28|03|23

① DATI I SEGUENTI SOTTOINSIEMI $V = \{(x,y,z) \in \mathbb{R}^3 : 2x = y\}$
$W = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$.

a) VERIFICARE CHE $V, W \subseteq \mathbb{R}^3$ E PER CIASCUNO DETERMINARE $\mathcal{B}_V$ E $\mathcal{B}_W$
e $\dim(V)$, $\dim(W)$.

ⓥ
- $2 \cdot 0 = 0$ ✓ $\Rightarrow \vec{0} \in V$ $\quad (0,0,0) = \vec{0}$
- $\vec{v}_1, \vec{v}_2 \in V \Rightarrow \vec{v}_1 + \vec{v}_2 \in V$

  VERIFICHIAMOLO

  $2x_1 \stackrel{\downarrow}{=} y_1$ e $2x_2 \stackrel{\downarrow}{=} y_2$ (per ipotesi)

  $2(x_1 + x_2) = y_1 + y_2$

  $2x_1 + 2x_2 = y_1 + y_2$ ✓
- $\vec{v}_1 \in V$, $c \in \mathbb{R}$

  $c\vec{v}_1 = (cx_1, cy_1, cz_1)$ per ipotesi $2x_1 = y_1$

  $2 \cdot cx_1 = cy_1 \Leftrightarrow c(2x_1 - y_1) = 0$ ✓ $\Rightarrow V \subseteq \mathbb{R}^3$

ⓦ
- $\vec{0} \in W$ ✓ $0+0+0=0$
- $\vec{v}_1 + \vec{v}_2 = (x_1+x_2, y_1+y_2, z_1+z_2)$

  $(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = 0$?

  POICHÉ $\vec{v}_1 \in W$, $x_1 + y_1 + z_1 = 0$

  $\vec{v}_2 \in W$, $x_2 + y_2 + z_2 = 0$

  $\underbrace{x_1 + y_1 + z_1} + \underbrace{x_2 + y_2 + z_2} = 0$ ✓
- $c\vec{v}_1 \in W$? SÌ PERCHÉ $cx_1 + cy_1 + cz_1 = 0$

  $c(x_1 + y_1 + z_1) = 0$ $\Rightarrow W \subseteq \mathbb{R}^3$

X ⇧ FARE LA STESSA COSA CON $cw_1 - w_2$ (2ª FORMA)

BASE V

$(x, 2x, z)$ $\quad \vec{u}_1 = (1,2,0)$, $(0,0,1) = \vec{u}_2$

$\mathcal{B}_V = \{(1,2,0), (0,0,1)\} \Rightarrow \dim V = 2$

$V: \quad 2x = y$

(x, y, z)
$\downarrow \ \downarrow \ \downarrow$
$x \quad 2x \quad z$

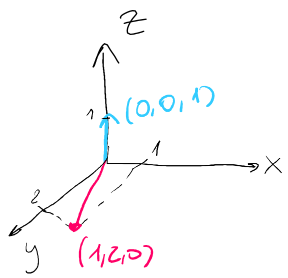
$\boxed{(x, 2x, z)}$

$(1, 2, 0) \quad (0,0,1)$

$\Rightarrow \underline{(x, 2x, z)} = (x, 2x, 0) + (0,0,z)$

BASE W : $W \rightsquigarrow x+y+z=0$

$x=-y-z$



$W=\{(-y-z,\ y,\ z) : y\in\mathbb{R},\ z\in\mathbb{R}\}$

$W=\{(-y,y,0)+(-z,0,z) : y,z\in\mathbb{R}\}$

$\mathcal{B}_W=\{(-1,1,0),\ (-1,0,1)\} \Rightarrow \dim(W)=2$

b) $V+W$

$V+W=\{u=v+w,\ v\in V,\ w\in W\}$

$V+W=[(1,2,0),(0,0,1),(-1,0,1),(-1,1,0)]$

$*\ (0,0,0)=\alpha(1,2,0)+\beta(0,0,1)+\gamma(-1,0,1)+\delta(-1,1,0)$

$$\begin{cases} 0=\alpha-\gamma-\delta \\ 0=2\alpha+\delta \\ 0=\beta+\gamma \end{cases} \Rightarrow \begin{cases} \alpha=-\frac{\beta}{3} \\ \delta=-2\alpha=\frac{2}{3}\beta \\ \gamma=-\beta \end{cases}$$

$\Rightarrow \left(-\frac{\beta}{3},\beta,-\beta,+\frac{2}{3}\beta\right)=\beta\left(-\frac{1}{3},1,-1,\frac{2}{3}\right) \Rightarrow$ LIN. DIP

INFATTI POSSO SCRIVERE $(1,2,0)$ COME COMBINAZIONE LIN. DEI RESTANTI VETTORI

$(1,2,0)=\alpha(0,0,1)+\beta(-1,0,1)+\gamma(-1,1,0)$

$\alpha=3,\ \beta=-3,\ \gamma=2$

$\Rightarrow V+W=[(0,0,1),(-1,0,1),(-1,1,0)] \to$ PER CASA: VERIFICARE CHE SONO LIN. INDIP.

$\Rightarrow$ SONO LIN. INDIP. $\Rightarrow \dim(V+W)=3$

c) $V\oplus W$?

