NTNU | Norwegian University of Science and Technology

TTK4135 – Lecture 7
Active Set Method for Quadratic programming

Lecturer: Lars Imsland

Overview of lecture

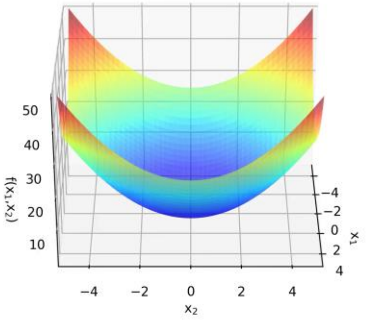
- Quadratic programming – used for control (MPC), in finance, …
- Recap last time – Equality-constrained QPs (EQPs)
- **Active set method for solving QPs**
  - For medium-sized problems – for large problems, interior point methods may be faster (not part of this course)
- Example 16.4

Reference: N&W Ch.15.3-15.5, **16.1-2,4-5**

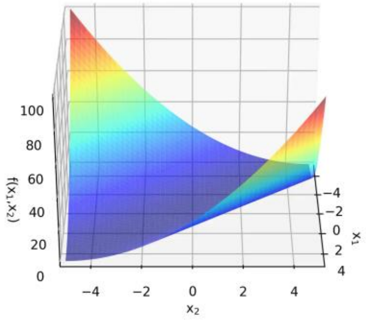
Quadratic programming
Solving (convex) quadratic programs, QPs

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

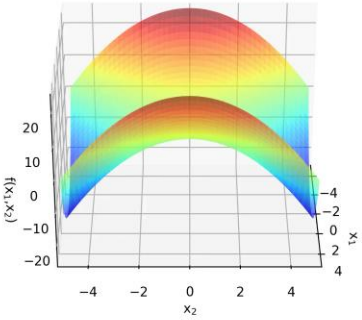
- Feasible set convex (as for LPs)
- The QP is convex if *G* ≥ *0* (strictly convex if *G* > *0*)



Strictly convex
$x^T P x = x_1^2 + x_2^2$



Convex
$x^T P x = x_1^2 + 2x_1x_2 + x_2^2$



Non-convex
$x^T P x = x_1^2 - x_2^2$

Equality-constrained QP (EQP)

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x^\top G x + c^\top x \\ \text{subject to} \quad & Ax = b, \quad A \in \mathbb{R}^{m\times n} \end{aligned}$$

Basic assumption:
A full row rank

- KKT-conditions (KKT system, KKT matrix):

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \quad \text{or, if we let } x^* = x + p, \quad \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

- Solvable when $Z^\top G Z > 0$ (columns of *Z* basis for nullspace of *A*):

$$Z^\top G Z > 0 \overset{\text{Lemma 16.1}}{\Rightarrow} K = \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \text{ non-singular} \Rightarrow \begin{pmatrix} x^* = x + p \\ \lambda^* \end{pmatrix} \text{ unique solution of KKT system}$$

$$\overset{\text{Theorem 16.2}}{\Rightarrow} x^* \text{ is the unique solution to EQP}$$

- How to solve KKT system (KKT matrix indefinite, but symmetric):
  - Full-space: Symmetric indefinite (LDL) factorization: $P^\top K P = LBL^\top$
  - Reduced-space: Use *Ax*=*b* to eliminate *m* variables. Requires computation of *Z*, which can be costly. Reduced space method faster than full-space when many constraints (if *n*-*m* $\ll$ *n*).

Active set method for QPs, simplified

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

1. Make a guess of which constraints are active at the optimal solution
2. Solve corresponding EQP
3. Check KKT-conditions
    1. IF KKT OK, then finished
    2. If not, update guess of active constraints in smart way, go to 2.

KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{cases}$$

Lagrangian: $$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && && \text{(stationarity)} \\ c_i(x^*) &= 0, && \forall i \in \mathcal{E}, && \\ c_i(x^*) &\geq 0, && \forall i \in \mathcal{I}, && \text{(primal feasibility)} \\ \lambda_i^* &\geq 0, && \forall i \in \mathcal{I}, && \text{(dual feasibility)} \\ \lambda_i^* c_i(x^*) &= 0, && \forall i \in \mathcal{E} \cup \mathcal{I}. && \text{(complementarity condition/ complementary slackness)} \end{aligned}$$

KKT for QP

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\
c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\
c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\
\lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\
\lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}.
\end{aligned}$$

Theorem 16.4: If x^* satisfies KKT and $G \geq 0$, then x^* is a global solution.

Degeneracy

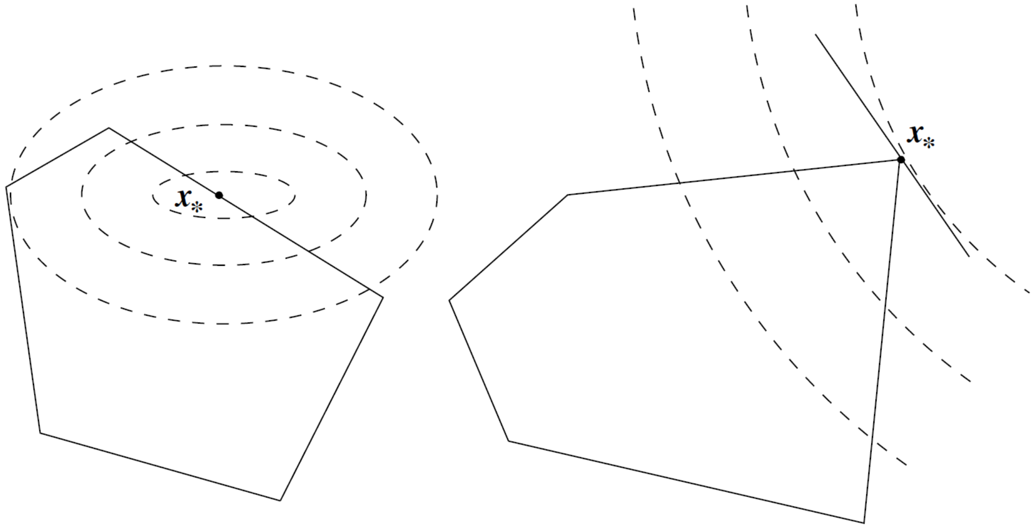


Figure 16.2 in Nocedal & Wright.

1) Strict complementarity does not hold
2) Constraints linearly dependent at solution

If active set known, QP can be solved as EQP

One step of active set method for QP

One step of active set method for QP, cont'd

General QP problem

$$\begin{aligned} \min_x \ & \frac{1}{2}x^\top Gx + x^\top c \\ \text{s.t.} \ & a_i^\top x = b_i, \qquad i \in \mathcal{E} \\ & a_i^\top x \geq b_i, \qquad i \in \mathcal{I} \end{aligned}$$

- Lagrangian

$$\mathcal{L}(x^*, \lambda^*) = \frac{1}{2}x^\top Gx + x^\top c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^\top x - b_i)$$

- KKT conditions

General:

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i, \qquad i \in \mathcal{E} \\ a_i^\top x^* &\geq b_i, \qquad i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \qquad i \in \mathcal{I} \\ \lambda_i^*(a_i^\top x^* - b_i) &= 0, \qquad i \in \mathcal{E} \cup \mathcal{I} \end{aligned}$$

Defined via active set:

$$\mathcal{A}(x^*) = \mathcal{E} \cup \left\{ i \in \mathcal{I} \middle| a_i^\top x^* = b_i \right\}$$

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i, \qquad i \in \mathcal{A}(x^*) \\ a_i^\top x^* &\geq b_i, \qquad i \in \mathcal{I} \backslash \mathcal{A}(x^*) \\ \lambda_i^* &\geq 0, \qquad i \in \mathcal{A}(x^*) \cap \mathcal{I} \end{aligned}$$

One step of active set method for QP, cont'd

Active set method for convex QP

Algorithm 16.3 (Active-Set Method for Convex QP).

