# Functional Analysis

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Lecture 13
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Topics:

- §6.3: Special classes of operators
- §6.4: Selfadjoint operators
- §6.6: Spectra of compact, selfadjoint operators

# Characterization of selfadjointness

**Lemma:** $T = T^* \Leftrightarrow \langle Tx, x\rangle \in \mathbb{R} \quad \forall x \in X$

**Proof ($\Rightarrow$):**

$$\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

**Note:** $\Leftarrow$ only true when $\mathbb{K} = \mathbb{C}$!

So from now on: $\mathbb{K} = \mathbb{C}$

# Characterization of selfadjointness

**Proof ($\Leftarrow$):** for all $x \in X$ we have

$$\langle Tx, x\rangle = \overline{\langle x, Tx\rangle} = \langle x, Tx\rangle$$

Imitating the proof of the polarization identity gives

$$\begin{aligned} 4\langle Tx, y\rangle &= \sum_{k=0}^{3} i^k \langle T(x + i^k y), x + i^k y\rangle \\ &= \sum_{k=0}^{3} i^k \langle x + i^k y, T(x + i^k y)\rangle = 4\langle x, Ty\rangle \end{aligned}$$

# Nonnegative operators

**Definition:** assume $X$ is a Hilbert space and $T \in B(X)$

$T$ is called nonnegative if $\langle Tx, x \rangle \geq 0 \quad \forall x \in X$

[In particular, $T$ is selfadjoint since $\mathbb{K} = \mathbb{C}$]

**Notation:** $T \geq 0$

# Orthogonal projections

**Example:** if $P$ is an orthogonal projection, then $P \geq 0$

For all $x \in X$ we have

$$\begin{aligned}\langle Px, x\rangle &= \langle P^2x, x\rangle \\ &= \langle Px, P^*x\rangle \\ &= \langle Px, Px\rangle \\ &= \|Px\|^2 \geq 0\end{aligned}$$

# The operator norm

**Lemma:** $T \geq 0 \quad \Rightarrow \quad \|Tx\|^2 \leq \|T\| \langle Tx, x \rangle \quad \forall x \in X$

**Proof:** imitating the proof of Cauchy-Schwarz gives

$$|\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle \quad \forall x, y \in X$$

Setting $y = Tx$ gives

$$\begin{aligned} \|Tx\|^4 &\leq \langle Tx, x \rangle \langle TTx, Tx \rangle \\ &\leq \langle Tx, x \rangle \|T^2 x\| \, \|Tx\| \qquad \text{[by Cauchy-Schwarz]} \\ &\leq \langle Tx, x \rangle \|T\| \, \|Tx\|^2 \end{aligned}$$

# The operator norm

**Lemma:** $T \geq 0 \quad \Rightarrow \quad \|T\| = \sup_{\|x\|=1} \langle Tx, x\rangle$

**Proof:** by Cauchy-Schwarz we have

$$\langle Tx, x\rangle \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 \quad \Rightarrow \quad \sup_{\|x\|=1} \langle Tx, x\rangle \leq \|T\|$$

But the previous lemma gives

$$\|T\|^2 = \left(\sup_{\|x\|=1} \|Tx\|\right)^2 = \sup_{\|x\|=1} \|Tx\|^2 \leq \|T\| \sup_{\|x\|=1} \langle Tx, x\rangle$$

# The numbers $a$ and $b$

**Definition:** for a selfadjoint $T \in B(X)$ we define

$$a := \inf_{\|x\|=1} \langle Tx, x \rangle \quad \text{and} \quad b := \sup_{\|x\|=1} \langle Tx, x \rangle$$

[Note: we do *not* have $\|T\| = b$ since we did not assume $T \geq 0$]

