Proof that Subgroups of Cyclic Groups are Cyclic

Proof

Let G be a cyclic group generated by an element a, i.e., $G = \langle a \rangle$. We aim to show that any subgroup H of G is also cyclic.

1. If $H = \{e\}$, where e is the identity element of G, then H is trivially cyclic.
2. Assume H is non-trivial. Let n be the smallest positive integer such that $a^n \in H$. We claim that $H = \langle a^n \rangle$.
3. Consider any other element $a^k \in H$. Since $k > n$ (as n is the smallest integer for which $a^n \in H$), we can write $k = nq + r$, where q and r are integers and $0 \leq r < n$.
4. We have $a^k = a^{nq+r} = (a^n)^q a^r$. Since a^k and $(a^n)^q$ are in H and H is a subgroup (and thus closed under group operation and inverses), it follows that a^r must also be in H.
5. However, because n was chosen as the smallest positive integer for which $a^n \in H$, the only way for a^r to also be in H (given that $0 \leq r < n$) is if $r = 0$.
6. Therefore, $k = nq$, which implies that every element in H can be written as $(a^n)^q$ for some integer q. Thus, H is generated by a^n and is cyclic.