# Mean-field theory and dynamical isometry of deep neural networks

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References list & slides can be find at

https://github.com/fwcore/mean-field-theory-deep-learning

# Outlines

- **Introduction**
- **Mean-field theory framework and its predictions**
  - Initialization strategies
    - MLP
    - CNN
  - Architectures
    - ResNet
    - Dropout
    - batch normalization
- **Details of the theory**
  - Assumptions
  - Possible pitfalls

# Introduction

**Initialization**
- He's & Xavier's

**Activation functions**

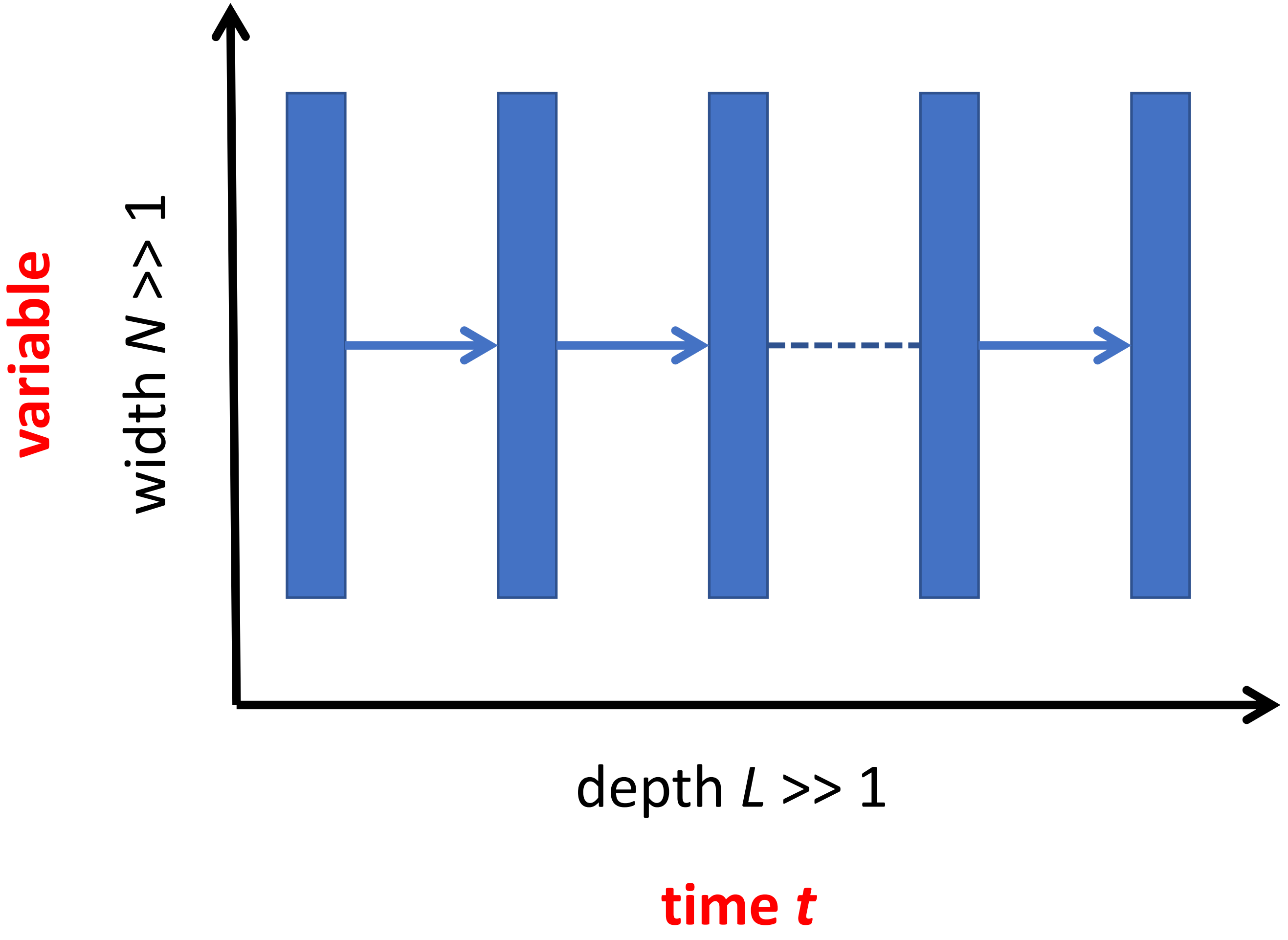
**Dropout**

**BatchNorm**

**LSTM for RNN**

**To enhance the information propagation**

# Is there a theory for deep learning?




**deep uniform neural networks**

**=**

**dynamical system**

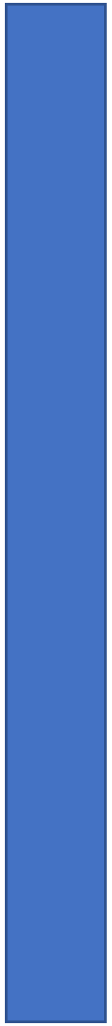
$$z_i^l = \sum_j W_{ij}^l y_j^l + b_i^l$$

$$y_i^{l+1} = \phi(z_i^l)$$

***But …***

***z*, *y* are random variables and hard to track**

# Mean-field theory for deep learning



width $N >> 1$

$$z_i^l = \sum_j W_{ij}^l y_j^l + b_i^l$$

sum of random numbers

random numbers

**Mean-field treatment**

- **Replace $z$ as Gaussian**
- **Match $z$'s mean and variance**

***Law of large numbers***

- mean($z$) = 0
- Hence we only need to track var($z$)

# Mean-field flow of var($z$)

$$\mathbb{E}[z^l_{i;a} z^l_{j;a}] = q^l_{aa}\delta_{ij}$$

$$q^l_{aa} = \sigma_w^2 \int \mathcal{D}z\,\phi^2\left(\sqrt{q^{l-1}_{aa}}\,z\right) + \sigma_b^2$$

*i, j*: neural index
*l*: layer index
*a, b*: sample index

**covariance**

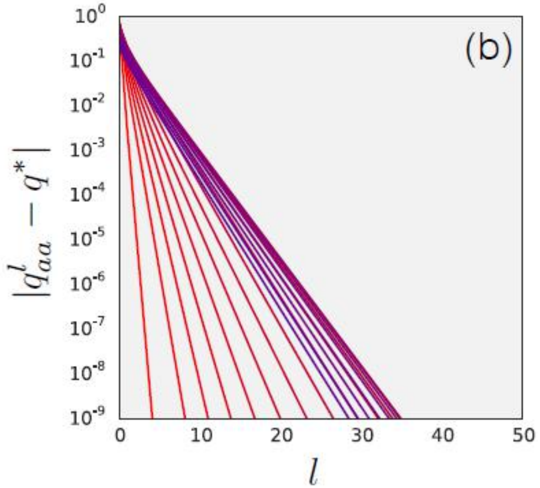
$$\mathbb{E}[z^l_{i;a} z^l_{j;b}] = q^l_{ab}\delta_{ij}$$

$$q^l_{ab} = \sigma_w^2 \int \mathcal{D}z_1 \mathcal{D}z_2\,\phi(u_1)\phi(u_2) + \sigma_b^2$$

$$u_1 = \sqrt{q^{l-1}_{aa}}\,z_1$$

$$u_2 = \sqrt{q^{l-1}_{bb}}\left(c^{l-1}_{ab} z_1 + \sqrt{1-(c^{l-1}_{ab})^2}\,z_2\right)$$

