NTNU | Norwegian University of Science and Technology

TTK4135 – Optimization and Control

Spring 2023

Lecturer: Lars Imsland
Teaching Assistant: Trym Arve Lund Gabrielsen
6 Student Assistants

Learning Objectives

- Optimization – important concepts and theory

- Formulating an engineering problem as a mathematical optimization problem (modeling for optimization)

- Solving an optimization problem numerically:
  - What type is the optimization problem?
  - What is the right algorithm/the right software?
  - Basics for implementation of algorithms

  for some important classes of optimization problems

- Applications of optimization in control engineering – model predictive control

Numerical optimization is an incredibly versatile tool across most engineering domains

Course Information: General

- Description:
  - All course information is provided through Blackboard
  - Course description: http://www.itk.ntnu.no/emner/ttk4135

- Assignments and assessment (More information on Blackboard):
  - Exercises: 7 of 10 assignments must be approved
    - No extra assignments will be given, deadlines are absolute
    - Pay attention and make sure your delivered assignments are approved
    - Do not copy (kok)!
  - Helicopter lab: must be approved
    - For approval: Must score 70% on lab report
  - Matlab assessments
    - 6 Matlab assessments, each counts 3.3% towards grade (pass/no pass)
  - Final exam (“school exam”)
    - Evaluation weighted 80% towards grade

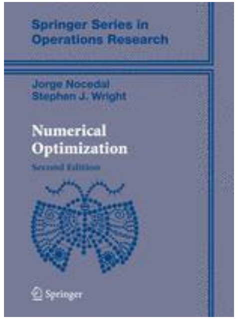
Matlab assessments

- Completed “inside” Blackboard
- You must implement and submit yourself, but you are allowed to cooperate, discuss and seek help
- Unlimited attempts

- The problems might seem more complex at a first glance, than they actually are
  - You only have to program a few lines inside a template

Course Information: Course Material

- Lectures:
  - Will not cover the full curriculum in lectures
  - Will focus on difficult parts and build intuition
  - Will be recorded, and video/PDF made available afterwards



- Course Material:
  - *Numerical Optimization*, J. Nocedal and S. J Wright, 2nd ed., Springer (ISBN-10: 0-387-30303-0 or ISBN-13: 987-0387-30303-1). Download here from campus or through VPN.
  - Errata on Blackboard
  - Note on *Merging Optimization and Control,* B. Foss and T. A. Heirung (Blackboard)
  - Note on *Matrix Calculus*, T. A. Heirung (Blackboard)

Course Information: Practical

- Grading
  - Final exam: 80%
  - Matlab assessments: 20% (6 individual tasks)
- Timetable
  - Lectures: Tuesday 14:15 – 16:00 EL5
Friday 08:15 – 10:00 A2
  - Assignment Sessions: Monday 17:15 - 18:00 EL5
- Exam: Written exam, June 8, 09:00 – 13:00
- Reference group!

Expected Background

- Linear algebra and real analysis
  - Quick recap next time (Also: Note on Blackboard + Exercise 0)
- Some numerical analysis (Newton's method)
- Basic control theory:
  - TTK4105 Control engineering
  - Advantage: TTK4115 Linear system theory

