Self-Organization in Space and Induced by Fluctuations

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Abstract[1]
We present a simple discrete model for the non-linear spatial interaction of different kinds of "subpopulations" composed of identical moving entities like particles, bacteria, individuals, etc. The model allows to mimic a variety of self-organized agglomeration and segregation phenomena. By relating it to game-theoretical ideas, it can be applied not only to attractive and repulsive interactions in physical and chemical systems, but also to the much richer combinations of positive and negative interactions found in biological and socio-economic systems. Apart from investigating symmetric interactions related to a continuous increase of the "overall success" within the system ("self-optimization"), we will focus on cases, where fluctuations further or induce self-organization, even though the initial conditions and the interactions are assumed homogeneous in space (translation invariant).

1. Introduction

Although the biological, social, and economic world are full of self-organization phenomena, many people believe that the dynamics behind them is too complex to be modelled in a mathematical way. Reasons for this are the huge number of interacting variables, most of which cannot be quantified, plus the assumed freedom of decision-making or large fluctuations within biological and socio-economic systems. However, in many situations, the living entities making up these systems decide for some (more

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or less) optimal behavior, which can make the latter describable or predictable to a certain extend []. This is even more the case for the behavior shown under certain constraints like, for example, in pedestrian or vehicle dynamics []. While pedestrians or vehicles can move freely at small traffic densities, at large densities the interactions with the others and with the boundaries of the street confines them to a small spectrum of moving behaviors. Consequently, empirical traffic dynamics can be reproduced by simulation models surprisingly well [].

In this connection, it is also interesting to mention some insights gained in statistical physics and complex systems theory: Nonlinearly interacting variables do not change independently of each other, and in many cases there is a separation of the time scales on which they evolve. This often allows to "forget" about the vast number of rapidly changing variables, which are usually determined by a small number of "order parameters" and treatable as fluctuations []. In the above mentioned examples of traffic dynamics, the order parameters are the traffic density and the average velocity of pedestrians or vehicles.

Another discovery is that, by proper transformations or scaling, many different models can by mapped onto each other, i.e. they behave basically the same []. That is, a certain class of models displays the same kinds of states, shows the same kinds of transitions among them, and can be described by the same "phase diagram", displaying the respective states as a function of some "control parameters" []. We call such a class of models a "universality class", since any of these models shows the same kind of "universal" behavior, i.e., the same phenomena. Consequently, one usually tries to find the simplest model having the properties of the universality class. While physicists like to call it a "minimal model", "prototype model", or "toy model", mathematicians named the corresponding mathematical equations "normal forms" [].

Universal behavior is the reason of the great success of systems theory [] in comparing phenomena in seemingly completely different systems, like physical, biological, or social ones. However, since these systems are composed of different entities and their corresponding interactions can be considerably different, it is not always easy to identify the variables and parameters behind their dynamics. It can be helpful to take up game-theoretical ideas, here, quantifying interactions in terms of payoffs []. This can be applied to positive (profitable, constructive, cooperative, symbiotic) or negative (competitive, destructive) interactions in socio-economic or biological systems, but to attractive and repulsive interactions in physical systems as well [].

In the following, we will investigate a simple model for interactive motion in space allowing to describe (i) various self-organized agglomeration phenomena, like settlement formation, and segregation phenomena, like ghetto formation, emerging from different kinds of interactions and (ii) fluctuation-induced ordering or self-organization phenomena.

Noise-related phenomena can be quite surprising and have, therefore, recently attracted the interest of many researchers. For example, we mention stochastic resonance [], noise-driven motion [], and "freezing by heating" [].

The issue of order through fluctuations has already a considerable history. Prigogine has discussed it in the context of structural

instability with respect to the appearance of a new species [], but this is not related to the approach considered in the following.

Moreover, since both, the initial conditions and the interaction strengths in our model are assumed independent of the position in space, the fluctuation-induced self-organization discussed later on must be distinguished from so-called "noise-induced transitions" as well, where we have a space-dependent diffusion coefficient which can induce a transition [].

Although our model is related to diffusive processes, it is also different from reaction-diffusion systems that can show fluctuation-induced self-organization phenomena known as Turing patterns [], which are usually periodic in space. The noise-induced self-organization that we find seems to have (i) no typical length scale and (ii) no attractor, since our model is translation-invariant. This, however, is not yet a final conclusion and still subject to investigations.

