③ Dati i seguenti sottoinsiemi $V = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{2x = y}_{2x - y = 0}\}$

$W = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$ si chiede di

a) Verificare che sono sottospazi di $\mathbb{R}^3$ e per ciascuno di essi determinare una base e la dimensione;

b) determinare $V + W$

c) stabilire se è somma diretta $V + W$.

a) Uso la I forma.

Ⓥ. $\vec{0} \in ? V$ sì perché $2 \cdot 0 = 0$ ✓

- $c \cdot \vec{v} \in ? V$ con $c \in \mathbb{R}$ e $\vec{v} \in V$

$c\vec{v} = c(x_1, y_1, z_1) = (cx_1, cy_1, cz_1)$

$2 \cdot cx_1 - cy_1 = 0 ? \Rightarrow c(2x_1 - y_1) = 0$ ✓

- $\vec{v}_1 + \vec{v}_2 \in ? V$ ove $\vec{v}_1, \vec{v}_2 \in V$

So che $2x_1 = y_1$ e $2x_2 = y_2 \Leftrightarrow 2x_1 - y_1 = 0$ e $2x_2 - y_2 = 0$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\Rightarrow 2(x_1 + x_2) - (y_1 + y_2) = ? 0$

$\Leftrightarrow 2x_1 + 2x_2 - y_1 - y_2 = 0$ ✓

Ⓦ. $\vec{0} \in ? W$ $0 + 0 + 0 = 0$ ✓

- $\vec{v}_1 + \vec{v}_2 \in ? W$

$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = \overbrace{(x_1 + y_1 + z_1)}^{=0} + \overbrace{(x_2 + y_2 + z_2)}^{=0} = 0$ ✓

- $c \cdot \vec{v}_1 \in ? W$

$cx_1 + cy_1 + cz_1 = c\overbrace{(x_1 + y_1 + z_1)}^{=0} = 0$ ✓

Base Ⓥ Basta trovare dei vettori di V che siano linearmente indip.

Ad esempio, $(1,2,0) \rightarrow 2 \cdot 1 - 2 \cdot 2 = 0$ ✓

$(0,0,1)$

Però notiamo che $2x = y \Rightarrow$ i vettori di V sono del tipo $(x, 2x, z)$ cioè non avrò una base di 3 vettori.

$\Rightarrow \mathcal{B} = \{(1,2,0), (0,0,1)\} \Rightarrow \dim V = 2$

Base (W) Da $x+y+z=0 \Leftrightarrow x=-y-z \Rightarrow (-y-z, y, z)$

$$\Rightarrow \mathcal{B} = \{ \underbrace{(-1,0,1)}_{\text{ho sostituito } y=0}, \underbrace{(-1,1,0)}_{\text{sostituito } z=0} \} \Rightarrow \dim W = 2$$

b) $V+W = [(1,2,0),(0,0,1),(-1,0,1),(-1,1,0)]$

$(0,0,0) = \alpha(1,2,0) + \beta(0,0,1) + \gamma(-1,0,1) + \delta(-1,1,0)$

$$\begin{cases} 0 = \alpha - \gamma - \delta \\ 0 = 2\alpha + \delta \\ 0 = \beta + \gamma \end{cases} \Rightarrow \begin{cases} \alpha = \gamma + \delta \\ \delta = -2\alpha \\ \gamma = -\beta \end{cases} \Rightarrow \begin{cases} \alpha = -\beta - 2\alpha \\ \delta = -2\alpha \\ \gamma = -\beta \end{cases} \Rightarrow \begin{cases} \alpha = -\frac{\beta}{3} \\ \delta = -2\alpha \\ \gamma = -\beta \end{cases}$$

$\Rightarrow \left(-\frac{\beta}{3}, \beta, -\beta, \frac{2}{3}\beta\right) \Rightarrow \beta\left(-\frac{1}{3}, 1, -1, \frac{2}{3}\right) \Rightarrow$ lin. dip.

