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TTK4135 – Lecture 12
Summing up MPC and LQ

Lecturer: Lars Imsland

Outline

- MPC example: Adaptive Cruise Control
  - MPC design with offset-free control (disturbance observer + target calculation)

- LQ-control (recap), LQG, stability and robustness, separation principle

Reference: F&H Ch. 4.5-4.6

Open-loop optimization with linear state-space model

QP

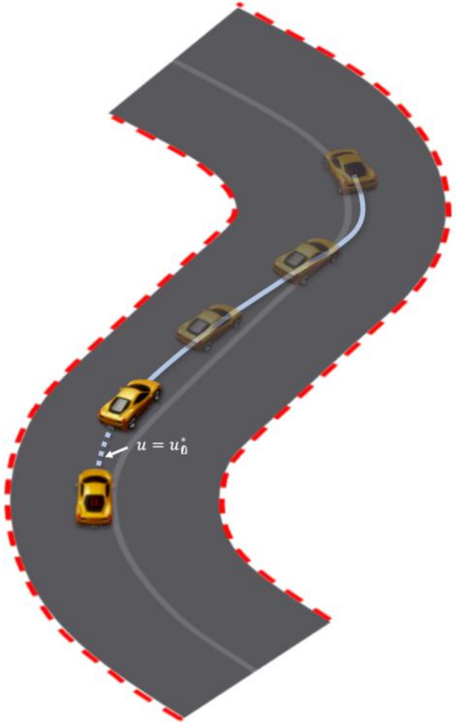
$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{x,t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{u,t} u_t + \frac{1}{2} \Delta u_t^\top S \Delta u_t$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\} \\
x^{\text{low}} &\le x_t \le x^{\text{high}}, \quad t = \{1, \dots, N\} \\
u^{\text{low}} &\le u_t \le u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
-\Delta u^{\text{high}} &\le \Delta u_t \le \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\}
\end{aligned}$$

where

$$\begin{aligned}
& x_0 \text{ and } u_{-1} \text{ is given} \\
& \Delta u_t := u_t - u_{t-1} \\
& z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
& n = N \cdot (n_x + n_u) \\
& Q_t \succeq 0 \quad t = \{1, \dots, N\} \\
& R_t \succ 0 \quad t = \{0, \dots, N-1\}
\end{aligned}$$



LQ: MPC open loop problem without inequality constraints

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t$$

subject to

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\ x_0 &= \text{given} \end{aligned}$$

where

$$\begin{aligned} z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\ n &= N \cdot (n_x + n_u) \\ Q_t &\succeq 0 \quad t = \{1, \dots, N\} \\ R_t &\succ 0 \quad t = \{0, \dots, N-1\} \end{aligned}$$

- Solution: LTV state feedback

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \\ P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

Linear quadratic control; some observations

- The optimal solution to LQ control is a linear, time-varying state feedback:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$K_t = R_t^{-1} B_t^{\top} P_{t+1} (I + B_t R_t^{-1} B_t^{\top} P_{t+1})^{-1} A_t, \qquad t = 0, \ldots, N-1$$
$$P_t = Q_t + A_t^{\top} P_{t+1} (I + B_t R_t^{-1} B_t^{\top} P_{t+1})^{-1} A_t, \qquad t = 0, \ldots, N-1$$
$$P_N = Q_N$$

- The matrix (difference) equation

$$P_t = Q_t + A_t^{\top} P_{t+1} (I + B_t R_t^{-1} B_t^{\top} P_{t+1})^{-1} A_t, \qquad t = 0, \ldots, N-1$$
$$P_N = Q_N$$

  is called the (discrete-time) *Riccati equation*.

- Note that the gain matrix K_t and the Riccati equation is independent of the states. It can therefore be computed in advance (knowing A_t, B_t, Q_t, R_t).

- Note that the "boundary condition" is given at the end of the horizon, and the P_t -matrices must be found iterating backwards in time.