We also point out that, in the case of spatial invariance, self-organization directly implies spontaneous symmetry-breaking, and we expect a pronounced history-dependence of the resulting state. Nevertheless, when averaging over a large ensemble of simulation runs with different random seeds, we again expect a homogeneous distribution, since this is the only result compatible with translation invariance.

Finally, we mention that our results do not fit into the concept of noise-induced transitions from a metastable disordered state (local optimum) to a stable ordered state (global optimum), which are, for example, found for undercooled fluids, metallic glasses, or some granular systems [].

2. Discrete Model of Interactive Motion in Space

Describing motion in space has the advantage that the essential variables like positions, densities, and velocities are well measurable, which allows to calibrate, test, and verify or falsify the model. Although we will focus on motion in "real" space like the motion of pedestrians or bacteria, our model may also be applied to changes of positions in abstract spaces, e.g. to opinion changes on an opinion scale []. There exist, of course, already plenty of models for motion in space, and we can mention only a few []. Most of them are, however, rather specific for certain systems, e.g., for fluids or for migration behavior.

For simplicity, we will restrict the following considerations to a one-dimensional space, but a generalization to higher dimensions is straightforward. The space is divided into I equal cells i which can be occupied by the entities. We will apply periodic boundary conditions, i.e. the space can be imagined as a circle. In our model, we group the N entities α in the system into homogeneously behaving subpopulations a. If $n_i^a(t)$ denotes the number of entities of subpopulation a in cell i at time t, we have the relations

$$\sum_i n_i^a(t) = N_a, \qquad \sum_a N_a = N. \quad (1)$$

We will assume that the numbers N_a of entities belonging to the subpopulations a do not change. It is, however, easy to take additional birth and death processes and/or transitions of individuals from one subpopulation to another into account [].

In order not to introduce any bias, we start our simulations with a completely uniform distribution of the entities in each sub-

population over the I cells of the system, i.e., $n_i^a(0) = n_{\text{hom}}^a = N_a/I$, for which we choose a natural number. At times $t \in \{1, 2, 3, ...\}$, we apply the following update steps, using a random sequential update (although a parallel update is possible as well, which is more efficient [], but normally less realistic [] due to the assumed synchronous updating):

1st step: For updating of the state of entity α, given it is a member of subpopulation a and located in cell i, determine the so-called (expected) "success" according to the formula

$$S_a(i, t) = \sum_b P_{ab}\, n_i^b(t) + \xi_\alpha(t)\,. \quad (2)$$

Here, P_{ab} is the "payoff" in interactions of an entity of subpopulation a with an entity of subpopulation b. The payoff P_{ab} is positive for attractive, profitable, constructive, or symbiotic interactions, while it is negative for repulsive, competitive, or destructive interactions. Notice that P_{ab} is assumed to be independent of the position (i.e., translation invariant), while the total payoff $\sum_b P_{ab}\, n_i^b(t)$ due to interactions depends on the distribution of entities over the system. The latter is an essential point for the possibility of fluctuation-induced self-organization. We also point out that, in formula (2), pair interactions are restricted to the cell in which the individual is located. Therefore, we do not assume spin-like or Ising-like interactions, in contrast to other quantitative models proposed for the description of social behavior [].

The quantities $\xi_\alpha(t)$ are random variables allowing to consider individual variations of the success, which may be "real" or due to uncertainty in the evaluation or estimation of success. In our simulation program, they are uniformly distributed in the interval $[0, D_a]$, where D_a is the fluctuation strength (not to be mixed up wich a diffusion constant). However, other specifications of the noise term are possible as well.

2nd step: Determine the (expected) successes $S_a(i \pm 1, t)$ for the nearest neighbors $(i \pm 1)$ as well.

3rd step: Keep entity α in its previous cell i, if $S_a(i, t) \geq \max\{S(i-1, t), S(i+1, t)\}$. Otherwise, move to cell $(i-1)$, if $S(i-1, t) > S(i+1, t)$, and move to cell $(i+1)$, if $S(i-1, t) < S(i+1, t)$. In the remaining case $S(i-1, t) = S(i+1, t)$, jump randomly to cell $(i-1)$ or $(i+1)$ with probability $1/2$.

