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TTK4135 – Lecture 18
Sequential Quadratic Programming (SQP)

Lecturer: Lars Imsland

Outline

- Recap: Newton's method for solving nonlinear equations
- Recap: Equality-constrained QPs

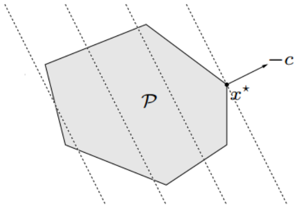
- SQP for *equality-constrained* nonlinear programming problems
  - Next time: SQP for general nonlinear programming problems

Reference: N&W Ch.18-18.1

Types of Constrained Optimization Problems

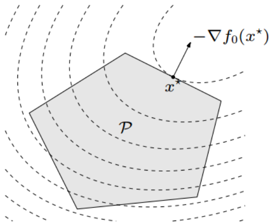
- Linear programming
  - Convex problem
  - Feasible set polyhedron

$$\begin{array}{ll} \text{min} & c^\top x \\ \text{subject to} & Ax \le b \\ & Cx = d \end{array}$$



- Quadratic programming
  - Convex problem if $P \geq 0$
  - Feasible set polyhedron

$$\begin{array}{ll} \text{min} & \frac{1}{2}x^\top P x + q^\top x \\ \text{subject to} & Ax \le b \\ & Cx = d \end{array}$$

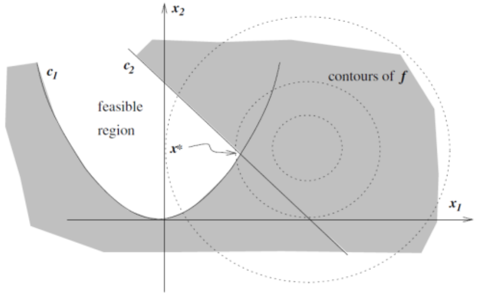


NLP

- Nonlinear programming
  - In general non-convex!

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & g(x) = 0 \\ & h(x) \ge 0 \end{array}$$

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{cases}$$

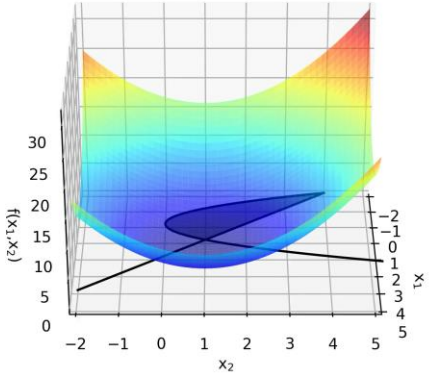


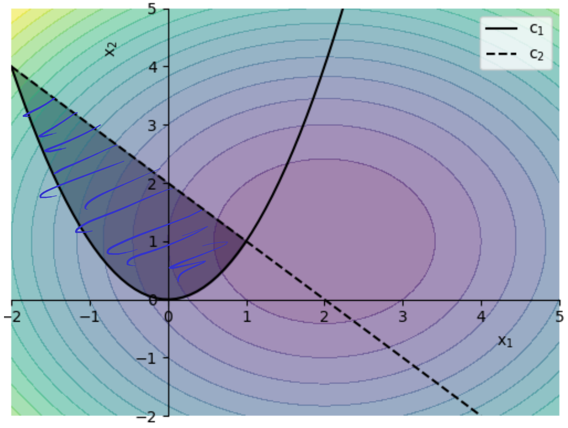
General Optimization Problem (NLP)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ \cancel{c_i(x) \geq 0}, & i \in \mathcal{I}. \end{array}$$

- Example:

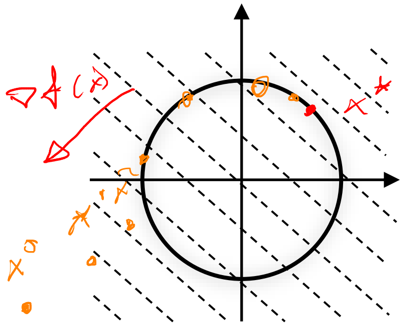
$$\min \; (x_1 - 2)^2 + (x_2 - 1)^2 \quad \text{subject to} \quad \begin{array}{l} x_1^2 - x_2 \leq 0, \\ x_1 + x_2 \leq 2. \end{array}$$





Today: Only equality constraints

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c_i(x) = 0, \quad i \in \mathcal{E}$$



Ex.: $\min_{x \in \mathbb{R}^2} -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$

