① $f: \mathbb{R}^3 \to \mathbb{R}^4$ 19/04/23

$$f(x,y,z) = (\underbrace{x-2y+z}, 3x+2y+11z, x+3z, 2x-y+5z)$$

a) f LINEARE

$$f(\alpha\vec{v} + \beta\vec{w}) = \alpha f(\vec{v}) + \beta f(\vec{w}) \quad \forall v, w \in \mathbb{R}^3 \quad \alpha,\beta \in \mathbb{R}$$

$\Rightarrow \vec{w}_1 = (x_1, y_1, z_1) \qquad \vec{w}_2 = (x_2, y_2, z_2) \quad \in \mathbb{R}^3$

Dimostro: $f(\alpha\vec{w}_1 + \beta\vec{w}_2) = \alpha f(\vec{w}_1) + \beta f(\vec{w}_2)$

$$f(\alpha\vec{w}_1 + \beta\vec{w}_2) = f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) =$$

$$= \Big(\underbrace{\alpha x_1 + \beta x_2 - 2\cdot(\alpha y_1 + \beta y_2) + \alpha z_1 + \beta z_2},$$
$$3(\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) + 11(\alpha z_1 + \beta z_2),$$
$$\alpha x_1 + \beta x_2 + 3\cdot(\alpha z_1 + \beta z_2),$$
$$2\cdot(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2) + 5(\alpha z_1 + \beta z_2)\Big) =$$

$$= \Big(\underline{\alpha x_1} + \underbrace{\beta x_2} \underline{- 2\alpha y_1} \underbrace{- 2\beta y_2} \underline{+ \alpha z_1} + \underbrace{\beta z_2},$$
$$3\alpha x_1 + 3\beta x_2 + 2\alpha y_1 + 2\beta y_2 + 11\alpha z_1 + 11\beta z_2,$$
$$\to \alpha x_1 + \beta x_2 + 3\alpha z_1 + 3\beta z_2,$$
$$2\alpha x_1 + 2\beta x_2 - \alpha y_1 - \beta y_2 + 5\alpha z_1 + 5\beta z_2\Big) =$$

$$= \Big(\alpha(x_1 - 2y_1 + z_1) + \beta(x_2 - 2y_2 + z_2),$$
$$\alpha(3x_1 + 2y_1 + 11z_1) + \beta(3x_2 + 2y_2 + 11z_2),$$
$$\alpha(x_1 + 3z_1) + \beta(x_2 + 3z_2),$$
$$\alpha(2x_1 - y_1 + 5z_1) + \beta(2x_2 - y_2 + 5z_2)\Big) =$$

$$= \alpha(x_1 - 2y_1 + z_1, 3x_1 + 2y_1 + 11z_1, x_1 + 3z_1, 2x_1 - y_1 + 5z_1)$$
$$+ \beta(x_2 - 2y_2 + z_2, 3x_2 + 2y_2 + 11z_2, x_2 + 3z_2, 2x_2 - y_2 + 5z_2) =$$
$$= \alpha f(\vec{w}_1) + \beta f(\vec{w}_2) \quad \Rightarrow f \text{ è LIN.}$$

② VERIFICARE SE $f(\underbrace{x,y,z}_{\in \mathbb{R}^3}) = (x+y+2z, x+2y+z, 2x+y+z)$ È LINEARE.

$\vec{v}_1 = (x_1, y_1, z_1)$ e $\vec{v}_2 = (x_2, y_2, z_2)$

DEVO VERIFICARE CHE $f(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha f(\vec{v}_1) + \beta f(\vec{v}_2)$

$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

$$f(\alpha\vec{v}_1 + \beta\vec{v}_2) = f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) =$$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + 2\alpha z_1 + 2\beta z_2,$$
$$\alpha x_1 + \beta x_2 + 2\alpha y_1 + 2\beta y_2 + \alpha z_1 + \beta z_2,$$
$$2\alpha x_1 + 2\beta x_2 + \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2) =$$

$$= (\alpha(x_1 + y_1 + 2z_1) + \beta(x_2 + y_2 + 2z_2),$$
$$\alpha(x_1 + 2y_1 + z_1) + \beta(x_2 + 2y_2 + z_2),$$
$$\alpha(2x_1 + y_1 + z_1) + \beta(2x_2 + y_2 + z_2)) =$$

