Sample Latex: Proof of Integration by Parts in Multivariable Calculus

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Introduction

Integration by parts in multivariable calculus can be demonstrated using the divergence theorem, which relates the flux of a vector field through a surface to the divergence of the field inside the volume bounded by the surface.

Theorem (Integration by Parts)

Let u and v be scalar functions defined on a domain D in $\mathbb{R}^3$ with smooth boundary ∂D, and let $\mathbf{F}$ be a vector field on D. If u, v, and $\mathbf{F}$ are differentiable, then

$$\int_D u\nabla \cdot (v\mathbf{F})\, dV = \int_{\partial D} uv\mathbf{F} \cdot d\mathbf{S} - \int_D v\nabla u \cdot \mathbf{F}\, dV$$

Proof

The proof employs the divergence theorem, which states

$$\int_D \nabla \cdot \mathbf{G}\, dV = \int_{\partial D} \mathbf{G} \cdot d\mathbf{S}$$

for any continuously differentiable vector field $\mathbf{G}$ on D.

Choosing $\mathbf{G} = uv\mathbf{F}$, we have

$$\nabla \cdot \mathbf{G} = \nabla \cdot (uv\mathbf{F}) = u(\nabla v \cdot \mathbf{F}) + v(\nabla u \cdot \mathbf{F}) + uv(\nabla \cdot \mathbf{F})$$

Applying the divergence theorem yields

$$\int_D \nabla \cdot (uv\mathbf{F})\, dV = \int_{\partial D} uv\mathbf{F} \cdot d\mathbf{S}$$

Expanding the divergence on the left side gives

$$\int_D u(\nabla v \cdot \mathbf{F}) + v(\nabla u \cdot \mathbf{F}) + uv(\nabla \cdot \mathbf{F})\, dV = \int_{\partial D} uv\mathbf{F} \cdot d\mathbf{S}$$

Rearranging terms provides the formula for integration by parts in multivariable calculus.

Conclusion

This theorem generalizes the concept of integration by parts to the realm of multivariable calculus, demonstrating the interplay between differential and integral calculus in higher dimensions.