Functional Analysis

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Lecture 7
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Topics:

- §4.1: Bounded linear operators
- §4.2: Spaces of bounded linear operators
- §4.3: Invertible operators

Bounded linear operators

Definition: let X, Y be a NLS and $T \in L(X, Y)$

T is called bounded if there exists $c > 0$ such that

$$\|Tx\| \leq c\|x\| \qquad \forall x \in X$$

Warning: it does NOT mean that $\|Tx\| \leq c$ for all $x \in X$!

Connection with continuity

Note: if $T \in L(X, Y)$ is bounded, then

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq c\|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in X$$

In particular, T is uniformly continuous on X

Connection with continuity

Lemma: if $T \in L(X, Y)$ is continuous at 0, then T is bounded

Proof: for $\varepsilon = 1$ there exists $\delta > 0$ such that

$$\|z - 0\| < \delta \quad \Rightarrow \quad \|Tz - T0\| < 1$$

If $x \neq 0$, then $z = \frac{1}{2}\delta x/\|x\|$ gives $\|z\| = \frac{1}{2}\delta < \delta$ so

$$\|Tz\| < 1 \quad \text{and thus} \quad \|Tx\| < \frac{2}{\delta}\|x\|$$

Replacing $<$ by $\leq$ makes the inequality valid for all $x \in X$

An integration operator

Example: consider $X = \mathcal{C}([a,b],\mathbb{K})$ with the sup-norm and

$$T : X \to X, \qquad Tf(x) = \int_a^x f(t)\,dt$$

For all $x \in [a,b]$ we have

$$|Tf(x)| = \left| \int_a^x f(t)\,dt \right| \le \int_a^x |f(t)|\,dt \le (b-a)\|f\|_\infty$$

It follows that T is bounded:

$$\|Tf\|_\infty = \sup_{x\in[a,b]} |Tf(x)| \le (b-a)\|f\|_\infty$$

A differentiation operator

Example: consider the space of polynomials

$$\begin{aligned} \mathcal{P} &= \left\{ \sum_{k=0}^{\infty} a_k x^k \ : \text{only finitely many } a_k \neq 0 \right\} \\ \|p\| &= \max_{k \geq 0} |a_k| \end{aligned}$$

The operator $T : \mathcal{P} \to \mathcal{P}$ given by $Tp = p'$ is NOT bounded:

$$p_n(x) = x^n \quad \Rightarrow \quad \|p_n\| = 1 \quad \text{but} \quad \|Tp_n\| = n$$

There is no constant $c > 0$ such that $\|Tp\| \leq c\|p\|$ for all $p \in \mathcal{P}$

The operator norm

Definition: let X, Y be a NLS and let $T \in L(X, Y)$

If T is bounded, then we define its operator norm by

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Note: we can also write

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \leq 1} \|Tx\|$$

[See lecture notes for proof]

Computing operator norms

Example: consider $X = \mathcal{C}([a, b], \mathbb{K})$ with the sup-norm and

$$T : X \to X, \qquad Tf(x) = \int_a^x f(t)\, dt$$

Claim: $\|T\| = b - a$

Key steps:

1. $\dfrac{\|Tf\|_\infty}{\|f\|_\infty} \leq b - a$ for all $f \neq 0$ [shown on slide 5]

2. equality holds for some $f \in X$ [see next slide]

Computing operator norms

Example (ctd): take f with $f(x) = 1$ for all $x \in [a, b]$

We have $\|f\|_\infty = 1$ and

$$Tf(x) = \int_a^x f(t)\,dt = x - a$$

This implies that $\|Tf\|_\infty = b - a$ and thus

$$\frac{\|Tf\|_\infty}{\|f\|_\infty} = b - a$$

Computing operator norms

Example: let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots) \in \ell^\infty$ and

$$T : \ell^1 \to \ell^1, \quad (x_1, x_2, x_3, \dots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

Claim: $\|T\| = \|\lambda\|_\infty$

Key steps:

1. $\dfrac{\|Tx\|_1}{\|x\|_1} \leq \|\lambda\|_\infty$ for all $x \neq 0$

2. $\forall \varepsilon > 0 \quad \exists x \in \ell^1 \quad$ such that $\quad \|\lambda\|_\infty - \varepsilon < \dfrac{\|Tx\|_1}{\|x\|_1}$