Algorithm:

- initial guess x^0
- iterative

The Lagrangian

For constrained optimization problems, introduce modification of objective function:

$$\mathcal{L}(x,\lambda) = f(x) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i c_i(x)$$

- Multipliers for *equality* constrains may have both signs in a solution
- Multipliers for *inequality* constraints cannot be negative (cf. shadow prices)
- For (inequality) constraints that are *inactive*, multipliers are zero

KKT conditions (Theorem 12.1)

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

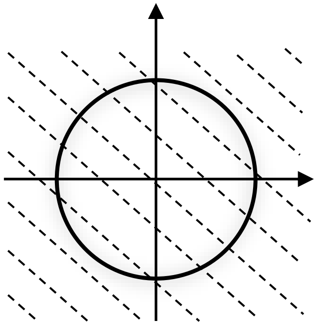
KKT conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && \text{(stationarity)} \\
c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, && \\
c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(primal feasibility)} \\
\lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(dual feasibility)} \\
\lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. && \text{(complementarity condition/ complementary slackness)}
\end{aligned}$$

Either $\lambda_i^* = 0$ *or* $c_i(x^*) = 0$

Example KKT system

$$\min_{x\in\mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\mathcal{L}(x,\lambda) = -x_1 - x_2 - \lambda_1 (x_1^2 + x_2^2 - 1)$$

KKT :

$$\left.\boxed{\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -1 - 2\lambda_1 x_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= -1 - 2\lambda_1 x_2 = 0 \\ x_1^2 + x_2^2 - 1 &= 0 \end{aligned}}\right\} F(x,\lambda) = 0$$

$$F(x,\lambda) = \begin{pmatrix} -1 - 2\lambda_1 x_1 \\ -1 - 2\lambda_1 x_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix} = 0$$

Today: Equality-constrained NLP

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c(x) = 0, \qquad c(x) = \begin{bmatrix} c_1(x) \\ \vdots \\ c_m(x) \end{bmatrix}$$

$$\mathcal{L}(x, \lambda) = f(x) - \lambda^T c(x), \qquad \lambda = (\lambda_1, \dots, \lambda_m)^T$$

KKT:

$$\boxed{\begin{aligned} \nabla_x \mathcal{L}(x, \lambda) &= \nabla f(x) - A(x)^T \lambda = 0, \\ c(x) &= 0 \end{aligned}}$$

$$A(x) = \begin{bmatrix} \nabla c_1(x)^T \\ \vdots \\ \nabla c_m(x)^T \end{bmatrix}$$

Jacobian of $c(x)$

$$\lambda^T c(x) = \sum_{i=1}^{m} \lambda_i c_i(x)$$

$$\nabla_x (\lambda^T c(x)) = \sum_{i=1}^{m} \lambda_i \nabla c_i(x) = \underbrace{\begin{bmatrix} \nabla c_1(x) & \dots & \nabla c_m(x) \end{bmatrix}}_{A(x)^T} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}}_{\lambda}$$

Today: Equality-constrained NLP

KKT:

$$F(x,\lambda) = \begin{bmatrix} \nabla f(x) - A(x)^T \lambda \\ c(x) \end{bmatrix} = 0$$

Observe:

- This is a set of nonlin. eq. in x and λ
- The solution is a "candidate solution" to opt. prob.
- Excellent method for solving $F(x,\lambda) = 0$:

Newton's method

Newton’s method for solving nonlinear equations (Ch. 11)

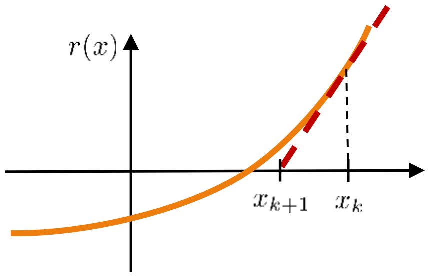
- Solve equation system $r(x) = 0, \quad r(x) : \mathbb{R}^n \to \mathbb{R}^n$
- Assume Jacobian $J(x) \in \mathbb{R}^{n \times n}$ exists and is continuous
- Taylor: $r(x+p) = r(x) + J(x)p + O(\|p\|^2)$

$$J(x) = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \cdots \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Algorithm 11.1 (Newton’s Method for Nonlinear Equations).
Choose x_0;
for $k = 0, 1, 2, \ldots$
Calculate a solution p_k to the Newton equations

$$J(x_k)p_k = -r(x_k);$$

$x_{k+1} \leftarrow x_k + p_k$;
end (for)



