2021 WSMO Speed Round Solutions

SMO Team

SR 1: Let $f^1(x) = (x-1)^2$, and let $f^n(x) = f^1(f^{n-1}(x))$. Find the value of $|f^7(2)|$.

Answer: $\boxed{1}$

Solution: We have

$$\begin{aligned}
|f^7(2)| &= f(f(f(f(f(f(f(2))))))) \\
&= f(f(f(f(f(f((2-1)^2)))))) \\
&= f(f(f(f(f(f(1)))))) \\
&= f(f(f(f(f((1-1)^2))))) \\
&= f(f(f(f(f(0))))) \\
&= f(f(f(f((0-1)^2)))) = f(f(f(f(1)))) \\
&= f(f(f((1-1)^2))) = f(f(f(0))) \\
&= f(f((0-1)^2)) = f(f(1)) = f((1-1)^2) \\
&= f(0) = (0-1)^2 = \boxed{1}.
\end{aligned}$$

SR 2: A square with side length of 4 units is rotated around one of its sides by 90°. The volume the square sweeps out can be expressed as $m\pi$. Find m.

Answer: $\boxed{16}$

Solution: When a square is rotated about one of its sides, it forms a cylinder. Thus, the answer is

$$4^3\pi \cdot \frac{90}{360} = 16\pi \implies \boxed{16}.$$

SR 3: Let $a@b = \frac{a^2-b^2}{a+b}$. Find the value of $1@(2@(\dots(2020@2021)\dots)$.

Answer: $\boxed{1011}$

Solution: We have

$$a@b = \frac{a^2-b^2}{a+b} = \frac{(a-b)(a+b)}{a+b} = a-b.$$

Now, $a@(b@c) = a-(b-c) = a-b+c$ and $a@(b@(c@d)) = a-(b-(c-d)) = a-b+c-d$. Using this pattern, we find that

$$1@(2@(\dots(2020@2021)\dots) = 1-2+3-4+\dots-2020+2021 =$$

$$(1-2)+(3-4)+\dots+(2019-2020)+2021 = -1010+2021 = \boxed{1011}.$$

SR 4: A square $ABCD$ with side length 10 is placed inside of a right isosceles triangle XYZ with $\angle XYZ = 90^\circ$ such that A and B are on XZ, C is on YZ, and D is on XY. Find the area of XYZ.

Answer: $\boxed{225}$

Solution: We are given that $AB = BC = CD = DA = 10$. So, $YD = \frac{10}{\sqrt{2}} = 5\sqrt{2}$, and $XD = \sqrt{2} \cdot AD = 10\sqrt{2}$, meaning $XY = 10\sqrt{2} + 5\sqrt{2} = 15\sqrt{2}$ and

$$[XYZ] = \frac{(15\sqrt{2})^2}{2} = \boxed{225}.$$

SR 5: The number of ways to arrange the characters in "delicious greenbeans" into two separate strings of letters can be expressed as $a \cdot b!$, where b is maximized and both a and b are positive integers. Find $a + b$. (A string of letters is defined as a group of consecutive letters with no spaces between them.)

Answer: $\boxed{8736}$

Solution: Note that "delicious greenbeans" has 19 letters. In addition, the "delicious greenbeans" has 1 d, 4 e's, 1 l, 2 i's, 1 c, 1 o, 1 u, 1 g, 1 r, 2 s's, 2 n's, 1 b, and 1 a. Thus, there are

$$\frac{19!}{1!4!1!2!1!1!1!1!@!2!1!1!} = \frac{19!}{4!2!2!2!}$$

distinct ways to arrange the letters in "delicious greenbeans". In addition, since there are 19 letters, there are 18 possible places to put the space (" "). Thus, the answer is

$$\frac{19!}{4!2!2!2!} \cdot 18 = \frac{18 \cdot 19!}{192} = \frac{18 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{192} = 8721 \cdot 15! \implies 8721 + 15 = \boxed{8736}.$$

SR 6: A bag weighs 1 pound and can hold 16 pounds of food at maximum. Danny buys 100 packages of tomatoes and 300 packages of potatoes. Tomatoes come in packages that are 12 ounces each and potatoes come in packages that are 24 ounces each. All of Danny's food must be packed into bags, and Danny will use only as many bags as he needs. What is the minimum possible weight, in pounds, of Danny's luggage, including the bags?

Answer: $\boxed{559}$

Solution: Since bags weigh multiples of 12 ounces, a bag can hold at most 252 ounces, meaning that Danny uses at least $\frac{8400}{252} = \frac{100}{3}$ bags, or 34 bags. A construction is to let the first 33 bags have 9 packages of potatoes and 3 packages of tomatoes, while the last bag has 3 packages of potatoes and 1 package of tomatoes. The total weight of Danny's luggage is

$$\frac{12 \cdot 100}{16} + \frac{24 \cdot 300}{16} + 34 = 75 + 450 + 34 = \boxed{559}.$$

SR 7: Consider triangle ABC with side lengths $AB = 13, AC = 14, BC = 15$ and incircle ω. A second circle ω_2 is drawn which is tangent to AB, AC and externally tangent to ω. The radius of ω_2 can be expressed as $\frac{a-b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a + b + c + d$.