If there is a maximum density $\rho_{\max} = N_{\max}/I$ of entities, overcrowding can be avoided by introducing a saturation factor

$$c(j, t) = 1 - \frac{N_j(t)}{N_{\max}}\,, \quad N_j(t) = \sum_a n_j^a(t)\,, \quad (3)$$

and performing the update steps with the generalized success

$$S_a'(j, t) = c(j, t) S_a(j, t) \quad (4)$$

instead of $S_a(j, t)$, where $j \in \{i-1, i, i+1\}$. The model can be also easily extended to include long distance interactions, jumps to more remote cells, etc. (cf. Section 5).

3. Simulation Results

We consider two subpopulations $a \in \{1, 2\}$ and $N_1 = N_2 = 100$ entities in each subpopulation, which are distributed over $I = 20$ cells. The payoff matrix (P_{ab}) will be represented by the vector $\boldsymbol{P} = (P_{11}, P_{12}, P_{21}, P_{22})$, where we will restrict ourselves to $|P_{ab}| \in \{1, 2\}$ for didactical reasons. For symmetric interactions between subpopulations, we have $P_{ab} = P_{ba}$, while for asymmetric interactions, there is $P_{ab} \neq$

P_{ba}, if $a \neq b$. For brevity, the interactions within the same population will be called self-interactions, those between different populations cross-interactions.

To characterize the level of self-organization in each subpopulation a, we can, for example, use the overall successes

$$S_a(t) = \frac{1}{I^2} \sum_i \sum_b n_i^a(t) \, P_{ab} \, n_i^b(t) \,, \quad (5)$$

the variances

$$V_a(t) = \frac{1}{I^2} \sum_i [n_i^a(t) - n_{\mathrm{hom}}^a]^2 \,, \quad (6)$$

or the alternation strengths

$$A_a(t) = \frac{1}{I^2} \sum_i [n_i^a(t) - n_{i-1}^a(t)]^2 \,. \quad (7)$$

3.1. Symmetric Interactions

By analogy with a more complicated model [] it is expected that the global overall success $S(t) = \sum_a S_a(t)$ is an increasing function in time, if the fluctuation strengths D_a are zero. However, what happens at finite noise amplitudes D_a is not exactly known. One would usually expect that finite noise tends to obstruct or suppress self-organization, which will be investigated in the following.

We start with the payoff matrix $\boldsymbol{P} = (2, -1, -1, 2)$ corresponding to positive (or attractive) self-interactions and negative (or repulsive) cross-interactions. That is, entities of the same subpopulation like each other, while entities of different subpopulations dislike each other. The result will naturally be segregation ("ghetto formation") [], if the noise amplitude is small. However, segregation is suppressed by large fluctuations, as expected (see Fig. 1).
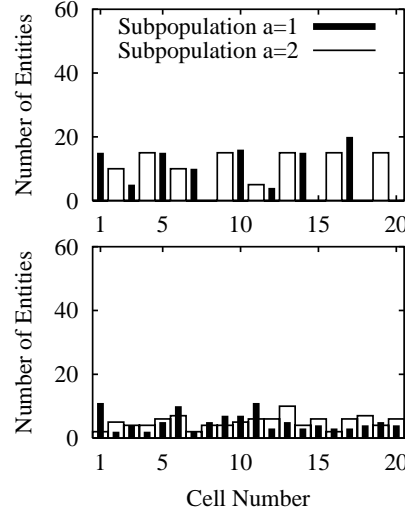

Figure 1: Resulting distribution of entities at $t = 4000$ for the payoff matrix $\boldsymbol{P} = (2, -1, -1, 2)$ at small fluctuation strength $D_a = 0.1$ (top) and large fluctuations strength $D_a = 5$ (bottom).

However, for medium noise amplitudes D_a, we find a much more pronounced self-organization (segregation) than for small ones (compare Fig. 2 with Fig. 1). The effect is systematic insofar as the degree of segregation (and, hence, the overall success) increases with increasing noise amplitude, until segregation breaks down above a certain critical noise level.

