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TTK4135 – Lecture 5
Solving LPs – the Simplex method

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Purpose of lecture

- Brief recap previous lecture
- The geometry of the feasible set
- Basic feasible points, “The fundamental theorem of linear programming”
- **The simplex method**
- Example 13.1
- Some implementation issues

Reference: N&W Ch.13.2-13.3, also 13.4-13.5

Linear programming, standard form and KKT: recap

LP: $\min_{x \in \mathbb{R}^n} c^T x$ subject to $\begin{cases} a_i x = b_i, & i \in \mathcal{E} \\ a_i x \geq b_i, & i \in \mathcal{I} \end{cases}$

LP, standard form: $\min_{x \in \mathbb{R}^n} c^T x$ subject to $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

Lagrangian: $\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$

KKT-conditions (LPs: necessary *and* sufficient for optimality):

$$
\begin{aligned}
A^T \lambda^* + s^* &= c, \\
Ax^* &= b, \\
x^* &\geq 0, \\
s^* &\geq 0, \\
x_i^* s_i^* &= 0, \quad i = 1, 2, \ldots, n
\end{aligned}
$$

Duality

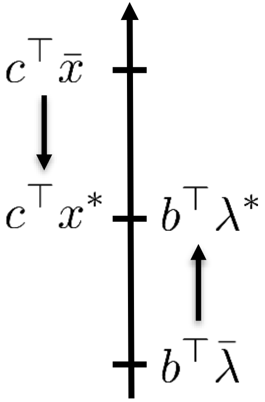
Primal problem

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{\lambda, s} \quad & b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda + s = c \\ & s \geq 0 \end{aligned}$$

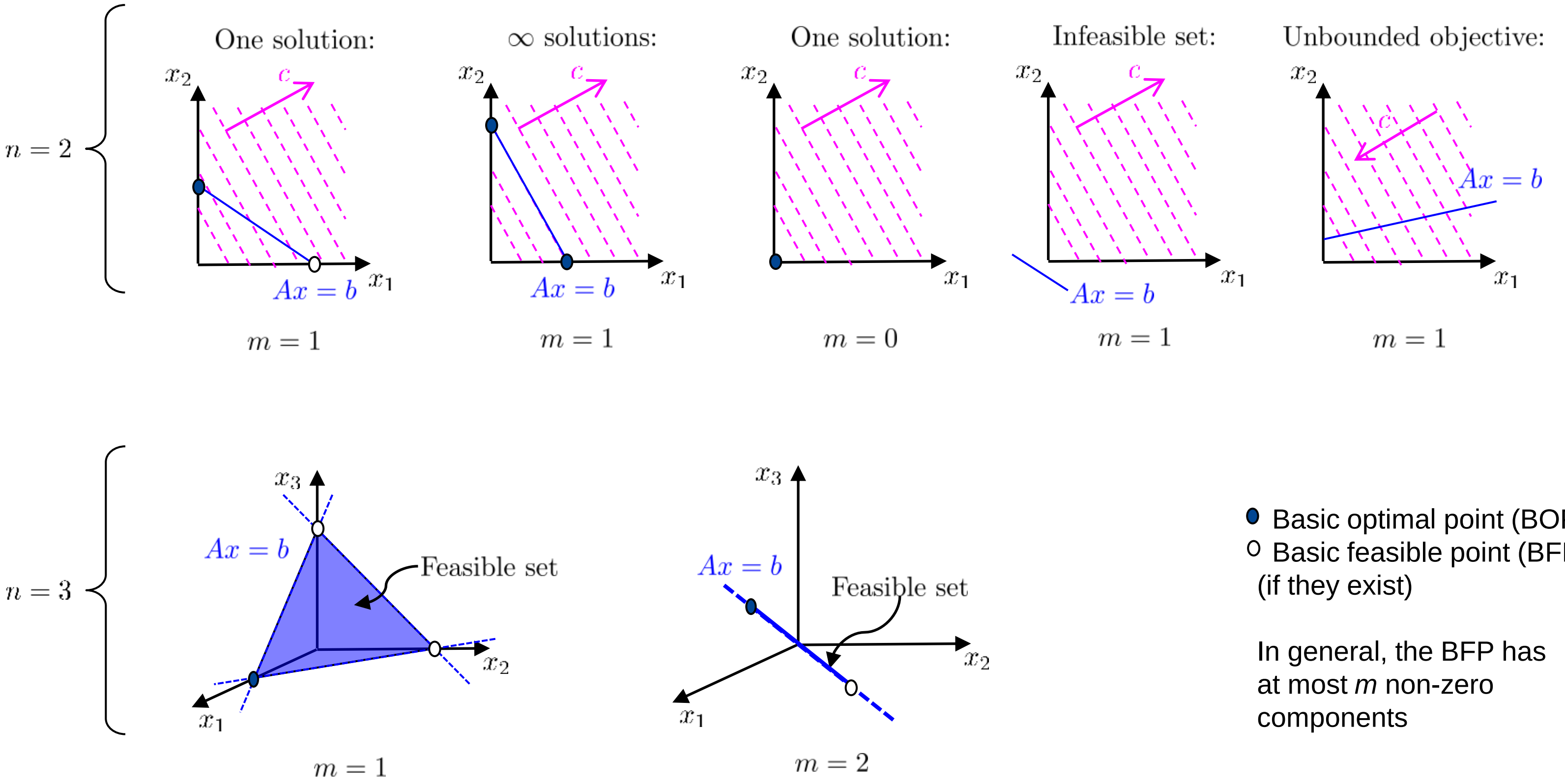
- Primal and dual have same KKT conditions!
- Equal optimal value: $c^\top x^* = b^\top \lambda^*$
- Weak duality: $c^\top \bar{x} \geq b^\top \bar{\lambda}$ ($\bar{x}, \bar{\lambda}$ feasible)
- Duality gap: $c^\top \bar{x} - b^\top \bar{\lambda}$
- Strong duality (Thm 13.1):
  - i) If primal or dual has finite solution, both are equal
  - ii) If primal or dual is unbounded, the other is infeasible



