7. Sia $v = (2,3)$; per quali valori di m, il vettore $w = (1, m)$ è ortogonale a v?

Si impone la condizione

$$0 = \langle v, w \rangle = \langle (2,3), (1,m) \rangle = 2 + 3m$$

$$\Rightarrow \quad m = -\frac{2}{3}$$

2. Sia $W = \{(a, b, 0),\ a, b \in R\}$. Si trovi il sottospazio $W^\perp$.

Si determina una base di W:

$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ generatori di W lin. indip.

Per determinare $W^\perp$ basta determinare elementi ortogonali a B, ossia

$$W^\perp = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ t.c. } \langle (x, y, z)^T, (1, 0, 0)^T \rangle = x = 0 ; \right.$$

$$\left. \langle (x, y, z)^T, (0, 1, 0)^T \rangle = y = 0 \right\}$$

$$= \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

11. Sia $v=(1,3)$ e $w=(4,2)$

Trovare il coefficiente di Fourier di v rispetto a w

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle} = \frac{4+6}{16+4} = \frac{1}{2}$$

Determinare la proiezione di v su w

$$(v)_w = c\,w = \frac{1}{2}(4,2) = (2,1)$$

Infatti $v - cw$ è ortogonale a w

$$\langle (1,3)-(2,1), (4,2)\rangle = \langle (-1,2),(4,2)\rangle$$
$$= -4+4=0$$

10. Trovare le coordinate di $v=(1,2)$ rispetto alla base

$$\mathcal{B}=\{v_1, v_2\}, \quad v_1=(1,1), \quad v_2=\left(\frac{3}{2}, -\frac{3}{2}\right)$$

$\mathcal{B}$ è base ortogonale $\langle v_1, v_2\rangle = \frac{3}{2}-\frac{3}{2}=0$

$$\Rightarrow \quad v=\underbrace{\frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle}}_{x_{\mathcal{B}}} v_1 + \underbrace{\frac{\langle v, v_2\rangle}{\langle v_2, v_2\rangle}}_{y_{\mathcal{B}}} v_2$$

$$x_{\mathcal{B}} = \frac{\langle (1,2), (1,1)\rangle}{\langle (1,1), (1,1)\rangle} = \frac{3}{2}$$

$$y_{\mathcal{B}} = \frac{\langle (1,2), \left(\frac{3}{2}, -\frac{3}{2}\right)\rangle}{\langle \left(\frac{3}{2}, -\frac{3}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}\right)\rangle} = \frac{\frac{3}{2}-3}{\frac{9}{4}+\frac{9}{4}} = -\frac{\cancel{3}}{\cancel{2}}\,\frac{\overset{2}{\cancel{4}}}{\underset{6}{\cancel{18}}} = -\frac{1}{3}$$

$$v=\left(\frac{3}{2}, -\frac{1}{3}\right)_{\mathcal{B}}$$

5. Calcolare il coseno dell'angolo formato dai vettori $(1,0,0,0)$ e $(2,2,2,2)$.

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\,\|w\|} = \frac{2}{4} = \frac{1}{2}$$

$\langle v, w\rangle = 1 \cdot 2 = 2$

$\|v\| = 1$

$\|w\| = \sqrt{4+4+4+4} = 4$

6. Calcolare il coseno dell'angolo formato dai vettori $(0,0,0,2)$ e $(1,1,1,1)$

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\,\|w\|} = \frac{2}{2 \cdot 2} = \frac{1}{2}$$

1. Determinare i vettori di R^4 ortogonali a $(1,2,3,0)^T$.
Trovare i versori.

Occorre determinare, dato $S=\{(1,2,3,0)^T\}$

$S^\perp$.

$S^\perp = \{(x,y,z,t)^T : x\cdot 1 + y\cdot 2 + z\cdot 3 + t\cdot 0 = 0\}$

$= \{(x,y,z,t) : x+2y+3z=0\}$

$= \{(-2y-3z, y, z, t)\} =$

$= [\underbrace{(-2,1,0,0)^T}_{v_1}, \underbrace{(-3,0,1,0)^T}_{v_2}, \underbrace{(0,0,0,1)^T}_{v_3}]$

Versori

$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0\right)^T$

$\|v_1\| = \sqrt{4+1} = \sqrt{5}$

$w_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}, 0\right)^T$

$\|v_2\| = \sqrt{9+1} = \sqrt{10}$

$w_3 = \frac{v_3}{\|v_3\|} = (0,0,0,1)^T$

3. Sia $\mathcal{B} = \{ (0,1,0,\cancel{2}1)^T, (2,1,0,1)^T, (-1,0,0,1)^T, (0,0,1,0)^T \}$ una base di $\mathbb{R}^4$. Costruire a partire da esso una base ortogonale. Costruire una base ortonormale.

