NTNU | Norwegian University of Science and Technology

TTK4135 – Lecture 8
Open-Loop Dynamic optimization

Lecturer: Lars Imsland

Outline

- Static vs dynamic optimization (and “quasi-dynamic”)
- Dynamic optimization = optimization of dynamic systems
- How to construct objective function for dynamic optimization
- Batch approach vs recursive approach for solving dynamic optimization problems

Reference: F&H Ch. 3,4

Course overview

Optimization problems we study	Use of optimization in control
• Class: Linear programming • Method: Simplex $$\begin{aligned} \min \quad & c^\top x \\ \text{subject to} \quad & Ax \le b \\ & Cx = d \end{aligned}$$	• (Some use in MPC, but not typical)

• Class: Quadratic programming
• Method: Active set
$$\begin{aligned} \min \quad & \frac{1}{2} x^\top G x + c^\top x \\ \text{subject to} \quad & Ax \le b \\ & Cx = d \end{aligned}$$
	• Dynamic optimization using linear models
	  • Open loop dynamic optimization
	  • Closed loop dynamic optimization/ Model Predictive Control
	  • (Linear Quadratic Control)

• Class: Nonlinear programming
• Methods
  • Without constraints: Linesearch methods
  • With constraints: SQP
$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & h(x) \ge 0 \end{aligned}$$
	• Dynamic optimization using nonlinear models
	  • Nonlinear Model Predictive Control

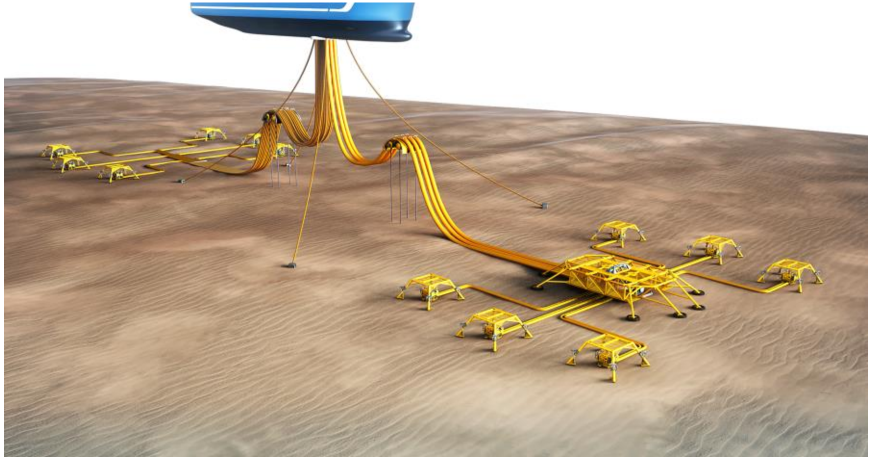
Static vs dynamic optimization

When using optimization for solving practical problems (that is, we optimize some *process*) we have two cases:

- The model of the process is time independent, resulting in static optimization
  - Common in finance, economic optimization, …
  - Recall farming example
- The model of the process is time dependent, resulting in dynamic optimization
  - The typical case in control engineering
  - The process is a mechanical system (boat, drone, robot, …), chemical process (e.g. chemical reactor, process plant, …), electrical grid, …

- F&H argues for a third option called quazi-dynamic optimization
  - The process is slowly time-varying, and can be assumed to be static for the purposes of optimization
  - We take care of the time-varying effects by re-solving regularly (or when the problem has changed sufficiently)

Oil production
(example of quasi-dynamic optimization, Ex. 2 in F&H)



Dynamic models, linearization

(in this course)

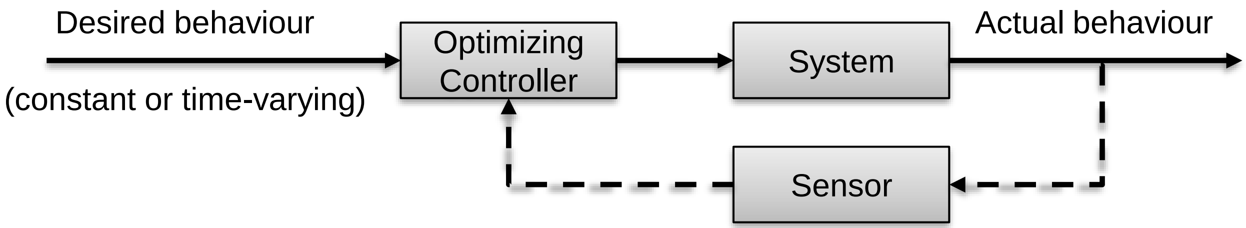
$$x_{t+1} = g(x_t, u_t) \qquad \text{(nonlinear)}$$

$$x_{t+1} = A_t x_t + B_t u_t \qquad \text{(LTV)}$$

$$x_{t+1} = A x_t + B u_t \qquad \text{(LTI)}$$

General dynamic optimization problem

Possible objectives (stage costs) in dynamic optimization



Typical objectives in control:

