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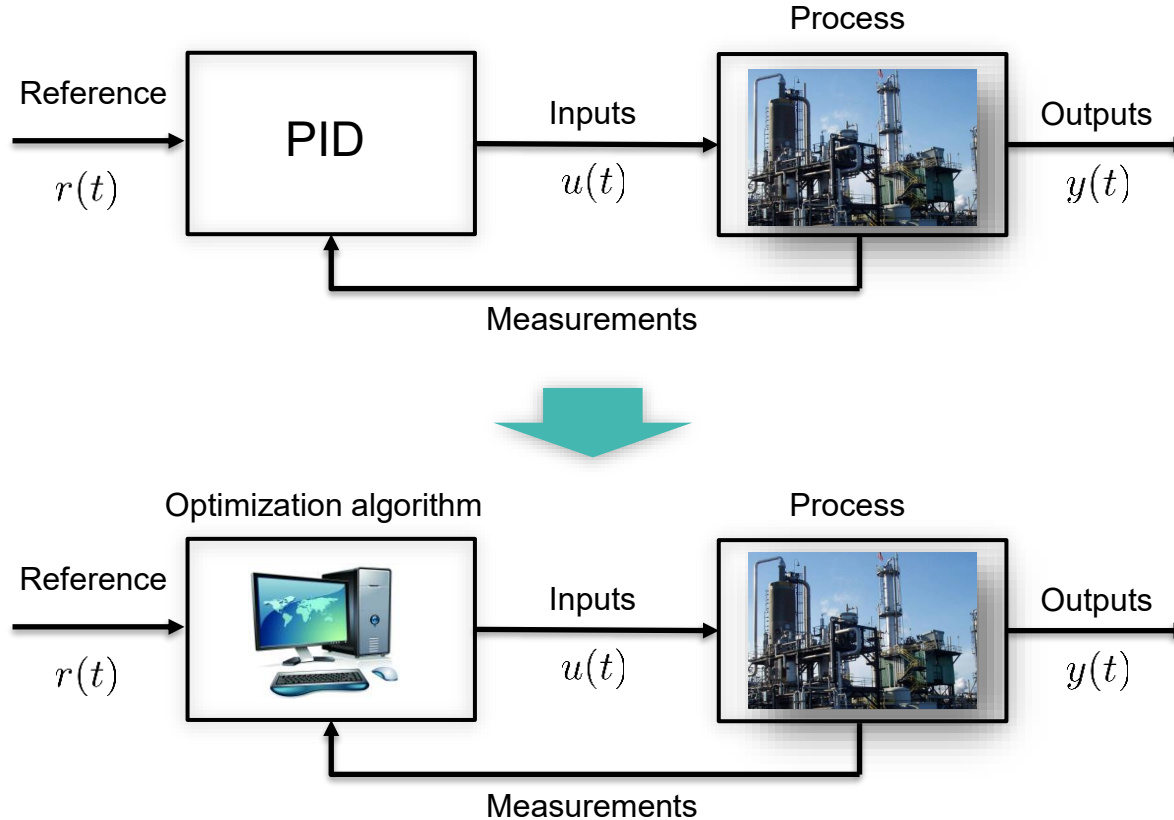
Norwegian University of
Science and Technology

TTK4135 – Lecture 10

Model Predictive Control

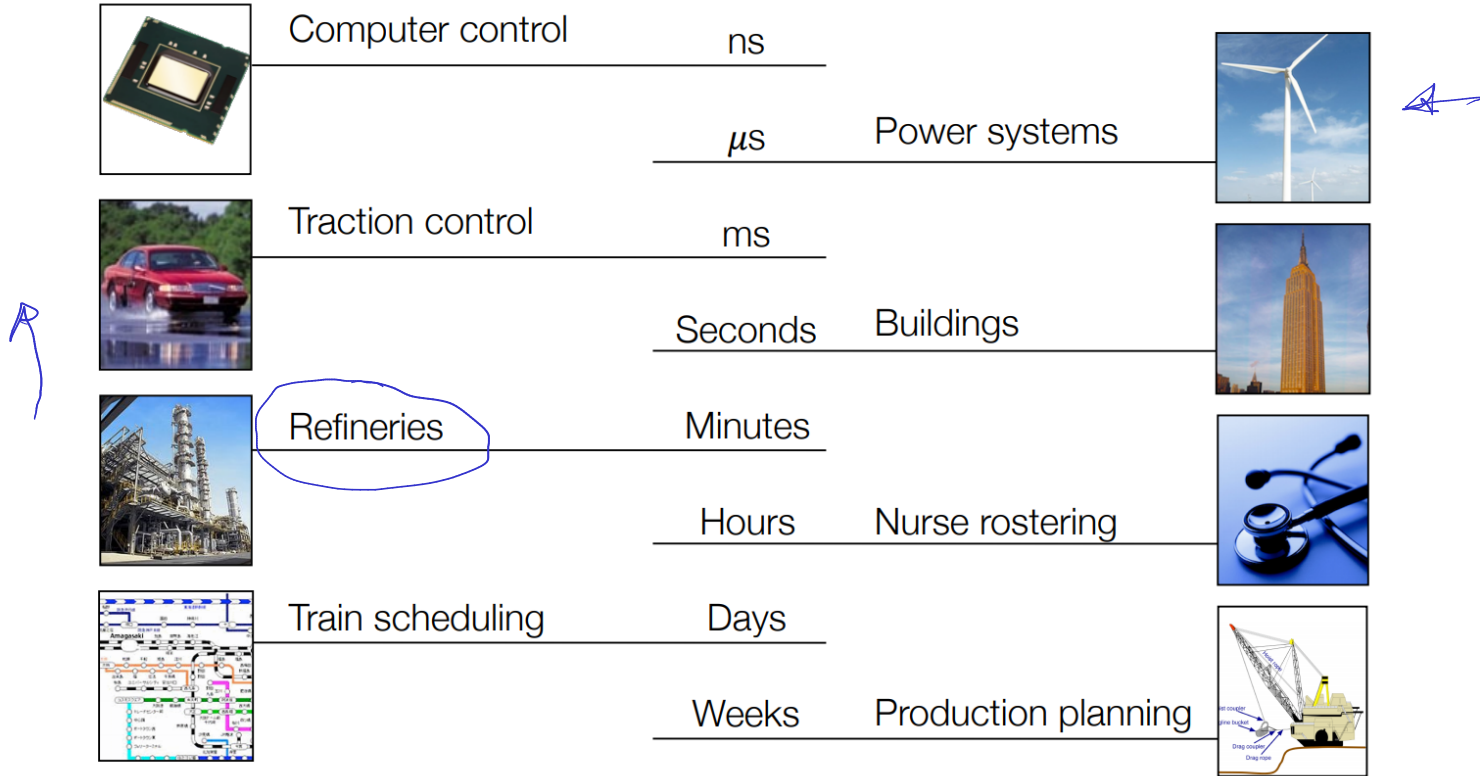
Lecturer: Lars Imsland

Model Predictive Control – control based on optimization



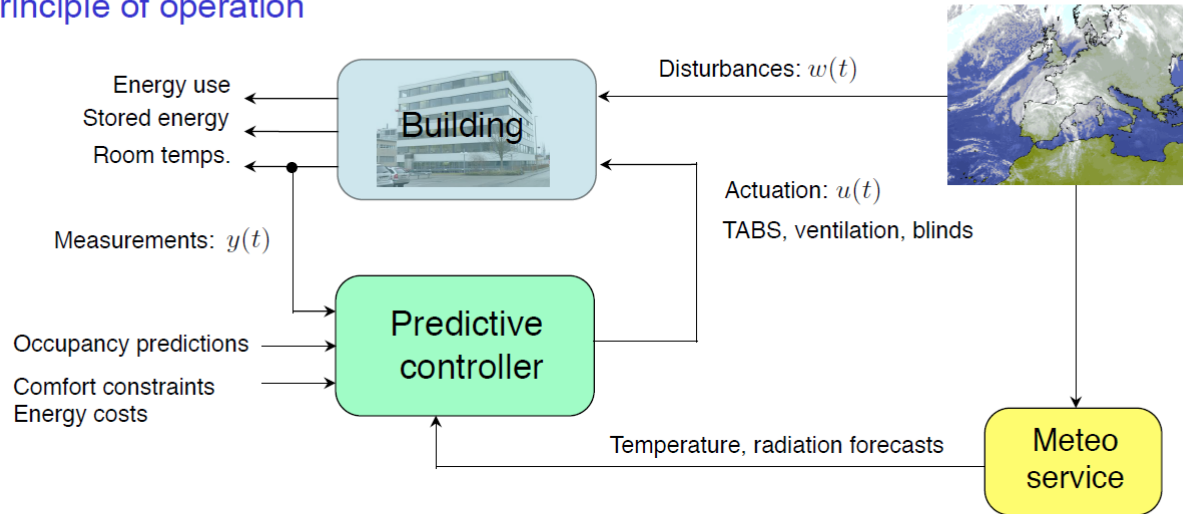
A **model** of the process is used to compute the **control** signals (inputs) that optimize **predicted** future process behavior

MPC: Applications



Model predictive control (MPC)

Principle of operation



Predicted Cost = minimize $\underset{u(t)}{\text{Expected}} \left(\sum_t^{t+N} \text{energy cost}(t) \right)$ ← Minimize the predicted energy cost

subject to $u(t) \in \mathcal{U}$ ← Actuation within limits
 $x(t) \in \mathcal{X}$ ← Predicted temperatures within limits
 $x(t+1) = f(x(t), u(t), w(t))$ ← Predicted dynamics of the building

From ETH

Open-loop optimization with linear state-space model

QP

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} \underline{x_{t+1}^\top} Q_{t+1} \underline{x_{t+1}} + \underline{d_{x,t+1}} x_{t+1} + \frac{1}{2} \underline{u_t^\top} R_t \underline{u_t} + \underline{d_{u,t}} u_t + \frac{1}{2} \underline{\Delta u_t^\top} S \underline{\Delta u_t}$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\}$$

$$x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\}$$

$$u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

$$-\Delta u^{\text{high}} \leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

where

x_0 and u_{-1} is given

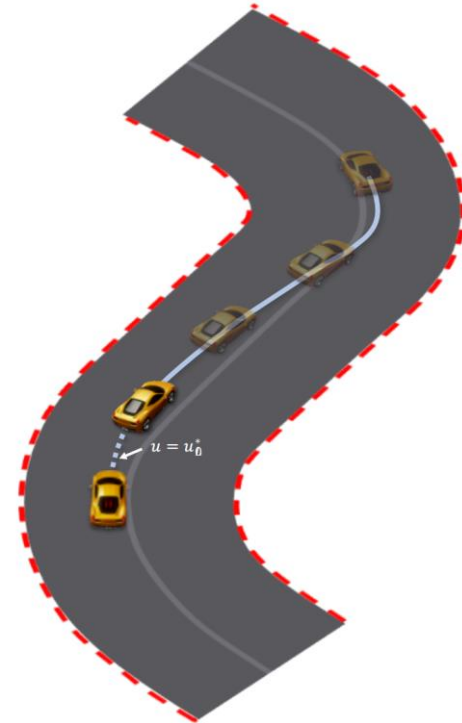
$$\Delta u_t := u_t - u_{t-1}$$

$$z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)$$

$$n = N \cdot (n_x + n_u)$$

$$Q_t \succeq 0 \quad t = \{1, \dots, N\}$$

$$R_t \succeq 0 \quad t = \{0, \dots, N-1\}$$



Open-loop dynamic optimization problem as QP

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

subject to

$$x_{t+1} = Ax_t + Bu_t, \quad t = \{0, \dots, N-1\}$$

$$x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\}$$

$$u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

where

x_0 is given

$$z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)$$

$$Q \succeq 0, \quad R \succ 0$$

$$\min_z z^\top G z$$

$$G = \begin{bmatrix} Q & K & 0 \\ 0 & R & \dots \end{bmatrix}$$

$$\text{s.t. } A_{\text{eq}} z = b_{\text{eq}}$$

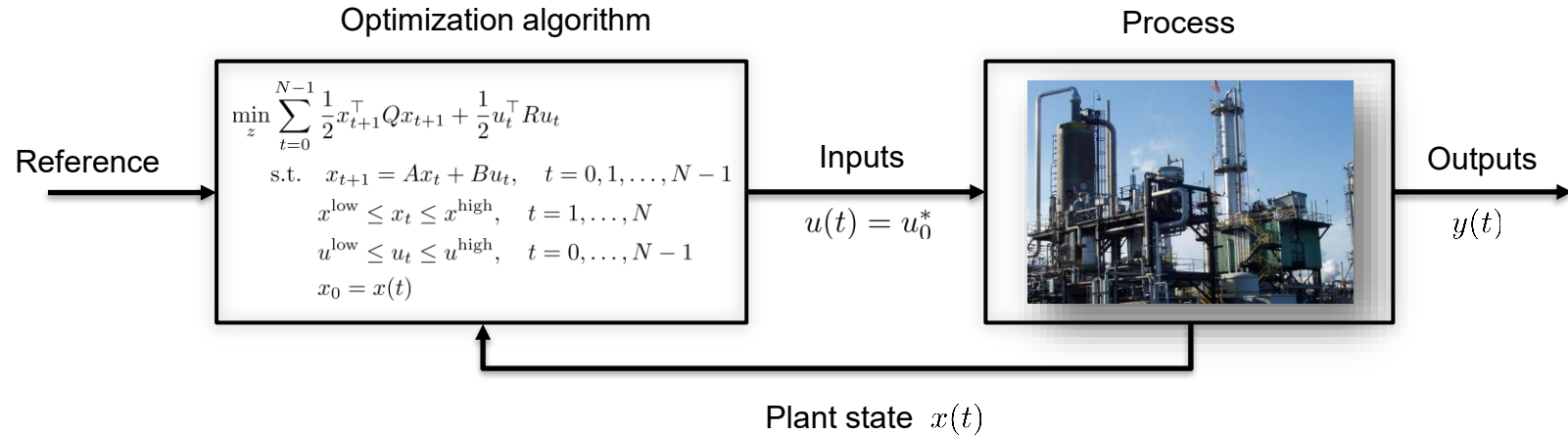
$$A_{\text{ineq}} z \geq b_{\text{ineq}}$$

$$A_{\text{eq}} = \begin{bmatrix} B & -I & 0 & 0 & 0 & 0 & \dots \\ 0 & A & B & -I & 0 & 0 & \dots \\ 0 & 0 & 0 & A & B & -I & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad z = \begin{bmatrix} -Ax_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$A_{\text{ineq}} = \begin{bmatrix} I & 0 & 0 & \dots \\ -I & 0 & 0 & \dots \\ 0 & I & 0 & \dots \\ 0 & -I & 0 & \dots \\ 0 & 0 & I & \dots \end{bmatrix} \quad z \geq \begin{bmatrix} u^{\text{low}} \\ -u^{\text{high}} \\ x^{\text{low}} \\ -x^{\text{high}} \\ u^{\text{low}} \\ \vdots \end{bmatrix}$$

Batch approach,
"full space"

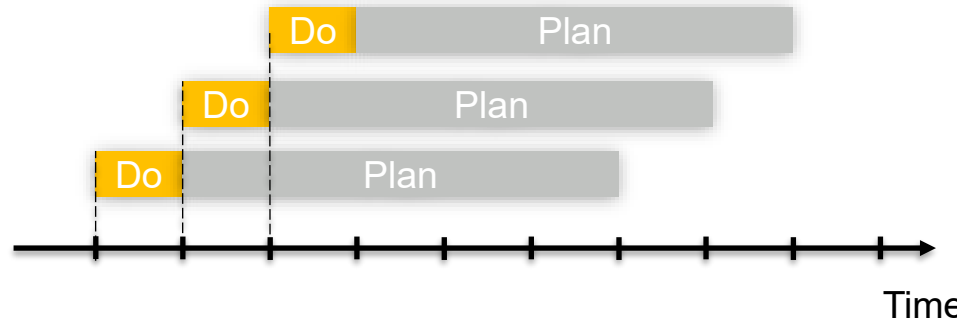
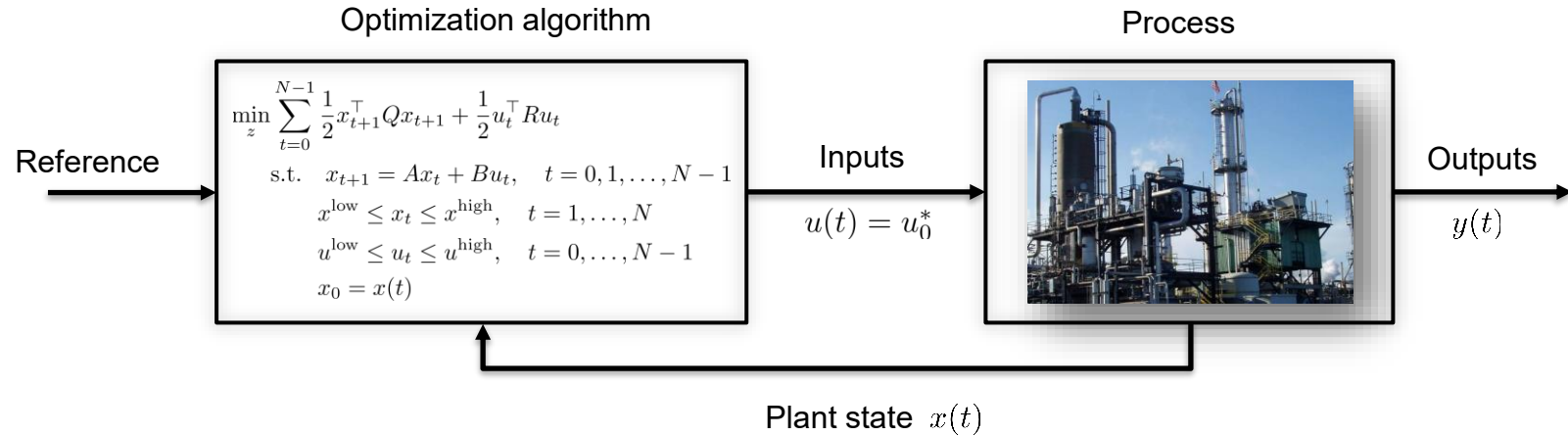
Model predictive control principle



