Functional Analysis

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Topics:

- §7.2: the dual space of ℓ^p
- §7.5: the second dual space and reflexive spaces

Uniform convergence

Let (φ_n) be a sequence in $\mathcal{C}([a,b],\mathbb{K})$

Recall: φ_n converges uniformly to $\varphi \in \mathcal{C}([a,b],\mathbb{K})$ if

$$\forall\, \varepsilon > 0 \quad \exists\, N_\varepsilon > 0 \ \text{ s.t. } \ n \geq N_\varepsilon \quad \Rightarrow \quad |\varphi_n(x) - \varphi(x)| \leq \varepsilon \ \ \forall\, x \in [a,b]$$

Equivalent: convergence in $\|\cdot\|_\infty$-norm:

$$\forall\, \varepsilon > 0 \quad \exists\, N_\varepsilon > 0 \ \text{ s.t. } \ n \geq N_\varepsilon \quad \Rightarrow \quad \|\varphi_n - \varphi\|_\infty \leq \varepsilon$$

Pointwise convergence

Recall: φ_n converges pointwise to $\varphi \in \mathcal{C}([a,b],\mathbb{K})$ if for all fixed $x \in [a,b]$ we have

$$\forall\, \varepsilon > 0 \;\; \exists\, N_{\varepsilon,x} > 0 \;\text{ s.t. }\; n \geq N_{\varepsilon,x} \quad \Rightarrow \quad |\varphi_n(x) - \varphi(x)| \leq \varepsilon$$

How to understand this in terms of normed linear spaces?

Dual spaces

Definition: let X be a NLS

The dual space of X is defined as

$$X' = B(X, \mathbb{K})$$

The norm on X' is given by

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

The dual of a Hilbert space

If X is a Hilbert space and $y \in X$, then the map

$$f_y : X \to \mathbb{K}, \qquad f_y(x) = \langle x, y \rangle$$

belongs to X' and $\|f_y\| = \|y\|$

Riesz-Fréchet theorem: for all $f \in X'$ there exists a unique $y \in X$ such that

$$f = f_y \quad \text{and} \quad \|f\| = \|y\|$$

The dual of a Hilbert space

Corollary: the map $T : X \to X'$ given by $Ty = f_y$ is:

- conjugate-linear: $T(\lambda y + \mu z) = \bar{\lambda} Ty + \bar{\mu} Tz$
- isometric: $\|Ty\| = \|y\|$ for all $y \in X$ (hence T is injective)
- surjective

Corollary: the dual space X' is a Hilbert space with inner product

$$\langle f_y, f_z \rangle := \langle z, y \rangle$$

[The order of y and z is indeed reversed; this is not a typo!]

The dual of ℓ^p for $1 < p < \infty$

Theorem: for $1 < p < \infty$, $1/p + 1/q = 1$, and $a \in \ell^q$ define

$$f_a : \ell^p \to \mathbb{K}, \quad f_a(x) = \sum_{i=1}^{\infty} x_i a_i$$

Then

1. $f_a \in (\ell^p)'$
2. $a \mapsto f_a$ is an isometric isomorphism from ℓ^q onto $(\ell^p)'$

Corollary: if $1 < p < \infty$, then $(\ell^p)' \simeq \ell^q$

The dual of ℓ^p for $1 < p < \infty$

Proof: for all $x \in \ell^p$ we have

$$\begin{aligned} |f_a(x)| &= \left| \sum_{i=1}^{\infty} x_i a_i \right| \\ &\leq \sum_{i=1}^{\infty} |x_i a_i| \\ &\leq \|x\|_p \, \|a\|_q \qquad \text{[Hölder's inequality]} \end{aligned}$$

This implies that $f_a \in (\ell^p)'$ and

$$\|f_a\| = \sup_{x \neq 0} \frac{|f_a(x)|}{\|x\|_p} \leq \|a\|_q$$

The dual of ℓ^p for $1 < p < \infty$

Proof (ctd): let $f \in (\ell^p)'$ and $a_i = f(e_i)$, where

$$e_i = (0, 0, \dots, 0, 1, 0, 0, \dots) \qquad \text{[1 at } i\text{-th entry]}$$

Fix $n \in \mathbb{N}$ and define

$$b = (b_1, \dots, b_n, 0, 0, 0, \dots) \qquad b_i = \begin{cases} |a_i|^q / a_i & \text{if } a_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\sum_{i=1}^n |a_i|^q = f(b) = |f(b)| \leq \|f\| \, \|b\|_p = \|f\| \left(\sum_{i=1}^n |a_i|^q \right)^{1/p}$$

The dual of ℓ^p for $1 < p < \infty$

Proof (ctd): for all $x \in \ell^p$ and $n \in \mathbb{N}$ we have

$$\left(\sum_{i=1}^n |a_i|^q\right)^{1/q} = \left(\sum_{i=1}^n |a_i|^q\right)^{1-1/p} \leq \|f\|$$

$$f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i a_i = f_a\left(\sum_{i=1}^n x_i e_i\right)$$

Hence we obtain

$$f = f_a \qquad \text{and} \qquad \|a\|_q \leq \|f\| = \|f_a\| \leq \|a\|_q$$

The dual of ℓ^1 and ℓ^∞

Theorem: $(\ell^1)' \simeq \ell^\infty$

[Proof as in ℓ^p-case]

