① Risolvere il seguente sistema lineare 18/04/23

$$\begin{cases} x+y-2z=1 \\ 2x+2y-z=-1 \\ x+y-z=0 \\ -x+3y-3z=0 \end{cases}$$

$Ax=b$

4 eq. 3 inc. $=n$

$$A=\begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & 3 & -3 \end{pmatrix} \qquad b=\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$rg(A) \le \min\{3,4\} = 3$

TEO Sist. lin. m equazioni e n incognite ha soluzione

$\Updownarrow$

$r(A) = r(A \,\vdots\, b) = k$

- se $k = n \Rightarrow$ sist. è determinato $\Rightarrow \exists!$ sol.
- se $k < n \Rightarrow$ sist. è sottodeterminato $\Rightarrow \infty^{n-k}$ sol

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2-2=0$$

$$\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \neq 0 \quad 2\times2 \qquad rg(A) \geq 2$$

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \text{conto} \dots = 0$$

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ -1 & 3 & -3 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 3 & -3 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} =$$

$$= -6+3-2(-3+6)-(-1+4) = -12 \neq 0$$

$\Rightarrow rg(A) = 3$

$$[A,b]=[A|b]=\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 3 & -3 & 0 \end{array}\right)$$

$R_2 \leftarrow R_2 + R_1$

$v = v + v_1$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -2 & 1 \\ 3 & 3 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 3 & -3 & 0 \end{vmatrix} = -1\begin{vmatrix} 3 & 3 & -3 \\ 1 & 1 & -1 \\ -1 & \underbrace{3 \quad -3}_{\text{lin. dip}} \end{vmatrix} = 0 \quad \Rightarrow 4\times4 = 0$$

$C_2 = -1 \cdot C_3$

$rg(A) = rg(A|b) = 3 \underset{\text{TEO}}{\Rightarrow}$ Determinato $\Rightarrow \exists!$

$$rg(A) = rg(A|b) = 3 \xrightarrow[\text{TEO}]{} \text{Determinato} \Rightarrow \exists!$$

$C_2 = -1 \cdot C_3$

$$\Rightarrow \begin{cases} x + y - 2z = 1 \\ 2x + 2y - z = -1 \\ -x + 3y - 3z = 0 \end{cases} \qquad M = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ -1 & 3 & -3 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & 1 & -2 \\ -1 & 2 & -1 \\ 0 & 3 & -3 \end{vmatrix}}{-12} \overset{R_3 = R_1 + R_2}{=} 0$$

$$\det(M) = \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 3 & -3 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} =$$

$$= -6 + \cancel{3} - 2 \cdot 3 - \cancel{3} = -12$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & -1 \\ -1 & 0 & -3 \end{vmatrix}}{-12} = -1$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ -1 & 3 & 0 \end{vmatrix}}{-12} = -1$$

$\Rightarrow (0, -1, -1)$ e' la sol.

② DISCUTERE E RISOLVERE AL VARIARE $\lambda \in \mathbb{R}$ IL SEGUENTE SISTEMA LINEARE.

$$\begin{cases} x + y + z = 0 \\ 2x - 3y + 4z = 0 \\ 3x - 2y + 5z = \lambda \end{cases} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 5 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

$1 \neq 0$

$\Rightarrow rg(A) \leq 3$

$$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 \neq 0 \Rightarrow rg(A) \geq 2$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 5 \end{vmatrix} = \dots = 0 \Rightarrow rg(A) = 2$$

$$[A|b] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -3 & 4 & 0 \\ 3 & -2 & 5 & \lambda \end{pmatrix} \qquad rg(A|b) \leq 3$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = \boxed{-5\lambda} \Rightarrow -5\lambda = 0 \Leftrightarrow \lambda = 0$$

- Se $\lambda = 0 \Rightarrow \left.\begin{array}{l} rg(A|b) = 2 \\ rg(A) = 2 \end{array}\right\} \Rightarrow$ COMPATIBILE

$$\begin{cases} x + y + z = 0 \\ 2x - 3y + 4z = 0 \\ \cancel{3x - 2y + 5z = 0} \end{cases} \Rightarrow$$

DETERMINATO $\exists!$ SOL

1 $3x - \cancel{2y} + 5z = 0$

UN SISTEMA PUO' ESSERE $\rightarrow$ COMPATIBILE $\exists$ ALMENO UNA SOL $\rightarrow$ DETERMINATO $\exists!$ SOL ; SOTTODETERMINATO ∞^{n-k} SOL

$\rightarrow$ INCOMPATIBILE (IMPOSSIBILE) $\nexists$ sol

$$\Rightarrow \begin{cases} x + y + z = 0 \\ 2x - 3y + 4z = 0 \end{cases} \qquad 2 \text{ EQ. } 3 \text{ INC.}$$

$$\begin{cases} x + y = -z \\ 2x - 3y = -4z \end{cases} \Rightarrow B = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \Rightarrow \det(B) = -5$$

$$b = \begin{pmatrix} -z \\ -4z \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} -z & 1 \\ -4z & -3 \end{vmatrix}}{-5} = \frac{+3z + 4z}{-5} = \frac{7z}{-5} = -\frac{7}{5}z$$

$$y = \frac{\begin{vmatrix} 1 & -z \\ 2 & -4z \end{vmatrix}}{-5} = \frac{-4z + 2z}{-5} = +\frac{2}{5}z$$

$$S = \left\{ \left(-\frac{7}{5}z, \frac{2}{5}z, z\right) : z \in \mathbb{R} \right\} = \left[\left(-\frac{7}{5}, \frac{2}{5}, 1\right) \right] =$$

$$= \left[\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{sol. } Ax=0} + \underbrace{\begin{pmatrix} -7/5 \\ 2/5 \\ 1 \end{pmatrix}}_{\text{sol. particolare}} z \right]$$