$\to \dim(V+W)=\dim(V)+\dim(W)-\dim(V\cap W)$

$$\left.\begin{aligned} 3 &= 2+2-\dim(V\cap W) \\ \Leftrightarrow \dim(V\cap W) &= 4-3=1 \end{aligned}\right\} \Rightarrow \text{NON E' SOMMA DIRETTA}$$

2) SIA $W=\left\{\left(x_2-\frac{1}{2}x_3+\frac{1}{5}x_1,\ 0,\ \frac{1}{2}x_3+x_1+2x_2,\ \frac{1}{2}x_1+\frac{1}{4}x_3+x_2\right) : x_1,x_2,x_3\in\mathbb{R}\right\}\subseteq\mathbb{R}^4$

a) TROVARE $\dim(W)$ E UNA BASE DI W:

$W=\left\{\left(\frac{1}{5}x_1+x_2-\frac{1}{2}x_3,\ 0,\ x_1+2x_2+\frac{1}{2}x_3,\ \frac{1}{2}x_1+x_2+\frac{1}{4}x_3\right) : x_1,x_2,x_3\in\mathbb{R}\right\}$

$W=\left[\left(\frac{1}{5},0,1,\frac{1}{2}\right),(1,0,2,1),\left(-\frac{1}{2},0,\frac{1}{2},\frac{1}{4}\right)\right]$

$$\begin{cases} 0 = \frac{1}{5}x_1 + x_2 - \frac{x_3}{2} \\ 0 = x_1 + 2x_2 + \frac{1}{2}x_3 \\ \left(0 = \frac{1}{2}x_1 + x_2 + \frac{1}{4}x_3\right)\cdot 2 \end{cases} \rightarrow 0 = x_1 + 2x_2 + \frac{1}{2}x_3$$

$$\iff \begin{cases} x_2 = -\frac{2}{5}x_1 \\ x_3 = -\frac{2}{5}x_1 \end{cases} \Rightarrow$$ COEFFICIENTI DELLA COMBINAZIONE LINEARE DEI VETTORI DI W

$\Rightarrow$ NON SONO LIN. INDIP.

$$\left[\left(\frac{1}{5}, 0, 1, \frac{1}{2}\right) = \alpha\,(1, 0, 2, 1) + \beta\left(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}\right) \right]$$
VERIFICA

VEDIAMO SE $(1,0,2,1)$ e $\left(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}\right)$ SONO LIN. IND.

$$\begin{cases} 0 = x_1 - \frac{1}{2}x_2 \\ 0 = 2x_1 + \frac{1}{2}x_2 \\ 0 = x_1 - \frac{1}{4}x_2 \end{cases} \iff \begin{cases} x_1 = \frac{1}{2}x_2 \\ 0 = 2\cdot\frac{1}{2}x_2 + \frac{1}{2}x_2 \end{cases} \iff \begin{cases} x_1 = \frac{1}{2}x_2 \\ \frac{3}{2}x_2 = 0 \end{cases}$$

$$\iff \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases} \Rightarrow \text{LIN. INDIP} \checkmark$$

$$\mathcal{B} = \left\{(1,0,2,1), \left(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}\right)\right\} \Rightarrow \dim(W) = 2$$

b) DATO $V = \{(x_2, x_1, 2x_2, x_1 + x_2) : x_1, x_2 \in \mathbb{R}\}$
TROVARE $V + W$

c) $V+W$ E' SOMMA DIRETTA ?

X 🏠

③ CALCOLARE IL DETERMINANTE DI

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 3 \\ 2 & 2 & \frac{1}{2} \end{pmatrix} \qquad \det(A) = |A| \rightarrow \text{NOTAZIONE}$$

3×3

$$\det(A) = \sum_{j=1}^{m} \left[a_{ij}(-1)^{i+j}\det(A_{ij})\right] \quad \text{SVILUPPO x RIGHE}$$

$$\det(A) = \sum_{i=1}^{m} \left[a_{ij}(-1)^{i+j}\det(A_{ij})\right] \quad \text{" COLONNE}$$

SOLO MATRICI QUADRATE!

$$\det(A) = \begin{vmatrix} +1 & -1 & -1 \\ -0 & 3^{+} & 3^{-} \\ +2 & 2^{-} & \frac{1}{2}^{+} \end{vmatrix} = -1\begin{vmatrix} 3 & 3 \\ 2 & \frac{1}{2} \end{vmatrix} + \cancel{(-0)\begin{vmatrix} -1 & -1 \\ 2 & \frac{1}{2} \end{vmatrix}} + 2\begin{vmatrix} -1 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= 1\left(3\cdot\frac{1}{2} - 3\cdot 2\right) + 2\left(-1\cdot 3 - (3\cdot(-1))\right) =$$

$$= \frac{3}{2} - 6 \cancel{-6+6} = \frac{3-12}{2} = -\frac{9}{2}$$

$$\det(A) = 0\begin{vmatrix} -1 & -1 \\ 2 & \frac{1}{2} \end{vmatrix} + 3\begin{vmatrix} +1 & -1 \\ 2 & \frac{1}{2} \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$$

④

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{pmatrix} = 7\begin{vmatrix} 3 & 0 \\ 0 & 10 \end{vmatrix} = 7\cdot(3\cdot 10 - 0) =$$

$$= 7\cdot 3\cdot 10 = 210$$

$$M = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 10 & 0 \\ 7 & 0 & 0 \end{pmatrix}$$

$$\det(M) = \begin{vmatrix} 0 & 0 & 3 \\ 0 & 10 & 0 \\ 7 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 10 \\ 7 & 0 \end{vmatrix} = -210$$

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \Rightarrow \det(D) = \prod_{i=1}^{n} d_i = d_1 \cdot d_2 \cdot d_3 \cdots d_n$$

NB ELEMENTI SULLA DIAGONALE PRINCIPALE

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$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} \Rightarrow \det(B) = -2 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} =$$

$$= -10 + 10 = 0$$