NTNU | Norwegian University of Science and Technology

# TTK4135 – Lecture 9
# Linear Quadratic Control

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# Outline

- Recap: Open-loop linear dynamic optimization problems, and the different ways of solving them
  - Two batch methods (-> QPs)
  - One recursive method

- Today: Linear Quadratic Control ( = "The recursive method")
  - Finite horizon
  - Infinite horizon

Reference: F&H Ch. 4.3-4.4

# Last time: Dynamic open-loop optimization (with linear state-space model)

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + d_{x,t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t + d_{u,t}u_t + \frac{1}{2}\Delta u_t^\top S \Delta u_t$$

$\geq 0$ $\quad\quad\quad\quad$ $> 0$

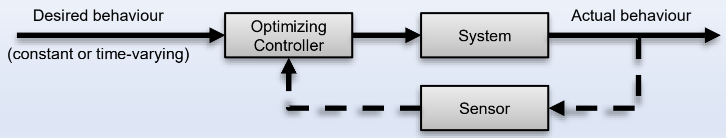
subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0,\dots,N-1\}\\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1,\dots,N\}\\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0,\dots,N-1\}\\
-\Delta u^{\text{high}} &\leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0,\dots,N-1\}
\end{aligned}$$

where

$$\begin{aligned}
&x_0 \text{ and } u_{-1} \text{ is given}\\
&\Delta u_t := u_t - u_{t-1}\\
&z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)\\
&n = N\cdot(n_x + n_u)\\
&Q_t \succeq 0 \quad t = \{1,\dots,N\}\\
&R_t \succ 0 \quad t = \{0,\dots,N-1\}
\end{aligned}$$

Open-loop vs closed-loop/feedback:

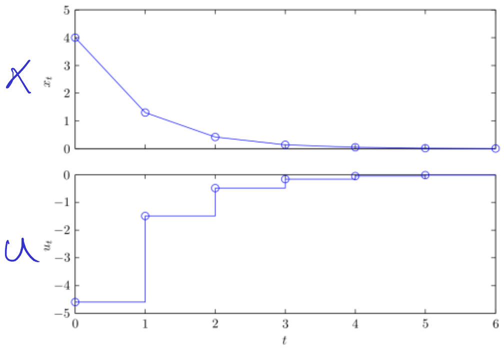


# The significance of weigths

$$\min \sum_{t=0}^{5} q\, x_{t+1}^2 + r\, u_t^2$$

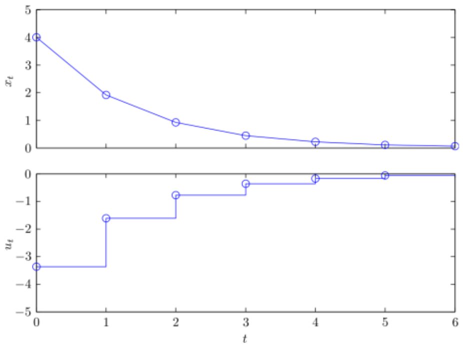
$$\text{s.t.} \quad x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \dots, 5$$

$q = 5,\ r = 1$



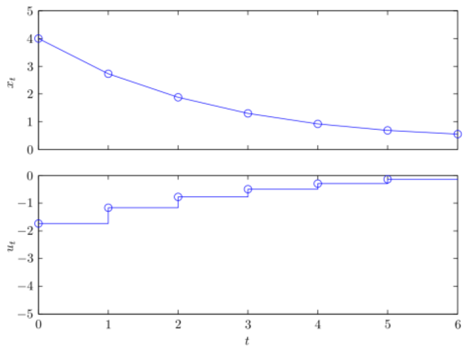
$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \quad \sum_{t=0}^{N-1} u_t^2 = 23.6$

$q = 2,\ r = 1$



$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \quad \sum_{t=0}^{N-1} u_t^2 = 14.7$

$q = 1,\ r = 2$



$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \quad \sum_{t=0}^{N-1} u_t^2 = 5.3$

# Linear quadratic control: Dynamic optimization without (inequality) constraints

$$\min_{z} \sum_{t=0}^{N-1} x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \dots, N-1$$
$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

Three approaches for solution:

- Batch approach v1, "full space" – solve as QP
- Batch approach v2, "reduced space" – solve as QP
- Recursive approach – solve as linear state feedback

Also work with input- and state constraints!

Only work without constraints!

