21/03/23

1) $\vec{v}_1 = (1,0,1)$, $\vec{v}_2 = (0,1,0)$, $\vec{v}_3 = (1,1,2) \in \mathbb{R}^3$

Determinare la proiezione ortogonale $\vec{v}_1'$ di $\vec{v}_1$ sul piano che contiene $\vec{v}_2$ e $\vec{v}_3$

$$\frac{\vec{v}_2 \times \vec{v}_3}{|\vec{v}_2 \times \vec{v}_3|} = \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)$$

$$\vec{v}_2 \times \vec{v}_3 = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = (2,0,-1) = 2i - k \qquad |\vec{v}_2 \times \vec{v}_3| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$\vec{w} = \left\langle \vec{v}_1, \frac{\vec{v}_2 \times \vec{v}_3}{|\vec{v}_2 \times \vec{v}_3|} \right\rangle \frac{\vec{v}_2 \times \vec{v}_3}{|\vec{v}_2 \times \vec{v}_3|} =$$

$$= \left\langle (1,0,1), \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right) \right\rangle \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right) =$$

$$= \left(\left(1 \cdot \frac{2}{\sqrt{5}}\right) + 0 + 1 \cdot \left(-\frac{1}{\sqrt{5}}\right)\right) \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right) =$$

$$= \left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right) \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right) = \left(\frac{2}{5}, 0, -\frac{1}{5}\right)$$

$$\vec{p} = \vec{v}_1 - \vec{w} = (1,0,1) - \left(\frac{2}{5}, 0, -\frac{1}{5}\right) = \left(\frac{3}{5}, 0, \frac{6}{5}\right)$$ proiezione

SPAZI VETTORIALI

2) STABILIRE QUALI TRA I SEGUENTI SOTTOINSIEMI DI $\mathbb{R}^3$ E' UN SOTTOSPAZIO VETTORIALE DI $\mathbb{R}^3$

$V \subseteq W$

a) $W_1 = \{ (x,y,z) \in \mathbb{R}^3 : 2x + 3y - z = 0 \}$

I
- $\vec{0} \in W_1$
- $\forall w_1, w_2 \in W_1 \Rightarrow w_1 + w_2 \in W_1$ CHIUSURA RISPETTO ALLA SOMMA
- $\forall c \in \mathbb{R}, \forall w_1 \in W_1 \Rightarrow c w_1 \in W_1$ ... PRODOTTO PER UNO SCALARE

II $\forall w_1, w_2 \in W_1, \forall c \in \mathbb{R} \Rightarrow c w_1 - w_2 \in W_1$

I
- $(0,0,0) \in W_1$?

  $2 \cdot 0 + 3 \cdot 0 - 0 = 0$ ✓

- $w_1 = (x_1, y_1, z_1)$ $w_2 = (x_2, y_2, z_2) \in W_1$ (IPOTESI)

  DD: $w_1 + w_2 \in W_1$

  $\Updownarrow$

  $\rightarrow (x_1+x_2, y_1+y_2, z_1+z_2) \in W_1$

$2(x_1+x_2) + 3(y_1+y_2) - 1(z_1+z_2) = 0$?

$$2x_1 + 2x_2 + 3y_1 + 3y_2 - z_1 - z_2 = 0 \quad ?$$

So che $\quad 2x_1 + 3y_1 - z_1 = 0 \quad , \quad 2x_2 + 3y_2 - z_2 = 0$

$$\underbrace{(2x_1 + 3y_1 - z_1)}_{=0} + \underbrace{(2x_2 + 3y_2 - z_2)}_{=0} = 0 \quad \checkmark$$

perché $w_1 \in W_1$ $\qquad\qquad$ $w_2 \in W_1$

- $\underbrace{\forall\, c \in \mathbb{R},\ \forall\, w_1 \in W_1}_{\text{IPOTESI}} \overset{?}{\Rightarrow} c w_1 \in W_1 \ ?$

