2021 WSMO Accuracy Round Solutions

SMO Team

AR 1: Find the sum of all the positive integers n such that n is $\frac{2n^2-5n+5}{n-5}$ an integer.

Answer: $\boxed{121}$

Solution: Note that $2n^2 - 5n + 5 = (n-5)(2n+5) + 30$. The given condition implies that

$$\frac{(n-5)(2n+5)+30}{n-5} = 2n + 5 + \frac{30}{n-5},$$

which implies that $n - 5$ is a factor of 30. Thus,

$$n - 5 = -30, -15, -10, -6, -5, -3, -2, -1, 1, 2, 3, 5, 6, 10, 15, 30.$$

Solving, we find that

$$n = -25, -10, -5, 0, 2, 3, 4, 6, 7, 8, 10, 11, 15, 20, 35.$$

Since n is positive, we find that

$$n = 2, 3, 4, 6, 7, 8, 10, 11, 15, 20, 35.$$

Summing, we find that the answer is

$$2 + 3 + 4 + 6 + 7 + 8 + 10 + 11 + 15 + 20 + 35 = \boxed{121}.$$

AR 2: A fair 20-sided die has faces labeled with the numbers $1, 3, 6, \ldots, 210$. Find the expected value of a single roll of this die.

Answer: $\boxed{77}$

Solution: Note that the numbers on the die are the first 20 triangular numbers. Thus, the expected value of a single roll of this die is

$$\frac{1 + 3 + 6 + \ldots + 210}{20} = \frac{\frac{20\cdot 21\cdot 22}{6}}{20} = \boxed{77}.$$

AR 3: Suppose f is a monic polynomial of minimal degree with rational coefficients satisfying $f\left(3+\sqrt{5}\right) = 0$ and $f\left(4-\sqrt{7}\right) = 0$. Find the value of $|f(1)|$.

Answer: $\boxed{2}$

Solution: Since $3+\sqrt{5}$ is a root of the polynomial, then $3-\sqrt{5}$ is too. Similarly, since $4-\sqrt{7}$ is a root of the polynomial, then so is $4+\sqrt{7}$. Thus,

$$f(x) = d(x)\left(x - \left(3+\sqrt{5}\right)\right)\left(x - \left(3-\sqrt{5}\right)\right)\left(x - \left(4-\sqrt{7}\right)\right)\left(x - \left(4+\sqrt{7}\right)\right),$$

for some polynomial d. To minimize the degree of f, we can set the degree of d to 0, which means that $d(x) = c$. This means that the leading coefficient of $f(x) = c$. Since f is monic, $c = 1$, which means that

$$f = \left(x - \left(3 + \sqrt{5}\right)\right)\left(x - \left(3 - \sqrt{5}\right)\right)\left(x - \left(4 - \sqrt{7}\right)\right)\left(x - \left(4 + \sqrt{7}\right)\right) =$$

$$\left((x-3) - \sqrt{5}\right)\left((x-3) + \sqrt{5}\right)\left((x-4) + \sqrt{7}\right)\left((x-4) - \sqrt{7}\right) =$$

$$\left((x-3)^2 - \left(\sqrt{5}\right)^2\right)\left((x-4)^2 - \left(\sqrt{7}\right)^2\right) =$$

$$\left(x^2 - 6x + 4\right)\left(x^2 - 8x + 9\right) = x^4 - 14x^3 + 61x^2 - 86x + 36.$$

We conclude that

$$|f(1)| = |1 - 14 + 61 - 86 + 36| = |-2| = \boxed{2}.$$

AR 4: A 12-hour clock has a minute hand that is the same length as the second hand, and an hour hand half the length of the minute hand. In a day, the tip of the minute hand travels a distance of m, the tip of the second hand travels a distance of s, and the tip of the hour hand travels a distance of h. The value of $\frac{m^2}{hs}$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Answer: $\boxed{7}$

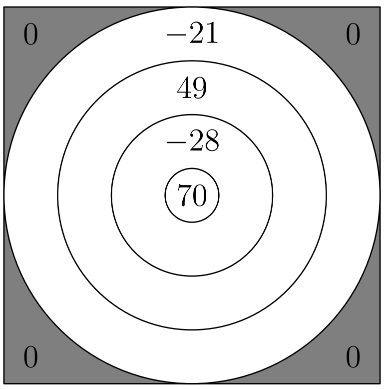
Solution: WLOG, assume that the length of the minute hand is 2. This means that the length of the second hand is 2 and the length of the hour hand is 1. In a day, there are 24 hours, which means that the minute hand travels $24 \cdot 4\pi$ each day. Also, since there are $24 = 2 \cdot 12$ hours, the hour hand travels $2 \cdot 2\pi = 4\pi$ each day. Finally, there are $24 \cdot 60$ minutes in a day, which means that the second hand travels $24 \cdot 60 \cdot 4\pi$ each day. Thus, the final answer is

$$\frac{(24 \cdot 4\pi) \cdot (24 \cdot 4\pi)}{(4\pi) \cdot (24 \cdot 60 \cdot 4\pi)} = \frac{2}{5} \implies 2 + 5 = \boxed{7}.$$

AR 5: Suppose regular octagon $ABCDEFGH$ has side length 5. The distance from the center of the octagon to one of the sides can be expressed as $\frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a + b + c + d$.