- Convergence rate (Thm 11.2): Quadratic convergence if $J(x)$ is invertible (quadratic convergence is very good, but only holds close to the solution)

Newton's method to solve $\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}$

$$F(x,\lambda) = \begin{pmatrix} \nabla f(x) - A^\top(x)\lambda \\ c(x) \end{pmatrix}$$

Jacobian of

Algorithm:

Given x_0, λ_0; $k = 0$

While $\|F(x,\lambda)\| > \varepsilon$,

Solve
$$\underbrace{\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x_k,\lambda_k) & -A(x_k)^T \\ A(x_k) & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_\lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) + A(x_k)^T\lambda \\ -c(x_k) \end{pmatrix}}_{\text{KKT system}}$$

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} + \begin{pmatrix} p_k \\ p_\lambda \end{pmatrix}$$

$k = k+1$

end

KKT matrix

Newton's method to solve $\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}$

$$F(x,\lambda) = \begin{pmatrix} \nabla f(x) - A^\top(x)\lambda \\ c(x) \end{pmatrix}$$

(excellent method when starting close to (x^*, λ^*))

Assumption 18.1 / Thm 16.2 : KKT-system has a solution if

a) $A(x_k)$ has full row rank (LICQ)

b) $d^T \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k) d > 0$ for all $d \neq 0$ s.t. $A(x_k) d = 0$

$\hookrightarrow$ pos. def. on tangent space of constraints

b) holds close to a solution which satisfies 2nd order sufficient conditions.

Equality-constrained QP (EQP)

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^\top Gx + c^\top x$$
$$\text{subject to} \quad Ax = b, \quad A \in \mathbb{R}^{m\times n}$$

Basic assumption:
A full row rank

- KKT-conditions (KKT system, KKT matrix):

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \quad \text{or, if we let } x^* = x + p, \quad \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

- Solvable when $Z^\top GZ > 0$ (columns of *Z* basis for nullspace of *A*)

- That is: QP with only equality constraints is solved by a solving a set of linear equations

Alternative "derivation" of KKT-system

$$(*) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c(x) = 0$$

Approximate $(*)$ at (x_k, λ_k) by EQP:

$$\min_p \; f(x_k) + \nabla f(x_k)^T p + \frac{1}{2} p^T \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k) p \approx f(x_k + p)$$

$$\text{s.t.} \; c(x_k) + A(x_k) p = 0 \qquad \approx c(x_k + p) = 0$$

Lagrangian of approximation:

$$\bar{\mathcal{L}}(p, l) = f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2_{xx} \mathcal{L}_k p - l^T (c(x_k) + A(x_k) p)$$

KKT for approximation:

$$\boxed{\begin{aligned} \nabla_p \bar{\mathcal{L}}(p, l) &= \nabla f_k + \nabla^2_{xx} \mathcal{L}_k p - A(x_k)^T l = 0 \\ c(x_k) &+ A(x_k) p = 0 \end{aligned}}$$

Alternative "derivation" of KKT-system, cont'd

KKT-system:

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L}_k & -A^T(x_k) \\ A(x_k) & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \ell_k \end{bmatrix} = \begin{bmatrix} -\nabla f_k \\ -c(x_k) \end{bmatrix}$$

Equivalent!

From Newton's method:
$$\underbrace{\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x_k,\lambda_k) & -A^\top(x_k) \\ A(x_k) & 0 \end{pmatrix}}_{\text{Jacobian of } F(x,\lambda) \text{ at } (x_k,\,\lambda_k)} \begin{pmatrix} p_k \\ p_{\lambda_k} \end{pmatrix} = \underbrace{\begin{pmatrix} -\nabla f(x_k) + A^\top(x_k)\lambda_k \\ -c(x_k) \end{pmatrix}}_{-F(x_k,\lambda_k)}$$

Subtract $A^T(x_k)\lambda_k$ from first row of

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}_k & -A^T(x_k) \\ A(x_k) & 0 \end{pmatrix} \begin{pmatrix} p_k \\ \lambda_k + p_\lambda \end{pmatrix} = \begin{bmatrix} -\nabla f_k \\ -c(x_k) \end{bmatrix}$$

$$\ell_k = \lambda_k + p_\lambda = \lambda_{k+1}$$

We see that one iteration of algorithm has two interpretations:

1. Newton's method to solve KKT of NLP
   - Analysis: Method has quadratic convergence
2. Sequentially solving QP approximations of NLP
   - Extension to inequalities
   - Practical implementation: Use QP-solvers

