Toffoli Depth Optimized Quantum Multiplication

장경배

https://youtu.be/35jFwt5cnmA

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ON THE DESIGN AND OPTIMIZATION OF A QUANTUM POLYNOMIAL-TIME ATTACK ON ELLIPTIC CURVE CRYPTOGRAPHY[a]

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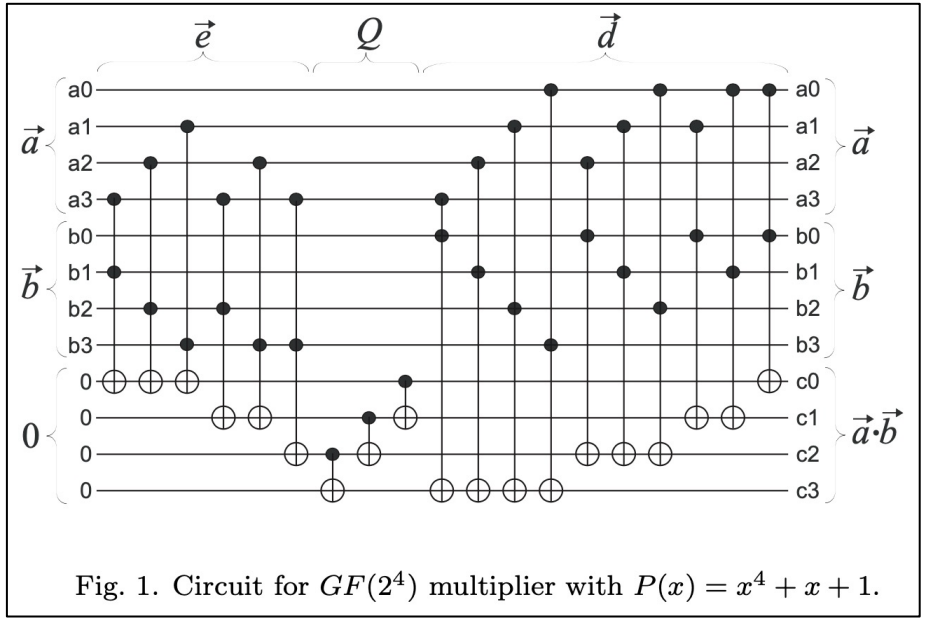

Fig. 1. Circuit for $GF(2^4)$ multiplier with $P(x) = x^4 + x + 1$.

- **일반적인 Schoolbook 곱셈**
- $h = f \cdot g$ $(size = n)$의 곱셈에 **$3n$개의 큐비트 사용**
- **Modular 연산 고려, 상위 곱셈부터 수행**
- **Toffoli 게이트 : n^2**

S. Kepley et al.

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Quantum circuits for $\mathbb{F}_{2^n}$-multiplication with subquadratic gate count

Shane Kepley[1] · Rainer Steinwandt[1]

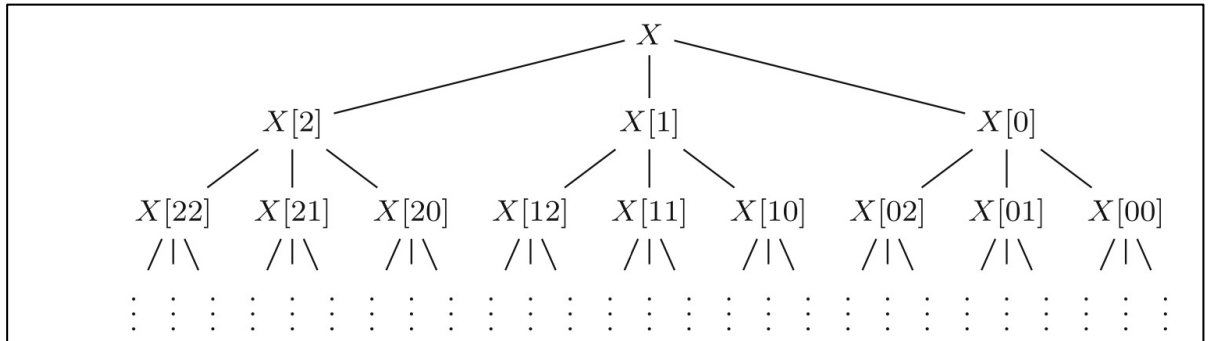

Fig. 1 Indexing in a ternary tree

- **Karatsuba 곱셈**
- $h = f \cdot g$ $(size = n)$의 곱셈에 **Karatsuba 알고리즘을 재귀적으로 적용**
- **곱셈 시 CNOT 게이트가 추가적으로 사용되지만, Toffoli 게이트를 최대한 감소시킬 수 있음**
- **큐비트를 추가 사용 → (2n + 사용 된 Toffoli 게이트 수)**
- **Toffoli 게이트 :** $n^2 \cdot \frac{3}{4}^{\log_2 n}$

