2021 WSMO Team Round Solutions

SMO Team

TR 1: How many strings can be formed by selecting, without replacement, three letters from the phrase "Winter Solstice" (ignore the space) and arranging them, such that the first letter is an uppercase letter and the next two letters are lowercase?

Answer: $\boxed{150}$

Solution: First, the capital letter must be either "W" or "S", giving 2 choices. After removing the capital, we're left with 12 lowercase letters from "interolstice", among which 3 letters ("i", "t", and "e") appear twice, and the rest appear once. This gives 9 unique lowercase letters. We can form

$$9 \cdot 8 = 72$$

ordered pairs of distinct lowercase letters, and 3 more using the repeated letters ("ii", "tt", "ee"). So each capital letter leads to

$$72 + 3 = 75$$

valid words, and the final answer is

$$2 \cdot 75 = \boxed{150}.$$

TR 2: Bobby has some pencils. When he tries to split them into 5 equal groups, he has 2 left over. When he tries to split them into groups of 8, he has 6 left over. What is the second smallest number of pencils that Bobby could have?

Answer: $\boxed{62}$

Solution: We want $x \equiv 2 \pmod 5$ and $x \equiv 6 \pmod 8$. The smallest such x is 22, and the next is 62. So, our answer is

$$\boxed{62}.$$

TR 3: Farmer Sam has n dollars. He knows that this is exactly enough to buy either 50 pounds of grass and 32 ounces of hay or 96 ounces of grass and 24 pounds of hay. However, he must save 4 dollars for tax. After some quick calculations, he finds that he has exactly enough to buy 18 pounds of grass and 16 pounds of hay (and still have money left over for tax!). Find n.

Answer: $\boxed{54}$

Solution: Let the cost of grass be g dollars per pound and hay be h dollars per pound. Converting units, we have

$$32 \text{ ounces} = 2 \text{ pounds} \quad \text{and} \quad 96 \text{ ounces} = 6 \text{ pounds},$$

so:

$$n = 50g + 2h = 6g + 24h$$
$$n = 18g + 16h + 4.$$

From the first equation,

$$50g + 2h = 6g + 24h \implies 44g = 22h \implies h = 2g.$$

Substituting, we have

$$n = 50g + 2(2g) = 54g$$
$$n = 18g + 16(2g) + 4 = 50g + 4.$$

So,

$$54g = 50g + 4 \implies g = 1 \implies n = \boxed{54}.$$

TR 4: Consider a triangle $A_1B_1C_1$ satisfying $A_1B_1 = 3, B_1C_1 = 3\sqrt{3}, A_1C_1 = 6$. For all successive triangles $A_nB_nC_n$, we have $A_nB_nC_n \sim B_{n-1}A_{n-1}C_{n-1}$ and $A_n = B_{n-1}, C_n = C_{n-1}$, where $A_nB_nC_n$ is outside of $A_{n-1}B_{n-1}C_{n-1}$. Find the value of

$$\left(\sum_{i=1}^{\infty}[A_iB_iC_i]\right)^2.$$

Answer: $\boxed{972}$

Solution: Note that $A_1B_1C_1$ is a right triangle with right angle at B_1, so its area is

$$[A_1B_1C_1] = \frac{1}{2} \cdot A_1B_1 \cdot B_1C_1 = \frac{9\sqrt{3}}{2}.$$

For all i, we have

$$\frac{A_iC_i}{A_{i-1}C_{i-1}} = \frac{B_{i-1}C_{i-1}}{A_{i-1}C_{i-1}} = \frac{B_1C_1}{A_1C_1} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}.$$

Thus, the ratio of areas is

$$\frac{[A_iB_iC_i]}{[A_{i-1}B_{i-1}C_{i-1}]} = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

This forms a geometric series:

$$\sum_{i=1}^{\infty}[A_iB_iC_i] = [A_1B_1C_1]\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i = \frac{9\sqrt{3}}{2} \cdot \frac{1}{1-\frac{3}{4}} = \frac{9\sqrt{3}}{2} \cdot 4 = 18\sqrt{3}.$$

The final answer is

$$\left(18\sqrt{3}\right)^2 = 324 \cdot 3 = \boxed{972}.$$

TR 5: Two runners are running at different speeds. The first runner runs at a consistent 12 miles per hour. The second runner runs at $t+4$ miles per hour, where t is the number of hours that have passed. After n hours, the runners have run the same distance. Find n.

