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TTK4135 – Lecture 7
Active Set Method for Quadratic programming

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Overview of lecture

- Quadratic programming – used for control (MPC), in finance, …
- Recap last time – Equality-constrained QPs (EQPs)
- **Active set method for solving QPs**
  - For medium-sized problems – for large problems, interior point methods may be faster (not part of this course)
- Example 16.4

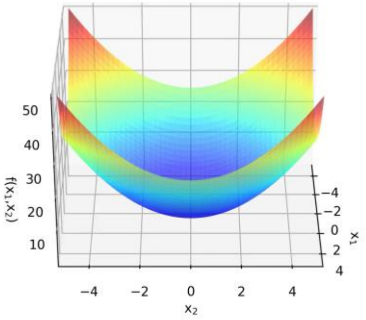
Reference: N&W Ch.15.3-15.5, **16.1-2,4-5**

Quadratic programming

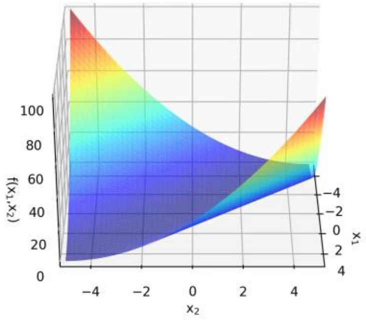
Solving (convex) quadratic programs, QPs

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

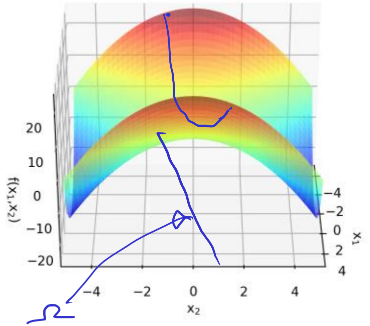
- Feasible set convex (as for LPs)
- The QP is convex if $G \geq 0$ (strictly convex if $G > 0$)



Strictly convex
$x^T P x = x_1^2 + x_2^2$



Convex
$x^T P x = x_1^2 + 2x_1x_2 + x_2^2$



Non-convex
$x^T P x = x_1^2 - x_2^2$

Equality-constrained QP (EQP)

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{2}x^\top Gx + c^\top x$$
$$\text{subject to} \quad Ax = b, \quad A \in \mathbb{R}^{m\times n}$$

Basic assumption:
A full row rank

- KKT-conditions (KKT system, KKT matrix):

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \quad \text{or, if we let } x^* = x + p, \quad \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

$G > 0$

- Solvable when $Z^\top GZ > 0$ (columns of *Z* basis for nullspace of *A*):

$$Z^\top GZ > 0 \overset{\text{Lemma 16.1}}{\Rightarrow} K = \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \text{ non-singular} \Rightarrow \begin{pmatrix} x^* = x + p \\ \lambda^* \end{pmatrix} \text{ unique solution of KKT system}$$

$$\overset{\text{Theorem 16.2}}{\Rightarrow} x^* \text{ is the unique solution to EQP}$$

- How to solve KKT system (KKT matrix indefinite, but symmetric):
  - Full-space: Symmetric indefinite (LDL) factorization: $P^\top KP = LBL^\top$
  - Reduced-space: Use *Ax=b* to eliminate *m* variables. Requires computation of *Z*, which can be costly. Reduced space method faster than full-space when many constraints (if *n-m* ≪ *n*).

Active set method for QPs, simplified

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

1. Make a guess of which constraints are active at the optimal solution
2. Solve corresponding EQP
3. Check KKT-conditions
   1. IF KKT OK, then finished
   2. If not, update guess of active constraints in smart way, go to 2.

KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

Lagrangian: $$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, & & \text{(stationarity)} \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, & & \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, & & \text{(primal feasibility)} \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, & & \text{(dual feasibility)} \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. & & \text{(complementarity condition/ complementary slackness)} \end{aligned}$$

KKT for QP

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

$f(x)$

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. \end{aligned}$$

$$\mathcal{L}(x, \lambda) = \frac{1}{2} x^T G x + c^T x - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^T x - b_i)$$

KKT:

$$G x^* + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i a_i = 0$$

$$a_i^T x^* = b_i, \quad i \in \mathcal{E}$$

$$a_i^T x^* \geq b_i, \quad i \in \mathcal{I}$$

$$\lambda_i^* \geq 0, \quad i \in \mathcal{I}$$

$$\rightarrow \lambda_i^* \cdot (a_i^T x^* - b_i) = 0, \quad i \in \mathcal{I} \cup \mathcal{E}$$

Alternative formulation:

$$\mathcal{A}(x^*) = \{ i \in \mathcal{E} \cup \mathcal{I} \mid a_i^T x^* = b_i \}$$

KKT:

$$\rightarrow \overbrace{G x^* + c}^{g} - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0$$

$$a_i^T x^* = b_i, \quad i \in \mathcal{A}(x^*)$$

$$a_i^T x^* \geq b_i, \quad i \in \mathcal{I} \setminus \mathcal{A}(x^*)$$

$$\rightarrow \lambda_i^* \geq 0, \quad i \in \mathcal{I} \cap \mathcal{A}(x^*)$$

Theorem 16.4: If x^* satisfies KKT and $G \geq 0$, then x^* is a global solution.

Proof: Assume x feasible, $x \neq x^*$.

