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TTK4135 – Lecture 11
Practical use of MPC: Output feedback, target calculation and offset-free control

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Outline

- Recap: Model Predictive Control (MPC), Feasibility&stability

Common (necessary) features in practical MPC implementations:

- Output feedback
- Target calculation
- Offset-free MPC (integral action in MPC)

Reference: F&H Ch. 4.2.3-4.2.4

(Two articles containing more information on Blackboard – not curriculum)

Open-loop optimization with linear state-space model

QP

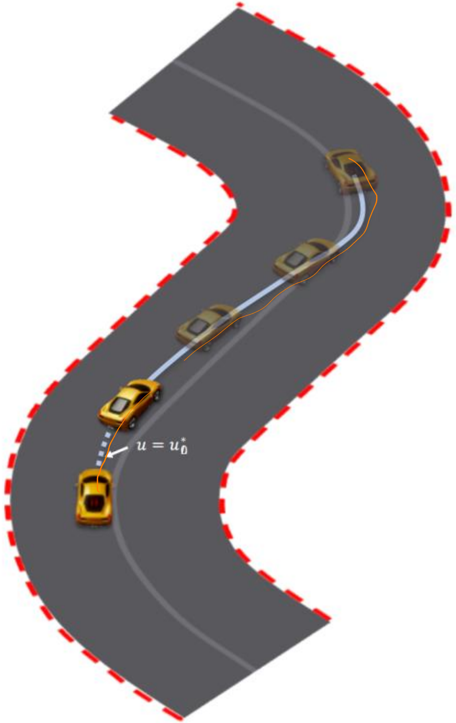
$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \cancel{d_{x,t+1} x_{t+1}} + \frac{1}{2} u_t^\top R_t u_t + \cancel{d_{u,t} u_t} + \frac{1}{2} \Delta u_t^\top S \Delta u_t$$

subject to

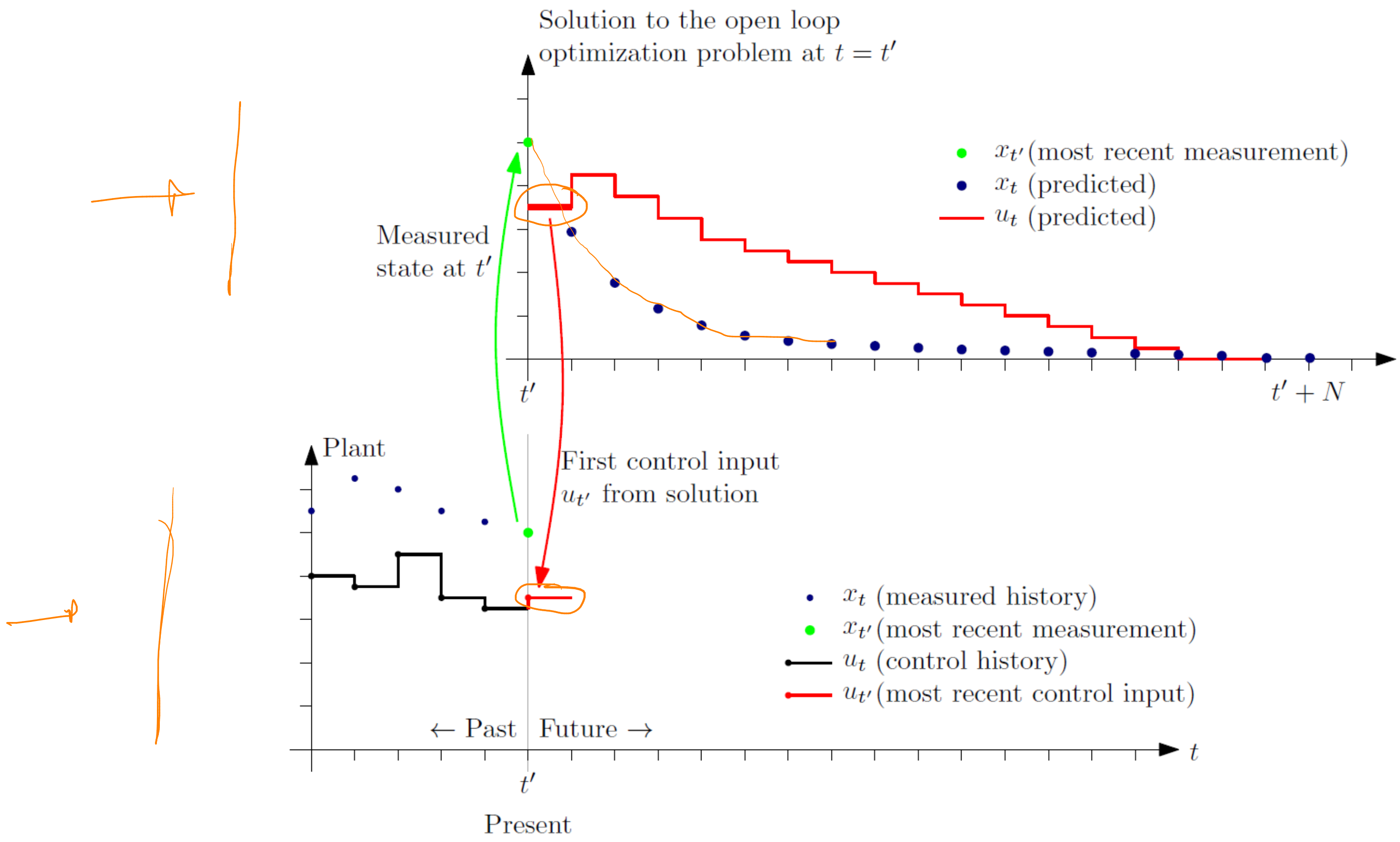
$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\} \\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\} \\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
-\Delta u^{\text{high}} &\leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\}
\end{aligned}$$

