Fourier Transform Example

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Fourier Transform of a Sine Wave

Given a sine wave in the time domain:

$$x(t) = \sin(2\pi f_0 t)$$

where $f_0 = 5\,\text{Hz}$, we compute its Fourier Transform using the formula:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

Step 1: Rewrite the Sine Wave Using Euler's Formula

Using Euler's formula:

$$\sin(2\pi f_0 t) = \frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)$$

Substitute this into the Fourier Transform formula:

$$X(f) = \int_{-\infty}^{\infty} \frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right) e^{-j2\pi f t}\, dt$$

Step 2: Simplify the Expression

Distribute $e^{-j2\pi f t}$:

$$X(f) = \frac{1}{2j}\left(\int_{-\infty}^{\infty} e^{j2\pi(f_0 - f)t}\, dt - \int_{-\infty}^{\infty} e^{-j2\pi(f_0 + f)t}\, dt\right)$$

Step 3: Solve Each Integral

The Fourier Transform of a complex exponential is:

$$\int_{-\infty}^{\infty} e^{j2\pi\alpha t}\, dt = \begin{cases} \infty, & \text{if } \alpha = 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. For $\int_{-\infty}^{\infty} e^{j2\pi(f_0 - f)t}\, dt$:

$$\text{This evaluates to } \delta(f - f_0).$$

2. For $\int_{-\infty}^{\infty} e^{-j2\pi(f_0 + f)t}\, dt$:

$$\text{This evaluates to } \delta(f + f_0).$$

Step 4: Combine Results

The Fourier Transform of $x(t) = \sin(2\pi f_0 t)$ is:

$$X(f) = \frac{1}{2j} \left(\delta(f - f_0) - \delta(f + f_0) \right)$$

Conclusion

In the frequency domain, the sine wave has two peaks:

- A positive peak at $f = f_0$ (5 Hz).
- A negative peak at $f = -f_0$ (-5 Hz).

This representation shows that the signal oscillates at these two frequencies, and the Fourier Transform breaks down the time-domain signal into its frequency components.