# Linear Quadratic Control

## Batch approach v1, "Full space" QP

- Formulate with model as equality constraints, all inputs and states as optimization variables

$$
\begin{aligned}
\min_{z} \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^{\top} Q x_{t+1} + \frac{1}{2} u_t^{\top} R u_t \\
\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\
z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^{\top}
\end{aligned}
$$

$$
\min_{z} \quad \frac{1}{2} z^{\top} \begin{pmatrix} R & & & & \\ & Q & & & \\ & & R & & \\ & & & \ddots & \\ & & & & Q \end{pmatrix} z
$$

$$
\text{s.t.} \quad \begin{pmatrix} -B & I & & & & & \\ & -A & -B & I & & & \\ & & & -A & -B & I & \\ & & & & \ddots & \ddots & \\ & & & & -A & -B & I \end{pmatrix} z = \begin{pmatrix} A x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

$$
z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^{\top}
$$

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

# Linear Quadratic Control

## Batch approach v2, "Reduced space" QP

- Use model to eliminate states as variables
  - Future states as function of inputs and initial state

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \overset{S^x}{\begin{pmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{pmatrix}} x_0 + \overset{S^u}{\begin{pmatrix} B & & & & \\ AB & B & & & \\ A^2 & AB & B & & \\ \vdots & \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \dots & B \end{pmatrix}} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = S^x x_0 + S^u U$$

  - Insert into objective (no constraints!)

$$\min_U \frac{1}{2} \left(S^x x_0 + S^u U\right)^\top \mathbf{Q} \left(S^x x_0 + S^u U\right) + \frac{1}{2} U^\top \mathbf{R} U$$

$$\mathbf{Q} = \begin{pmatrix} Q & & \\ & Q & \\ & & \ddots \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R & & \\ & R & \\ & & \ddots \end{pmatrix}$$

  - Solution (when no inequality constraints) found by setting gradient equal to zero:

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -\underbrace{\left((S^u)^\top \mathbf{Q} S^u + \mathbf{R}\right)^{-1} (S^u)^\top \mathbf{Q} S^x}_{F} x_0 = -F x_0$$

# Linear Quadratic Control
## Recursive approach

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$
$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

- By writing up the KKT-conditions, we can show (we will do this today) that the solution can be formulated as:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned}
K_t &= R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, && t = 0, \dots, N-1 \\
P_t &= Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, && t = 0, \dots, N-1 \\
P_N &= Q
\end{aligned}$$

# Comments to the three solution approaches

- All give same numerical solution
  - If problem is strictly convex (Q psd, R pd), solution is unique
- The batch approaches give an open-loop solution, the recursive approach give a closed-loop (feedback) solution

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -Fx_0 \qquad \text{vs} \qquad u_t = -K_t x_t, \quad t = 0, \dots, N-1$$

- Constraints on inputs and states:
  - Easy for batch approaches (both becomes convex QPs)
  - Difficult for the recursive approach

- How to add feedback (and thereby robustness) to batch approaches?
  - Model predictive control! (Next time)

# Today: The recursive solution (LQ control)

# KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

**Lagrangian:** $\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$

**KKT-conditions** (First-order necessary conditions): If $x^*$ is a local solution and LICQ holds, then there exist $\lambda^*$ such that

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \qquad \text{(stationarity)}$$

$$\left.\begin{array}{ll} c_i(x^*) = 0, & \forall i \in \mathcal{E}, \\ \cancel{c_i(x^*) \geq 0,} & \cancel{\forall i \in \mathcal{I},} \end{array}\right\} \qquad \text{(primal feasibility)}$$

$$\cancel{\lambda_i^* \geq 0,} \quad \cancel{\forall i \in \mathcal{I},} \qquad \text{(dual feasibility)}$$

$$\cancel{\lambda_i^* c_i(x^*) = 0,} \quad \cancel{\forall i \in \mathcal{E} \cup \mathcal{I}.} \qquad \text{(complementarity condition/ complementary slackness)}$$

LQ~~R~~C:

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$\rightarrow z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

KKT: $\rightarrow$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0,$$
$$c_i(x^*) = 0, \quad \forall i \in \mathcal{E},$$
$$\cancel{c_i(x^*) \geq 0}, \quad \forall i \in \mathcal{I},$$
$$\cancel{\lambda_i^* \geq 0, \quad \forall i \in \mathcal{I}},$$
$$\cancel{\lambda_i^* c_i(x^*) = 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}}.$$

$$\mathcal{L}(z, \lambda_1, \lambda_2, \dots, \lambda_N) = \sum_{t=0}^{N-1} \left( \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \right) - \sum_{t=0}^{N-1} \lambda_{t+1}^\top (x_{t+1} - A x_t - B u_t)$$