where

$$\begin{aligned}
&x_0 \text{ and } u_{-1} \text{ is given} \\
\Delta u_t &:= u_t - u_{t-1} \\
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u) \\
Q_t &\succeq 0 \quad t = \{1, \dots, N\} \\
R_t &\succ 0 \quad t = \{0, \dots, N-1\}
\end{aligned}$$



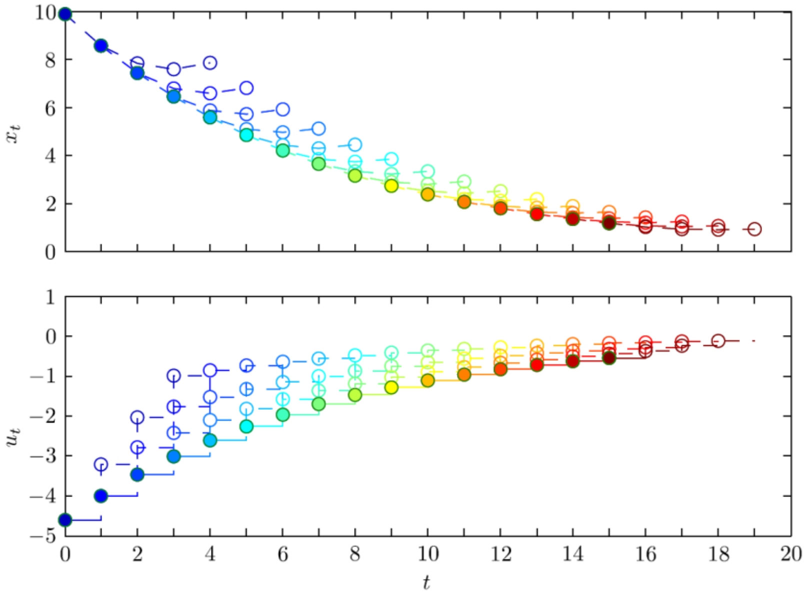
Model predictive control principle



Open-loop vs closed-loop trajectories

$$\min \sum_{t=0}^{4} x_{t+1}^2 + 4\, u_t^2$$

$$\text{s.t.} \qquad x_{t+1} = 1.2x_t + 0.5u_t, \quad t = 0, \ldots, 4$$



- Closed-loop trajectories different from open-loop (optimized) trajectories!
- It is the closed-loop trajectories that must analyzed for feasibility and stability.

