Minimum Jerk Trajectory Generation for Differential Wheeled Mobile Robots

R. Palamakumbura, D.H.S. Maithripala, C. F. Martin

***Abstract*—In this paper we present a trajectory generation scheme for approximate way point tracking for a nonholonomic differential wheeled mobile robot. The generated trajectories are guaranteed to satisfy the nonholonomic dynamics and minimizes the jerk of the center of mass motion of the robot. Minimizing jerk is desirable to ensure the smoothness of the trajectories and is often necessary to ensure physical hardware limitations of the electromagnetic actuators, prevent undesirable induced vibrations, and ensure comfort of the passengers. The problem is naturally formulated in the setting of a constrained optimal control problem. The paper presents a way of reducing the computationally intensive infinite dimensional optimal control solution to that of a finite dimensional nonlinear programming problem that can be easily solved using any of the existing fast numerical algorithms and thus making the scheme realtime implementable.**

Keywords*—*Jerk, Mobile Robots, Motion Planning, Optimal Trajectory

I. Introduction

Trajectory planning is a crucial aspect in robotics, aeronautics, space traveling, navigation and biomechanics. For instance when a mobile robot needs to move from one place to another, it may not be possible to reach the destination by traveling in a straight line due to the existence of obstacles or due to the requirement that it stop at some intermediate point to perform a certain task. This is usually not an easy task since there may be limitations on available resources, speed, acceleration and jerk that need to be considered as well. The term *motion planning* is used to refer to this type of computational process of moving an object from one place to another subject to one or more constraints. The collection of article presented in [1] and the texts [2], [3], [4] provide a historical overview and a collection of approaches to motion planning of mobile robots.

In order to avoid confusion between terms often used as synonyms, the difference between a *path* and a *trajectory* needs to be clarified. A path denotes the locus of points in space that the object has to follow in the execution of the assigned motion. Thus a path is a purely geometric description of motion. On the other hand, a trajectory is a path on which a time law is specified. For example, the velocity and the acceleration information along a path may be specified. Thus path planning is the determination of the geometry of the motion and the trajectory planning is the determination of the time history of the motion.

Typically the path planning phase precedes the trajectory planning phase and the offline planned paths are fed into a trajectory planning algorithm that converts the path information into trajectory information. The most general version of the problem is that the complete path is specified. The other, and simpler version is that, we are given a sequence of target points as the path constraints and we are required to reach target points at specified target times in an approximate sense. This approach is most suitable for trajectory planning in the presence of obstacles. In this paper we focus on the realtime trajectory planning problem where we seek to convert the path information prescribed in terms of path constraints directly to that of a control law. Such a control law can either be used in realtime to ensure that the motion of the robot satisfies the path constraints or to generate dynamically feasible trajectories that satisfy the path constraints. Here dynamic feasibility means that the trajectories satisfy the nonlinear dynamics of the system. Such problems are naturally formulated as a *constrained optimization problem*. We refer the reader to the texts of [2], [5], [3], [4] and [6] for a review of these ideas.

A review of the recent state of the art in trajectory planning for robotic manipulators can be found in [5]. Much of the hitherto existing literature is concerned with time-optimality criteria. Consideration of minimum energy criteria is less common with [7], [6], [5] providing an overview of the existing methods. Smoothness of trajectories are also preferred since actuators, which are typically electro-mechanical systems, are incapable of generating large velocities, accelerations, and large sudden changes in accelerations (jerk) [8]. Jerk, the third time derivative of the position also plays a crucial role in trajectory planning in robotic manipulators where the magnitude of jerk is related to joint errors [9], structural vibrations of the robots [8], and passenger comfort [10]. An excellent review of the existing methods of path planning for non-holonomic mobile robots with such curvature constraints can be found in [11]. They also propose a continuous curvature steering method for car-like vehicles that upper bounds the curvature and its derivative. Being a path planning method the approach requires a subsequent trajectory planner to convert the path information into trajectory information. To the best of our knowledge there exists no direct trajectory planning scheme for mobile robots that consider minimum jerk of the center of mass motion of the robot.