$\min_x f(x)$ s.t. $c(x)=0$

Algorithm for local NLP:

Given x_0, λ_0; $k=0$

While not converged,

solve QP: $\min_p f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2_{xx} \mathcal{L}_k p$

s.t. $\nabla c_i(x_k)^T p + c_i(x_k) = 0, \; i \in \mathcal{E}$

$\nabla c_i(x_k)^T p + c_i(x_k) \geq 0, \; i \in \mathcal{I}$

$x_{k+1} = x_k + p_k$

end

Local SQP-algorithm for solving equality-constrained NLPs

$$\min f(x)$$
$$\text{subject to } c(x) = 0$$

Algorithm 18.1 (Local SQP Algorithm for solving (18.1)).

Choose an initial pair (x_0, λ_0); set $k \leftarrow 0$;

repeat until a convergence test is satisfied

Evaluate f_k, ∇f_k, $\nabla^2_{xx}\mathcal{L}_k$, c_k, and A_k;

Solve (18.7) to obtain p_k and l_k;

Set $x_{k+1} \leftarrow x_k + p_k$ and $\lambda_{k+1} \leftarrow l_k$;

end (repeat)

EQP:

$$\min_p \quad f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla^2_{xx}\mathcal{L}_k p$$
$$\text{subject to} \quad A_k p + c_k = 0.$$

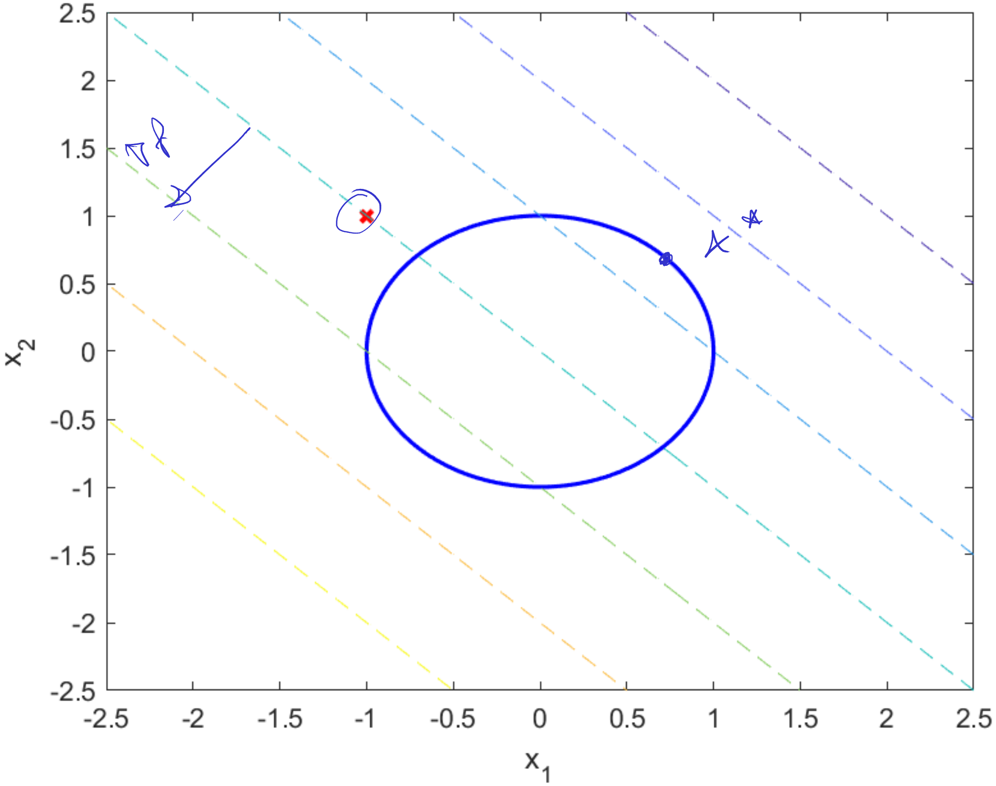
$$\min_{x\in\mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$

```
% min -x1 - x2 s.t. x1^2 + x2^2 = 1

f = @(x) - x(1) - x(2);
df = @(x) [-1; -1];

c = @(x) x(1)^2 + x(2)^2 - 1;
A = @(x) [2*x(1), 2*x(2)];

HL = @(x,lambda) diag([- 2*lambda, -2*lambda]);

x0 = [-1;1];lambda0 = -1;
x(:,1) = x0; lambda(1,:) = lambda0;

for i = 1:10,
    [p,fval,exitflag,output,lo] = quadprog(HL(x(:,i),lambda(i)),df(x(:,i))',[],[], A(x(:,i)),-c(x(:,i)));
    l = -lo.eqlin;

    % z = [ HL(x(:,i),lambda(i)), -A(x(:,i))'; A(x(:,i)), 0] \ [-df(x(:,i)); -c(x(:,i))];
    % p = z(1:2);
    % l = z(3);

    x(:,i+1) = x(:,i) + p;
    lambda(:,i+1) = l;
end
```