Example

$$\min \sum_{t=0}^{10} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r u_t^2$$

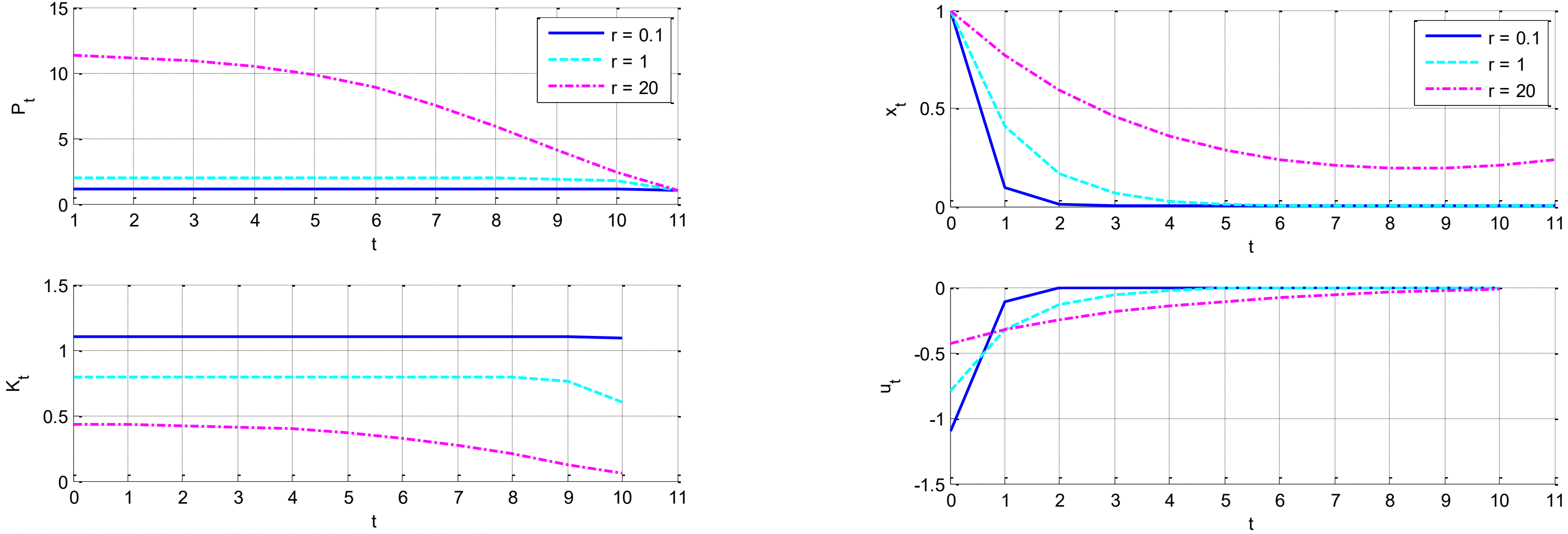
$$\text{s.t.} \quad x_{t+1} = 1.2 x_t + u_t, \quad t = 0, 1, \ldots, 10$$

LQ solution

$$P_t = 1 + \frac{1.44 r P_{t+1}}{P_{t+1} + r}, \quad t = 10, \ldots, 1$$

$$P_{11} = 1$$

$$K_t = 1.2 \frac{P_{t+1}}{P_{t+1} + r}, \quad t = 0, \ldots, 10$$



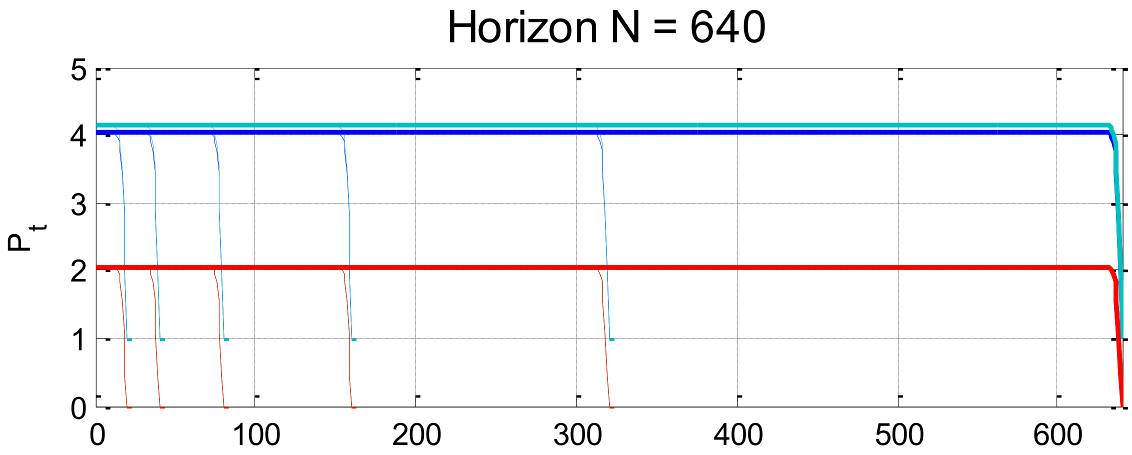
Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$



Infinite horizon LQ solution is steady-state finite horizon LQ solution!