$$c^\top \bar{x} \geq c^\top x^* = b^\top \lambda^* \geq b^\top \bar{\lambda}$$

LP: Geometry of the feasible set

$$\min_x \; c^\top x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

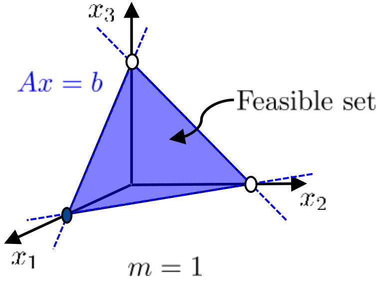


- Basic optimal point (BOP)
- Basic feasible point (BFP)

(if they exist)

In general, the BFP has at most *m* non-zero components

Basic feasible point (BFP)



$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

A point x is a basic feasible point if

- x is feasible
- There is an index set $\mathcal{B}(x) \subset \{1, \dots, n\}$ such that
  - $\mathcal{B}(x)$ contains *m* indices
  - $i \notin \mathcal{B}(x) \quad \Rightarrow \quad x_i = 0$
  - $B = [A_i]_{i \in \mathcal{B}(x)}$ is non-singular, $B \in \mathbb{R}^{m \times m}$

Always holds if A is full row rank

- $\mathcal{B}(x)$ is called a basis for the LP
- The indices not in $\mathcal{B}(x)$ are called $\mathcal{N}(x)$

Facts about Simplex method

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

The Simplex method generates iterates that are BFP, and converge to a solution if

1) there are BFPs, and
2) one of them is a solution (Basic Optimal Point, BOP)

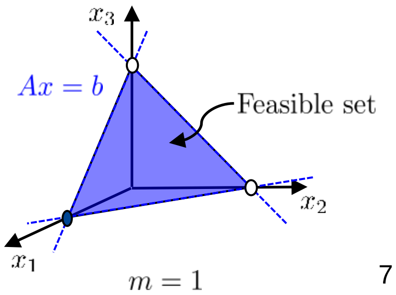
Theorem 13.2 (Fundamental theorem of Linear Programming): For standard form LP

