Matematica discreta - a.a. 2021-22
Ogni risposta deve essere giustificata.
All answers must be motivated.

1. (3 punti) Determinare il vettore proiezione w del vettore $u = (1, 2, 1)$ sul piano contenente i vettori $a = (-1, 0, 1)$ e $b = (2, 1, 0)$.

   Find the vector w which is the projection of the vector $u = (1, 2, 1)$ on the plane containing the vectors $a = (-1, 0, 1)$ and $b = (2, 1, 0)$.

2. (4 punti) Dati i sottospazi $A = [(2, 0, 0, 1), (0, 0, -2, 0), (0, 0, 1, -1)]$ e $B = [(0, 1, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0)]$, trovare la loro dimensione, la loro somma e la dimensione. Si tratta di una somma diretta?

   Let $A = [(2, 0, 0, 1), (0, 0, -2, 0), (0, 0, 1, -1)]$ and $B = [(0, 1, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0)]$ be two subspaces of $\mathbb{R}^4$. Find $\dim(A)$, $\dim(B)$, the subspace $A + B$ and $\dim(A + B)$. Is $A + B$ a direct sum?

3. (4 punti) Per quali valori di k la matrice $A = \begin{pmatrix} 4 & 1-k \\ 2k & -1 \end{pmatrix}$ è invertibile? Per tali valori trovare la sua inversa.

   Find the values of k (if there exist) such that the matrix $A = \begin{pmatrix} 4 & 1-k \\ 2k & -1 \end{pmatrix}$ is invertible. For this/these value/values, Find the inverse of A.

4. (4 punti) Discutere, al variare del parametro reale $k \in \mathbb{R}$, la risolubilità del seguente sistema e calcolarne le soluzioni, quando esistono:

$$\begin{aligned} -2x + y - z &= k \\ 2x - 2y + kz &= 0 \\ 5x - 3y + z &= 0 \end{aligned}$$

   Find for $k \in \mathbb{R}$ if the following system is solvable and compute the solutions when they exist:

$$\begin{aligned} -2x + y - z &= k \\ 2x - 2y + kz &= 0 \\ 5x - 3y + z &= 0 \end{aligned}$$

5. (4 punti) Sia $F : \mathbb{R}^2 \to \mathbb{R}^3$ l'applicazione lineare definita da

$$F(x, y) = (x - y, 0, 3x - 2y)$$

   e sia $G : \mathbb{R}^3 \to \mathbb{R}^3$ l'applicazione lineare definita da

$$G(x, y, z) = (x, 0, x - z)$$

   - Trovare la dimensione e una base di $\mathrm{Imm}(F)$

- Trovare la dimensione e una base di $\ker(F)$
- Scrivere la matrice che rappresenta $G \circ F$ rispetto alle basi canoniche.

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear application, defined as

$$F(x, y) = (x - y, 0, 3x - 2y).$$

Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear application, defined as

$$G(x, y, z) = (x, 0, x - z)$$

- Find $\dim(\mathrm{Imm}(F))$ and a basis of $\mathrm{Imm}(F)$
- Find $\dim(\ker(F))$ and a basis of $\ker(F)$
- Find the matrix representing $G \circ F$ with respect to the canonical bases.

6. (4 punti) Sia $f : \mathbb{R}^3 \to \mathbb{R}^3$ l'applicazione lineare così definita: $f(x, y, z) = (x - z, 2x + y, 2y + z)$. Si scriva la matrice rappresentativa di f rispetto alla base $\mathcal{B} = \{(1, 0, 1), (2, 0, 0), (-3, 1, 1)\}$ di $\mathbb{R}^3$ rispetto a dominio e codominio.

   Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear application defined as: $f(x, y, z) = (x - z, 2x + y, 2y + z)$. Find the matrix representing f with respect to the basis $\mathcal{B} = \{(1, 0, 1), (2, 0, 0), (-3, 1, 1)\}$ of R^3 with respect to the domain and the codomain.