Infinite horizon LQ (LQR – The Linear Quadratic Regulator)

$$\begin{aligned} \min_{z \in \mathbb{R}^\infty} f(z) &= \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{subject to} \quad x_{t+1} &= A x_t + B u_t, \quad t = 0, 1, \dots \\ x_0 &= \text{given} \\ Q &\succeq 0, \quad R \succ 0 \end{aligned}$$

- This has a solution provided (A,B) is stabilizable
- Then the optimal solution is the LTI state feedback

$$u_t = -K x_t$$

where the feedback gain matrix is derived by

DARE

$$K = R^{-1} B^\top P (I + B R^{-1} B^\top P)^{-1} A,$$

$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A; \quad P = P^\top \succ 0$$

- This solution is guaranteed to be closed-loop stable (eigenvalues of A-BK stable) if (A,D) is detectable, where $Q = D^T D$
- Being a state feedback solution, it implies some robustness (more on this later)

Controllability vs stabilizability
Observability vs detectability

- Stabilizable: All unstable modes are controllable
  (that is: all uncontrollable modes are stable)

- Detectability: All unstable modes are observable
  (that is: all unobservable modes are stable)

- Controllability implies stabilizability
- Observability implies detectability

LQR vs MPC

- LQR can be thought of as MPC without constraints -> solution is “linear state feedback”
  - MPC solution is “online optimization” (QP)

- Often: Constraints can be active when far from setpoint, but become irrelevant close to setpoint
  - In other words: MPC “reduces” to (same solution as) LQR when close to setpoint

- Consider double integrator example:

The double integrator, two integrators in series, discretized with sample interval T_s, can be written in state-space form as

$$\frac{y}{u}(s) = \frac{1}{s^2}$$

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix}.$$

Consider an MPC cost function with

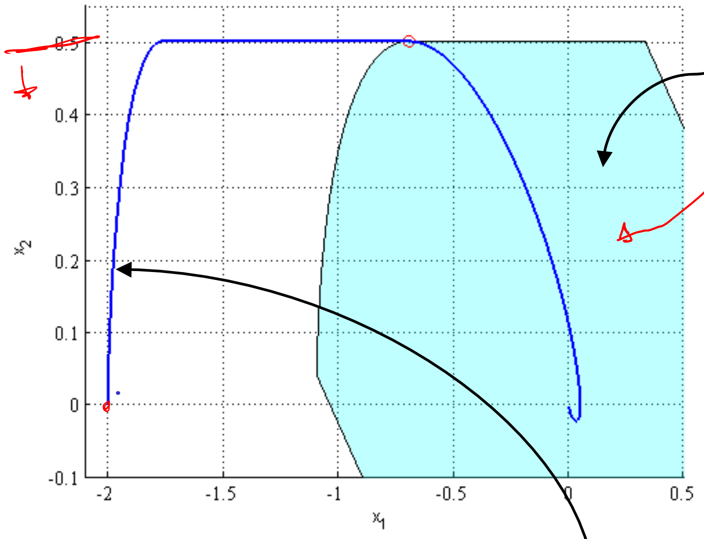
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1,$$

The constraints are $-0.5 \leq x_2 \leq 0.5$, and $-1 \leq u \leq 1$.

