① SENZA USARE LA DEFINIZIONE DI LINEARE INDIP., STABILIRE SE I SEGUENTI SOTTOINSIEMI DI $\mathbb{R}^3$ SONO LIN. INDIP. OPPURE DIP.

- $S_1 = \{(1,2,3), (-1,1,0), (0,1,-1)\}$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 5 = -6 \neq 0 \Rightarrow \text{LIN. INDIP.}$$

- $S_2 = \{(1,2,1), (1,0,1)(2,2,2)\}$

$$(2,2,2) = (1,2,1) + (1,0,1) \Rightarrow \text{LIN. DIP.}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2 + 2 = 0$$

② SIA A LA MATRICE REALE $A = \begin{pmatrix} k & k-1 & k \\ 0 & 2k-2 & 0 \\ 1 & k-1 & 2-k \end{pmatrix}$ $k \in \mathbb{R}$

a) DETERMINARE PER QUALI k LA MATRICE E' INVERTIBILE

$$(AA^{-1} = I)$$

TEOREMA INVERTIBILITA' A INVERTIBILE $\Leftrightarrow \det(A) \neq 0$

$$\det(A) = (2k-2) \begin{vmatrix} k & k \\ 1 & 2-k \end{vmatrix} = \dots = -2k(1-k)^2$$

$$-2k(1-k)^2 \neq 0 \Leftrightarrow \boxed{k \neq 0 \wedge k \neq 1} \Rightarrow A \text{ E' INVERTIBILE}$$

b) CALCOLARE L'INVERSA DI A PER $k = -1$

$$A = \begin{pmatrix} -1 & -2 & -1 \\ 0 & -4 & 0 \\ 1 & -2 & 3 \end{pmatrix}$$

- $\det(A)$
- $\text{cof}(A)$

$$A_{ij} = \begin{pmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ \dots & \dots & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot (\text{cof}(A))^T = \frac{1}{\det(A)} \text{Adj}(A)$$

$$\rightarrow \boxed{\text{cof}(A)_{ij}} = (-1)^{i+j} \cdot \det(A_{ij})$$

$\det(A) = 8 \neq 0$

$a_{11} = (-1)^{1+1} \begin{vmatrix} -4 & 0 \\ -2 & 3 \end{vmatrix} = -12$ $\qquad a_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} = 0$

$a_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -4 \\ 1 & -2 \end{vmatrix} = 4$

$a_{21} = (-1)^{2+1} \begin{vmatrix} -2 & -1 \\ -2 & 3 \end{vmatrix} = 8$ $\qquad a_{22} = \begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} = -3+1 = -2$

$a_{23} = -\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} = -4$

$a_{31} = \begin{vmatrix} -2 & -1 \\ -4 & 0 \end{vmatrix} = -4$ $\qquad a_{32} = -\begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix} = 0$

$a_{33} = \begin{vmatrix} -1 & -2 \\ 0 & -4 \end{vmatrix} = 4$

$\Rightarrow \mathrm{Cof}(A) = \begin{pmatrix} -12 & 0 & 4 \\ 8 & -2 & -4 \\ -4 & 0 & 4 \end{pmatrix} \Rightarrow \mathrm{Adj}(A) = (\mathrm{Cof}(A))^T = \begin{pmatrix} -12 & 8 & -4 \\ 0 & -2 & 0 \\ 4 & -4 & 4 \end{pmatrix}$

$A^{-1} = \frac{1}{8}\begin{pmatrix} -12 & 8 & -4 \\ 0 & -2 & 0 \\ 4 & -4 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\Rightarrow$ È L'INVERSA

③ CALCOLARE IL RANGO DI

$$A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 4 \\ 6 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad 5\times 4$$

$\neq 0$ $\quad 2\times 2$

$$\begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 6 & 3 & 0 \end{vmatrix} = 0$$

DEF
- IL RANGO DI UNA MATRICE $A \in \mathbb{R}^{m \times n}$ È PARI AL MASSIMO ORDINE DI UN MINORE NON NULLO.
- IL RANGO ... È LA DIMENSIONE SOTTOSPAZIO DELLE RIGHE/COLONNE DI A.

NOTAZIONE $\quad r(A) = rg(A) = rk(A)$

$r(A) \leq \min\{\#\text{righe}, \#\text{colonne}\}$

Sia $A \in \mathbb{R}^{m \times n} \Rightarrow r(A) \leq \min\{m, n\}$ $\qquad m = 5, \; n = 4$

$\Rightarrow r(A) \leq \min\{5, 4\} \iff r(A) \leq 4$

2x2
$\Rightarrow \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow r(A) \geq 2$

3x3
$\Rightarrow \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = -4 - (-2-2) = 0$

$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & 4 \end{vmatrix} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$

$\begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} +2 & 1 \\ -1 & 0 \end{vmatrix} = +1 \neq 0 \Rightarrow r(A) \geq 3$

$\Rightarrow \begin{vmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \dots = 0 \Rightarrow r(A) \neq 4$