Answer: $\boxed{16}$

Solution: After n hours, the first runner has run $12n$ miles. The second runner runs at $t+4$ miles per hour, so his average speed over n hours is

$$\frac{(n+4)+4}{2} = \frac{n+8}{2}.$$

Thus, the second runner travels

$$\frac{n+8}{2} \cdot n = \frac{n^2+8n}{2} \text{ miles.}$$

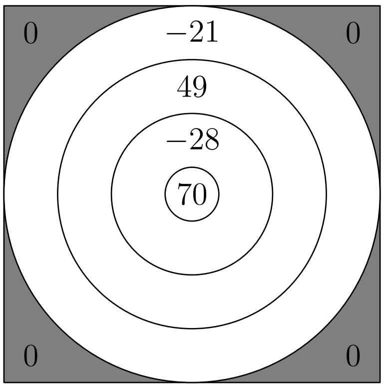
Setting the distances equal:

$$\frac{n^2+8n}{2} = 12n \implies n^2+8n = 24n \implies n^2 = 16n \implies n = \boxed{16}.$$

TR 6: Suppose that regular dodecagon $ABCDEFGHIJKL$ has side length 5. The area of the shaded region can be expressed as $a + b\sqrt{c}$, where c is a squarefree positive integer. Find $a+b+c$.

Answer: $\boxed{78}$

Solution:



Note that the shaded region consists of four identical trapezoids and a regular dodecagon has interior angles of

$$\frac{(12-2)\cdot 180^\circ}{12} = 150^\circ.$$

Consider the trapezoid $\triangle ALCB$. Let the foot of the perpendicular from A to LC be X. Then $\angle LAX = 60^\circ$. This means

$$AX = 5\cos(60^\circ) = \frac{5}{2} \quad \text{and} \quad LX = 5\sin(60^\circ) = \frac{5\sqrt{3}}{2}$$

So,

$$LC = AX + LX \cdot 2 = 5 + 2 \cdot \frac{5\sqrt{3}}{2} = 5 + 5\sqrt{3}.$$

The height of the trapezoid is $AX = \frac{5}{2}$, and the two bases are 5 and $5 + 5\sqrt{3}$. So the area of one trapezoid is

$$\frac{5}{2} \cdot \frac{5 + (5 + 5\sqrt{3})}{2} = \frac{50 + 25\sqrt{3}}{4}.$$

Multiplying by 4 trapezoids, we get

$$4 \cdot \frac{50 + 25\sqrt{3}}{4} = 50 + 25\sqrt{3} \implies 50 + 25 + 3 = \boxed{78}.$$

TR 7: A frog makes one hop every minute on the first quadrant of the coordinate plane (this means that the frog's x and y coordinates are positive). The frog can hop up one unit, right one unit, left one unit, down one unit, or it can stay in place, and will always randomly choose a valid hop from these 5 directions (a valid hop is a hop that does not place the frog outside the first quadrant). Given that the frog starts at $(1, 1)$, the expected number of minutes until the frog reaches the line $x + y = 5$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{66}$

Solution: Let $E_{(x,y)}$ be the expected time it takes to get from (x, y) to $x + y = 5$. By symmetry, we have $E_{(1,2)} = E_{(2,1)}$ and $E_{(1,3)} = E_{(3,1)}$. Now, note that for all points satisfying $x + y = 5$, we have $E_{(x,y)} = 0$. Notably, $E_{(1,4)} = E_{(2,3)} = E_{(3,2)} = E_{(4,1)} = 0$. Now, we apply probabilistic states. For $(1, 1)$, we have

$$\begin{aligned}
E_{(1,1)} &= \frac{1}{3}E_{(1,1)} + \frac{1}{3}E_{(1,2)} + \frac{1}{3}E_{(2,1)} + 1 \implies \\
\frac{2}{3}E_{(1,1)} &= \frac{2}{3}E_{(1,2)} + 1 \implies \\
E_{(1,2)} &= E_{(1,1)} - \frac{3}{2}.
\end{aligned}$$

For $(1, 2)$, we have,

$$\begin{aligned}
E_{(1,2)} &= \frac{1}{4}E_{(1,2)} + \frac{1}{4}E_{(1,1)} + \frac{1}{4}E_{(2,2)} + \frac{1}{4}E_{(1,3)} + 1 \implies \\
\frac{3}{4}E_{(1,2)} &= \frac{1}{4}E_{(1,1)} + \frac{1}{4}E_{(2,2)} + \frac{1}{4}E_{(1,3)} + 1 \implies \\
E_{(1,2)} &= \frac{1}{3}E_{(1,1)} + \frac{1}{3}E_{(2,2)} + \frac{1}{3}E_{(1,3)} + \frac{4}{3} \qquad (1)
\end{aligned}$$