Let us investigate some other cases: For the structurally similar payoff matrix $(1, -2, -2, 1)$, we find segregation as well, which is not surprising. In contrast, we find agglomeration for the payoff matrices $(1, 2, 2, 1)$ and $(2, 1, 1, 2)$. This agrees with intuition, since all entities like each other in these cases, which makes them move to the same places, like in the formation of settle-
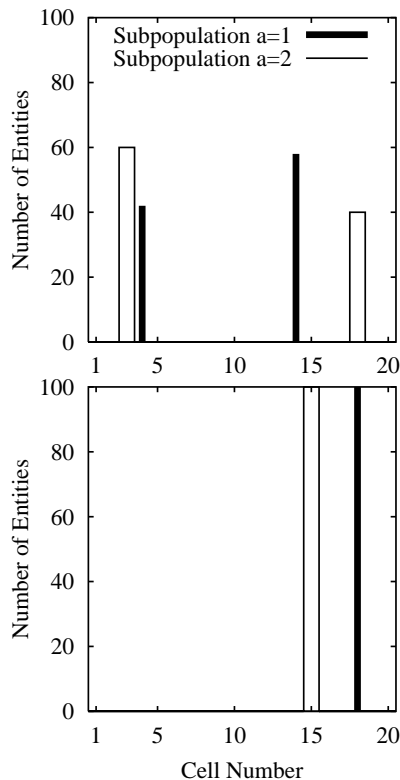
Figure 2: As Fig. 1, but with medium fluctuation strength $D_a = 2$ (top) and $D_a = 3$ (bottom).

ments [], the development of trail systems [], or the build up of slime molds []. More interesting is the case corresponding to the payoff matrix $(-1, 2, 2, -1)$, where the cross-interactions are positive (attractive), while the self-interactions are negative (repulsive). One may think that this causes the entities of the same subpopulation to spread homogeneously over the system, and in all cells would result an equal number of entities of both subpopulations, which is compatible with mutual attraction. However, this ho-

mogeneous distribution turns out to be unstable with respect to fluctuations. Instead, we find agglomeration! This result is more intuitive if we imagine one subpopulation to represent women and the other one men (without taking this example too serious). While the interaction between women and men is normally strongly attractive, the interactions among men or among women may be considered to be weakly competitive. As we all know, the result is a tendency of young men and women to move into cities. Corresponding simulation results for different noise strengths are depicted in Fig. 3. Again, we find that the self-organized pattern is destroyed by strong fluctuations in favour of a more or less homogeneous distribution, while medium noise strengths further self-organization.

For the payoff matrices $(-2, 1, 1, -2)$ and $(-2, -1, -1, -2)$, i.e. cases of strong negative self-interactions, we find a more or less homogeneous distribution of entities in both subpopulations, irrespective of the noise amplitude. In contrast, the payoff matrix $(-1, -2, -2, -1)$ corresponding to negative self-interactions but even stronger negative cross-interactions, leads to another self-organized pattern. We may describe it as the formation of lanes, as it is observed in pedestrian counterflows [] or in sheared granular media with different kinds of grains []. While both subpopulations tend to separate from each other, at the same time they tend to spread over all the available space (see Fig. 4), in contrast to the situation depicted in Figs. 1 and 2. Astonishingly enough, a medium level of noise again supports self-organized ordering, since it helps the subpopulations to separate from each other.

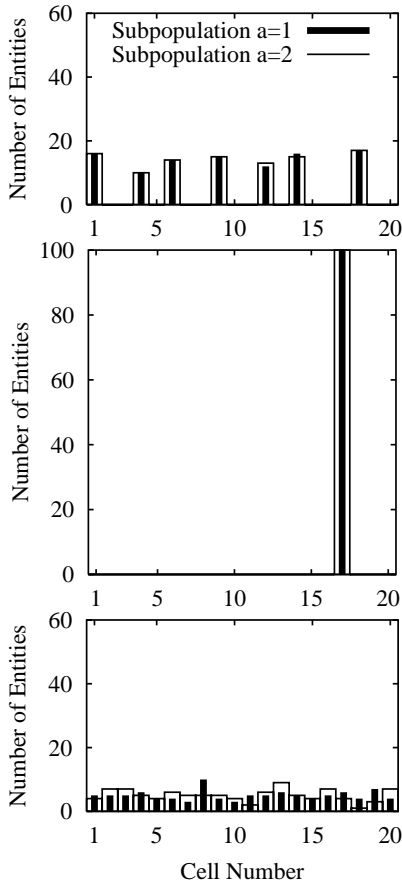We finally mention that a finite saturation level suppresses self-organization in