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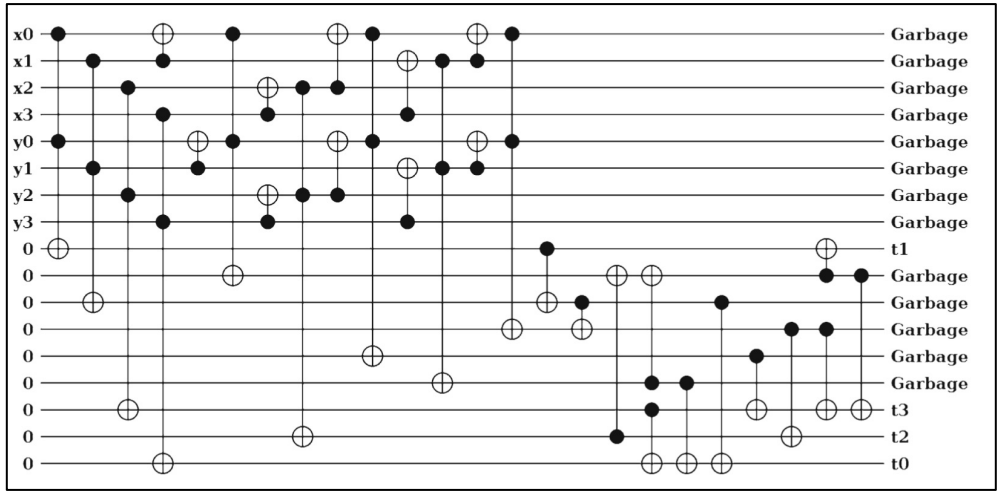
Example 3 For $\mathbb{F}_2[t]/(f)$ with $f = t^4 + t + 1$, before reduction we can form the appropriate coefficients in the product as follows:

j	z_j
0	C_0
1	$C_0 + C_1 + C_2$
2	$C_0 + \quad C_2 + C_3 + \quad C_6$
3	$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8$
4	$\quad C_2 + \quad C_5 + C_6 + \quad C_8$
5	$\quad C_6 + C_7 + C_8$
6	$\quad C_8$

j	z_j
0	$C_0 + \quad C_2 + \quad C_5 + C_6 + \quad C_8$
1	$C_0 + C_1 + \quad + \quad C_5 + \quad C_7$
2	$C_0 + \quad C_2 + C_3 + \quad C_7$
3	$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7$



I. van Hoof

Space-efficient quantum multiplication of polynomials for binary finite fields with sub-quadratic Toffoli gate count

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Concrete quantum cryptanalysis of binary elliptic curves

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- S. Kepley et al.와 동일하게 Karatsuba 알고리즘을 재귀적으로 적용하여 Toffoli 게이트를 최대한 줄임
- **차이점은 큐비트 수**
  - **LUP 분해를 활용하여**, Schoolbook 곱셈과 동일하게 **$3n$의 큐비트만을 사용**
  - CNOT 게이트를 조금 더 사용하긴 하지만, 그렇게 큰 단점은 아님

- Banegas, Bernstein의 **Binary ECC에 대한 Shor 알고리즘 적용 논문에서** 해당 곱셈 기법을 사용
  - S. Kepley의 곱셈보다 Toffoli Depth와 Full depth는 더 높을 것으로 보임

Ours

- **기존 연구들은 Depth를 신경 쓰지 않음**
- **이번 곱셈은 Toffoli depth 최적화 → Full depth 또한 감소**
- I. van Hoof의 곱셈(LUP 분해)을 사용하는 것은 X
  - Toffoli depth 최적화가 불가능
- 동일하게 Karatsuba 알고리즘을 재귀적으로 적용
  - 곱셈이 나눠지기 때문에 이것 만으로도 효과는 있음
  - 하지만 여전히 **종속성이 존재**
- **Karatsuba를 적용할 때 마다 생기는 종속성을 다 제거**
  - **추가 큐비트**
  - 끝까지 Karatsuba를 적용하고, 한 번에 곱셈
- 모든 Field size 2^n에 대해 **Toffoli depth 1** 곱셈 가능