At each sample time:

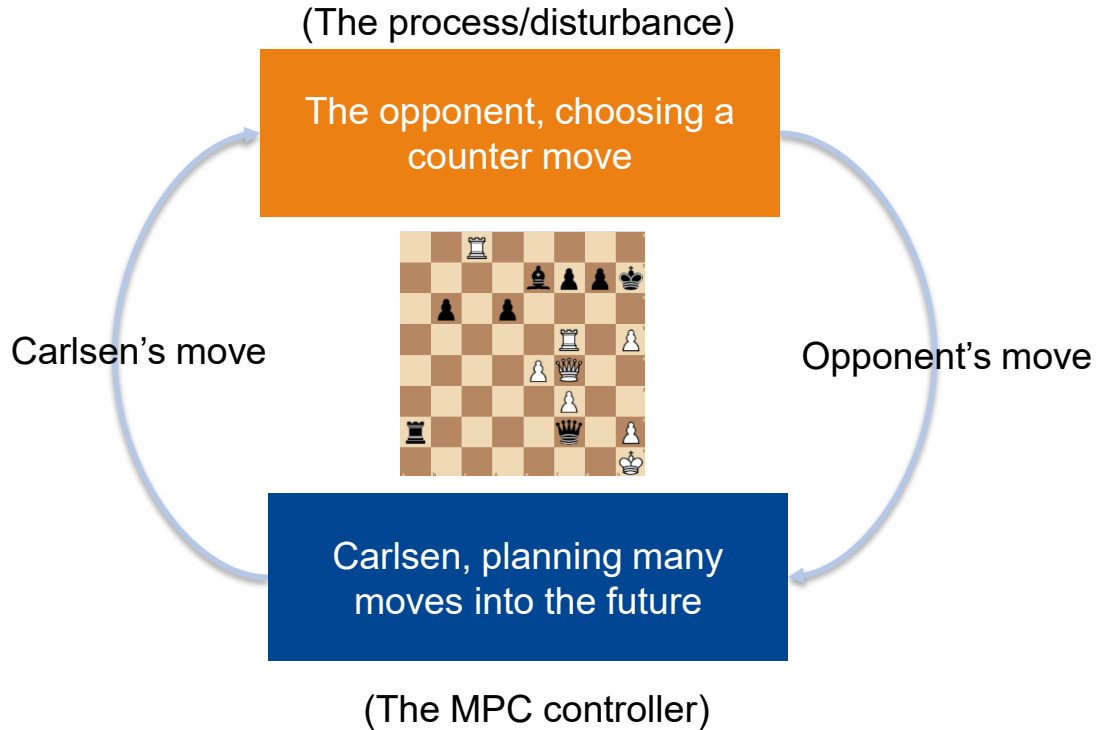
- Measure or estimate current state $x(t)$
- Find optimal open-loop input sequence $U^* = (u_0^*, u_1^*, \dots, u_{N-1}^*)$
- Implement only the first element of sequence: $u(t) = u_0^*$

Model predictive control principle

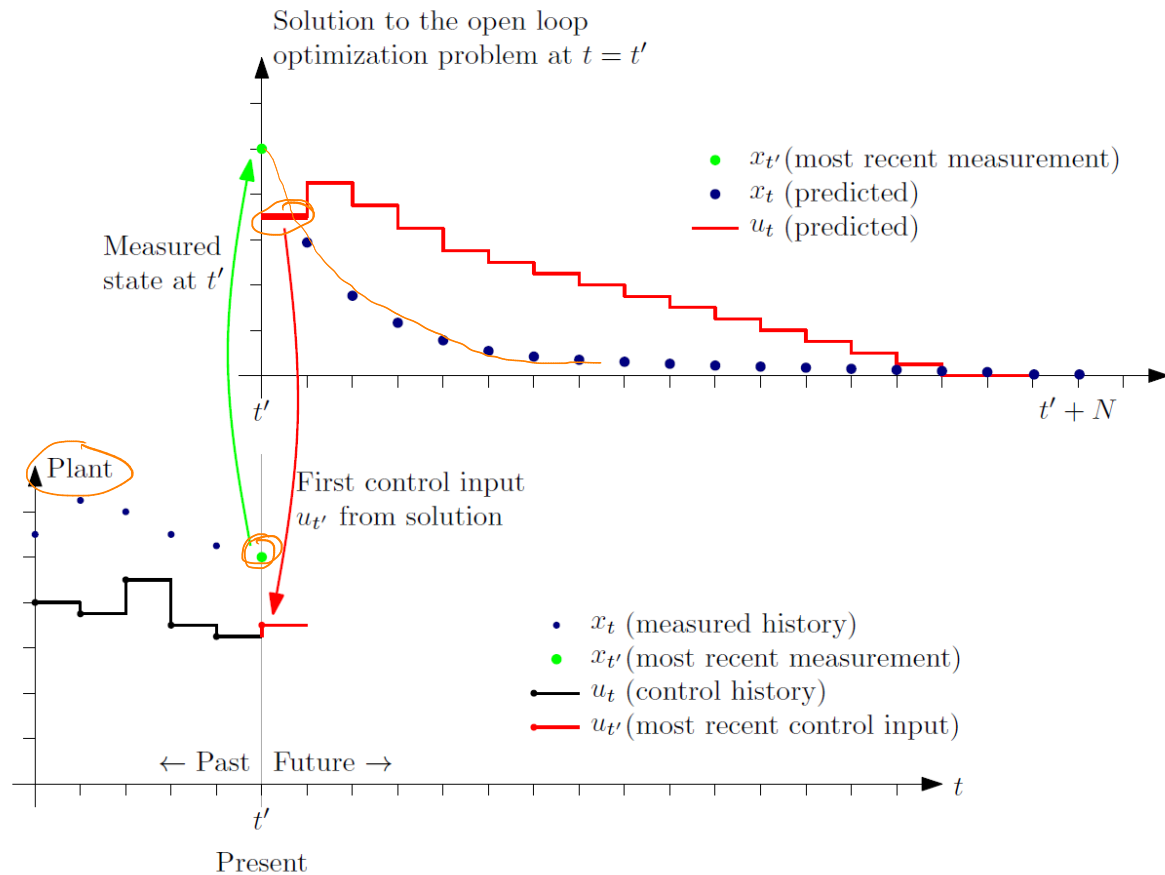


Why? This introduces feedback!