Figure 3: As Fig. 1, but for the pay-off matrix $\boldsymbol{P} = (-1, 2, 2, -1)$ and $D_a = 0.05$ (top), $D_a = 1.5$ (middle), and $D_a = 5$ (bottom).
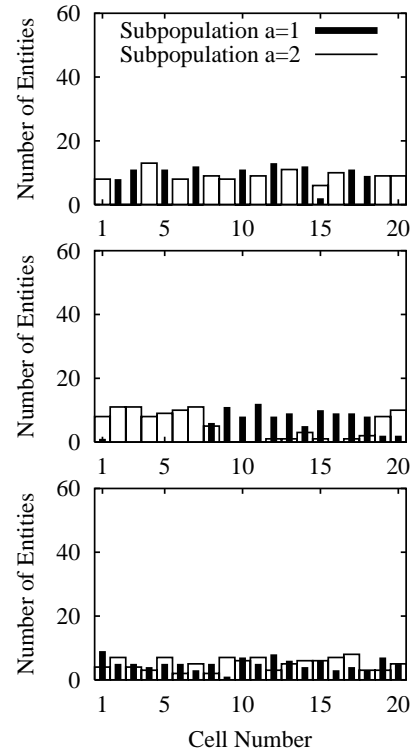

Figure 4: As Fig. 1, but for the payoff matrix $\boldsymbol{P} = (-1, -2, -2, -1)$ and $D_a = 0.05$ (top), $D_a = 0.5$ (middle), and $D_a = 5$ (bottom).

a surprisingly strong way, as is shown in Fig. 5. Instead of pronounced segregation, we will find a result similar to lane formation, and even strong agglomeration will be replaced by an almost homogeneous distribution.

Noise-induced ordering

A possible interpretation for noise-induced ordering would be that fluctuations allow the system to leave local minima (corresponding to partial agglomeration or segregation only). This could trigger a transition to a more stable state with more pronounced ordering. However, although this interpretation is consistent with a related example discussed in Ref. [], the idea of a step-wise coarsening process is not supported by the temporal evolution of the distribution of en-
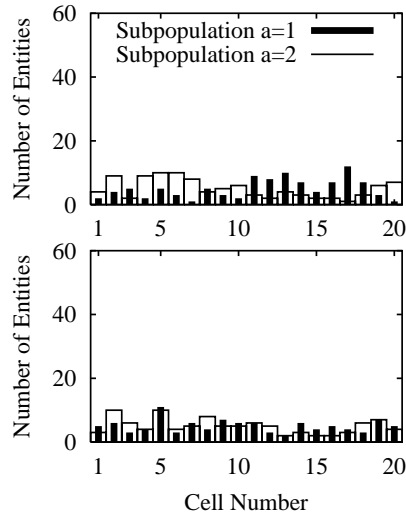
Figure 5: Resulting distribution of entities at $t = 4000$ with saturation level $N_{\max} = 50$. Top: $\boldsymbol{P} = (2, -1, -1, 2)$ and $D_a = 3$. Bottom: $\boldsymbol{P} = (-1, 2, 2, -1)$ and $D_a = 1.5$.

tities (see Fig. 6) and the time-dependence of the overall success within the subpopulations (see Fig. 7). This idea is anyway not applicable to segregation, since, in the one-dimensional case, the repulsive clusters of different subpopulations cannot simply pass each other in order to join others of the same subpopulation.

According to Figs. 6 and 7, segregation and agglomeration rather take place in three phases: First, there is a certain time interval, during which the distribution of entities remains more or less homogeneous. Second, there is a short period of rapid self-organization. Third, there is a continuing period, during which the distribution and overall success do not change anymore. The latter is a consequence of the short-range interactions within our model, which

