y = x$

Lemma: let $X \neq \varnothing$ be partially ordered

If every totally ordered subset of X has an upper bound in X, then X has a maximal element

Linear spaces

Hahn-Banach Theorem: assume that

- X is a linear space
- $V \subset X$ is a proper linear subspace
- $p : X \to [0, \infty)$ is a semi-norm
- $f \in L(V, \mathbb{K})$ satisfies the bound

$$|f(x)| \leq p(x) \qquad \forall\, x \in V$$

Then there exists $F \in L(X, \mathbb{K})$ such that

$$F \restriction V = f \qquad \text{and} \qquad |F(x)| \leq p(x) \qquad \forall\, x \in X$$

[Note: F is not necessarily unique!]

Linear spaces

Proof: assume $\mathbb{K} = \mathbb{R}$

Pick $x_0 \in X \setminus V$, $y_0 \in \mathbb{R}$ and define

$$F_0(x + \lambda x_0) = f(x) + \lambda y_0, \qquad x \in V, \qquad \lambda \in \mathbb{R}$$

Hence F_0 linear and $F_0 \restriction V = f$

The choice $y_0 = \inf\{\, p(x + x_0) - f(x) : \; x \in V\}$ gives

$$|F_0(x + \lambda x_0)| \leq p(x + \lambda x_0), \qquad \forall\, x \in V, \qquad \forall\, \lambda \in \mathbb{R}$$

[See lecture notes]

Linear spaces

Proof (ctd): consider the following partially ordered set:

$$P = \left\{ g \in L(\operatorname{dom} g, \mathbb{R}) \quad : \quad \begin{array}{l} V \subset \operatorname{dom} g \text{ properly} \\ g \restriction V = f \\ |g(x)| \leq p(x) \quad \forall\, x \in \operatorname{dom} g \end{array} \right\}$$

$$P \neq \varnothing \quad \text{as} \quad F_0 \in P$$

$$g \preceq h \quad \Leftrightarrow \quad \operatorname{dom} g \subset \operatorname{dom} h \quad \text{and} \quad h \restriction \operatorname{dom} g = g$$

Linear spaces

Proof (ctd): assume $Q \subset P$ is totally ordered

Define the map

$$h : \bigcup_{g \in Q} \operatorname{dom} g \to \mathbb{R}, \qquad h(x) := g(x) \text{ if } x \in \operatorname{dom} g, \quad g \in Q$$

$$\Rightarrow \quad |h(x)| = |g(x)| \leq p(x)$$

Linear spaces

Proof (ctd): h is well-defined

Assume: $x \in \operatorname{dom} g_1$ and $x \in \operatorname{dom} g_2$ $(g_1, g_2 \in Q)$

Q totally ordered $\Rightarrow$ w.l.o.g. $g_1 \preceq g_2$

$\Rightarrow$ $\operatorname{dom} g_1 \subset \operatorname{dom} g_2$ and $g_2 \restriction \operatorname{dom} g_1 = g_1$

$\Rightarrow$ $g_1(x) = g_2(x) = h(x)$

Linear spaces

Proof (ctd): $h \in L(\operatorname{dom} h, \mathbb{R})$ since

$$
\begin{aligned}
x_1, x_2 \in \operatorname{dom} h \quad &\Rightarrow \quad \exists\, g_1, g_2 \in Q \text{ such that } \begin{cases} x_1 \in \operatorname{dom} g_1 \\ x_2 \in \operatorname{dom} g_2 \end{cases} \\
Q \text{ totally ordered} \quad &\Rightarrow \quad \text{w.l.o.g. } g_1 \preceq g_2 \\
&\Rightarrow \quad \operatorname{dom} g_1 \subset \operatorname{dom} g_2 \\
&\Rightarrow \quad x_1, x_2 \in \operatorname{dom} g_2 \\
&\Rightarrow \quad \lambda x_1 + \mu x_2 \in \operatorname{dom} g_2 \subset \operatorname{dom} h \\
&\Rightarrow \quad h(\lambda x_1 + \mu x_2) = g_2(\lambda x_1 + \mu x_2) \\
&\qquad\qquad\qquad\qquad\; = \lambda g_2(x_1) + \mu g_2(x_2) \\
&\qquad\qquad\qquad\qquad\; = \lambda h(x_1) + \mu h(x_2)
\end{aligned}
$$