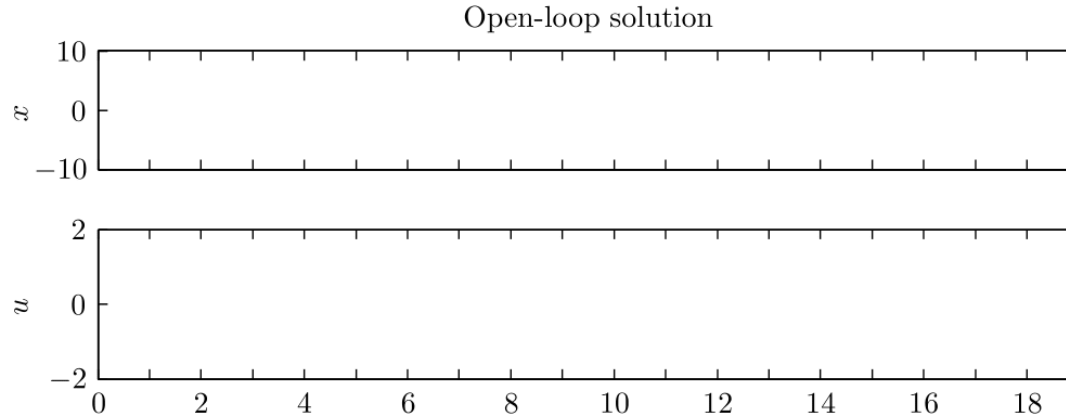
Chess analogy



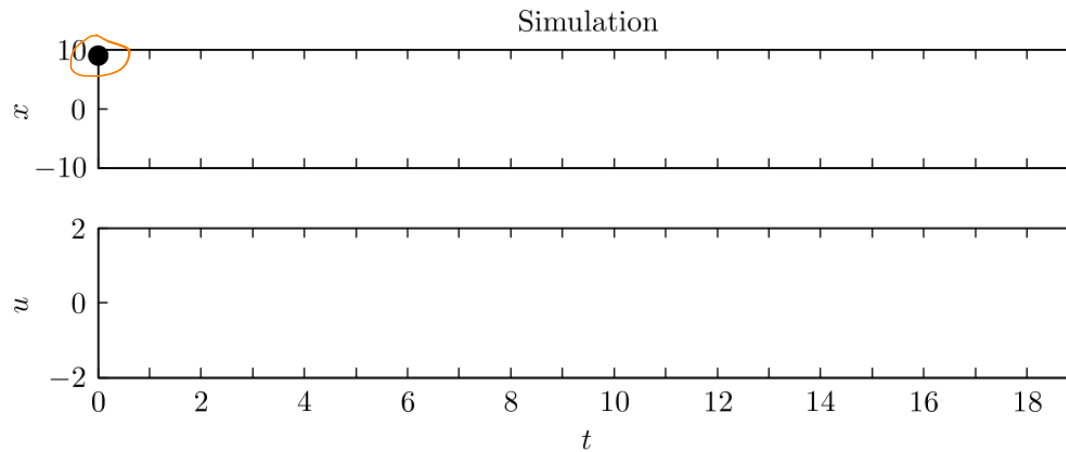
Model predictive control principle

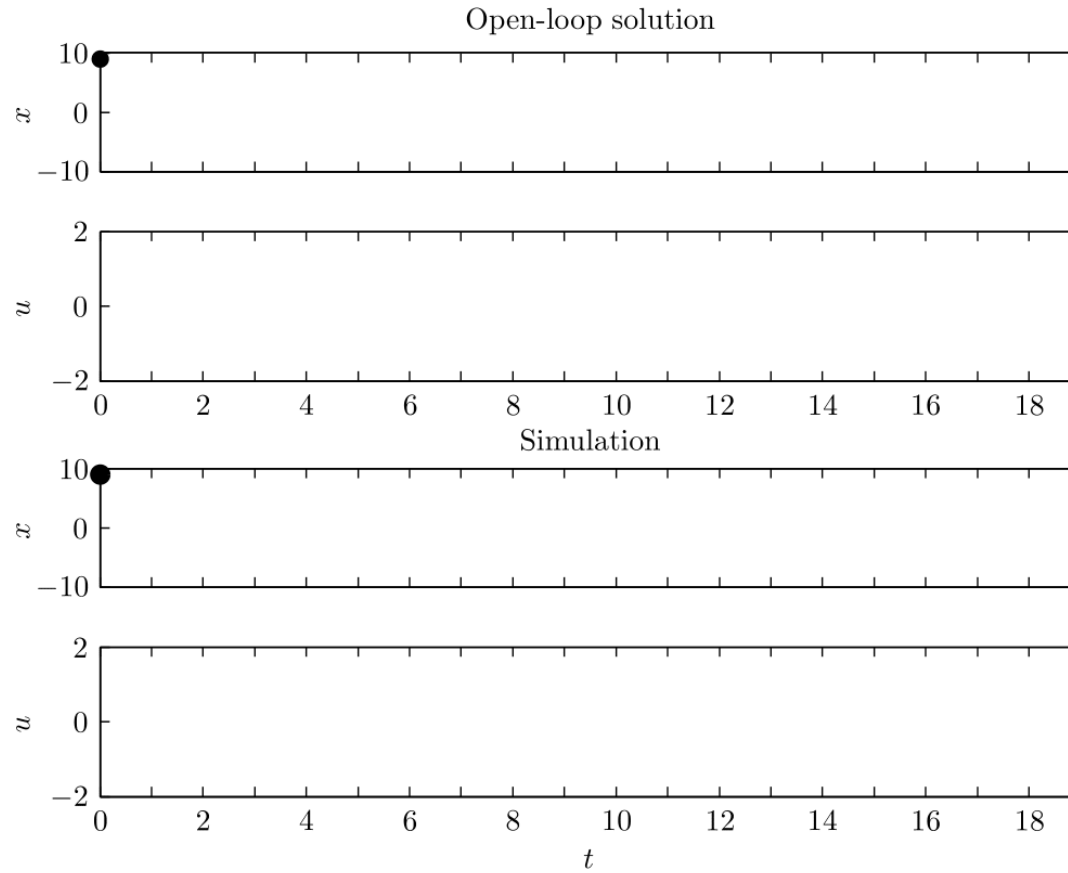


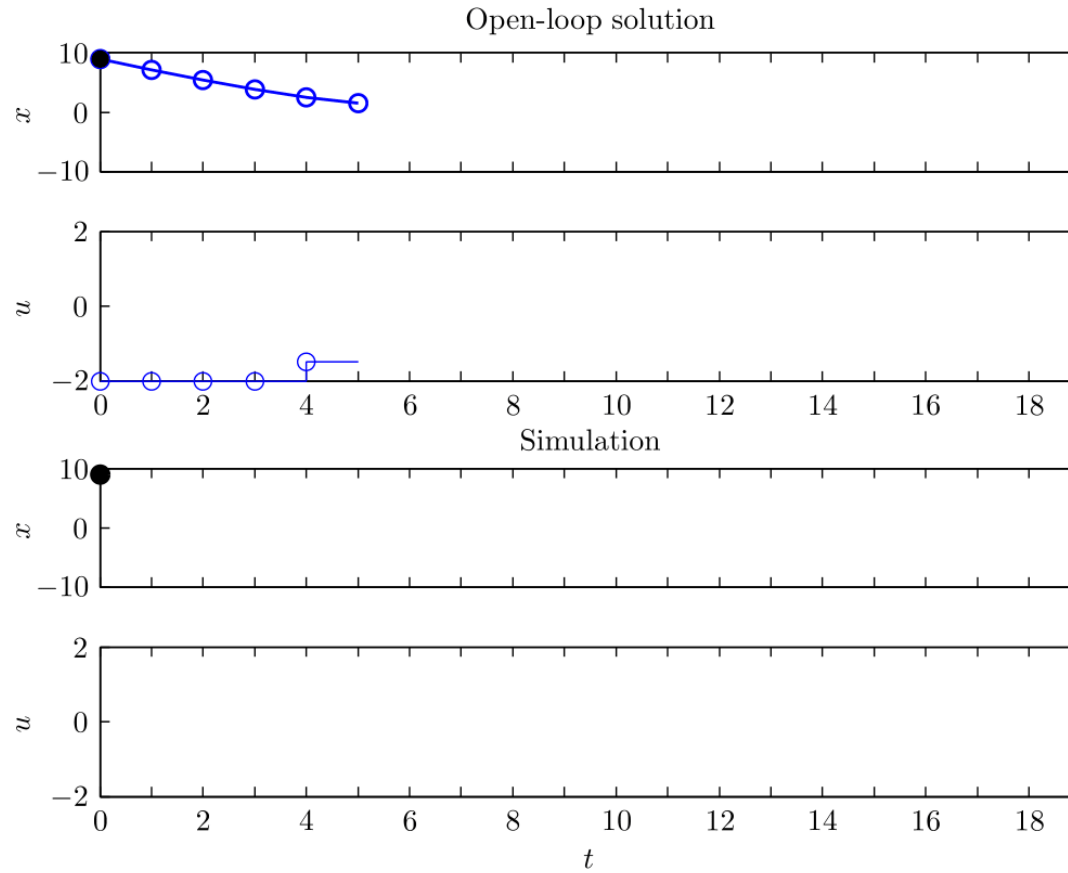
Controller

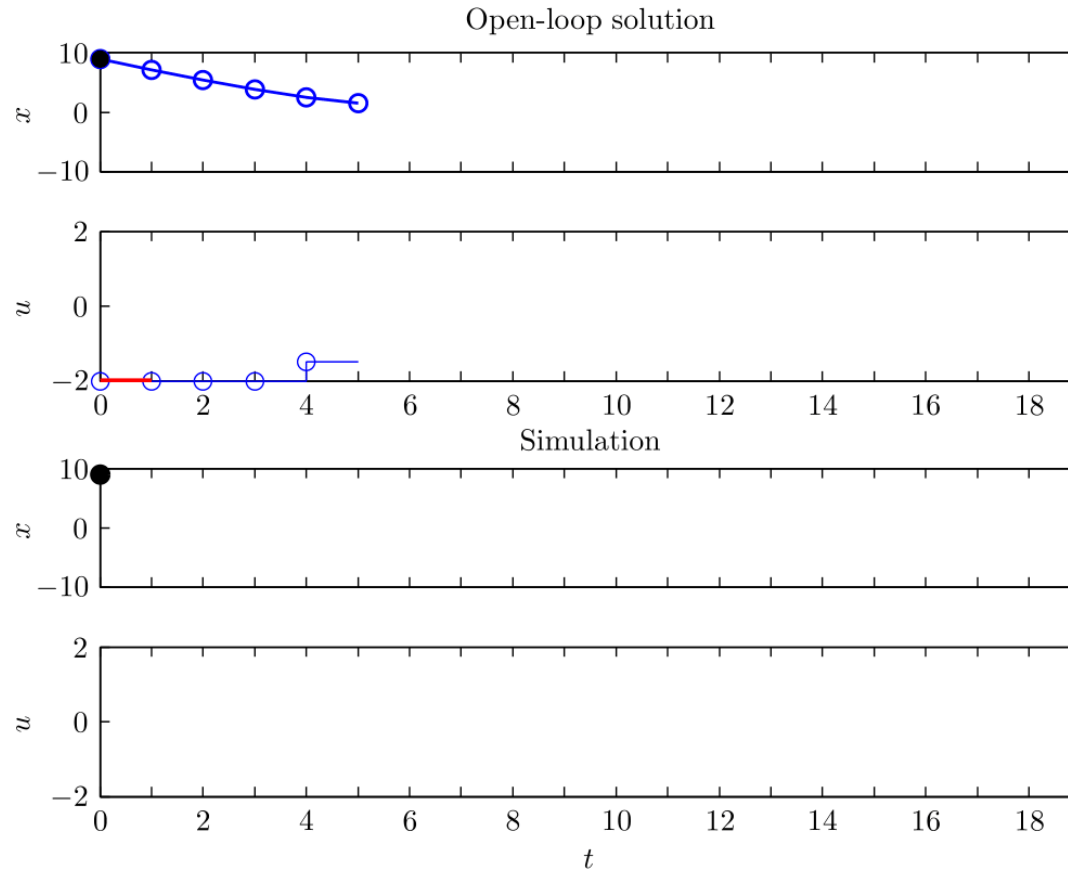


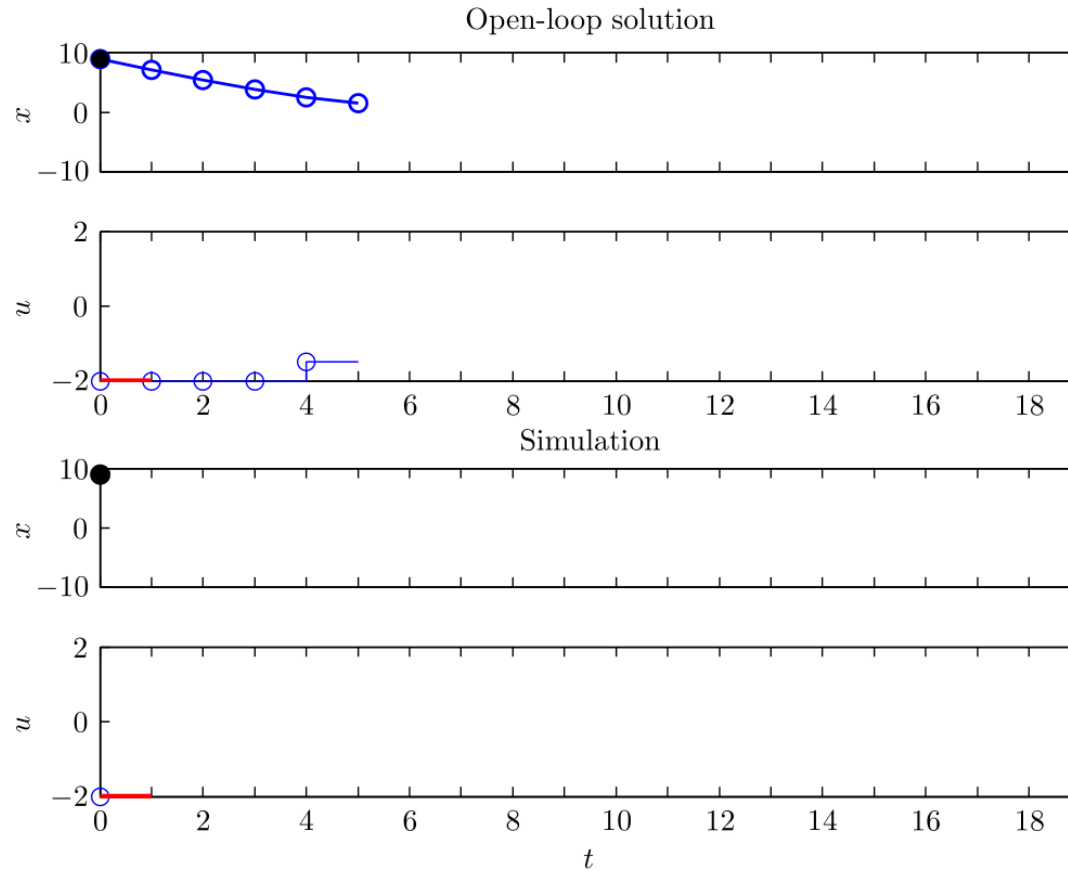
Plant

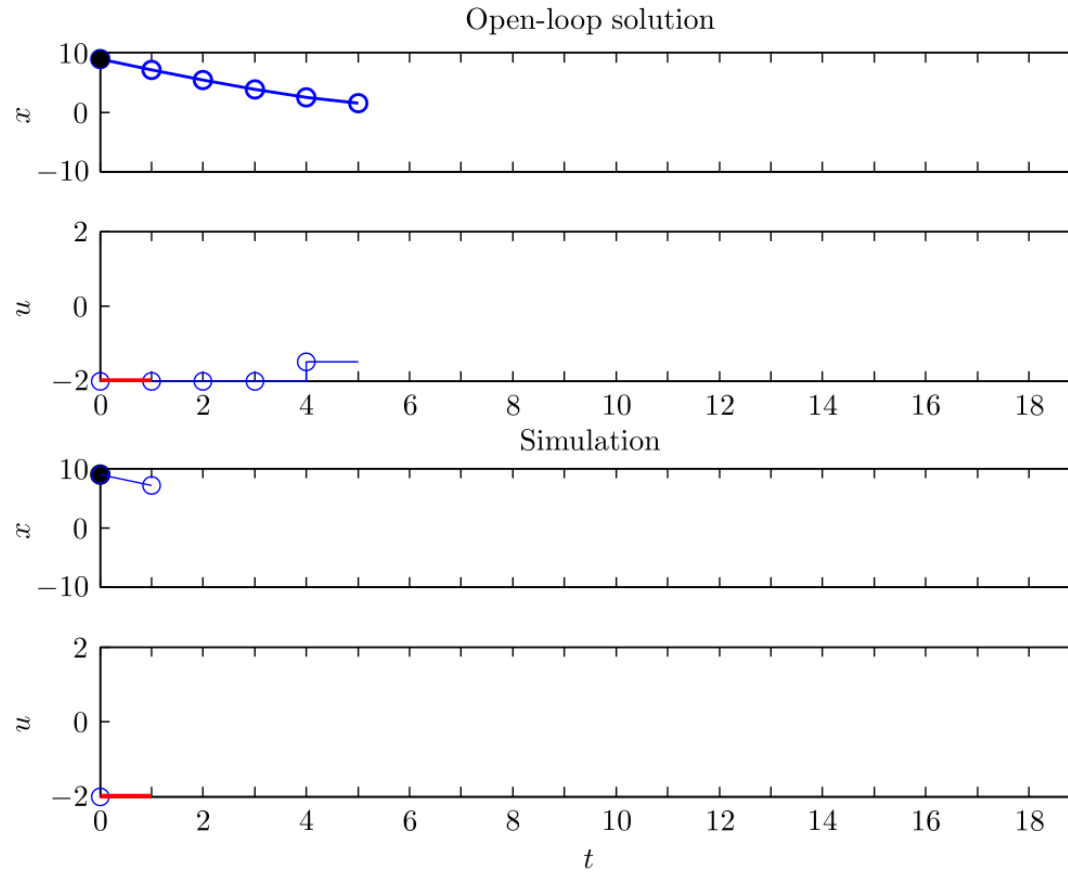


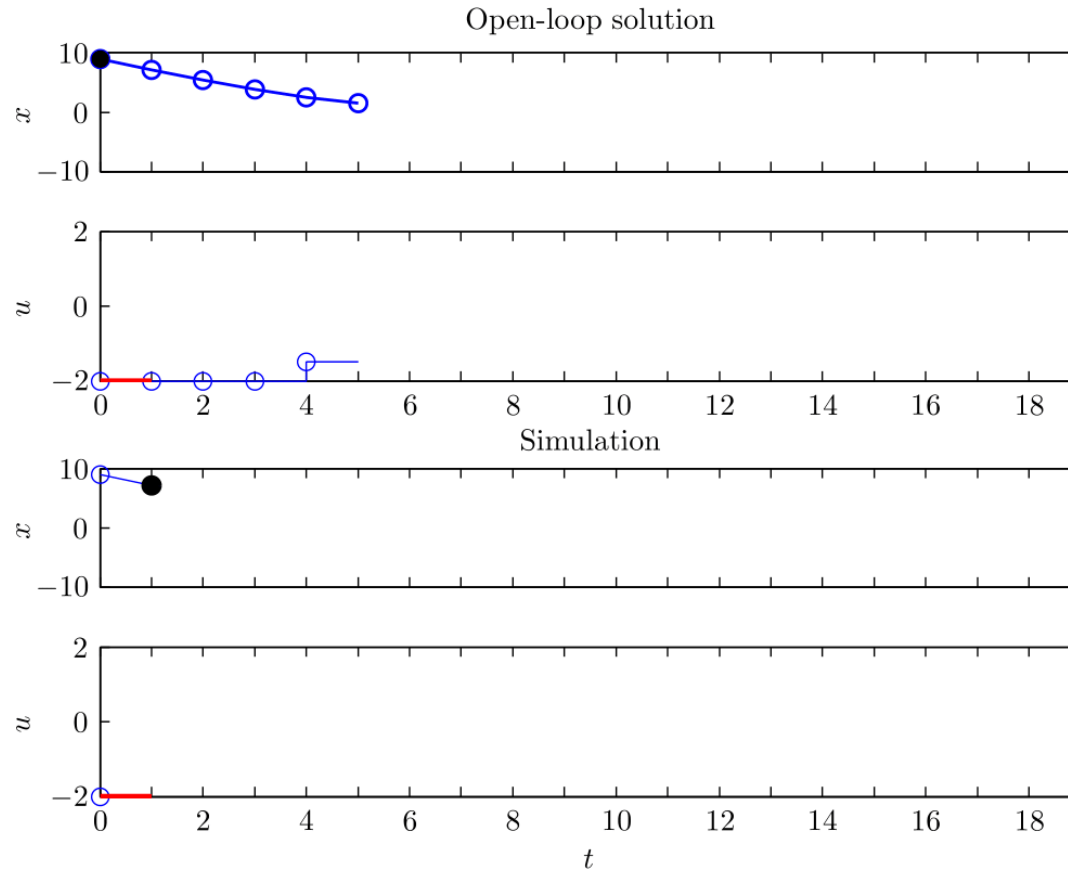


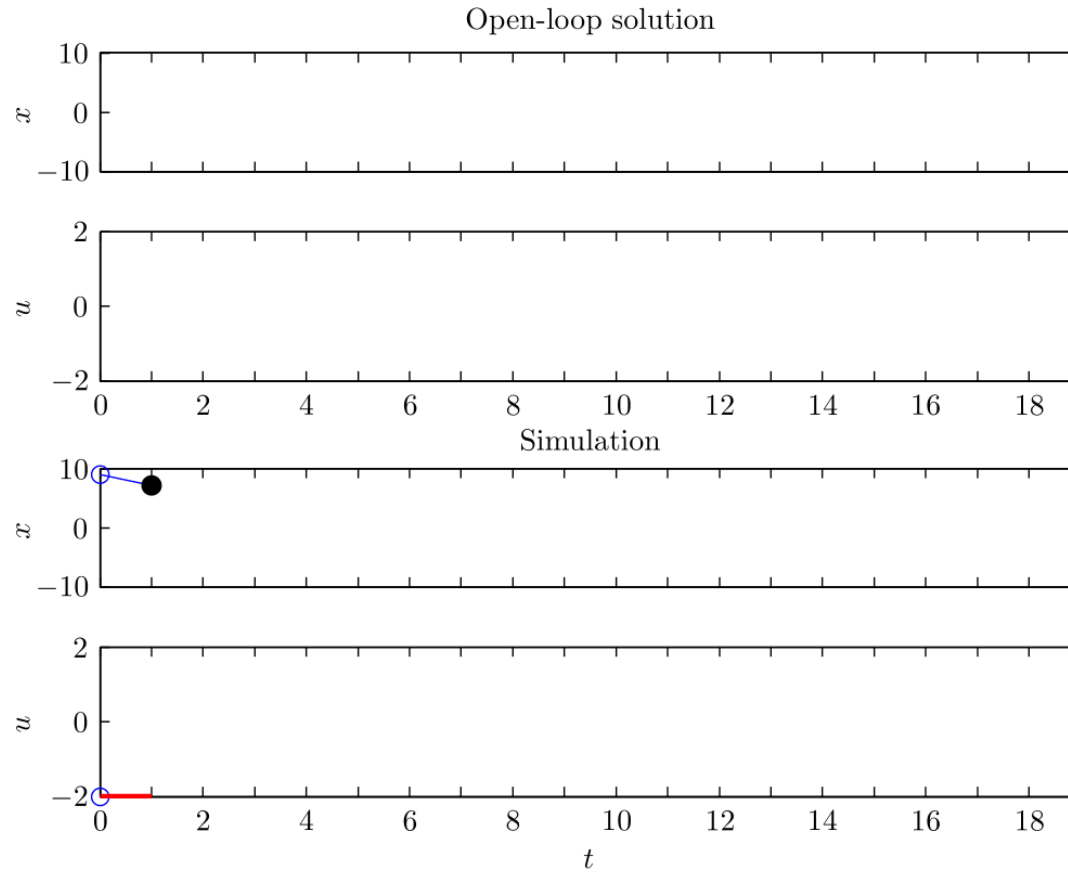


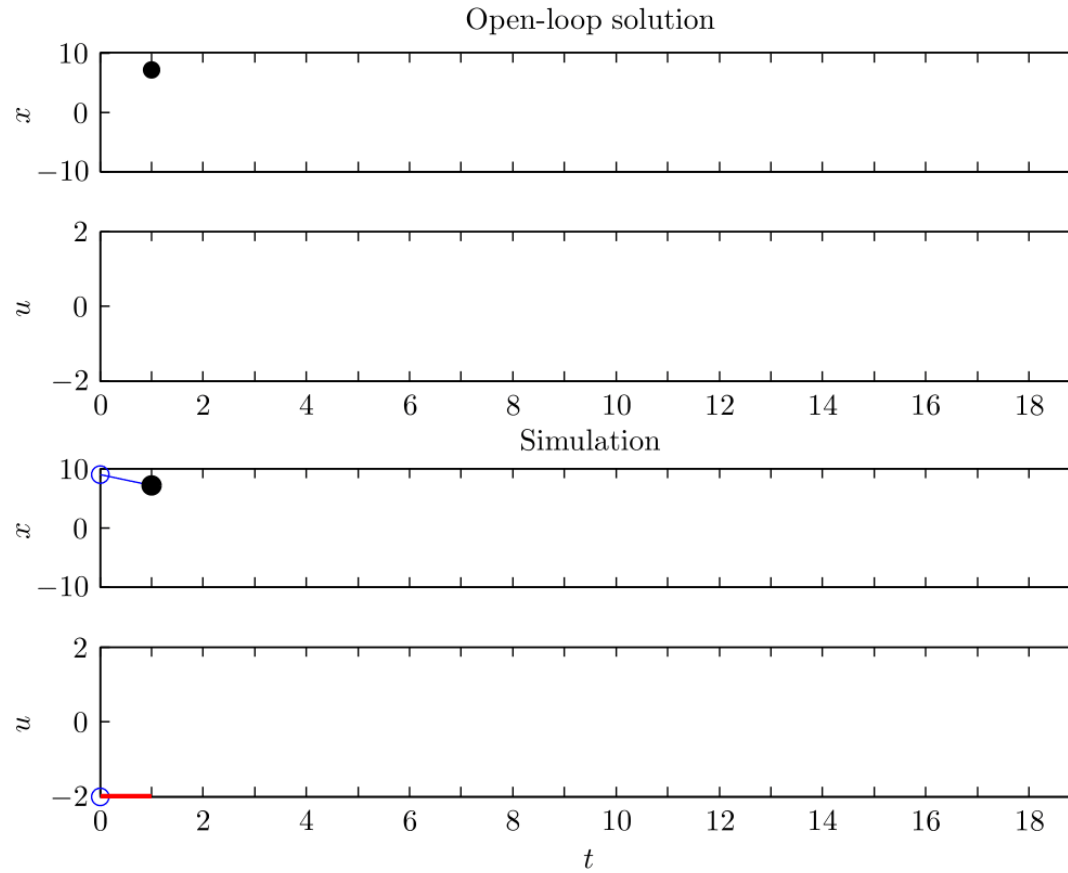


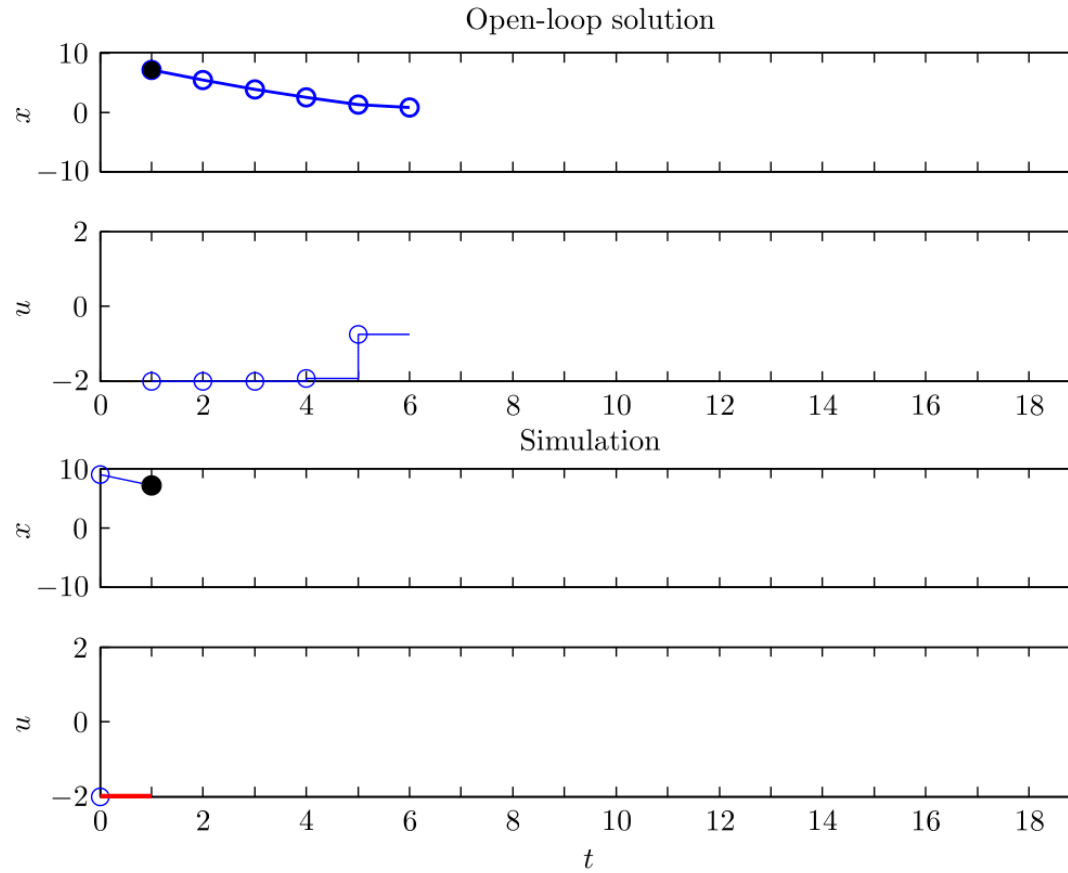


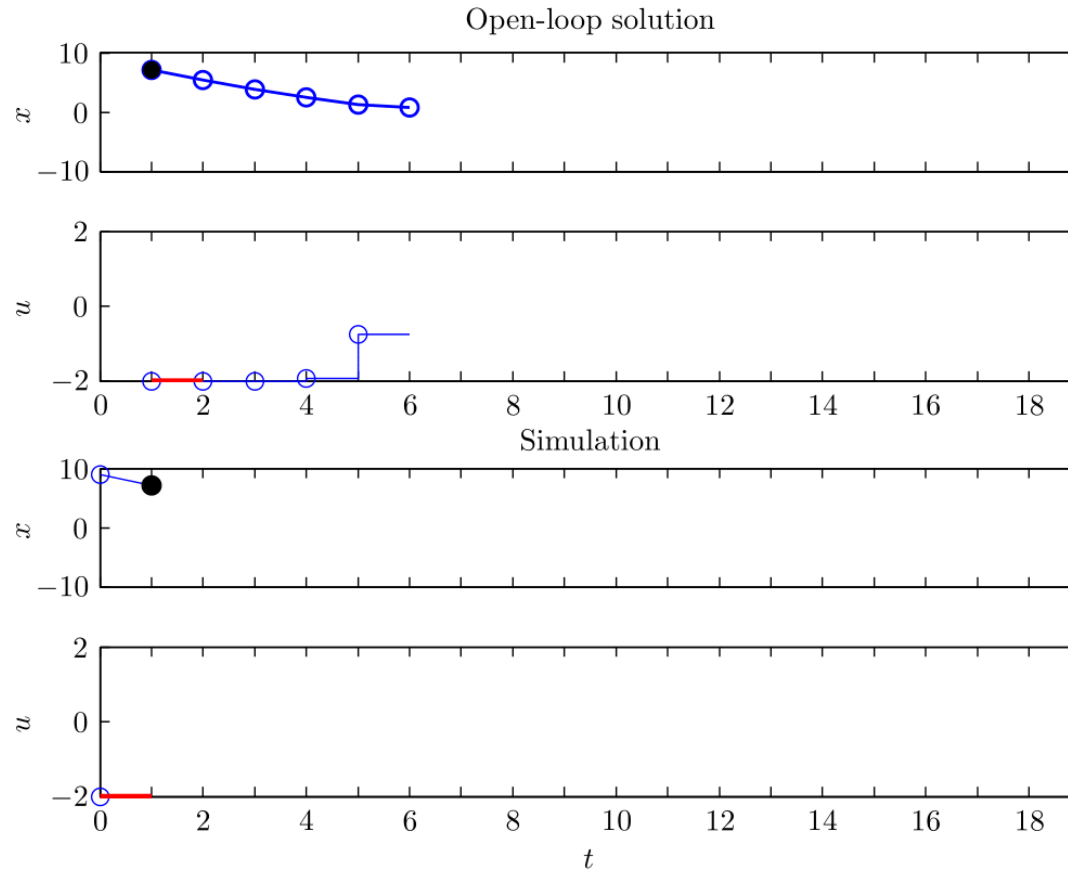


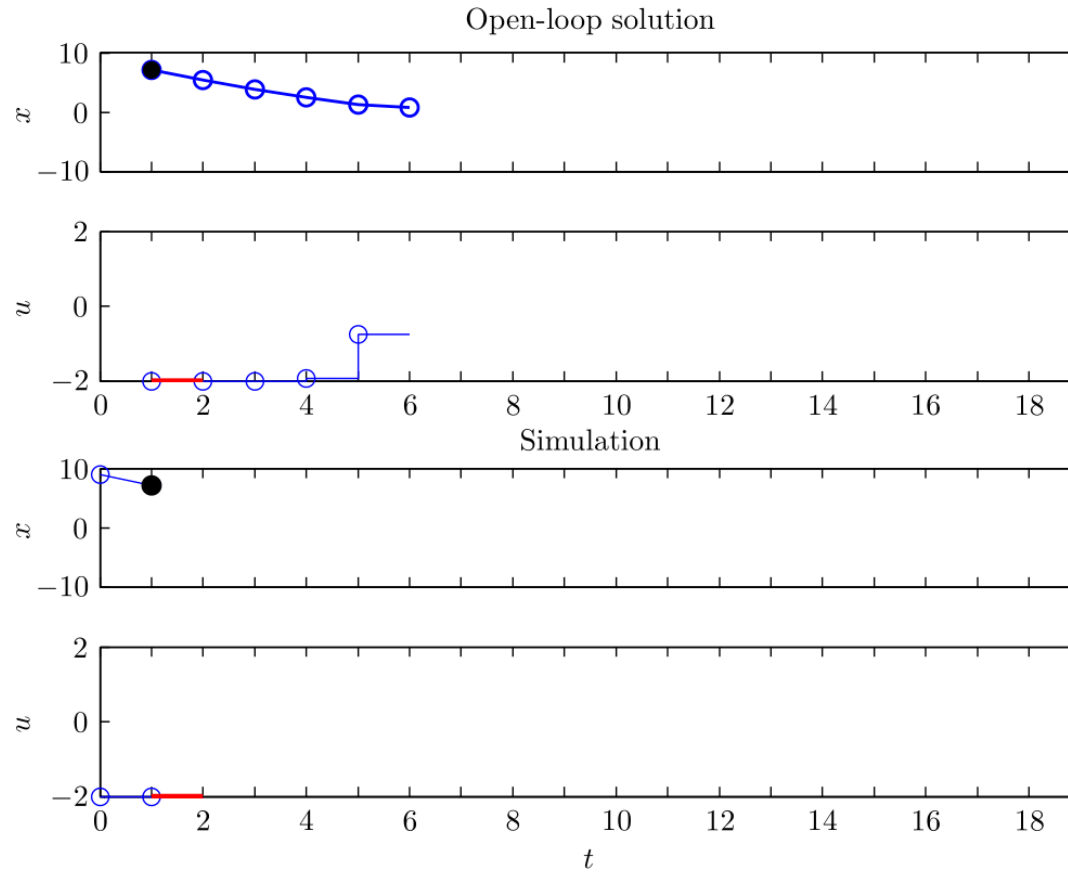


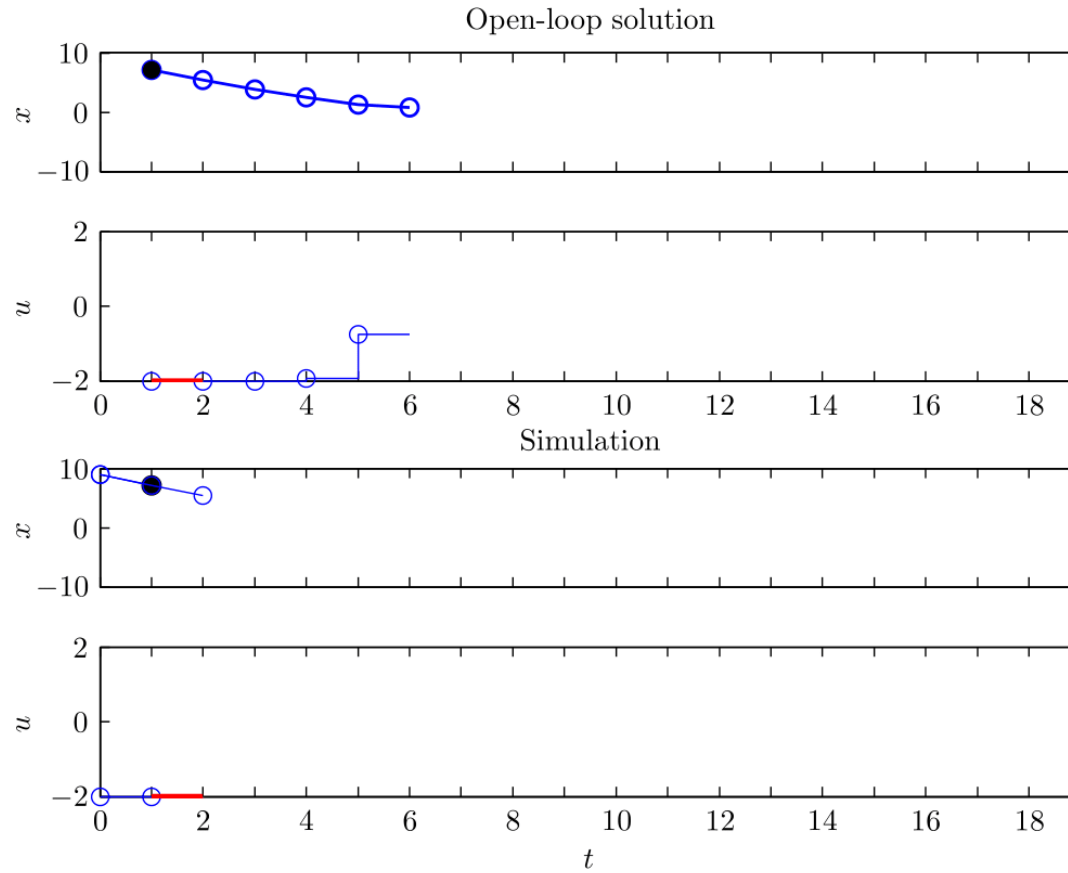


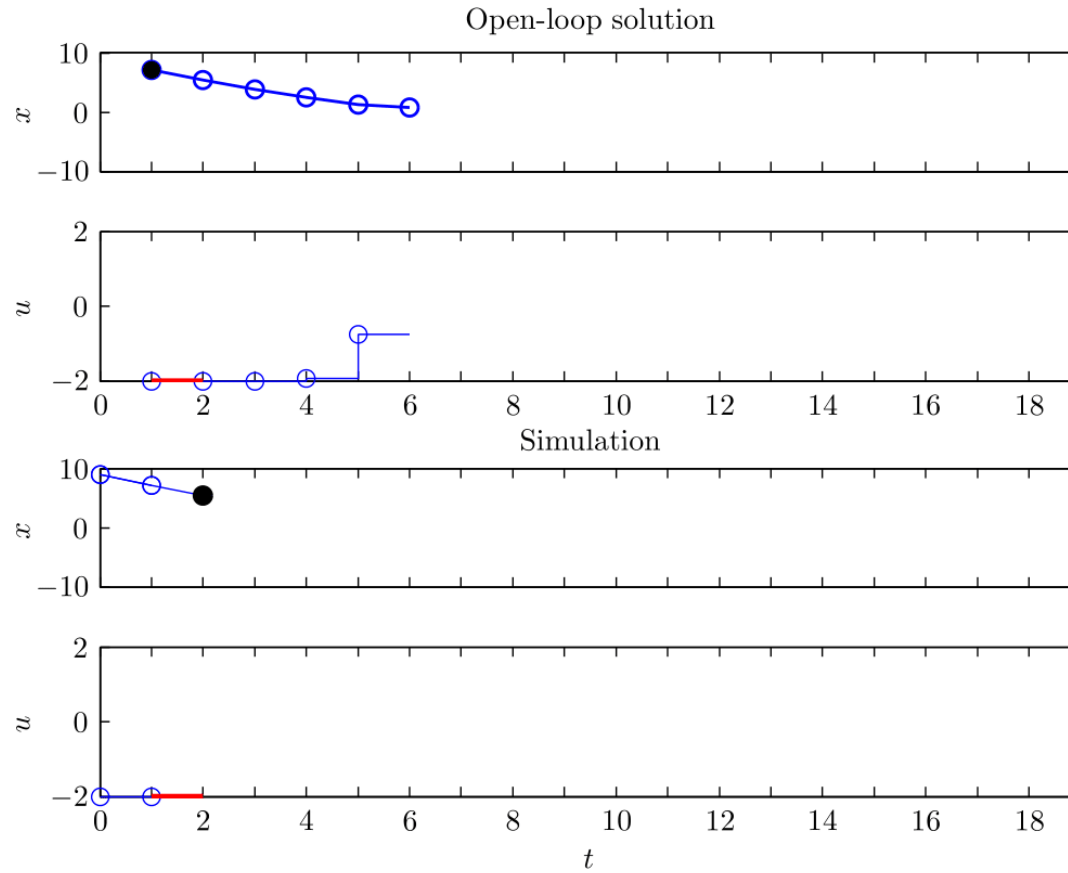


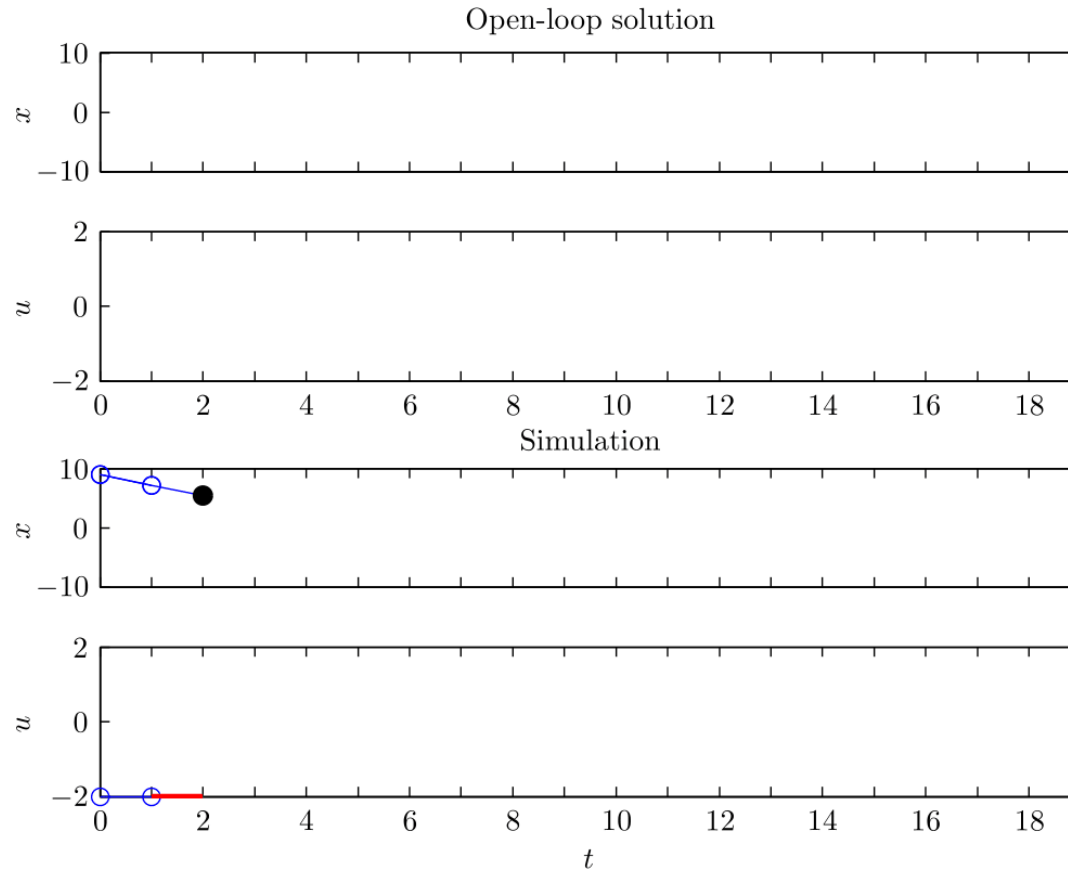


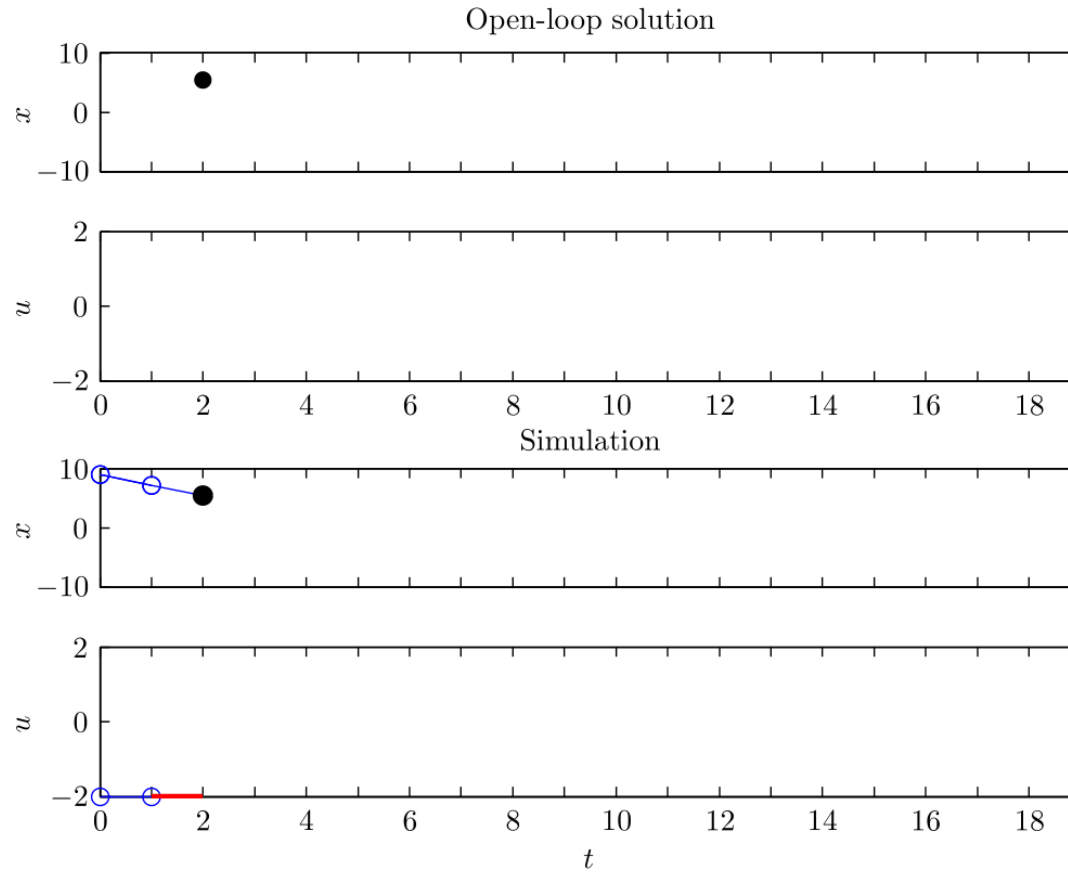


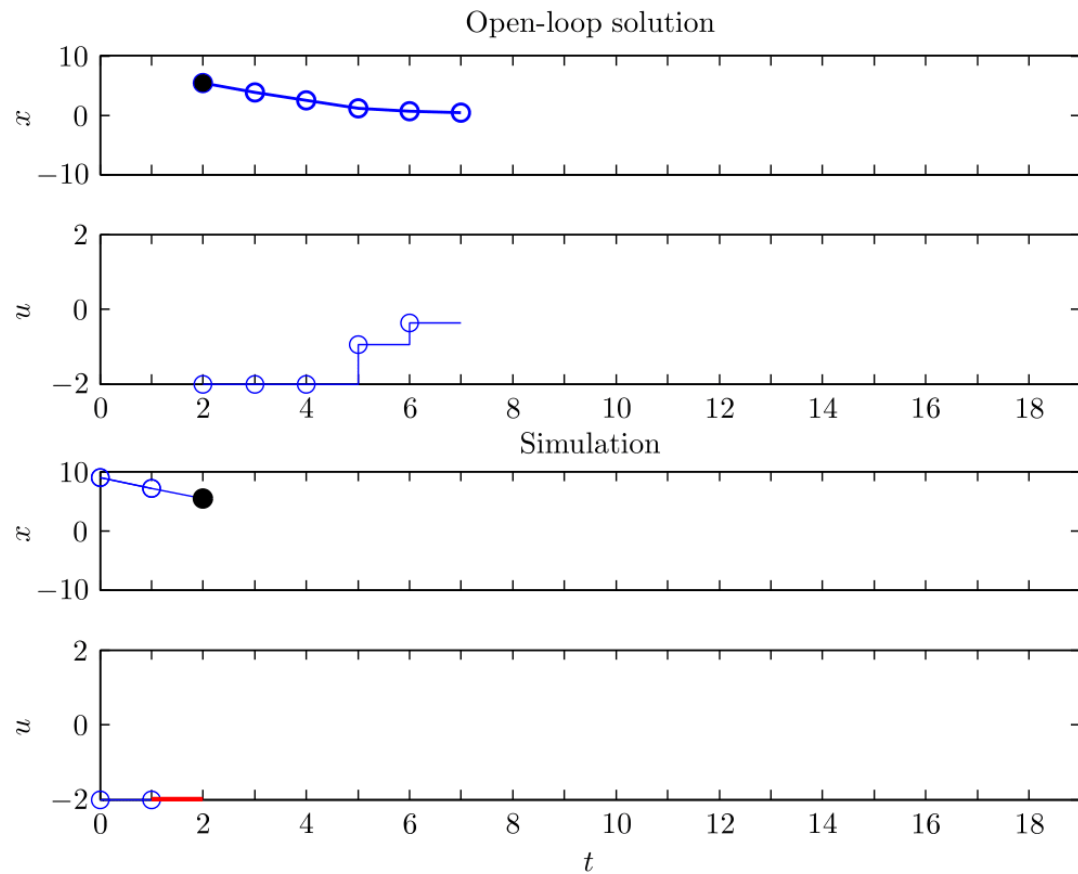


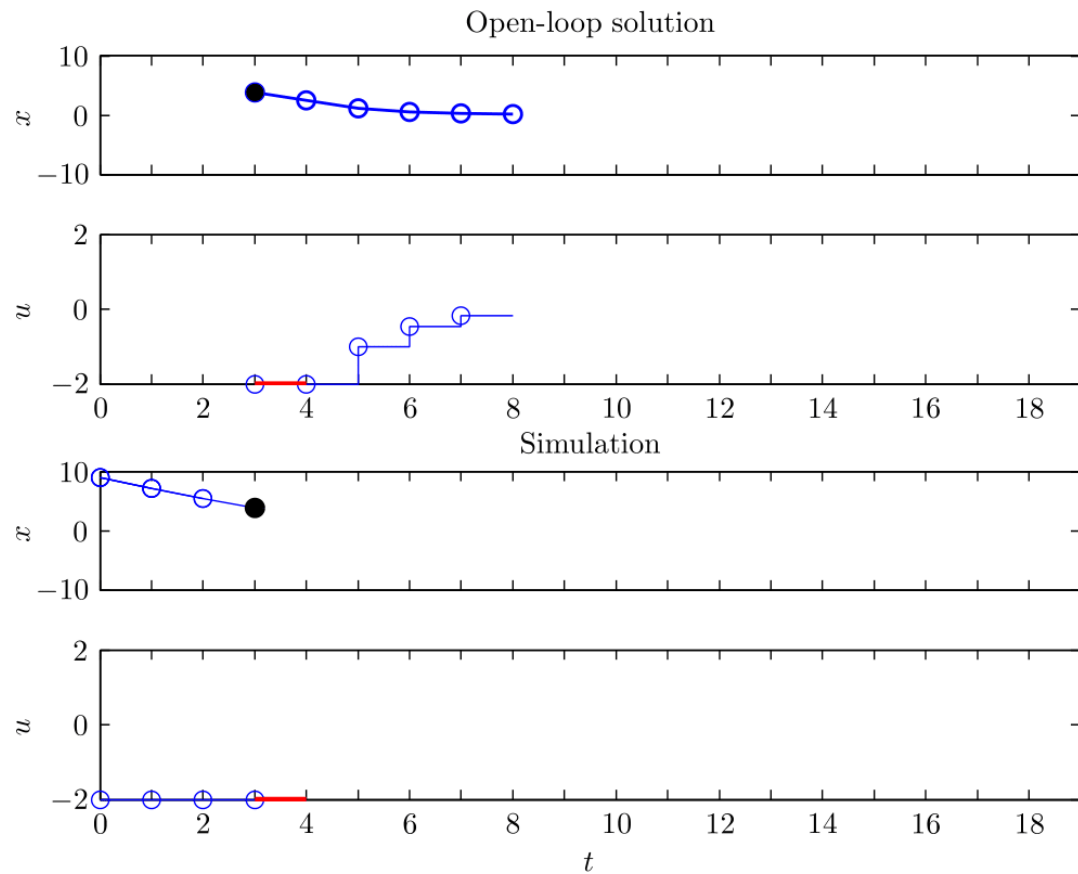