Compute a feasible starting point x_0;
Set $\mathcal{W}_0$ to be a subset of the active constraints at x_0;
for $k = 0, 1, 2, \ldots$
  Solve (16.39) to find p_k;
  if $p_k = 0$
    Compute Lagrange multipliers $\hat{\lambda}_i$ that satisfy (16.42),
      with $\hat{\mathcal{W}} = \mathcal{W}_k$;
    if $\hat{\lambda}_i \geq 0$ for all $i \in \mathcal{W}_k \cap \mathcal{I}$
      stop with solution $x^* = x_k$;
    else
      $j \leftarrow \arg\min_{j \in \mathcal{W}_k \cap \mathcal{I}} \hat{\lambda}_j$;
      $x_{k+1} \leftarrow x_k$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \backslash \{j\}$;
  else (* $p_k \neq 0$ *)
    Compute α_k from (16.41);
    $x_{k+1} \leftarrow x_k + \alpha_k p_k$;
    if there are blocking constraints
      Obtain $\mathcal{W}_{k+1}$ by adding one of the blocking
        constraints to $\mathcal{W}_k$;
    else
      $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k$;
end (for)

$$\min_p \quad \tfrac{1}{2} p^T G p + g_k^T p \tag{16.39a}$$

$$\text{subject to} \quad a_i^T p = 0, \quad i \in \mathcal{W}_k. \tag{16.39b}$$

$$\sum_{i \in \hat{\mathcal{W}}} a_i \hat{\lambda}_i = g = G\hat{x} + c, \tag{16.42}$$

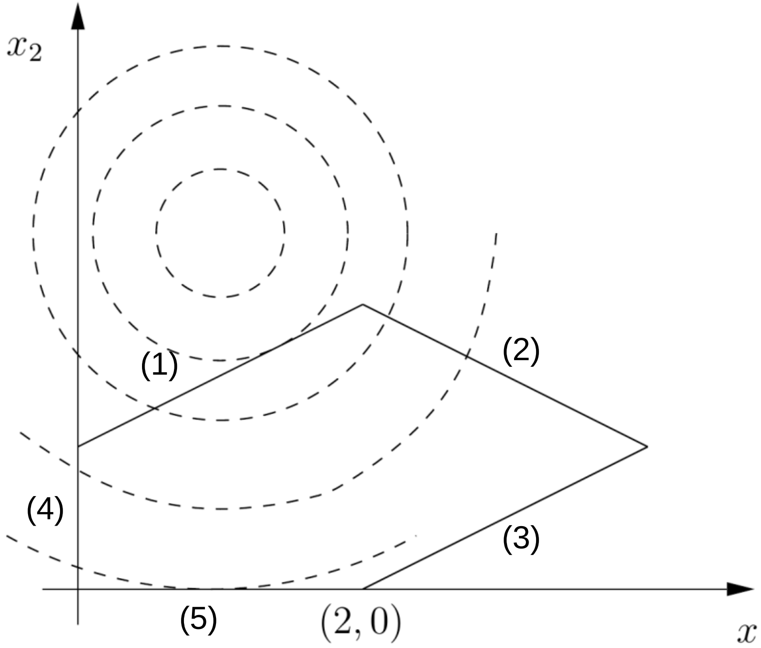
$$\alpha_k \stackrel{\text{def}}{=} \min\left(1, \min_{i \notin \mathcal{W}_k,\, a_i^T p_k < 0} \frac{b_i - a_i^T x_k}{a_i^T p_k}\right). \tag{16.41}$$

No degeneracy and *G>0*: Active set method converges in finite number of iterations.