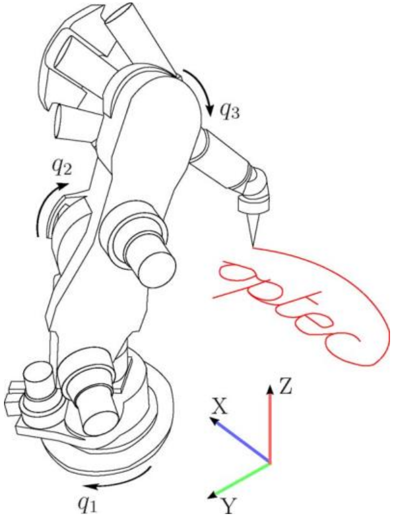
- Penalize deviations from a constant reference/setpoint (*regulation*), or deviations from a reference trajectory (*tracking*).

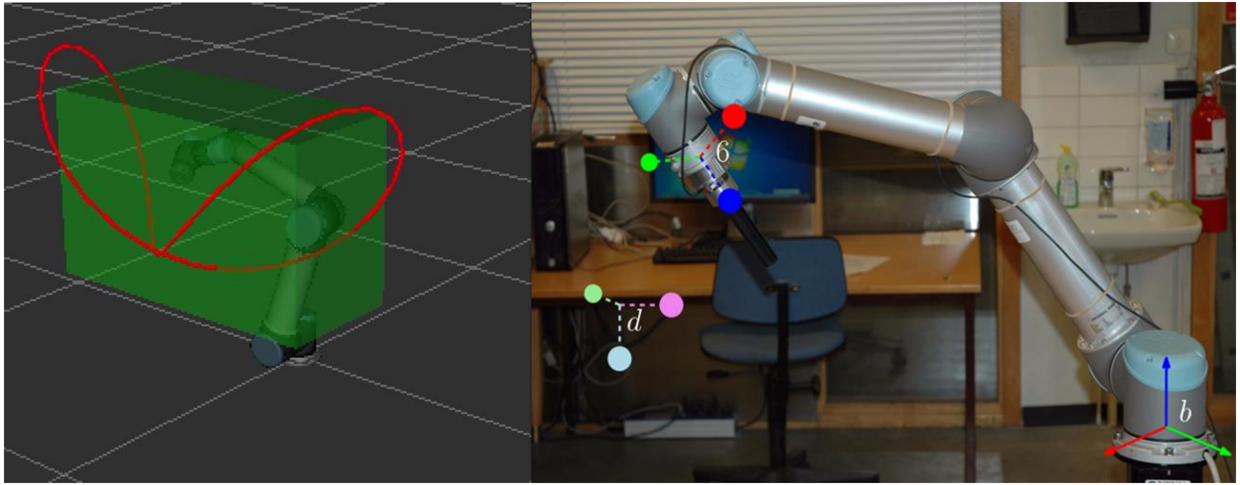
Other types of objectives:

- Economic objectives. Optimize economic profit: maximize production (e.g. oil), and/or minimize costs (e.g energy or raw material)
- Limit tear and wear of equipment (e.g. valves)
- Reach a specific endpoint as fast as possible
- Reach a specific endpoint, possibly avoiding obstacles

“Standard” stage costs in dynamic optimization

Examples tracking





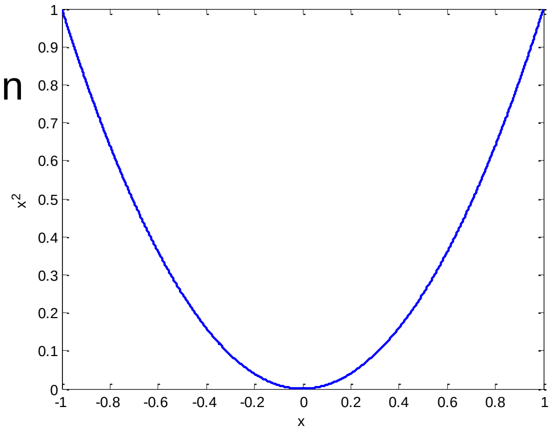
Race car trajectory tracking (Formula student 2020, ETH)

https://www.youtube.com/watch?v=vkVBi9LWJo0
https://arxiv.org/abs/2003.04882

Why quadratic objective?

