



NTNU

Norwegian University of
Science and Technology

TTK4135 – Lecture 9

Linear Quadratic Control

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Outline

- Recap: Open-loop linear dynamic optimization problems, and the different ways of solving them
 - Two batch methods (-> QPs)
 - One recursive method
- Today: Linear Quadratic Control (= “The recursive method”)
 - Finite horizon
 - Infinite horizon

Reference: F&H Ch. 4.3-4.4

Last time: Dynamic open-loop optimization (with linear state-space model)

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{x,t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{u,t} u_t + \frac{1}{2} \Delta u_t^\top S \Delta u_t$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\}$$

$$x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\}$$

$$u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

$$-\Delta u^{\text{high}} \leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

where

x_0 and u_{-1} is given

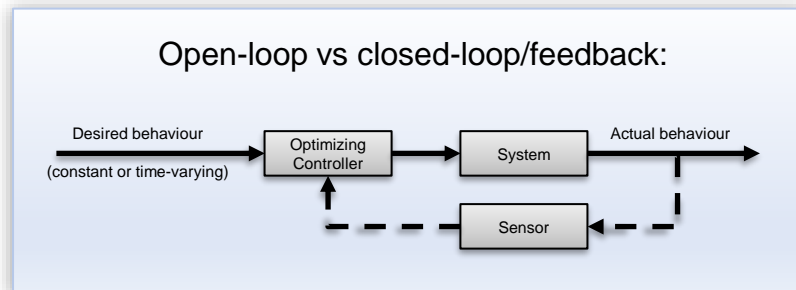
$$\Delta u_t := u_t - u_{t-1}$$

$$z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)$$

$$n = N \cdot (n_x + n_u)$$

$$Q_t \succeq 0 \quad t = \{1, \dots, N\}$$

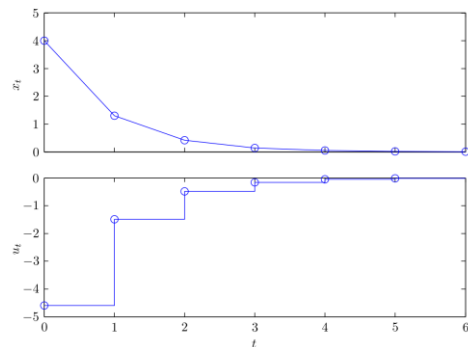
$$R_t \succ 0 \quad t = \{0, \dots, N-1\}$$



The significance of weights

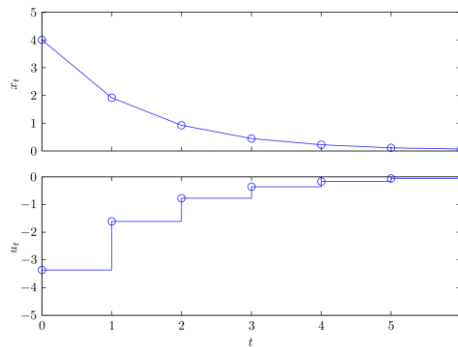
$$\begin{aligned} \min \quad & \sum_{t=0}^5 q x_{t+1}^2 + r u_t^2 \\ \text{s.t.} \quad & x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \dots, 5 \end{aligned}$$

$$q = 5, r = 1$$



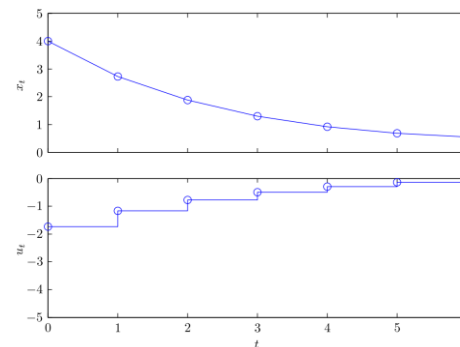
$$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \quad \sum_{t=0}^{N-1} u_t^2 = 23.6$$

$$q = 2, r = 1$$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \quad \sum_{t=0}^{N-1} u_t^2 = 14.7$$

$$q = 1, r = 2$$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \quad \sum_{t=0}^{N-1} u_t^2 = 5.3$$

Linear quadratic control: Dynamic optimization without (inequality) constraints

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top \end{aligned}$$

Three approaches for solution:

- Batch approach v1, “full space” – solve as QP
- Batch approach v2, “reduced space” – solve as QP
- Recursive approach – solve as linear state feedback



Also work with input- and state constraints!