Note: $(x-x^*)^T(Gx^*+c) = (x-x^*)^T \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = \sum_{i \in \mathcal{E}} \lambda_i^* \underbrace{\underbrace{a_i^T x}_{b} - \underbrace{a_i^T x^*}_{b}}_{0} + \sum_{i \in \mathcal{A}(x^*) \cap \mathcal{I}} \lambda_i^* \underbrace{\underbrace{a_i^T x}_{\geq b} - \underbrace{a_i^T x^*}_{= b}}_{\geq 0} \geq 0$

$$q(x) = \frac{1}{2}(x^* + (x-x^*))^T G(x^* + (x-x^*)) + c^T(x^* + (x-x^*))$$

$$= \frac{1}{2} x^{*T} G x^* + c^T x^* + \frac{1}{2}(x-x^*)^T G (x-x^*) + (x-x^*)^T G x^* + c^T(x-x^*)$$

$$= q(x^*) + \underbrace{\frac{1}{2}(x-x^*)^T G (x-x^*)}_{\geq 0} + \underbrace{(x-x^*)^T(Gx^*+c)}_{\geq 0}$$

$$\geq q(x^*) \qquad \square$$

$G > 0 \rightarrow z^T G z > 0 \Rightarrow x^*$ is strict global solution

Degeneracy

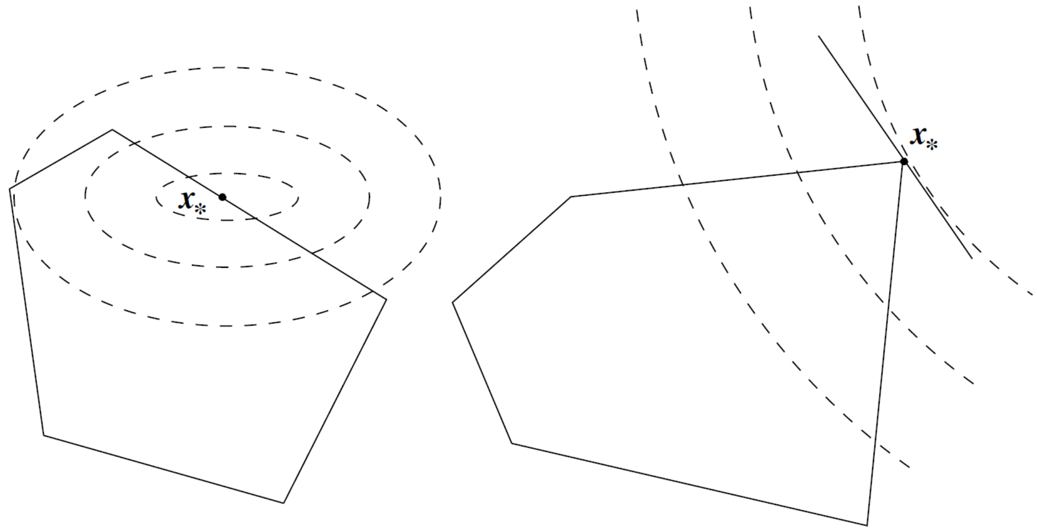


Figure 16.2 in Nocedal & Wright.

1) Strict complementarity does not hold
2) Constraints linearly dependent at solution

If active set known, QP can be solved as EQP

$\mathcal{A}(x^*)$ known: $\min\limits_{x \in \mathbb{R}^n} \frac{1}{2} x^T G x + c^T x$ s.t. $a_i^T x = b_i$, $i \in \mathcal{A}(x^*)$

can be solved by solving (e.g.)

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

One step of active set method for QP

In iteration $\underline{k}$: W_k : Current estimate of $\mathcal{A}(x^*)$

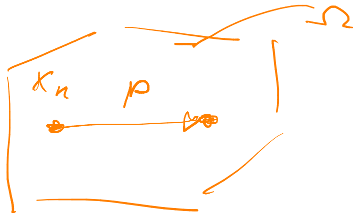
x_k : Current $\underline{\text{feasible}}$ estimate of x^*

Define $A_k = \begin{bmatrix} \vdots \\ a_i^T \\ \vdots \end{bmatrix}, \; b_k = \begin{bmatrix} \vdots \\ b_i \\ \vdots \end{bmatrix}, \; i \in W_k$

Consider EQP

$$\min_{p \in \mathbb{R}^n} \frac{1}{2}(x_k+p)^T G (x_k+p) + c^T(x_k+p)$$

$$\text{s.t.} \quad A_k(x_k+p) = b_k \rightarrow A_p = 0 \text{ since } A_k x_k = b$$



$$\Leftrightarrow \min_{p \in \mathbb{R}^n} \frac{1}{2} p^T G p + \underbrace{(G x_k + c)^T}_{g_k} p \quad \text{s.t.} \quad A_k p = 0$$

Call solution p_k

One step of active set method for QP, cont'd

If $p_k = 0$: solve $\sum_{i \in W_k} a_i \hat{\lambda}_i = g_k$ for $\hat{\lambda}_i$

→ If $\hat{\lambda}_i \geq 0$, KKT fulfilled!

→ If $\hat{\lambda}_i \ngeq 0$: pick index of one negative $\hat{\lambda}_i$

Remove this index from W_k.