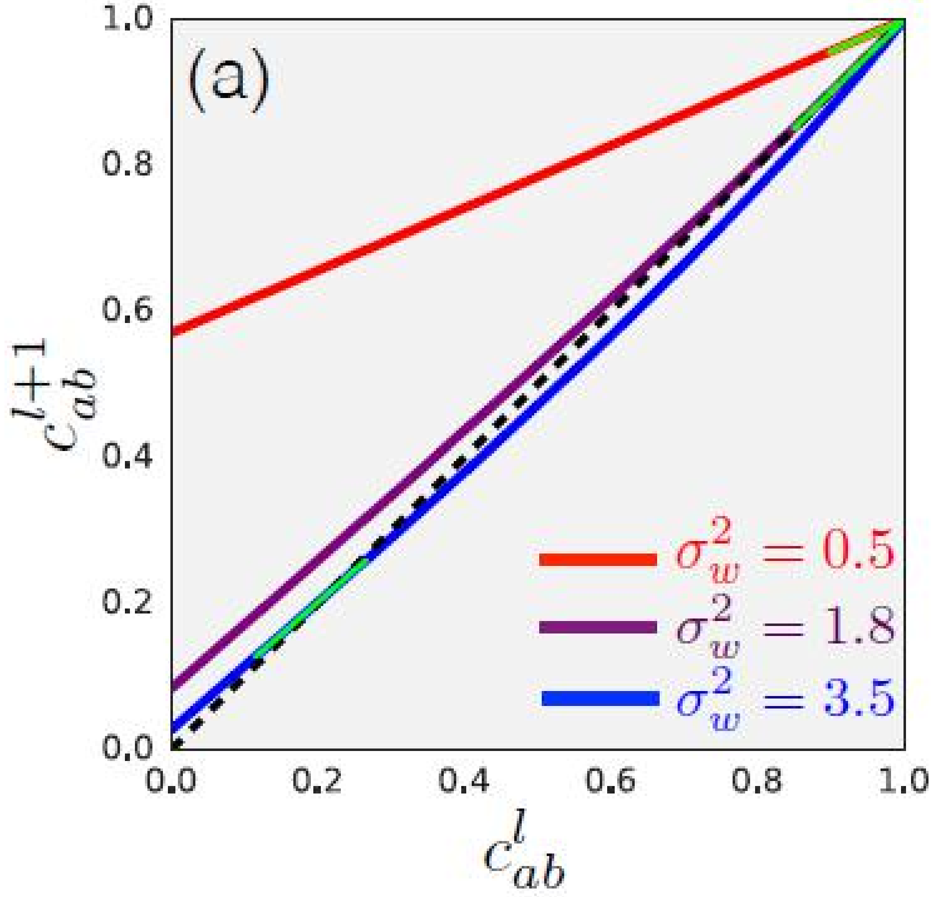
$$c^l_{ab} = q^l_{ab}/\sqrt{q^l_{aa} q^l_{bb}}$$

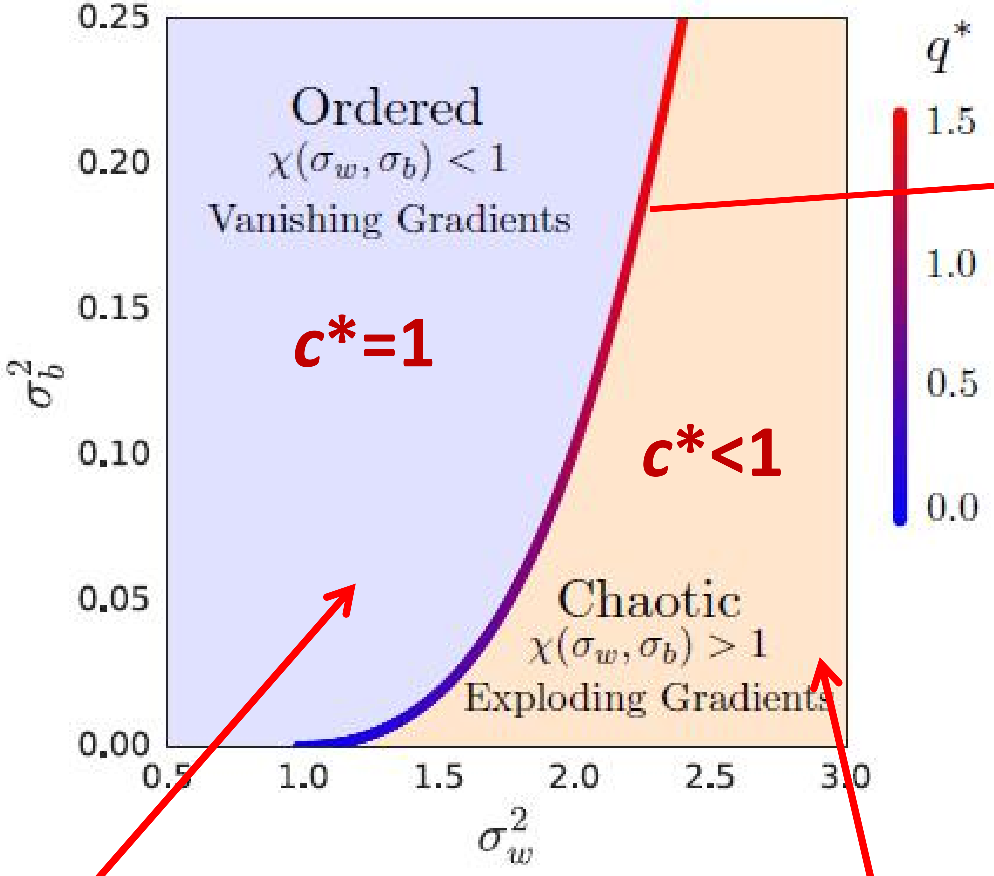



$q^* = \lim_{l\to\infty} q^l_{aa}$ is a smooth function.

However, ***limit c is differen***t

# Mean-field flow of *c*: a phase transition



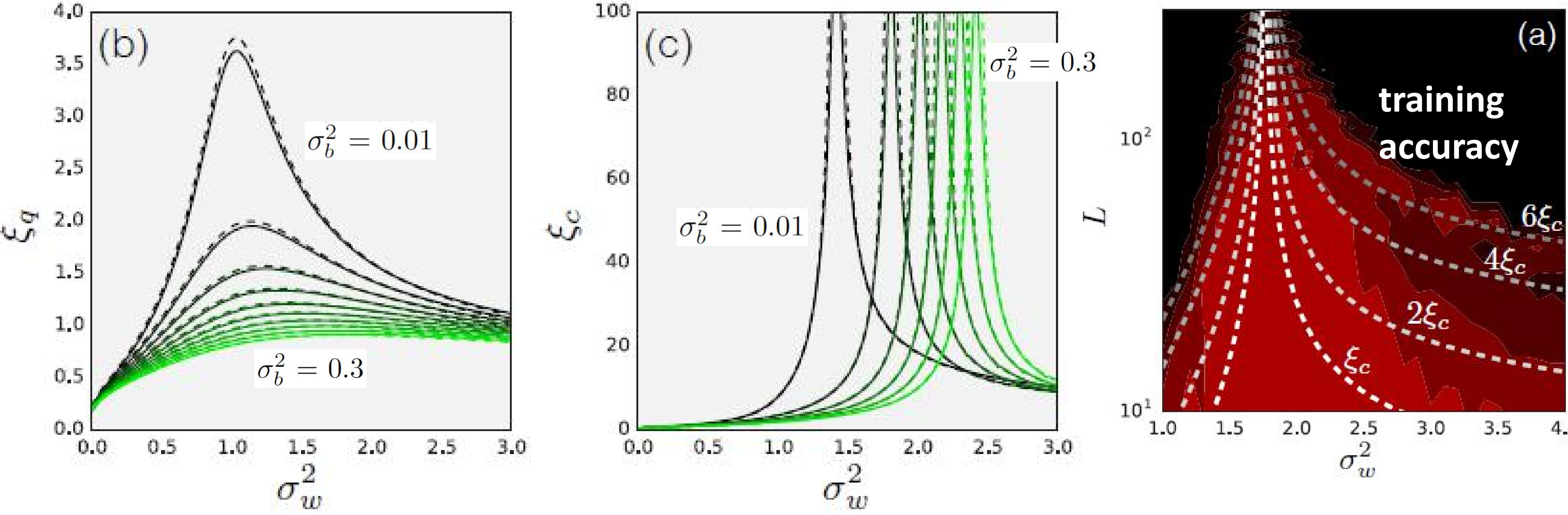

Edge of chaos

- Information collapse.
- Gradients vanish.