MPC and feasibility

Is there always a solution to the MPC open-loop optimization problem?

- Not necessarily – state constraints may become infeasible, for example after a disturbance
- Practical solution: Soft constraints (aka “exact penalty” formulations)
  - “Soften” state constraints by adding “slack variables”

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n, \epsilon} \; & f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + \rho^\top \epsilon \\
\text{s.t.} \quad & x_{t+1} = A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\} \\
& x^{\text{low}} - \epsilon \leq x_t \leq x^{\text{high}} + \epsilon, \quad t = \{1, \dots, N\}, \qquad \epsilon > 0 \\
& \vdots
\end{aligned}
$$

MPC optimality implies stability?

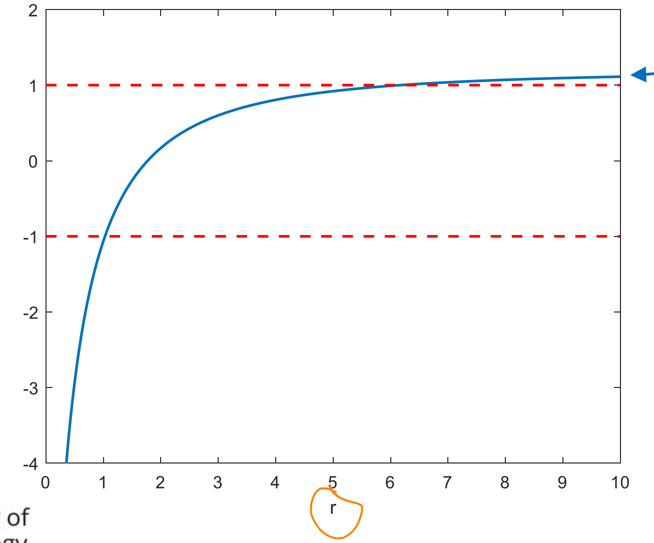
$$\min \sum_{t=0}^{1} x_{t+1}^2 + r\,u_t^2$$
$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1$$

MPC solution

$$u_t = -\frac{1.2 + 2.64r}{1 + 3.2r + r^2} x_t$$

MPC closed loop

$$x_{t+1} = \underbrace{\left(1.2 - \frac{1.2 + 2.64r}{1 + 3.2r + r^2}\right)}\, x_t$$



MPC and stability

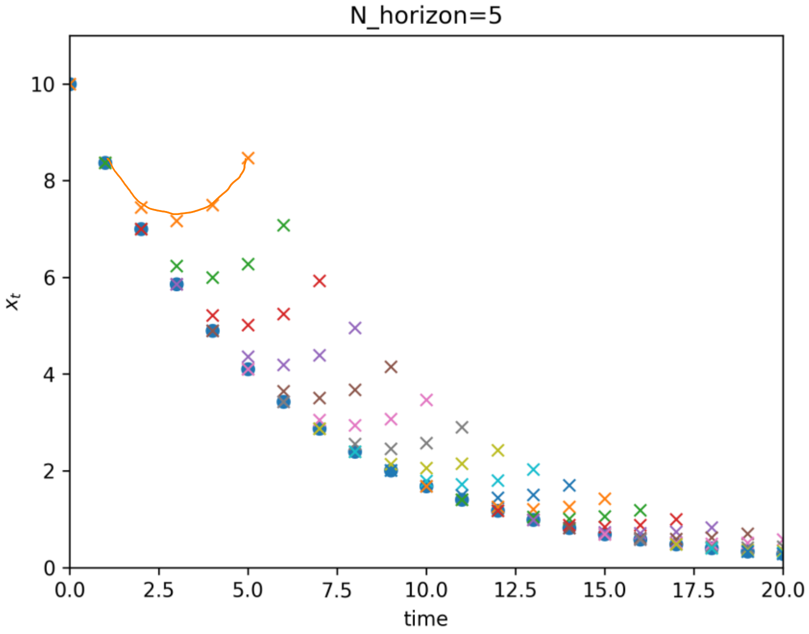
Requirements for stability:

- Stabilizability ((A,B) stabilizable)
- Detectability ((A,D) detectable)
  - D is a matrix such that $Q = D^{\mathsf{T}}D$ (that is, "D is matrix square root of Q")
  - Detectability: No modes can grow to infinity without being "visible" through Q

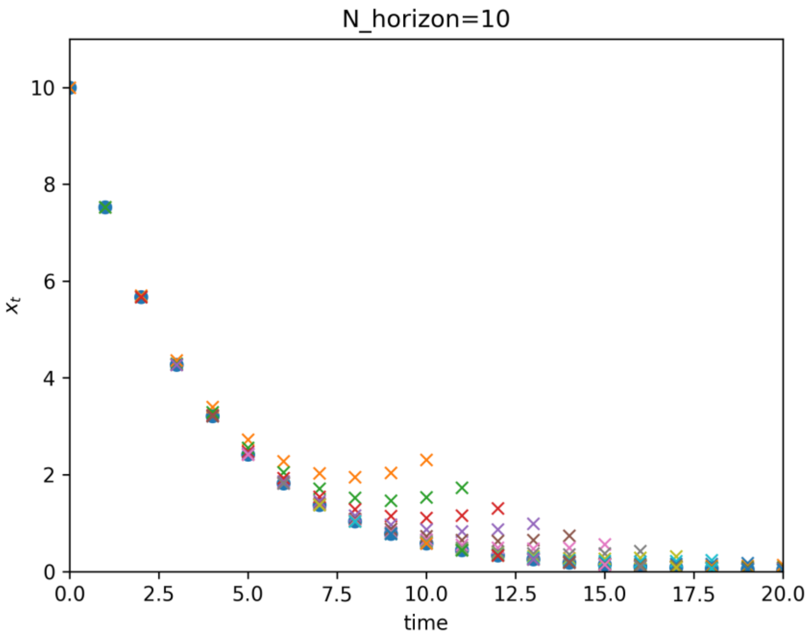
How to design MPC schemes with guaranteed *nominal* stability:

- Choose prediction horizon equal to infinity (usually not possible)
- For given N, choose Q and R such that MPC is stable (cf. example)
  - Difficult, and not always possible!
- Change the optimization problem – add terminal cost/terminal constraints – such that
  - The new problem is an "upper approximation" of infinite horizon problem
  - The constraints holds after the prediction horizon
- Typically, in practice: Choose horizon N "large enough"
  - Usually works well!
  - What is "large enough"? Longer than dominating dynamics, but shorter can be OK.
  - Good practice: Choose N large enough such that open-loop predictions resembles closed-loop (test in simulations!)