Computing operator norms

Example (ctd): for all $x \in \ell^1$ we have

$$
\begin{aligned}
\|Tx\|_1 &= \sum_{n=1}^{\infty} |\lambda_n x_n| \\
&= \sum_{n=1}^{\infty} |\lambda_n|\,|x_n| \\
&\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sum_{n=1}^{\infty} |x_n| \\
&= \|\lambda\|_\infty \, \|x\|_1
\end{aligned}
$$

Computing operator norms

Example (ctd): since $\|\lambda\|_\infty = \sup_{n\in\mathbb{N}} |\lambda_n|$ we have

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \text{such that} \quad \|\lambda\|_\infty - \varepsilon < |\lambda_N|$$

Take $x = (0, 0, \dots, 0, 1, 0, 0, \dots) \in \ell^1$, with 1 in the N-th component

Then $\dfrac{\|Tx\|_1}{\|x\|_1} = |\lambda_N| > \|\lambda\|_\infty - \varepsilon$

The space of bounded linear operators

Lemma: the linear space

$$B(X, Y) = \{T \in L(X, Y) \,:\, T \text{ bounded}\}$$

becomes a NLS with the operator norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Proof: see lecture notes

Product properties

Lemma: if X and Y are a NLS and $T \in B(X, Y)$, then

$$\|Tx\| \leq \|T\| \, \|x\| \quad \text{for all} \quad x \in X$$

Proof: obvious for $x = 0$

For all nonzero $x \in X$ we have

$$\frac{\|Tx\|}{\|x\|} \leq \|T\|$$

Product properties

Lemma: if X, Y, and Z are a NLS, then

$$\left.\begin{array}{l} T \in B(X,Y) \\[2em] S \in B(Y,Z) \end{array}\right\} \quad \Rightarrow \quad ST \in B(X,Z) \quad \text{and} \quad \|ST\| \leq \|S\|\,\|T\|$$

Proof: for all $x \in X$ we have

$$\|(ST)x\| = \|S(Tx)\| \leq \|S\|\,\|Tx\| \leq \|S\|\,\|T\|\,\|x\|$$

This gives

$$\|ST\| = \sup_{x \neq 0} \frac{\|(ST)x\|}{\|x\|} \leq \|S\|\,\|T\|$$

Product properties

Lemma: if $T_n \in B(X, Y)$ and $S_n \in B(Y, Z)$ for all $n \in \mathbb{N}$, then

$$T_n \to T \quad \text{and} \quad S_n \to S \quad \Rightarrow \quad S_n T_n \to ST$$

Proof: with $M = \sup_{n \in \mathbb{N}} \|T_n\|$ we have

$$\begin{aligned} \|S_n T_n - ST\| &\leq \|S_n T_n - ST_n\| + \|ST_n - ST\| \\ &\leq \|S_n - S\| \, \|T_n\| + \|S\| \, \|T_n - T\| \\ &\leq M\|S_n - S\| + \|S\| \, \|T_n - T\| \to 0 \end{aligned}$$

Completeness

Theorem: X NLS and Y Banach $\Rightarrow B(X,Y)$ Banach

Proof: if (T_n) is Cauchy in $B(X,Y)$ then for all $x \in X$

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\| \to 0 \qquad m, n \to \infty$$

Since $(T_n x)$ is Cauchy in Y we can define $T : X \to Y$ by

$$Tx := \lim_{n \to \infty} T_n x \qquad \text{(limit in } Y\text{)}$$

To show: $T \in B(X,Y)$ and $\|T_n - T\| \to 0$

Completeness

Proof (ctd): T is linear since for all $x, z \in X$ and $\lambda \in \mathbb{K}$ we have

$$T(x+z) = \lim_{n\to\infty} T_n(x+z) = \lim_{n\to\infty} (T_n x + T_n z) = Tx + Tz$$

$$T(\lambda x) = \lambda Tx$$ [via similar argument]

T is bounded since $\|T_n\| \le M := \sup_{n\in\mathbb{N}} \|T_n\|$ for all $n \in \mathbb{N}$ and thus

