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COURSE: MAT 251

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(1) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Solution

$\lim_{x \to 0} \frac{\sin x}{x}$

differentiate

$\lim_{x \to 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x}$

$\lim_{x \to 0} \frac{\cos x}{1}$

$\lim_{x \to 0} \frac{\cos(0)}{1} = \frac{1}{1} = 1$

(2) Show that $\begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

is continuous but not differentiable at the origin

Solution

to solve $\lim_{x \to 0} x \sin(\frac{1}{x})$

Since Sin function Osculate between -1 and 1

we can say $-1 \leq \sin f(x) \leq 1$ also

$-1 \leq \sin(\frac{1}{x}) \leq 1$

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Multiply through by x to get
$$-x \leq x \sin(1/x) \leq x$$

Considering $\lim_{x \to 0} x \sin(1/x)$

$$\lim_{x \to 0} (-x) \leq \lim_{x \to 0} x \sin(1/x) \leq \lim_{x \to 0} (x)$$

$$0 \leq \lim_{x \to 0} x \sin(1/x) \leq 0$$

Using squeeze theorem
$$\lim_{x \to 0} x \sin(1/x) = 0$$

Since $\lim_{x \to 0} x \sin(1/x) = f(0) = 0$ (are equal)

hence this shows that function $f(x)$ defined as $x \sin 1/x$ is continuous at $x = 0$.

Same principle apply with: 0; $x = 0$ such that $\lim_{x \to 0} [0] = f(0) = 0$ which shows that $f(x)$ is continuous

Now to Show that the function is not differen-
tiable at the origin ie $x = x_0 = 0$

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

Then it follows that at the Origin where $x = 0$

hence $f'(0) = \lim_{x \to 0} \frac{f(0+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

recall $f(x) = x \sin(1/x)$ hence $f(h) = h \sin(1/h)$

$$\lim_{h \to 0} \frac{h \sin(1/h) + 0 \sin(1/0)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h} \sin(1/h) - 0}{\cancel{h}}$$

$$\lim_{x \to 0} \sin(1/h)$$

as h approach 0, the expression $\sin(1/h)$ oscillate infinitely between -1 and 1, therefore

$\lim_{x \to 0} \sin(1/h)$ does not exist

Since it does not exist,

hence $\underline{f(x) \text{ is not differentiable}}$

(3) Use the mean value theorm to prove that if $x > 0$, then $\ln(1+x) < x$

Solution

$\ln(1+x) < x$

$\ln(\cancel{1}+x) - x < 0$

Let $f(x) = \ln(1+x) - x$ Suchthat

$f(0) = \ln(1+x) - 0 = 0$

Since $f(0) = \ln(1+0) - 0 = \ln(1) - 0 = 0 - 0 = 0$

hence $f'(c) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x) - 0}{x}$

$f'(c) = \dfrac{f(x)}{x}$

Recall: $f(x) = \ln(1+x) - x$

$f'(x) = \dfrac{1}{1+x} - 1 = \dfrac{1-1-x}{1+x} = \dfrac{-x}{1+x}$

$f'(c) = -x/1+x$

$f'(c) = \dfrac{-c}{1+c} = f(x)$ Where $f(x)' < f(x)'$

$f'(c) = \dfrac{f(x)}{x} = f'(x)$

$\dfrac{-c}{1+c} = \dfrac{f(x)}{x} = \dfrac{-x}{1+x}$

$$\frac{-c}{1+c} = \frac{-x}{1+x}$$

$$\frac{-c}{1+c} < \frac{-x}{1+x}$$

$$-c < \frac{-x}{1+x}(1+c)$$

$$-c < \frac{-x}{1+x} - \frac{xc}{1+x}$$

$$-c(1+x) < -x - xc$$

$$-c - \cancel{cx} < -x - \cancel{xc}$$

$$-c < -x$$

$$c < x$$

hence $f'(c) < 0 \ \forall x > 0$,

We have shown that $f(x)$ is a decreasing function on the interval $(0, x)$ since $f'(c) < 0$ then $f(x) < 0$ where $f(x) = \ln(1+x) - x \Rightarrow \ln(1+x) - x < 0$

$\Rightarrow \underline{\underline{\ln(1+x) < x}}$