NTNU | Norwegian University of Science and Technology

TTK4135 – Lecture 6
Quadratic Programming
Equality-constrained QPs

Lecturer: Lars Imsland

Purpose of Lecture

- (Very) brief recap LPs

- QPs

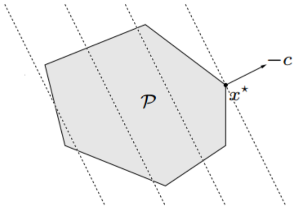
- Solving <u>equality-constrained</u> QPs
  - Can be solved *directly* by solving system of linear equations
  - Two formulations:
    - Full space method: One large equation system
    - Reduced space method: Two smaller equation systems

Reference: Chapter 16.1, 16.2 (15.3) in N&W

Types of Constrained Optimization Problems

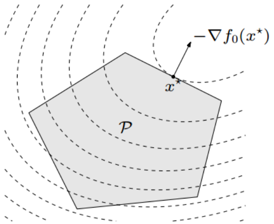
- Linear programming
  - Convex problem
  - Feasible set polyhedron

$$
\begin{aligned}
\text{min} \quad & c^\top x \\
\text{subject to} \quad & Ax \leq b \\
& Cx = d
\end{aligned}
$$



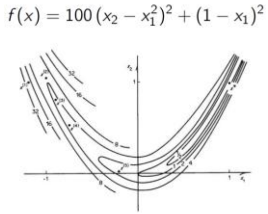
- Quadratic programming
  - Convex problem if $P \geq 0$
  - Feasible set polyhedron

$$
\begin{aligned}
\text{min} \quad & \frac{1}{2} x^\top P x + q^\top x \\
\text{subject to} \quad & Ax \leq b \\
& Cx = d
\end{aligned}
$$



- Nonlinear programming
  - In general non-convex!

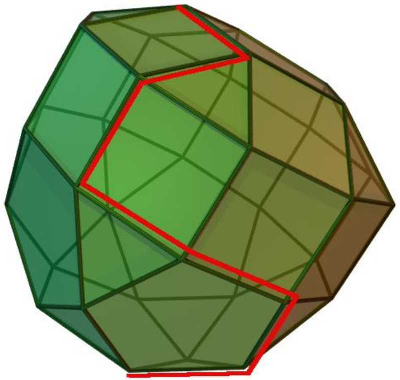
$$
\begin{aligned}
\text{min} \quad & f(x) \\
\text{subject to} \quad & g(x) = 0 \\
& h(x) \geq 0
\end{aligned}
$$



$$
\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{cases}
$$

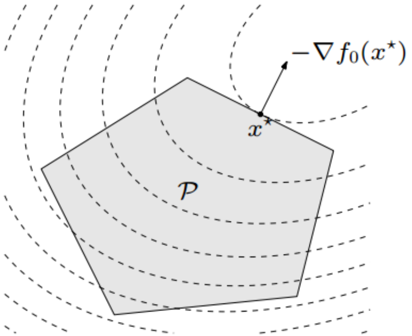
Last time: The simplex method for LP

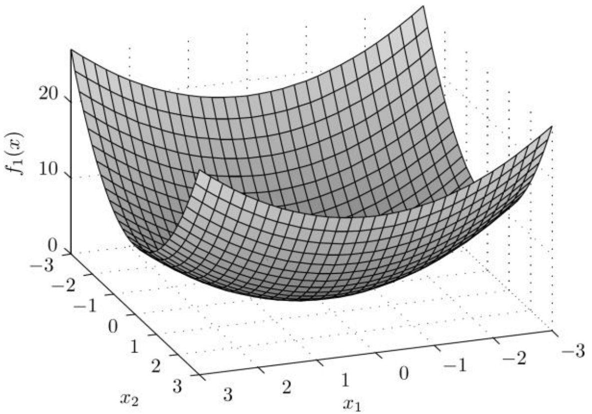
$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$