- Information decorrelates.
- Gradients explode.

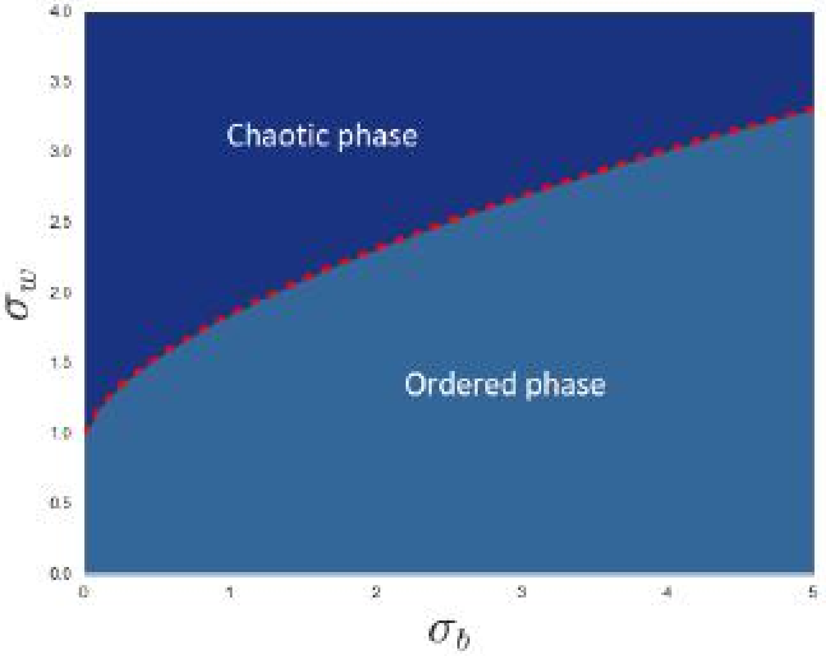
# Perturbation around $c^*$
# = asymptotic behavior near the edge of chaos

$$\epsilon^{l+1} = \epsilon^l \left[ \sigma_w^2 \int \mathcal{D}z_1 \mathcal{D}z_2 \phi'(u_1^*)\phi'(u_2^*) \right] + \mathcal{O}((\epsilon^l)^2) \sim e^{-l/\xi_c}$$

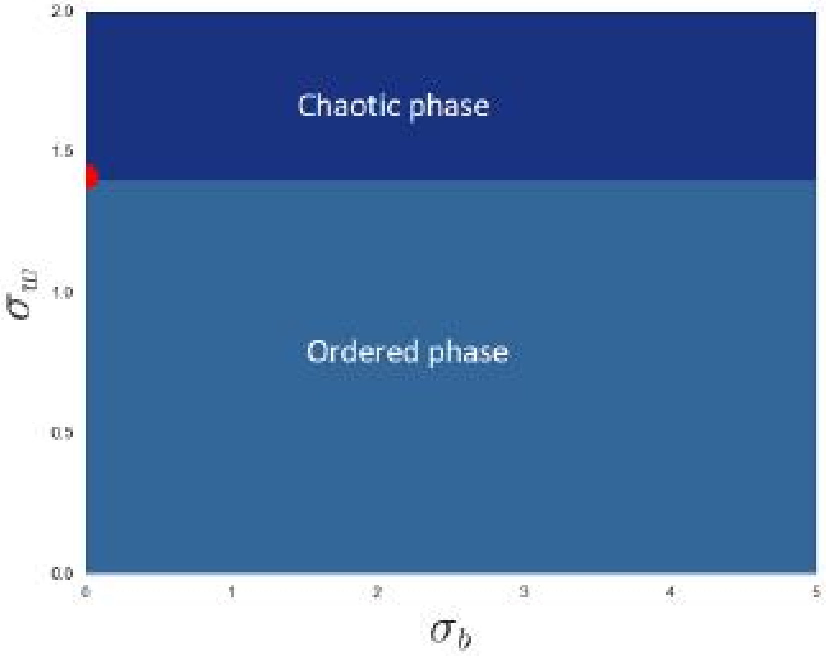



Deep Information Propagation | arXiv:1611.01232

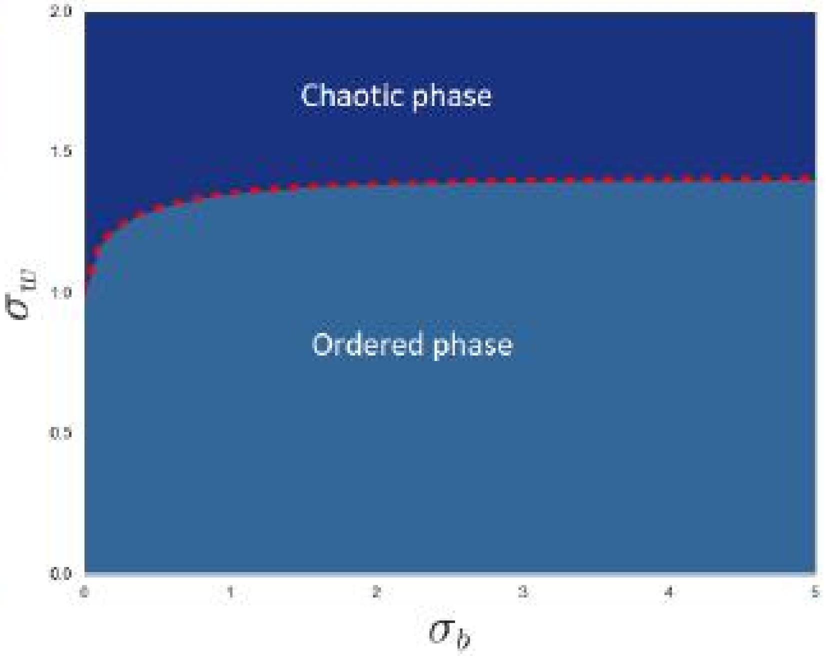
# Activation functions




tanh




ReLU




ELU

# Gaussian initialization is enough?

$\xi_c$ controls information propagation depth, and determines training accuracy,

How about ***training speed*** & ***generalization***?