KKT:

stationarity: $\nabla_z \mathcal{L} = 0$

① $$\frac{\partial \mathcal{L}}{\partial u_t} = R u_t + B^\top \lambda_{t+1} = 0, \quad t = 0, \dots, N-1$$

② $$\frac{\partial \mathcal{L}}{\partial x_t} = Q x_t - \lambda_t + A^\top \lambda_{t+1} = 0, \quad t = 1, \dots, N-1$$

③ $$\frac{\partial \mathcal{L}}{\partial x_N} = Q x_N - \lambda_N = 0$$

Feasibility:

④ $$x_{t+1} = A x_t + B u_t, \quad t = 0, \dots, N-1$$

(1) $\Rightarrow \quad u_t = -R^{-1} B^T \lambda_{t+1}$

(3) $\Rightarrow \quad \lambda_N = Q x_N$

"Guess": $\lambda_t = P_t x_t$ , $P_t = P_t^T > 0$

(4) $$\underline{x_{t+1}} = A x_t + B\left(-R^{-1} B^T P_{t+1} \underline{x_{t+1}}\right)$$

solve for $x_{t+1}$:

$$\left(I + B R^{-1} B^T P_{t+1}\right) x_{t+1} = A x_t$$

$$x_{t+1} = \underline{\left(I + B R^{-1} B^T P_{t+1}\right)^{-1} A} \; x_t$$

$$② \Rightarrow \quad Q x_t - \underbrace{\lambda_t}_{= P_t x_t} + A^T \underbrace{\lambda_{t+1}}_{= P_{t+1} x_{t+1}} = 0 , \quad t = 1, \dots, N-1$$

$$Q x_t - P_t x_t + A^T P_{t+1} \left( I + B R^{-1} B^T P_{t+1} \right)^{-1} A x_t = 0 , \quad t = 1, \dots, N-1$$

$$\underbrace{\left[ Q - P_t + A^T P_{t+1} \left( I + B R^{-1} B^T P_{t+1} \right)^{-1} A \right]}_{= 0, \; t = 1, \dots, N-1} x_t = 0 , \quad t = 1, \dots, N-1$$

must hold for all $x_t$

$$① \Rightarrow \quad u_t = - R^{-1} B^T \lambda_{t+1} = \underbrace{- R^{-1} B^T P_{t+1} \left( I + B R^{-1} B^T P_{t+1} \right)^{-1} A}_{:= K_t , \quad t = 0, \dots, N-1} x_t$$

Summing up: KKT satisfied if

$$u_t = -K_t x_t$$

where

$$K_t = R^{-1} B^T P_{t+1} (I + B R^{-1} B^T P_{t+1})^{-1} A, \quad t = 0, \dots, N-1$$

Discrete-time Riccati eq. $\longrightarrow$

$$P_t = Q + A^T P_{t+1} (I + B R^{-1} B^T P_{t+1}) A, \quad t = 1, \dots, N-1$$

$\uparrow$ 0

$$P_N = Q$$

# Second-order conditions

**Theorem 12.6** (Second-Order Sufficient Conditions).

*Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector $\lambda^*$ such that the KKT conditions (12.34) are satisfied. Suppose also that*

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0, \quad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0. \tag{12.65}$$

*Then $x^*$ is a strict local solution for (12.1).*

- Critical directions:

$$w \in \mathcal{C}(x^*, \lambda^*) \ \Leftrightarrow \ \begin{cases} \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{E}, \\ \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0, \\ \nabla c_i(x^*)^T w \geq 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0. \end{cases} \tag{12.53}$$

- The critical directions are the “allowed” directions where it is not clear from KKT-conditions whether the objective will decrease or increase

# Thm 16.4: For convex QP, KKT is sufficient

- From N&W, p. 464:

KKT conditions

> For convex QP, when $G$ is positive semidefinite, the conditions (16.37) are in fact sufficient for $x^*$ to be a global solution, as we now prove.
>
> **Theorem 16.4.**
>
> *If $x^*$ satisfies the conditions (16.37) for some $\lambda_i^*$, $i \in \mathcal{A}(x^*)$, and $G$ is positive semidefinite, then $x^*$ is a global solution of (16.1).*

- That is: Since the solution of the Riccati equation implies the KKT conditions are fulfilled, Thm 16.4 means that the Riccati equation gives the global solution

  – Side-remark: It is, in fact, the *unique* global solution. If G is positive definite (implied by Q positive definite), this follows from the proof of Thm 16.4. If Q positive semidefinite, further arguments are necessary (for instance using Thm 12.6 as in the note).