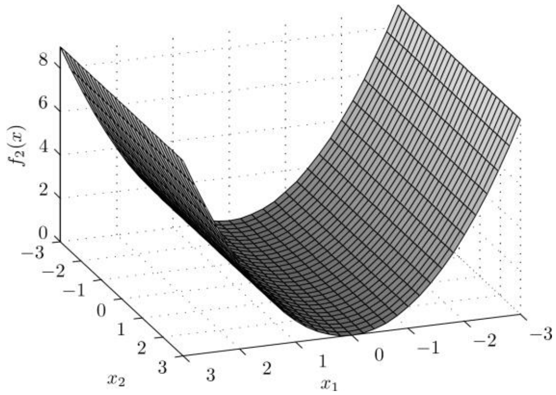
- The Simplex algorithm
  - The feasible set of LPs are (convex) polytopes
  - LP solution is a vertex/“corner”/**BFP** of the feasible set
  - Simplex works by going from vertex to neighbouring vertex in such a manner that the objective decreases in each iteration
  - In each iteration, we solve a linear system to find which component in the **basis** (set of “not active constraints”) we should change
  - “Almost” guaranteed convergence (if LP not unbounded or infeasible)
- Complexity:
  - Typically, at most *2m* to *3m* iterations
  - Worst case: All vertices must be visited (exponential complexity in *n*)
- Active set methods (such as simplex method):
  - Maintains explicitly an estimate of the set of inequality constraints that are active at the solution (the set $\mathcal{N}$ for the Simplex method)
  - Makes small changes to the set in each iteration (a single index in Simplex)

- Next lecture: Active set method for QP





$G > 0$, strictly convex



$G \geq 0$, convex

Why are we interested in (convex) QPs?

- It is the “easiest” nonlinear programming problem
  - “easy”: efficient algorithms exist for convex QPs, when local solutions are global

- The QP is the basic building block of SQP (“sequential quadratic programming”), a common method for solving general nonlinear programs
  - Topic in end of course (N&W Ch. 18)

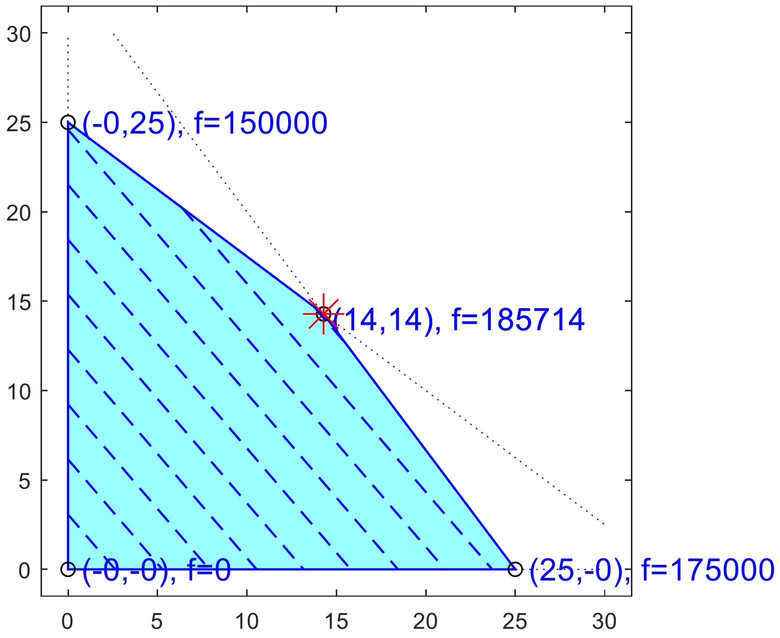
- QPs are often used in control, especially as solvers in Model Predictive Control
  - Topic in a few weeks
  - Also used in finance (“Portifolio optimization”), some types of Machine Learning/regression problems, control allocation, economics, …

QP Example: Farming example with changing prices

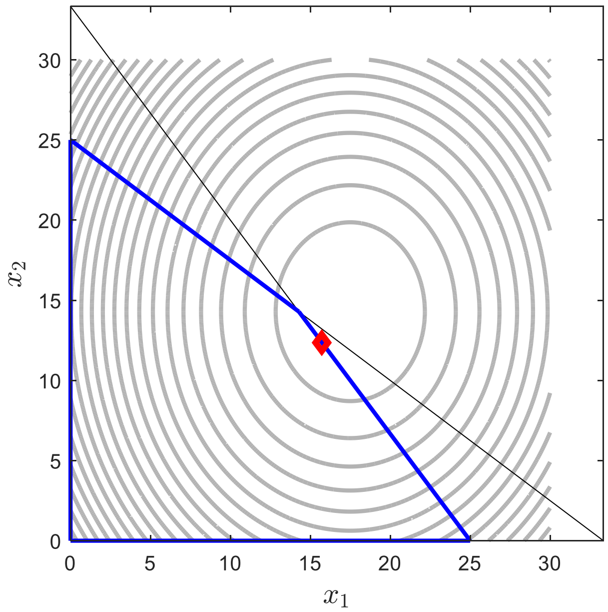
- A farmer wants to grow apples (A) and bananas (B)
- He has a field of size 100 000 m^2
- Growing 1 tonne of A requires an area of 4 000 m^2, growing 1 tonne of B requires an area of 3 000 m^2
- A requires 60 kg fertilizer per tonne grown, B requires 80 kg fertilizer per tonne grown
- The profit for A is 7000 – 200 x_1 per tonne (including fertilizer cost), the profit for B is 6000 – 140 x_1 per tonne (including fertilizer cost)
- The farmer can legally use up to 2000 kg of fertilizer
- The farmer wants to maximize his profits