$\Updownarrow$

$(cx_1, cy_1, cz_1) \in W_1 \ ?$

$2x + 3y - z = 0$

$2(cx_1) + 3(cy_1) - (cz_1) = 0 \ ?$

$\cancel{c}(\underbrace{2x_1 + 3y_1 - z_1}) = 0 \ ?$

$w_1 \in W_1 \Leftrightarrow 2x_1 + 3y_1 - z_1 = 0$

b) $W_2 = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y - z + 1 = 0\}$

- $\vec{0} \in W_2$? NO $\quad 1 \cancel{=} 0$

$\Rightarrow W_2 \not\subseteq \mathbb{R}^3$

$W_2 \equiv E$

c) $W_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x = 2y,\ y = 2z\} = (2y, y, 2z)$

$w_1 = (x_1, y_1, z_1) \quad w_2 = (x_2, y_2, z_2) \in W_3,\ c \in \mathbb{R}$

$c w_1 - w_2 \in W_3$?

$(cx_1 - x_2,\ cy_1 - y_2,\ cz_1 - z_2) \in W_3$?

$x = 2y$

$\boxed{\begin{array}{l} x_1 = 2y_1 \\ x_2 = 2y_2 \end{array}}$ IPOTESI

$cx_1 - x_2 = c\,2y_1 - 2y_2 = 2(cy_1 - y_2) \ \checkmark$

$cy_1 - y_2 = c \cdot 2z_1 - 2z_2 = 2(cz_1 - z_2) \ \checkmark$

$\Rightarrow W_3 \subseteq \mathbb{R}^3$

ALT. $\ast$ $(2y_1, y_1, \frac{1}{2}y_1) \in W_3$

$W_3 = \{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x = 2y \\ y = 2z \end{cases}\} =$

$= \{ \ldots \mid (2y, y, \frac{1}{2}y)\}$

$y = 2z$

$\frac{1}{2}y = z$ $\quad$ 3ª comp.

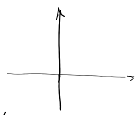
$w_1 \in W_3 \Leftrightarrow (2y_1, y_1, \frac{1}{2}y_1) \in W_3$

$w_2 \in W_3 \Leftrightarrow (2y_2, y_2, \frac{1}{2}y_2) \in W_3$

$c w_1 - w_2 \in W_3$

3) STABILIRE SE I SEGUENTI SOTTOINSIEMI SONO SOTTOSPAZI VETTORIALI

$\mathbb{R}^2$



a) $W = \{(x,y) \in \mathbb{R}^2 : xy \geq 0\}$

I. $(0,0) \in \mathbb{R}^2 \Rightarrow \vec{0} \in W \ \checkmark \quad 0 \cdot 0 \geq 0 \ \checkmark$

- $w_1 = (x_1, y_1) \quad w_2 = (x_2, y_2) \in W \Rightarrow w_1 + w_2 \in W$?

$x_1 y_1 \geq 0$ e $x_2 y_2 \geq 0$

$(-1, -2) \in W$

$(-1)\cdot(-2) = 2 \geq 0$

$(x_1 + x_2)\cdot(y_1 + y_2) \geq 0$?

$\underbrace{x_1 y_1}_{\geq 0} + x_1 y_2 + x_2 y_1 + \underbrace{x_2 y_2}_{\geq 0} \geq 0$?