Infatti posso togliere dai vettori generatori scritti inizialmente $(1,2,0)$ perché $-\frac{1}{3}(1,2,0) + 1(0,0,1) + (-1)(-1,0,1) + \frac{2}{3}(-1,1,0) = \vec{0}$

$(1,2,0) = 3(0,0,1) - 3(-1,0,1) + 2(-1,1,0)$ ✓

$\Rightarrow V+W = [(0,0,1), (-1,0,1), (-1,1,0)]$

$(0,0,0) = \alpha(0,0,1) + \beta(-1,0,1) + \gamma(-1,1,0)$

$$\begin{cases} 0 = -\beta - \gamma \\ 0 = \gamma \\ 0 = \alpha + \beta \end{cases} \Rightarrow \begin{cases} \gamma = 0 \\ \beta = 0 \\ \alpha = 0 \end{cases} \Rightarrow \text{sono lin. indip.} \Rightarrow \dim(V+W) = 3$$

Poiché $\dim(V+W) = 3$ e $V+W \subseteq \mathbb{R}^3 \Rightarrow V+W = \mathbb{R}^3$

c) Uso la relazione di Grassmann:

$$\dim(V+W) = \dim V + \dim W - \dim(V \cap W)$$

$$3 = 2 + 2 - \dim(V \cap W) \Rightarrow \dim(V \cap W) = 1$$

$\Rightarrow$ Non è somma diretta.

4) Sia $W = \{ (\overset{\frac{1}{5}x_1 + x_2 - \frac{1}{2}x_3}{x_2 - \frac{1}{2}x_3 + \frac{1}{5}x_1}, \overset{0}{0}, \overset{x_1 + 2x_2 + \frac{1}{2}x_3}{\frac{1}{2}x_3 + 1x_1 + 2x_2}, \overset{\frac{1}{2}x_1 + x_2 + \frac{1}{4}x_3}{\frac{1}{2}x_1 + \frac{1}{4}x_3 + x_2}) : x_1, x_2, x_3 \in \mathbb{R} \} \subseteq \mathbb{R}^4$

a) trovare la dimensione e una base di W.

$$W = \left[\left(\frac{1}{5}, 0, 1, \frac{1}{2}\right), (1,0,2,1), \left(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}\right) \right]$$

Verifico se sono linearmente indip.:

$$(0,0,0,0)=\alpha\left(\frac{1}{5},0,1,\frac{1}{2}\right)+\beta(1,0,2,1)+\gamma\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)$$

$$\begin{cases} 0=\frac{1}{5}\alpha+\beta-\frac{\gamma}{2} \\ \cancel{0=0} \checkmark \\ 0=\alpha+2\beta+\frac{\gamma}{2} \\ \cancel{0=\frac{\alpha}{2}+\beta+\frac{\gamma}{4}} \end{cases} \Rightarrow \begin{cases} \beta=-\frac{\alpha}{5}+\frac{\gamma}{2} \\ \frac{\gamma}{2}=-\alpha-2\beta \end{cases} \Rightarrow \begin{cases} \beta=-\frac{\alpha}{5}-\alpha-2\beta \\ \gamma=-2\alpha-4\beta \end{cases}$$

$$\Rightarrow \begin{cases} 3\beta=\frac{-\alpha-5\alpha}{5} \\ \gamma=-2\alpha-4\beta \end{cases} \Rightarrow \begin{cases} \beta=-\frac{6}{15}\alpha=-\frac{2}{5}\alpha \\ \gamma=-2\alpha+4\cdot\frac{2}{5}\alpha=-\frac{2}{5}\alpha \end{cases}$$

$\alpha\left(1,-\frac{2}{5},-\frac{2}{5}\right)$

$\Rightarrow \left(\alpha,-\frac{2}{5}\alpha,-\frac{2}{5}\alpha\right)$ non sono lin indip

$$\left(\frac{1}{5},0,1,\frac{1}{2}\right)=\frac{2}{5}(1,0,2,1)+\frac{2}{5}\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)$$