Furthermore all the above methods for path and trajectory planning only consider equality type constraints on the output. This turns out to be too restrictive when either it is not necessary for the output to exactly be equal to a given desired

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output and/or if the desired final output is not known exactly. The former condition typically arises in trajectory planning for obstacle avoidance, where a trajectory that avoids the obstacles can be defined using a set of waypoints the robot needs to pass at given time instances. In this case passing exactly though the given waypoints is not necessary and one can relax it by requiring that the robot passes only through a given neighborhood of the given waypoint at the given time instance. For linear single input systems [12], [13] considers inequality constraints on the final state where they propose the use of standard Lagrange multiplier methods to solve a energy optimal problem subject to the constraints of linear dynamics and nonlinear inequality constraints on the output at pre-defined time instances. This work was generalized by the authors to multi-input-multi-output (MIMO) linear systems in [14] where necessary and sufficient conditions are proved for the existence of a unique solution. The computationally intensive infinite dimensional constrained optimal control problem is converted in [12], [13], [14] to that of a much simpler finite dimensional nonlinear programming problem that can be easily solved, on a time scale that is much smaller than the time scale at which the reference gets updated, using any of the existing algorithms such as those presented in [15].

In this work we show how our previous results of [14] for linear MIMO systems with cost on control energy can be extended to generate realtime trajectories and controls for a nonlinear non-holonomic mobile robot that minimizes the jerk of the center of mass motion of the robot while ensuring that the robot passes through pre-defined neighborhoods of a given set of waypoints at given specific time instances. To the best of our knowledge it is the first time that such a result has been presented.

In section-II we will formally formulate and solve the problem of minimum jerk trajectory planning problem for nonholonomic differential wheeled mobile robots with approximate constraints on the trajectory. The scheme is demonstrated using simulations in section-III and the conclusions are presented in section-IV.

II. Optimal Jerk Trajectory Planning for Mobile Robots

Consider the differential wheeled mobile robot with configuration co-ordinates $q = (x, y, \theta) \in \mathrm{SE}(2)$. Here (x, y) denote the co-ordinates of the center of mass of the mobile robot and θ denotes the orientation of the robot with respect to an inertial frame. Let $m = (m_b + m_w), I = (m_b d^2 + 2m_w l^2 + I_b + 2I_p)$ where m_b is the mass of the robot structure, m_w is the mass of a wheel, I_b is the moment of inertia of the robot structure about the vertical axis through the center of mass of the robot, r_w is the radius of a wheel, l is the distance between the vertical axis of the robot through the center of mass of the robot and the vertical axis though the wheel, and I_p, I_w are the moments of inertia of a wheel about its symmetry axis. The objective is to design an optimal waypoint tracking controller that will minimize a quadratic cost on the jerk of the center of mass motion of the robot while constrained to be within a certain given disk around the waypoint at a given specific time.

The constrained equations of motion of the nonholonomic robot expressed in the body frame is

$$\begin{aligned} \dot{x} &= v\cos\theta, && (1)\\ \dot{y} &= v\sin\theta, && (2)\\ \dot{\theta} &= \omega, && (3)\\ \dot{v} &= \frac{1}{m} f, && (4)\\ \dot{\omega} &= \frac{1}{I} u_\omega && (5)\\ \dot{f} &= u_{\dot{f}}, && (6) \end{aligned}$$

The output of the system is the position of the center of mass $Y = [x \quad y]^T$ and the inputs are $u = [u_{\dot{f}} \quad u_\omega]^T$.

Given a desired set of trajectory points

$$\mathcal{S} = \{(t_i, \alpha_i); i = 1 \cdots N\}$$

and corresponding disks about the waypoints given by $B_i = \{z : \|z - \alpha_i\| \le a_i\}$ with radius a_i, we seek to find $u = [u_{\dot{f}} \quad u_\omega]^T$ such that it minimizes a quadratic cost function of the jerk of the center of mass motion of the robot given by

$$J(u) \triangleq \int_0^T ||\dddot{Y}(t)||^2 \, dt, \quad (7)$$

subjected to the quadratic constraint

$$Y(t_i) \in B_i \Leftrightarrow (Y(t_i) - \alpha_i)^T (Y(t_i) - \alpha_i) \le a_i^2. \quad (8)$$

This constraint ensures that the mobile robot will pass through the disk of radius a_i cantered at the waypoint α_i at the time instant t_i. We will refer to the above problem as the *minimum jerk approximate way point tracking problem for non-holonomic mobile robots*.