Procedimento di Gram-Schmidt

$$v_1' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \langle v_1', v_1' \rangle = 2 \qquad \| v_1' \| = \sqrt{2}$$

$$v_2' = v_2 - \frac{\langle v_1', v_2 \rangle}{\langle v_1', v_1' \rangle} v_1' = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\cancel{2}}{\cancel{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-0 \\ 1-1 \\ 0-0 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle v_2', v_2' \rangle = 4 \qquad \| v_2' \| = 2$$

$$v_3' = v_3 - \frac{\langle v_1', v_3 \rangle}{\langle v_1', v_1' \rangle} v_1' - \frac{\langle v_2', v_3 \rangle}{\langle v_2', v_2' \rangle} v_2' =$$

$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{-\cancel{2}}{\cancel{4}_2} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{1}{2} 2 \\ 0 - \frac{1}{2} + 0 \\ 0 + 0 + 0 \\ 1 - \frac{1}{2} + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\langle v_3', v_3' \rangle = \frac{1}{2} \qquad \| v_3' \| = \frac{1}{\sqrt{2}}$$

$$v_4' = v_4 - \frac{\langle v_1', v_4\rangle}{\langle v_1', v_1'\rangle} v_1' - \frac{\langle v_2', v_4\rangle}{\langle v_2', v_2'\rangle} v_2' - \frac{\langle v_3', v_4\rangle}{\langle v_3', v_3'\rangle} v_3'$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{4}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{0}{\frac{1}{2}}\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\langle v_4', v_4'\rangle = 1$

$\| v_4' \| = 1$

Base ortogonale:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Base ortonormale $\quad v_i' / \| v_i' \|$

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

4. Sia $\mathcal{B} = \{(1,1,1)^T, (0,0,1)^T, (1,0,1)^T\}$ una base di $\mathbb{R}^3$. Costruire a partire da essa una base ortogonale. Costruire una base ortonormale.

Procedimento di Gram Schmidt

$$v_1' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle v_1', v_1' \rangle = 3 \qquad \|v_1'\| = \sqrt{3}$$

$$v_2' = v_2 - \frac{\langle v_1', v_2 \rangle}{\langle v_1', v_1' \rangle} v_1' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1-\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\langle v_2', v_2' \rangle = \frac{1}{9} + \frac{1}{9} + \frac{4}{9} = \frac{\cancel{6}\,2}{\cancel{9}\,3} \qquad \|v_2'\| = \frac{\sqrt{2}}{\sqrt{3}}$$

$$v_3' = v_3 - \frac{\langle v_1', v_3 \rangle}{\langle v_1', v_1' \rangle} v_1' - \frac{\langle v_2', v_3 \rangle}{\langle v_2', v_2' \rangle} v_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\cancel{1/3}}{\frac{2}{\cancel{3}}}\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{2}{3} + \frac{1}{2}\frac{1}{3} \\ 0 - \frac{2}{3} + \frac{1}{2}\frac{1}{3} \\ 1 - \frac{2}{3} - \frac{1}{2}\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{6-4+1}{6} \\ \frac{-4+1}{6} \\ \frac{6-4-2}{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \qquad \langle v_3', v_3' \rangle = 1/2 \qquad \|v_3'\| = 1/\mathrm{sqrt}(2)$$

Base ortogonale: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$

Base ortonormale $\quad v_i' / \|v_i'\|$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

8. Dati $(1,1)$ e $(0,1)$ costruire una base ortonormale di R^2.
Costruire la matrice del cambiamento di base dalla base canonica di R^2 a questa nuova base.

$v_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \langle v_1', v_1' \rangle = 2$

$$v_2' = v_2 - \frac{\langle v_1', v_2 \rangle}{\langle v_1', v_1' \rangle} v_1' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ \frac{1}{2} \end{pmatrix}$$

Base ortogonale

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\qquad \langle v_2', v_2' \rangle = \frac{1}{2}$

Base ortonormale $\qquad v_i / \|v_i\|$

$$\mathcal{B} = \left\{ \underset{w_1}{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} \quad \underset{w_2}{\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} \right\} \qquad \mathcal{B} = \{ w_1, w_2 \}$$

Si vuole $M^{C}_{\mathcal{B}}(i_{R^2})$ $\qquad C = \{ \underset{e_1}{(1,0)}, \underset{e_2}{(0,1)} \}$

$$M^{C}_{\mathcal{B}}(i_{R^2}) = \begin{pmatrix} \langle e_1, w_1 \rangle & \langle e_2, w_1 \rangle \\ \langle e_1, w_2 \rangle & \langle e_2, w_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Si osservo che $M_C^B(i_{\mathbb{R}^2}) = \left(M_B^C(i_{\mathbb{R}^2})\right)^T$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \langle w_1, e_1 \rangle & \langle w_2, e_1 \rangle \\ \langle w_1, e_2 \rangle & \langle w_2, e_2 \rangle \end{pmatrix}$$

9. Dato $b_1=(2,3)$, trovare un vettore b_2 ortogonale a b_1. Scrivere le componenti di un vettore (x,y) rispetto alla base b_1, b_2. Normalizzare gli elementi della base.

Si determina b_2 t.c. $\langle b_1, b_2 \rangle = 0$

$b_2 = (x, y)$

$2x + 3y = 0$

$\Downarrow$

$b_2 = (-3, 2)$

Dato $v = (x, y)_{\mathcal{B}}$ le componenti rispetto a $\mathcal{B} = \{ b_1, b_2 \}$ sono

$$x = \frac{\langle v, b_1 \rangle}{\langle b_1, b_1 \rangle} \qquad y = \frac{\langle v, b_2 \rangle}{\langle b_2, b_2 \rangle}$$

coefficienti di Fourier

$$v = \frac{\langle v, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 + \frac{\langle v, b_2 \rangle}{\langle b_2, b_2 \rangle} b_2$$

$$b_1' = \frac{b_1}{\|b_1\|} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right) \qquad b_2' = \frac{b_2}{\|b_2\|} = \left(-\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right)$$