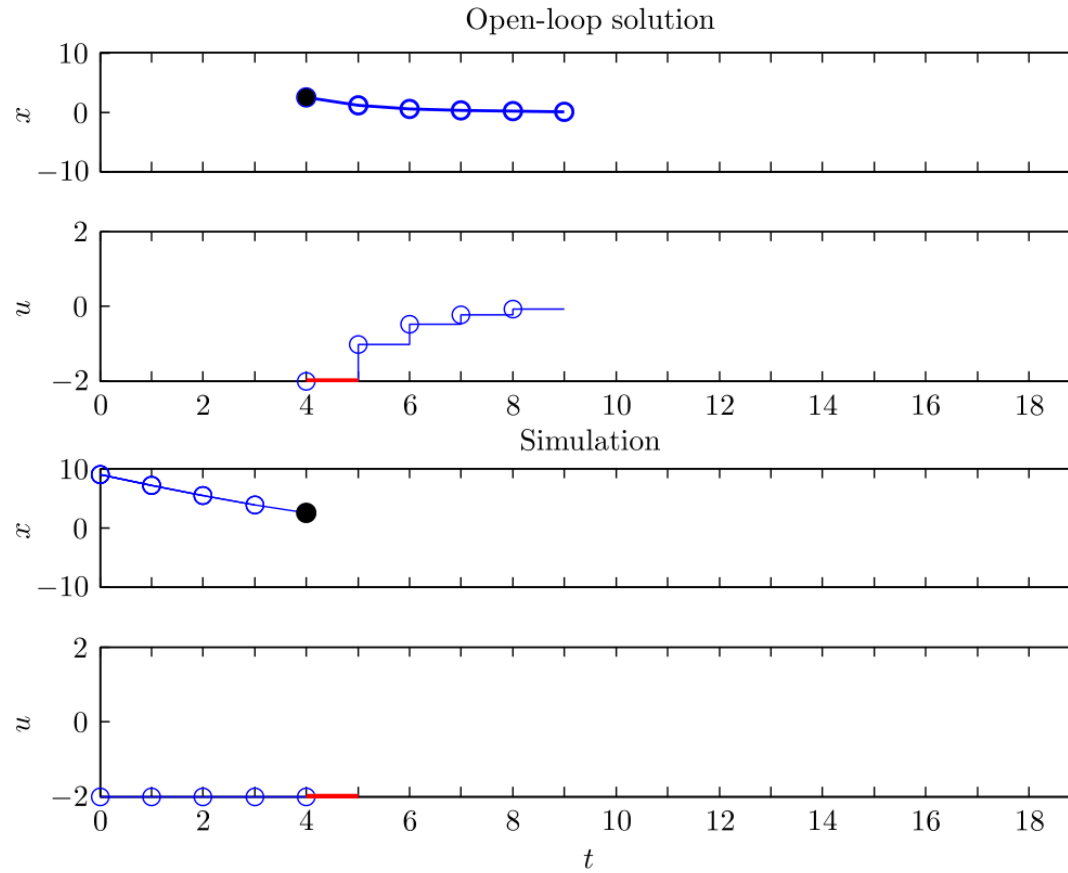


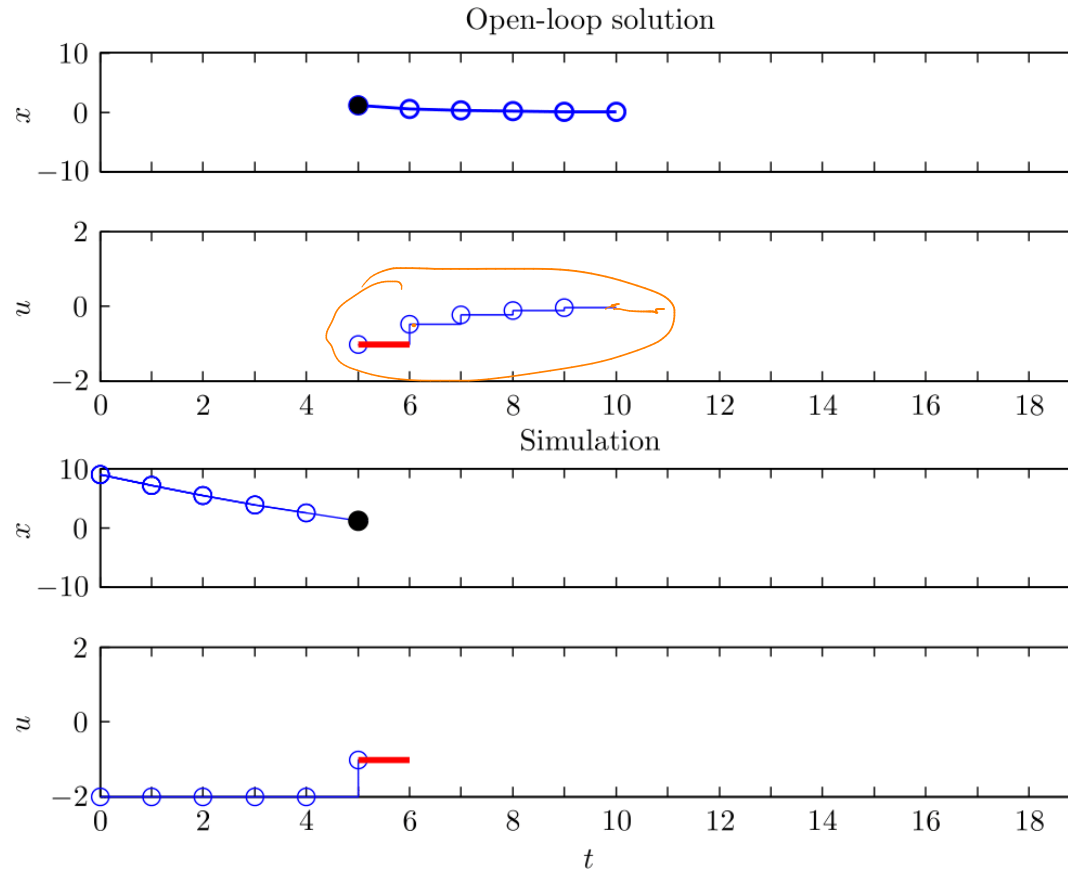


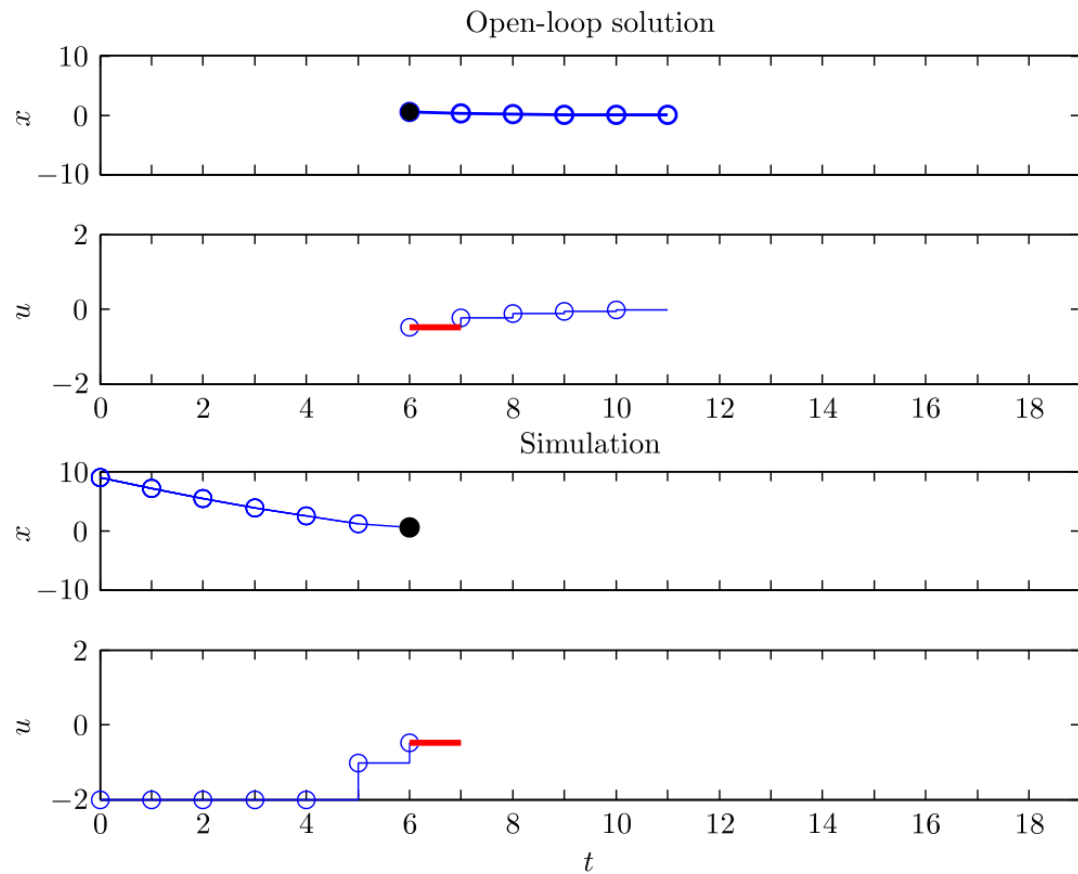


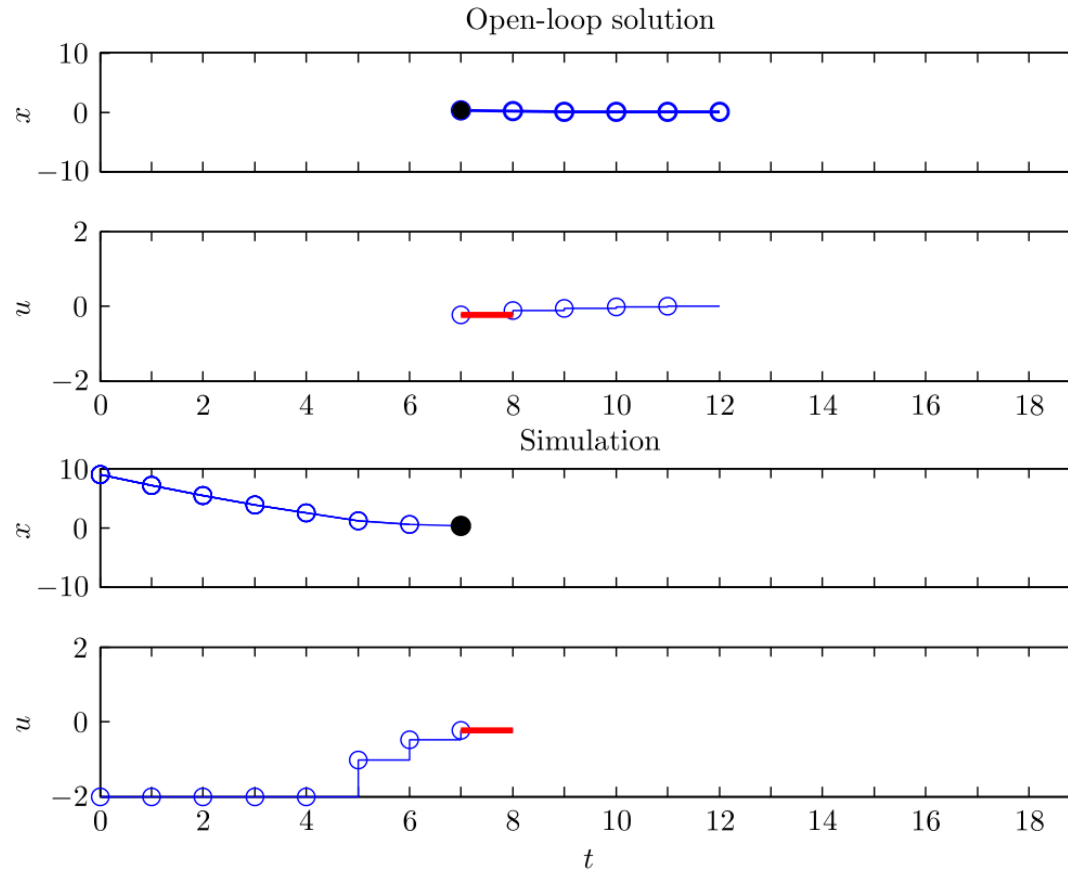


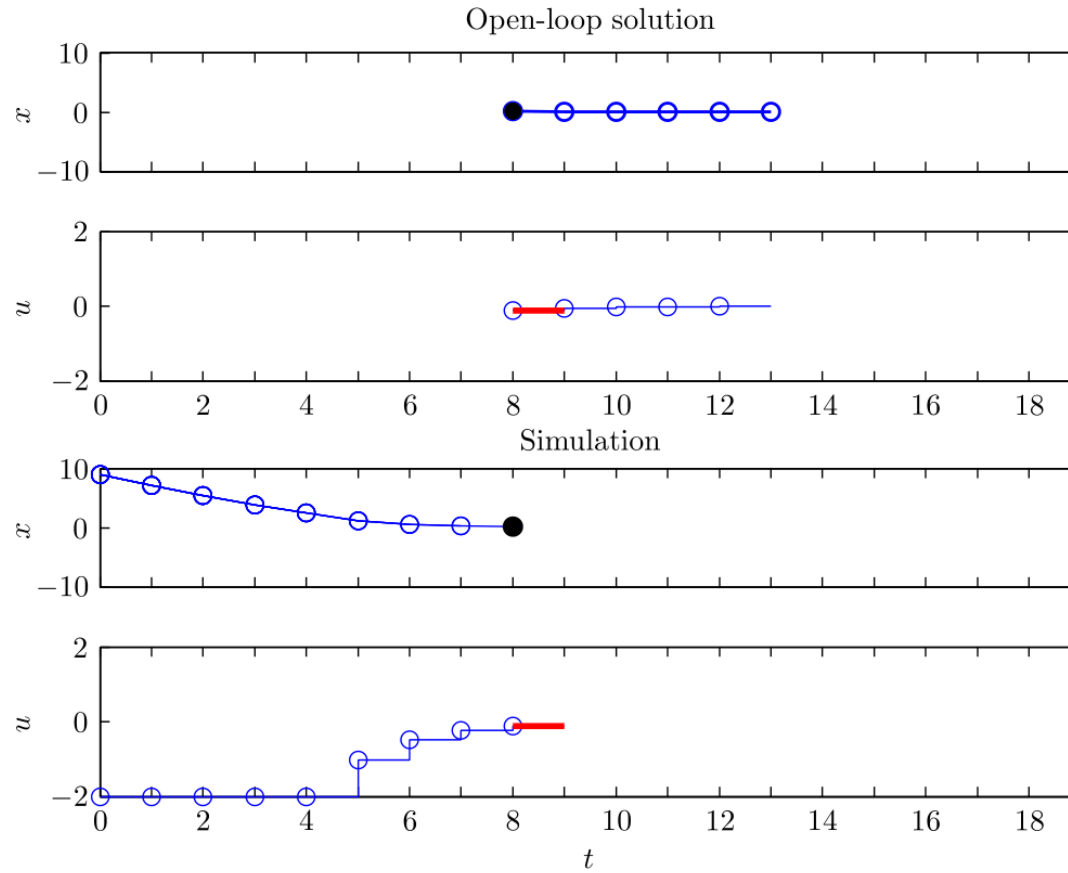


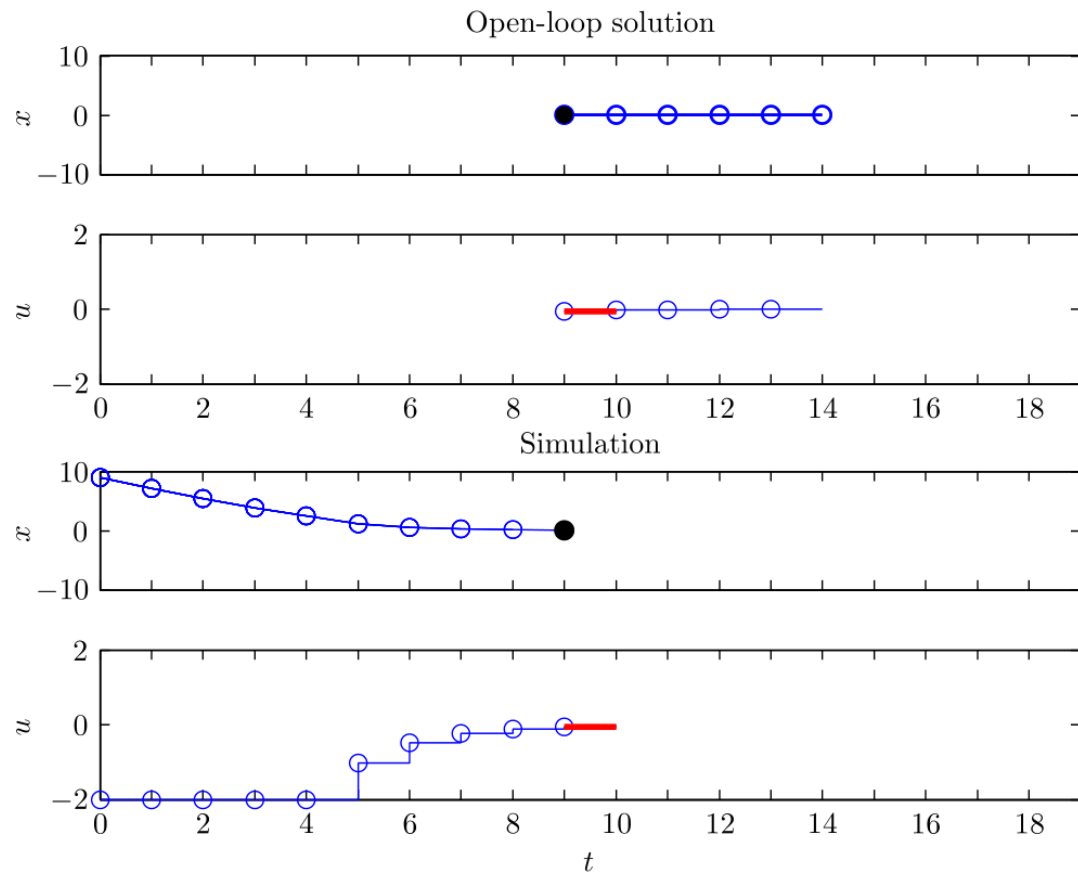


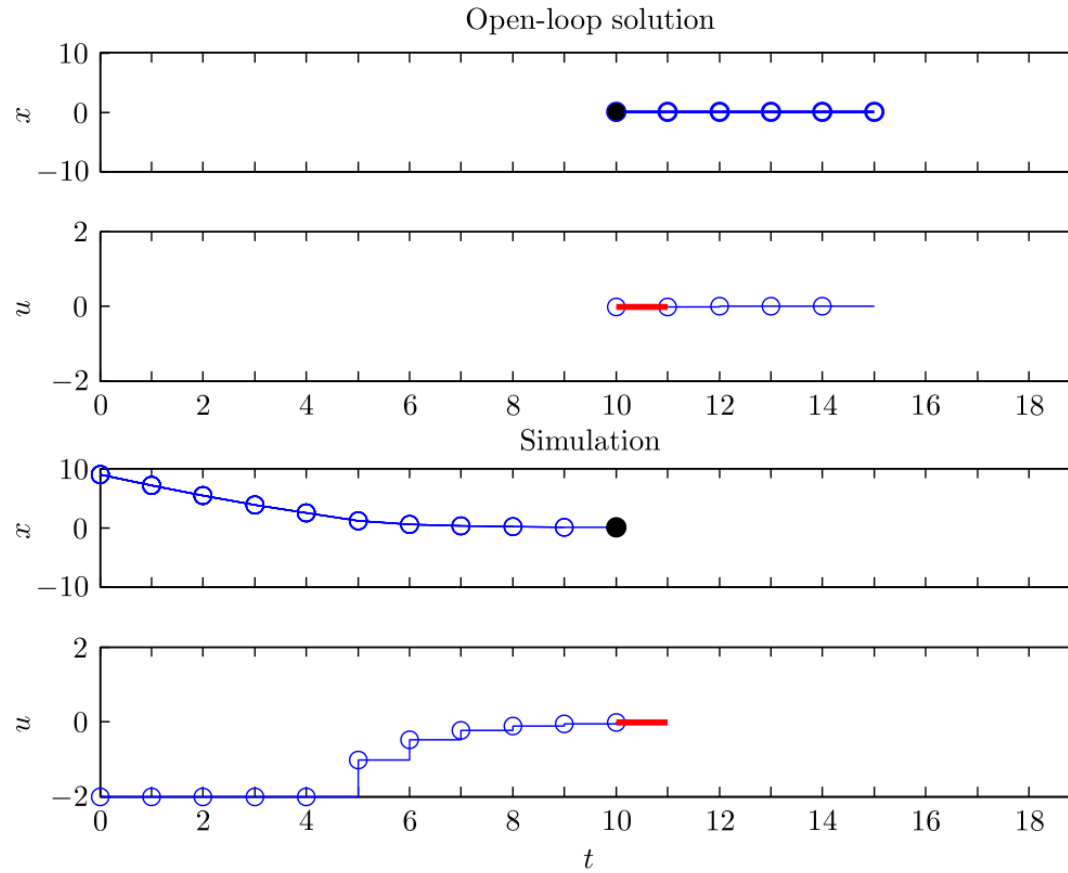


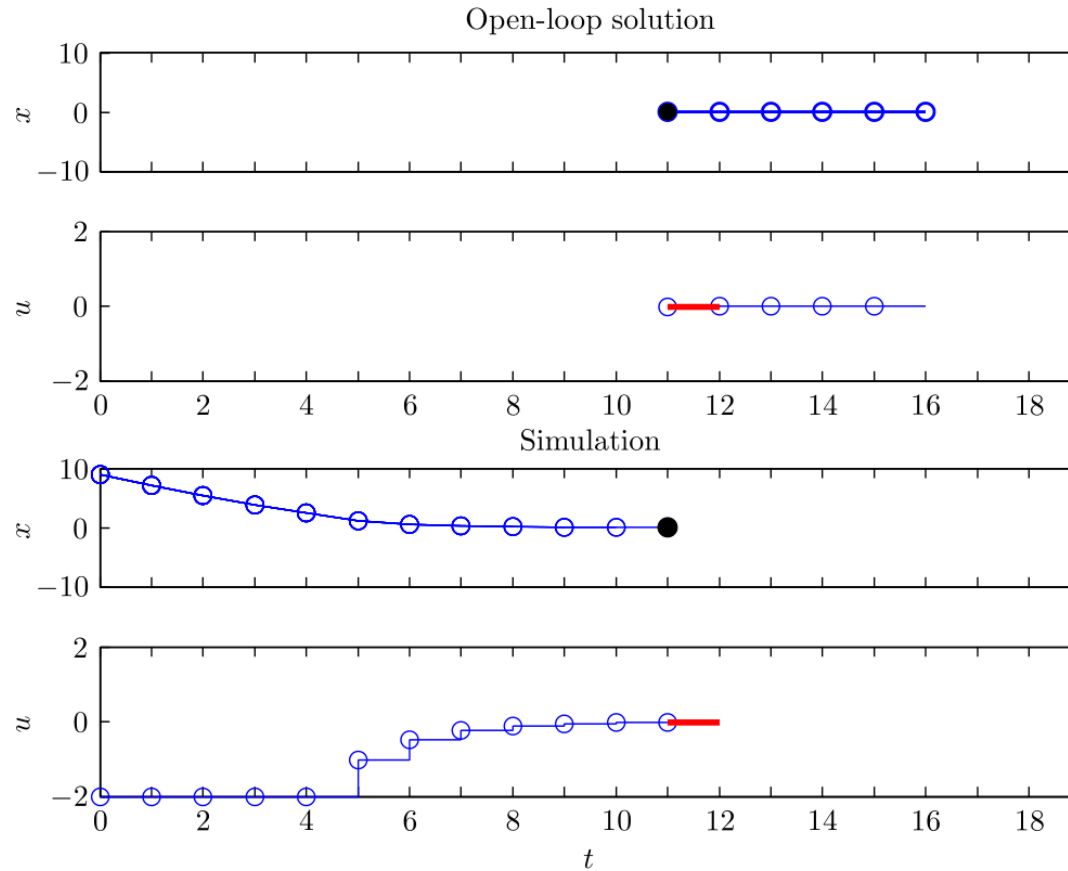


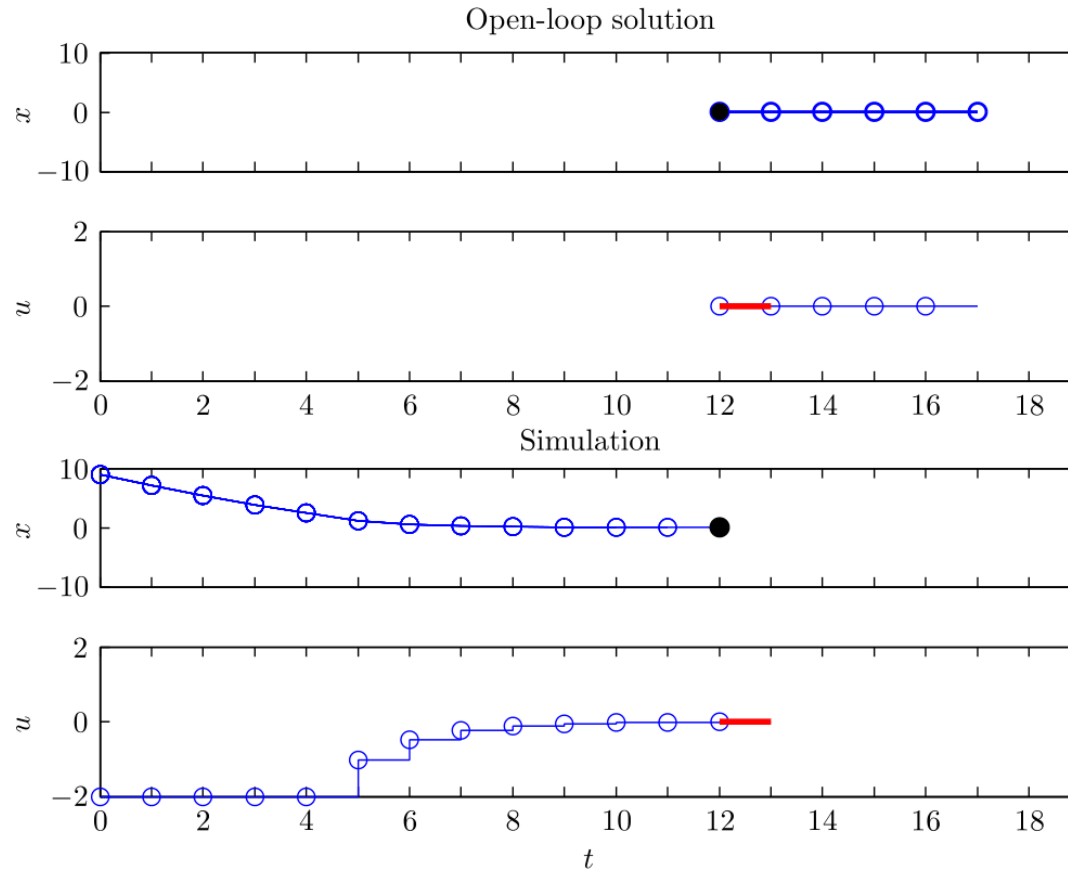


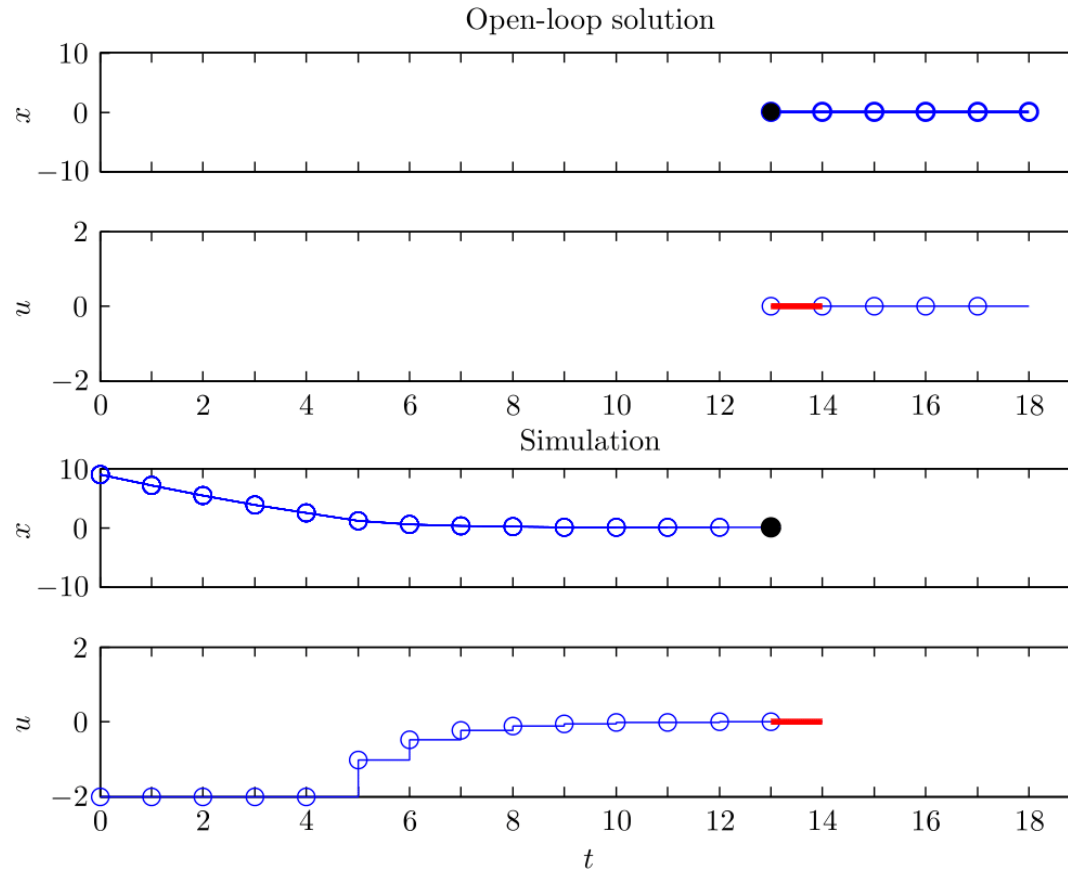


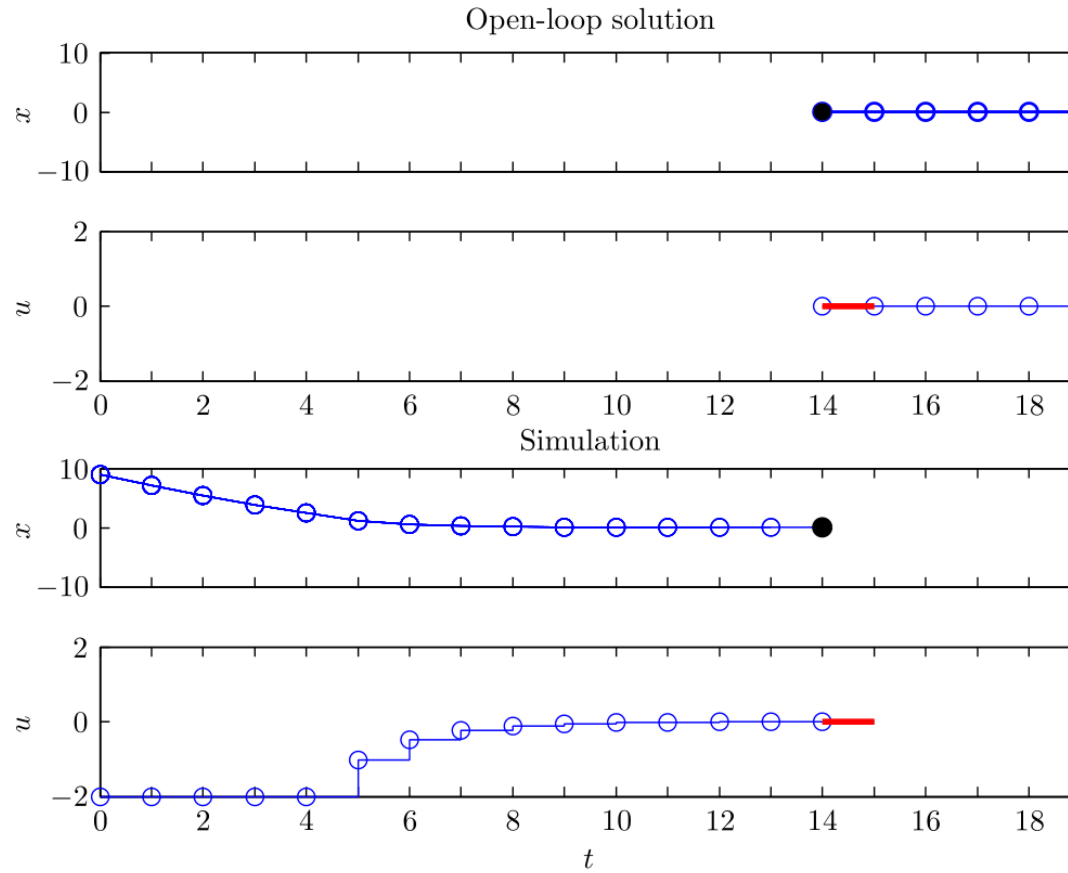












Open-loop optimization with linear state-space model

QP

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{x,t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{u,t} u_t + \frac{1}{2} \Delta u_t^\top S \Delta u_t$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\}$$

$$x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\} \quad \text{Is this always possible?}$$