**Edge of chaos**

guarantees slow changes of correlation

**Dynamical isometry**

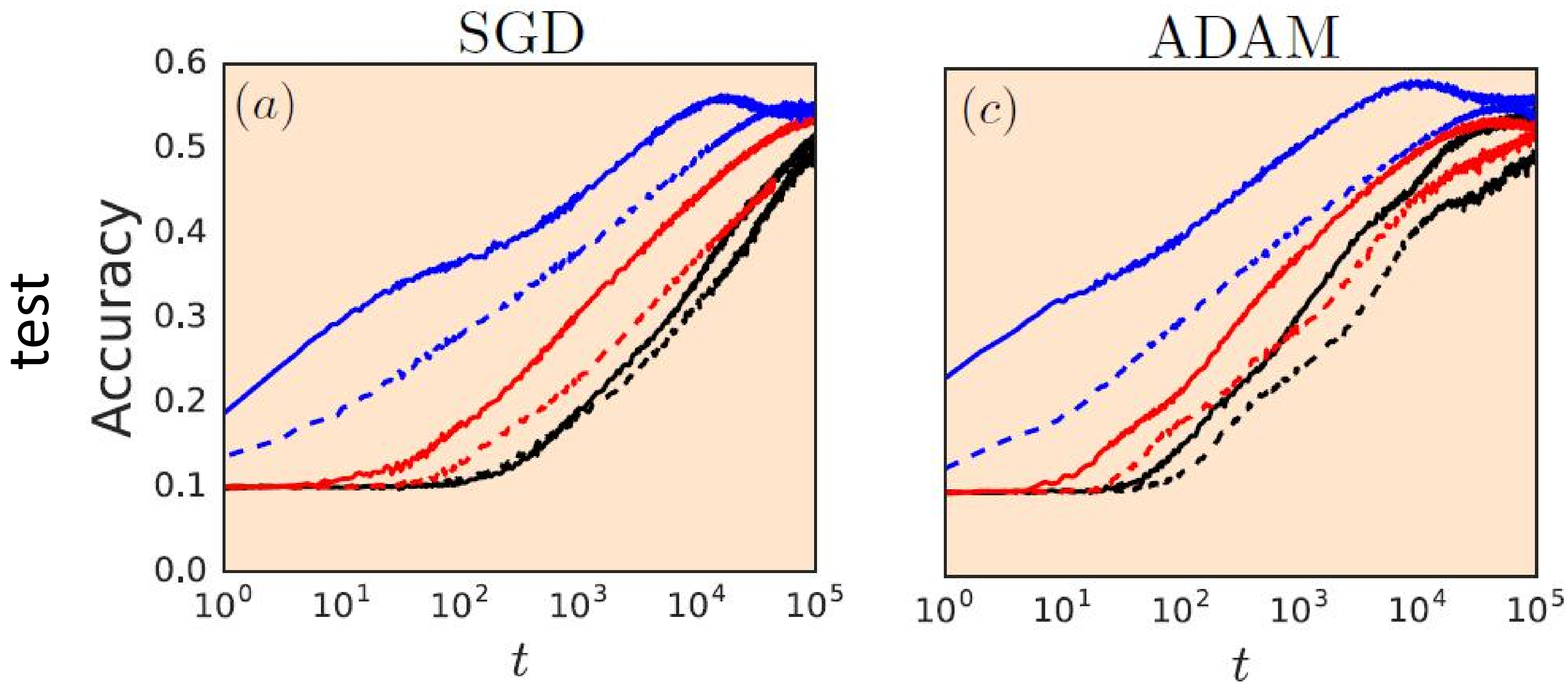
guarantees correlation preservation

$$\mathbf{J} = \frac{\partial \mathbf{x}^L}{\partial \mathbf{h}^0} = \prod_{l=1}^{L} \mathbf{D}^l \mathbf{W}^l \qquad D_{ij}^l = \phi'(h_i^l)\,\delta_{ij}$$

All singular values have norm 1

Resurrecting the sigmoid in deep learning through dynamical isometry: theory and practice | arXiv:1711.04735

# Dynamical isometry initialization speeds up training



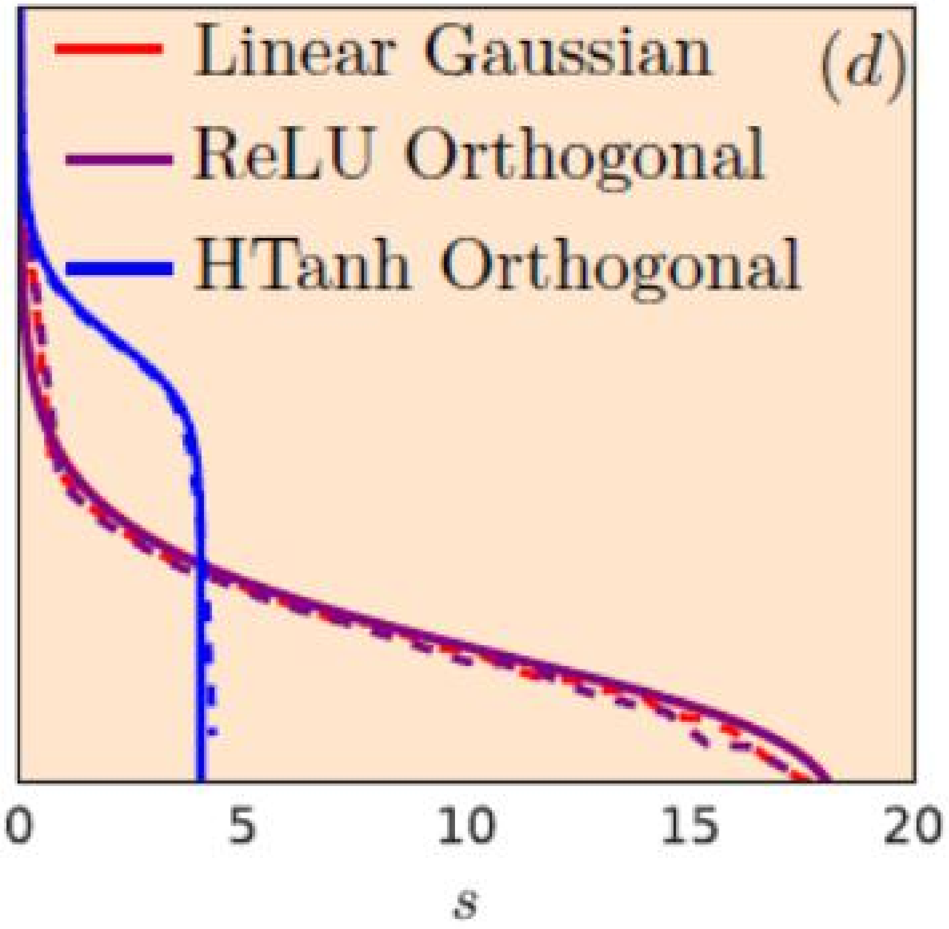

dashed line: Gaussian
solid line: orthogonal

tanh with $\sigma_w^2$= 1.05 (close to EOC)
tanh with $\sigma_w^2$= 2 (far away from EOC)
ReLU with $\sigma_w^2$= 2 (at critical point)

Resurrecting the sigmoid in deep learning through dynamical isometry: theory and practice | arXiv:1711.04735

# Activation functions

p(s)