$$= \alpha f(\vec{v}_1) + \beta f(\vec{v}_2)$$

RIPRENDO ES. 1

b) SCRIVERE LA MATRICE A ASSOCIATA A f RISPETTO ALLE BASI CANONICHE

$$A \overset{\text{def}}{=} [f(\vec{e}_1), f(\vec{e}_2), f(\vec{e}_3)] \overset{\text{NOTAZIONE}}{=} [f(\vec{i}), f(\vec{j}), f(\vec{k})]$$

$$f(\vec{e}_1) = f(\vec{i}) = f((1,0,0)) = (1,3,1,2)$$

$$f(\vec{e}_2) = f(\vec{j}) = f((0,1,0)) = (-2,2,0,-1)$$

$$f(\vec{e}_3) = f(\vec{k}) = f((0,0,1)) = (1,11,3,5)$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{bmatrix} \Rightarrow f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\vec{x} = \begin{pmatrix} x - 2y + z \\ 3x + 2y + 11z \\ x + 3z \\ 2x - y + 5z \end{pmatrix}$$

c) TROVARE UNA BASE PER $\operatorname{Imm}(f)$ E LA DIMENSIONE DEL $\operatorname{Ker}(f)$.

$$\operatorname{Imm}(f) = \{ f(v) : v \in V \} \subseteq W \qquad f: V \to W$$

$$\operatorname{Ker}(f) = \{ v \in V : f(v) = \vec{0}_W \} \subseteq V$$

$$\operatorname{Imm}(f) = \left\{ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x,y,z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x - 2y + z \\ 3x + 2y + 11z \\ x + 3z \\ 2x - y + 5z \end{pmatrix} : x,y,z \in \mathbb{R} \right\} =$$

$2x - y + 5z$,

$$= \left[\underbrace{\begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}}_{v_1} x + \underbrace{\begin{pmatrix} -2 \\ 2 \\ 0 \\ -1 \end{pmatrix}}_{v_2} y + \underbrace{\begin{pmatrix} 1 \\ 11 \\ 3 \\ 5 \end{pmatrix}}_{v_3} z \right]$$

$\vec{v}_3 = 3\vec{v}_1 + \vec{v}_2$

LIN. DIP

$\Rightarrow \mathcal{B} = \{(1,3,1,2), (-2,2,0,-1)\} \Rightarrow \dim(\mathrm{Imm}(f)) = 2$

TEO $\quad \dim(\mathrm{Imm}(f)) + \dim(\mathrm{Ker}(f)) = \dim V \qquad f: V \to W$

$\Rightarrow 2 + \dim(\mathrm{Ker}(f)) = 3 \Rightarrow \dim(\mathrm{Ker}(f)) = 1$

d) DISCUTERE APPARTENENZA DEL VETTORE $(\alpha+1, \alpha, \alpha, \alpha)$ all'$\mathrm{Imm}(f)$ AL VARIARE DI $\alpha \in \mathbb{R}$

$$A = \begin{pmatrix} \alpha+1 & \alpha & \alpha & \alpha \\ 1 & 3 & 1 & 2 \\ -2 & 2 & 0 & -1 \end{pmatrix} \Rightarrow 2 \le rg(A) \le 3$$

$$\begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 3 & 1 \\ -2 & 2 & 0 \end{vmatrix} = \dots = 2(2\alpha - 1) = 0 \Leftrightarrow \boxed{\alpha = \frac{1}{2}}$$

$$\begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 3 & 2 \\ -2 & 2 & -1 \end{vmatrix} = \dots = -2\alpha - 7 = 0 \Leftrightarrow \boxed{\alpha = -\frac{7}{2}}$$

- Se $\alpha = -\frac{7}{2} \Rightarrow$ il 1° minore $3\times3 \neq 0 \Rightarrow rg(A) = 3$
- Se $\alpha = \frac{1}{2} \Rightarrow rg(A) = 3$
- Se $\alpha \neq \frac{1}{2} \wedge \alpha \neq -\frac{7}{2} \Rightarrow rg(A) = 3$

$\Rightarrow (\alpha+1, \alpha, \alpha, \alpha) \notin \mathrm{Imm}(f) \quad \forall \alpha \in \mathbb{R}$