2021 WSMO Team Round

SMO Team

TR 1: How many ways are there to pick 3 letters without replacement from "Winter Solstice" and arrange them to make a word such that the word is a capital letter followed by two lowercase letters? (A word does not have to be an English word.)

TR 2: Bobby has some pencils. When he tries to split them into 5 equal groups, he has 2 left over. When he tries to split them into groups of 8, he has 6 left over. What is the second smallest number of pencils that Bobby could have?

TR 3: Farmer Sam has n dollars. He knows that this is exactly enough to buy either 50 pounds of grass and 32 ounces of hay or 96 ounces of grass and 24 pounds of hay. However, he must save 4 dollars for tax. After some quick calculations, he finds that he has exactly enough to buy 18 pounds of grass and 16 pounds of hay (and still have money left over for tax!). Find n.

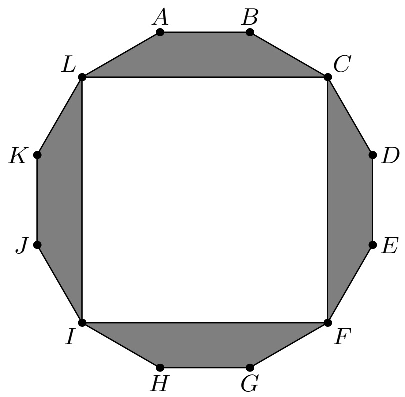
TR 4: Consider a triangle $A_1B_1C_1$ satisfying $A_1B_1 = 3, B_1C_1 = 3\sqrt{3}, A_1C_1 = 6$. For all successive triangles $A_nB_nC_n$, we have $A_nB_nC_n \sim B_{n-1}A_{n-1}C_{n-1}$ and $A_n = B_{n-1}, C_n = C_{n-1}$, where $A_nB_nC_n$ is outside of $A_{n-1}B_{n-1}C_{n-1}$. Find the value of

$$\left(\sum_{i=1}^{\infty}[A_iB_iC_i]\right)^2,$$

where $[A_iB_iC_i]$ is the area of $A_iB_iC_i$.

TR 5: Two runners are running at different speeds. The first runner runs at a consistent 12 miles per hour. The second runner runs at $t + 4$ miles per hour, where t is the number of hours that have passed. After n hours, the runners have run the same distance, where n is positive. Find n.

TR 6: Suppose that regular dodecagon $ABCDEFGHIJKL$ has side length 5. The area of the shaded region can be expressed as $a + b\sqrt{c}$, where c is not divisible by the square of any prime. Find $a + b + c$.



TR 7: A frog makes one hop every minute on the first quadrant of the coordinate plane (this means that the frog's x and y coordinates are positive). The frog can hop up one unit, right one unit, left one unit, down one unit, or it can stay in place, and will always randomly choose a valid hop from these 5 directions (a valid hop is a hop that does not place the frog outside the first quadrant). Given that the frog starts at $(1,1)$, the expected number of minutes until the frog reaches the line $x+y=5$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

TR 8: Isaac, Gottfried, Carl, Euclid, Albert, Srinivasa, René, Adihaya, and Euler sit around a round table (not necessarily in that order). Then, Hypatia takes a seat. There are $a \cdot b!$ possible seatings where Euler doesn't sit next to Hypatia and Isaac doesn't sit next to Gottfried, where b is maximized. Find $a+b$. (Rotations are not distinct, but reflections are)

TR 9: In triangle ABC, points D and E trisect side BC such that D is closer to C than E. If $\angle CAD = \angle EAD, ED = 3$, and $[AEB] = 6$, then find $[ABC]$, where $[ABC]$ is the area of ABC.

TR 10: The minimum possible value of

$$\sqrt{m^2+n^2}+\sqrt{3m^2+3n^2-6m+12n+15}$$

can be expressed as a. Find a^2.

TR 11: Find the remainder when

$$\sum_{x+y+z\leq 10} \frac{(x+y+z)!}{x!y!z!}$$

is divided by 100.

TR 12: Choose three integers x, y, z randomly and independently from the nonnegative integers. The probability that the sum of the factors of $2^x 3^y 5^z$ is divisible by 6 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

TR 13: Square $BCDE$ is drawn outside of equilateral triangle ABC. Regular hexagon $DEFGHI$ is drawn outside of square $BCDE$. If the area of triangle AED is 3, then the area of triangle AGH can be expressed as $a\sqrt{b}-c$, where b is not divisible by the square of any prime. Find $a+b+c$.

TR 14: Suppose that x is a complex number such that $x+\frac{1}{x}=\frac{\sqrt{6}+\sqrt{2}}{2}$ and the imaginary part of x is nonnegative. Find the sum of the five smallest nonnegative integers n such that $x^n+\frac{1}{x^n}$ is an integer.

TR 15: Let $ABCD$ and $DEFG$ be squares that intersect exactly once and with areas 1011^2 and 69^2 respectively. There exists a constant M such that $CE+AG>M$ where M is maximized. Find M.