Start over (solve new EQP)

General QP problem

$$\min_x \frac{1}{2}x^\top Gx + x^\top c$$
$$\text{s.t. } a_i^\top x = b_i, \qquad i \in \mathcal{E}$$
$$a_i^\top x \geq b_i, \qquad i \in \mathcal{I}$$

- Lagrangian

$$\mathcal{L}(x^*, \lambda^*) = \frac{1}{2}x^\top Gx + x^\top c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^\top x - b_i)$$

- KKT conditions

General:

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i, \qquad i \in \mathcal{E} \\ a_i^\top x^* &\geq b_i, \qquad i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \qquad i \in \mathcal{I} \\ \lambda_i^* (a_i^\top x^* - b_i) &= 0, \qquad i \in \mathcal{E} \cup \mathcal{I} \end{aligned}$$

Defined via active set:

$$\mathcal{A}(x^*) = \mathcal{E} \cup \left\{ i \in \mathcal{I} \middle| a_i^\top x^* = b_i \right\}$$

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i, \qquad i \in \mathcal{A}(x^*) \\ a_i^\top x^* &\geq b_i, \qquad i \in \mathcal{I} \backslash \mathcal{A}(x^*) \\ \lambda_i^* &\geq 0, \qquad i \in \mathcal{A}(x^*) \cap \mathcal{I} \end{aligned}$$

One step of active set method for QP, cont'd

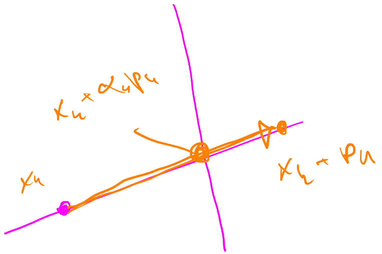
If $p_k \neq 0$:

- If $x_{k+1} = x_k + p_k$ is feasible: Set $W_{k+1} = W_k$, start over
- If $x_{k+1} = x_k + p_k$ is not feasible. Find blocking constraint

For $i \in \{ i \mid a_i^T p_k < 0 \}$:

Want $a_i^T (x_k + \alpha_k p_k) \geq b_i \Rightarrow \alpha_k \leq \dfrac{b_i - a_i^T x_k}{a_i^T p_k}$

Set j = "i with smallest α_k"



$x_{k+1} = x_k + \alpha_k p_k$, $W_{k+1} = W_k + \{ j \}$

start over!

Active set method for convex QP

$g_k = G x_k + c$

Algorithm 16.3 (Active-Set Method for Convex QP).

Compute a feasible starting point x_0;
Set $\mathcal{W}_0$ to be a subset of the active constraints at x_0;
for $k = 0, 1, 2, \ldots$
  Solve (16.39) to find p_k;
  if $p_k = 0$
    Compute Lagrange multipliers $\hat{\lambda}_i$ that satisfy (16.42),
      with $\hat{\mathcal{W}} = \mathcal{W}_k$;
    if $\hat{\lambda}_i \geq 0$ for all $i \in \mathcal{W}_k \cap \mathcal{I}$
      stop with solution $x^* = x_k$;
    else
      $j \leftarrow \arg\min_{j \in \mathcal{W}_k \cap \mathcal{I}} \hat{\lambda}_j$;
      $x_{k+1} \leftarrow x_k$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \backslash \{j\}$;
  else (* $p_k \neq 0$ *)
    Compute α_k from (16.41);
    $x_{k+1} \leftarrow x_k + \alpha_k p_k$;
    if there are blocking constraints
      Obtain $\mathcal{W}_{k+1}$ by adding one of the blocking
        constraints to $\mathcal{W}_k$;
    else
      $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k$;
end (for)

$$\min_p \quad \frac{1}{2} p^T G p + g_k^T p \tag{16.39a}$$

$$\text{subject to} \quad a_i^T p = 0, \quad i \in \mathcal{W}_k. \tag{16.39b}$$

$$\sum_{i \in \hat{\mathcal{W}}} a_i \hat{\lambda}_i = g = G\hat{x} + c, \tag{16.42}$$

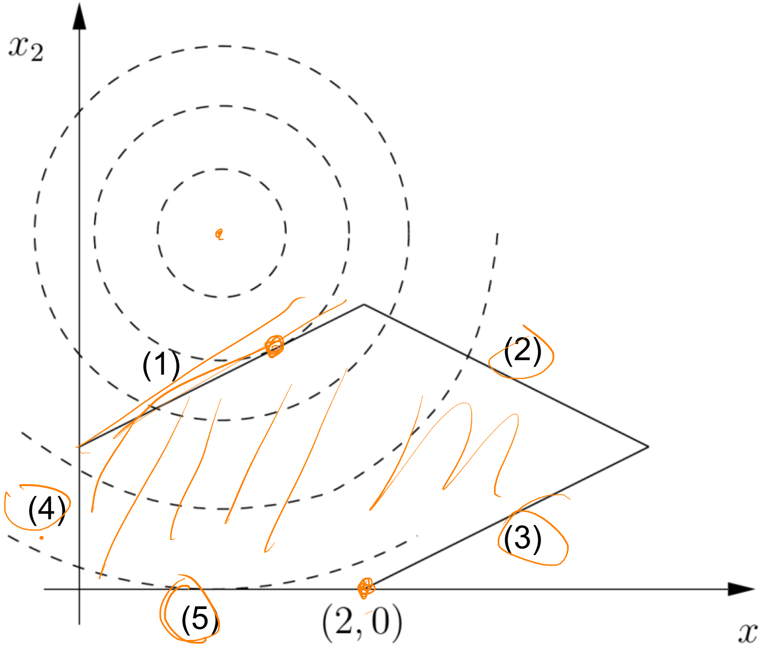
$$\alpha_k \stackrel{\text{def}}{=} \min \left(1, \min_{i \notin \mathcal{W}_k,\, a_i^T p_k < 0} \frac{b_i - a_i^T x_k}{a_i^T p_k}\right). \tag{16.41}$$

No degeneracy and $G>0$: Active set method converges in finite number of iterations.