$$u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

$$-\Delta u^{\text{high}} \leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\}$$

where

x_0 and u_{-1} is given

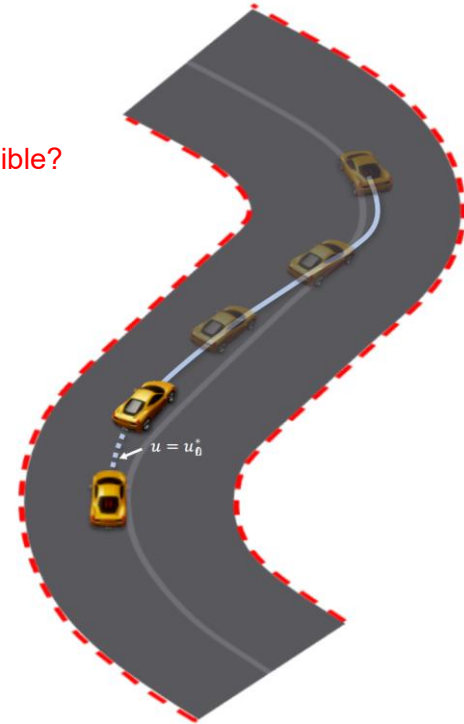
$$\Delta u_t := u_t - u_{t-1}$$

$$z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)$$

$$n = N \cdot (n_x + n_u)$$

$$Q_t \succeq 0 \quad t = \{1, \dots, N\}$$

$$R_t \succeq 0 \quad t = \{0, \dots, N-1\}$$



The **feasibility** problem: Inequality constraints on states may imply that for some x_0 , there are no solutions to the MPC QP

$$\min_{z, \varepsilon} \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^T Q x_{t+1} + \frac{1}{2} u_t^T R u_t + \underbrace{s^T \varepsilon}_{\text{penalty}} + \frac{1}{2} \varepsilon^T W \varepsilon$$

$$\text{s.t. } x_{t+1} = A x_t + B u_t, \quad t = 0, \dots, N-1$$