LQR vs MPC, II

Region (polytopic set) where LQR solution is optimal
(where we can assume problem unconstrained)

$\boxed{u_t = -K x_t}$

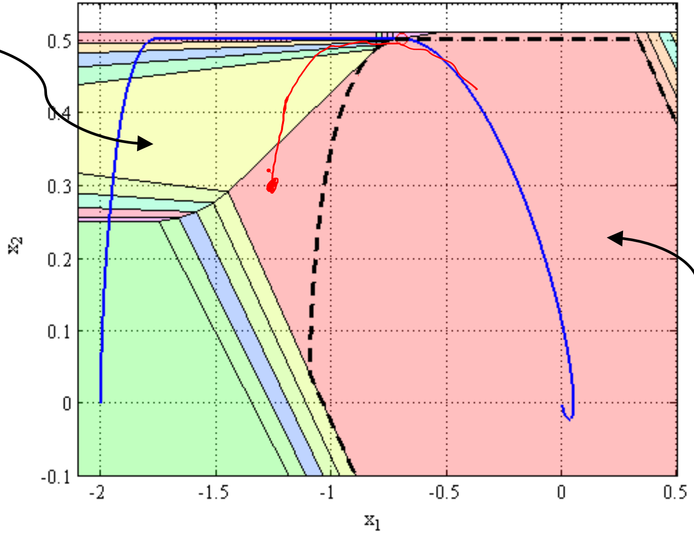


MPC solution has larger feasible region than LQR solution!

LQR vs MPC, III

- In fact, the MPC solution is piecewise linear, defined on polytopic regions

$$\boxed{u_k = K^i x_k + k^i}$$



$$u_k = -K x_k$$

- Proof/computation of this is an exercise in studying KKT conditions
  - (Not very difficult, but was not realized before ca. 2000)
  - But: solution quickly becomes very complex (many regions), except for very small systems.

Bemporad, A., Morari, M., Dua, V., Pistikopoulos, E.N.: The explicit linear quadratic regulator for constrained systems. Automatica 38(1), 3–20 (2002)
Tøndel, P., Johansen, T.A., Bemporad, A.: An algorithm for multi-parametric quadratic programming and explicit MPC solutions. Automatica 39(3), 489–497 (2003)

LQ regulator (LQR)

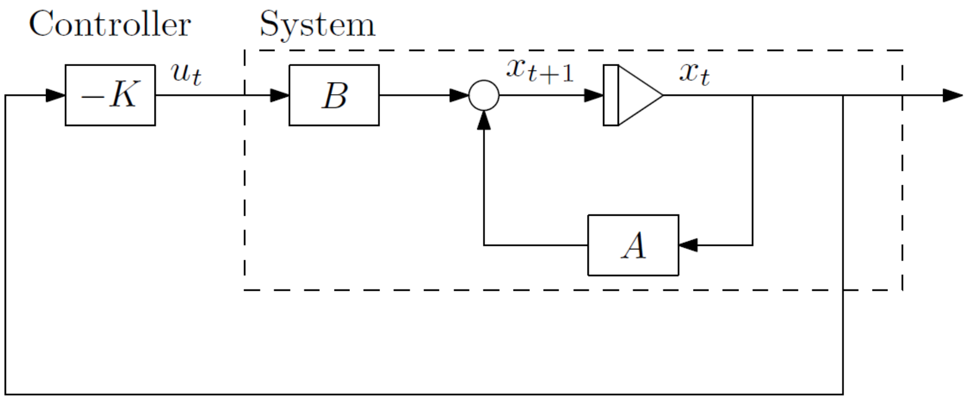


Figure 4.4: Solution of the LQ control problem, i.e., with state feedback.

LQ and robustness

- SISO LQ regulators have 60 degrees phase margin and 6dB gain margin
- Can be extended to MIMO systems

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Gain and Phase Margin for Multiloop ~~LQG~~ Regulators
LQ

MICHAEL G. SAFONOV, STUDENT MEMBER, IEEE, AND MICHAEL ATHANS, FELLOW, IEEE

Abstract—Multiloop linear-quadratic state-feedback (LQSF) regulators are shown to be robust against a variety of large dynamical linear time-invariant and memoryless nonlinear time-varying variations in open-loop dynamics. The results are interpreted in terms of the classical concepts of gain and phase margin, thus strengthening the link between classical and modern feedback theory.

measured in terms of multiloop generalizations of the classical notions of *gain and phase margin*. Like classical gain and phase margin, the present results consider robustness as an input-output property characterizing variations in open-loop transfer functions which will not