are limited to the nearest neighbors. Therefore, the segregation or aggregation process practically stops, after separate peaks have evolved. This is not the case for lane formation, where the entities redistribute, but all cells remain occupied, so that we have ongoing interactions. This is reflected in the non-stationarity of the lanes and by the oscillations of the overall success.
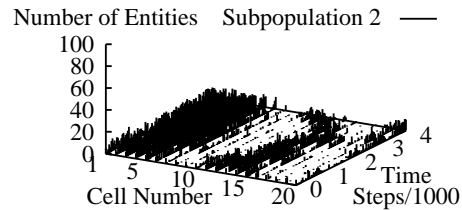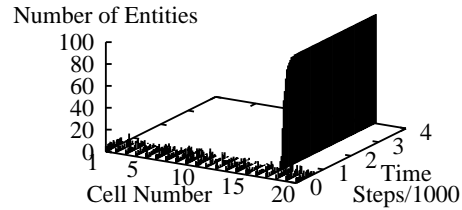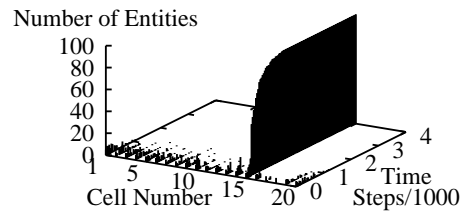

Figure 6: Temporal evolution of the distribution of entities within subpopulation $a = 2$ for $\boldsymbol{P} = (2, -1, -1, 2)$ and $D_a = 3$ (top), $\boldsymbol{P} = (-1, 2, 2, -1)$ and $D_a = 1.5$ (middle), and $\boldsymbol{P} = (-1, -2, -2, -1)$ and $D_a = 0.5$ (bottom).

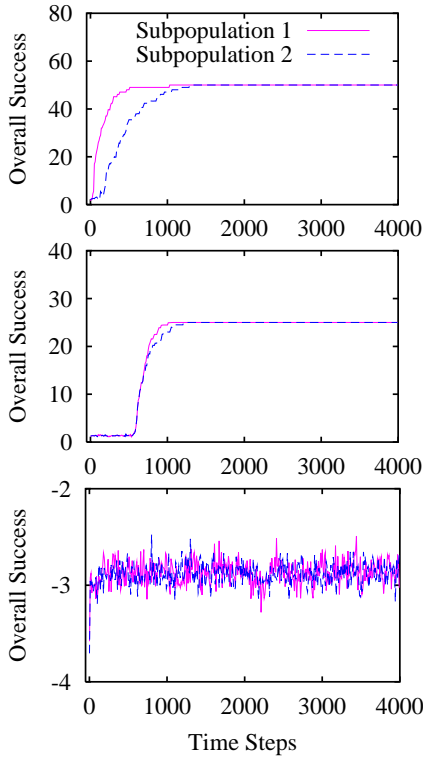We suggest the following interpretation

Figure 7: Temporal evolution of the overall success within both subpopulations for $\boldsymbol{P} = (2, -1, -1, 2)$ and $D_a = 3$ (top), $\boldsymbol{P} = (-1, 2, 2, -1)$ and $D_a = 1.5$ (middle), and $\boldsymbol{P} = (-1, -2, -2, -1)$ and $D_a = 0.5$ (bottom).

for the three phases mentioned above: During the first time interval, which is characterized by a quasi-continuous distribution of entities over space, a long-range pre-ordering process takes place. After this "phase of preparation", order develops in the second phase similar to crystallization, and it persists in the third phase. The role of fluctuations seems to be the following:

An increased noise level avoids a rash local self-organization by keeping up a quasi-continuous distribution of entities, which is required for a redistribution of entities over larger distances. In this way, a higher noise level increases the effective interaction range by extending the first phase, the "interaction phase". As a consequence, the resulting structures are more extended in space (but probably without a characteristic length scale, see Introduction).

It would be interesting to investigate, whether this mechanism has something to do with the recently discovered phenomenon of "freezing by heating", where a medium noise level causes a transition to a highly ordered (but energetically less stable) state, while extreme noise levels produce a disordered, homogeneous state again [].