So che : $x_1 y_1 \geq 0 \rightarrow x_1 \geq 0 \quad y_1 \geq 0$

$\qquad\qquad\qquad\quad \searrow x_1 \leq 0 \quad y_1 \leq 0$

$x_2 y_2 \geq 0 \rightarrow x_2 \geq 0 \quad y_2 \geq 0$

$\qquad\qquad \searrow x_2 \leq 0 \quad y_2 \leq 0$

$w_1 = (-3,-2) \qquad w_2 = (5,1)$

$w_1 \in W$ perche' $-3\cdot(-2) = 6 \geq 0$

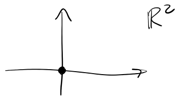
$w_2 \in W$ perche' $5\cdot 1 = 5 \geq 0$

$w_1 + w_2 = (-3,-2) + (5,1) = \underline{(2,-2)} \Rightarrow 2\cdot(-2) = -4 \not\geq 0$

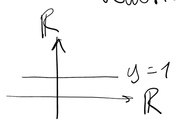
$\Rightarrow$ W <u>non</u> e' sottosp. vettoriale di $\mathbb{R}^2$

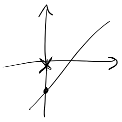
ⓑ $W = \{(x,y) \in \mathbb{R}^2 : \underbrace{x^2}_{\geq 0} + \underbrace{4y^2}_{\geq 0} = 0\} =$

$= \{(x,y) \in \mathbb{R}^2 : x=0,\ y=0\} = \{\vec{0}\}$



$\rightarrow$ W e' sottospazio vettoriale di $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$





~~X~~ ⓒ $W = \{(x,y,z) \in \mathbb{R}^3 : (x-y+3z)^2 + (2x-y+z)^2 = 0\}$

④ DATI I SEGUENTI SOTTOINSIEMI DI $\mathbb{R}^3$ DIRE QUALI FRA QUESTI SONO RISPETTIVAMENTE

- INSIEME DI GENERATORI DI $\mathbb{R}^3$
- UN SOTTOINSIEME LINEARMENTE INDIPENDENTE DI $\mathbb{R}^3$ — lin. indip.
- UNA BASE DI $\mathbb{R}^3$

a) $S_1 = \{(2,1,0), (\frac{1}{2},1,1), (0,1,0)\}$

$(x,y,z) = \alpha(2,1,0) + \beta(\frac{1}{2},1,1) + \gamma(0,1,0)$

$$\begin{cases} x = 2\alpha + \frac{1}{2}\beta \\ y = \alpha + \beta + \gamma \\ z = \beta \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{2}x - \frac{1}{4}z \\ \gamma = y - \alpha - \beta \\ \beta = z \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = \frac{1}{2}x - \frac{1}{4}z \\ \gamma = y - \frac{1}{2}x + \frac{1}{4}z - z \\ \beta = z \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{2}x - \frac{1}{4}z \\ \beta = z \\ \gamma = -\frac{1}{2}x + y - \frac{3}{4}z \end{cases}$$

$\Rightarrow$ sono generatori di $\mathbb{R}^3$

- $(0,0,0) = \alpha(2,1,0) + \beta(\frac{1}{2},1,1) + \gamma(0,1,0)$

$$\begin{cases} 0 = 2\alpha + \frac{\beta}{2} \\ 0 = \alpha + \beta + \gamma \\ 0 = \beta \end{cases} \Leftrightarrow \begin{cases} 2\alpha = -\cancel{\frac{\beta}{2}} \\ \beta = 0 \\ \gamma = -\alpha - \cancel{\beta} \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

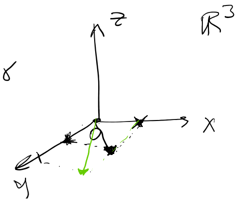
$\Rightarrow$ lin. indip.

- Generatori + lin. indip $\Rightarrow$ base di $\mathbb{R}^3$

b) $S_2 = \{(1,3,1), (1,1,1), (1,-1,1)\}$

$(x,y,z) = \alpha(1,3,1) + \beta(1,1,1) + \gamma(1,-1,1)$

$$\begin{cases} x = \underline{\alpha + \beta + \gamma} \\ y = 3\alpha + \beta - \gamma \\ z = \underline{\alpha + \beta + \gamma} \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = 3\alpha + \beta - \gamma \end{cases}$$