7. (4 punti) Determinare gli autovalori e una base per gli autospazi per la seguente matrice:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

   Stabilire se la matrice è diagonalizzabile.

   Find the eigenvalues and the eigenspaces of the following matrix:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

   Find if the matrix is diagonalizable.

8. (4 punti) A partire dei vettori $(1, 0, 0), (2, 1, 1), (0, 1, -2)$, calcolare una base ortonormale. Calcolare le componenti del vettore $v = (2, 1, 4)$ rispetto alla base ortonormale.

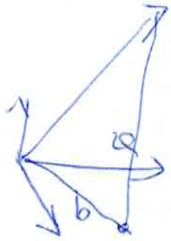
   Given the vectors $(1, 0, 0), (2, 1, 1), (0, 1, -2)$, Find an orthonormal basis. Find the coordinates of the vector $v = (2, 1, 4)$ with respect the orthonormal basis.

9. (4 punti) Determinare la matrice che rappresenta la forma quadratica $q(x, y, z) = x^2 + 3y^2 - 4xz + 4z^2$ e stabilire il segno della forma quadratica.

   Find the matrix associated to the quadratic form $q(x, y, z) = x^2 + 3y^2 - 4xz + 4z^2$ and find the sign of the quadratic form.

(1)

$u = (1, 2, 1)$



$$a \times b = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k} = (-1, 2, -1)$$

$$|a \times b|^2 = 1 + 4 + 1 = 6$$

$$w = u - \left\langle u, \frac{a \times b}{|a \times b|} \right\rangle \frac{(a \times b)}{|a \times b|} =$$

$$= (1, 2, 1) - \frac{(-1 + 4 - 1)}{6} (-1, 2, -1)$$

$$= (1, 2, 1) - \frac{1}{3}(-1, 2, -1) = \left(1 + \frac{1}{3}, 2 - \frac{2}{3}, 1 + \frac{1}{3}\right)$$

$$= \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

② $A = \left[\underbrace{\begin{pmatrix}2\\0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\-2\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}}_{\text{base}}\right]$

$\dim A = 3$ per chè $\operatorname{rk}\left(\begin{pmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & -2 & 1\\1 & 0 & -1\end{pmatrix}\right) = 3$

$$\det\begin{pmatrix}2 & 0 & 0\\0 & -2 & 1\\1 & 0 & -1\end{pmatrix} = 2\begin{vmatrix}-2 & 1\\0 & -1\end{vmatrix} = 4 \neq 0$$

$B = \left[\underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{\text{base}} \begin{pmatrix}1\\2\\0\\0\end{pmatrix}\right]$ $\dim B = 2$

$$\begin{pmatrix}\boxed{\begin{matrix}0 & 1\\1 & 1\end{matrix}} & \begin{matrix}1\\2\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 0\\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix}$$

$\det\begin{pmatrix}0 & 1\\1 & 1\end{pmatrix} = -1 \neq 0$

$\begin{vmatrix}0 & 1 & 1\\1 & 1 & 2\\0 & 0 & 0\end{vmatrix} = 0$

$$A+B = \left[\underbrace{\begin{pmatrix}2\\0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\-2\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{\text{generatori}}\right] = \left[\underbrace{\begin{pmatrix}2\\0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\-2\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{\text{base}}\right]$$

$\dim A+B = 4$

$$\det\begin{pmatrix}2 & 0 & 0 & 1\\0 & 0 & 1 & 1\\0 & -2 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix} = -\begin{vmatrix}2 & 0 & 1\\0 & -2 & 0\\1 & 0 & 0\end{vmatrix} = 2\begin{vmatrix}2 & 1\\1 & 0\end{vmatrix} = -2 \neq 0$$

$$4 = \dim(A+B) = \underset{3}{\dim A} + \underset{2}{\dim B} - \dim(A \cap B)$$

$\Rightarrow \dim(A \cap B) = 1 \quad \Rightarrow$ $A+B$ non è somma diretta

(3)

$$A = \begin{pmatrix} 4 & 1-k \\ 2k & -1 \end{pmatrix}$$

$$\begin{vmatrix} 4 & 1-k \\ 2k & -1 \end{vmatrix} = -4-(1-k)2k = -4-2k+2k^2$$

$$= 2k^2-2k-4$$

$$k = \frac{2 \pm \sqrt{4+32}}{4} = \begin{cases} \frac{2+6}{4} = 2 \\ \frac{2-6}{4} = -1 \end{cases}$$