3.2. Asymmetric Interactions

Even more intriguing transitions than in the symmetric case can be found for asymmetric interactions between the subpopulations. Here, we will focus on the payoff matrix $(-1, 2, -2, 1)$, only. This example corresponds to the curious case, where individuals of subpopulation 1 weakly dislike each other, but strongly like individuals of the other subpopulation. In contrast, individuals of subpopulation 2 weakly like each other, but they strongly dislike the other subpopulation. A good example for this is hard to find. With some good will, one may imagine subpopulation 1 to represent poor people, while subpopulation 2 corresponds to rich people. What will be the outcome? In simple terms, the rich are expected to agglomerate in a few areas, if the poor are moving too nervously (see Fig. 8). In detail, however, the situation is quite complex, as discussed in the next paragraph.
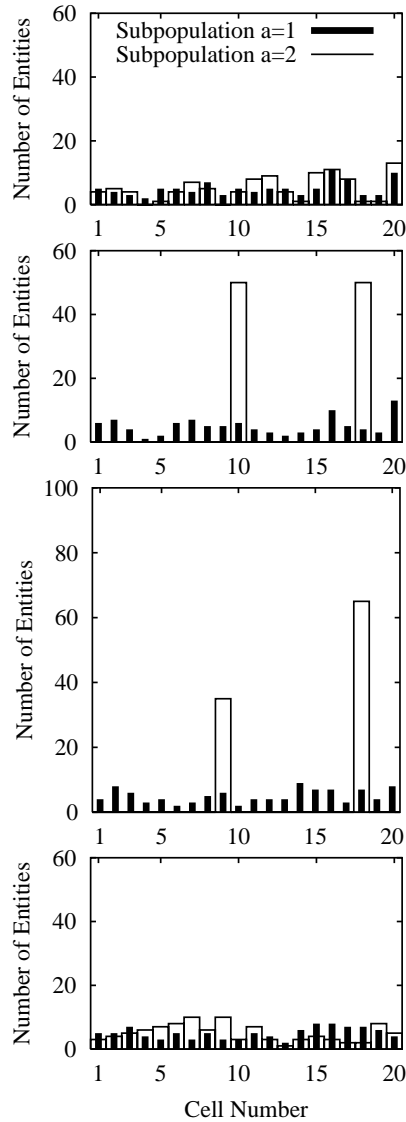
Figure 8: Distributions for $\boldsymbol{P} = (-1, 2, -2, 1)$ and $D_1 = D_2 = 0.5$ (top), $D_1 = 50$, $D_2 = 0.5$ (second), $D_1 = 5000$, $D_2 = 0.5$ (third), $D_1 = 5000$, $D_2 = 5$ (bottom).

Noise-induced self-organization

At small noise levels D_a, we will just find more or less homogeneous distributions of the entities. This is already different from the cases of agglomeration, segregation, and lane formation we have discussed before. Self-organization is also not found at higher noise amplitudes D_a, as long as we assume that they are the same in both subpopulations (i.e., $D_1 = D_2$). However, given that the fluctuation amplitude D_2 in subpopulation 2 is small, we find an agglomeration in subpopulation 2, if the noise level D_1 in subpopulation 1 is medium or high, so that subpopulation 1 remains homogeneously distributed. The order in subpopulation 2 breaks down, as soon as we have a relevant (but still small) noise level D_2 in subpopulation 2 (see Fig. 8).

Hence, we have a situation where asymmetric noise with $D_1 \neq D_2$ can facilitate self-organization in a system with completely homogeneous initial conditions and interaction laws, where we would not have ordering without any noise. We call this phenomenon noise-induced self-organization. It is to be distinguished from the noise-induced increase in the degree of ordering discussed above, where we have self-organization even without noise, if only the initial conditions are not fully homogeneous.

The role of the noise in subpopulation 1 seems to be the following: Despite of the attractive interaction with subpopulation 2, it suppresses an agglomeration in subpopulation 1, in particular at the places where subpopulation 2 agglomerates. Therefore, the repulsive interaction of subpopulation 2 with subpopulation 1 is effectively reduced. As a consequence, the attractive self-interaction within subpopulation 2 dom-

inates, which gives rise to the observed agglomeration.

The temporal development of the distribution of entities and of the overall success in the subpopulations gives additional information (see Fig. 9). As in the case of lane formation, the overall success fluctuates strongly, because the subpopulations do not separate from each other, causing ongoing interactions. Hence, the resulting distribution is not stable, but changes continuously. It can, therefore, happen, that clusters of subpopulation 2 merge, which is associated with an increase of overall success in subpopulation 2 (see Fig. 9).
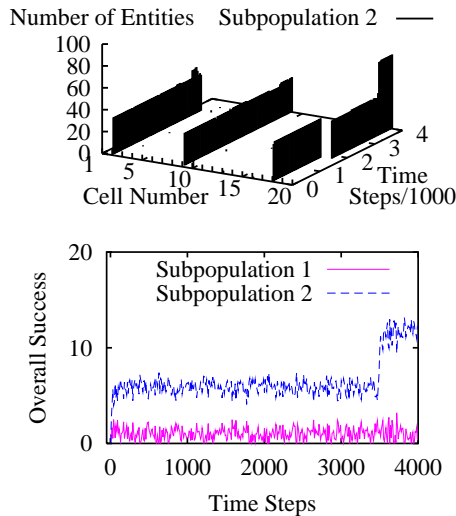