$$\min_p \; \cancel{f_k} + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla^2_{xx}\mathcal{L}_k p$$

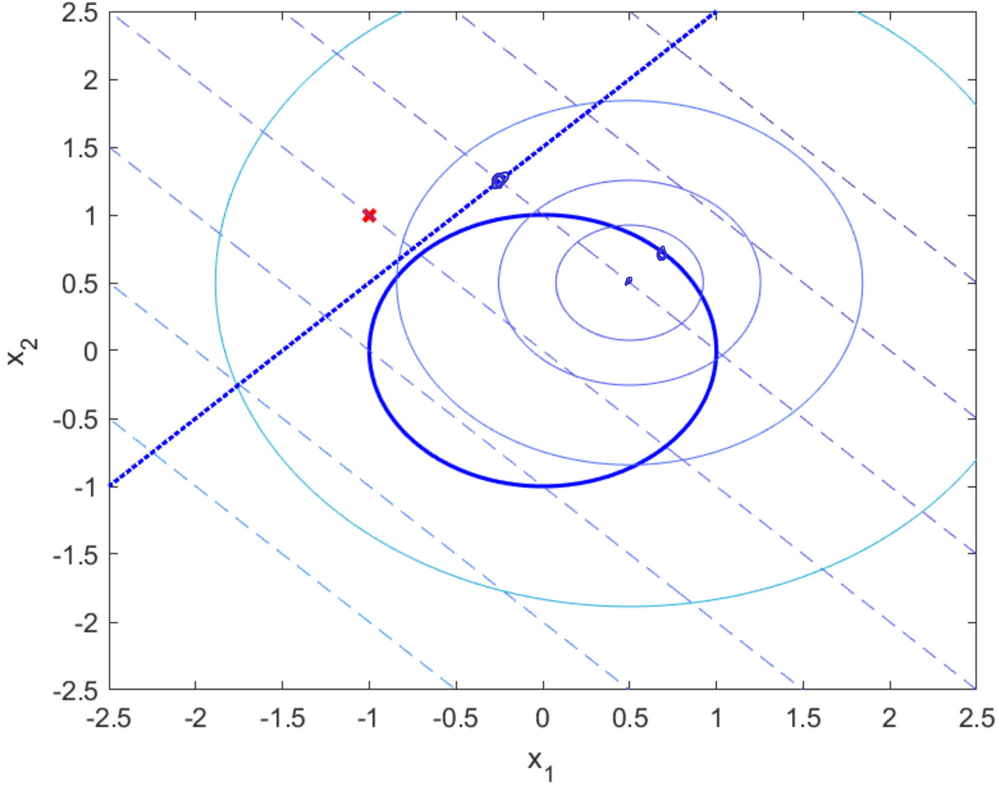
$$\text{subject to} \quad A_k p + c_k = 0.$$