Per $k \neq 2$ e -1 la matrice è invertibile

$$A^{-1} = \frac{1}{2k^2-2k-4} \begin{pmatrix} -1 & k-1 \\ -2k & 4 \end{pmatrix}$$

(4) $\begin{cases} -2x + y - z = k \\ 2x - 2y + kz = 0 \\ 5x - 3y + z = 0 \end{cases}$

$$A = \begin{pmatrix} -2 & 1 & -1 \\ 2 & -2 & k \\ 5 & -3 & 1 \end{pmatrix}$$

$$\det A = -2(-2+3k) - (2-5k) - 1(-6+10)$$
$$= -k - 2$$

- per $k \neq -2$ $\quad r(A) = 3 \quad r(A|b) = 3$
sistema di Cramer $\Rightarrow$ una sola soluzione

- per $k = -2$ $\quad \begin{vmatrix} -2 & 1 \\ 2 & -2 \end{vmatrix} = 4 - 2 \neq 0 \quad r(A) = 2$

$$r(A|b) = 3$$

$$(A|b) = \begin{pmatrix} -2 & 1 & -1 & k \\ 2 & -2 & k & 0 \\ 5 & -3 & 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -2 & 1 & -2 \\ 2 & -2 & 0 \\ 5 & -3 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 \\ 5 & -3 \end{vmatrix} =$$
$$= -2(4) = -8 \neq 0$$

sistema impossibile per $k = 2$

Nel caso $k \neq -2$

$$x = \frac{\begin{vmatrix} k & 1 & -1 \\ 0 & -2 & k \\ 0 & -3 & 1 \end{vmatrix}}{-k-2} = \frac{k(2-3k)}{k+2}$$

$$y = \frac{\begin{vmatrix} -2 & k & -1 \\ 2 & 0 & k \\ 5 & 0 & 1 \end{vmatrix}}{-k-2} = \frac{k(2-5k)}{k+2}$$

$$z = \frac{\begin{vmatrix} -2 & 1 & k \\ 2 & -2 & 0 \\ 5 & -3 & 0 \end{vmatrix}}{-k-2} = \frac{k(4)}{-k-2} = \frac{-4k}{k+2}$$

5

$F: R^2 \to R^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x-y \\ 0 \\ 3x-2y \end{pmatrix}$$

$$A_F = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -2 + 3 \neq 0$$

$\dim \operatorname{Imm} F = 2$

base di $\operatorname{Imm} F$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \right\}$$

$\dim \operatorname{Ker} f = 2 - 2 = 0$

$\operatorname{Ker} f = \{0\}$

$G: R^3 \to R^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ 0 \\ x-z \end{pmatrix}$$

$$A_G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$G \circ F: R^2 \to R^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to C \begin{pmatrix} x \\ y \end{pmatrix}$$

$$C = A_G \, A_F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -2 & 1 \end{pmatrix}$$