• $\lambda \neq 0 \quad rg(A|b) = 3 \neq rg(A) \Rightarrow$ sistema incompatibile $\Leftrightarrow \nexists$ sol

$$\mathrm{Ker}(A) \overset{DEF}{:=} \{ \vec{x} \in \mathbb{R}^n : \underline{A\vec{x} = \vec{0}} \}$$

3 SIA DATA $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & k^2 \\ 1 & 2 & -1 \end{pmatrix} \quad \forall k \in \mathbb{R}$

a) $\mathrm{Ker}(A)$ e $\dim(\mathrm{Ker}(A))$ al VARIARE DI k

b) DISCUTERE E RISOLVERE $Ax = b$, $b^T = (k+1, 4, 1)$

a) $A\vec{x} = 0 \qquad \vec{x} = (x, y, z)$

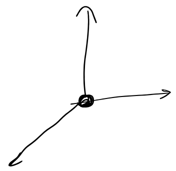
$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & k^2 \\ 1 & 2 & -1 \end{pmatrix} \overbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}^{\vec{x}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + z = 0 \\ 2x + k^2 z = 0 \\ x + 2y - z = 0 \end{cases}$$

$$\begin{vmatrix} 2 & 2 & 1 \\ 2 & 0 & k^2 \\ 1 & 2 & -1 \end{vmatrix} = \dots = 2(2-k)(2+k) = 0 \iff k = \pm 2$$

- Se $k \neq \pm 2 \Rightarrow rg(A) = 3 \Rightarrow \exists!$ sol.
  Siccome $Ax = 0$ è omog. $\Rightarrow$ sol. è $\vec{0}$ $\Big\} \Rightarrow \vec{0}$ è l'unica sol.

  $Ker(A) = \{\vec{0}\} \Rightarrow dim(Ker(A)) = 0$



- Se $k = \pm 2 \Rightarrow rg(A) = 2$

  $dim(Ker(A)) = n - rg(A) = 3 - 2 = 1$

SOSTITUISCO $k = \pm 2$, $k^2 = 4 \Rightarrow \begin{cases} 2x + 2y + z = 0 \\ 2x + 4z = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2y = -z \\ 2x = -4z \end{cases}$

CONTI $\Rightarrow$ $x = -2z \qquad y = \frac{3}{2}z$

$Ker(A) = \{(-2z, \frac{3}{2}z, z) : z \in \mathbb{R}\} \Rightarrow dim(Ker(A)) = 1$

ALTERNATIVA:

$\begin{cases} y + \frac{z}{2} = -x \\ 2z = -x \end{cases} \qquad B = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}$

$det(B) = 2 \qquad b = \begin{pmatrix} -x \\ -x \end{pmatrix}$

$y = \dfrac{\begin{vmatrix} -x & \frac{1}{2} \\ -x & 2 \end{vmatrix}}{2} = \dfrac{-\frac{3}{2}x}{2} = -\frac{3}{4}x$

$z = \dfrac{\begin{vmatrix} 1 & -x \\ 0 & -x \end{vmatrix}}{2} = -\frac{x}{2}$

$\Rightarrow y = -\frac{3}{4}x \qquad z = -\frac{x}{2}$

$Ker(A) = \{(x, -\frac{3}{4}x, -\frac{x}{2}) : x \in \mathbb{R}\} = [(1, -\frac{3}{4}, -\frac{1}{2})]$

$\Rightarrow dim(Ker(A)) = 1$

b) $[A|b] = \left(\begin{array}{ccc|c} 2 & 2 & 1 & k+1 \\ 2 & 0 & k^2 & 4 \\ 1 & 2 & -1 & 1 \end{array}\right) \qquad rg(A|b) \leq 3$

$$\begin{vmatrix} 2 & 2 & k+1 \\ 2 & 0 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & k+1 \\ 2 & 4 \end{vmatrix} =$$

$$= -2(2-4) - 2(8 - 2k - 2) = 4(k-2)$$

- $\boxed{\text{Se } k \neq \pm 2} \Rightarrow 4(k-2) \neq 0 \Rightarrow rg(A|b) = 3$
  Siccome $rg(A) = 3$ $\Big] \Rightarrow \exists!$ sol

  $\Rightarrow$ USARE KRAMER

- $\boxed{\text{Se } k = 2} \Rightarrow rg(A|b) = 2 = rg(A) \Rightarrow \infty^{3-2}$ sol, cioè ∞^1 sol.

  $\Rightarrow \begin{cases} 2x + 2y + z = 3 \\ 2x + 4z = 4 \end{cases} \longrightarrow$ qui DEVO sostituire $k = 2$ in $\begin{cases} 2x + 2y + z = k+1 \\ 2x + 4z = 4 \end{cases}$

- $\boxed{\text{Se } k = -2} \Rightarrow \begin{matrix} rg(A|b) = 3 \\ rg(A) = 2 \end{matrix} \Big] \neq \Rightarrow$ INCOMPATIBILE $\Rightarrow \nexists$ SOL