$$\min_{x \in \mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



Iteration 1

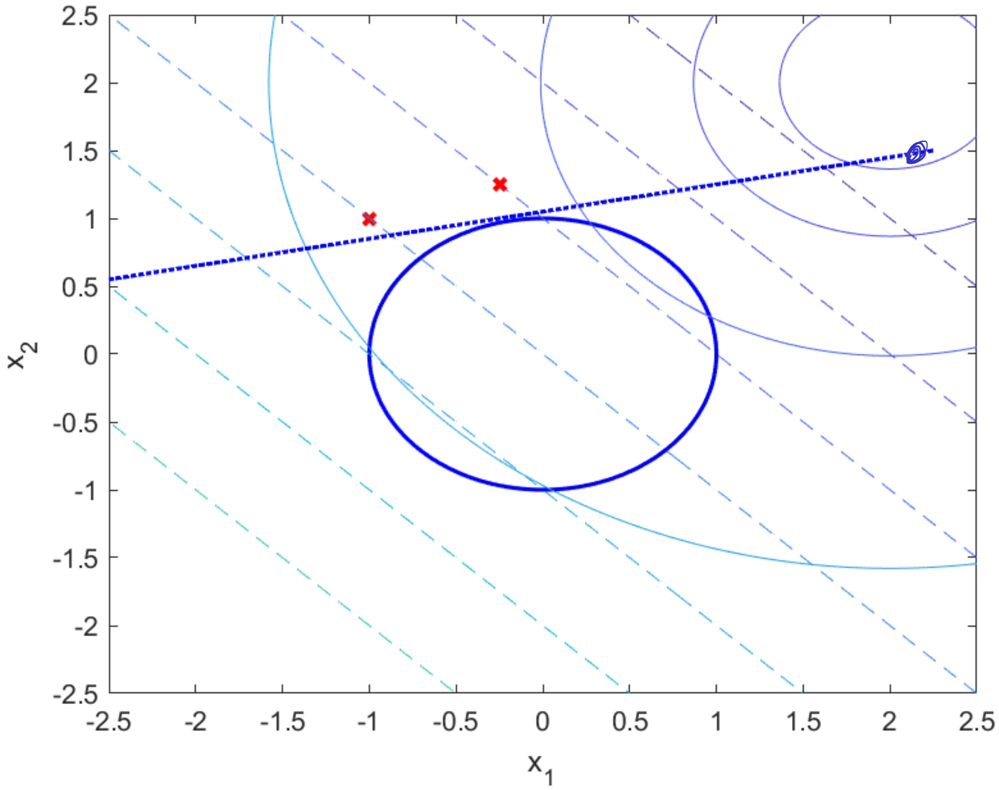
$$\min_{x \in \mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

Iteration 1

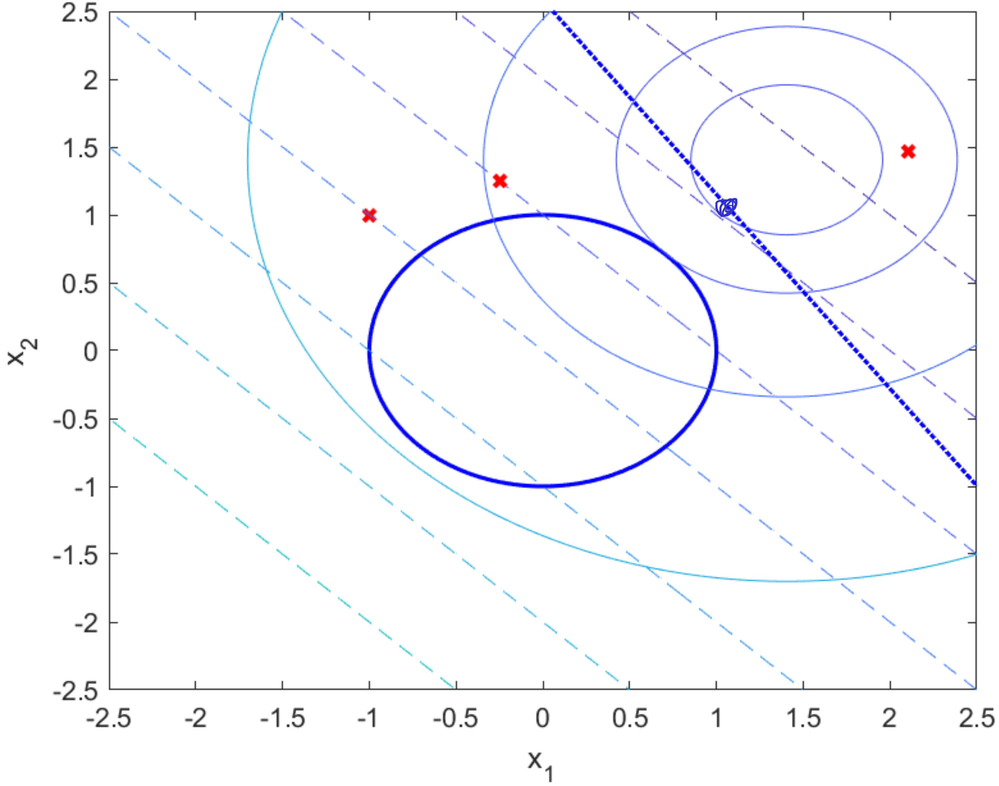
$$\min_{x \in \mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