Tentative lecture schedule

	TTK4135 Plan for Spring 2023						
Week no.	Lectures Tuesday 14:15-16:00	Lectures Friday 8:15-10:00	Helicopter project	Matlab assessment	Exercise out (Mon 15:00)	Help session Monday 17:15-18:00	Exercise in (Thu 23:59)
2	Lecture 1 Introduction on optimization - N&W Ch.1	Lecture 2 Optimality conditions - N&W Ch. 12.1-12.2			0: Matrix Calculus, 1: KKT		
3	Lecture 3 Optimality conditons and linear algebra - N&W Ch.12.3, 12.5 (12.8, 12.9)	Lecture 4 Linear Programming - N&W Ch.13.1-13.5		Assessment 1 out	2: LP	0, 1, 2	
4	Lecture 5 Linear Programming - N&W Ch.13.1-13.5	Lecture 6 Quadratic programming - N&W Ch.15.3-15.5, 16.1-2,4-5		Assessment 1 in	3: LPQP	2, 3	0, 1
5	Lecture 7 Quadratic Programming - N&W Ch.15.3-15.5, 16.1-2,4-5	Lecture 8 Open loop dynamic optimization - MPC note Ch.3-3.2	Helicopter Lab week	Assessment 2 out	4: QP	3, 4	2
6	Lecture 9 Linear quadratic control - MPC note Ch.4.3.2-4.4	Lecture 10 Model predictive control - MPC note Ch.3.3-4.2.1	Helicopter Lab week	Assessment 2 in	5: OLMPC	4, 5	3
7	Lecture 11 Model predictive control - MPC note Ch.4.2.2-4.3.1	Lecture 12 Linear quadratic control - MPC note repetition and 4.6	Helicopter Lab week	Assessment 3 out	6: MPCLQR	5	4
8	No lecture	No lecture	Helicopter Lab week	Assessment 3 in		5, 6	
9	Lecture 13 Unconstrained optimization - N&W Ch.2.1-2.2	No lecture	Helicopter Lab week	Assessment 4 out	7: RICATTI	6, 7	5
10	No lecture	Lecture 14 Line search methods - N&W Ch.3-3.1, 3.4, 3.5	Helicopter Lab week	Assessment 4 in	8: UNCON	7, 8	6
11	Lecture 15 Quasi Newton methods - N&W Ch.6-6.1, 8-8.1	Lecture 16 Derivative free optimization - Ch.9, 9.5	Helicopter Lab week	Assessment 5 out	9: OPTALG	8, 9	7
12	Lecture 17 Newton's method for nonlinear equations - N&W 11-11.1, 11.2	Excursion?	Helicopter Lab week	Assessment 5 in	10: SQP		8
13	Excursion	Excursion					
14	Easter vacation						
15	Easter vacation	Lecture 18 Sequential quadratic programming (SQP) - N&W Ch.18-18.2					
16	Lecture 19 Sequential quadratic programming (SQP) - Ch.18.3-18.4, 15.4, 15.5	Lecture 20 Summing up + Nonlinear MPC - MPC note 4.5/4.6		Assessment 6 out		9, 10	
17		End of lecturing		Assessment 6 in		10	9
18							10
	Submit heli report	21.04.2022, 23:59					
	Q&A before exam						
	Final written exam	08.06.2023 (?)					

- Updated schedule will be available on Blackboard

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TTK4135 – Lecture 1
Optimization: What and Why?

Spring 2022

Lecturer: Lars Imsland

Purpose of Lecture

- Brief Timeline & Motivation
- Formulation of optimization problems, classes of optimization problems
- Definition of important terms
  - Convexity and non-convexity
  - Global vs. local solution
  - Constrained vs. unconstrained problems
  - Feasible set

Reference: Chapter 1 Nocedal & Wright

Brief Timeline

~1600 BC	Ancient Geometry: Babylonian method for solving $x^2 + bx = c$
~300 BC	Ancient Geometry: Euclid's minimal distance between point and line
~200s	Iterative approaches: Han Dynasty methods for solving $\sum_{i=0}^{3} a_i x^i = 0$
~900s	Modern algebra and arithmetics: Muhammad Al-Khwarizmi ("Algorismi") gives various root solving methods
1600s	Basis of Calculus of Variations: Newton's Body of minimal resistance, Bernoulli's Brachistochrone problem
1700s	Calculus of Variations and combinatorial optimization: Maupertius' Principle of Least Action, Samuel König's optimal honeycomb
1800s	First "Optimization algorithms": Hamilton-Jacobi Equation, Extreme Value Theorem, Rolle's Theorem, Cauchy's Gradient Descent
1900-1957	Rigorous theory and applications: Minkowski's Convex Sets, Hancock's Theory of Minima and Maxima, Kantorevich's Linear Optimization Problems, Dantzig's Simplex method, Neumann and Morgenstern's Dynamic Programming, Karush-Kuhn-Tucker's Optimality Conditions, Bellman's Optimality principle, Pontryagin's Maximum Principle
1950+	Optimization is applied to economics, agriculture, space travel, social media, robots, manufacturing, art and everything in between

Examples of optimization problems

In finance

- How to best invest NOK 500 000?
  - To maximize return
  - To minimize risk

  - No more than 20% of money in one stock
  - Return rate > 4%

In control engineering

- How to drive an autonomous vehicle from A to B?
  - To minimize fuel consumption
  - To minimize travel time

  - Distance to closest obstacle > 2 meters
  - Speed < 50 km/h.
  - Path needs to be smooth (no sudden changes in direction)

In machine learning

- How to assign likelihoods to transactions being fraudulent?
  - To minimize probability of a false negative
  - To penalize overfitting on training set

  - Probability of false positive < .15
  - Misclassification error on training set < 5%

