Proof for Orthogonality of Eigenvectors

Solomon Tessema
Zen Math

March 31, 2024

Proof

Given a real symmetric matrix A where $A = A^T$, let v_1 and v_2 be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 respectively, such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

To prove that v_1 and v_2 are orthogonal, consider the following steps:

1. Start with the scalar product $v_1^T(Av_2)$. Since $Av_2 = \lambda_2 v_2$, we can substitute to get:

$$v_1^T(Av_2) = v_1^T(\lambda_2 v_2) = \lambda_2(v_1^T v_2)$$

2. Similarly, consider the scalar product $(Av_1)^T v_2$. Since $Av_1 = \lambda_1 v_1$, we have:

$$(Av_1)^T v_2 = (\lambda_1 v_1)^T v_2 = \lambda_1(v_1^T v_2)$$

3. By the property of symmetry of A ($A = A^T$), it follows that $v_1^T(Av_2)$ is equal to $(Av_1)^T v_2$, thus:

$$\lambda_2(v_1^T v_2) = \lambda_1(v_1^T v_2)$$

4. Given that $\lambda_1 \neq \lambda_2$, the equation above can only hold if $v_1^T v_2 = 0$, which means that v_1 and v_2 are orthogonal.

This concludes the proof that eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal to each other.