$\Rightarrow r(A) = 3$

4 CALCOLARE AL VARIARE DI $\lambda \in \mathbb{R}$ IL RANGO DI:

i) $A = \begin{pmatrix} \lambda & 1 & 3 \\ 1 & 0 & -1 \\ 2 & \lambda & 2 \end{pmatrix}$ $\qquad r(A) \leq \min\{3, 3\} = 3$

2x2
$\Rightarrow \begin{vmatrix} \lambda & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow r(A) \geq 2$

3x3
$\Rightarrow \begin{vmatrix} \lambda & 1 & 3 \\ 1 & 0 & -1 \\ 2 & \lambda & 2 \end{vmatrix} = \lambda\begin{vmatrix} 0 & -1 \\ \lambda & 2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ \lambda & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} =$

$= \lambda^2 - 2 + 3\lambda - 2 = \lambda^2 + 3\lambda - 4 = (\lambda - 1)(\lambda + 4)$

$r(A) = 3 \iff \det(A) \neq 0 \iff (\lambda-1)(\lambda+4) \neq 0 \iff \lambda \neq 1 \wedge \lambda \neq -4$

- Se $\lambda = 1 \Rightarrow rg(A) = 2$ $\qquad A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} \rightarrow \det = 0$
- Se $\lambda = -4 \Rightarrow rg(A) = 2$

ii) $B = \begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -2 \\ 1 & 3 & -1 & \lambda \end{pmatrix}_{3\times 4}$ $\neq 0$

$r(B) \le \min\{3,4\} = 3$
$\Rightarrow r(B) \le 3$

$2\times2 \Rightarrow \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1 \neq 0 \Rightarrow r(B) \ge 2$

$3\times3 \Rightarrow \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix} = \underset{-1-6}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} + \underset{4+3}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = -7+7 = 0$

$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & \lambda \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = \lambda + 6 - 4 - 3 = \lambda - 1$

$r(B) = 3 \Leftrightarrow \lambda - 1 \neq 0 \Leftrightarrow \lambda \neq 1$

- se $\lambda = 1 \Rightarrow r(B) = 2$

(5) SIANO $\vec{v}_1 = (0, 1, -2, \beta)$ $\vec{v}_2 = (\beta+1, -1, 0, 1)$ $\in \mathbb{R}^4$
$\vec{v}_3 = (\beta+3, 2, \beta, -3)$ $\vec{v}_4 = (\beta^2-4, 0, 0, 0)$

$\beta \in \mathbb{R}$.

SIA $W = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4]$ SOTTOSPAZIO DI $\mathbb{R}^4$

DETERMINARE LA DIMENSIONE DI W AL VARIARE DI β

C

$\begin{vmatrix} 0 & 1 & -2 & \beta \\ \beta+1 & -1 & 0 & 1 \\ \beta+3 & 2 & \beta & -3 \\ \beta^2-4 & 0 & 0 & 0 \end{vmatrix} = -(\beta^2-4)\begin{vmatrix} 1 & -2 & \beta \\ -1 & 0 & 1 \\ 2 & \beta & -3 \end{vmatrix} = \underset{\text{CONTI} \ldots}{} = (\beta-2)^2(\beta+2)(\beta+1)$ PER CASA

$\det(C) \neq 0 \Leftrightarrow \beta \neq 2, \beta \neq -2, \beta \neq -1 \Rightarrow v_1, v_2 \ldots v_4$ SONO LIN INDIP.

- $\dim(W) = 4$ SE $\beta \neq \pm 2 \wedge \beta \neq -1$

- Se $\beta = \pm 2 \vee \beta = 1 \Rightarrow \det(C) = 0 \Rightarrow r(C) \le 3$

▶ $\boxed{\text{Se } \beta = 2}$ $A_2 = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ 5 & 2 & 2 & -3 \end{bmatrix}$

$\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} \neq 0 \Rightarrow r(A_2) \geq 2$

$\begin{vmatrix} 0 & 1 & -2 \\ 3 & -1 & 0 \\ 5 & 2 & 2 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} + 5 \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -3(2+4) + 5 \cdot (-2) = -28 \neq 0$

$\Rightarrow r(A_2) = 3 \Rightarrow \dim W = 3$

- Se $\beta = -2$ $\quad A_{-2} = \begin{bmatrix} 0 & 1 & -2 & -2 \\ -1 & -1 & 0 & 1 \\ 1 & 2 & -2 & -3 \end{bmatrix}$ $\quad R_1 = R_2 + R_3$

$(0, 1, -2, -2) = (-1, -1, 0, 1) + (1, 2, -2, -3)$

$\Rightarrow r(A_{-2})$ NON PUO' ESSERE 3 $\Rightarrow r(A_{-2}) \leq 2$

$\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \neq 0 \Rightarrow r(A_{-2}) \geq 2$ $\quad \Rightarrow r(A_{-2}) = 2$

$\Downarrow$

$\dim W = 2$

- $\beta = 1$ ... X CASA

b) $\exists$ VALORI DI β PER I QUALI $W = \mathbb{R}^4$?

Se SÌ QUALI?

$\forall \beta \in \mathbb{R} \setminus \{\pm 2, 1\}$, $\dim W = 4$, $W \subseteq \mathbb{R}^4 \overset{\text{TEO}}{\Longrightarrow} W \equiv \mathbb{R}^4$