common solution:

$$- \varepsilon + x^{\text{low}} \leq x_t \leq x^{\text{high}} + \varepsilon, \quad t = 1, \dots, N$$

$$u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1$$

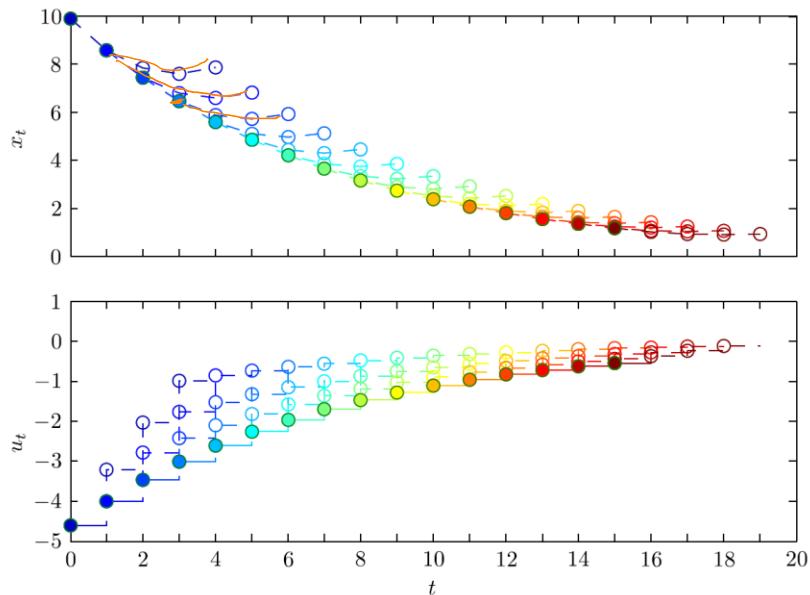
$$\varepsilon \geq 0$$

Add "slack variable" ε

"Exact penalty": s large enough $\Rightarrow \varepsilon = 0$ whenever possible

Open-loop vs closed-loop trajectories

$$\begin{aligned} \min \quad & \sum_{t=0}^4 x_{t+1}^2 + 4 u_t^2 \\ \text{s.t.} \quad & \underline{x_{t+1} = 1.2x_t + 0.5u_t, \quad t = 0, \dots, 4} \end{aligned}$$



- Closed-loop trajectories different from open-loop (optimized) trajectories!
- It is the closed-loop trajectories that must analyzed for stability

Example: Is MPC alway stable?

Design MPC for $x_{t+1} = 1.2x_t + u_t$, no constraints, $N = 2$

$$\min_{\substack{u_0, u_1 \\ x_1, x_2}} \sum_{t=0}^1 \frac{1}{2} x_{t+1}^2 + \frac{r}{2} u_t^2$$