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \lim_{n\to\infty} \|T_n\| \, \|x\| \le M\|x\|$$

Completeness

Proof (ctd): for all $\varepsilon > 0$ there exists $N > 0$ such that

$$m, n \geq N \quad \Rightarrow \quad \|T_n - T_m\| \leq \varepsilon$$

$$\Rightarrow \quad \|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\| \leq \varepsilon \quad \forall \|x\| \leq 1$$

$$n \geq N \quad \Rightarrow \quad \|T_n x - Tx\| \leq \varepsilon \qquad \forall \|x\| \leq 1 \qquad [\text{take } m \to \infty]$$

$$\Rightarrow \quad \|T_n - T\| = \sup_{\|x\| \leq 1} \|T_n x - Tx\| \leq \varepsilon$$

Conclusion: $T_n \to T$ in $B(X, Y)$

Invertible operators

Definition: $T \in B(X, Y)$ is called invertible if

1. $T : X \to Y$ is a bijection
2. $T^{-1} \in B(Y, X)$

Warning: (1) does NOT imply (2)!

A counter example

Exercise: consider the sup-norm on

$$s = \left\{ (x_1, x_2, x_3, \dots) \ : \ \text{only finitely many } x_i \in \mathbb{K} \text{ nonzero} \right\}$$

Define $T : s \to s$ by

$$(x_1, x_2, \dots, x_n, \dots) \mapsto (x_1, \tfrac{1}{2} x_2, \dots, \tfrac{1}{n} x_n, \dots)$$

Show that T is bounded and bijective but T^{-1} is NOT bounded!

Characterization

Lemma:

$T \in B(X, Y)$ invertible $\Leftrightarrow \exists\, S \in B(Y, X)$ such that

$$ST = I_X \quad \text{and} \quad TS = I_Y$$

Characterization

Proof (⇒): T is bijective and $S = T^{-1} \in B(Y, X)$ satisfies

$$ST = I_X \quad \text{and} \quad TS = I_Y$$

Proof (⇐):

T injective: $\qquad Tx = 0 \quad \Rightarrow \quad x = STx = 0$

T surjective: $\qquad y \in Y \quad \Rightarrow \quad y = TSy \in \operatorname{ran} T$

$T^{-1} = S \in B(Y, X)$

Inversion by geometric series

Theorem: if X is Banach and $T \in B(X)$, then

$$\sum_{k=0}^{\infty} \|T^k\| < \infty \quad \Rightarrow \quad (I - T)^{-1} = \sum_{k=0}^{\infty} T^k \in B(X)$$

Proof: $B(X) := B(X, X)$ is Banach since X itself is Banach, so

$$\sum_{k=0}^{\infty} \|T^k\| < \infty \quad \Rightarrow \quad \sum_{k=0}^{\infty} T^k \quad \text{converges and} \quad \|T^k\| \to 0$$

Inversion by geometric series

Proof (ctd):

$$S_n = \sum_{k=0}^{n} T^k \to S = \sum_{k=0}^{\infty} T^k$$

$$\left.\begin{array}{l} (I-T)S_n = I - T^{n+1} \to I \\[2ex] (I-T)S_n \to (I-T)S \end{array}\right\} \quad \Rightarrow \quad (I-T)S = I$$

$S(I-T) = I$ follows similarly

Solving integral equations

Example: find $f \in \mathcal{C}([0,1], \mathbb{K})$ such that

$$3f(x) = 1 + \int_0^x f(t)\,dt$$

Equivalent formulation:

$$(I - T)f = g \quad \text{where} \quad Tf(x) = \frac{1}{3}\int_0^x f(t)\,dt \quad \text{and} \quad g(x) = 1/3$$

Since $\|T\| = 1/3$ we have

$$f(x) = (I - T)^{-1}g(x) = \sum_{k=0}^{\infty} T^k g(x) = \tfrac{1}{3}e^{x/3}$$

Exercise: verify the last equality by computing $T^k g$ for all $k \geq 0$

Note: $\|T^k\| \leq \|T\|^k = 1/3^k$ so $\sum_{k=0}^{\infty} \|T^k\| < \infty$