$\Rightarrow$ NON sono generatori

$\Rightarrow$ Non formano una base di $\mathbb{R}^3$

X II Vedere se sono linear. indipendenti

22/03/23

SPAZI VETTORIALI

① SIA $W = \{ (x-y+z, 2x+y-4z, x-z) : x,y,z \in \mathbb{R} \}$ è sottospazio di $\mathbb{R}^3$

a) TROVARE BASE E DIMENSIONE DI W

$$W = \{ (x, 2x, x) + (-y, y, 0) + (z, -4z, -z) : x,y,z \in \mathbb{R} \}$$
$$= \{ x(1,2,1) + y(-1,1,0) + z(1,-4,-1) : x,y,z \in \mathbb{R} \} =$$
$$= \langle (1,2,1), (-1,1,0), (1,-4,-1) \rangle$$

$$(0,0,0) = \alpha(1,2,1) + \beta(-1,1,0) + \gamma(1,-4,-1)$$

$$\begin{cases} 0 = \alpha - \beta + \gamma \\ 0 = 2\alpha + \beta - 4\gamma \\ 0 = \alpha - \gamma \end{cases} \Leftrightarrow \begin{cases} \beta = \alpha + \gamma = 2\gamma \\ \alpha = \gamma \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = \gamma \\ \beta = 2\gamma \end{cases} \Rightarrow (\gamma, 2\gamma, \gamma) \Rightarrow \boxed{(1,2,1)}$$

$$\gamma = -\alpha + \beta \qquad \gamma(1,2,1)$$

$$\sum_{i=1}^{m} a_i v_i = 0 \not\Rightarrow a_i = 0 \quad \forall i = 1, \dots, m$$

$\Rightarrow$ I VETTORI NON SONO LIN. INDIP.

$$(0,0,0) = \alpha(1,2,1) + \beta(-1,1,0)$$

$$\begin{cases} 0 = \alpha - \beta \\ 0 = 2\alpha + \beta \end{cases} \Rightarrow \begin{cases} \alpha = \beta \\ 0 = 3\beta \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases} \Rightarrow$$ I VETTORI SONO LIN INDIP.

$\Rightarrow (1,2,1)$ e $(-1,1,0)$ FORMANO UNA BASE PER W $\Rightarrow \dim(W) = 2$

b) VERIFICARE CHE $\vec{w} = (2,-5,-1) \in W$ e TROVARE LE COORDINATE DEL VETTORE RISPETTO ALLA BASE TROVATA

$$(2, -5, -1) = \alpha(1,2,1) + \beta(-1,1,0)$$

$$\begin{cases} 2 = \alpha - \beta \\ -5 = 2\alpha + \beta \\ -1 = \alpha \end{cases} \Rightarrow \begin{cases} \alpha = -1 \\ \beta = -3 \end{cases}$$

$(\alpha, \beta) = (-1, -3)$ COORDINATE DEL VETTORE w RISPETTO ALLA BASE

$\underline{OSS}$ $(1,-4,-1) = a(1,2,1) + b(-1,1,0)$

$$\begin{cases} 1 = a-b \\ -4 = 2a+b \\ -1 = a \end{cases} \Rightarrow \begin{cases} a=-1 \\ b = a-1 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=-2 \end{cases}$$