1) If there is a feasible point, there is a BFP
2) If the LP has a solution, one solution is a BOP
3) If LP is feasible and bounded, there is a solution

Theorem 13.3: All vertices of the feasible polytope

$$\{x \mid Ax = b,\ x \geq 0\}$$

are BFPs (and all BFPs are vertices)



Facts about Simplex method, cont'd

Degeneracy: A LP is *degenerate* if there is a BFP x with $x_i = 0$ for some $i \in \mathcal{B}(x)$

Theorem 13.4: If an LP is bounded and non-degenerate, the Simplex method terminates at a BOP

LP KKT conditions (necessary&sufficient)

$$\begin{aligned} \min_{x} \quad & c^{\top} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Simplex method iterates BFPs until one that fulfills KKT is found.

$$\begin{aligned} A^T \lambda + s &= c, && \text{(KKT-1)} \\ Ax &= b, && \text{(KKT-2)} \\ x &\geq 0, && \text{(KKT-3)} \\ s &\geq 0, && \text{(KKT-4)} \\ x_i s_i &= 0, \quad i = 1, 2, \ldots, n && \text{(KKT-5)} \end{aligned}$$

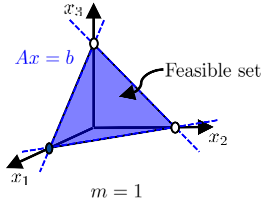
- Each step is a move from a vertex to a neighboring vertex *(one change in the basis),* that decreases the objective

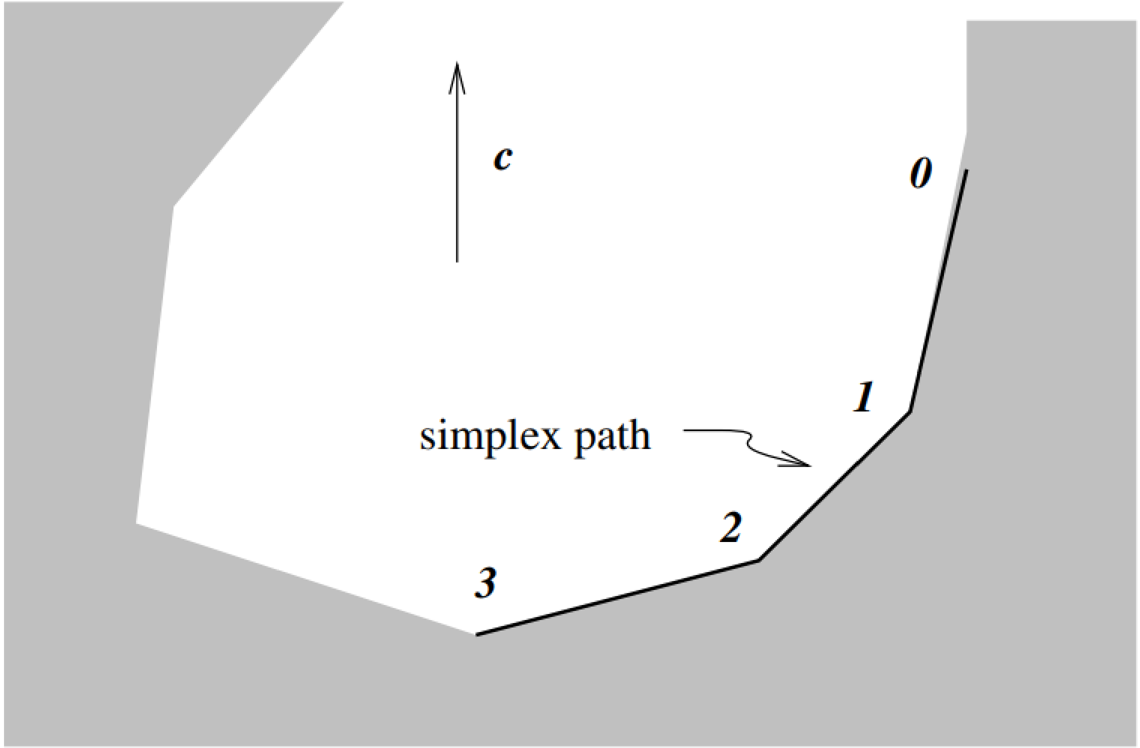
Simplex definitions

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \end{pmatrix} \begin{pmatrix} 0 \\ * \\ * \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