Example 16.4

$$\min_x q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 + 2 &\geq 0 && (1) \\ -x_1 - 2x_2 + 6 &\geq 0 && (2) \\ -x_1 + 2x_2 + 2 &\geq 0 && (3) \\ x_1 &\geq 0 && (4) \\ x_2 &\geq 0 && (5) \end{aligned}$$



$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_1 = -2$$

$$a_2 = \begin{bmatrix} -1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_2 = -6$$

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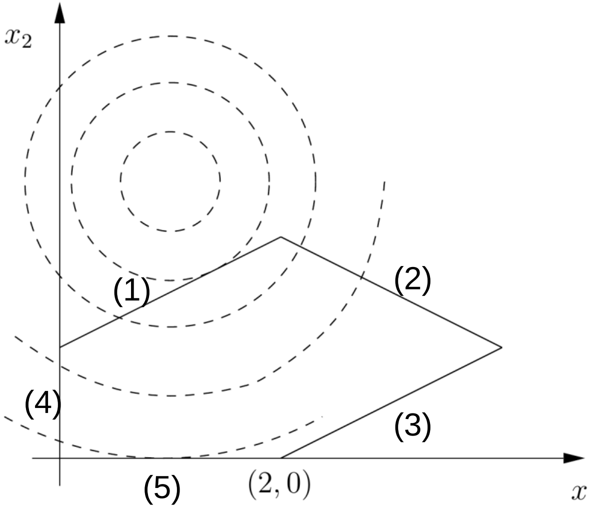

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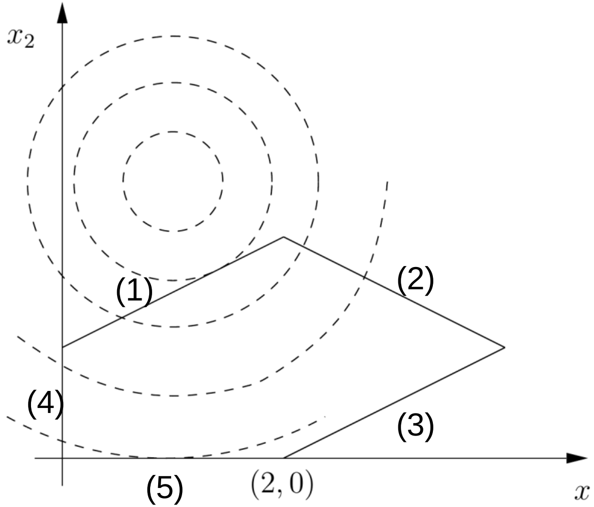

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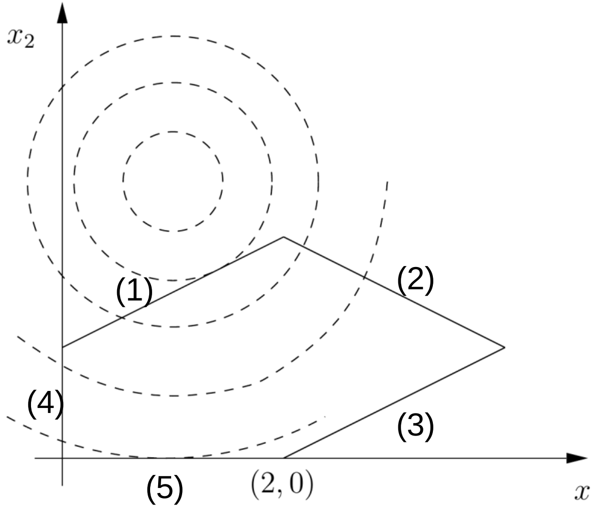