② SIANO DATI I SEGUENTI SOTTOSPAZI DI $\mathbb{R}^3$

$W_1 = \{(x,y,z) \in \mathbb{R}^3 : x+z=0,\ x-y=0\}$

$W_2 = \{(x,y,z) \in \mathbb{R}^3 : x-2y=0\}$

a) DETERMINARE UNA BASE PER W_1 E W_2

$W_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x=-z,\ x=y\} = (x,y,z)$

$\Updownarrow$

$z=-x$

$(x, x, -x)$

$\Downarrow$

$\underbrace{x(1,1,-1)}$

$= [(1,1,-1)]$

$\mathcal{B}_1 = \{(1,1,-1)\}$

$W_2 = \{(x,y,z) \in \mathbb{R}^3 : x=2y\}$ $\quad (2y,y,z)$

$= \{y,z \in \mathbb{R} : y(2,1,0)+z(0,0,1)\} = [(2,1,0),(0,0,1)]$

$(2y,y,z) = (2y,y,0)+(0,0,z) =$

$= y(2,1,0) + z(0,0,1)$

RICORDO

SISTEMA DI GENERATORI + LIN. INDIP. $\Rightarrow$ BASE

$$\begin{cases} 0 = 2a \\ \boxed{\begin{matrix} 0 = a \\ 0 = b \end{matrix}} \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases} \Rightarrow \text{LIN. INDIP.} \Rightarrow \mathcal{B}_2 = \{(2,1,0),(0,0,1)\}$$

b) DETERMINARE $W_1 + W_2$

$W_1 + W_2 = [(1,1,-1),(2,1,0),(0,0,1)]$

$$\begin{cases} 0 = \alpha + 2\beta \\ 0 = \alpha + \beta \\ 0 = -\alpha + \gamma \end{cases} \Rightarrow \begin{cases} \alpha = -2\beta \\ 0 = -2\beta + \beta \\ \gamma = \alpha \end{cases} \Leftrightarrow \beta = 0$$

$$\Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \Rightarrow \text{LIN. INDIP.} \Rightarrow \text{BASE}$$

$\dim(W_1+W_2) = 3 \quad \xRightarrow{\text{TEO}} \quad W_1+W_2 = \mathbb{R}^3$

$W_1 + W_2 \subseteq \mathbb{R}^3$ $\downarrow$

TEOREMA Sia $W \subseteq V$.

- $\dim W \leq \dim V$
- $\dim W = \dim V \iff W = V$

c) STABILIRE SE È SOMMA DIRETTA

RICORDO $W_1, W_2 \subseteq V$, $W_1 \cap W_2 = \{0\}$ e $W_1 + W_2 = V$

$\Rightarrow V = W_1 \oplus W_2$ SOMMA DIRETTA

RELAZIONE DI GRASSMANN

$W_1, W_2 \subseteq V$ DI DIMENSIONE FINITA.

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$3 = 1 + 2 - \dim(W_1 \cap W_2)$$

$$\iff \dim(W_1 \cap W_2) = 0$$

$$\Rightarrow W_1 + W_2 = W_1 \oplus W_2 = \mathbb{R}^3$$

4) $S \subseteq \mathbb{R}^3$ DETERMINARE SE S È UN INSIEME DI GENERATORI, LIN. INDIP, BASE?

$$S = \{(0,0,1), (2, \tfrac{1}{2}, 0), (2, \tfrac{1}{2}, 1), (2, \tfrac{1}{2}, 2)\}$$

$$(x,y,z) = \alpha(0,0,1) + \beta(2, \tfrac{1}{2}, 0) + \gamma(2, \tfrac{1}{2}, 1) + \delta(2, \tfrac{1}{2}, 2)$$

$$\begin{cases} x = 2\beta + 2\gamma + 2\delta \\ y = \frac{1}{2}\beta + \frac{1}{2}\gamma + \frac{1}{2}\delta \\ z = \alpha + \gamma + 2\delta \end{cases} \iff \begin{cases} x = 2(\beta + \gamma + \delta) \\ y = \frac{1}{2}(\beta + \gamma + \delta) \\ z = \alpha + \gamma + 2\delta \end{cases} \iff$$

$$\iff \begin{cases} \boxed{x = 4y} \\ y = \frac{1}{2}(\beta + \gamma + \delta) \\ z = \alpha + \gamma + 2\delta \end{cases}$$

$\Rightarrow$ I VETTORI SONO DEL TIPO $(4y, y, z)$

$\Rightarrow$ NON GENERA TUTTO $\mathbb{R}^3$.