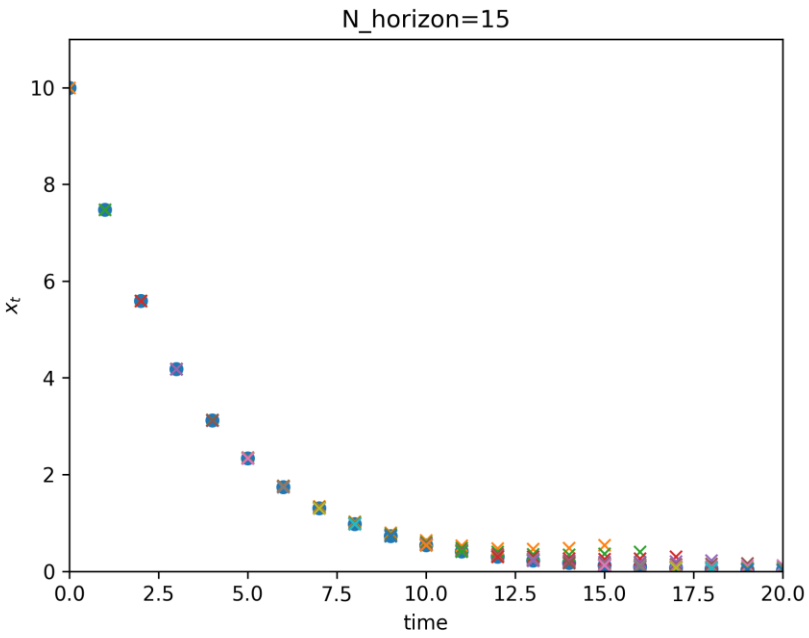
Open-Loop vs Closed-Loop: N = 5



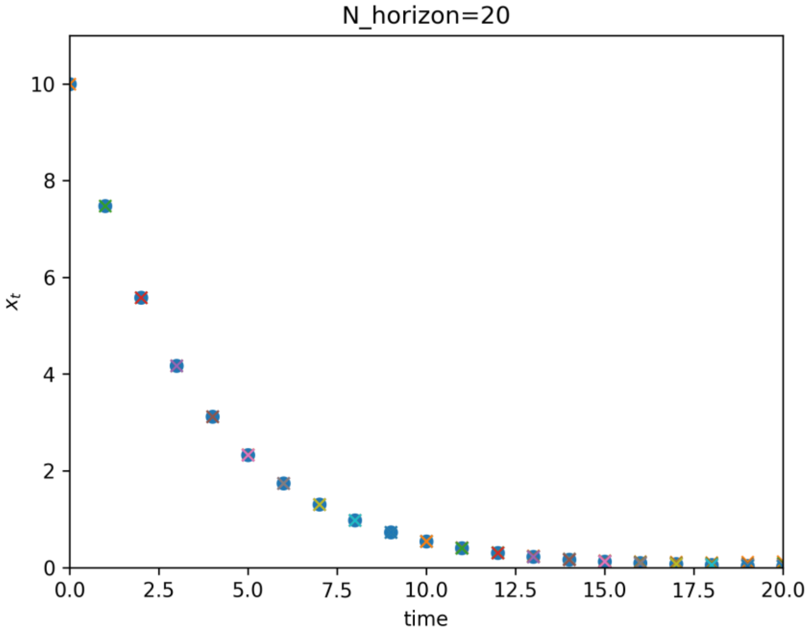
Open-Loop vs Closed-Loop: N = 10



Open-Loop vs Closed-Loop: N = 15



Open-Loop vs Closed-Loop: N = 20



MPC controller – state feedback

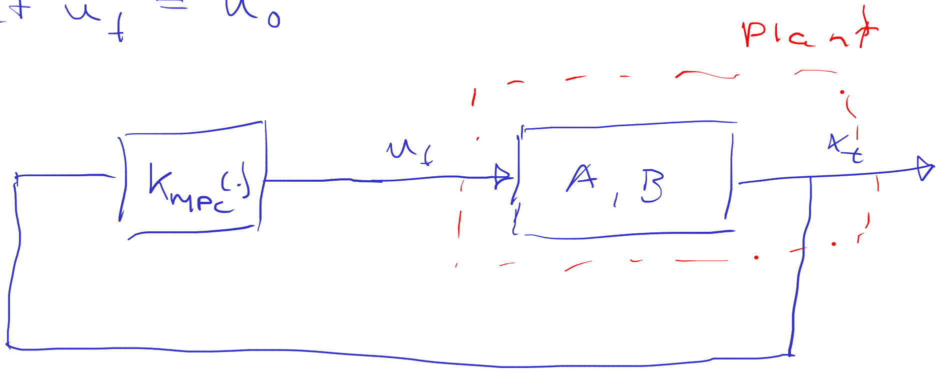
For $t = 0, 1, 2, \ldots, \infty$ do

- Determine x_t
- Solve open loop opt. problem with $x_0 = x_t$

  Obtain $u_0, u_1, u_2, \ldots$
- Set $u_t = u_0$

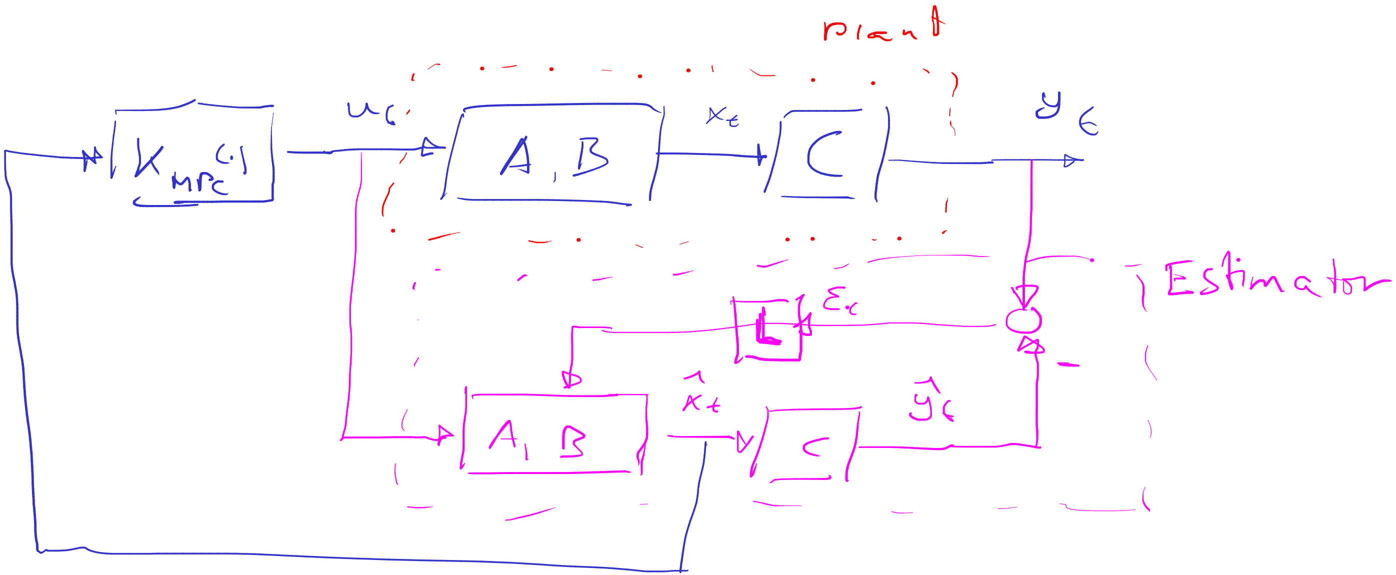
$u_t = K_{MPC}(x_t)$

end



Observation: MPC controller $K_{MPC}(\cdot)$ is a nonlinear <u>state</u> feedback

Output feedback MPC controller



Linear state estimators:

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t)$$

Reference tracking (regulation)

Typically, we want $y_t = y_{t,ref}$, $y_t = H x_t$

controlled output

Note:
Controlled output y_t
$\neq$ measured output y_t
in general.

But often $y_t = y_t$ $(H=C)$

Assume (for simplicity) that $y_{t,ref} = y_{ref} =$ const.

Steady state

$$x_s = A x_s + B u_s \Rightarrow x_s = (I-A)^{-1} B u_s$$

$$y_s = H x_s = H (I-A)^{-1} B u_s$$

Ex.

$$A = \begin{bmatrix} 0.8 & 0.4 \\ -0.1 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 & 0.5 \\ 0 & 2.0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\Rightarrow y_s = \begin{bmatrix} 3.33 & 8.33 \end{bmatrix} u_s$$

Reference tracking, cont'd

Observe:

① Input constraints limit the possible y_s:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq u_t \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 0 \leq y_s \leq 11.66$$

② Several u_s give same y_s

$$\left.\begin{aligned} u_s &= \begin{bmatrix} 0.0 \\ 0.14 \end{bmatrix} \\ u_s &= \begin{bmatrix} 0.17 \\ 0.17 \end{bmatrix} \\ u_s &= \begin{bmatrix} 0.6 \\ 0 \end{bmatrix} \end{aligned}\right\} \Rightarrow y_s = 2.0$$

Which u_s should we choose?

