2024 SSMO Team Round

SMO Team

TR 1: How many ordered triples of positive integers (a, b, c) satisfy the equation $2(a^b)^c+1 = 513$?

TR 2: Find the sum of the three smallest positive integers n where the last two digits of n^4 are 01.

TR 3: Consider positive integers N such that when N's units digit and leading nonzero digit are removed, what remains is a two-digit perfect square. The average of all N can be expressed as $\frac{m}{n}$, for relatively prime positive integers m and n. Find $m + n$.

TR 4: Let ABC be a right triangle with circumcenter O and incenter I such that $\angle ABC = 90^\circ$ and $\frac{AB}{BC} = \frac{3}{4}$. Let D the projection of O onto AB, and let E be the projection of O onto BC. Denote ω_1 be the incenter of ADO and ω_2 as the incenter of OEC. If $\frac{[\omega_1\omega_2 I]}{[ABC]} = \frac{m}{n}$, for relatively prime positive integers m and n, find $m + n$.

TR 5: Let ABC be a triangle with $AB = AC = 5$ and $BC = 6$. Let ω_1 be the circumcircle of ABC and let ω_2 be the circle externally tangent to ω_1 and tangent to rays AB and AC. If the distance between the centers of ω_1 and ω_2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

TR 6: Let α, β, and γ be the roots of the polynomial $x^3 - 6x^2 - 19x - n$. If n is an integer, what is the least possible positive value of $\alpha^3 + \beta^3 + \gamma^3$?

TR 7: Let a and b be real numbers that satisfy $a^3 + 8ab^2 = 8b^3 + 4a^2b = 375$. Find $\lfloor ab \rfloor$.

TR 8: Three integers $0 \le a \le b \le c < 229$ satisfy the congruence $n^3 \equiv 1 \pmod{229}$. Given that $71^2 - 3$ and $107^2 + 1$ are both multiples of 229, find the value of b.

TR 9: Let $ABCDEFGH$ be an equiangular octagon such that $AB = 6, BC = 8, CD = 10, DE = 12, EF = 6, FG = 8, GH = 10$, and $AH = 12$. The radius of the largest circle that fits inside the octagon can be expressed as $a + b\sqrt{c}$, where a, b, and c are integers and c is squarefree. Find $a + b + c$.

TR 10: The side-lengths of a convex cyclic quadrilateral $ABCD$ are integers and $(AB \cdot AD + BC \cdot CD)^2 = AC^2 \cdot BD^2 - 72$. Find all possible values of the perimeter of $ABCD$.

TR 11: Let S denote the set of positive divisors of 5400. Let

$$S_i = \{d \mid d \in S,\, d \equiv i \pmod{}\}$$

and let s_i denote the sum of all elements of S_i. Find the value of

$$s_0^2 + s_1^2 + s_2^2 + s_3^2 - 2s_0s_2 - 2s_1s_3.$$

TR 12: What is the smallest positive integer n with 3 positive prime factors such that for all integers k, $k^n \equiv k \pmod{n}$?

TR 13: In a deck of 54 cards (2 identical jokers, 4 identical cards with $1, 2, 3, \ldots, 13$), each card is dealt to one of 3 people, each having a $\frac{1}{3}$ chance of receiving each card. If the expected sum of the number of unique cards the three of them have can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

TR 14: Let $a_1, a_2, \ldots, a_7$ be the roots of the polynomial

$$x^7 + 5x^6 + 9x^5 + x^4 + x^3 + 10x^2 + 5x + 1.$$

Find the value of

$$\left| \prod_{n=1}^{7} \prod_{m=n+1}^{7} (a_n a_m - 1) \right|.$$

TR 15: In triangle ABC inscribed in circle ω, let M be the midpoint of BC. Denote P as the intersection of AM with ω. If $BP = 9, CP = 13$, and $AM = 20$, find the perimeter of triangle ABC.