⑥ $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x - z \\ 2x + y \\ 2y + z \end{pmatrix}$$

$$\mathcal{B} = \{(1,0,1),(2,0,0)(-3,1,1)\}$$

$$M_C^C(f) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad M_C^B(i_{\mathbb{R}^3}) = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_B^B(f) = M_B^C(i_{\mathbb{R}^3})\, M_C^C(f)\, M_C^B(i_{\mathbb{R}^3})$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \quad \begin{aligned} x &= a + 2b - 3c \\ y &= c \\ z &= a + c \end{aligned}$$

$$\begin{cases} a = z - y \\ b = \frac{1}{2}(x - z + y + 3y) \\ c = y \end{cases} \qquad \begin{cases} a = z - y \\ b = \frac{x}{2} - \frac{z}{2} + 2y \\ c = y \end{cases}$$

$$M_B^C(i_{\mathbb{R}^3}) = \begin{pmatrix} 0 & -1 & 1 \\ \frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & -1 & 1 \\ \frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -4 & 8 \\ 3.5 & 9 & -13.5 \\ 2 & 4 & -5 \end{pmatrix}$$

(7)

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & -1 & 0 \\ 0 & \lambda-3 & 0 \\ -3 & 0 & \lambda \end{vmatrix} = \lambda\,(\lambda-3)^2$$

$\lambda = 0 \qquad m.a = 1$

$\lambda = 3 \qquad m.a = 2$

$$V_0 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} -3x - y = 0 \\ -3y = 0 \\ -3x = 0 \end{matrix} \right\} = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \quad m.g.\ 1$$

$$V_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{matrix} -y = 0 \\ -3x + 3z = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} y = 0 \\ x = z \end{matrix} \right\} = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

$m.g.\ 1$

matrice non diagonalizzabile

(8) $v_1 = (1,0,0) \quad v_2 = (2,1,1) \quad v_3 = (0,1,-2)$

$v_1' = v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |v_1'| = 1$

$v_2' = v_2 - \frac{\langle v_1', v_2\rangle}{\langle v_1', v_1'\rangle} v_1' = \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{2}{1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} \qquad |v_2'| = \sqrt{2}$

$v_3' = v_3 - \frac{\langle v_1' v_3\rangle}{\langle v_1' v_1'\rangle} v_1' - \frac{\langle v_2' v_3\rangle}{\langle v_2' v_2'\rangle} v_2' =$

$= \begin{pmatrix}0\\1\\-2\end{pmatrix} - \frac{0}{1}\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\3/2\\-2+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\3/2\\-3/2\end{pmatrix}$

$|v_3'| = \sqrt{\frac{9}{4}+\frac{9}{4}} = \frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{2}}$

Base ortonormale

$\underset{v_1''}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \quad \underset{v_2''}{\begin{pmatrix}0\\1/\sqrt{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix}} \quad \underset{v_3''}{\begin{pmatrix}0\\ \frac{\cancel{3}\sqrt{2}}{2\cdot\cancel{3}}\\ \frac{-\cancel{3}\sqrt{2}}{2\cancel{3}}\end{pmatrix}} \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

$v = (2,1,4)$

$v_x = \langle v, v_1''\rangle = 2 \qquad v_y = \langle v, v_2''\rangle = \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

$v_z = \langle v, v_3''\rangle = \frac{1}{\sqrt{2}} - \frac{4}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$

(8) $q(x,y,z) = \cancel{8}x^2 + 3y^2 - 4xz + 4z^2$

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 2 & 0 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-3)\left[(\lambda-1)(\lambda-4) - 4\right]$$

$$= (\lambda-3)(\lambda^2 - 5\lambda + 4 - 4) =$$

$$= (\lambda-3)\,\lambda\,(\lambda-5)$$

$\lambda = 3$
$\lambda = 0$
$\lambda = 5$

forma semidefinito positivo

$$V_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} 2x + 2z = 0 \\ 2x + z = 0 \end{matrix} \right\} = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \qquad |v_1| = 1$$

$$V_0 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} -x + 2z = 0 \\ 3y = 0 \\ 2x - 4z = 0 \end{matrix} \right\} = \left[\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right] \qquad |v_2| = \sqrt{5}$$

$$V_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} 4x + 2z = 0 \\ 2y = 0 \\ 2x + z = 0 \end{matrix} \right\} = \left[\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right]$$

Base

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$