Differentiating Y three times we obtain the following expression for the jerk of the **center** of mass motion of the robot

$$\dddot{Y} = \begin{bmatrix} -\frac{2}{m} f\omega\sin\theta - v\omega^2\cos\theta \\ \frac{2}{m} f\omega\cos\theta - v\omega^2\sin\theta \end{bmatrix} + \begin{bmatrix} \frac{\cos\theta}{m} & -\frac{v\sin\theta}{I} \\ \frac{\sin\theta}{m} & \frac{v\cos\theta}{I} \end{bmatrix} \begin{bmatrix} u_{\dot{f}} \\ u_\omega \end{bmatrix}. (9)$$

Then it follows that the linearizing controls $u = [u_{\dot{f}} \quad u_\omega]^T$ given by

$$\begin{bmatrix} u_{\dot{f}} \\ u_\omega \end{bmatrix} = \begin{bmatrix} mv\omega^2 \\ -\frac{2I}{m}\frac{f\omega}{v} \end{bmatrix} + \begin{bmatrix} m\cos\theta & m\sin\theta \\ -\frac{I}{v}\sin\theta & \frac{I}{v}\cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (10)$$

transforms (1)—(6) into the linear system

$$\begin{aligned} \dot{X} &= AX + Bu_L, && (11)\\ Y &= CX, && (12) \end{aligned}$$

where

$$A = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\1&0\\0&1 \end{bmatrix}$$

$$X = \begin{bmatrix} Y \\ \dot{Y} \\ \ddot{Y} \end{bmatrix}, \; C = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{bmatrix}, \; u_L = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Thus it follows easily that, when the heading velocity of the robot v is non-zero, the controller (10) with $u_L = [u_1 \quad u_2]^T$ chosen such that it minimizes the cost function

$$J_L(u) \triangleq \int_0^T ||u_L(t)||^2 dt, \tag{13}$$

subjected to the linear dynamics (11) — (12) and the quadratic constraint (8) will minimize (7) subject to the nonlinear dynamics (1)—(6) and the quadratic constraints (8).

The construction of the optimal control for the linear system will be briefly presented in the next section.

A. Construction of the Linear Optimal Controller

The problem of finding the control u_L such that the cost function (13) is minimized subject to the linear dynamics (11) — (12) and the quadratic constraint (8) is in the form of a *constrained optimization problem*. It can be shown that the cost functional $J(u)$, given by (13), is a convex functional on $\mathcal{H}$ and the constraint (8) is a convex functional from $\mathcal{H}$ to $\mathbb{R}$ where $\mathcal{H}$ is the Hilbert space of measurable functions $u : [0,T] \mapsto \mathbb{R}$ such that the inner product $< u,u >$ defined by

$$< u, w >= \int_0^T u^T(t) w(t) dt,$$

for $u, w \ \in \mathcal{H}$ is finite. The convexity implies the existence and uniqueness of the solution and the Lagrange multiplier method turns out to be the most suitable method to solve such problems [16]. The problem was first solved explicitly by the authors in [14] when the initial conditions were zero. Since the minimum jerk optimal trajectory planning problem for the differential wheeled mobile robot is equivalent to the above optimal control problem for the linear system only when the heading velocity is non-zero the solution provided in [14] needs to be suitably altered to allow non-zero initial conditions. Since the derivation is similar to the case with zero initial condition we briefly outline the derivation and refer the reader to [14] for the details of the derivation.

Using the method of Lagrange Multipliers, the optimal control problem can be reformulated by constructing the Lagrangian [16],

$$H(u,\lambda) = \int_0^T u(t)^T u(t) dt + \sum_{i=1}^N \lambda_i \left(|| \left(Y(t_i) - \alpha_i\right) ||^2 - a_i^2 \right) \tag{14}$$

where λ_i's are the Lagrange multipliers. Then solving the constrained optimal control problem defined above is equivalent to solving the unconstrained optimal control problem

$$\max_{\lambda \geq 0} \min_u H(u,\lambda). \tag{15}$$

Here $\lambda \geq 0$ means that for each i, $\lambda_i \geq 0$.

It can be shown that using the following substitutions one can re-formulate the above infinite dimensional optimization problem as a finite dimensional optimization problem. Let Λ be the block diagonal matrix of Lagrange multipliers given by

$$\Lambda = \begin{pmatrix} \lambda_1 I_m & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 I_m & 0 & \cdots & 0 \\ & & & & \\ 0 & \cdots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda_N I_m \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix,

$$L_i(s) = \begin{cases} Ce^{(t_i - s)A} B & t_i - s > 0 \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

and

$$\tau_i \triangleq Y(t_i) - \alpha_i = Ce^{At_i} x_0 + \int_0^T L_i(s) u^*(s) ds - \alpha_i.$$