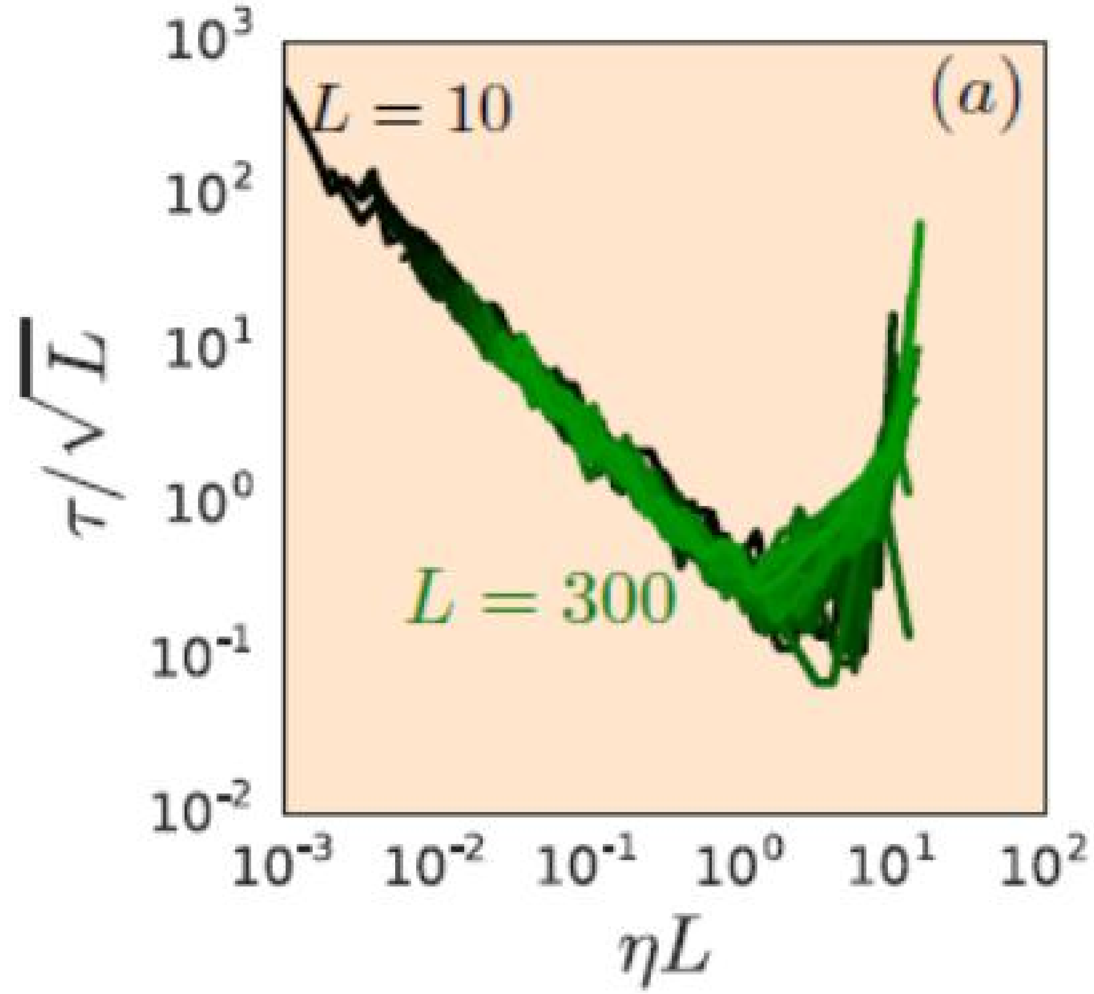



singular values of *J*

- ReLU has no dynamical isometry
- tanh is better to be initialized by Orthogonal scheme with a slightly small $\sigma_b^2$

Resurrecting the sigmoid in deep learning through dynamical isometry: theory and practice | arXiv:1711.04735

# Scaling of training time

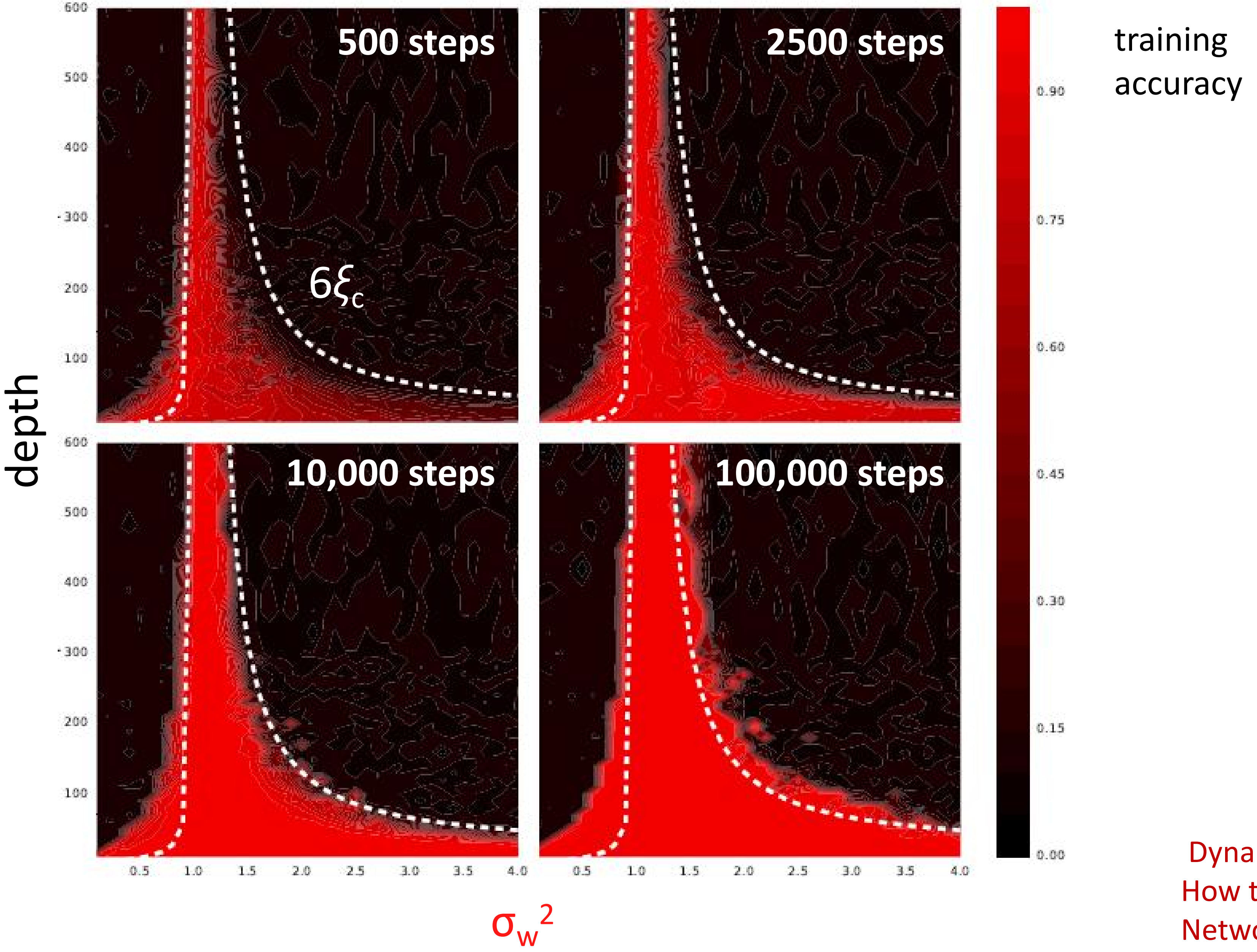



**Under dynamical isometry**

- ➢ training time grows slower than *L*
- ➢ optimal training rate η ~ 1/*L*

Resurrecting the sigmoid in deep learning through dynamical isometry: theory and practice | arXiv:1711.04735

