① SIA DATA LA MATRICE $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ DET. SE POSSIBILE UNA MATRICE ORTOGONALE U CHE DIAGONALIZZA A.

$A^T = A^{-1}$ $\qquad AA^T = I$

TEOREMA SPETTALE

A è simmetrica $\Rightarrow \exists\, U$ ort. : $U^T A \underset{\text{autovettori}}{U} = D \rightarrow$ autovalori

$\Longleftrightarrow A = UDU^T$

Infatti: $\underbrace{U \cdot U^T}_{I} A U = U \cdot D$

$AU = UD$

$A \underbrace{U U^T}_{I} = UDU^T$

$A = UDU^T$

Poiché A è sim. $\underset{\text{TEO. SPETTRALE}}{\Longrightarrow} \exists\, U$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = (\lambda-1)\left[(\lambda-1)^2 - 1\right] + (-\lambda + \cancel{1} - \cancel{1}) - (\cancel{1} + \lambda - \cancel{1}) =$$

$$\overset{!}{=} (\lambda-1)(\lambda^2 + \cancel{1} - 2\lambda - \cancel{1}) - 2\lambda =$$

$$= \lambda^3 - 2\lambda^2 - \lambda^2 + \cancel{2\lambda} - \cancel{2\lambda} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$$

$|\lambda I - A| = 0 \iff \lambda^2(\lambda-3) = 0$

$\lambda_1 = 0 \qquad m.a(\lambda_1) = 2$

$\lambda_2 = 3 \qquad m.a(\lambda_2) = 1 \quad \overset{\text{TEO}}{\Longrightarrow} \quad m.g(\lambda_2) = 1$

$\boxed{\lambda_1 = 0}$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -x - y - z = 0 \\ \cancel{-x - y - z = 0} \\ \cancel{-x - y - z = 0} \end{cases}$$

$\iff x = -y - z$

$V_{\lambda_1} = \{(-y-z, y, z) : y, z \in \mathbb{R}\} = \{(-y, y, 0) + (-z, 0, z) : y, z \in \mathbb{R}\}$

$\overset{!}{=} [\underset{v_1}{(-1, 1, 0)}, \underset{v_2}{(-1, 0, 1)}] \Rightarrow mg(\lambda_1) = 2$

$\boxed{\lambda_2 = 3}$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 2z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} z = -x + 2y \\ 2x - y - z = 0 \\ -x - y - 2x + 4y = 0 \end{cases} \underset{\dots}{\Leftrightarrow} \begin{cases} x = y \\ z = y \end{cases}$$

$$V_{\lambda_2} = [\underbrace{(1,1,1)}_{v_3}] \Rightarrow mg(\lambda_2) = 1$$

$\langle v_1, v_2 \rangle \neq 0 \Rightarrow$ USO GRAHM - SCHMIDT POICHE' NON SONO $\perp$

$$v_1' = (-1, 1, 0)$$

$$v_2' = v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' =$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\langle (-1,0,1), (-1,1,0) \rangle}{\langle (-1,1,0), (-1,1,0) \rangle} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$v_3' = v_3 - \cancel{\frac{\langle v_3, v_2' \rangle}{\langle v_2', v_2' \rangle}} v_2' - \cancel{\frac{\langle v_3, v_1' \rangle}{\langle v_1', v_1' \rangle}} v_1' = (1,1,1)$$

$$U = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \text{ORTOGONALE}$$

$$\langle v_3, v_2' \rangle = \langle (1,1,1), \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right) \rangle =$$

$$= \underbrace{-\frac{1}{2} + \left(-\frac{1}{2}\right)} + 1 = -1 + 1 = 0$$

$$\langle v_3, v_1' \rangle = \langle (1,1,1), (-1,1,0) \rangle = -1 + 1 = 0$$

b) U ORTONORMALE

$$v_1'' = \frac{v_1'}{|v_1'|} = \frac{(-1, -1, 0)}{\sqrt{(-1)^2 + 1^2}} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$v_2'' = \frac{v_2'}{|v_2'|} = \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$v_3'' = \frac{v_3'}{|v_3'|} = \frac{(1,1,1)}{\sqrt{1^2 + 1^2 + 1^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

2 DATA LA FORMA QUADRATICA $q(x,y,z) = x^2 + 3y^2 - 4xz + 4z^2$

a) SCRIVERE LA MATRICE A CHE RAPPRESENTA q

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$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

b) STABILIRE IL SEGNO DI q

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 2 & 0 & \lambda-4 \end{vmatrix} = (\lambda-3)[(\lambda-1)(\lambda-4)-4] =$$

$$= (\lambda-3)(\lambda^2-4\lambda-\lambda+\cancel{4}-\cancel{4}) = (\lambda-3)(\lambda^2-5\lambda) =$$

$$= (\lambda-3)\lambda(\lambda-5)$$

$\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 5$ $\Longrightarrow$ TUTTI I λ_i PER $i=1,2,3$ SONO NON NEGATIVI