Here u^* is the optimal control minimizing the cost (13). Further define the grammian matrix, G,

$$G = \begin{pmatrix} \int_0^T L_1(t) L_1^T(t) dt & \cdots & \int_0^T L_1(t) L_N^T(t) dt \\ \vdots & \ddots & \vdots \\ \int_0^T L_N(t) L_1^T(t) dt & \cdots & \int_0^T L_N(t) L_N^T(t) dt \end{pmatrix}, \tag{17}$$

and

$$G_i = \left(\int_0^T L_i(t) L_1^T(t) dt \cdots \int_0^T L_i(t) L_N^T(t) dt \right). \tag{18}$$

Using these notations, it can be shown [14] that one can re-write the Lagrangian of the problem as

$$\begin{aligned} H(\tau,\lambda) \quad &= \quad \tau^T \Lambda G \Lambda \tau \\ &\quad + \sum_{i=1}^N \lambda_i \Big(\tau^T \Lambda G_i^T G_i \Lambda \tau + 2\alpha_i^T G_i \Lambda \tau + \alpha_i^T \alpha_i - a_i^2 \\ &\quad + ((Ce^{At_i} x_0)^T (2 G_i \Lambda \tau + \alpha_i + Ce^{At_i} x_0)) \Big). \end{aligned} \tag{19}$$

From the necessary condition [16] that $H(\tau,\lambda)$ be stationary at the optimal solution with respect to τ and λ, we have that the optimal τ^* is given by

$$\begin{aligned} \tau^* \quad &= \quad -[\sum_{i=1}^N \lambda_i \Lambda G_i^T G_i \Lambda + \Lambda G \Lambda]^{-1} \\ &\quad \sum_{i=1}^N \lambda_i [\Lambda G_i^T \alpha_i + \Lambda G_i^T C e^{At_i} x_0]. \end{aligned} \tag{20}$$

and that the optimal λ^* must necessarily satisfy

$$\begin{aligned} &2\tau^T \Lambda G I_i \tau + 2 \sum_{j=1}^N \lambda_j \left(\tau^T I_i G_j^T (G_j \Lambda \tau + \alpha_j) + (Ce^{At_j} x_0)^T G_j I_i \tau \right) \\ &+ (G_i \Lambda \tau + \alpha_i)^T (G_i \Lambda \tau + \alpha_i) - a_i^2 \\ &+ (Ce^{At_i} x_0)^T (2 G_i \Lambda \tau + 2\alpha_i + Ce^{At_i} x_0) \\ &= 0, \end{aligned} \tag{21}$$

where I_i is the $Nm \times Nm$ block diagonal matrix whose i^{th} $m \times m$ diagonal block is the $m \times m$ identity matrix I_m for $i = 1, 2, \cdots, N$ and other entries are 0. Hence we have a system with N equations for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ and solving the system (21) with the expression for optimal τ we can find the optimal λ and τ. Then the optimal controller is given by

$$u^*(t) = -\sum_{i=1}^{N} \lambda_i^* L_i^T(t) \tau_i^*. \tag{22}$$

We formally state the above result as a lemma below and refer the reader to [14] for details of the proof.

Lemma 1. *Let (τ^*, λ^*) be the solution of the coupled set of equations (20) and (21) such that $H(\tau, \lambda)$ given by (19) is maximized. Then the optimal control (22) corresponding to (τ^*, λ^*) solves the optimal trajectory planning problem of finding u_L such that it minimizes the cost function (13) subjected to the linear dynamics (11) — (12) and the quadratic constraint (8).*

Thus finally we have that the controls (10) and (22) will ensure that the output $Y = [x \;\; y]^T$ of the differential wheeled mobile robot system (1)—(5) will solve the optimal control problem of approximate way point tracking while minimizing the jerk of the center of mass motion. The above lemma reduces the computationally extensive infinite dimensional constrained optimization problem to that of a much simpler finite dimensional nonlinear programming problem that can be solved using any one of the many numerical methods available in the literature. We refer the reader to [15] for a review of some of the existing methods. The finite dimensionality of the problem allows the solution to be computed in a much shorter time scale than the reference update rendering the scheme realtime implementable.