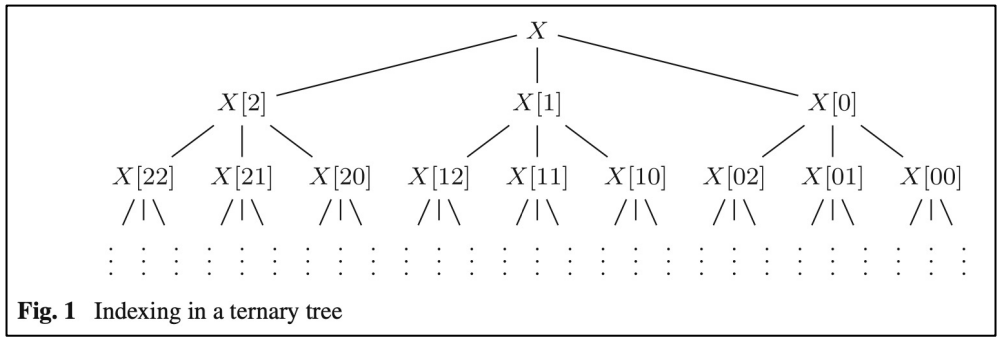
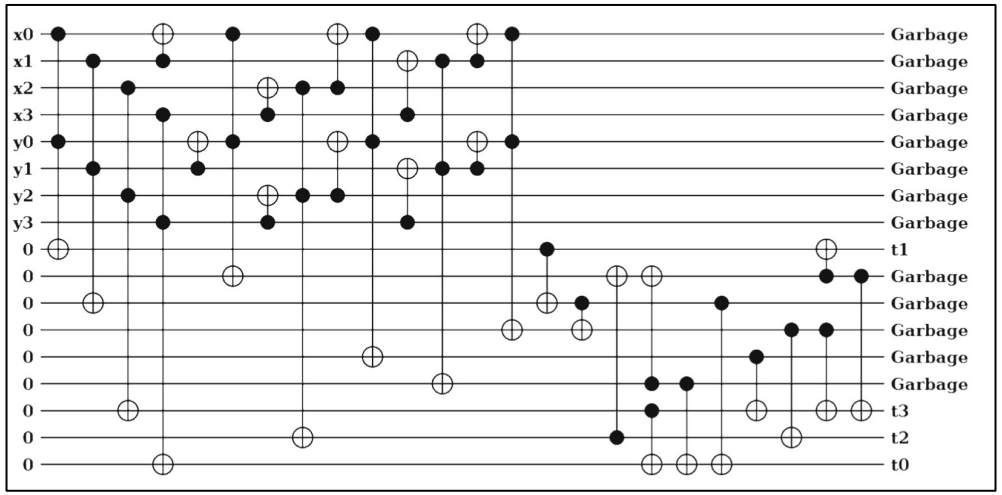