Only work without constraints!

Linear Quadratic Control

Batch approach v1, “Full space” QP

- Formulate with model as equality constraints, all inputs and states as optimization variables

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t. } x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$



$$\min_z \frac{1}{2} z^\top \begin{pmatrix} R & & & \\ & Q & & \\ & & R & \\ & & & \ddots \\ & & & & Q \end{pmatrix} z$$

$$\text{s.t. } \begin{pmatrix} -B & I & & & \\ & -A & -B & I & \\ & & -A & -B & I \\ & & & \ddots & \ddots \\ & & & & -A & -B & I \end{pmatrix} z = \begin{pmatrix} A x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

Linear Quadratic Control

Batch approach v2, “Reduced space” QP

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top \end{aligned}$$

- Use model to eliminate states as variables
 - Future states as function of inputs and initial state

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{pmatrix} x_0 + \begin{pmatrix} B & & & \\ AB & B & & \\ A^2 & AB & B & \\ \vdots & \vdots & \vdots & \ddots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \dots & B \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = S^x x_0 + S^u U$$

- Insert into objective (no constraints!)

$$\min_U \frac{1}{2} (S^x x_0 + S^u U)^\top \mathbf{Q} (S^x x_0 + S^u U) + \frac{1}{2} U^\top \mathbf{R} U$$

$$\mathbf{Q} = \begin{pmatrix} Q & & \\ & Q & \\ & & \ddots \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R & & \\ & R & \\ & & \ddots \end{pmatrix}$$

- Solution (when no inequality constraints) found by setting gradient equal to zero:

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = - \underbrace{((S^u)^\top \mathbf{Q} S^u + \mathbf{R})^{-1}}_F (S^u)^\top \mathbf{Q} S^x x_0 = -F x_0$$

Linear Quadratic Control

Recursive approach

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top \end{aligned}$$

- By writing up the KKT-conditions, we can show (we will do this today) that the solution can be formulated as:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, & t = 0, \dots, N-1 \\ P_t &= Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, & t = 0, \dots, N-1 \\ P_N &= Q \end{aligned}$$

Comments to the three solution approaches

- All give same numerical solution
 - If problem is strictly convex (Q psd, R pd), solution is unique
- The batch approaches give an open-loop solution, the recursive approach give a closed-loop (feedback) solution

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -Fx_0 \quad \text{vs} \quad u_t = -K_t x_t, \quad t = 0, \dots, N-1$$

- Constraints on inputs and states:
 - Easy for batch approaches (both becomes convex QPs)
 - Difficult for the recursive approach
- How to add feedback (and thereby robustness) to batch approaches?
 - Model predictive control! (Next time)

Today: The recursive solution (LQ control)

KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{aligned} c_i(x) &= 0, & i \in \mathcal{E}, \\ c_i(x) &\geq 0, & i \in \mathcal{I}. \end{aligned}$$

Lagrangian:
$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, & (\text{stationarity}) \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, & \left. \begin{array}{l} \\ \end{array} \right\} (\text{primal feasibility}) \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, & (\text{dual feasibility}) \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. & (\text{complementarity condition/} \\ & & \text{complementary slackness}) \end{aligned}$$

LQR:

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top \end{aligned}$$

KKT:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. \end{aligned}$$

Second-order conditions

Theorem 12.6 (Second-Order Sufficient Conditions).