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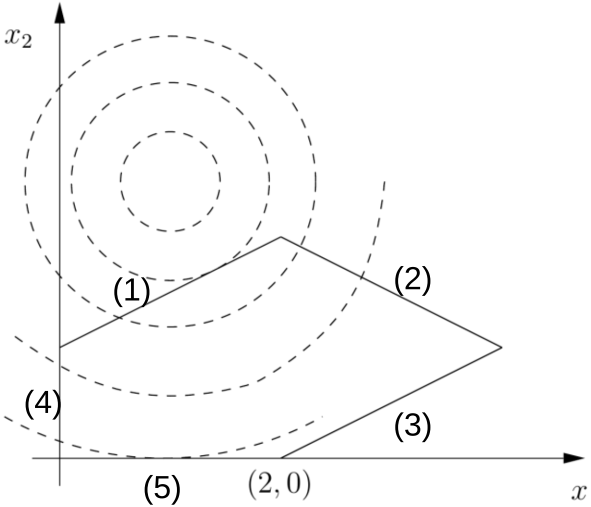

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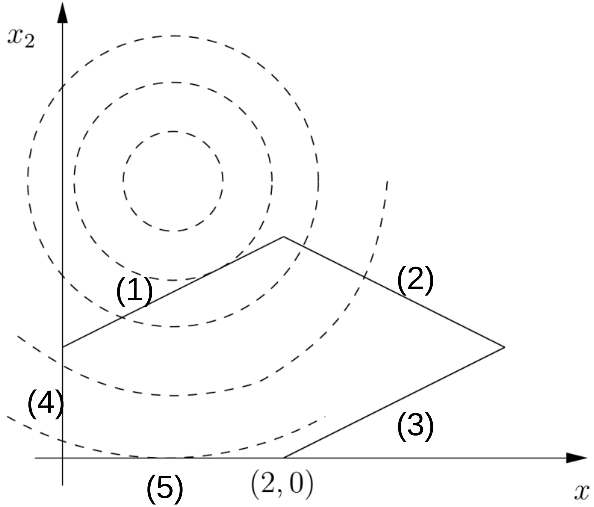

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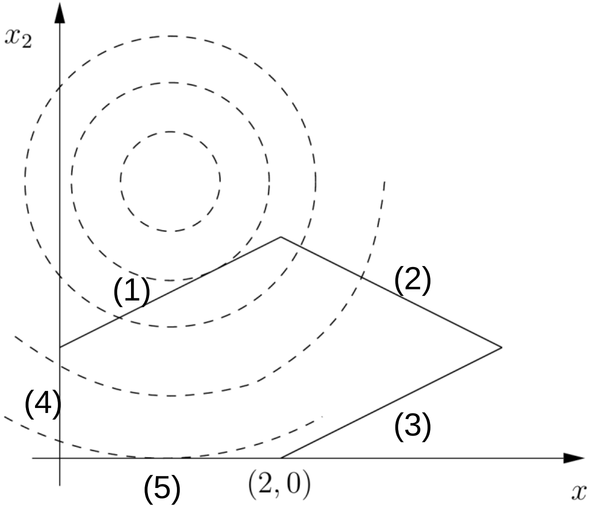

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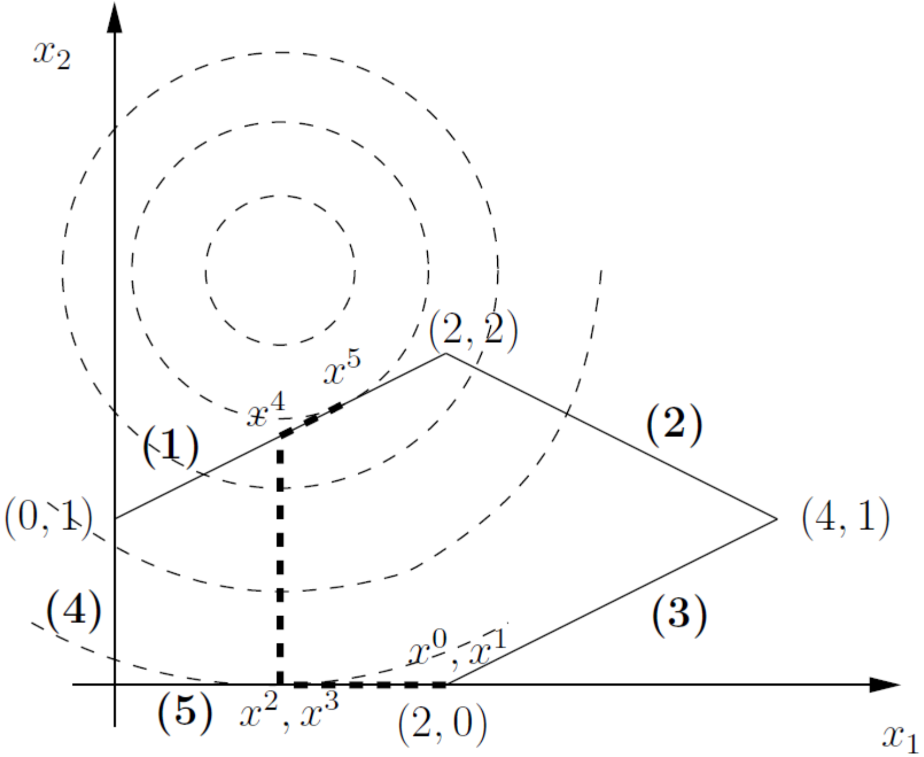