- Finite horizon LQ optimal control problem:

$$\begin{aligned}
\min_{z \in \mathbb{R}^n} \; & f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t \\
\text{subject to} \quad & x_{t+1} = A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\
& x_0 = \text{given} \\
& Q_t \succeq 0 \quad t = 1, \dots, N \\
& R_t \succ 0 \quad t = 0, \dots, N-1
\end{aligned}$$

where

$$\begin{aligned}
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u)
\end{aligned}$$

- State feedback solution

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

(discrete) Riccati equation

$$\begin{aligned}
K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \\
P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \\
P_N &= Q_N
\end{aligned}$$

# Linear quadratic control (finite horizon)

- The optimal solution to LQ control is a linear, time-varying state feedback:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 \\ P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

- Note that the gain matrix $K_t$ is independent of the states, and can therefore be computed in advance (knowing $A_t$, $B_t$, $Q_t$, $R_t$)

- The matrix (difference) equation

$$\begin{aligned} P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

is called the (discrete) *Riccati equation*

- Note that the "boundary condition" is given at the end of the horizon, and the $P_t$ -matrices must be found iterating backwards in time

# Example

$$\min \sum_{t=0}^{10} \frac{1}{2}x_{t+1}^2 + \frac{1}{2}r\ u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \ldots, 10$$

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$K_t = R^{-1}B^\top P_{t+1}(I + BR^{-1}B^\top P_{t+1})^{-1}A, \qquad t = 0, \ldots, N-1$$

$$P_t = Q + A^\top P_{t+1}(I + BR^{-1}B^\top P_{t+1})^{-1}A, \qquad t = 0, \ldots, N-1$$

$$P_N = Q$$

$$P_N = Q = 1$$

$$P_t = 1 + 1.2\, P_{t+1}\left(1 + \frac{1}{r}P_{t+1}\right)^{-1} 1.2 = 1 + 1.44\, r \frac{P_{t+1}}{r + P_{t+1}}, \quad t = 0, \ldots, N-1$$

$$K_t = \frac{1}{r}P_{t+1}\left(1 + \frac{1}{r}P_{t+1}\right)^{-1} 1.2 = 1.2 \frac{P_{t+1}}{r + P_{t+1}}$$

$$P_N = 1$$

$P_{10} = \ldots, \quad K_{10} = \ldots \quad \Rightarrow u_{10} = -K_{10}x_{10}$

$P_9 = \ldots, \quad K_9 = \quad \Rightarrow u_9 = -K_9 x_9$

$\vdots$

$P_1 = \ldots, \quad K_1 = \ldots \quad \Rightarrow u_1 = -K_1 x_1$

$K_0 = \ldots \quad \Rightarrow u_0 = -K_0 x_0$

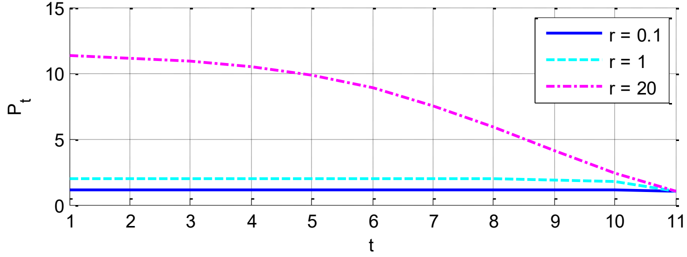
# Example

$$\min \sum_{t=0}^{10} \frac{1}{2}x_{t+1}^2 + \frac{1}{2}r\ u_t^2$$
$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \ldots, 10$$