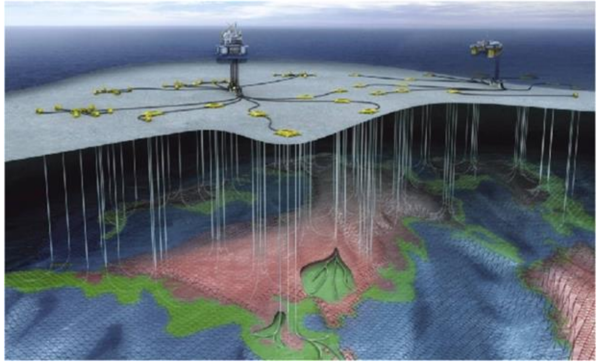
The question is **not**

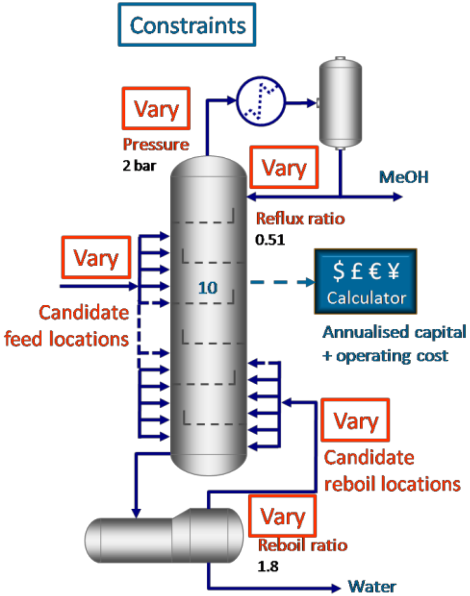
which problems are optimization problems (since all are),

but

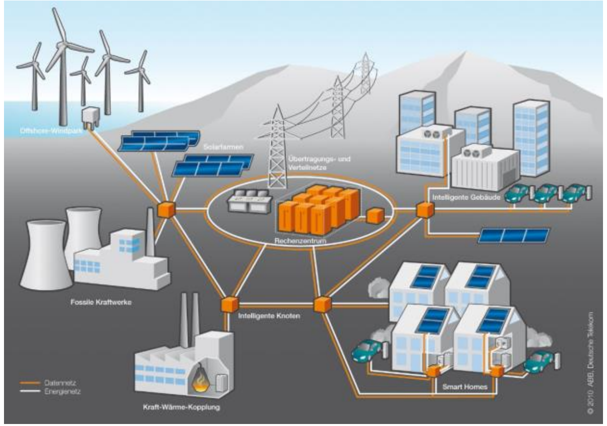
which optimization problems can we solve!

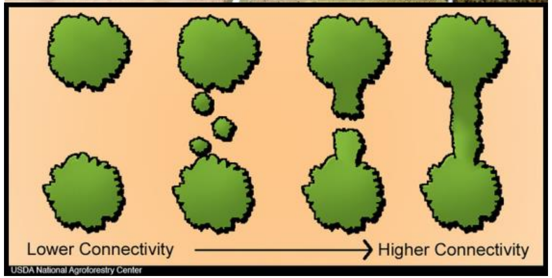
Control Applications: Model Predictive Control for all domains





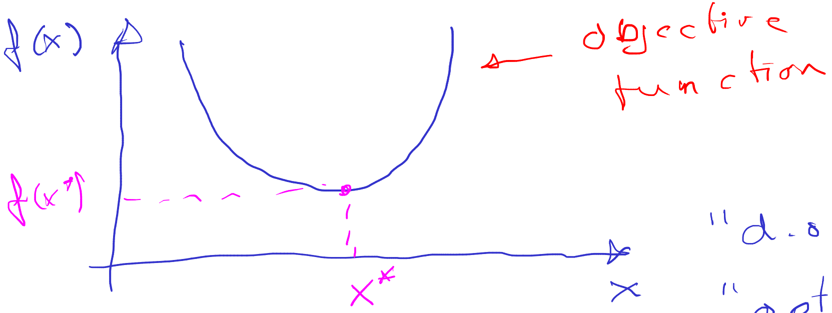
Lots of other applications





What is optimization?

Search for the best solution



"d.o.f."

