

Fourier Transform Example

SOLOMON TESSEMA

December 19, 2024

Fourier Transform of a Sine Wave

Given a sine wave in the time domain:

$$x(t) = \sin(2\pi f_0 t)$$

where $f_0 = 5$ Hz, we compute its Fourier Transform using the formula:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Step 1: Rewrite the Sine Wave Using Euler's Formula

Using Euler's formula:

$$\sin(2\pi f_0 t) = \frac{1}{2j} \left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t} \right)$$

Substitute this into the Fourier Transform formula:

$$X(f) = \int_{-\infty}^{\infty} \frac{1}{2j} \left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t} \right) e^{-j2\pi f t} dt$$

Step 2: Simplify the Expression

Distribute $e^{-j2\pi f t}$:

$$X(f) = \frac{1}{2j} \left(\int_{-\infty}^{\infty} e^{j2\pi(f_0-f)t} dt - \int_{-\infty}^{\infty} e^{-j2\pi(f_0+f)t} dt \right)$$

Step 3: Solve Each Integral

The Fourier Transform of a complex exponential is:

$$\int_{-\infty}^{\infty} e^{j2\pi\alpha t} dt = \begin{cases} \infty, & \text{if } \alpha = 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. For $\int_{-\infty}^{\infty} e^{j2\pi(f_0-f)t} dt$:

This evaluates to $\delta(f - f_0)$.

2. For $\int_{-\infty}^{\infty} e^{-j2\pi(f_0+f)t} dt$:

This evaluates to $\delta(f + f_0)$.

Step 4: Combine Results

The Fourier Transform of $x(t) = \sin(2\pi f_0 t)$ is:

$$X(f) = \frac{1}{2j} (\delta(f - f_0) - \delta(f + f_0))$$

Conclusion

In the frequency domain, the sine wave has two peaks:

- A positive peak at $f = f_0$ (5 Hz).
- A negative peak at $f = -f_0$ (-5 Hz).

This representation shows that the signal oscillates at these two frequencies, and the Fourier Transform breaks down the time-domain signal into its frequency components.