$\Rightarrow$ q È SEMIDEFINITA POSITIVA

c) DETERMINARE UNA BASE CHE DIAGONALIZZA LA FORMA QUADRATICA

- $\boxed{\lambda_1 = 3}$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} 2x + 2z = 0 \\ 2x - z = 0 \end{cases} \quad (2x+4z=0) \longrightarrow \begin{cases} 6z = 0 \\ z = 2x \end{cases}$$

$$\Rightarrow \begin{cases} z = 0 \\ x = 0 \end{cases} \Rightarrow V_{\lambda_1} = [(0,1,0)^T] = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \rightarrow \text{GIÀ NORMALIZZATO}$$

- $\boxed{\lambda_2 = 0}$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} -x + 2z = 0 \\ -3y = 0 \\ 2x - 4z = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x = +2z \\ y = 0 \end{cases} \Rightarrow V_{\lambda_2} = [\underbrace{(+2,0,1)}_{v_2}]$$

- $\boxed{\lambda_3 = 5}$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} 4x + 2z = 0 \\ 2y = 0 \\ 2x + z = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y = 0 \\ z = -2x \end{cases} \Rightarrow V_{\lambda_3} = [\underbrace{(1,0,-2)}_{v_3}]$$

$$\Rightarrow \tilde{v}_2 = \frac{v_2}{|v_2|} = \frac{(2,0,1)}{\sqrt{4+1}} = \left(\frac{2}{\sqrt5}, 0, \frac{1}{\sqrt5}\right)$$

$$\tilde{v}_3 = \frac{v_3}{|v_3|} = \frac{(1,0,-2)}{\sqrt{4+1}} = \left(\frac{1}{\sqrt5}, 0, -\frac{2}{\sqrt5}\right)$$

$$\tilde{v}_1 = v_1$$

$B_1 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ base

3) SIA r LA RETTA DEL PIANO $z=0$ DI EQ. $3x+6y-1=0$

a) SCRIVERE LE EQ. PARAMETRICHE E CARTESIANE DELLA RETTA s ORTOGONALE A r E PASSANTE PER $P=(1,0)$

$$r: \begin{cases} 3x+6y-1=0 \\ z=0 \end{cases} \Rightarrow A = \begin{pmatrix} 3 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \longrightarrow$$ ci serve x i parametri direttori della retta

$$v_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad v_2 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \qquad v_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$v_1 = \begin{vmatrix} 6 & 0 \\ 0 & 1 \end{vmatrix} = 6 \qquad v_2 = -\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = -3 \qquad v_3 = \begin{vmatrix} 3 & 6 \\ 0 & 0 \end{vmatrix} = 0$$

$$s \perp r \iff u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$$
$$\iff 6u_1 - 3u_2 = 0 \iff u_2 = 2u_1$$

$P(1,0)$

$$\frac{x-x_0}{u_1} = \frac{y-y_0}{u_2} \iff \frac{x-1}{\cancel{u_1}} = \frac{y-0}{2\cancel{u_1}} \iff x-1 = \frac{y}{2}$$

$$\iff \boxed{x - \frac{y}{2} - 1 = 0}$$ eq. di s

EQ. PARAM: PONGO $y=t$

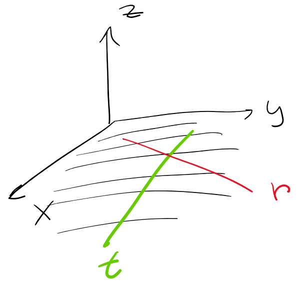
$$\Rightarrow \begin{cases} x - \frac{t}{2} - 1 = 0 \\ y = t \end{cases} \iff \begin{cases} x-1 = \frac{t}{2} \\ y = t \end{cases} \iff$$

$$\iff \begin{cases} x = \frac{t}{2} + 1 \\ y = t \end{cases} \quad t \in \mathbb{R}$$

b) $Q = \left(-\frac{1}{2}, -3\right) \in s$?

$$-\frac{1}{2} - \frac{(-3)}{2} - 1 = 0 \,? \quad \text{sì} \Rightarrow Q \in s$$

c) sia t la retta nel piano $z=0$ passante per i punti $R=(0,1)$ e $S=(3,5)$. $t \parallel s$?