LP farming example: Geometric interpretation and solution

$$
\begin{aligned}
\max_{x_1,\, x_2} \quad & 7000x_1 + 6000x_2 \\
\text{subject to:} \quad & 4000x_1 + 3000x_2 \leq 100000 \\
& 60x_1 + 80x_2 \leq 2000 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$



QP farming example: Geometric interpretation and solution



$$\max_{x_1,\, x_2} \quad (7000 - 200x_1)x_1 + (6000 - 140x_2)x_2$$

$$\begin{aligned} \text{subject to:} \quad 4000x_1 + 3000x_2 &\leq 100000 \\ 60x_1 + 80x_2 &\leq 2000 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{cases}$$

Lagrangian: $$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && \text{(stationarity)} \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, && \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(primal feasibility)} \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(dual feasibility)} \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. && \text{(complementarity condition/ complementary slackness)} \end{aligned}$$

KKT for Equality-constrained QP (EQP)

Solving EQPs

$$\min_x \quad \frac{1}{2}x^\top G x + c^\top x$$
$$\text{subject to} \quad Ax = b$$

KKT $\longrightarrow$

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix}$$

Solve linear system $\longrightarrow$ $x^*, \quad \lambda^*$

or

$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

Solve linear system $\longrightarrow$ $x^* = x + p, \quad \lambda^*$

When is the KKT solution the solution to the EQP?

Nullspace

When can EQP be solved?

“Proof” Theorem 16.2

Example 16.2

$$\min_{x} \quad \frac{1}{2}x^\top G x + c^\top x$$
$$\text{subject to} \quad Ax = b$$

$$\min_{x_1,x_2,x_3} \; 3x_1^2 + 2x_1x_2 + x_1x_3 + 2.5x_2^2 + 2x_2x_3 + 2x_3^2 - 8x_1 - 3x_2 - 3x_3$$
$$\text{subject to} \quad x_1 + x_3 = 3, \quad x_2 + x_3 = 0$$

$$\text{Matrices:} \quad G = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} -8 \\ -3 \\ -3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Note symmetry of G.
Always possible!

```
>> G = [6 2 1; 2 5 2; 1 2 4]; c = [-8; -3; -3]; A = [1 0 1;0 1 1]; b = [3;0];
>> K = [G, -A'; A, zeros(2,2)];
>> K\[-c;b]     % X = A\B is the solution to the equation A*X = B

ans =

    2.0000
   -1.0000
    1.0000
    3.0000
   -2.0000
```

x^*

λ^*

Example 16.2

$$\min_{x} \quad \frac{1}{2}x^\top G x + c^\top x$$
$$\text{subject to} \quad Ax = b$$

$$\min_{x_1,x_2,x_3} 3x_1^2 + 2x_1x_2 + x_1x_3 + 2.5x_2^2 + 2x_2x_3 + 2x_3^2 - 8x_1 - 3x_2 - 3x_3$$
$$\text{subject to} \quad x_1 + x_3 = 3, \quad x_2 + x_3 = 0$$

$$\text{Matrices:} \quad G = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} -8 \\ -3 \\ -3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

```
>> G = [6 2 1; 2 5 2; 1 2 4]; c = [-8; -3; -3]; A = [1 0 1;0 1 1]; b = [3;0];
>> K = [G, -A'; A, zeros(2,2)];
>> K\[-c;b]     % X = A\B is the solution to the equation A*X = B

ans =

    2.0000
   -1.0000
    1.0000
    3.0000
   -2.0000
```

x^*

λ^*

```
>> [Q,R,P] = qr(A')

Q =

   -0.7071    0.4082   -0.5774
         0   -0.8165   -0.5774
   -0.7071   -0.4082    0.5774

R =

   -1.4142   -0.7071
         0   -1.2247
         0         0

P =

     1     0
     0     1
```