Example 16.4

$$\min_x q(x) = (x_1-1)^2 + (x_2-2.5)^2$$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 + 2 &\geq 0 && (1)\\ -x_1 - 2x_2 + 6 &\geq 0 && (2)\\ -x_1 + 2x_2 + 2 &\geq 0 && (3)\\ x_1 &\geq 0 && (4)\\ x_2 &\geq 0 && (5) \end{aligned}$$



$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^T, \quad b_1 = -2$$

$$a_2 = \begin{bmatrix} -1 & -2 \end{bmatrix}^T, \quad b_2 = -6$$

$$a_3 = \begin{bmatrix} -1 & 2 \end{bmatrix}^T, \quad b_3 = -2$$

$$a_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad b_4 = 0$$

$$a_5 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad b_5 = 0$$

Example 16.4

$x^0 = \begin{pmatrix}2\\0\end{pmatrix}$, $W_0 = \{3,5\}$, $g_0 = Gx_0 + c = \begin{pmatrix}2\\-5\end{pmatrix}$

$$\min_{p \in \mathbb{R}^n} \ \frac{1}{2} p^T G p + g_k^T p \iff \min_{p_1, p_2} \ p_1^2 + p_2^2 + 2p_1 - 5p_2$$

$$\text{s.t. } A_k p = 0 \qquad \text{s.t. } -p_1 + p_2 = 0$$

$$p_2 = 0$$

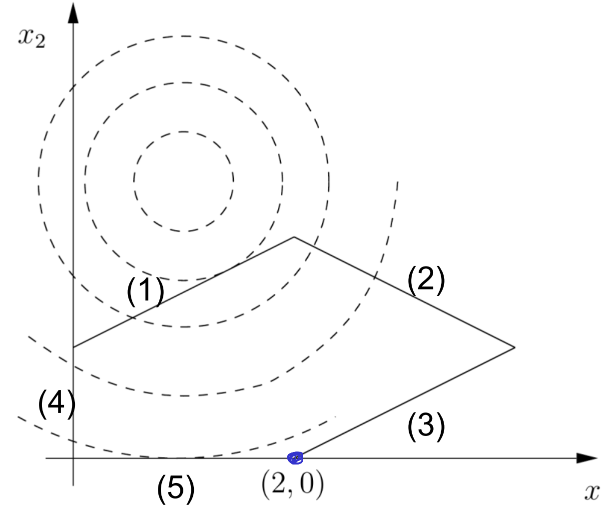
$$\Rightarrow p^0 = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\text{Check } \sum_{i \in W_0} a_i \hat{\lambda}_i = g_0 \qquad \begin{bmatrix}-1\\2\end{bmatrix}\hat{\lambda}_3 + \begin{bmatrix}0\\1\end{bmatrix}\hat{\lambda}_5 = \begin{bmatrix}2\\-5\end{bmatrix}$$

$$\Rightarrow \hat{\lambda}_3 = -2$$

$$\hat{\lambda}_5 = -1$$

Remove $\{3\}$ from W_0



$$G = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \quad c = \begin{bmatrix}-2\\-5\end{bmatrix}$$

$$a_1 = \begin{bmatrix}1 & -2\end{bmatrix}^T, \quad b_1 = -2$$

$$a_2 = \begin{bmatrix}-1 & -2\end{bmatrix}^T, \quad b_2 = -6$$

$$a_3 = \begin{bmatrix}-1 & 2\end{bmatrix}^T, \quad b_3 = -2$$

$$a_4 = \begin{bmatrix}1 & 0\end{bmatrix}^T, \quad b_4 = 0$$

$$a_5 = \begin{bmatrix}0 & 1\end{bmatrix}^T, \quad b_5 = 0$$

Example 16.4

$x^1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $W_1 = \{5\}$, $g_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$\frac{1}{2} p^T G p + g_1^T p$

$$\min_p \; p_1^2 + p_2^2 + 2p_1 - 5p_2$$

$$\text{s.t.} \quad p_2 = 0$$

$\min \; \underbrace{p_1^2 + 2p_1}$

$\nabla f = 0$

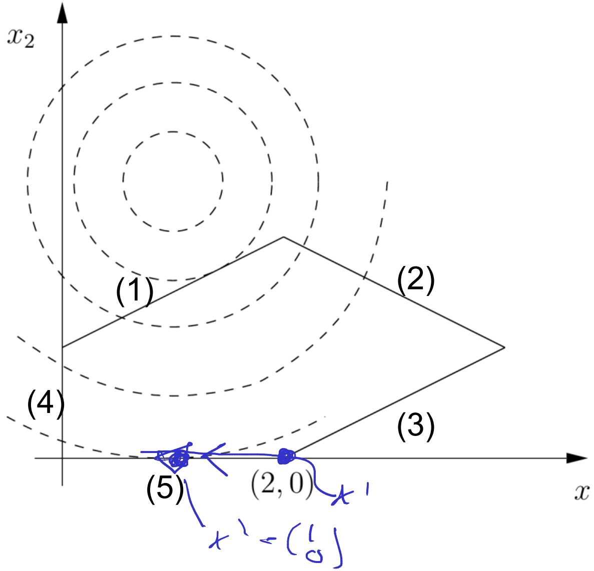
$2p_1 + 2 = 0$

$p_1 = -1$

$\Rightarrow p^1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$x^2 = x^1 + p^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ feasible!