$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \iff \frac{x-0}{3-0} = \frac{y-1}{5-1}$$

$$\iff \frac{x}{3} = \frac{y-1}{4} \iff y = \frac{4}{3}x + 1$$

$$t: \begin{cases} y = \frac{4}{3}x + 1 \\ z = 0 \end{cases} \iff \frac{4}{3}x - y + 1 = 0$$

$$\Rightarrow A = \begin{pmatrix} \frac{4}{3} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$v_1 = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \qquad v_2 = -\begin{vmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{vmatrix} = -\frac{4}{3} \qquad v_3 = \begin{vmatrix} \frac{4}{3} & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$v_1 = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \qquad v_2 = \begin{vmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{vmatrix} = \frac{4}{3} \qquad v_3 = \begin{vmatrix} \frac{4}{3} & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$u_2 = 2u_1$ DALLA RETTA s $\qquad u_1 = -1 \Rightarrow u_2 = -2$

$\Rightarrow$ PONGO $u_1 = 1 \Rightarrow u_2 = 2$

$$\boxed{t \parallel s \Leftrightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2}} \qquad \frac{u_1}{v_1} = \frac{1}{-1} = -1 \neq \frac{u_2}{v_2} = \frac{2}{-\frac{4}{3}}$$

$\Rightarrow t \not\parallel s$

d) IN CASO ESISTANO DETERMINARE I PUNTI DI INTERSEZIONE DELLE DUE RETTE

$$\begin{cases} x - \frac{y}{2} - 1 = 0 \\ \frac{4}{3}x - y + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} -\frac{y}{2} = -x + 1 \\ 4x - 3y + 3 = \end{cases} \Leftrightarrow \begin{cases} y = 2x - 2 \\ 4x - 6x + 6 + 3 = 0 \\ -2x = -9 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{9}{2} \\ y = \cancel{2} \cdot \frac{9}{\cancel{2}} - 2 = 7 \end{cases} \Rightarrow P\left(\frac{9}{2}, 7\right)$$ E' IL PUNTO DI INTERSEZIONE

SIA $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\} \subseteq \mathbb{R}^3$

a) SI DETERMINI UNA BASE DI U CHE SIA ORTONORMALE

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = -2x_2 - 3x_3\} =$$
$$= \{(-2x_2 - 3x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} =$$
$$= [\underbrace{(-2, 1, 0)}_{v_1}, \underbrace{(-3, 0, 1)}_{v_2}]$$

$v_1 \perp v_2$? $\langle v_1, v_2 \rangle = +6 \neq 0 \Rightarrow v_1 \not\perp v_2$

G.S. $\Rightarrow$ $v_1' = v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$v_2' = v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' =$$
$$= \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{\langle (-3,0,1), (-2,1,0) \rangle}{\langle (-2,1,0), (-2,1,0) \rangle} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} =$$
$$= \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ -\frac{6}{5} \\ 1 \end{pmatrix}$$

$$\Rightarrow v_1' = \frac{v_1}{|v_1|} = \frac{(-2,1,0)}{\sqrt{4+1}} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$v_2' = \frac{v_2}{|v_2|} = \frac{\left(-\frac{3}{5}, -\frac{6}{5}, 1\right)}{\sqrt{\frac{9}{25} + \frac{36}{25} + 1}} = \left(-\frac{3}{\sqrt{70}}, -\frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right)$$

$$\frac{\quad}{\sqrt{\frac{9}{25}+\frac{36}{25}+1}} \quad \left(\frac{\quad}{\sqrt{70}}, \frac{\quad}{\sqrt{70}}, \frac{\quad}{\sqrt{70}}\right)$$

$$\mathcal{B} = \{v_1', v_2'\}$$

b) COMPLETARE $\mathcal{B}$ AD UNA BASE ORTONORMALE $\mathcal{B}'$ DI $\mathbb{R}^3$

$$\begin{cases} \left\langle (x,y,z), \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \right\rangle = 0 \\ \left\langle (x,y,z), \left(-\frac{3}{\sqrt{70}}, -\frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right) \right\rangle = 0 \end{cases}$$

$$\begin{cases} -\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = 0 \\ -\frac{3}{\sqrt{70}}x - \frac{6}{\sqrt{70}}y + \frac{5}{\sqrt{70}}z = 0 \end{cases} \iff \begin{cases} -2x + y = 0 \\ -3x - 6y + 5z = 0 \end{cases}$$

$$\begin{cases} y = 2x \\ -3x - 12x + 5z = 0 \end{cases} \iff \begin{aligned} 5z &= +15x \\ z &= 3x \end{aligned}$$

$$\begin{cases} y = 2x \\ z = 3x \end{cases} \Rightarrow v_3 = (1, 2, 3) \text{ vettore ortogonale a } v_1' \text{ e } v_2'$$

$$\Rightarrow v_3' = \frac{v_3}{|v_3|} = \frac{(1,2,3)}{\sqrt{1+4+9}} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \qquad \mathcal{B}' = \{v_1', v_2', v_3'\}$$