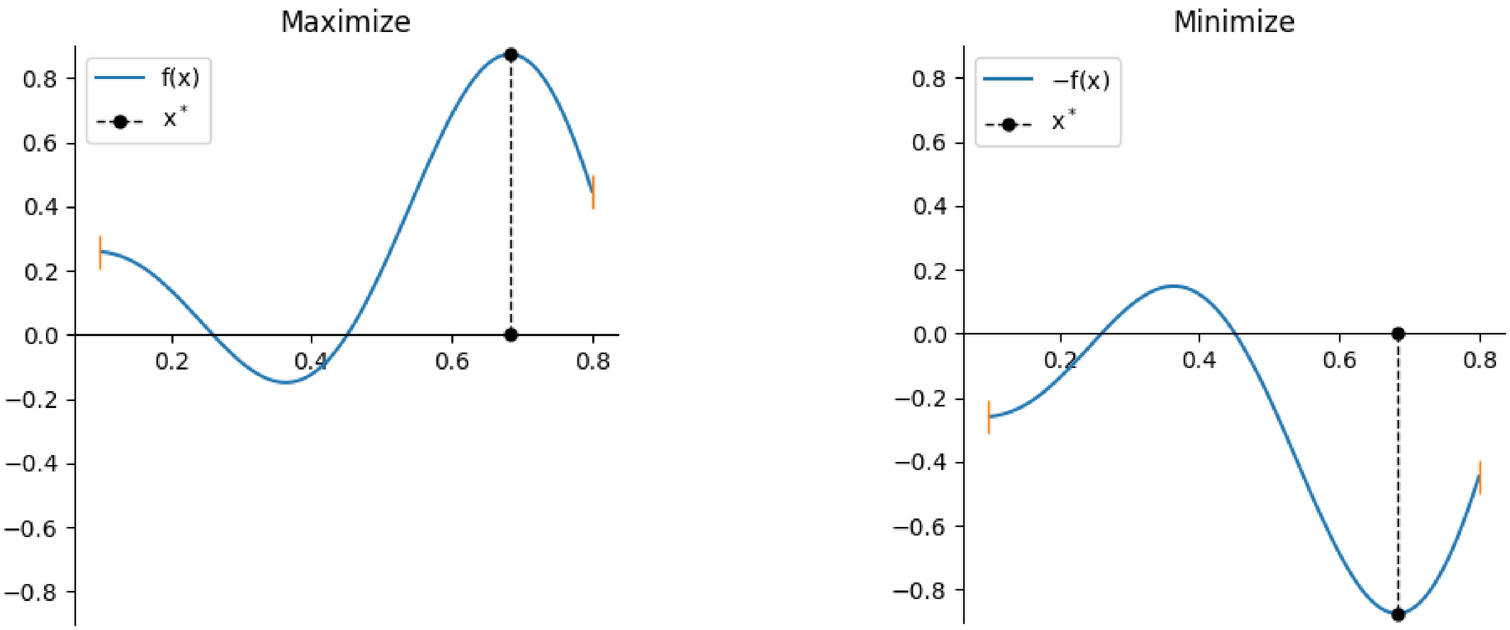
"opt. vars"

Math:

$$\min_{x \in \mathbb{R}^n} f(x)$$

[unconstrained opt.]

Minimization or Maximization?



Convention this course: Minimization!

What characterizes an optimum?

$\min_{x \in \mathbb{R}^n} f(x)$

$f \in C^2$

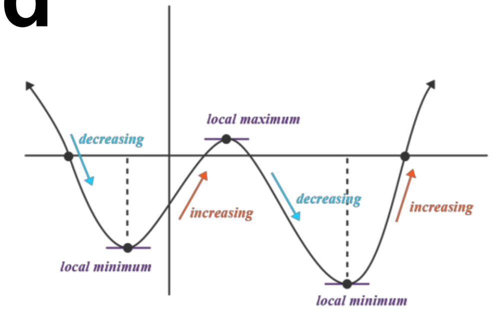
What are necessary cond. for x^* to be an optimum?

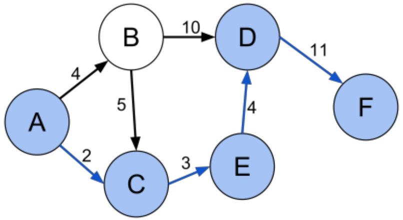
1. order : $f'(x^*) = 0$ $\quad [\nabla f(x) = 0]$

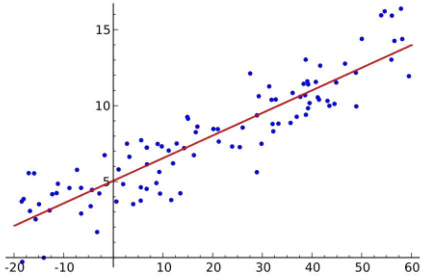
2. order : $f''(x^*) \geq 0$ $\quad [\nabla^2 f(x) \succeq 0]$

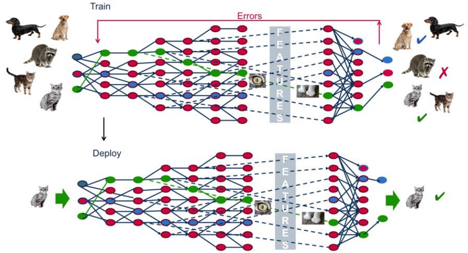
Optimization – A recurring friend

- Finding max and min of a function (Calc 1)
- The Lagrange Method (Calc 2)
- Algorithms course (Shortest path, dynamic programming, max flow, travelling salesman, etc)
- Statistics (Least-squares, data fitting)
- Machine Learning (Gradient descent)
- (And many applications in control…)