$W_2 = W_1$



$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$a_1 = [1 \quad -2]^T, \quad b_1 = -2$

$a_2 = [-1 \quad -2]^T, \quad b_2 = -6$

$a_3 = [-1 \quad 2]^T, \quad b_3 = -2$

$a_4 = [1 \quad 0]^T, \quad b_4 = 0$

$\rightarrow a_5 = [0 \quad 1]^T, \quad b_5 = 0$

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Example 16.4

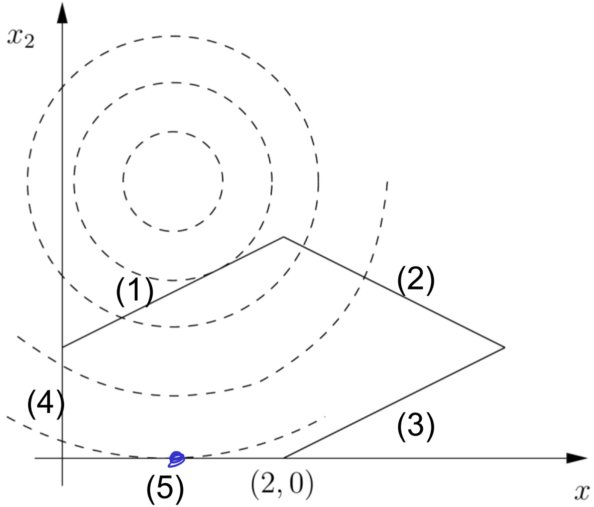
$$x^2 = \begin{pmatrix}1\\0\end{pmatrix}, \quad W_2 = \{5\}, \quad g_2 = Gx^2 + c = \begin{bmatrix}0\\-5\end{bmatrix}$$

$$\min_p \; p_1^2 + p_2^2 - 5p_2 \quad \Rightarrow \quad p^2 = \begin{pmatrix}0\\0\end{pmatrix}$$
$$\text{s.t. } p_2 = 0$$

$$\underset{a_5}{\begin{bmatrix}0\\1\end{bmatrix}} \hat{\lambda}_5 = \underset{g^2}{\begin{bmatrix}0\\-5\end{bmatrix}} \Rightarrow \hat{\lambda}_5 = -5$$

Remove $\{5\}$ from W_2

$$W_3 = W_2 - \{5\} = \emptyset$$



$$G = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \quad c = \begin{bmatrix}-2\\-5\end{bmatrix}$$

$$a_1 = \begin{bmatrix}1 & -2\end{bmatrix}^T, \quad b_1 = -2$$
$$a_2 = \begin{bmatrix}-1 & -2\end{bmatrix}^T, \quad b_2 = -6$$
$$a_3 = \begin{bmatrix}-1 & 2\end{bmatrix}^T, \quad b_3 = -2$$
$$a_4 = \begin{bmatrix}1 & 0\end{bmatrix}^T, \quad b_4 = 0$$
$$a_5 = \begin{bmatrix}0 & 1\end{bmatrix}^T, \quad b_5 = 0$$

Example 16.4

$x^3 = \begin{pmatrix}1\\0\end{pmatrix}, \; W_3 = \emptyset, \; g_3 = Gx^3 + c = \begin{pmatrix}0\\-5\end{pmatrix}$

$\min_p \; p_1^2 + p_2^2 - 5p_2 \;\Rightarrow\; p^3 = \begin{bmatrix}0\\2{,}5\end{bmatrix}$

But: $x^3 + p^3$ not feasible!

Check $i \notin W_3$: $a_1^T p^3 = -5$: $\alpha_3^1 = \dfrac{b_1 - a_1^T x^3}{a_1^T p^3} = 0.6$

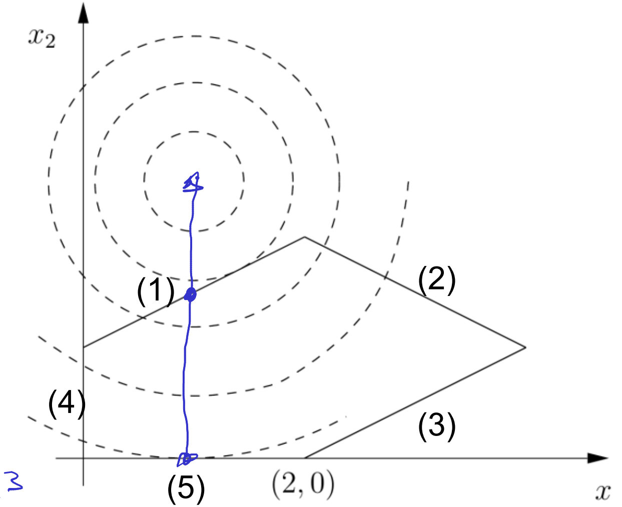
$a_2^T p^3 = -5$: $\alpha_3^2 = \dots = 1.4$