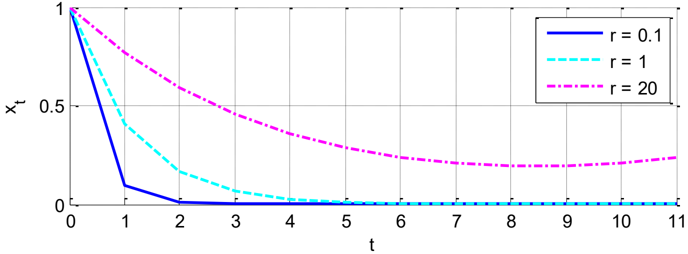
LQ solution

$$P_t = 1 + \frac{1.44 r P_{t+1}}{P_{t+1} + r}, \quad t = 10, \ldots, 1$$
$$P_{11} = 1$$
$$K_t = 1.2\frac{P_{t+1}}{P_{t+1} + r}, \quad t = 0, \ldots, 10$$

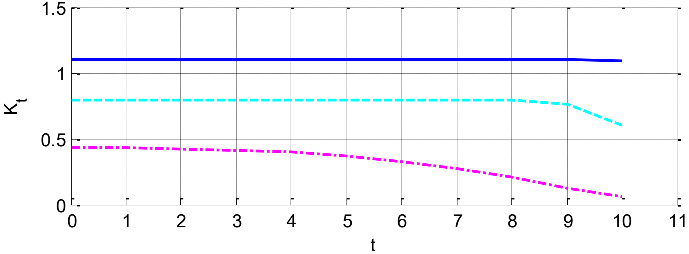
$P_t$



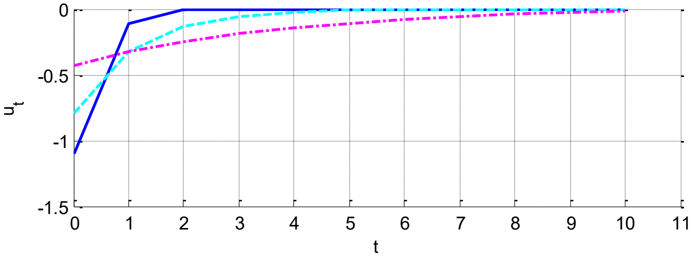

$x_t$




$K_t$




$u_t$

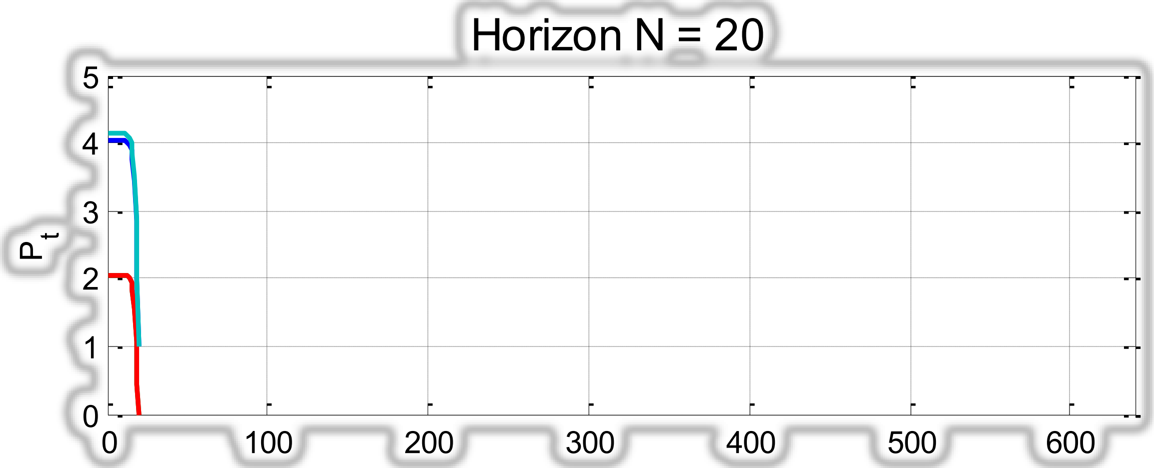


# What if the horizon goes to infinity?

# Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