One step of Simplex-algorithm

$$
\begin{aligned}
A^T\lambda + s &= c, && \text{(KKT-1)}\\
Ax &= b, && \text{(KKT-2)}\\
x &\geq 0, && \text{(KKT-3)}\\
s &\geq 0, && \text{(KKT-4)}\\
x_i s_i &= 0, \quad i = 1, \ldots, n && \text{(KKT-5)}
\end{aligned}
$$




c
0
1
simplex path
2
3

Check KKT-conditions for BFP

- Given BFP x, and corresponding basis $\mathcal{B}(x)$. Define

$$\mathcal{N}(x) = \{1, 2, \dots, n\} \backslash \mathcal{B}(x)$$

- Partition x, s and c :

$$x_B = [x_i]_{i \in \mathcal{B}(x)} \qquad x_N = [x_i]_{i \in \mathcal{N}(x)}$$

- KKT conditions

  KKT-2: $Ax = Bx_B + Nx_N = Bx_B = b$ (since *x* is BFP)

  KKT-3: $x_B = B^{-1}b \geq 0, \quad x_N = 0$ (since *x* is BFP)

  KKT-5: $x^\top s = x_B^\top s_B + x_N^\top s_N = 0$ if we choose $s_B = 0$

  KKT-1: $\begin{bmatrix} B^T \\ N^T \end{bmatrix} \lambda + \begin{bmatrix} s_B \\ s_N \end{bmatrix} = \begin{bmatrix} c_B \\ c_N \end{bmatrix} \Rightarrow \begin{cases} \lambda = B^{-T} c_B \\ s_N = c_N - N^T \lambda \end{cases}$

  KKT-4: Is $s_N \geq 0$?

- If $s_N \geq 0$, then the BFP x fulfills KKT and is a solution
- If not, change basis, and try again
  - E.g. pick smallest element of s_N (index *q*), increase x_q along $Ax=b$ until x_p becomes zero.
    Move *q* from $\mathcal{N}$ to $\mathcal{B}$, and *p* from $\mathcal{B}$ to $\mathcal{N}$.
    This guarantees decrease of objective, and no “cycling” (if non-degenerate).

Ex. 13.1

Ex. 13.1, first iteration

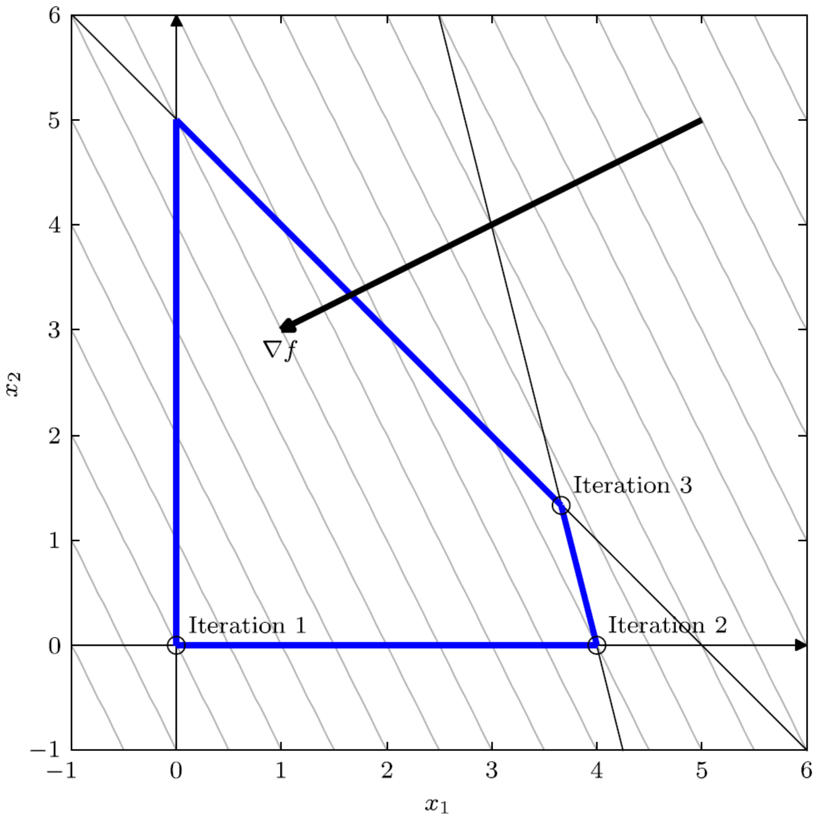
$$c^\top = \begin{pmatrix} -4 & -2 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & .5 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