Suppose that for some feasible point $x^ \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions (12.34) are satisfied. Suppose also that*

$$w^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) w > 0, \quad \text{for all } w \in \mathcal{C}(x^*, \lambda^*), w \neq 0. \quad (12.65)$$

Then x^ is a strict local solution for (12.1).*

- Critical directions:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{E}, \\ \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0, \\ \nabla c_i(x^*)^T w \geq 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0. \end{cases} \quad (12.53)$$

- The critical directions are the “allowed” directions where it is not clear from KKT-conditions whether the objective will decrease or increase

Thm 16.4: For convex QP, KKT is sufficient

- From N&W, p. 464:

For convex QP, when G is positive semidefinite, the conditions (16.37) are in fact sufficient for x^* to be a global solution, as we now prove.

Theorem 16.4.

If x^ satisfies the conditions (16.37) for some $\lambda_i^*, i \in \mathcal{A}(x^*)$, and G is positive semidefinite, then x^* is a global solution of (16.1).*

- That is: Since the solution of the Riccati equation implies the KKT conditions are fulfilled, Thm 16.4 means that the Riccati equation gives the global solution
 - Side-remark: It is, in fact, the *unique* global solution. If G is positive definite (implied by Q positive definite), this follows from the proof of Thm 16.4. If Q positive semidefinite, further arguments are necessary (for instance using Thm 12.6 as in the note).

KKT conditions



- Finite horizon LQ optimal control problem:

$$\begin{aligned} \min_{z \in \mathbb{R}^n} f(z) &= \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t \\ \text{subject to } x_{t+1} &= A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\ x_0 &= \text{given} \\ Q_t &\succeq 0 \quad t = 1, \dots, N \\ R_t &\succ 0 \quad t = 0, \dots, N-1 \end{aligned}$$

where

$$\begin{aligned} z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\ n &= N \cdot (n_x + n_u) \end{aligned}$$

- State feedback solution

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, & t = 0, \dots, N-1 \\ P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, & t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

(discrete) Riccati equation



Linear quadratic control (finite horizon)

- The optimal solution to LQ control is a linear, time-varying state feedback:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$K_t = R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad t = 0, \dots, N-1$$

$$P_t = Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad t = 0, \dots, N-1$$

$$P_N = Q_N$$

- Note that the gain matrix K_t is independent of the states, and can therefore be computed in advance (knowing A_t , B_t , Q_t , R_t)
- The matrix (difference) equation

$$P_t = Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad t = 0, \dots, N-1$$

$$P_N = Q_N$$

is called the (discrete) *Riccati equation*

- Note that the “boundary condition” is given at the end of the horizon, and the P_t -matrices must be found iterating backwards in time

Example

$$\min \sum_{t=0}^{10} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \dots, 10$$

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$K_t = R^{-1} B^{\top} P_{t+1} (I + B R^{-1} B^{\top} P_{t+1})^{-1} A, \quad t = 0, \dots, N-1$$

$$P_t = Q + A^{\top} P_{t+1} (I + B R^{-1} B^{\top} P_{t+1})^{-1} A, \quad t = 0, \dots, N-1$$

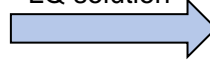
$$P_N = Q$$

Example

$$\min \sum_{t=0}^{10} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \dots, 10$$