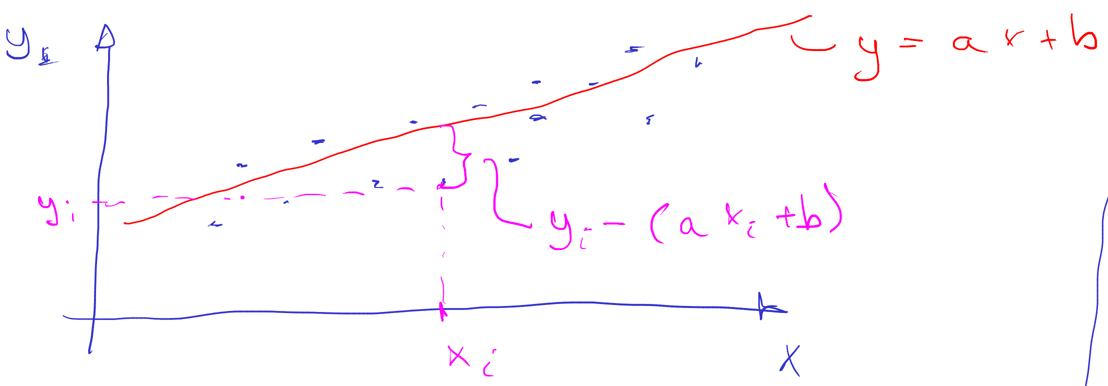


Example optimization problem: Least Squares

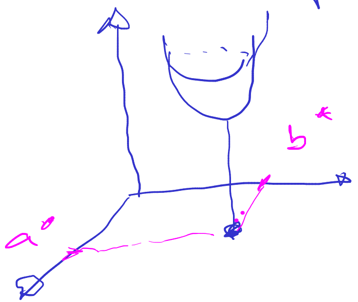
Problem: Predict how many ice creams to make, based on temperature forecast.

Data: x_i = Temperature day i

y_i = # ice creams sold day i



$$\min_{a,b} \underbrace{\sum_{i=1}^{n} (y_i - a x_i - b)^2}_{f(a,b)}$$

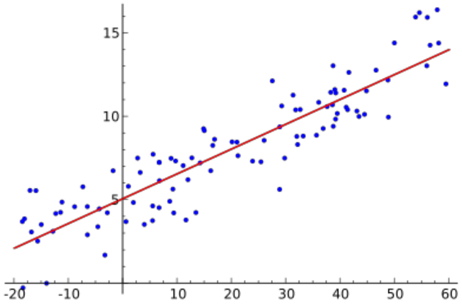


Example: Machine Learning

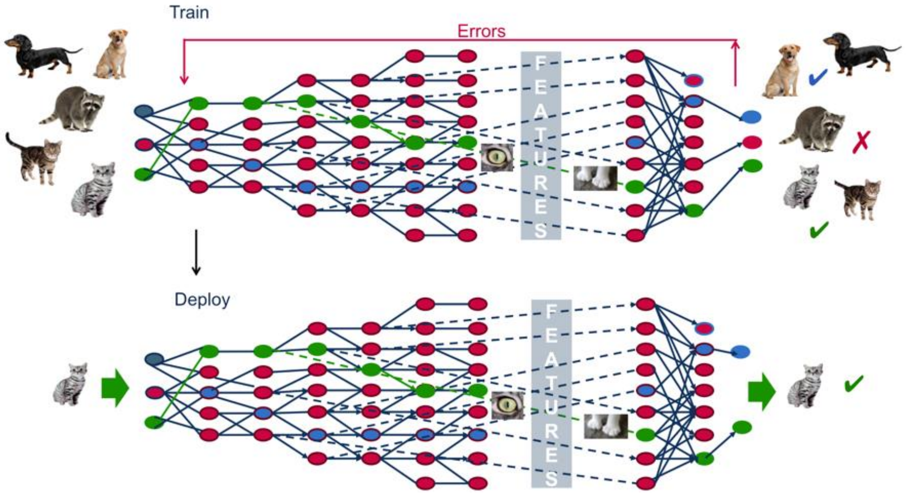
- Learn, and make predictions, from data

- Linear regression is the most basic ML algorithm, solved using optimization
  - Linear least squares: Explicit solution
  - Nonlinear least squares: Ch. 10, N&W

- In a similar fashion: ML, neural networks, deep learning etc. are “trained” using “gradient descent” algorithms
  - Gradient descent topic of Ch. 2-10, N&W



Wikipedia.org



Constrained optimization problems

Unconstrained

$$\min_{x \in \mathbb{R}^n} f(x)$$

Constrained opt.