$$\text{s.t. } x_1 = A x_0 + B u_0$$

$$x_2 = A x_1 + B u_1 = A (A x_0 + B u_0) + B u_1 = A^2 x_0 + A B u_0 + B u_1$$

$$A = 1.2$$

$$B = 1$$

Re formulate by substitution:

$$\min_{u_0, u_1} \frac{1}{2} (1.2 x_0 + u_0)^2 + \frac{1}{2} (1.44 x_0 + 1.2 u_0 + u_1)^2 + \frac{r}{2} u_0^2 + \frac{r}{2} u_1^2$$

$$f(u_0, u_1)$$

Example: Is MPC alway stable?

Design MPC for $x_{t+1} = 1.2x_t + u_t$, no constraints, $N = 2$

Since convex + no constraints: solution given by $\frac{\partial f}{\partial u_i} = 0, i=1,2$

$$\frac{\partial f}{\partial u_0} = 1.2x_0 + u_0 + 1.2(1.44x_0 + 1.2u_0 + u_1) + ru_0 = 0$$

$$\frac{\partial f}{\partial u_1} = (1.44x_0 + 1.2u_0 + u_1) + ru_1 = 0 \Rightarrow u_1 = \frac{-1}{1+r} (1.44x_0 + 1.2u_0)$$

$$\Rightarrow u_0 = - \frac{1.2 + 2.64r}{1 + 3.2r + r^2} x_0$$

→ MPC controller!