$a_3^T p_3 = 5$: not relevant

(same for 4 and 5)

$\Rightarrow \alpha_3 = 0.6 \;\Rightarrow\; x^4 = x^3 + \alpha_3 p^3 = \begin{pmatrix}1\\1{,}5\end{pmatrix}$

$W_4 = W_3 + \{1\} = \{1\}$



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$$a_4 = \begin{bmatrix}1 & 0\end{bmatrix}^T, \quad b_4 = 0$$

$$a_5 = \begin{bmatrix}0 & 1\end{bmatrix}^T, \quad b_5 = 0$$

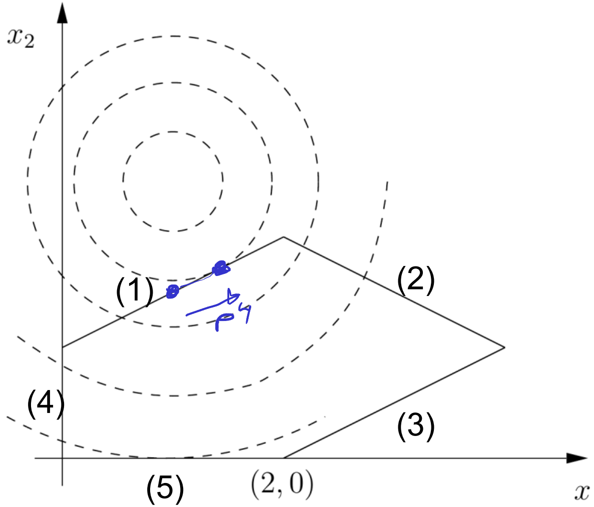
Example 16.4

$x^4 = \begin{pmatrix} 1 \\ 1.7 \end{pmatrix}$, $W_4 = \{1\}$, $g_4 = Gx^4 + c = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$$\left.\begin{aligned} \min_{p} \quad & p_1^2 + p_2^2 - 2p_2 \\ \text{s.t.} \quad & p_1 - 2p_2 = 0 \end{aligned}\right\} \quad p^4 = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix}$$

$$x^5 = x^4 + p^4 = \begin{pmatrix} 1.4 \\ 1.7 \end{pmatrix}$$

feasible



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$a_2 = [-1 \;\; -2]^T, \quad b_2 = -6$

$a_3 = [-1 \;\; 2]^T, \quad b_3 = -2$

$a_4 = [1 \;\; 0]^T, \quad b_4 = 0$

$a_5 = [0 \;\; 1]^T, \quad b_5 = 0$

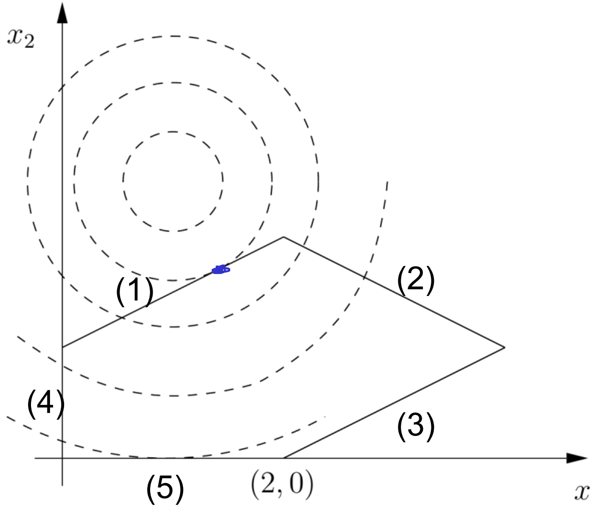
Example 16.4

$$x^5 = \begin{pmatrix}1.4\\1.7\end{pmatrix},\quad W_5 = \{1\},\quad g_5 = \begin{pmatrix}0.8\\-1.6\end{pmatrix}$$

$$\left.\begin{aligned}\min_p \quad & p_1^2 + p_2^2 + 0.8p_1 - 1.6p_2 \\ \text{s.t.} \quad & p_1 - 2p_2 = 0\end{aligned}\right\} \Rightarrow p^5 = \begin{pmatrix}0\\0\end{pmatrix}$$

$$a_i^T \hat{\lambda} = g_5 \Rightarrow \begin{bmatrix}1\\-2\end{bmatrix}\hat{\lambda}_1 = \begin{bmatrix}0.8\\-1.6\end{bmatrix} \Rightarrow \hat{\lambda}_1 = 0.8 > 0$$

$$\Rightarrow \boxed{x^* = \begin{pmatrix}1.4\\1.7\end{pmatrix}}$$



$$G = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix},\quad c = \begin{bmatrix}-2\\-5\end{bmatrix}$$

