Nell'ultimo esercizio c'è una correzione che non è presente nella registrazione del tutorato del 22/05/23.
Fate riferimento a questo PDF.

① Data $A = \begin{pmatrix} 1 & \boxed{-2} & 1 & -2 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

a) Dire se A è diagonalizzabile

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & \boxed{2} & -1 & 2 \\ 1 & \lambda & -1 & -1 \\ 0 & 0 & \lambda-2 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} \lambda-1 & 2 & -1 \\ 1 & \lambda & -1 \\ 0 & 0 & \lambda-2 \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-2)\begin{vmatrix} \lambda-1 & 2 \\ 1 & \lambda \end{vmatrix} = (\lambda+1)(\lambda-2)(\lambda^2-\lambda-2) =$$

$-2, +1$

$$= (\lambda-2)^2(\lambda+1)^2$$

$\lambda+1=0 \Leftrightarrow \lambda=-1$

$\lambda_1 = 2 \rightarrow m.a(\lambda_1) = 2$

$\lambda_2 = -1 \rightarrow m.a(\lambda_2) = 2$

- $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x+2y-z+\cancel{2t}=0 \\ x+2y-z\cancel{-t}=0 \\ 3t=0 \Leftrightarrow t=0 \end{cases}$$

$$\begin{cases} t=0 \\ x=-2y+z \end{cases} \rightsquigarrow (-2y+z, y, z, 0)$$

$$V_{\lambda_1} = \{(-2y, y, 0, 0)+(z, 0, z, 0) : y \text{ e } z \in \mathbb{R}\} =$$

$$= [(-2,1,0,0),(1,0,1,0)] \Rightarrow mg(\lambda_1)=2$$

- $\boxed{\lambda_2 = -1}$

$$\begin{pmatrix} -2 & 2 & -1 & 2 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2x+2y-\cancel{z}+2t=0 \\ x-y-\cancel{z}-t=0 \\ -3z=0 \Leftrightarrow z=0 \end{cases}$$

$$\begin{cases} z=0 \\ x=y+t \end{cases} \Rightarrow V_{\lambda_2} = \{(y+t, y, 0, t) : y,t \in \mathbb{R}\} =$$

$$= [(1,1,0,0),(1,0,0,1)] \Rightarrow m.g(\lambda_2)=2$$

$\overset{\text{TEO}}{\Rightarrow}$ A è diagonalizzabile.

b) Se sì, determinare una matrice diagonale D che sia simile ad A e la matrice invertibile E tale che $D = E^{-1} A E$

$$D = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad E = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underbrace{}_{\lambda_1=2}$ $\underbrace{}_{\lambda_2=-1}$

In alternativa:

$$D = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\underbrace{}_{\lambda_2=-1}$ $\underbrace{}_{\lambda_1=2}$

2) Dato l'endomorfismo $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ definito da

$$T(x,y,z,t) = (2x, y+z, z, y+t)$$

a) Determinare gli autovalori di T

b) T è diagonalizzabile?

$$T = \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \qquad |\lambda I - T| = \begin{vmatrix} \lambda-2 & 0 & 0 & 0 \\ 0 & \lambda-1 & -1 & 0 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & -1 & 0 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda-1)\begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda-1)(\lambda-1)\begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda-1)(\lambda-1)[(\lambda-2)(\lambda-1)] =$$

$$= (\lambda-1)^3(\lambda-2)$$

$$= (\lambda-1)^3(\lambda-2) \Rightarrow \begin{array}{l} \lambda_1 = 1 \rightarrow m.a(\lambda_1) = 3 \\ \lambda_2 = 2 \rightarrow m.a(\lambda_2) = 1 \end{array}$$

- $\lambda_1 = 1$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x = 0 \\ -z = 0 \\ -y = 0 \end{cases} \rightsquigarrow (0,0,0,t)$$

$$V_{\lambda_1} = [(0,0,0,1)] \rightarrow \boxed{m.g(\lambda_1) = 1} \neq m.a(\lambda_1) = 3$$

Poiché $m.a(\lambda_1) \neq m.g(\lambda_1) \Rightarrow T$ non è diagonalizzabile

③ DISCUTERE AL VARIARE DI $k \in \mathbb{R}$ SE A È DIAGONALIZZABILE.

$$A = \begin{pmatrix} -9 & k & 3 \\ 0 & k & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+9 & -k & -3 \\ 0 & \lambda-k & 0 \\ -3 & 0 & \lambda+1 \end{vmatrix} = (\lambda-k)\begin{vmatrix} \lambda+9 & -3 \\ -3 & \lambda+1 \end{vmatrix} =$$

$$= (\lambda-k)[(\lambda+9)(\lambda+1) - 9] = (\lambda-k)(\lambda^2+\lambda+9\lambda+\cancel{9}-\cancel{9}) =$$

$$= (\lambda-k)(\lambda^2+10\lambda) = \lambda(\lambda-k)(\lambda+10)$$

$$|\lambda I - A| = 0 \Leftrightarrow \lambda(\lambda-k)(\lambda+10) = 0$$

$$\Leftrightarrow \lambda = -10 \lor \lambda = k \lor \lambda = 0 \qquad k \in \mathbb{R}$$