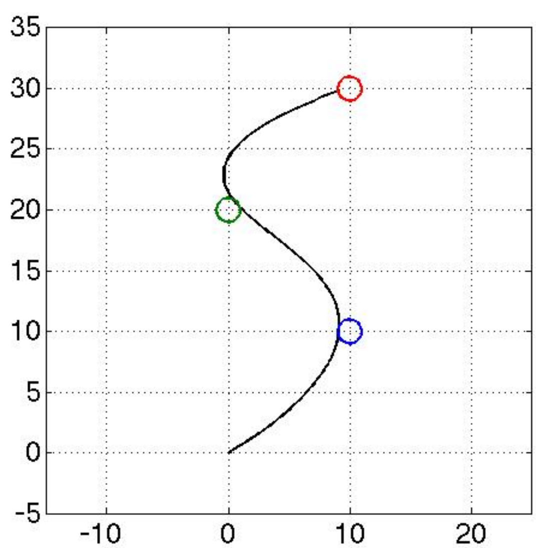

Fig. 1. The Trajectory of the Position of the center of mass of the mobile robot $(x(t), y(t))$ for $\mathcal{S} = \{(1, (10, 10)), (2, (0, 20)), (3, (10, 30)\}$ with radii $a_1 = a_2 = a_3 = 1$.

III. Simulation Results

For simulation purposes we chose three waypoints positioned at $(10, 10)$, $(0, 20)$, and $(10, 30)$. We require that the mobile robot be within a disk of radius 1 centered at each of the way points at the time instances $t_1 = 1, t_2 = 2, t_3 = 3$ respectively. That is we choose $N = 3$, $t_1 = 1, t_2 = 2, t_3 = 3$, $\alpha_1 = (10, 10), \alpha_2 = (0, 20), \alpha_3 = (10, 30)$, and $a_1 = a_2 = a_3 = 1$. Following the notation given earlier we thus have $\mathcal{S} = \{(1, (10, 10)), (2, (0, 20)), (3, (10, 30)\}$. We choose $m = 1$ and $I = 1$ and $R = \text{diag}(1, 1)$, and initial condition $(x(0), y(0), \theta(0), v(0), \omega(0), f(0)) = (0, 0, \pi/6, 10, 0, 0)$. For these values a unique optimal value for λ given by $\lambda^* = (1254, 525, 119)$ is found using a *trust region reflective* algorithm to solve the nonlinear programming problem described in Lemma-1 above. The trajectory of the output $Y(t) = (x(t), y(t))$, the co-ordinates of the center of mass $x(t)$ Vs t and $y(t)$ Vs t, the jerk of the center of mass $\dddot{Y}(t) = u_L(t)$, and the corresponding optimal control $u^*(t)$ are plotted in Figure 1—3 respectively.

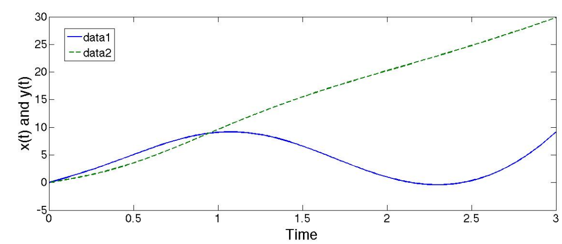

Fig. 2. The coordinates of the center of mass of the of the mobile robot Vs time t.

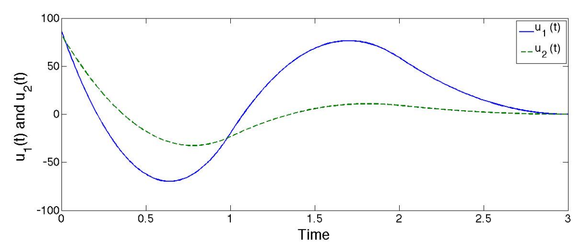

Fig. 3. The jerk of the center of mass of the mobile robot $u_L(t) = (u_1(t), u_2(t))$.

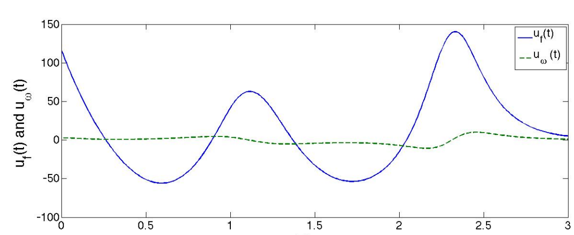

Fig. 4. The optimal control inputs $u(t) = (u_{\dot{f}}(t), u_\omega(t))$.

IV. CONCLUSION

In this paper we provide a scheme to generate optimal trajectories for nonholonomic mobile robots. The trajectories minimize the jerk of the trajectory of the center of mass of the robot. Furthermore the trajectories are constrained to pass within certain predefined neighborhoods at certain given predefined time instances. The solution to the problem is found by formulating the problem as an infinite dimensional constrained optimization problem and then transforming it to a much less computationally intensive finite dimensional nonlinear programming problem. To the best of our knowledge it is the first time that such a result has been presented. The effectiveness of the proposed scheme is demonstrated using numerical simulations.

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