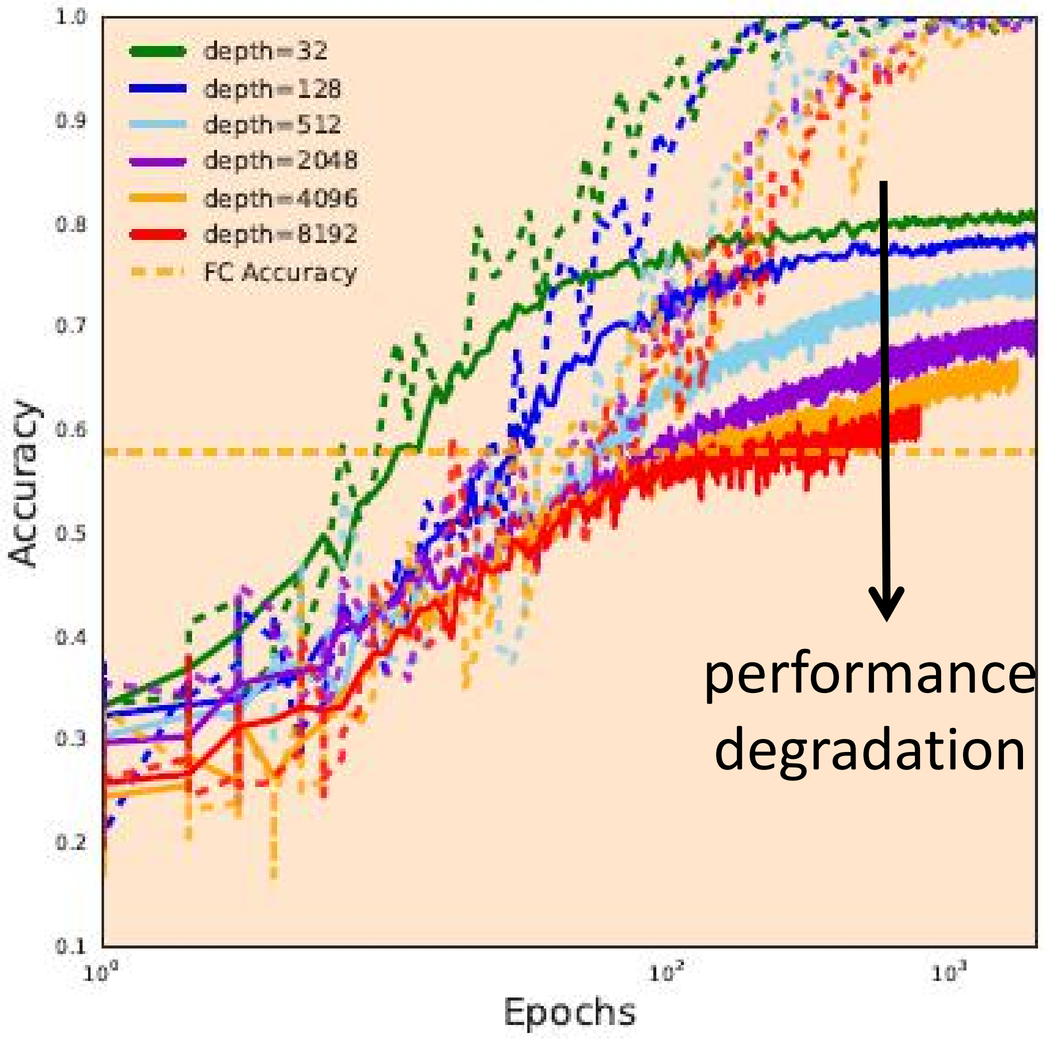
# Deep ConvNet



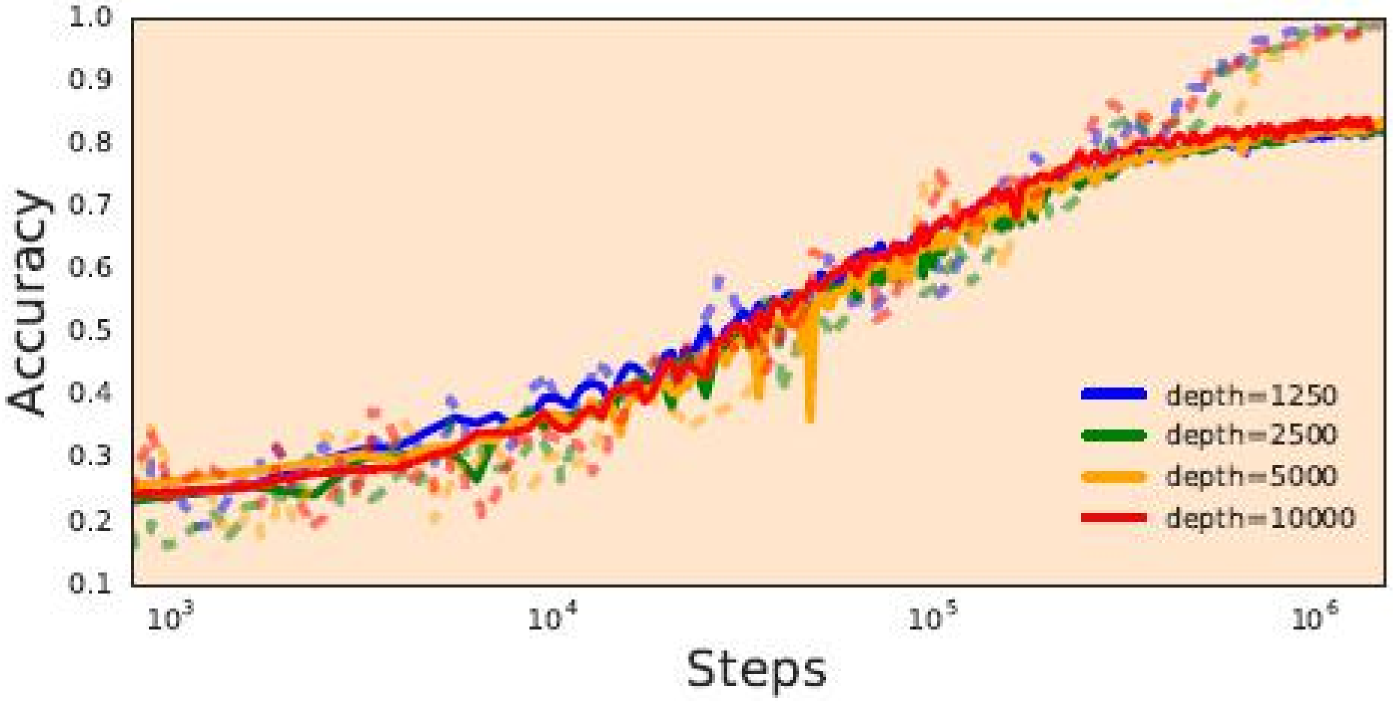

Dynamical Isometry and a Mean Field Theory of CNNs: How to Train 10,000-Layer Vanilla Convolutional Neural Networks | arXiv:1806.05393