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t.} \quad c_i(x) = 0, \quad i \in \mathcal{E}$$

$$c_i(x) \geq 0, \quad i \in \mathcal{I}$$

Feasible set

$$\Omega = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{array} \right\}$$

General Constrained Optimization Problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

$\mathcal{E} = \emptyset$

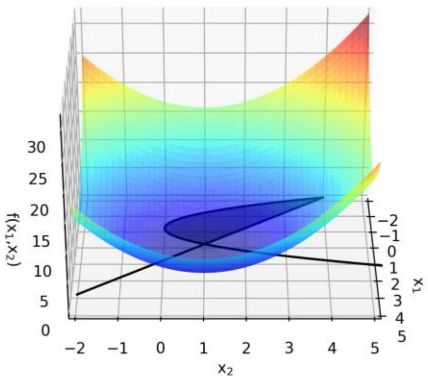
$\mathcal{I} = \{1, 2\}$

$-c_1(x)$

- Example:

$$\min \ (x_1 - 2)^2 + (x_2 - 1)^2 \quad \text{subject to} \quad \begin{array}{l} x_1^2 - x_2 \leq 0, \\ x_1 + x_2 \leq 2. \end{array}$$

$c_2(x) = 2 - x_1 - x_2 \geq 0$



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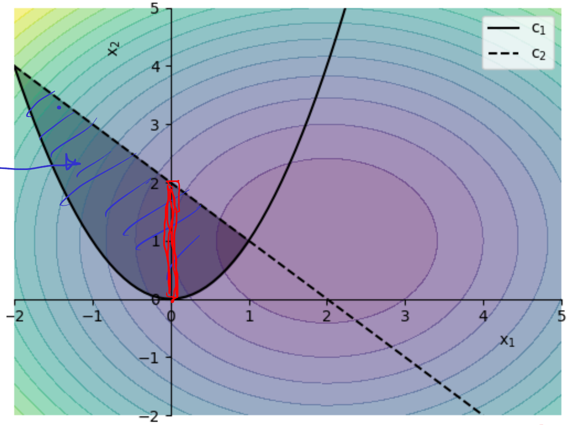


- What if we add equality-constraint $x_1 = 0$?