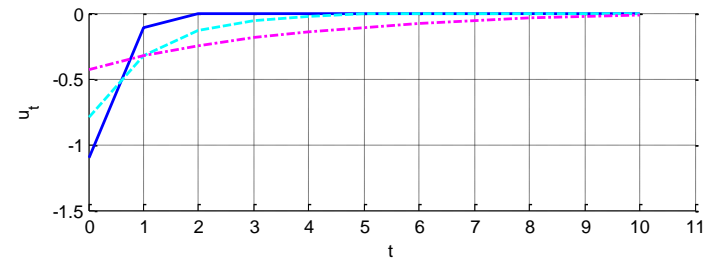
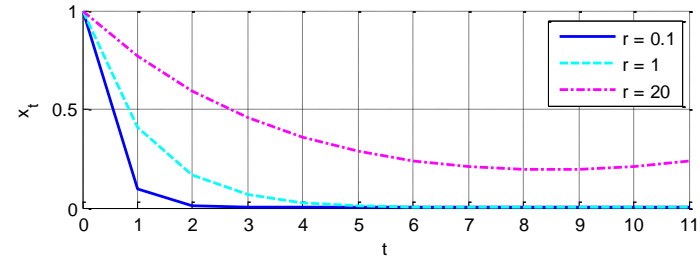
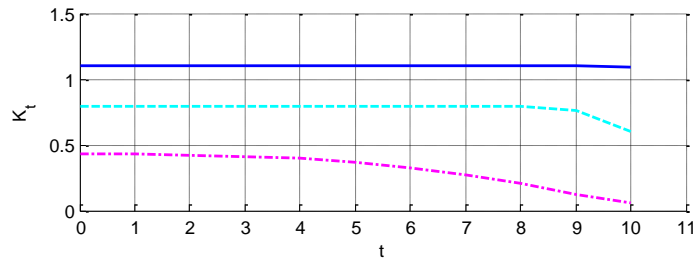
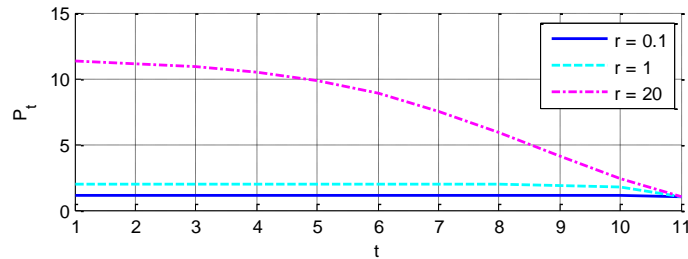
LQ solution



$$P_t = 1 + \frac{1.44rP_{t+1}}{P_{t+1} + r}, \quad t = 10, \dots, 1$$

$$P_{11} = 1$$

$$K_t = 1.2 \frac{P_{t+1}}{P_{t+1} + r}, \quad t = 0, \dots, 10$$



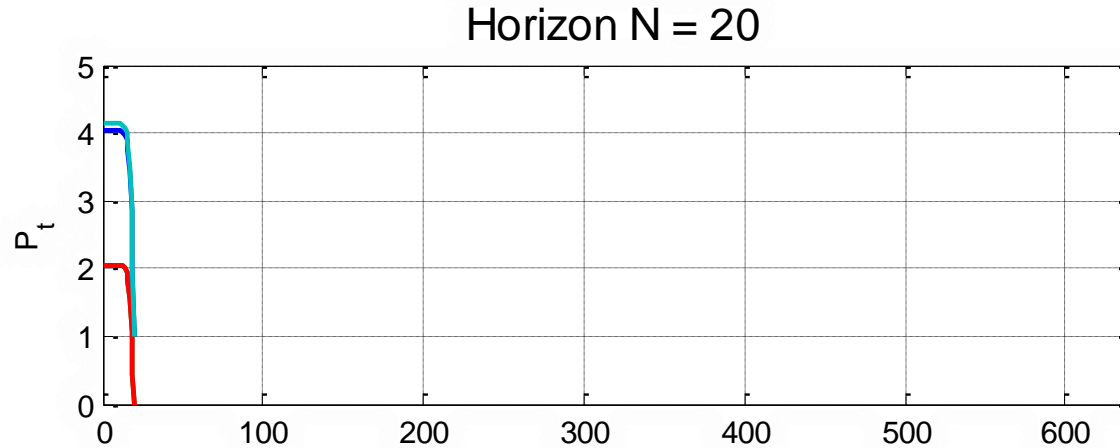
What if the horizon goes to infinity?

Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$

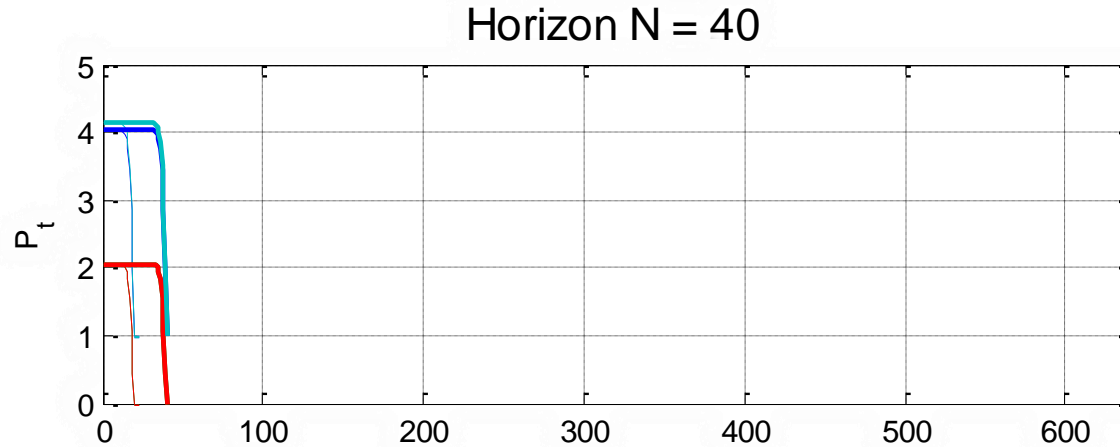


Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

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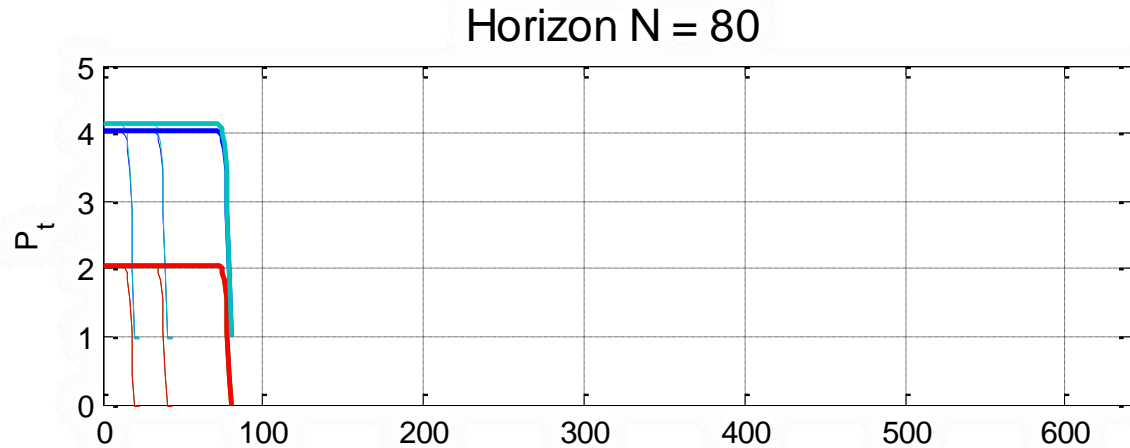


Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

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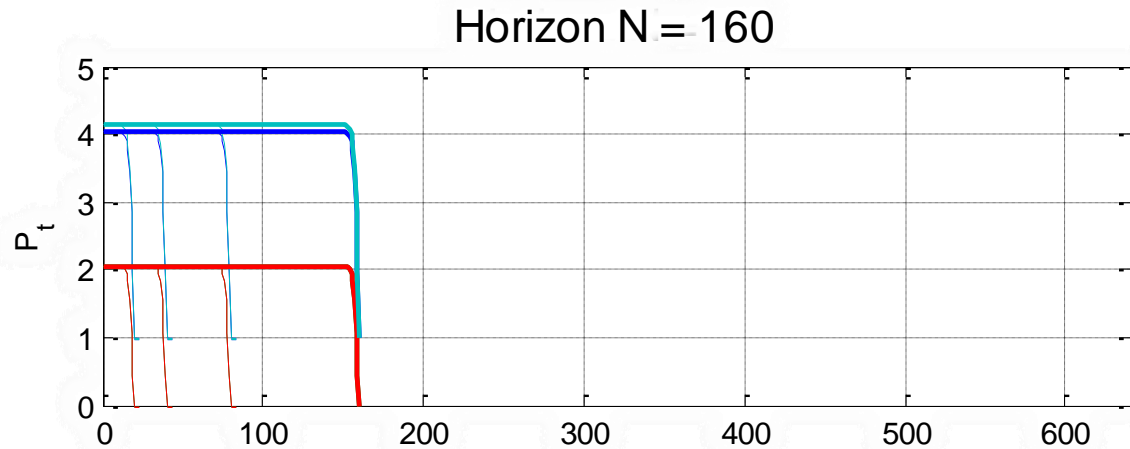


Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$

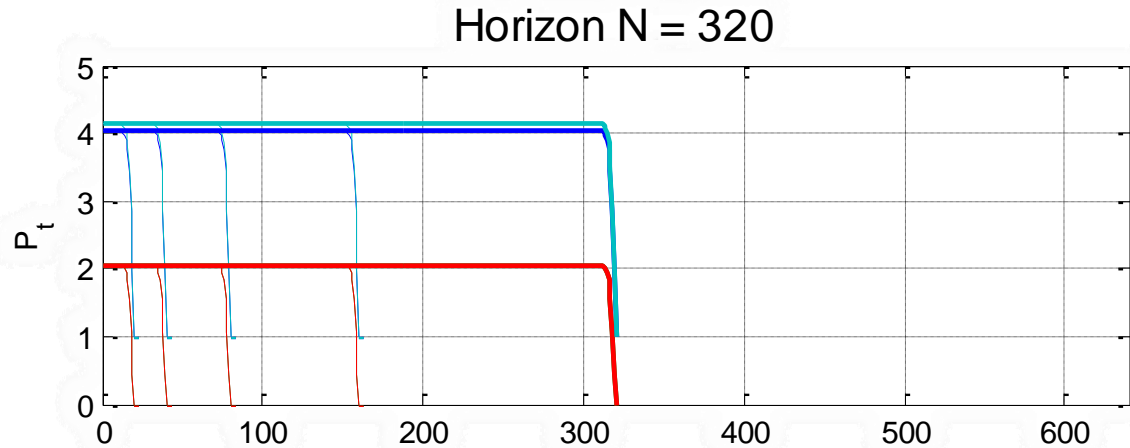


Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$

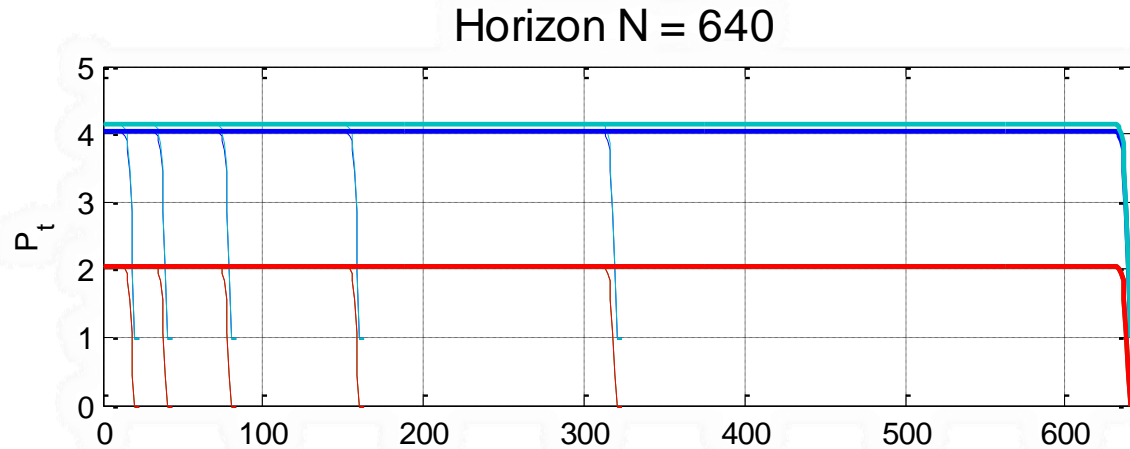


Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$



Infinite horizon LQ control

$$\begin{aligned} \min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^{\top} Q x_{t+1} + \frac{1}{2} u_t^{\top} R u_t \\ \text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots \end{aligned}$$

Fact: Steady-state ($P_{t+1} = P_t$) backwards-in-time solution of Riccati equation is **infinite horizon solution**

$$u_t = -K_t x_t$$

where

$$\begin{aligned} K_t &= R^{-1} B^{\top} P_{t+1} (I + B R^{-1} B^{\top} P_{t+1})^{-1} A, & t &= 0, \dots, N-1 \\ P_t &= Q + A^{\top} P_{t+1} (I + B R^{-1} B^{\top} P_{t+1})^{-1} A, & t &= 0, \dots, N-1 \\ P_N &= Q \end{aligned}$$



$$u_t = -K x_t$$

where

$$\begin{aligned} K &= R^{-1} B^{\top} P (I + B R^{-1} B^{\top} P)^{-1} A \\ P &= Q + A^{\top} P (I + B R^{-1} B^{\top} P)^{-1} A \end{aligned}$$

Infinite horizon LQ control

Theorem: The solution (when one exists) to

$$\min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^{\top} Q x_{t+1} + \frac{1}{2} u_t^{\top} R u_t$$

s.t. $x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots$

is given by

$$u_t = -K x_t$$

where

$$K = R^{-1} B^{\top} P (I + B R^{-1} B^{\top} P)^{-1} A$$

$$P = Q + A^{\top} P (I + B R^{-1} B^{\top} P)^{-1} A$$

Two central questions:

- When does a solution exist?
- When is the closed-loop stable?

(Discrete-time Algebraic Riccati Equation, DARE)



Controllability vs stabilizability

Observability vs detectability

- Stabilizable: All unstable modes are controllable
(that is: all uncontrollable modes are stable)
- Detectability: All unstable modes are observable
(that is: all unobservable modes are stable)
- Controllability implies stabilizability
- Observability implies detectability

Riccati equations

- Discrete-time Riccati equation in the note (and lecture)

$$P_t = Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad P_N = Q_N$$

- However, another, equivalent, form is found in other sources:

$$P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (R_t + B_t^\top P_{t+1} B_t)^{-1} B_t^\top P_{t+1} A_t, \quad P_N = Q_N$$

- The latter is more numerically stable due to “enforced symmetry”
- The trick used to get the different formulas is the “Matrix Inversion Lemma” (a very useful Lemma in control theory, optimization, ...)
- Discrete-time Algebraic Riccati equation (DARE) in the note (and lecture)

$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A$$

- Equivalent form (e.g. Matlab)

$$P = Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A$$

- Note: This is a quadratic equation with two solutions. The one we want is the positive definite solution (the “stabilizing” solution).

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>> help dare
dare Solve discrete-time algebraic Riccati equations.
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[X,L,G] = dare(A,B,Q,R,S,E) computes the unique
stabilizing solution X of the discrete-time
algebraic Riccati equation
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Example

$$\min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \dots$$

$$u_t = -Kx_t$$

where

$$K = R^{-1}B^{\top}P(I + BR^{-1}B^{\top}P)^{-1}A$$

$$P = Q + A^{\top}P(I + BR^{-1}B^{\top}P)^{-1}A$$