$$a_1 = \begin{bmatrix}1 & -2\end{bmatrix}^T,\quad b_1 = -2$$

$$a_2 = \begin{bmatrix}-1 & -2\end{bmatrix}^T,\quad b_2 = -6$$

$$a_3 = \begin{bmatrix}-1 & 2\end{bmatrix}^T,\quad b_3 = -2$$

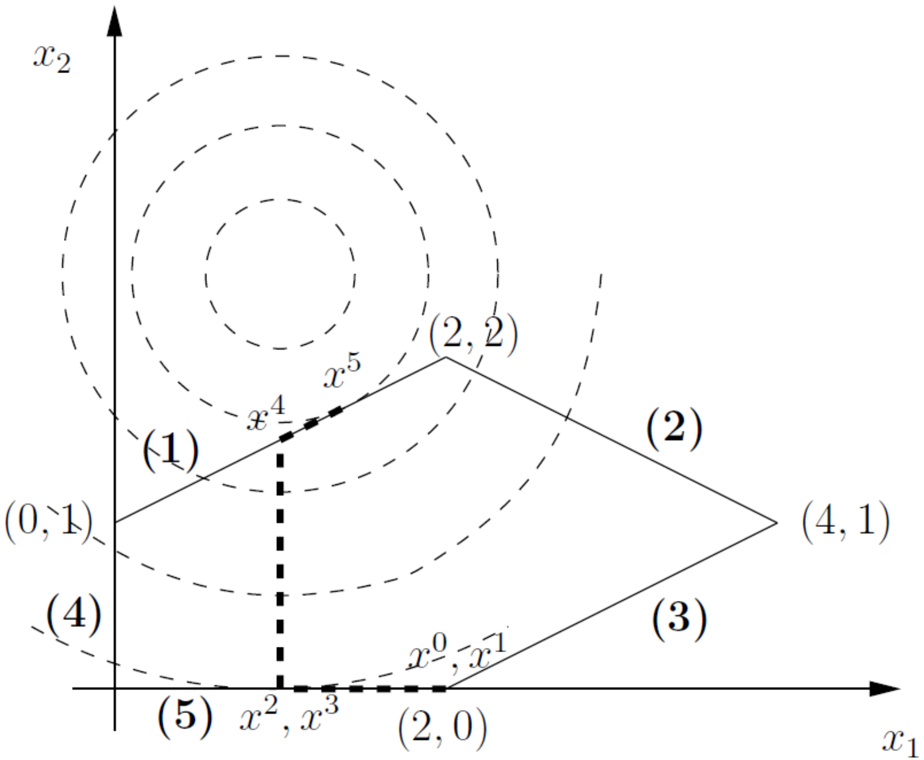
$$a_4 = \begin{bmatrix}1 & 0\end{bmatrix}^T,\quad b_4 = 0$$

$$a_5 = \begin{bmatrix}0 & 1\end{bmatrix}^T,\quad b_5 = 0$$

Example 16.4

$$\min_x q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 + 2 &\geq 0 && (1) \\ -x_1 - 2x_2 + 6 &\geq 0 && (2) \\ -x_1 + 2x_2 + 2 &\geq 0 && (3) \\ x_1 &\geq 0 && (4) \\ x_2 &\geq 0 && (5) \end{aligned}$$



$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_1 = -2$$

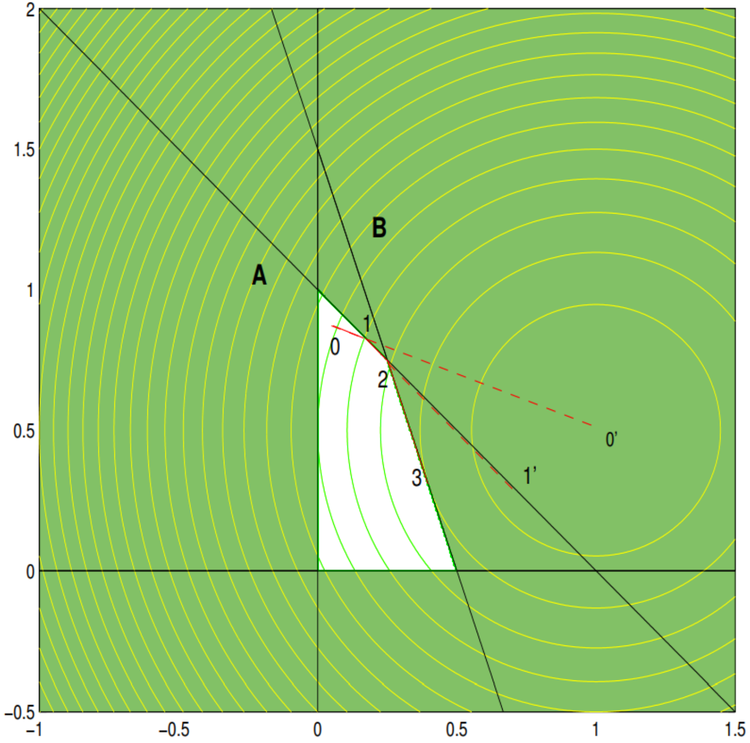
$$a_2 = \begin{bmatrix} -1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_2 = -6$$

$$a_3 = \begin{bmatrix} -1 & 2 \end{bmatrix}^\mathsf{T}, \quad b_3 = -2$$

$$a_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\mathsf{T}, \quad b_4 = 0$$

$$a_5 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\mathsf{T}, \quad b_5 = 0$$

Another example (N. Gould)



$$\min (x_1 - 1)^2 + (x_2 - 0.5)^2$$
$$\text{subject to } \; x_1 + x_2 \le 1$$
$$3x_1 + x_2 \le 1.5$$
$$(x_1, x_2) \ge 0$$

0. Starting point
0'. Unconstrained minimizer
1. Encounter constraint A
1'. Minimizer on constraint A
2. Encounter constraint B, move off constraint A
3. Minimizer on constraint B = required solution

How to find feasible initial point?