For $(1, 3)$, we have,

$$\begin{aligned}
E_{(1,3)} &= \frac{1}{4}E_{(1,3)} + \frac{1}{4}E_{(1,2)} + \frac{1}{4}E_{(3,2)} + \frac{1}{4}E_{(1,4)} + 1 \implies \\
\frac{3}{4}E_{(1,3)} &= \frac{1}{4}E_{(1,2)} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + 1 \implies \\
E_{(1,3)} &= \frac{1}{3}E_{(1,2)} + \frac{4}{3} \\
&= \frac{1}{3}\left(E_{(1,1)} - \frac{3}{2}\right) + \frac{4}{3} \implies \\
&= \frac{1}{3}E_{(1,1)} + \frac{5}{6}.
\end{aligned}$$

For $(2,2)$, we have,

$$\begin{aligned}
E_{(2,2)} &= \frac{1}{5}E_{(2,2)} + \frac{1}{5}E_{(1,2)} + \frac{1}{5}E_{(2,1)} + \frac{1}{5}E_{(2,3)} + \frac{1}{5}E_{(3,2)} + 1. \implies \\
\frac{4}{5}E_{(2,2)} &= \frac{1}{5}E_{(1,2)} + \frac{1}{5}E_{(1,2)} + \frac{1}{5}\cdot 0 + \frac{1}{5}\cdot 0 + 1 \implies \\
\frac{4}{5}E_{(2,2)} &= \frac{2}{5}E_{(1,2)} + 1 \implies \\
E_{(2,2)} &= \frac{1}{2}E_{(1,2)} + \frac{5}{4} \\
&= \frac{1}{2}\left(E_{(1,1)} - \frac{3}{2}\right) + \frac{5}{4} \\
&= \frac{1}{2}E_{(1,1)} + \frac{1}{2}.
\end{aligned}$$

Substituting the values of $E_{(1,2)}, E_{(1,3)}$, and $E_{(2,2)}$ into equation (1), we have

$$\begin{aligned}
E_{(1,1)} - \frac{3}{2} &= \frac{1}{3}E_{(1,1)} + \frac{1}{3}\left(\frac{1}{2}E_{(1,1)} + \frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{3}E_{(1,1)} + \frac{5}{6}\right) + \frac{4}{3} \\
E_{(1,1)} - \frac{3}{2} &= \frac{11}{18}E_{(1,1)} + \frac{16}{9} \implies \\
\frac{7}{18}E_{(1,1)} &= \frac{59}{18} \implies \\
E_{(1,1)} &= \frac{59}{7} \implies 59 + 7 = \boxed{66}.
\end{aligned}$$

TR 8: Isaac, Gottfried, Carl, Euclid, Albert, Srinivasa, Rene, Adihaya, Euler, and Hypatia sit around a round table. There are $a \cdot b!$ possible seatings (disregarding rotations) where Euler isn't seated next to Hypatia and Isaac isn't seated next to Gottfried, where b is maximized. Find $a + b$.

Answer: $\boxed{56}$

Solution: First, we deal with Euler and Hypatia. Out of the $\binom{10}{2}$ possible seatings for the two, there are only $\frac{10\cdot 7}{2}$ that are valid. Now, given Euler and Hypatia aren't together, we'll calculate the chance Isaac and Gottfried don't sit together. After Euler and Hypatia are seated, note that there are $\binom{8}{2}$ possible seatings of Isaac and Gottfried. Excluding Euler and Hypatia, there are $8 \cdot 5$ possible arragements of Isaac and Gottfried that are valid, since there are always 2 empty neighboring seats next to Isaac that Gottfried cant sit in. However, over all 8 possible seatings are Isaac, there are 4 neighboring seats that are already occupied. Therefore, including Euclid and Hypatia, there are $8 \cdot 5 + 4 = 44$ possible valid arragements, where ordering matters. Therefore, our answer is

$$\left(\frac{10!}{10}\right)\left(\frac{\frac{10\cdot 7}{2}}{\binom{10}{2}}\right)\left(\frac{\frac{44}{2}}{\binom{8}{2}}\right) = 51 \cdot 7! \implies 51 + 7 = \boxed{56}.$$

TR 9: In triangle ABC, points D and E trisect side BC such that D is closer to C than E. If $\angle CAD = \angle EAD, ED = 3$, and $[AEB] = 6$, then find $[ABC]$.