Two main reasons:

- Because it is convenient, mathematically
  - Smooth is good, both for analysis and numerical optimization
  - Give linear gradients



- Because it is natural; the effect is often desirable
  - Tends to ignore small deviations
  - Tends to punish large deviations

- However, other types of objective functions are also used

Standard linear dynamic optimization problem

Standard linear dynamic optimization problem

Batch approach v1, “Full space”

Standard linear dynamic optimization problem

Batch approach v2, “Reduced space”

Standard linear dynamic optimization problem

Batch approach v2, “Reduced space”

Linear quadratic control:
Dynamic optimization without (inequality) constraints

$$\begin{aligned}
\min_{z} \sum_{t=0}^{N-1} & x_{t+1}^{\top} Q x_{t+1} + u_t^{\top} R u_t \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\
& z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^{\top}
\end{aligned}$$

Three approaches for implementation:

- Batch approach v1, “full space” – solve as QP
- Batch approach v2, “reduced space” – solve as QP
- Recursive approach – solve as linear state feedback

Linear Quadratic Control
Batch approach v1, “Full space” QP

- Formulate with model as equality constraints, all inputs and states as optimization variables

$$
\begin{aligned}
\min_z \ & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.} \ & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1 \\
& z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top
\end{aligned}
$$

$$
\begin{aligned}
\min_z \quad & \frac{1}{2} z^\top \begin{pmatrix} R & & & & \\ & Q & & & \\ & & R & & \\ & & & \ddots & \\ & & & & Q \end{pmatrix} z \\
\text{s.t.} \quad & \begin{pmatrix} -B & I & & & & & \\ & -A & -B & I & & & \\ & & & -A & -B & I & \\ & & & & \ddots & \ddots & \\ & & & & -A & -B & I \end{pmatrix} z = \begin{pmatrix} A x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
& z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top
\end{aligned}
$$

Linear Quadratic Control
Batch approach v2, "Reduced space" QP

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$
$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

- Use model to eliminate states as variables
  - Future states as function of inputs and initial state

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{pmatrix} x_0 + \begin{pmatrix} B & & & & \\ AB & B & & & \\ A^2 & AB & B & & \\ \vdots & \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \dots & B \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = S^x x_0 + S^u U$$

  - Insert into objective (no constraints!)

$$\min_U \frac{1}{2} \left(S^x x_0 + S^u U\right)^\top \mathbf{Q} \left(S^x x_0 + S^u U\right) + \frac{1}{2} U^\top \mathbf{R} U$$

$$\mathbf{Q} = \begin{pmatrix} Q & & \\ & Q & \\ & & \ddots \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R & & \\ & R & \\ & & \ddots \end{pmatrix}$$

  - Solution found by setting gradient equal to zero:

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -\left((S^u)^\top \mathbf{Q} S^u + \mathbf{R}\right)^{-1} (S^u)^\top \mathbf{Q} S^x x_0 = -F x_0$$

Linear Quadratic Control
Recursive approach

$$
\begin{aligned}
\min_{z} \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1 \\
& z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top
\end{aligned}
$$

- By writing up the KKT-conditions, we can show (we will do this later) that the solution can be formulated as:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$
\begin{aligned}
K_t &= R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad && t = 0, \ldots, N-1 \\
P_t &= Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad && t = 0, \ldots, N-1 \\
P_N &= Q
\end{aligned}
$$

Comments to the three solution approaches

- All give same numerical solution
  - If problem is strictly convex (Q psd, R pd), solution is unique
- Batch approaches give open-loop solution, recursive approach give feedback solution
  - Means the recursive solution is more robust in implementation -> always use this for LQC

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -Fx_0 \qquad \text{vs} \qquad u_t = -K_t x_t$$

- But what if we have constraints on inputs and states:
  - **Straightforward to add constraints as inequalities to batch approaches (both becomes convex QPs)**
  - Much more difficult to add constraints to the recursive approach