# Dynamical isometry for deep ConvNet

Orthogonal initialization



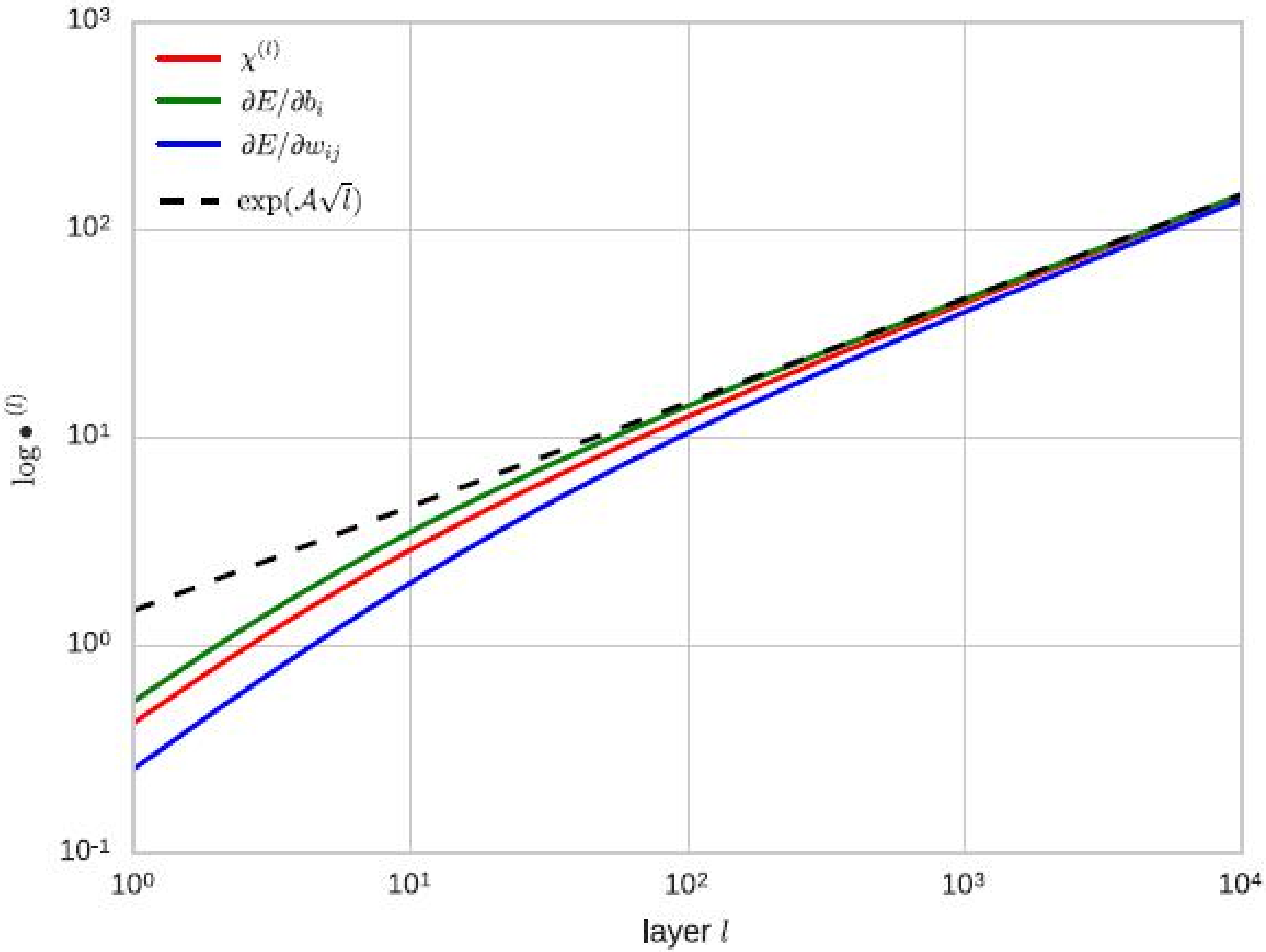

Delta-Orthogonal initialization




Dynamical isometry prevents performance degradation caused by increasing depth.

*Reminder: ResNet is an alternative*

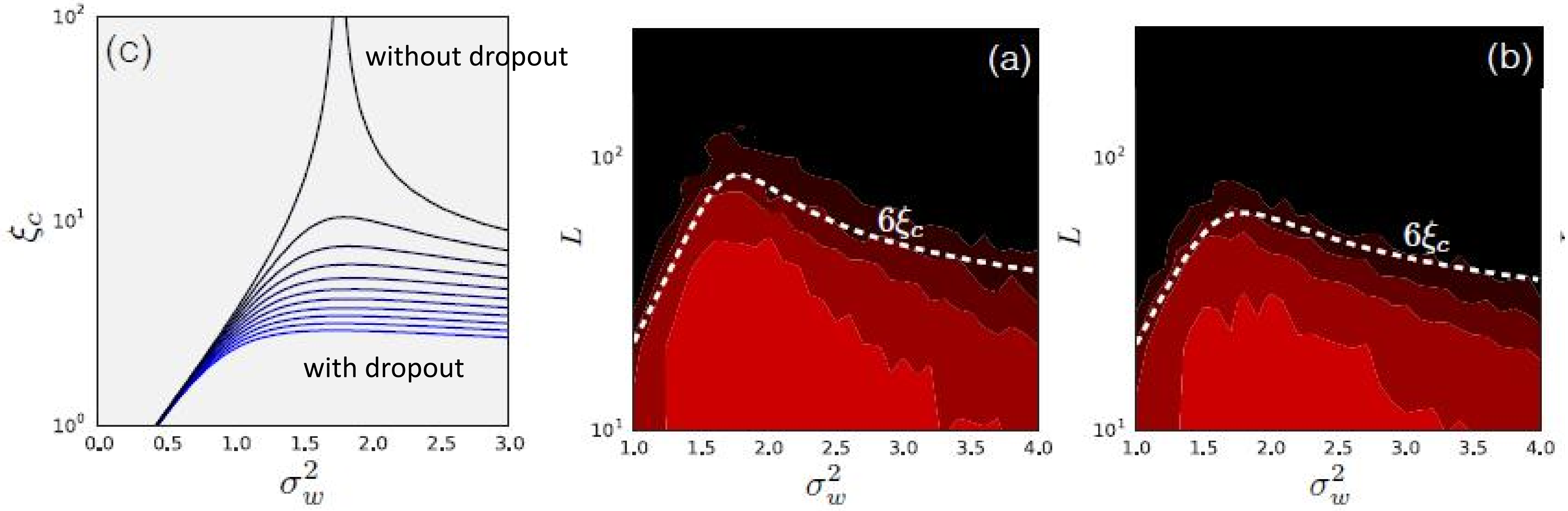
# Why ResNet?