**Exercise:** show that

$$T - a \geq 0 \qquad \text{and} \qquad b - T \geq 0$$

# Properties of the spectrum

**Theorem:** if $T$ is selfadjoint, then

1. $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$ implies $\langle x, y \rangle = 0$
2. $\sigma(T)$ only contains approximate eigenvalues
3. $\sigma(T) \subset [a, b]$
4. $a, b \in \sigma(T)$

**Proof (1,2):** follows from $T$ being normal

# Properties of the spectrum

**Proof (3):** Let $\lambda \in \sigma(T)$

There exists $(x_n)$ with $\|x_n\| = 1$ such that $(T - \lambda)x_n \to 0$

$$\begin{aligned} |\langle Tx_n, x_n\rangle - \lambda| &= |\langle Tx_n, x_n\rangle - \lambda\langle x_n, x_n\rangle| \\ &= |\langle (T-\lambda)x_n, x_n\rangle| \\ &\leq \|(T-\lambda)x_n\| \to 0 \end{aligned}$$

$\langle Tx_n, x_n\rangle \in [a, b]$ for all $n \ \Rightarrow\ \lambda \in [a, b]$

# Properties of the spectrum

**Proof (4):** there exists $(x_n)$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \langle Tx_n, x_n\rangle \to a$$

Since $T - a \geq 0$ we have

$$\begin{aligned} \|(T-a)x_n\|^2 &\leq \|T-a\|\langle (T-a)x_n, x_n\rangle \\ &= \|T-a\|\{\langle Tx_n, x_n\rangle - a\} \to 0 \end{aligned}$$

Hence $a \in \sigma(T)$

Similarly, $b \in \sigma(T)$

# The operator norm

**Theorem:** if $T$ is selfadjoint, then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \max\{|a|, |b|\}$$

**Proof:**

$$\|x\| = 1 \quad \Rightarrow \quad |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$$

$$\Rightarrow \quad M := \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$$

# The operator norm

**Proof (ctd):**

$$
\begin{aligned}
4\,\mathrm{Re}\langle Tx, y\rangle &= 2[\langle Tx, y\rangle + \overline{\langle Tx, y\rangle}] \\
&= 2[\langle Tx, y\rangle + \langle Ty, x\rangle] \qquad (T = T^*) \\
&= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\
&\leq |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| \\
&\leq M(\|x+y\|^2 + \|x-y\|^2) \\
&= 2M(\|x\|^2 + \|y\|^2)
\end{aligned}
$$

# The operator norm

**Proof (ctd):**

$$\left.\begin{array}{l} 4\,\mathrm{Re}\langle Tx, y\rangle \leq 2M(\|x\|^2 + \|y\|^2) \\[2ex] y = \dfrac{\|x\|}{\|Tx\|}\, Tx \end{array}\right\} \quad \Rightarrow \quad 4\|Tx\|\,\|x\| \leq 4M\|x\|^2$$

Hence $\|T\| \leq M$

# Existence of an eigenvalue

**Proposition:** if $T$ is compact and selfadjoint, then

$$-\|T\| \quad \text{or} \quad \|T\| \quad \text{is an eigenvalue}$$

**Proof:** assume $\|T\| \neq 0$

$\|T\| = \max\{|a|, |b|\}$ so one of the following cases applies:

$$a = \|T\|, \quad a = -\|T\|, \quad b = \|T\|, \quad b = -\|T\|$$

# Existence of an eigenvalue

**Proof (ctd):** assume that $a = \|T\|$

Since $a \in \sigma(T)$ there exists $(x_n)$ such that

$$\|x_n\| = 1 \quad \text{and} \quad (T - a)x_n \to 0$$

**Claim:** there exists $y \neq 0$ with $Ty = ay$

# Existence of an eigenvalue

**Proof (ctd):**

$$
\begin{aligned}
T \text{ compact} \quad &\Rightarrow \quad Tx_{n_k} \to y \text{ for some subsequence of } (x_n) \\
&\Rightarrow \quad ax_{n_k} = Tx_{n_k} - (T - a)x_{n_k} \to y \quad (\Rightarrow \|y\| = a \neq 0) \\
&\Rightarrow \quad x_{n_k} \to y/a \\
&\Rightarrow \quad Tx_{n_k} \to Ty/a \\
&\Rightarrow \quad Ty/a = y \\
&\Rightarrow \quad Ty = ay
\end{aligned}
$$