Reference tracking – target calculation

QP

$$\min_{x_s, u_s, \gamma} \; \frac{1}{2} u_s^T R_s u_s + \frac{1}{2} \gamma^T Q_s \gamma$$

s.t.

$$x_s = A x_s + B u_s$$

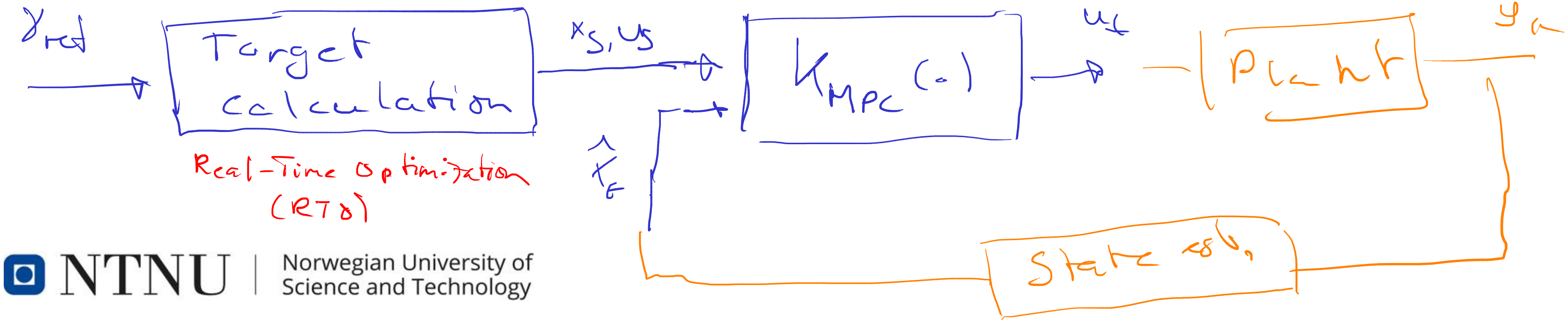
$$\to \; H x_s = y_{ref} \; \to \; -\gamma \le H x_s - y_{ref} \le \gamma, \quad \gamma \ge 0$$

$$x^{low} \le x_s \le x^{high}$$

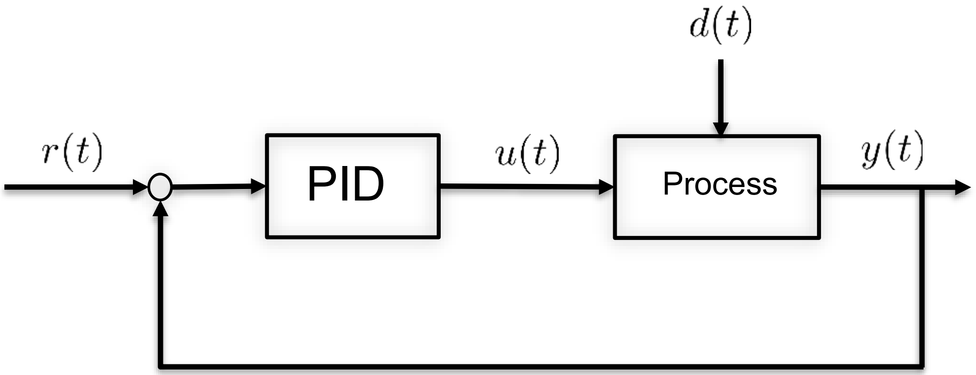
$$u^{low} \le u_s \le u^{high}$$

$$\min \sum_{t=0}^{N-1} \frac{1}{2} (y_{t+1} - H x_s)^T Q (y_{t+1} - H x_s) + \frac{1}{2} (u_t - u_s)^T R (u_t - u_s)$$

s.t. ...