MPC optimality implies stability?

$$\min \sum_{t=0}^1 x_{t+1}^2 + r u_t^2$$

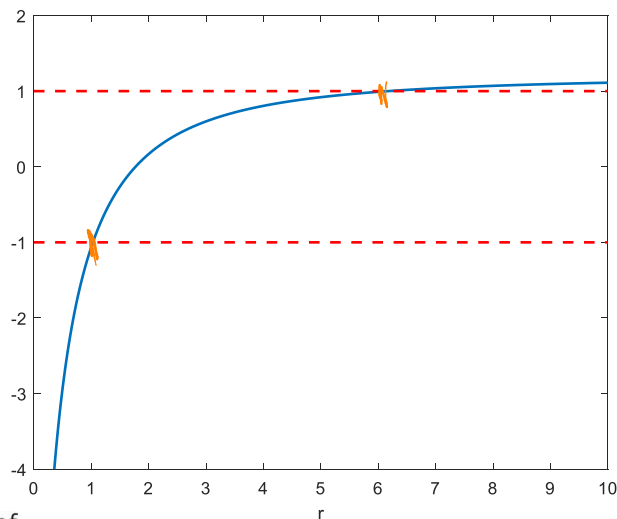
$$\text{s.t. } x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1$$

MPC solution

$$u_t = -\frac{1.2 + 2.64r}{1 + 3.2r + r^2} x_t$$

MPC closed loop

$$x_{t+1} = \left(1.2 - \frac{1.2 + 2.64r}{1 + 3.2r + r^2} \right) x_t$$



stable for
 $1 \leq r \leq 6$

MPC and stability

Nominal vs robust stability

- “Nominal stability”: Stability when optimization model = plant model
 - No “model-plant mismatch”, no disturbances
- “Robust stability”: Stability when optimization model \neq plant model
 - “Model-plant mismatch” and/or disturbances (more difficult to analyze, not part of this course)

Requirements for nominal stability:

- Stabilizability ((A,B) stabilizable)
- Detectability ((A,D) detectable)
 - D is a matrix such that $Q = D^T D$ (that is, “ D is matrix square root of Q ”)
 - Detectability: No modes can grow to infinity without being “visible” through Q
- But more is needed to guarantee stability...

How to achieve nominal stability?

Three different theoretical solutions:

1. Choose prediction horizon equal to infinity ($N = \infty$)
 - Usually not possible (unless no constraints: LQR)

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\ & u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1 \end{aligned}$$

How to achieve nominal stability?

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\ & u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1 \end{aligned}$$

Three different theoretical solutions:

1. Choose prediction horizon equal to infinity ($N = \infty$)
 - Usually not possible (unless no constraints: LQR)
2. For given N , design Q and R such that MPC is stable (cf. example)
 - Difficult in general! And usually we want to design (“tune”) Q and R for performance, not for stability

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Three different theoretical solutions:

1. Choose prediction horizon equal to infinity ($N = \infty$)
 - Usually not possible (unless no constraints: LQR)
2. For given N , design Q and R such that MPC is stable (cf. example)
 - Difficult in general! And usually we want to design (“tune”) Q and R for performance
3. Change the optimization problem: Terminal cost + terminal constraint
 - Choose terminal cost + terminal constraint such that cost of new problem is a feasible upper bound on cost of infinite horizon problem
 - General theory for finding such terminal cost & terminal constraint exist, not difficult, but may be somewhat impractical for “real” problems

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\ & u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1 \end{aligned}$$

Terminal cost

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \left(\frac{1}{2} x_t^\top Q x_t + \frac{1}{2} u_t^\top R u_t \right) + \underbrace{\frac{1}{2} x_N^\top P x_N}_{\text{Terminal cost}} \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\ & u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1 \\ & \underbrace{x_N \in \mathcal{S}}_{\text{Terminal constraint}} \end{aligned}$$

Terminal constraint

How to achieve nominal stability?

Three different theoretical solutions:

1. Choose prediction horizon equal to infinity ($N = \infty$)
 - Usually not possible (unless no constraints: LQR)
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What is the ~~usual practical solution?~~

- Choose N “large enough”
 - Can show: stability guaranteed for N large enough, but difficult/conservative to compute this limit
 - So what is “large enough” in practice? Rule of thumb: longer than “dominating dynamics”
 - (...but not too large, as large N might give robustness issues...)

$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x_t^{\text{low}} \leq x_t \leq x_t^{\text{high}}, \quad t = 1, \dots, N \\ & u_t^{\text{low}} \leq u_t \leq u_t^{\text{high}}, \quad t = 0, \dots, N-1 \end{aligned}$$

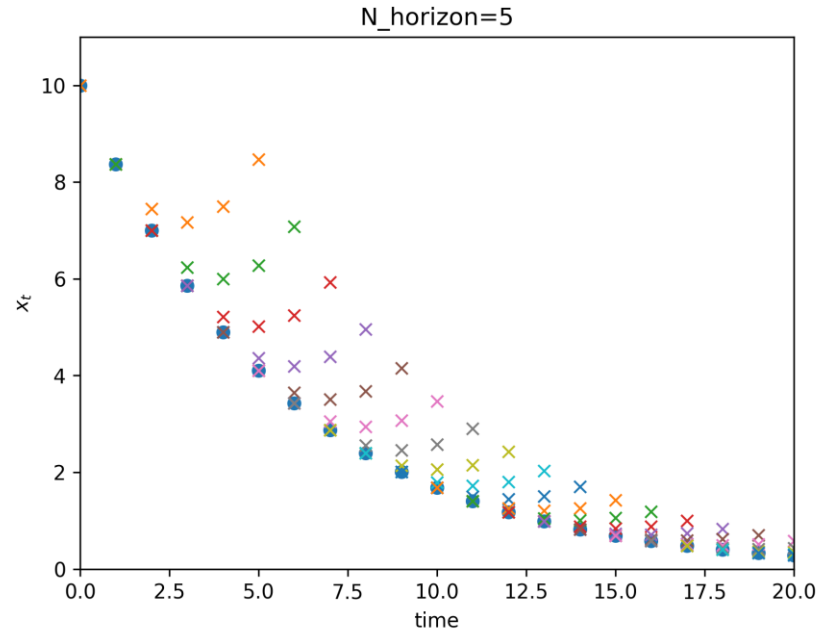
$$\begin{aligned} \min_z \quad & \sum_{t=0}^{N-1} \left(\frac{1}{2} x_t^\top Q x_t + \frac{1}{2} u_t^\top R u_t \right) + \frac{1}{2} x_N^\top P x_N \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\ & x_t^{\text{low}} \leq x_t \leq x_t^{\text{high}}, \quad t = 1, \dots, N \\ & u_t^{\text{low}} \leq u_t \leq u_t^{\text{high}}, \quad t = 0, \dots, N-1 \\ & x_N \in \mathcal{S} \end{aligned}$$

Terminal cost
↓

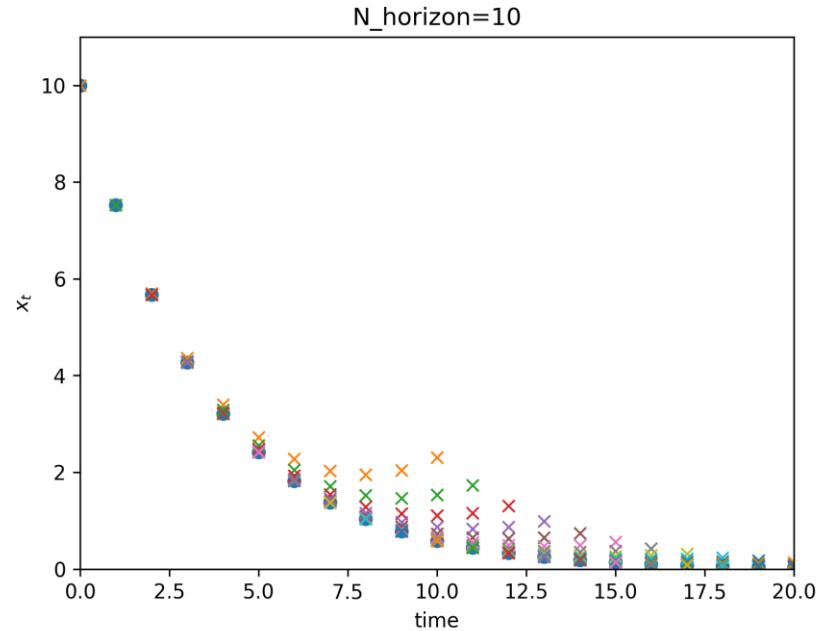
↑
Terminal constraint

“is great time constant”

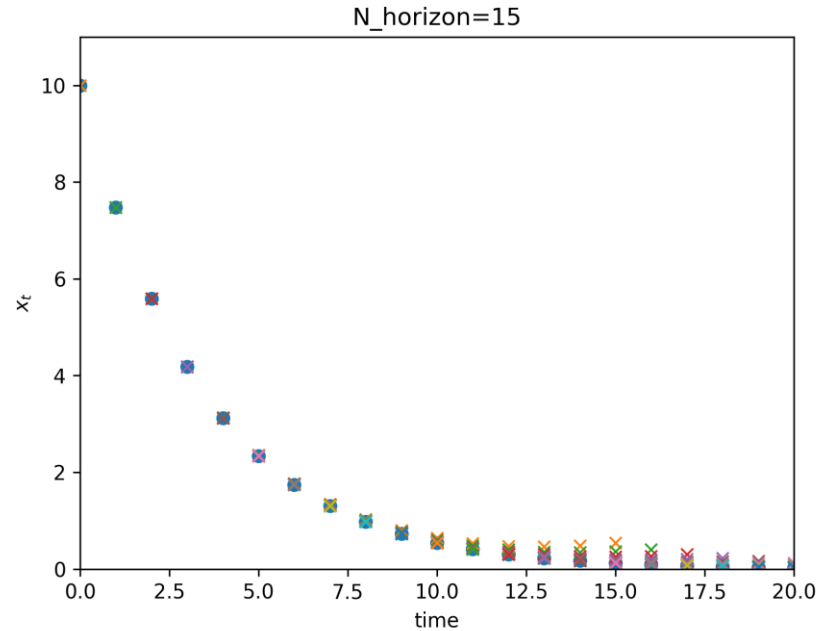
Open-Loop vs Closed-Loop: $N = 5$



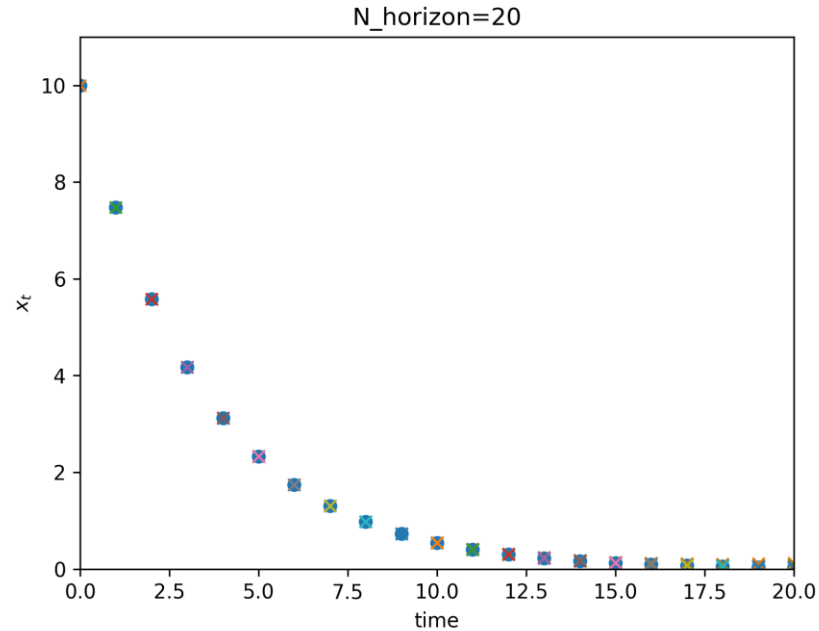
Open-Loop vs Closed-Loop: N = 10



Open-Loop vs Closed-Loop: N = 15



Open-Loop vs Closed-Loop: N = 20



Note: This assumes no model-plant mismatch!

Why MPC over PID control?

Advantages of MPC:

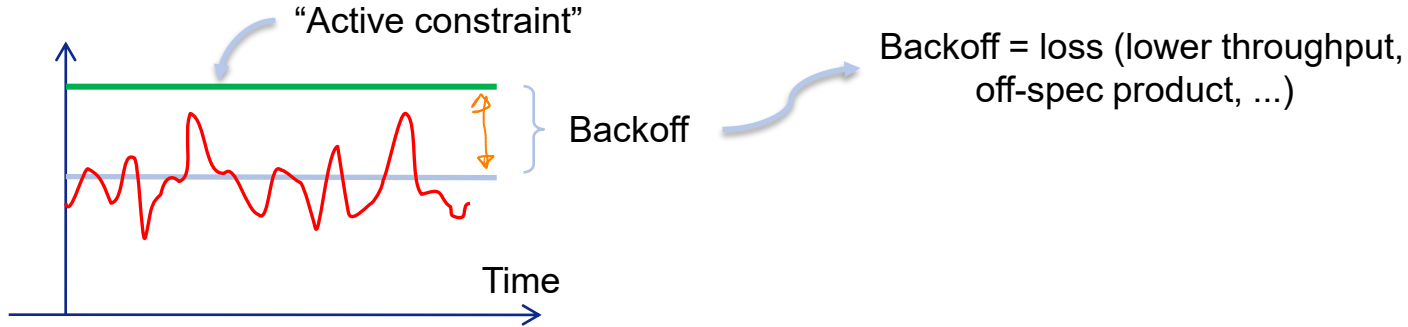
- **MPC handles constraints in a transparent way**
 - Physical constraints (actuator limits), performance constraints, safety limits, ...
- Intuitive and easy to tune (...relatively, at least)
- MPC is by design multivariable (MIMO)
- MPC gives “optimal” performance (but what is the optimal objective?)

Disadvantage with MPC

- Online complexity (but only solving a QP, so not so bad)
- Requires models! Increased commissioning cost?
- Difficult to maintain?

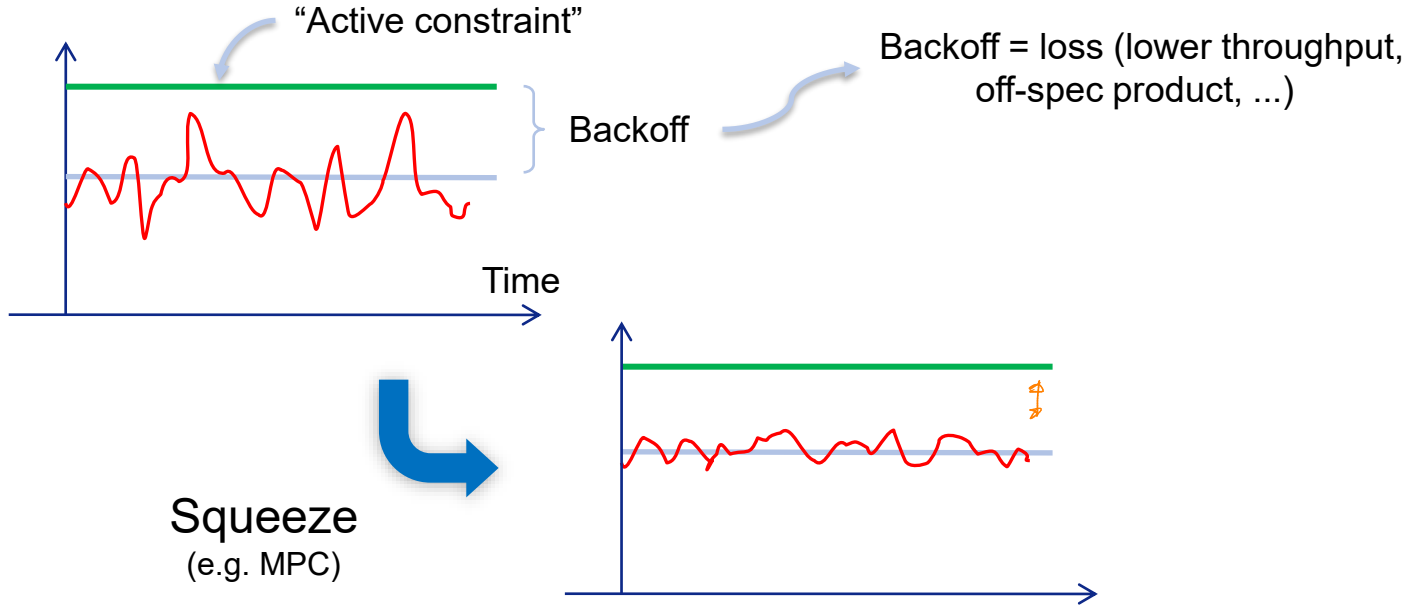
“Squeeze and shift”

How MPC (or improved/advanced control in general) improves profitability



“Squeeze and shift”

How MPC (or improved/advanced control in general) improves profitability



“Squeeze and shift”

How MPC (or improved/advanced control in general) improves profitability