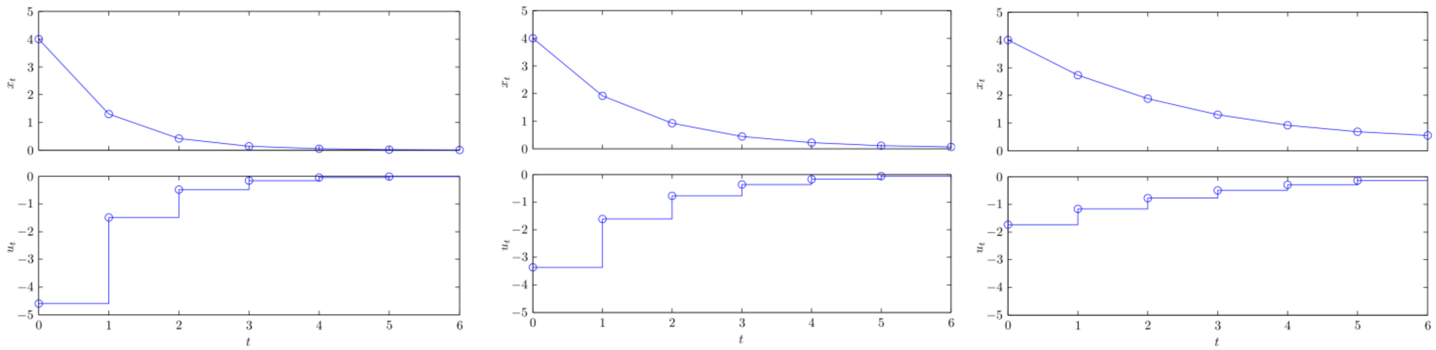
- How to to add feedback (and thereby robustness) to batch approaches?
  - **Model predictive control!**

The significance of weights

$$\min \sum_{t=0}^{5} q\, x_{t+1}^2 + r\, u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \dots, 5$$

$q = 5,\ r = 1$ | $q = 2,\ r = 1$ | $q = 1,\ r = 2$



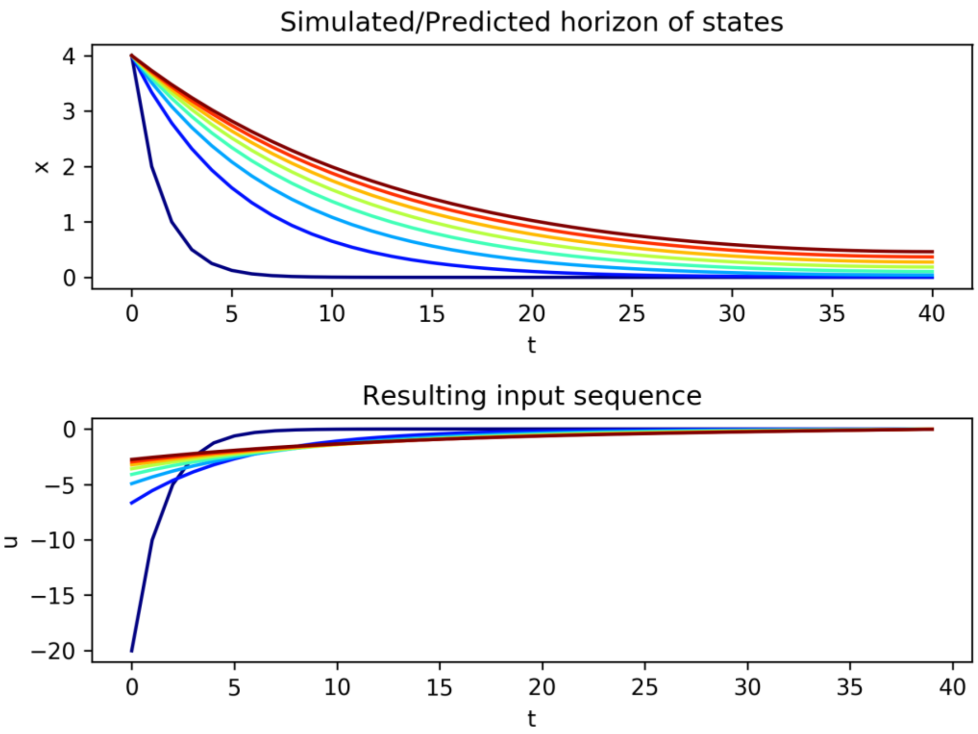
$$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \quad \sum_{t=0}^{N-1} u_t^2 = 23.6$$

$$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \quad \sum_{t=0}^{N-1} u_t^2 = 14.7$$

$$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \quad \sum_{t=0}^{N-1} u_t^2 = 5.3$$

Significance of weights – Ratios

$$x_{t+1} = 1.001x_t + 0.1u_t, q = 5, r \in [0.1, \dots, 10]$$



Open loop vs closed loop

- Next week: How to use open-loop optimization for closed-loop (feedback!)
  - This is called Model Predictive Control