Iteration 3

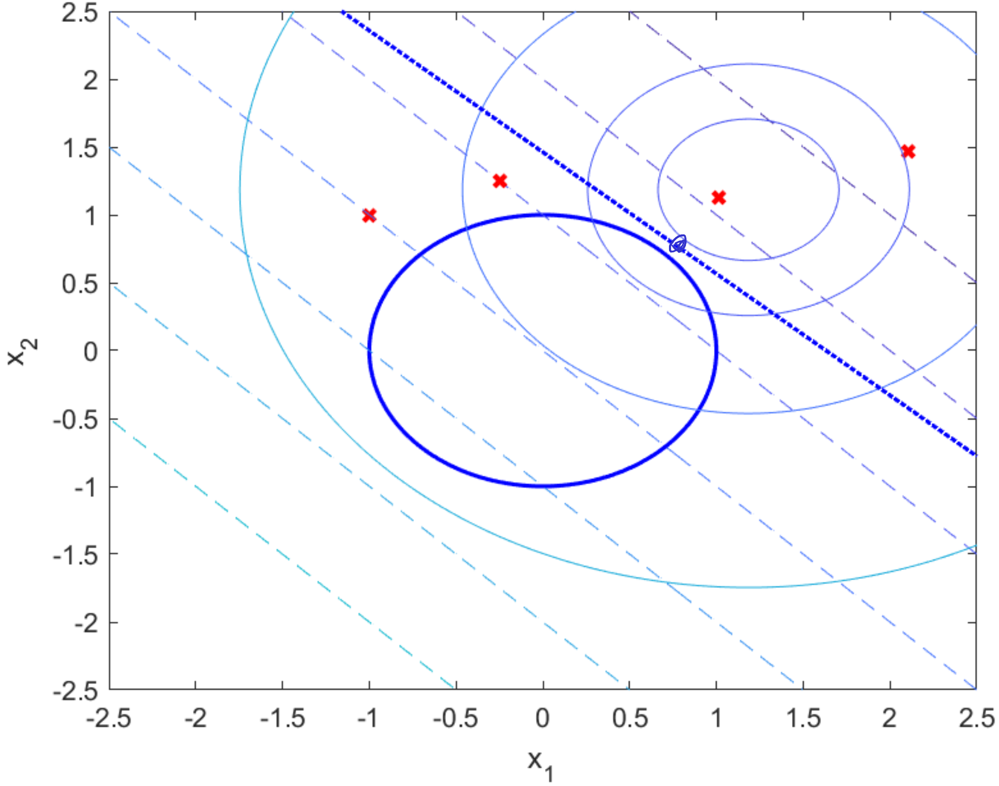
$$\min_{x \in \mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

Iteration 4

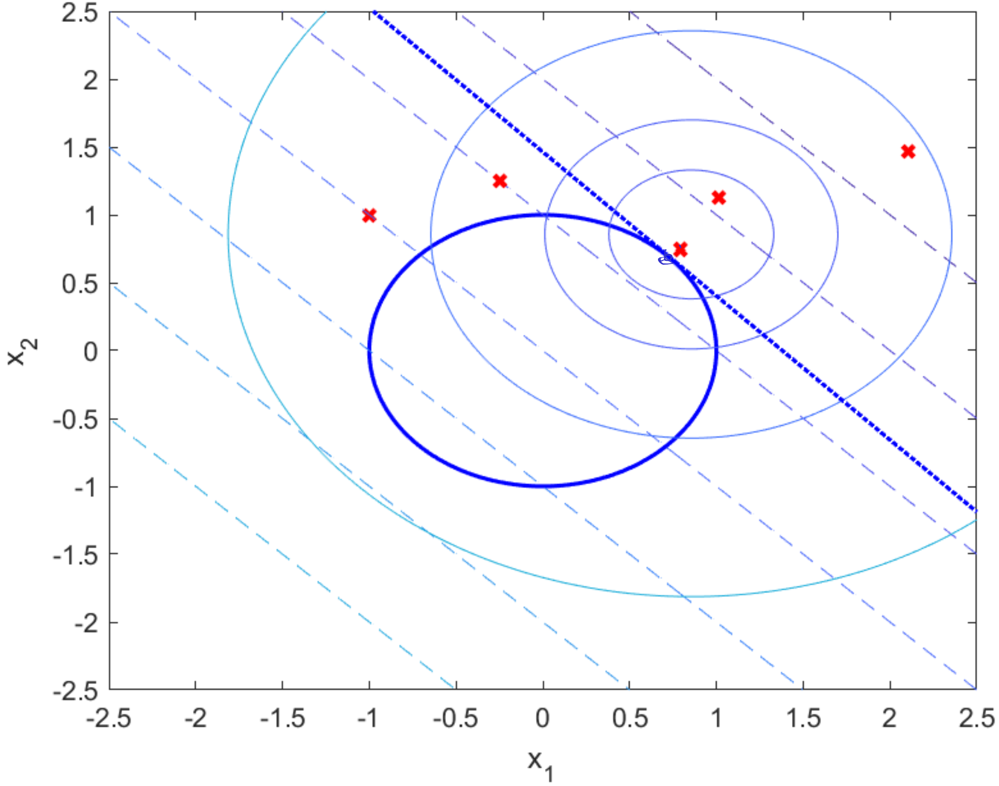
$$\min_{x\in\mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