- Se $k \neq -10$ e $k \neq 0$ $\Rightarrow$ $\lambda_1 = -10$ $m.a(\lambda_1) = 1$ $\xRightarrow{TEO}$ $m.g(\lambda_1) = 1$
$\lambda_2 = 0$ $m.a(\lambda_2) = 1$ $m.g(\lambda_2) = 1$
$\lambda_3 = k$ $m.a(\lambda_3) = 1$ $m.g(\lambda_3) = 1$

$\Rightarrow$ A e' diagonalizzabile

- $\boxed{\text{Se } k = 0}$ $\Rightarrow$ $\lambda_1 = 0$ $m.a(\lambda_1) = 2$
$\lambda_2 = -10$ $m.a(\lambda_2) = 1$

$$\begin{pmatrix} 9 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} \cancel{9x - 3z = 0} \\ -3x + z = 0 \end{cases} \iff \begin{cases} z = 3x \end{cases}$$

$$V_{\lambda_1} = \{(x, y, 3x), x,y \in \mathbb{R}\} = [(1,0,3), (0,1,0)] \Rightarrow m.g(\lambda_1) = 2$$

$\lambda_2 = -10$, $k = 0$

$$\begin{pmatrix} -1 & 0 & -3 \\ 0 & -10 & 0 \\ -3 & 0 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -x - 3z = 0 \\ -10y = 0 \\ -3x - 9z = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ x = -3z \end{cases} \rightsquigarrow (-3z, 0, z) \qquad V_{\lambda_2} = [(-3, 0, 1)] \qquad m.g(\lambda_2) = 1$$

Per $k=0$ la matrice A e' diagonaliz.

- Se $k = \lambda_2$ cioe' se $k = -10$ $\Rightarrow$ $\lambda_1 = 0$ $m.a(\lambda_1) = 1$
$\lambda_2 = -10$ $m.a(\lambda_2) = 2$

Sostituisco $\lambda_1 = 0$ e $k = -10$ in $\lambda I - A$

$$\rightarrow \begin{pmatrix} 9 & 10 & -3 \\ 0 & 10 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 9x + \cancel{10y} - 3z = 0 \\ 10y = 0 \\ -3x + z = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ z = 3x \end{cases} \qquad V_{\lambda_1} = \{(x, 0, 3x) : x \in \mathbb{R}\} = [(1,0,3)]$$

$$m.g(\lambda_1) = 1$$

Ora sostituisco $\lambda_2 = -10$ e $k = -10$ in $\lambda I - A$

$$\begin{pmatrix} -1 & 10 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -x + 10y - 3z = 0 \\ -3x - 9z = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} -3x = 9z \\ -x + 10y - 3z = 0 \end{cases} \Longleftrightarrow \begin{cases} x = -3z \\ \cancel{3z} + 10y \cancel{-3z} = 0 \end{cases} \Longleftrightarrow \begin{cases} x = -3z \\ y = 0 \end{cases}$$

$$V_{\lambda_2} = \{ (-3z, 0, z) : z \in \mathbb{R} \} = [(-3, 0, 1)] \Longrightarrow m.g(\lambda_2) = 1$$

Poiché $\lambda_2 = 2$ ha $m.a(\lambda_2) = 2$ ma $m.g(\lambda_2) = 1$, A non è diagonalizzabile.

(1) $k \neq 0$, $k \neq -10 \Rightarrow \lambda_1 = 0$, $\lambda_2 = -10$, $\lambda_3 = k$ $k \in \mathbb{R}$
A è diag.

(2) $k = 0$, $\lambda_1 = 0$ $m.a(\lambda_1) = 2$ $m.g(\lambda_1) = 2$, $\lambda_2 = -10$ $m.a(\lambda_2) = 1$ $m.g(\lambda_2) = 1$
A è diag.

(3) $k = -10$, $\lambda_1 = 0$ $m.a(\lambda_1) = 1$ $m.g(\lambda_1) = 1$, $\lambda_2 = -10$ $\boxed{m.a(\lambda_2) = 2 \quad m.g(\lambda_2) = 1}$
A è NON è diag.