Fig. 1 Indexing in a ternary tree



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Ours

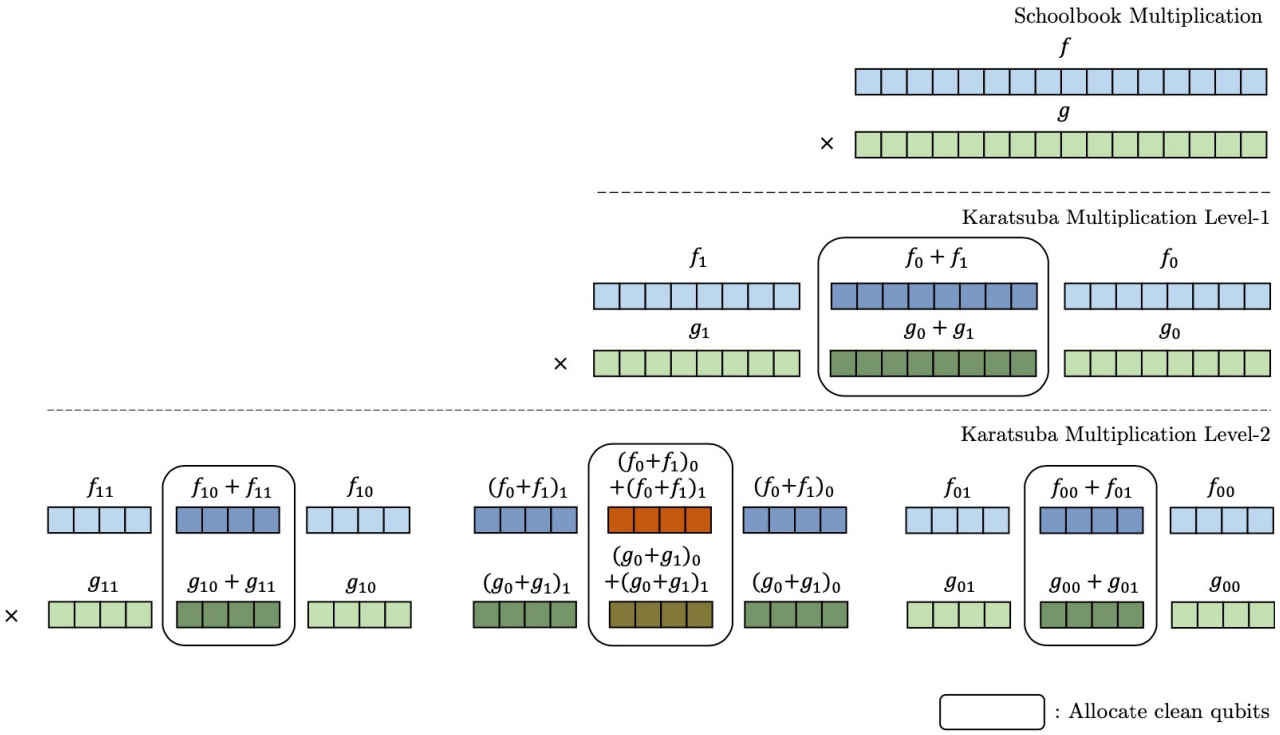

Table 1. Quantum resources required for $f \cdot g$ with Karatsuba Level-1

Field size 2^n	#CNOT	#Toffoli	Toffoli depth	#qubits	Full depth
Schoolbook	·	n^2	n^2	$4n-1$	$8n^2$
Karatsuba Level-1	$5n-4$	$3 \cdot (n/2)^2$	$(n/2)^2$	$3 \cdot (2n-1)$	$8 \cdot (n/2)^2 + 5$
Karatsuba Level-2	$(5n-4)+$ $3 \cdot (5n/2-4)$	$3^2 \cdot (n/2^2)^2$	$(n/2^2)^2$	$3^2 \cdot (n-1)$	$8 \cdot (n/4)^2 + 10$
Karatsuba Level-3	$(5n-4)+3 \cdot (5n/2-4)$ $+9 \cdot (5n/4-4)$	$3^3 \cdot (n/2^3)^2$	$(n/2^3)^2$	$3^3 \cdot (n/2-1)$	$8 \cdot (n/8)^2 + 15$

(Before Modular)

Ours

```python
def Karatsuba_Toffoli_Depth_1(eng) :

    n = 8
    a = eng.allocate_qureg(n)
    b = eng.allocate_qureg(n)
    r_a = eng.allocate_qureg(n)
    r_b = eng.allocate_qureg(n)
    c = eng.allocate_qureg(27)

    if (not resource_check):
        Round_constant_XOR(eng, a, 0xff, n)
        Round_constant_XOR(eng, b, 0xff, n)

    new_a = []
    new_b = []
    new_r = []

    for i in range(int(n/2)):
        CNOT | (a[i], r_a[i])
        CNOT | (a[int(n/2)+i], r_a[i])
        CNOT | (b[i], r_b[i])
        CNOT | (b[int(n/2) + i], r_b[i])

    new_a = karatsuba_mul_4bit(eng, a[0:4], b[0:4], c[0:9])
    new_b = karatsuba_mul_4bit(eng, a[4:8], b[4:8], c[18:27])
    new_r = karatsuba_mul_4bit(eng, r_a, r_b, c[9:18])

    # Combine
    result = []
    result = combine(eng, new_a, new_b, new_r, n)

    # modular

    if(not resource_check):
        print_state(eng, result, 2*n-1)
```

- Performance
  - Field 2^8 곱셈 (T + Clifford Level)

```
/Users/kb/PycharmProjects/
Result : 101010101010101

Gate counts:
    Allocate : 81
    CX : 300
    Deallocate : 81
    H : 54
    T : 81
    T^\dagger : 108

Depth : 23.
```

감사합니다