- However, usually one does not measure all the states…

Output feedback MPC

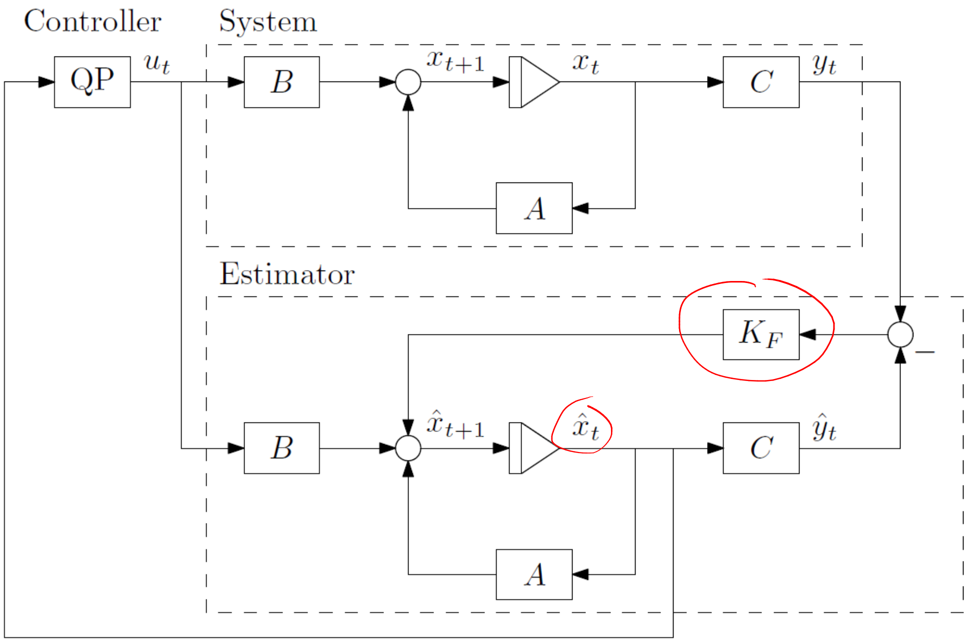


Figure 4.3: The structure of an output feedback linear MPC.

LQG: Linear Quadratic Gaussian (= LQR + KF)

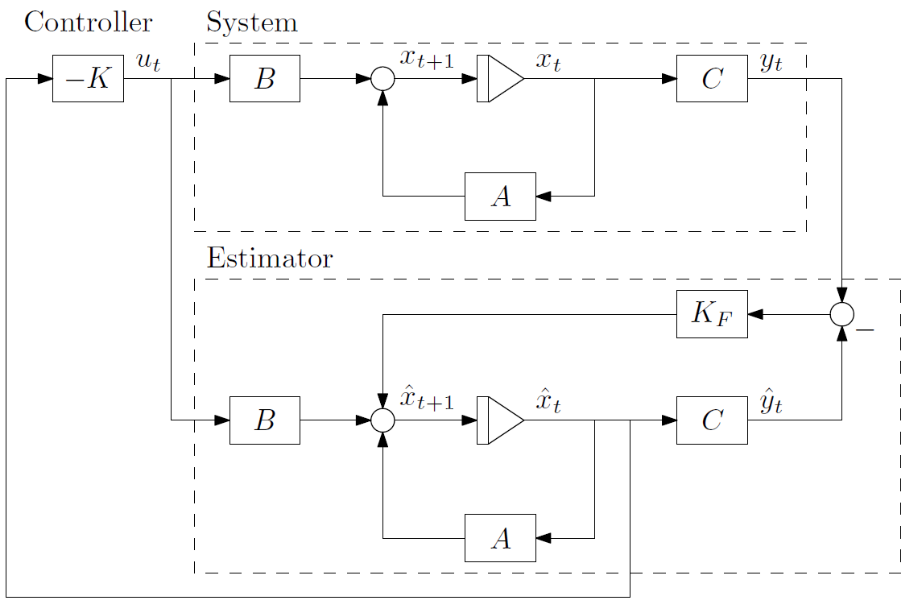


Figure 4.7: Structure of the LQG controller, i.e., output feedback LQ control.