Linear spaces

Proof (ctd): h is an upper bound for Q

$$\begin{aligned} g \in Q \quad &\Rightarrow \quad \operatorname{dom} g \subset \operatorname{dom} h \text{ and} \\ &\qquad g(x) = h(x) \quad \forall\, x \in \operatorname{dom} g \text{ by definition} \\ &\Rightarrow \quad \operatorname{dom} g \subset \operatorname{dom} h \quad \text{and} \quad h\restriction \operatorname{dom} g = g \\ &\Rightarrow \quad g \preceq h \end{aligned}$$

Linear spaces

Proof (ctd): assume $Q \subset P$ totally ordered and define

$$h : \bigcup_{g \in Q} \operatorname{dom} g \to \mathbb{R}, \qquad h(x) := g(x) \text{ if } x \in \operatorname{dom} g$$

Conclusion:

- $h \in L(\operatorname{dom} h, \mathbb{R})$ is well-defined
- $h \in P$ is an upper bound of Q

Zorn's lemma $\Rightarrow$ P has a maximal element F

Linear spaces

Proof (ctd): assume $\operatorname{dom} F \neq X$

Pick $x_0 \in X \setminus \operatorname{dom} F$

Make 1-dimensional extension $\widetilde{F}$ of F to $\operatorname{span}\{\operatorname{dom} F, x_0\}$

$\widetilde{F} \in P$ and $F \preceq \widetilde{F}$ but $F \neq \widetilde{F} \quad \Rightarrow \quad F$ not maximal!

Contradiction, so $\operatorname{dom} F = X$

Linear spaces

Proof (ctd): assume $\mathbb{K} = \mathbb{C}$, then

$$f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix)$$

$$|\operatorname{Re} f(x)| \leq |f(x)| \leq p(x) \qquad \forall\, x \in V$$

There exists an $\mathbb{R}$-linear map $\Phi : X \to \mathbb{R}$ such that

$$\Phi\restriction V = \operatorname{Re} f \qquad \text{and} \qquad |\Phi(x)| \leq p(x) \qquad \forall\, x \in X$$

Define the $\mathbb{C}$-linear map $F : X \to \mathbb{C}$ by

$$F(x) = \Phi(x) - i\Phi(ix)$$

Linear spaces

Proof (ctd): $F(x) = \Phi(x) - i\Phi(ix)$

Verification that $F : X \to \mathbb{C}$ is $\mathbb{C}$-linear:

$$F(x+y) = F(x) + F(y) \qquad \text{[exercise]}$$

$$\begin{aligned} F((a+bi)x) &= \Phi(ax + ibx) - i\Phi(-bx + iax) \\ &= \Phi(ax) + \Phi(ibx) - i\Phi(-bx) - i\Phi(iax) \\ &= a\Phi(x) + b\Phi(ix) + bi\Phi(x) - ai\Phi(ix) \\ &= (a+bi)(\Phi(x) - i\Phi(ix)) \\ &= (a+bi)F(x) \end{aligned}$$

Linear spaces

Proof (ctd): for some $\theta \in \mathbb{R}$ we have

$$F(x) = |F(x)|e^{i\theta}$$

Verification that F is bounded by p:

$$\begin{aligned}
|F(x)| &= F(e^{-i\theta}x) \\
&= \operatorname{Re} F(e^{-i\theta}x) \\
&= \Phi(e^{-i\theta}x) \\
&\leq p(e^{-i\theta}x) \\
&\leq |e^{-i\theta}|p(x) \;=\; p(x) \qquad \forall\, x \in X
\end{aligned}$$

Normed linear spaces

Hahn-Banach Theorem: if X is a NLS and $V \subset X$ is a linear subspace, then for all $f \in V'$ there exists $F \in X'$ such that

$$F \restriction V = f \qquad \text{and} \qquad \|F\| = \|f\|$$

[This version is used in most examples and exercises]