Iteration 5

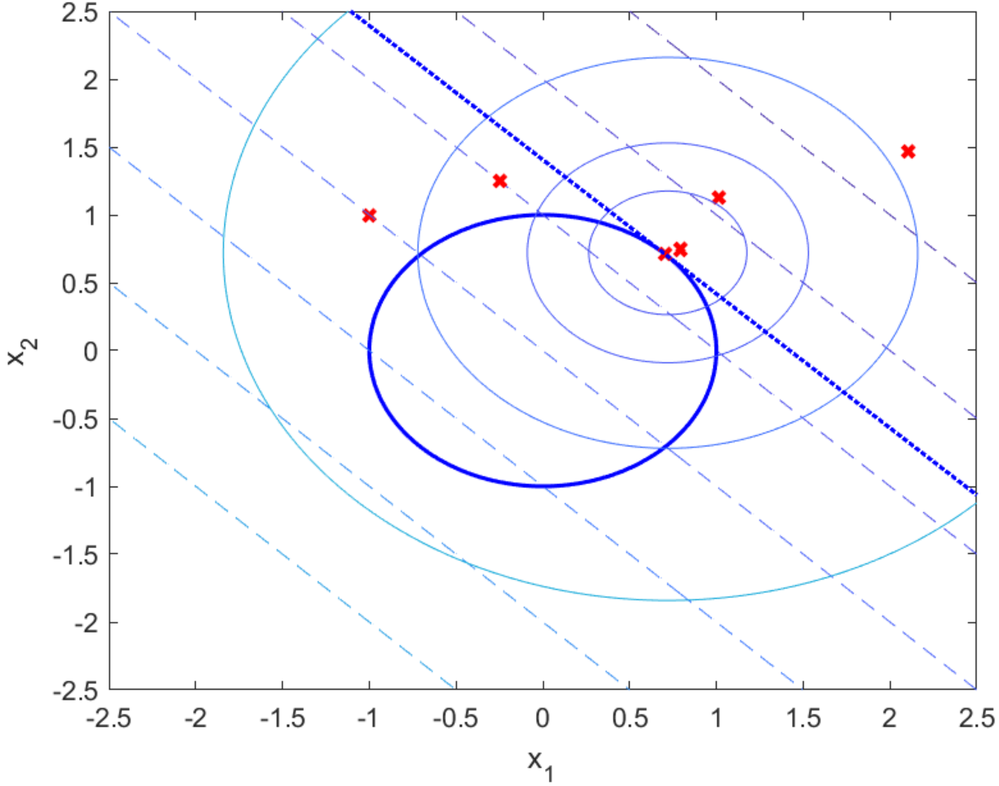
$$\min_{x\in\mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

Iteration 6

$$\min_{x\in\mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



$$\begin{aligned} \min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\ \text{subject to} \quad & A_k p + c_k = 0. \end{aligned}$$

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QP approximation can be seen as approximation of Lagrangian

$$\min_{p} \quad f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla^2_{xx} \mathcal{L}_k p$$

$$\text{subject to} \quad A_k p + c_k = 0.$$

Note: $f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2_{xx} \mathcal{L}_k p$

$$= f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla_{xx} \mathcal{L}_k p + \lambda_k^T \overbrace{(A_k p + c_k)}^{=0}$$

$$= \mathcal{L}(x_k, \lambda_k) + \nabla_x \mathcal{L}(x_k, \lambda_k)^T p + \frac{1}{2} p^T \nabla_{xx} \mathcal{L}_k p$$

$$\approx \mathcal{L}(x_k + p, \lambda_k)$$