Answer: $\boxed{106}$

Solution: Note that the length of the A-angle bisector is $\frac{\sqrt{(b+c-a)(b+c+a)bc}}{b+c}$. Now, let I be the incenter of triangle ABC. This means that

$$AI = \frac{b+c}{a+b+c} \cdot \frac{\sqrt{b+c-a}(b+c+a)bc}{b+c} =$$

$$\frac{\sqrt{(b+c-a)(b+c+a)bc}}{a+b+c} = \frac{\sqrt{12 \cdot 42 \cdot 13 \cdot 14}}{42} = 2\sqrt{13}.$$

Now, from Heron's formula, we find that the area of triangle ABC is 84. Since the area of a triangle is the product of the semi-perimeter and the inradius, we find that the length of the inradius of triangle ABC is $\frac{84}{\frac{13+14+15}{2}} = \frac{84}{21} = 4$. Now, let the radius of ω_2 be r. From similar triangles, we find that

$$\begin{aligned}
&\frac{r}{4} = \frac{2\sqrt{13} - 4 - r}{2\sqrt{13}} \implies \\
&r = \frac{4(2\sqrt{13} - 4)}{2\sqrt{13} + 4} = \frac{68 - 16\sqrt{13}}{9} \implies \\
&68 + 16 + 13 + 9 = \boxed{106}.
\end{aligned}$$

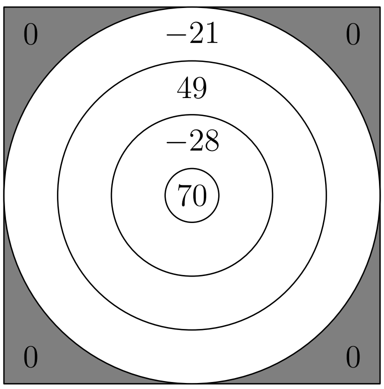
SR 8: Let n be the number of ways to seat 12 distinguishable people around a hexagon with two people seated on each side, where the order of the two people on each side does not matter and rotations are disregarded. Find the number of divisors of n.

Answer: $\boxed{240}$

Solution: Note that there are 12! ways to seat the people. Since rotations are considered the same, we have to divide by 6. In addition, since the order in which the two people are seated on each side does not matter, we have to divide by $2^6 = 64$. Thus, given the conditions, there are $\frac{12!}{64 \cdot 6} = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7^1 \cdot 11^1$. In conclusion, the answer is

$$(3+1)(4+1)(2+1)(1+1)(1+1) = \boxed{240}.$$

SR 9: Bobby is going to throw 20 darts at the dartboard shown below. It is formed by 4 concentric circles, with radii of $1, 3, 5$, and 7, with the largest circle being inscribed in a square. Each point on the dartboard has an equally likely chance of being hit by a dart, and Bobby is guaranteed to hit the dartboard. Each region is labeled with its point value (the number of points Bobby will get if he hits that region). The expected number of points Bobby will get after throwing the 20 darts can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: $\boxed{97}$

Solution: The probability of a dart getting 70 points is $\frac{\pi}{196}$, the probability of a dart getting -28 points is $\frac{8\pi}{196}$, the probability of a dart getting 49 points is $\frac{16\pi}{196}$, and the probability of a dart getting -21 points is $\frac{24\pi}{196}$. This means that on a random throw, the expected number of points Bobby gets is

$$70 \cdot \frac{\pi}{196} - 28 \cdot \frac{8\pi}{196} + 49 \cdot \frac{16\pi}{196} - 21 \cdot \frac{24\pi}{196} = \frac{9}{14}\pi.$$

After 20 throws, Bobby is expected to get

$$20 \cdot \frac{9}{14}\pi = \frac{90}{7}\pi \implies 90 + 7 = \boxed{97}.$$

SR 10: Find the remainder when $\underbrace{2021^{2022^{\cdots^{2022^{2021}}}}}_{2021\ 2021\text{'}s} \cdot \underbrace{2022^{2021^{\cdots^{2021^{2022}}}}}_{2022\ 2022\text{'}s}$ is divided by 11.

Answer: $\boxed{4}$

Solution: First, note that

$$\underbrace{2021^{2022^{\cdots^{2022^{2021}}}}}_{2021\ 2021\text{'}s} \cdot \underbrace{2022^{2021^{\cdots^{2021^{2022}}}}}_{2022\ 2022\text{'}s} \equiv \underbrace{8^{2022^{\cdots^{2022^{2021}}}}}_{2021\ 2021\text{'}s} \cdot \underbrace{9^{2021^{\cdots^{2021^{2022}}}}}_{2022\ 2022\text{'}s} \pmod{11}.$$

From Fermat's little theorem, we have $a^b \equiv a^{b \pmod{10}} \pmod{11}$ for all a and b. We have

$$\underbrace{2021^{\cdots^{2021^{2022}}}}_{2020\ 2021\text{'}s} \equiv 1 \pmod 4 \implies \underbrace{2022^{2021^{\cdots^{2021^{2022}}}}}_{2020\ 2021\text{'}s} \equiv 2^1 \equiv 2 \pmod{10} \quad \text{and}$$

$$\underbrace{2021^{2022^{\cdots^{2022^{2021}}}}}_{2021\ 2022\text{'}s} \equiv 1 \pmod{10}.$$

Thus,

$$\underbrace{2021^{2022^{\cdots^{2022^{2021}}}}}_{2021\ 2021\text{'}s} \cdot \underbrace{2022^{2021^{\cdots^{2021^{2022}}}}}_{2022\ 2022\text{'}s} \equiv 8^2 \cdot 9 \equiv \boxed{4} \pmod{11}.$$