# Invariant subspaces

**Lemma:** if $V$ is a linear subspace of $X$ and $T \in B(X)$, then

$$T(V) \subset V \quad \Rightarrow \quad T^*(V^\perp) \subset V^\perp$$

**Proof:**

$$x \in V^\perp \;\; \Rightarrow \;\; \langle T^*x, y\rangle = \langle x, Ty\rangle = 0 \qquad \forall\, y \in V$$

$$\Rightarrow \;\; T^*x \in V^\perp$$

# Diagonalization theorem

**Theorem:** if $X$ is a Hilbert space and $T = T^* \in K(X)$, then

there exist:

- countably many real eigenvalues $\lambda_i$

  [In case of infinitely many eigenvalues we have $\lambda_i \to 0$]

- countably many orthonormal eigenvectors $e_i$

such that

$$Tx = \sum_i \lambda_i \langle x, e_i \rangle e_i$$

# Diagonalization theorem

**Proof:**

$T_1 := T = T^* \in K(X) \quad \Rightarrow \quad \lambda_1 = \pm \| T_1 \| \in \sigma_p(T_1)$

$X_1 := \ker(T_1 - \lambda_1) = \mathrm{span}\{e_i^1 \,:\, i = 1, \ldots, m_1\}$ ONB

$X_1$ invariant under $T_1 \quad \Rightarrow \quad X_1^\perp$ invariant under $T_1^* = T_1$

If $T_2 := T_1 \restriction X_1^\perp$ is nonzero, then continue

# Diagonalization theorem

**Proof:**

$$T_2 = T_2^* \in K(X_1^\perp) \quad \Rightarrow \quad \lambda_2 = \pm \| T_2 \| \in \sigma_p(T_2)$$

$$|\lambda_2| = \| T_2 \| \leq \| T_1 \| = |\lambda_1|$$

$$X_2 := \ker(T_2 - \lambda_2) = \operatorname{span}\{e_i^2 \,:\, i = 1, \ldots, m_2\} \quad \text{ONB}$$

$X_2$ invariant under $T_2 \quad \Rightarrow \quad (X_1 \oplus X_2)^\perp$ invariant under $T_2^* = T_2$

$T_3 := T_2 \restriction (X_1 \oplus X_2)^\perp$ et cetera...

# Diagonalization theorem

**Proof (ctd):** after $n$ steps we have for all $x \in X$ that

$$P_n x := x - \sum_{j=1}^{n} \left( \sum_{i=1}^{m_j} \langle x, e_i^j \rangle e_i^j \right) \in (X_1 \oplus \cdots \oplus X_n)^{\perp}$$

Note that $P_n$ is an orthogonal projection and

$$x = P_n x + (I - P_n)x$$

This gives

$$\|x\|^2 = \|P_n x\|^2 + \|(I - P_n)x\|^2 \quad \text{and thus} \quad \|P_n x\| \leq \|x\|$$

## Diagonalization theorem

**Proof (ctd):** observe that

$$Tx - \sum_{j=1}^{n} \left( \sum_{i=1}^{m_j} \lambda_j \langle x, e_i^j \rangle e_i^j \right) = TP_n x$$

If for some $n \in \mathbb{N}$ we have $TP_n x = 0$ for all $x \in X$, then stop

Otherwise, observe that

$$\begin{aligned} \|TP_n x\| &= \|T_n P_n x\| \\ &\leq \|T_n\| \|P_n x\| \\ &\leq \|T_n\| \|x\| \\ &= |\lambda_n| \|x\| \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$