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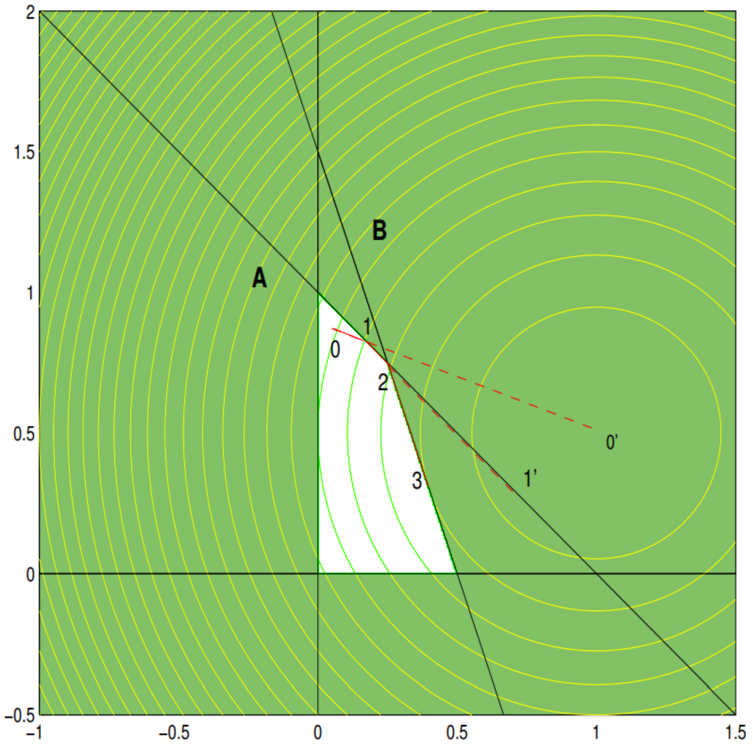
Another example (N. Gould)



$$\min (x_1 - 1)^2 + (x_2 - 0.5)^2$$
$$\text{subject to } \; x_1 + x_2 \leq 1$$
$$3x_1 + x_2 \leq 1.5$$
$$(x_1, x_2) \geq 0$$

0. Starting point
0'. Unconstrained minimizer
1. Encounter constraint A
1'. Minimizer on constraint A
2. Encounter constraint B,
   move off constraint A
3. Minimizer on constraint B
   = required solution

How to find feasible initial point?

- Same way as for LP:
  - Phase I: Define a LP with known feasible initial point, where solution is feasible for original QP.
  - Phase II: Solve original QP.