$\mathcal{E} = \{3\}$

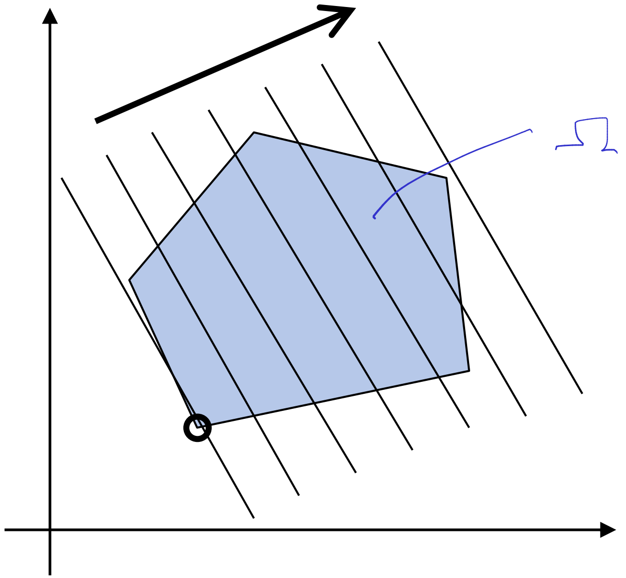
$c_3(x) = x_1 = 0$

Linear Programming

$$\min_x \; c^T x$$

$$\text{s.t.} \quad a_i^T x = b_i, \; i \in \mathcal{E}$$

$$a_i^T x \geq b_i, \; i \in \mathcal{I}$$

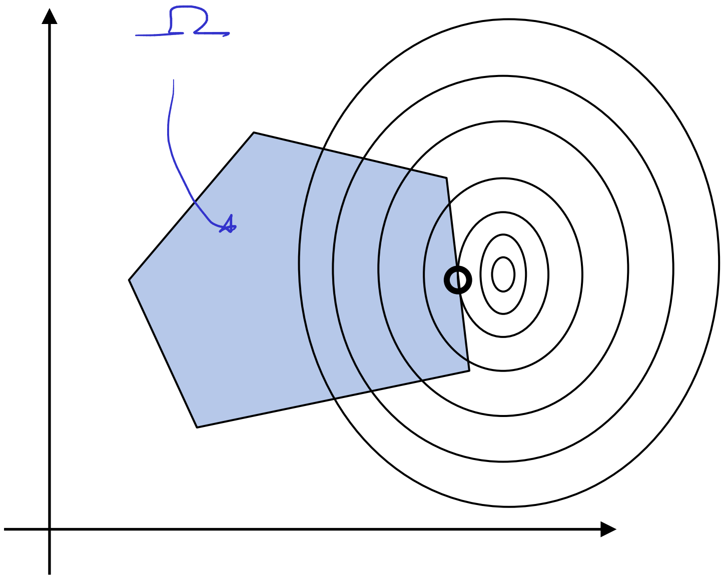


Quadratic Programming

$$\min_{x} \overbrace{\frac{1}{2} x^T G x + d^T x}^{q(x)}$$

$$\text{s.t.} \quad a_i^T x = b_i \, , \; i \in \mathcal{E}$$

$$a_i^T x \geq b_i \, , \; i \in \mathcal{I}$$



LP Example: Farming

- A farmer wants to grow apples (A) and bananas (B)
- He has a field of size 100 000 m^2
- Growing 1 tonne of A requires an area of 4 000 m^2, growing 1 tonne of B requires an area of 3 000 m^2
- A requires 60 kg fertilizer per tonne grown, B requires 80 kg fertilizer per tonne grown
- The profit for A is 7000 per tonne (including fertilizer cost), the profit for B is 6000 per tonne (including fertilizer cost)
- The farmer can legally use up to 2000 kg of fertilizer

- The farmer wants to maximize his profits

Formulating a LP optimization problem

Variables: x_1: # tonnes A

x_2: # tonnes B

Obj. fun: $7000 x_1 + 6000 x_2$

Constraints: $4000 x_1 + 3000 x_2 \leq 100\,000$

$60 x_1 + 80 x_2 \leq 2000$

$x_1 \geq 0$

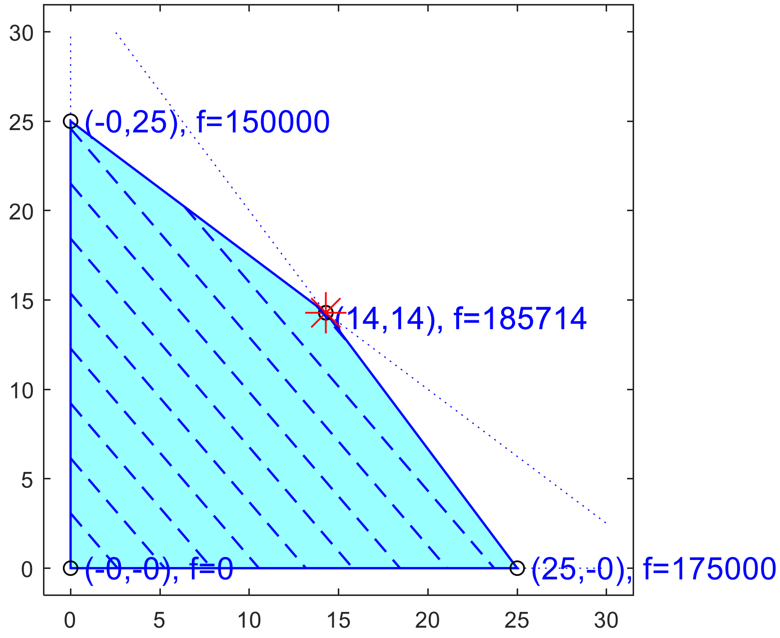
$x_2 \geq 0$

$$\min_x \; -7000 x_1 - 6000 x_2$$

s.t.

$c = [-7000, -6000]$

Farming Example: Geometric Interpretation and Solution



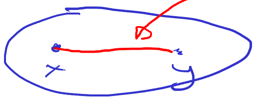
$$
\begin{aligned}
\max_{x_1,\, x_2} \quad & 7000x_1 + 6000x_2 \\
\text{subject to:} \quad & 4000x_1 + 3000x_2 \leq 100000 \\
& 60x_1 + 80x_2 \leq 2000 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

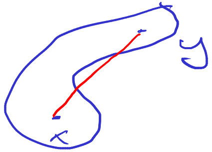
Matlab linprog input

Convexity

<u>Convex set</u>: A set S is convex if:

$$x, y \in S \Rightarrow \underbrace{\alpha x + (1-\alpha) y} \in S, \quad \alpha = [0, 1]$$