Separation principle for linear output feedback control

$$x_{t+1} = A x_t + B u_t = A x_t - BK\hat{x}_t$$

$$\hat{x}_{t+1} = A\hat{x}_t + B u_t + K_F(y_t - \hat{y}_t)$$

$$= A\hat{x}_t - BK\hat{x}_t + K_F C x_t - K_F C \hat{x}_t$$

Define $\tilde{x}_t = x_t - \hat{x}_t$:

$$\tilde{x}_{t+1} = A x_t - \cancel{BK\hat{x}_t} - A\hat{x}_t + \cancel{BK\hat{x}_t} - K_F C x_t + K_F C \hat{x}_t$$

$$= A(x_t - \hat{x}_t) - K_F C(x_t - \hat{x}_t)$$

$$= (A - K_F C)\tilde{x}_t$$

→ Observer error dynamics

$$x_{t+1} = A x_t - BK(x_t - \tilde{x}_t) = (A - BK)x_t + BK\tilde{x}_t$$

state feedback closed-loop dynamics

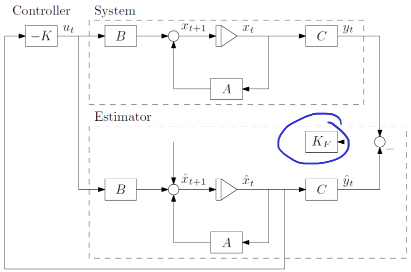


Figure 4.7: Structure of the LQG controller, i.e., output feedback LQ control.

Separation principle for linear output feedback control

Overall system:

$$\begin{bmatrix} x_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-K_F C \end{bmatrix}}_{\Psi} \begin{bmatrix} x_t \\ \tilde{x}_t \end{bmatrix}$$

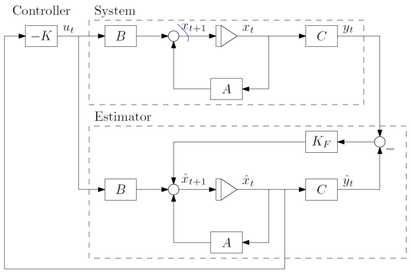


Figure 4.7: Structure of the LQG controller, i.e., output feedback LQ control.

Eigenvalues of Ψ are eigenvalues of $\underline{A-BK}$ and $\underline{A-K_F C}$.

Separation principle (linear systems):

Stable state feedback + stable observer

$\Rightarrow$ stable output feedback!

LQG and robustness

- Doyle, 1978:

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

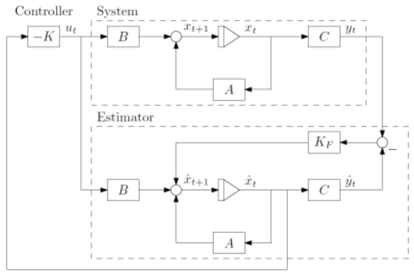


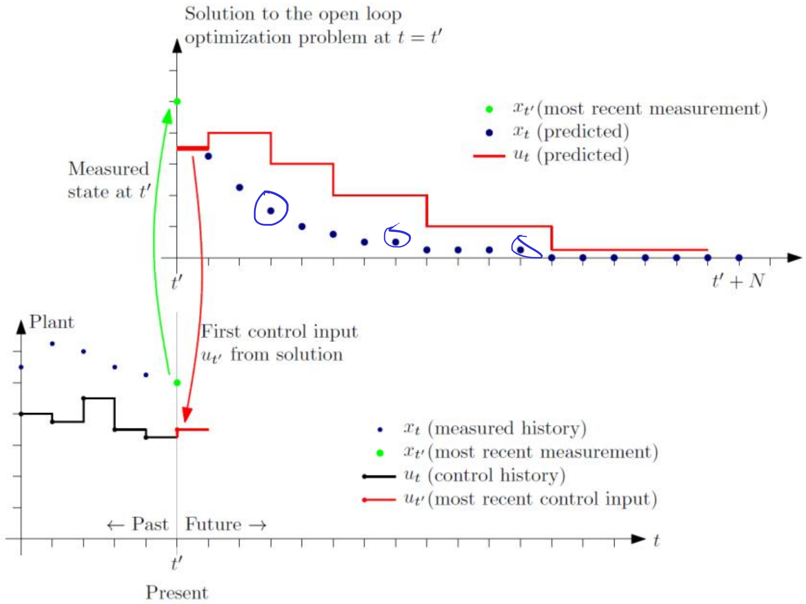
Figure 4.7: Structure of the LQG controller, i.e., output feedback LQ control.

- Lead to a lot of research in robust control in the 80's (and later), not topic of this course

Complexity reduction strategies in MPC

- Input blocking (or move blocking) – reduce number of QP variables

3-5 blocks



- "Incident points" – reduce number of QP constraints
  - Only check constraints at certain time instants, rather than at all times on horizon