- Same way as for LP:
  - Phase I: Define a LP with known feasible initial point, where solution is feasible for original QP.
  - Phase II: Solve original QP.

- Alternative method: “Big M”
  - Relax all constraints; penalize constraint violations in objective

Initialization methods

Phase 1

$$
\begin{aligned}
\min_{(x,z)} \ & e^T z \\
\text{subject to} \quad & a_i^T x + \gamma_i z_i = b_i, & i \in \mathcal{E}, \\
& a_i^T x + \gamma_i z_i \geq b_i, & i \in \mathcal{I}, \\
& z \geq 0,
\end{aligned}
$$

$e = (1, 1, \dots, 1)^T$, $\gamma_i = -\text{sign}(a_i^T \tilde{x} - b_i)$ for $i \in \mathcal{E}$
$\gamma_i = 1$ for $i \in \mathcal{I}$

- Feasible initial guess for LP problem:

$$
\begin{aligned}
x &= \tilde{x} \\
z_i &= |a_i^T \tilde{x} - b_i| \quad (i \in \mathcal{E}) \\
z_i &= \max(b_i - a_i^T \tilde{x}, 0) \quad (i \in \mathcal{I})
\end{aligned}
$$

Big M

$$
\begin{aligned}
\min_{(x,\eta)} \ & \tfrac{1}{2} x^T G x + x^T c + M\eta, \\
\text{subject to} \quad & (a_i^T x - b_i) \leq \eta, & i \in \mathcal{E}, \\
& -(a_i^T x - b_i) \leq \eta, & i \in \mathcal{E}, \\
& b_i - a_i^T x \leq \eta, & i \in \mathcal{I}, \\
& 0 \leq \eta,
\end{aligned}
$$

- Feasible initial guess for Big M: Whatever.
- η nonzero? Increase M and try again.

Concluding remarks

- Solves similar EQPs iteratively: recalculate only what's needed
- Active set method: Potentially slow, but with good initial guess will be FAST
- Alternative to Active Set: Interior Point (not curriculum)

Nonconvex QP

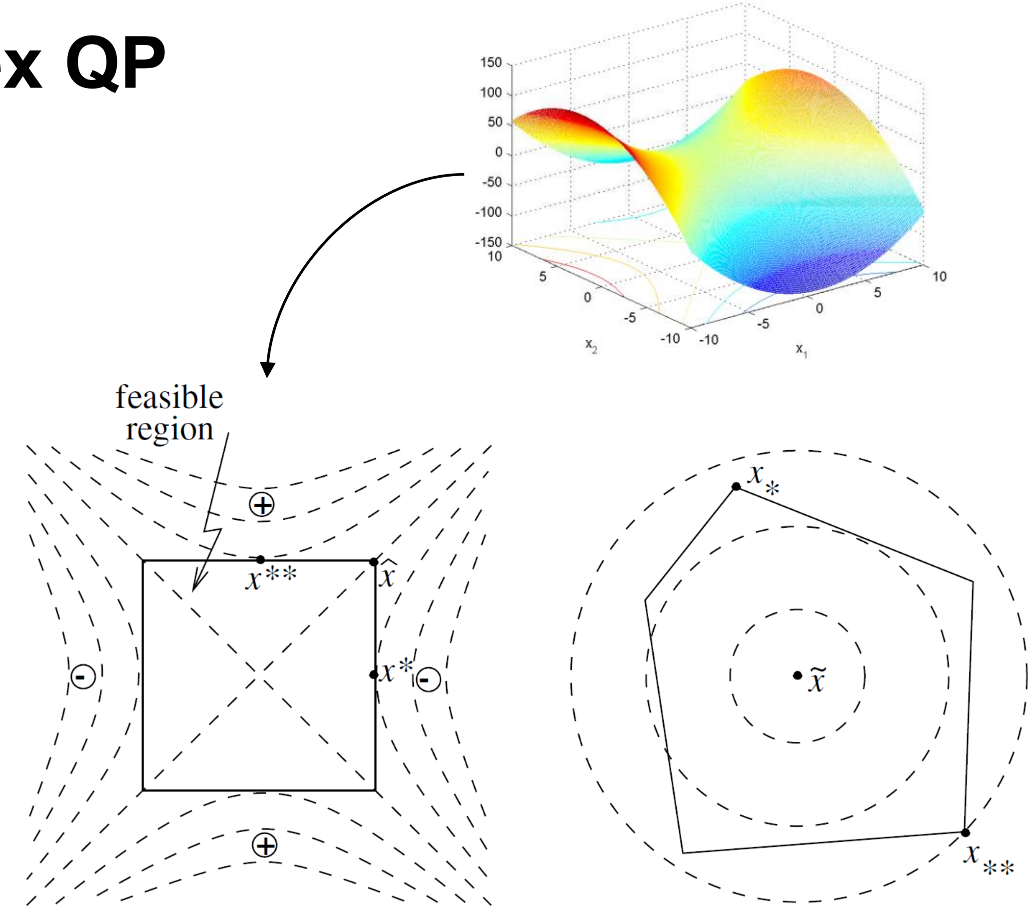


Figure 16.1 in Nocedal & Wright.