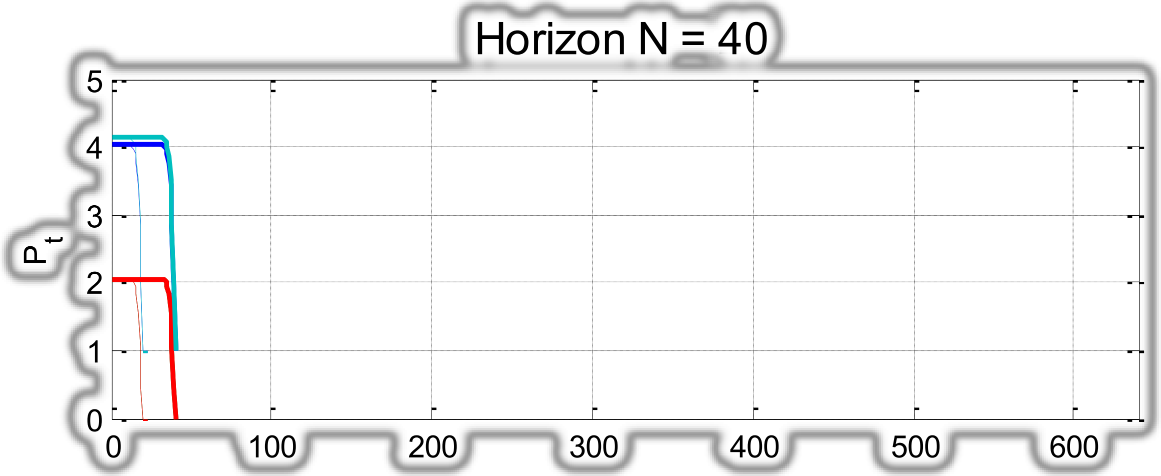
$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$



# Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$

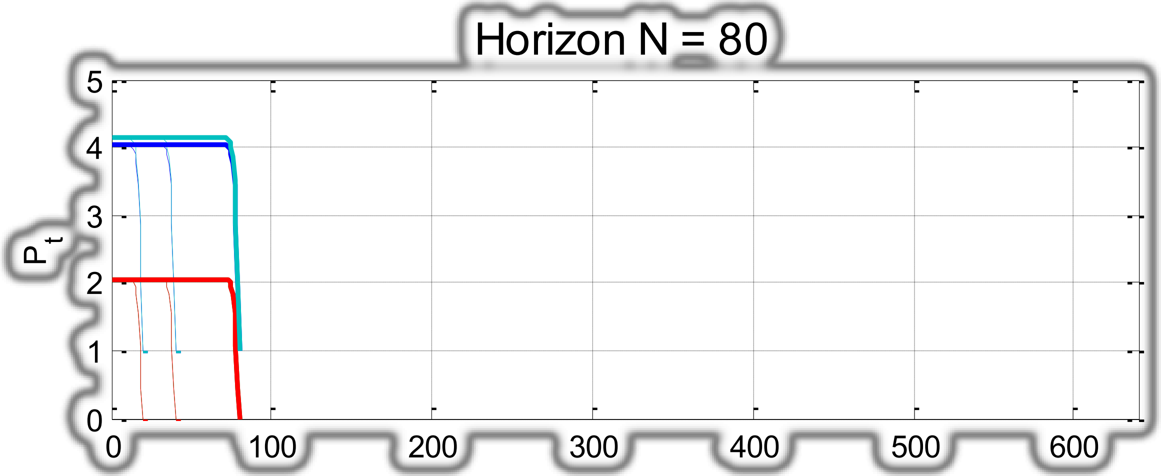


# Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$



# Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

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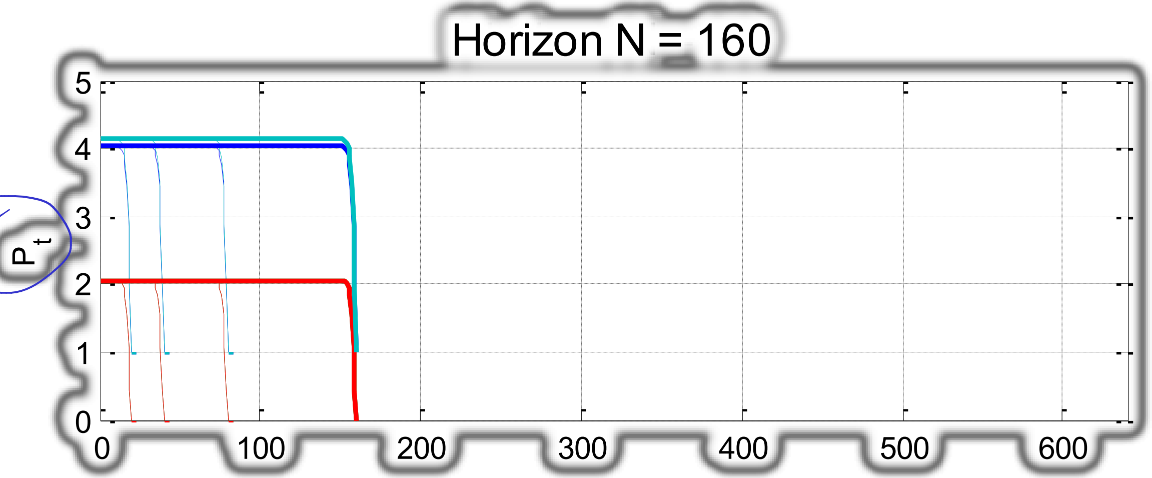