Answer: $\boxed{18}$

Solution: We have $BE = ED = DC$, meaning $[AEB] = [AED] = [ADC] = 6$. So,

$$[ABC] = [BAE] + [EAD] + [DAC] = 6 + 6 + 6 = \boxed{18}.$$

TR 10: The minimum possible value of

$$\sqrt{m^2+n^2}+\sqrt{3m^2+3n^2-6m+12n+15}$$

can be expressed as $\sqrt{m}$. Find m.

Answer: $\boxed{5}$

Solution: We rewrite

$$\begin{aligned}&\sqrt{m^2+n^2}+\sqrt{3m^2+3n^2-6m+12n+15}\\=&\sqrt{m^2+n^2}+\left(\sqrt{3}\right)\sqrt{(m-1)^2+(n+2)^2}.\end{aligned}$$

Let $A=(0,0)$ and $B=(1,-2)$. We have

$$\begin{aligned}&\sqrt{m^2+n^2}+\left(\sqrt{3}\right)\sqrt{(m-1)^2+(n+2)^2}\\=&AX+\left(\sqrt{3}\right)BX\end{aligned}$$

where $X=(m,n)$. From the triangle inequality, we have

$$AX+\left(\sqrt{3}\right)BX\geq(AX+BX)+\left(\sqrt{3}-1\right)BX\geq AB=\sqrt{5}\implies\boxed{5},$$

where equality occurs when $X=B$.

TR 11: Find the remainder when

$$\sum_{x+y+z\leq 10}\frac{(x+y+z)!}{x!y!z!}$$

is divided by 100.

Answer: $\boxed{73}$

Solution: Note that

$$\frac{(x+y+z)!}{x!y!z!}$$

is equivalent the number of ways to order x A's, y B's, and z C's. So,

$$\sum_{x+y+z=k}\frac{(x+y+z)!}{x!y!z!}$$

is equivalent to the number of orderings of length k made of A's, B's, and C's, which is equal to 3^k. We have

$$\sum_{x+y+z\leq 10}\frac{(x+y+z)!}{x!y!z!}=\sum_{k=0}^{10}\sum_{x+y+z=k}\frac{(x+y+z)!}{x!y!z!}=\sum_{k=0}^{10}3^k=\frac{3^11-1}{3-1}\equiv\boxed{73}\pmod{100}.$$

TR 12: Choose three integers x,y,z randomly and independently from the nonnegative integers. The probability that the sum of the factors of $2^x3^y5^z$ is divisible by 6 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{13}$

Solution: The sum of the factors of $2^x3^y5^z$ is

$$S = \sum_{i=0}^{x} 2^i \sum_{j=0}^{y} 3^j \sum_{k=0}^{z} 5^k.$$

For $S \equiv 0 \pmod 6$, we must have $S \equiv 0 \pmod 2$ and $S \equiv 0 \pmod 3$. In order for $S \equiv 0 \pmod 2$ to hold, we must have at least one of

$$\begin{aligned} \sum_{i=0}^{x} 2^i \equiv 0 \pmod 2 \quad &\implies \quad \text{impossible} \\ \sum_{j=0}^{y} 3^k \equiv 0 \pmod 2 \quad &\implies \quad y \equiv 0 \pmod 2 \\ \sum_{k=0}^{z} 5^k \equiv 0 \pmod 2 \quad &\implies \quad z \equiv 0 \pmod 2 \end{aligned}$$

true. In order for $S \equiv 9 \pmod 3$ to hold, we must have at least one of

$$\begin{aligned} \sum_{i=0}^{x} 2^i \equiv 0 \pmod 2 \quad &\implies \quad x \equiv 0 \pmod 2 \\ \sum_{j=0}^{y} 3^k \equiv 0 \pmod 2 \quad &\implies \quad \text{impossible} \\ \sum_{k=0}^{z} 5^k \equiv 0 \pmod 2 \quad &\implies \quad z \equiv 0 \pmod 2 \end{aligned}$$

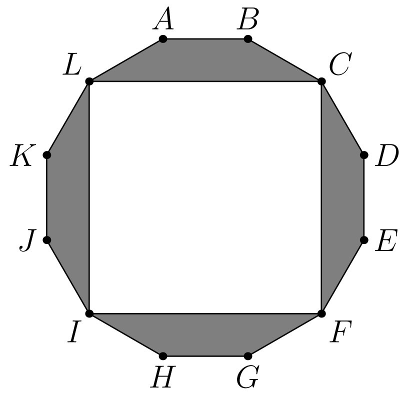
true. So, in order for $S \equiv 0 \pmod 6$ to hold, we must have either $z \equiv 0 \pmod 2$ or $x, y \equiv 0 \pmod 2$. There is a $\frac{1}{2}$ chance that $z \equiv 0 \pmod 2$. For the $\frac{1}{2}$ chance that $z \not\equiv 0 \pmod 2$, then we must have $x, y \equiv 0 \pmod 2$, which occurs with a $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ chance. This gives us a final probability of