Figure 9: Temporal evolution of the distribution of entitities within subpopulation $a = 2$ (top) and of the overall successes (bottom) for $\boldsymbol{P} = (-1, 2, -2, 1)$ and $D_1 = 50$, $D_2 = 0.5$.

4. Conclusions

We have proposed a game theoretical model for self-organization in space, which is applicable to many kinds of biological, economic, and social systems with various types of profitable or competitive self- and cross-interactions between subpopulations of the system. Depending on the structure of the payoff matrix, we found several different self-organization phenomena like agglomeration, segregation, or lane formation. It turned out that medium noise strengths can increase the resulting level of order, while a high noise level leads to more or less homogeneous distributions of entities over the available space. The mechanism of noise-induced ordering in the above discussed systems with short-range interactions seems to be the following: Noise extends a "pre-ordering" phase by keeping up a quasi-continuous distribution of entities, which allows a long-range ordering. For asymmetric payoff matrices, we can even have the phenomenon of noise-induced self-organization, although we start with completely homogeneous distributions and homogeneous (translation-invariant) payoffs. However, the phenomenon requires different noise amplitudes in both subpopulations. The role of noise is to suppress agglomeration in one of the subpopulations, in this way reducing repulsive effects that would suppress agglomeration in the other subpopulation.

We point out that all the above results can be semi-quantitatively understood by means of a linear stability analysis of a related continuous version of the model []. This continuous version indicates that the linearly most unstable modes are the ones with the shortest wave length, so that

one does not expect a characteristic length scale in the system. This is different from reaction-diffusion systems, where the most unstable mode has a finite wave length, which gives rise to the formation of periodic patterns. Nevertheless, the structures evolving in our model are spatially extended, but non-periodic. The spatial extension is increasing with the fluctuation strength, unless a critical noise amplitude is exceeded.

For a better agreement with real systems, the model can be generalized in many ways. The entities may perform a biased or unbiased random walk in space. One can allow random jumps to neigboring cells with some prescribed probability. This probability may depend on the subpopulation, and thus we can imitate different mobilities of the considered subpopulations. Evolution is slowed down by introducing a threshold, fixed or random, so that the entities change to other cells only if the differences in the relevant successes are bigger than the imposed threshold. The model can be also generalized to higher dimensions, with expected interesting patterns of self-organized structures.

In general, the random variables $\xi_\alpha(t)$ in the definition of the success functions can be allowed to have different variances for the considered cell i and the neighboring cells, with the interpretation that the uncertainty in the evaluation of the success in the considered cell is different (e.g. smaller) than that in the neighboring cells. Moreover, the uncertainities can be different for various subpopulations, which could reflect to some extent their different knowledge and behavior.

One can as well study systems with more than two subpopulations, the influence of long-range interactions, etc. The entities can also be allowed to jump to more remote cells.

As an example, the following update rule could be implemented: Move entity α from cell i to the cell $(i + l)$ for which

$$S_a''(i + l, t) = d^{|l|} c(i + l, t) S_a(i + l, t) \quad (8)$$

is maximal ($|l| = 0, 1, ..., l_{\max}$). If there are m cells in the range $\{(x - l_{\max}), \ldots, (x + l_{\max})\}$ with the same maximal value, choose one of them randomly with probability $1/m$. According to this, when spontaneously moving to another cell, the entity prefers cells in the neighborhood with higher success. The indirect interaction behind this transition, which is based on the observation or estimation of the success in the neighborhood, is short-ranged if $l_{\max} \ll I$, otherwise long-ranged. Herein, $l_{\max}$ denotes the maximum number of cells which an entity can move within one time step. The factor containing d with $0 < d < 1$ allows to consider that it is less likely to move for large distances, if this is not motivated by a higher success. A value $d < 1$ may also reflect the fact that the observation or estimation of the success over large distances becomes more difficult and less reliable.

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