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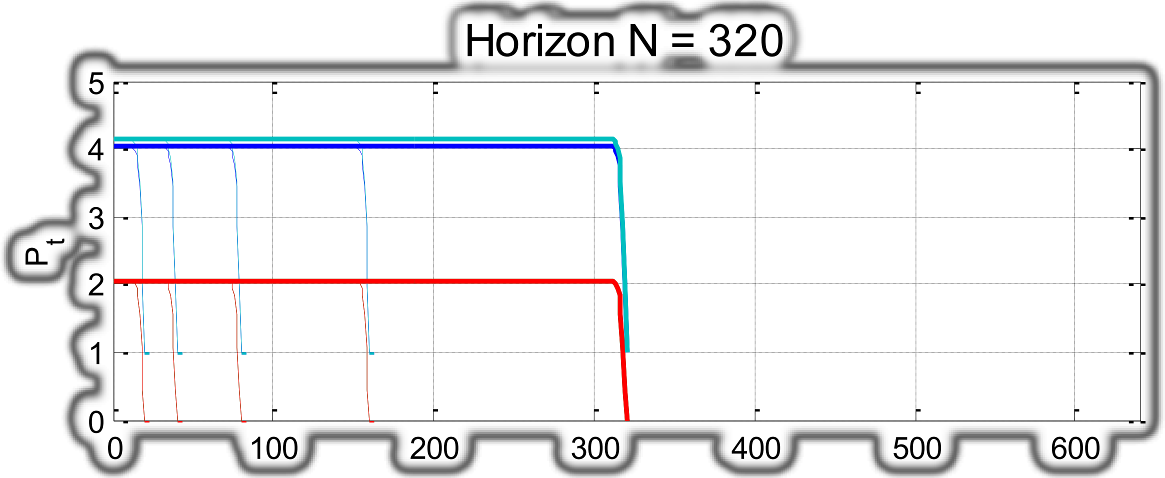

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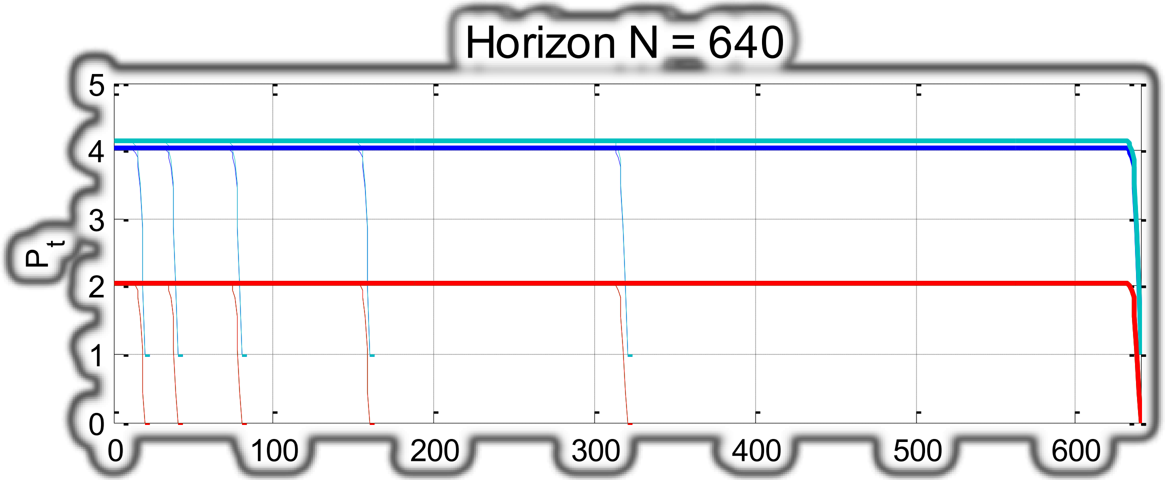