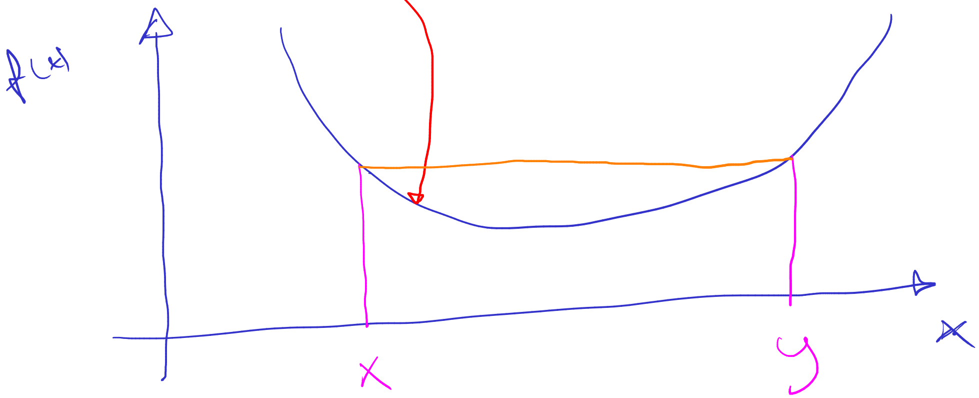




Convexity

Convex function: The function $f: S \to \mathbb{R}$ is convex if S is a convex set, and

$$\forall x, y \quad f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y), \quad \alpha \in [0,1]$$



Convex optimization problems

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad \underbrace{\begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{cases}}_{\Omega}$$

- An optimization problem is convex if
  1. $f(x)$ is a convex function, and
  2. the feasible set is a convex set

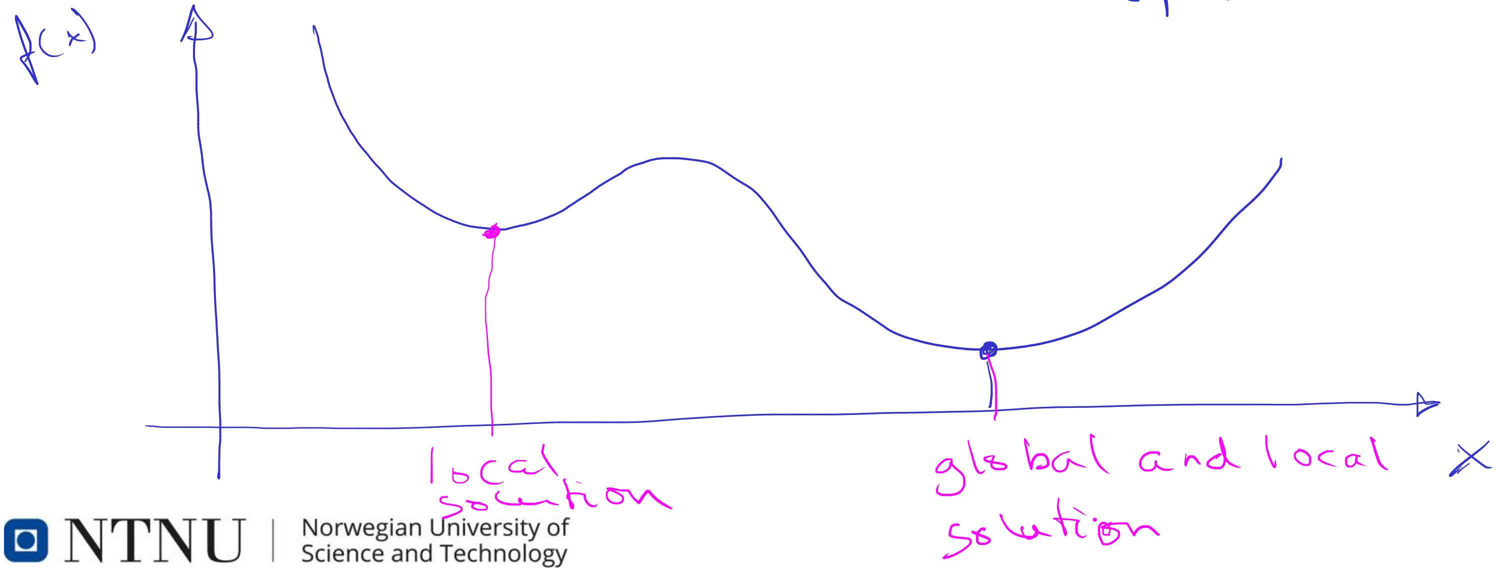
Ω

Local and global solutions

x^* is a global solution : $f(x^*) \leq f(x), \ \forall x \in \Omega$

x^* is a local solution : $f(x^*) \leq f(x), \ \forall x \in \mathcal{N} \cap \Omega$

where $\mathcal{N}$ is a neighborhood of x^*



Importance of convexity

Facts:

- Finding local solution: “easy”
- Finding global solution: “hard”

- Convex problems: All local solutions are global!
  - Consequence: Convex problems are “easy”

Optimization Taxonomy

Convex		Non-convex	Solution algorithms
Linear	Non-linear		
Linear programming			Simplex, interior point
	Quadratic programming		Active set, interior point
	Semi-definite programming (LMI, second-order cone p.)		Interior point
		Non-linear programming	SQP
		Mixed-integer programming	