Y Z

```
>> Z = Q(:,3);
>> Z'*G*Z

ans =

    4.3333
```

Fundamental Theorem of Linear Algebra

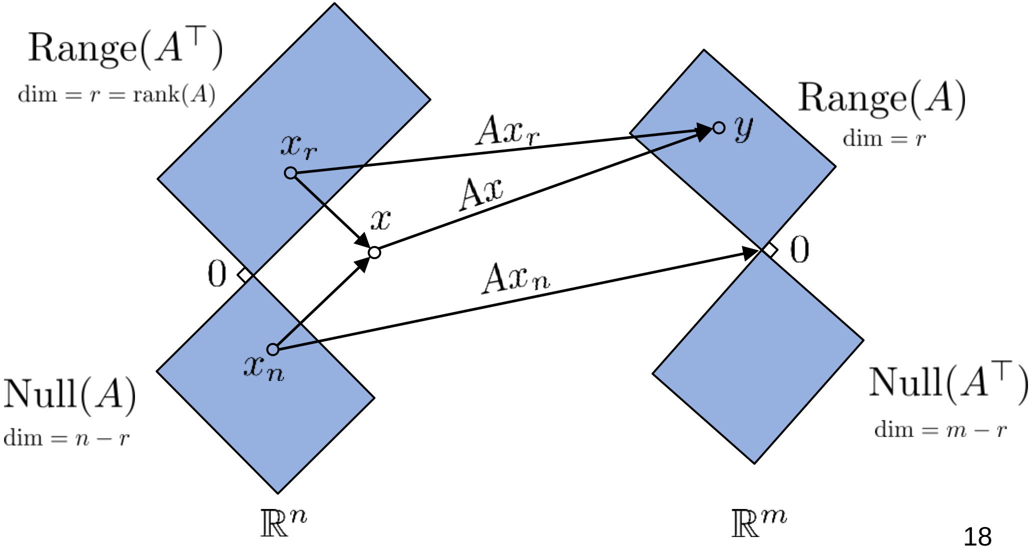
A matrix $A \in \mathbb{R}^{m \times n}$ is a mapping:

Nullspace of A: $\mathrm{Null}(A) = \{v \in \mathbb{R}^n \mid Av = 0\}$

Rangespace (columnspace) of A: $\mathrm{Range}(A) = \{w \in \mathbb{R}^m \mid w = Av, \text{ for some } v \in \mathbb{R}^n\}$

Fundamental theorem of linear algebra:

$$\boxed{\mathrm{Null}(A) \oplus \mathrm{Range}(A^\top) = \mathbb{R}^n}$$



Nullspace method/Elimination of variables (N&W 16.2/15.3)

Given $A \in \mathbb{R}^{m \times n}$, $m \leq n$, $\text{rank}(A) = m$

- Let Z be basis for $\text{Null}(A)$, $Z \in \mathbb{R}^{n \times (n-m)}$
- Let Y be basis for $\text{Range}(A^\top)$, $y \in \mathbb{R}^{n \times m}$

Nullspace method/Elimination of variables (N&W 16.2/15.3)

Summing up: Direct solutions of KKT system (16.2)

Solution of KKT system when $Z^\top G Z > 0$

- Full space:

$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

  - Use LU (or better, LDL-method, since KKT-matrix is symmetric)

- Reduced space, efficient if *n-m* ≪ *n*:

$$\begin{aligned} (AY)p_Y &= b - Ax \\ (Z^\top G Z)p_Z &= -Z^\top G Y p_Y - Z^\top (c + Gx) \\ p &= Y p_Y + Z p_Z \end{aligned}$$

  - Solve two much smaller systems using LU and Cholesky (both with complexity that scales with n^3)
  - Main complexity is calculating basis for nullspace. Usual method is using QR.

- Alternative to direct methods: Iterative methods for linear equation systems (16.3)
  - For very large systems, can be parallelized

Next time

- Active set method for general (convex) QPs
- Solving EQPs are key ingredient in active set method