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{5}{8} \implies 5 + 8 = \boxed{13}.$$

TR 13: Square $BCDE$ is drawn outside of equilateral triangle ABC. Regular hexagon $DEFGHI$ is drawn outside of square $BCDE$. If the area of triangle AED is 3, then the area of triangle AGH can be expressed as $a\sqrt{b} - c$, where b is a squarefree positive integer. Find $a + b + c$.

Answer: $\boxed{30}$

Solution:



Let $BC = s$. Firstly, $ED = GH$ since both are sides of a regular hexagon. Now, the distance from A to ED is equal to $s\left(\frac{\sqrt{3}}{2}+1\right) = \frac{2+\sqrt{3}}{2}$ and the distance from A to ED is equal to $s\left(\frac{\sqrt{3}}{2}+1+\sqrt{3}\right) = \frac{2+3\sqrt{3}}{2}$. So,

$$[AGH] = \frac{\frac{2+3\sqrt{3}}{2}}{\frac{2+\sqrt{3}}{2}}[AED] = (4\sqrt{3}-5)(3) = 12\sqrt{3}-15 \implies 12+3+15 = \boxed{30}.$$

TR 14: Suppose that x is a complex number such that $x + \frac{1}{x} = \frac{\sqrt{6}+\sqrt{2}}{2}$ and the imaginary part of x is nonnegative. Find the sum of the five smallest nonnegative integers n such that $x^n + \frac{1}{x^n}$ is an integer.

Answer: $\boxed{30}$

Solution: From the quadratic formula, we have

$$x = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) \pm i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = e^{\pm\frac{\pi}{12}i}.$$

So,

$$x^n + \frac{1}{x^n} = e^{\frac{\pi n}{12}} + e^{-\frac{\pi n}{12}} = 2\cos\left(\frac{\pi n}{12}\right).$$

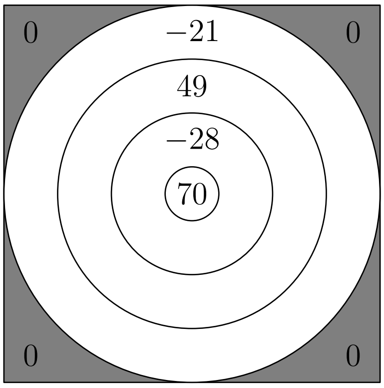
The five smallest values of n occur when

$$\frac{\pi n}{12} = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi \implies n = 0, 4, 6, 8, 12 \implies 0+4+6+8+12 = \boxed{30}.$$

TR 15: Let $ABCD$ and $DEFG$ be squares that intersect exactly once and with areas 1011^2 and 69^2 respectively. There exists a constant M such that $CE + AG > M$ where M is maximized. Find M.

Answer: $\boxed{2022}$

Solution:



We wish to the find the minimum value M of $CE + AG$. For simplicity, let the sidelength of $ABCD$ be $a = 1011$ and the sidelength of $DEFG$ be $b = 69$. Let $\cos(\angle GDA) = -\cos(\angle EDC) = x$. Note that $x \in [-1, 1]$. From the Law of Cosines, we have

$$CE + AG = \sqrt{a^2 + b^2 + 2abx} + \sqrt{a^2 + b^2 - 2abx}.$$

We have

$$\begin{aligned}(CE + AG)^2 &= (a^2 + b^2 + 2abx) + (a^2 + b^2 - 2abx) + \sqrt{(a^2 + b^2 + 2abx)(a^2 + b^2 - 2abx)} \\ &= 2(a^2 + b^2) + \sqrt{(a^2 + b^2)^2 - (2abx)^2}.\end{aligned}$$

since $a^2 + b^2$ is fixed, this expression is minimized when $(2abx)^2$ is maximized, or when $x = \pm 1$. When $x = \pm 1$, we have

$$CE + AG = \sqrt{a^2 + b^2 + 2ab} + \sqrt{a^2 + b^2 - 2ab} = (a + b) + (a - b) = 2a = \boxed{2022}.$$