Offset-free control (= “integral action”)



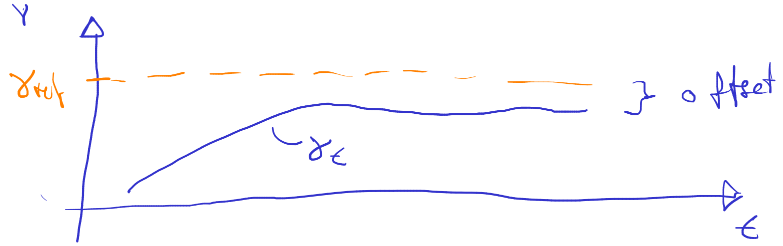
Recall role of “I” in PID control:

Removing effect of unknown (constant) disturbances

No such “integral action” in MPC so far. How can we achieve the same?

Offset-free control (= "integral action")

An unmodelled disturbance will give offset in y_t



Assume $y_t = y_t$ ($H = C$) for simplicity

Model with disturbance:

$$x_{t+1} = A x_t + B u_t + A_d d_t$$

$$y_t = C x_t + C_d d_t$$

Disturbance model

$$d_{t+1} = d_t$$

Offset-free control (= "integral action"), cont'd

Idea: Augment control model

$$(*) \quad \begin{bmatrix} x_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} A & A_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t , \quad y_t = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix}$$

Offset free MPC by:

1. Use state estimator to estimate $\hat{x}_t$ and $\hat{d}_t$
2. Use $(*)$ as control model in MPC

Note:

1. Require $\left(\begin{bmatrix} A & A_d \\ 0 & I \end{bmatrix}, \begin{bmatrix} C & C_d \end{bmatrix} \right)$ to be observable

   $\rightarrow$ Implies in practice that $\dim(d_t) \leq \dim(y_t)$

Offset-free control (= “integral action”), cont’d

Note, cont'd:

(2) Target calculation must be modified to depend on $\hat{d}_t$

(3) Industrial practice: $A_d = 0$, $C_d = I$ ("bias update")

→ do not need state estimation

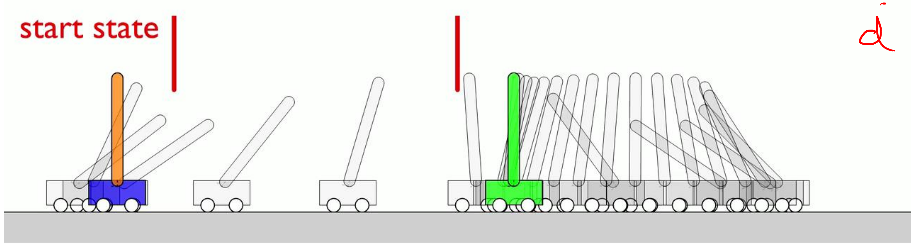
→ Often works (well), but not always

→ e.g. plants with pure integrators problematic

Offset-free MPC (or MPC with integral action)

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \begin{bmatrix} x_3\\ x_4\\ f_1(x_2,x_4)\\ f_2(x_2,x_4)\end{bmatrix} + \begin{bmatrix}0\\0\\b_1(x_2)\\b_2(x_2)\end{bmatrix}u + \begin{bmatrix}0\\0\\b_1(0)\\b_2(0)\end{bmatrix}d$$

$$\dot d = 0$$



From description:

- MPC with nonlinear model and a linear (input) disturbance model with one disturbance state: $x_t = f(x_t,u_t) + A_d\, d_t$. All states are measured ($y_t = x_t$).
- A linear observer is designed as a steady-state Kalman filter for the linearized augmented model at the final equilibrium.
- The forward-looking nature of the MPC controller allows to react to disturbances by considering obstacles in the environment and drastic replanning when necessary.

- From “Offset-free MPC explained: novelties, subtleties, and applications” - G. Pannocchia, M. Gabiccini, A. Artoni, NMPC 2015 - Seville, Spain September 17 - 20, 2015.