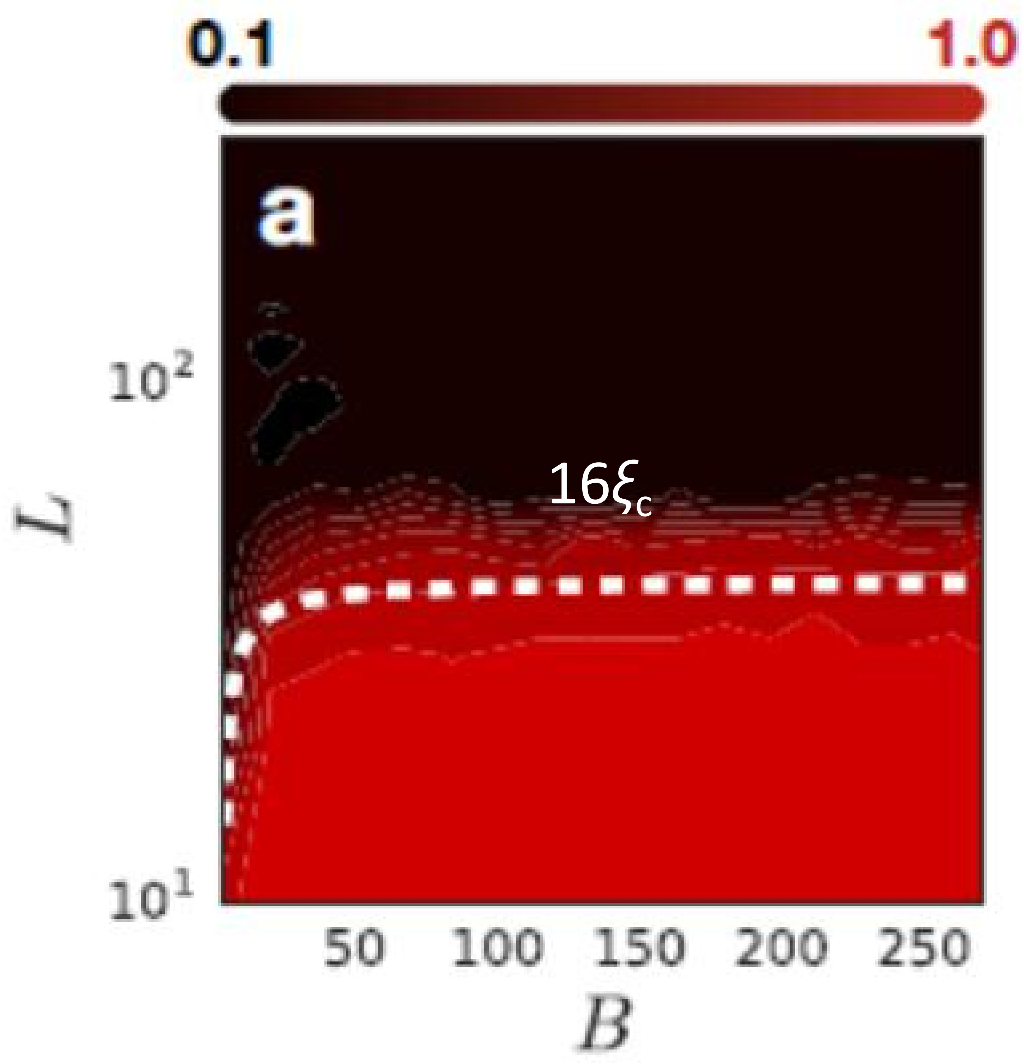

- **Xavier or the He initializations are not optimal for residual networks**
- **the optimal initialization variances depend on the depth**

Mean Field Residual Networks: On the Edge of Chaos | arXiv:1712.08969

# Dropout limits the depth



# Batch normalization limits the depth




**Method to overcome**

*Gradient explosion can be **reduced** by tuning the network close to the **linear** regime*

A Mean Field Theory of Batch Normalization | [arXiv:1902.08129]

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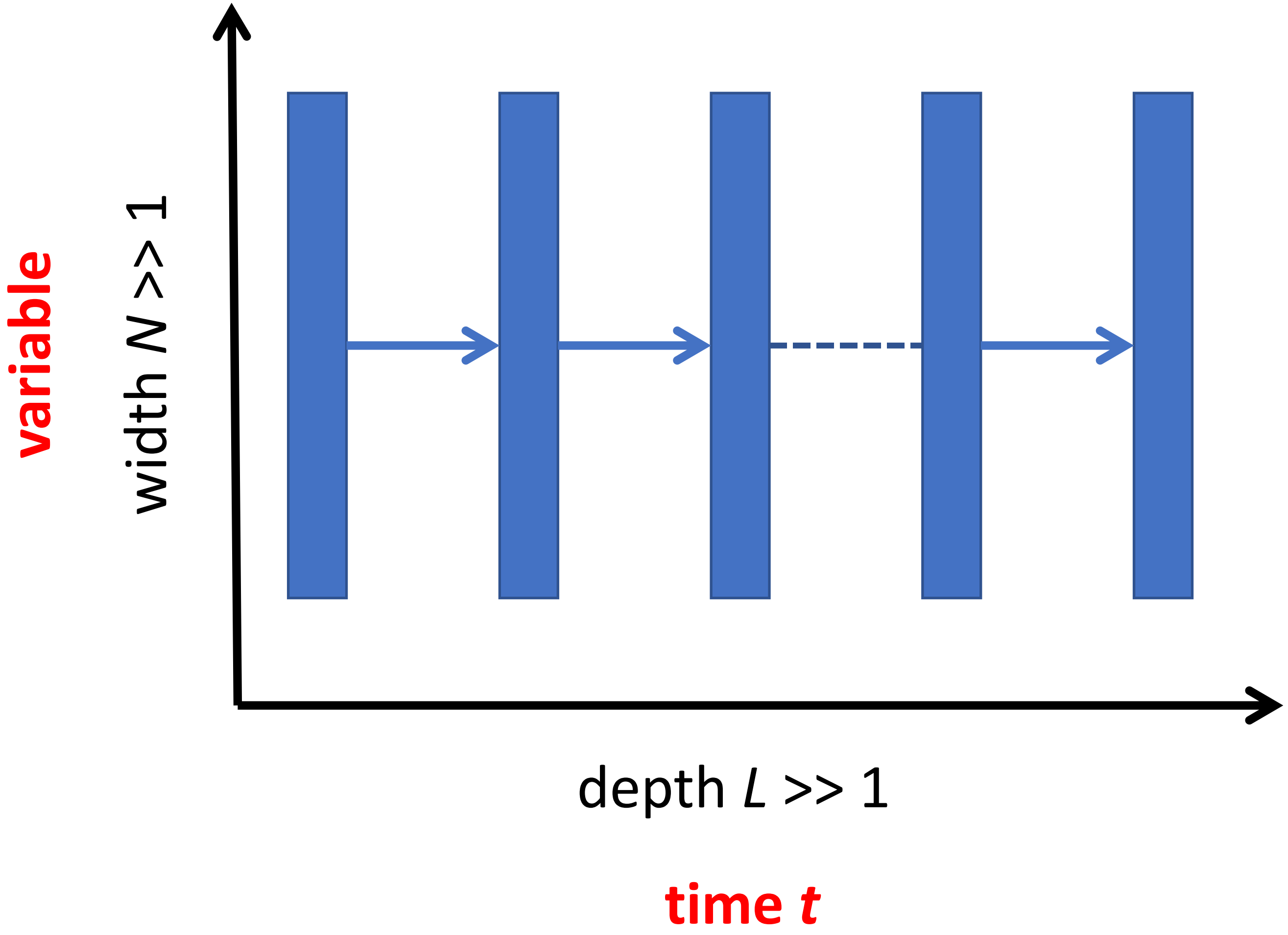
# Mean-field theory Assumptions




**deep uniform neural networks**

**=**

**dynamical system**

**Assumptions**

- Uniform layers (constant N)
- independent weights for forward & back propogation
- $N >> 1$
- $L >> 1$
- theory works *only* for untrained model (time t = layer number ≠ epoch)

# Mean-field theory limitations/pitfalls

- **Theory works *only* for untrained model**
  (time t = layer number ≠ epoch)

- **Orthogonal initialization ≠ dynamical isometry**
  necessary but not sufficient

- **May not work if width is small**

- **May not work for shallow networks**

- *Not theory on generalization/test performance, training speed*
  only on information propogation

# Summary

- Mean-field theory (MFT) describes information propogation.
- MFT determine the maximal training depth.
- Initialization at the edge of chaos unlimits the training depth.
- Dynamical isometry speeds up training, and prevents test accuracy degradation with depth.
- Orthogonal initialization is a powerful scheme to reach dynamical isometry.
- BatchNorm and dropout limit the training depth.
- ReLU has no dynamical isometry.