Ex. 13.1, second iteration $c^\top = \begin{pmatrix} -4 & -2 & 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & .5 & 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

Ex. 13.1, third iteration

$$c^\top = \begin{pmatrix}-4 & -2 & 0 & 0\end{pmatrix}, \quad A = \begin{pmatrix}1 & 1 & 1 & 0 \\ 2 & .5 & 0 & 1\end{pmatrix}, \quad b = \begin{pmatrix}5 \\ 8\end{pmatrix}$$

Example 13.1 – figure



Linear algebra – LU factorization

- Two linear systems must be solved in each iteration:
  - $B^\top \lambda = c_B$
  - $Bd = A_q$ (to find the direction to check when increasing x_q)
  - We also have $Bx_B = b$. Since x_B is not needed in the iterations, we don't need to solve this (apart from in the final iteration)
  - This is the major work per iteration of simplex, efficiency is important!

- *B* is a general, non-singular matrix
  - Guaranteed a solution to the linear systems
  - LU factorization is the appropriate method to use (same for both systems)
  - Don't use matrix inversion!

- In each step of Simplex method, one column of *B* is replaced:
  - Can update ("maintain") the LU factorization of *B* in a smart and efficient fashion
  - No need to do a new LU factorization in each step, save time!

Starting the Simplex method

- We assumed an initial BFP available – but finding this is as difficult as solving the LP
- Normally, simplex algorithms have two phases:
  - Phase I: Find BFP
  - Phase II: Solve LP

- Phase I: Design other LP with trivial initial BFP, and whose solution is BFP for original problem

$$\min e^\top z \text{ subject to } Ax + Ez = b, \quad (x, z) \geq 0$$

$$e = (1, 1, \dots, 1)^\top, \quad E \text{ diagonal matrix with } \begin{cases} E_{jj} = 1 \text{ if } b_j \geq 0 \\ E_{jj} = -1 \text{ if } b_j < 0 \end{cases}$$

Other practical implementation issues (Ch. 13.5)

- Selection of “entering index” q
  - Dantzig’s rule: Select the index of the most negative element in s_N
  - Other rules have proved to be more efficient in practice
- Handling of degenerate bases/degenerate steps (when increasing x_q is not possible)
  - If no degeneracy, each step leads to decrease in objective and convergence in finite number of iterations is guaranteed (Thm 13.4)
  - Degenerate steps lead to no decrease in objective. Not necessarily a problem, but can lead to cycling (we end up in the same basis as before)
  - Practical algorithms use perturbation strategies to avoid this
- Presolving (Ch. 13.7)
  - Reducing the size of the problem before solving, by various tricks to eliminate variables and constraints. Size reduction can be huge. Can also detect infeasibility.

Simplex complexity

- Typically, at most *2m* to *3m* iterations
- Worst case: All vertices must be visited (exponential complexity in *n*)
- Compare interior point method: Guaranteed polynomial complexity, but in practice hard to beat simplex on many problems

Simplex – an active set method

- Active set methods (such as simplex method):
  - Are iterative algorithms: Needs a starting point, and iterates several steps before it (hopefully) ends up in a solution: a point that fulfills the KKT conditions
  - Maintains explicitly an estimate of the set of inequality constraints that are active at the solution (the set $\mathcal{N}$ for the simplex method)
  - Makes small changes to the set in each iteration (a single index in simplex)

- Next time: Active set method for QP