Verifico che $(1,0,2,1)$ e $\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)$ sono lin. indip.

$$(0,0,0,0)=\alpha(1,0,2,1)+\beta\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)$$

$$\begin{cases} 0=\alpha-\frac{1}{2}\beta \\ 0=2\alpha+\frac{1}{2}\beta \\ 0=\alpha+\frac{1}{4}\beta \end{cases} \Rightarrow \begin{cases} \alpha=\frac{1}{2}\beta \\ 0=2\frac{1}{2}\cancel{\beta}\beta+\frac{1}{2}\beta \\ \alpha=-\frac{1}{4}\beta \end{cases} \Rightarrow \begin{cases} \alpha=0 \\ \beta=0 \end{cases} \Rightarrow \text{sono lin. indip.}$$

$\Rightarrow \dim W = 2$ poiché $\mathcal{B}_W=\left\{(1,0,2,1),\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)\right\}$

b) Dato $V=\{(x_2,x_1,2x_2,x_1+x_2) : x_1,x_2\in\mathbb{R}\}$ trovare $V+W$.

$V=[(0,1,0,1),(1,0,2,1)]$

$(x_2,0,2x_2,x_2)+(0,x_1,0,x_1)$

$x_2(1,0,2,1)+x_1(0,1,0,1)$

$$\begin{cases} 0=\beta \\ 0=\alpha \end{cases} \checkmark \Rightarrow \text{sono lin. indip} \Rightarrow \mathcal{B}_V=\{(0,1,0,1),(1,0,2,1)\}$$

$$\Rightarrow V+W=\left[(0,1,0,1),\underbrace{(1,0,2,1)}_{\text{compare 2 volte}},\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)\right]$$

$$\begin{cases} 0=\beta-\frac{\gamma}{2} \\ 0=\alpha \\ 0=2\beta+\frac{\gamma}{2} \\ 0=\cancel{\alpha}+\beta+\frac{1}{4}\gamma \end{cases} \Rightarrow \begin{cases} \beta=\frac{\gamma}{2} \\ \alpha=0 \\ \beta=-\frac{\gamma}{4} \end{cases} \Rightarrow \begin{cases} -\frac{\gamma}{4}-\frac{\gamma}{2}=0 \\ \alpha=0 \\ \beta=-\frac{\gamma}{4} \end{cases} \Rightarrow \begin{cases} \gamma=0 \\ \alpha=0 \\ \beta=0 \end{cases} \Rightarrow \text{LIN INDIP} \Downarrow \text{BASE}$$

$\Rightarrow \dim(V+W)=3$

c) E' $V+W$ somma diretta?

$\dim(V+W)=\dim V+\dim W-\dim(V\cap W) \Rightarrow 3=2+2-\underbrace{\dim(V\cap W)}_{\neq 0} \Rightarrow$ NO

5) Calcolare il determinante di $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 3 \\ 2 & 2 & \frac{1}{2} \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$

$|A| = \det(A) = \begin{vmatrix} 3 & 3 \\ 2 & 1/2 \end{vmatrix} + 2 \begin{vmatrix} -1 & -1 \\ 3 & 3 \end{vmatrix} =$

$= \left(\frac{3}{2} - 6\right) + 2\left(\cancel{-3+3}\right) = \frac{3-12}{2} = -\frac{9}{2}$

$|B| = \det(B) = -2 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -2(8-3) - 2(1-6) = -10 + 10 = 0$

RICORDO **REGOLA DI LAPLACE** (TEOREMA)

Sviluppo di Laplace per righe: $\det A = \sum_{j=1}^{n} \left[a_{ij} \cdot (-1)^{i+j} \cdot \det(A_{ij}) \right]$

Sviluppo di Laplace per colonne: $\det A = \sum_{i=1}^{n} \left[a_{ij} (-1)^{i+j} \det(A_{ij}) \right]$