- Alternative method: “Big M”
  - Relax all constraints; penalize constraint violations in objective

Initialization methods

Phase 1

$$
\begin{aligned}
\min_{(x,z)} \ & e^T z \\
\text{subject to} \quad & a_i^T x + \gamma_i z_i = b_i, && i \in \mathcal{E}, \\
& a_i^T x + \gamma_i z_i \geq b_i, && i \in \mathcal{I}, \\
& z \geq 0,
\end{aligned}
$$

$e = (1, 1, \ldots, 1)^T$, $\gamma_i = -\text{sign}(a_i^T \tilde{x} - b_i)$ for $i \in \mathcal{E}$

$\gamma_i = 1$ for $i \in \mathcal{I}$

- Feasible initial guess for LP problem:

$$
\begin{aligned}
x &= \tilde{x} \\
z_i &= |a_i^T \tilde{x} - b_i| \quad (i \in \mathcal{E}) \\
z_i &= \max(b_i - a_i^T \tilde{x}, 0) \quad (i \in \mathcal{I})
\end{aligned}
$$

Big M

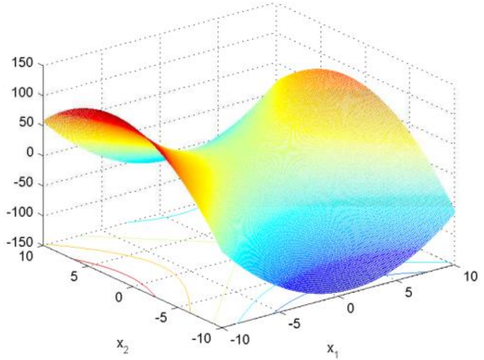
$$
\begin{aligned}
\min_{(x,\eta)} \ & \tfrac{1}{2} x^T G x + x^T c + M\eta, \\
\text{subject to} \quad & (a_i^T x - b_i) \leq \eta, && i \in \mathcal{E}, \\
& -(a_i^T x - b_i) \leq \eta, && i \in \mathcal{E}, \\
& b_i - a_i^T x \leq \eta, && i \in \mathcal{I}, \\
& 0 \leq \eta,
\end{aligned}
$$

- Feasible initial guess for Big M: Whatever.
- η nonzero? Increase M and try again.

Concluding remarks

- Solves similar EQPs iteratively: recalculate only what's needed
- Active set method: Potentially slow, but with good initial guess will be FAST
- Alternative to Active Set: Interior Point (not curriculum)

Nonconvex QP



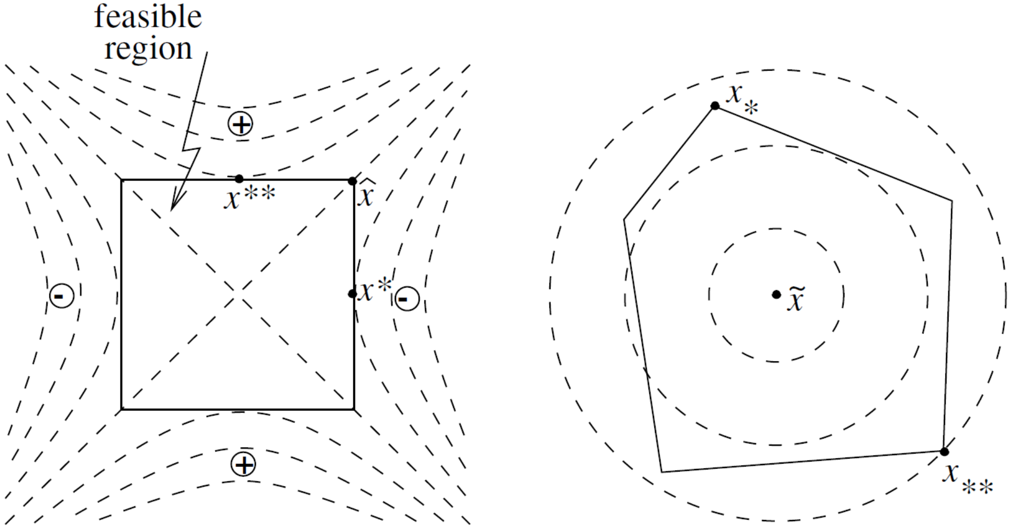


Figure 16.1 in Nocedal & Wright.