Theorem: X' separable $\Rightarrow$ X separable

[See lecture notes; proof uses Hahn-Banach]

Corollary: $(\ell^\infty)' \simeq \ell^1$ is NOT true since ℓ^∞ is not separable

Second dual space

Definition: let X be a NLS

The second dual space of X is defined as

$$X'' := (X')' = B(X', \mathbb{K})$$

The norm on X'' is given by

$$\|\varphi\| = \sup_{f \in X', f \neq 0} \frac{|\varphi(f)|}{\|f\|}$$

Evaluation map

Lemma: assume X is a NLS

Define for a fixed $x \in X$ the evaluation map

$$F_x : X' \to \mathbb{K}, \qquad F_x(f) = f(x)$$

Then $F_x \in X''$ and $\|F_x\| = \|x\|$

Evaluation map

Proof: for all $f, g \in X'$ and $\lambda, \mu \in \mathbb{K}$ we have

$$\begin{aligned} F_x(\lambda f + \mu g) &= (\lambda f + \mu g)(x) \\ &= \lambda f(x) + \mu g(x) \\ &= \lambda F_x(f) + \mu F_x(g) \end{aligned}$$

In addition, we have

$$\|F_x\| = \sup_{f \neq 0} \frac{|F_x(f)|}{\|f\|} = \sup_{f \neq 0} \frac{|f(x)|}{\|f\|} = \sup_{\|f\|=1} |f(x)| = \|x\|$$

Natural map

Definition: for a NLS X we define the natural map by

$$J : X \to X'', \qquad x \mapsto F_x \qquad \text{i.e.} \qquad J(x)(f) = f(x)$$

Note: J is isometric (and thus injective) since

$$\|J(x)\| = \|F_x\| = \|x\|$$

Natural map

Example: assume X is a NLS, but not Banach

- X is isometrically isomorphic to $J(X)$
- X'' is a Banach space
- $\overline{J(X)}$ is closed in X'' and hence a Banach space
- $\overline{J(X)}$ is a completion of X

Reflexive spaces

Definition: X is called reflexive if $J : X \to X''$ is surjective

In this case, $J : X \to X''$ is an isometric isomorphism

Remark: there exists a Banach space X such that

- X is isometrically isomorphic X''
- but $J : X \to X''$ is not surjective

Reflexive spaces

Examples:

- every finite-dimensional space is reflexive
- every Hilbert space is reflexive
- ℓ^p is reflexive for $1 < p < \infty$
- ℓ^1 and ℓ^∞ are NOT reflexive

Strong and weak convergence

Definition: a sequence (x_n) in X converges to $x \in X$

- in the strong sense if:

$$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty$$

- in the weak sense if:

$$f(x_n) \to f(x) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad f \in X'$$

Strong and weak convergence

Remark: strong convergence implies weak convergence since

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n - x)| \\ &\leq \|f\| \, \|x_n - x\| \end{aligned}$$

On a finite-dimensional space the converse is also true

In general, the converse is NOT true!

Weak convergence

Proposition: if X is a NLS and (x_n) converges weakly to x, then (x_n) is bounded

Proof: for every $n \in \mathbb{N}$ define the map

$$T_n : X' \to \mathbb{K}, \quad T_n(f) = f(x_n)$$

Apply the uniform boundedness principle:

$$\sup_{n\in\mathbb{N}} |T_n(f)| < \infty \quad \forall f \in X' \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} \|T_n\| < \infty$$

Finally, note that $T_n = J(x_n)$ and thus $\|T_n\| = \|x_n\|$

Pointwise convergence in $\mathcal{C}([a,b],\mathbb{K})$

Proposition: if φ_n converges weakly to φ in $\mathcal{C}([a,b],\mathbb{K})$ then

1. φ_n converges pointwise to φ
2. $\sup\limits_{n\in\mathbb{N}} \|\varphi_n\|_\infty < \infty$

Proof: for (1) pick a fixed $x \in [a,b]$ and consider

$$f_x : \mathcal{C}([a,b],\mathbb{K}) \to \mathbb{K}, \quad f_x(\varphi) = \varphi(x)$$

This gives

$$|\varphi_n(x) - \varphi(x)| = |f_x(\varphi_n) - f_x(\varphi)| \to 0 \quad \text{as} \quad n \to \infty$$

For (2) recall that weakly convergent sequences are bounded

Strong, weak, and weak* convergence on the dual

Definition: a sequence (f_n) in X' converges to $f \in X'$

- in the strong sense if:

$$\|f_n - f\| \to 0 \quad \text{as} \quad n \to \infty$$

- in the weak sense if:

$$g(f_n) \to g(f) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad g \in X''$$

- in the weak* sense if:

$$f_n(x) \to f(x) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x \in X$$

Weaker forms of compactness

Theorem: if X is reflexive, then

every bounded sequence in X has a weakly convergent subsequence

Theorem: if X is a separable Banach space, then

every bounded sequence in X' has a weak* convergent subsequence

Application

Weak and weak* convergence are used in existence proofs for solutions of nonlinear partial differential equations, see

James C. Robinson

Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors

Cambridge University Press, 2001