# Infinite horizon LQ control

$$\min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots$$

**Fact**: Steady-state ($P_{t+1} = P_t$) backwards-in-time solution of Riccati equation is infinite horizon solution

$\overset{P}{=}$

$$u_t = -K_t x_t$$

where

$$K_t = R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad t = 0, \dots, N-1$$
$$P_t = Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad t = 0, \dots, N-1$$
$$P_N = Q$$

$$u_t = -K x_t$$

where

$$K = R^{-1} B^\top P (I + B R^{-1} B^\top P)^{-1} A$$
$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A$$

# Infinite horizon LQ control

**Theorem**: The solution (when one exists) to

$$\min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots$$

is given by

$$u_t = -K x_t$$

where

$$K = R^{-1} B^\top P (I + B R^{-1} B^\top P)^{-1} A$$
$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A$$

(Discrete-time Algebraic Riccati Equation, DARE)

Two central questions:

- When does a solution exist?

$(A, B)$ stabilizable

- When is the closed-loop stable?

$x_{t+1} = A x_t + B u_t, \; u_t = -K x_t$

$= \underbrace{(A - BK)}_{} x_t$

stable : $|\lambda_i(A-BK)| \leq 1$

$(A, D)$ detectable

$D$ "square root" of $Q$ : $Q = D^\top D$

# Controllability vs stabilizability
# Observability vs detectability

- Stabilizable: All unstable modes are controllable
  (that is: all uncontrollable modes are stable)

- Detectability: All unstable modes are observable
  (that is: all unobservable modes are stable)

- Controllability implies stabilizability
- Observability implies detectability

Ex:

$$x_{t+1} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix} x_t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_t$$

| | Controllable | Stabilizable |
|---|---|---|
| $b_1 \neq 0,\ b_2 \neq 0$ | Y | Y |
| $b_1 \neq 0,\ b_2 = 0$ | N | Y |
| $b_1 = 0,\ b_2 \neq 0$ | N | N |

$a_1 = \sqrt{q_{11}}$

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}, \quad Q = D^T D$$

| | | Observable | Detectable |
|---|---|---|---|
| $D = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ ← | $q_{11} > 0,\ q_{22} > 0$ | Y | Y |
| $D = \begin{bmatrix} \sqrt{q_{11}} & 0 \end{bmatrix}$ ← | $q_{11} > 0,\ q_{22} = 0$ | N | Y |
| $D = \begin{bmatrix} 0 & \sqrt{q_{22}} \end{bmatrix}$ ← | $q_{11} = 0,\ q_{22} > 0$ | N | N |

# Riccati equations

$P_t = P_t^\top > 0$

- Discrete-time Riccati equation in the note (and lecture)

$$P_t = Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad P_N = Q_N$$

- However, another, equivalent, form is found in other sources:

$$P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (R_t + B_t^\top P_{t+1} B_t)^{-1} B_t^\top P_{t+1} A_t, \quad P_N = Q_N$$

- The latter is more numerically stable due to "enforced symmetry"

- The trick used to get the different formulas is the "Matrix Inversion Lemma" (a very useful Lemma in control theory, optimization, …)

- Discrete-time Algebraic Riccati equation (DARE) in the note (and lecture)

$$P = Q + A^\top P(I + BR^{-1}B^\top P)^{-1} A$$

- Equivalent form (e.g. Matlab)

$$P = Q + A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA$$

  - Note: This is a quadratic equation with two solutions. The one we want is the positive definite solution (the "stabilizing" solution).

```
>> help dare
dare  Solve discrete-time algebraic Riccati equations.

   [X,L,G] = dare(A,B,Q,R,S,E) computes the unique
            stabilizing solution X of the discrete-time
            algebraic Riccati equation
```

# Example

$$\min \sum_{t=0}^{\infty} \frac{1}{2}x_{t+1}^2 + \frac{1}{2}r\, u_t^2$$

$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \ldots$$

$$u_t = -Kx_t$$

where

$$K = R^{-1}B^\top P(I + BR^{-1}B^\top P)^{-1}A$$

$$\rightarrow \quad P = Q + A^\top P(I + BR^{-1}B^\top P)^{-1}A$$

$$P = 1 + 1.44\, r \frac{P}{r+P} \Rightarrow P(r+P) = r + P + 1.44\, rP$$

$$\Rightarrow P^2 + (r - 1 - 1.44\, r)P - r = 0$$

quadratic eq. !

$$\Rightarrow \quad P = \frac{-(\quad) \pm \sqrt{\cdots}}{2}$$

$\Rightarrow$ Two solutions

Choose the positive one