(4) Sia $\mathcal{B} = \{ (1,1,0), \underset{v_2}{(0,1,1)}, (0,0,1) \}$ una base di $\mathbb{R}^3$

a) Costruire a partire da $\mathcal{B}$ una base ortogonale $\mathcal{B}_1$ di $\mathbb{R}^3$

$$v_1' = v_1 = (1,1,0)$$

$$v_2' = v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' = (0,1,1) - \frac{\langle (0,1,1), (1,1,0) \rangle}{\langle (1,1,0), (1,1,0) \rangle} (1,1,0)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$v_3' = v_3 - \frac{\langle v_3, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' - \frac{\langle v_3, v_2' \rangle}{\langle v_2', v_2' \rangle} v_2' =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \cancel{\frac{\overbrace{\langle (0,0,1), (1,1,0) \rangle}^{=0}}{\langle (1,1,0), (1,1,0) \rangle} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} - \frac{\langle (0,0,1), (-\frac{1}{2}, \frac{1}{2}, 1) \rangle}{\langle (-\frac{1}{2}, \frac{1}{2}, 1), (-\frac{1}{2}, \frac{1}{2}, 1) \rangle} \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\frac{1}{4} + \frac{1}{4} + 1} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\frac{3}{2}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\mathcal{B}_1 = \left\{ (1,1,0), \left(-\frac{1}{2}, \frac{1}{2}, 1\right), \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \right.$$

b) COSTRUIRE A PARTIRE DA $\mathcal{B}_1$ UNA BASE ORTONORMALE $\mathcal{B}_2$ PER $\mathbb{R}^3$

$$v_1'' = \frac{v_1'}{|v_1'|} = \frac{(1,1,0)}{\sqrt{1^2+1^2+0^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$v_2'' = \frac{v_2'}{|v_2'|} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)}{\sqrt{\frac{6}{4}}} = \left(-\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$v_3'' = \frac{v_3'}{|v_3'|} = \frac{\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}}} = \frac{\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)}{\frac{1}{\sqrt{3}}} = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

$$\mathcal{B}_2 = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \right.$$

c) DATO $\vec{v} = (2, 1, 4)$, SCRIVERE LE COMPONENTI DI $\vec{v}$ RISPETTO A $\mathcal{B}_1$ E $\mathcal{B}_2$

$\mathcal{B}_1$ $\quad a_i = \dfrac{\langle v, v_i' \rangle}{\langle v_i', v_i' \rangle} \quad \forall\ i = 1, 2, 3$

$$a_1 = \frac{\langle (2,1,4), (1,1,0) \rangle}{\langle (1,1,0), (1,1,0) \rangle} = \frac{3}{2}$$

$$a_2 = \frac{\langle (2,1,4), \left(-\frac{1}{2}, \frac{1}{2}, 1\right) \rangle}{\langle \left(-\frac{1}{2}, \frac{1}{2}, 1\right), \left(-\frac{1}{2}, \frac{1}{2}, 1\right) \rangle} = \dots = \frac{7}{3}$$

$$a_3 = \frac{\langle (2,1,4), \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \rangle}{\langle \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \rangle} = 5$$

LE COORDINATE DI $\vec{v}$ RISPETTO A $\mathcal{B}_1$ SONO

$$\left(\frac{3}{2}, \frac{7}{3}, 5\right)$$

$$a_i = \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} \quad \forall\, i = 1, \dots 3$$

OSS $\langle v_1'', v_1'' \rangle =$

$$= \left\langle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \right\rangle$$

$$= \frac{1}{2} + \frac{1}{2} + 0 = 1$$

PER UNA BASE ORTONORMALE USO $a_i = \langle v, v_i \rangle$

$$a_i = \langle v, v_i'' \rangle$$

$$a_1 = \left\langle (2,1,4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \right\rangle = \frac{3}{\sqrt{2}}$$

$$a_2 = \left\langle (2,1,4), \left(-\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right) \right\rangle = \dots = \frac{7}{\sqrt{6}}$$

$$a_3 = \left\langle (2,1,4), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \right\rangle = \dots = \frac{5}{\sqrt{3}}$$

LE COORDINATE DI $\vec{v}$ RISPETTO ALLA BASE $\mathcal{B}_2$ SONO $\left(\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{6}}, \frac{5}{\sqrt{3}}\right)$

5) SIA $W = \left[(1,1,0,1), (1,-2,0,0), (1,0,-1,2) \right]$ SOTTOSPAZIO DI $\mathbb{R}^4$. TROVARE UNA BASE PER IL COMPLEMENTO ORTOGONALE $W^\perp$

RICORDO $W^\perp := \{ v \in V : \langle v, w \rangle = 0 \ \forall\, w \in W \}$

$$\langle (x,y,z,t), (1,1,0,1) \rangle = 0$$

$$\langle (x,y,z,t), (1,-2,0,0) \rangle = 0$$

$$\langle (x,y,z,t), (1,0,-1,2) \rangle = 0$$

$$\Rightarrow \begin{cases} x + y + t = 0 \\ x - 2y = 0 \\ x - z + 2t = 0 \end{cases} \iff \begin{cases} t = -3y \\ x = 2y \\ z = -4y \end{cases}$$

$$W^\perp = \{ (2y, y, -4y, -3y) : y \in \mathbb{R} \}$$