Note: the norms are meant as follows

$$\|F\| = \sup_{x \in X,\, x \neq 0} \frac{|F(x)|}{\|x\|} \quad \text{and} \quad \|f\| = \sup_{x \in V,\, x \neq 0} \frac{|f(x)|}{\|x\|}$$

Normed linear spaces

Proof: define $p : X \to [0, \infty)$ by $p(x) = \|f\| \, \|x\|$ then

$$|f(x)| \quad \leq \quad p(x) \qquad \forall \, x \in V$$

By Hahn-Banach there exists $F \in L(X, \mathbb{K})$ such that

$$F \restriction V = f \quad \text{and} \quad |F(x)| \leq p(x) = \|f\| \, \|x\| \quad \forall \, x \in X$$

Hence $\|F\| \leq \|f\|$ and $\|f\| \leq \|F\|$ is trivial

Linear functionals separate points

Proposition: if X is a NLS and $x, y \in X$ are distinct, then there exists $f \in X'$ such that

$$\|f\| = 1 \quad \text{and} \quad f(x) \neq f(y)$$

Proof: let $V = \operatorname{span}\{z\}$, where $z = x - y \neq 0$, and consider

$$f : V \to \mathbb{K}, \quad f(\lambda z) = \lambda \|z\|$$

We have

$$\|f\| = \sup_{v \in V, v \neq 0} \frac{|f(v)|}{\|v\|} = \sup_{\lambda \neq 0} \frac{|f(\lambda z)|}{\|\lambda z\|} = \sup_{\lambda \neq 0} \frac{|\lambda| \, \|z\|}{|\lambda| \, \|z\|} = 1$$

Apply Hahn-Banach

Existence of specific functionals

Proposition: if X is a NLS and $x_0 \in X$ is nonzero, then there exists $f \in X'$ such that

$$f(x_0) = \|x_0\| \quad \text{and} \quad \|f\| = 1$$

Proof: define $f : \mathrm{span}\{x_0\} \to \mathbb{K}$ by $f(\lambda x_0) = \lambda \|x_0\|$, then

$$\|f\| = \sup_{\lambda x_0,\, \lambda \neq 0} \frac{|f(\lambda x_0)|}{\|\lambda x_0\|} = 1$$

Apply Hahn-Banach

The norm in terms of functionals

Corollary: the norm of any nonzero $x_0 \in X$ can be written as

$$\|x_0\| = \sup\{|f(x_0)| : f \in X', \|f\| = 1\}$$

Proof: for any $f \in X'$ we have $|f(x_0)| \leq \|f\|\|x_0\|$, and thus

$$\sup\{|f(x_0)| : f \in X', \|f\| = 1\} \leq \|x_0\|$$

Equality follows from the existence of $f \in X'$ with

$$f(x_0) = \|x_0\| \quad \text{and} \quad \|f\| = 1$$

Existence of bounded projections

Proposition: if X is a NLS and $V \subset X$ is a finite-dimensional linear subspace, then there exists $P \in B(X)$ such that

$$P^2 = P, \quad \operatorname{ran} P = V$$

Proof: let $V = \operatorname{span}\{e_1, \ldots, e_n\}$

By Hahn-Banach there exist $f_i \in X'$ such that $f_i(e_j) = \delta_{ij}$

Define $P : X \to X$ by $Px = \sum_{i=1}^{n} f_i(x) e_i$

Then $\operatorname{ran} P = V$ since $P(e_j) = e_j$ for all $j = 1, \ldots, n$

Existence of bounded projections

Proof (ctd): for all $x \in X$ we have

$$
\begin{aligned}
Px &= \sum_{i=1}^{n} f_i(x) e_i \\
P^2 x &= \sum_{i=1}^{n} f_i(x) P e_i = \sum_{i=1}^{n} f_i(x) e_i = Px \\
\|Px\| &\leq \sum_{i=1}^{n} |f_i(x)| \|e_i\| \leq \left(\sum_{i=1}^{n} \|f_i\| \|e_i\| \right) \|x\|
\end{aligned}
$$