Answer: $\boxed{14}$

Solution:



Let the center of the octagon be O. We will focus on triangle AOB. Let $AO = OB = x$. From the Law of Cosines on triangle AOB, we find that

$$x^2 + x^2 - 2x^2 \cdot \cos(45^\circ) = 5^2 = 25 \implies$$

$$(2-\sqrt{2})x^2 = 25 \implies x^2 = \frac{25}{2-\sqrt{2}} = \frac{25(2+\sqrt{2})}{2}.$$

Now, let the distance from the center of the octagon to one of its sides be h. This means that

$$[AOB] = \frac{5h}{2}.$$

In addition, from the sine area formula,

$$[AOB] = \frac{1}{2}\sin\angle AOB \cdot AO \cdot BO = \frac{x^2\sqrt{2}}{4} = \frac{25(\sqrt{2}+1)}{4}.$$

Therefore, we have

$$\frac{5h}{2} = \frac{25(\sqrt{2}+1)}{4} \implies h = \frac{5+5\sqrt{2}}{2} \implies 5+5+2+2 = \boxed{14}.$$

AR 6: Roy is baking a circular three tier cake. All of the tiers are centered around the same point. Each tier's radius is $\frac{3}{4}$ of the radius of the tier below it, but the height of each tier stays constant. Roy wants to ice the cake, but only on the curved surfaces of the cake and the top of the smallest tier. The diameter of the lowest tier is 128 centimeters and its height is 10 centimeters. The surface area that is iced can be expressed as $m\pi$. Find m.

Answer: $\boxed{4256}$

Solution: We will consider the three tiers separately. The bottom tier has a diameter of 128 and and a height of 10. Therefore, the surface area of the iced part of the bottom tier is $128\pi \cdot 10 = 1280\pi$. The middle tier has a diameter of $128 \cdot \frac{3}{4} = 96$. Therefore, the surface area of the iced part of the middle tier is $96\pi \cdot 10 = 960\pi$. The top tier has a diameter of $96 \cdot \frac{3}{4} = 72$ and a height of 10. Therefore, the surface area of the iced part of the side of the top tier is $72\pi \cdot 10 = 720\pi$. Also, the radius of the top tier is $\frac{72}{2} = 36$. Thus, the surface area of the iced part of the top of the top tier is $36^2\pi = 1296\pi$. We conclude that the final answer is

$$1280\pi + 960\pi + 720\pi + 1296\pi = 4256\pi \implies \boxed{4256}.$$

AR 7: Find the value of $\sum_{n=1}^{100}\left(\sum_{i=1}^{n} r_i\right)$, where r_i is the remainder when $2^i + 3^i$ is divided by 10.

Answer: $\boxed{25150}$

Solution: From Fermat's Little Theorem, we find that $a^n \pmod{10}$ is periodic in cycles of 4. So,

$$\begin{aligned}
2^n + 3^n &\equiv 7 \pmod{10} &\quad \text{for } n \equiv 0 \pmod{4}\\
2^n + 3^n &\equiv 5 \pmod{10} &\quad \text{for } n \equiv 1 \pmod{4}\\
2^n + 3^n &\equiv 3 \pmod{10} &\quad \text{for } n \equiv 2 \pmod{4}\\
2^n + 3^n &\equiv 5 \pmod{10} &\quad \text{for } n \equiv 3 \pmod{4}
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i=1}^{4a+1} &= (7+5+3+5)a + (5) = 20a+5 \\
\sum_{i=1}^{4a+2} &= (7+5+3+5)a + (5+3) = 20a+8 \\
\sum_{i=1}^{4a+3} &= (7+5+3+5)a + (5+3+5) = 20a+13 \\
\sum_{i=1}^{4a+4} &= (7+5+3+5)a + (5+3+5+7) = 20a+20
\end{aligned}$$

This means that

$$\begin{aligned}
\sum_{n=1}^{100}\left(\sum_{i=1}^{n} r_i\right) &= \sum_{a=0}^{24}\left(\sum_{i=1}^{4a+1}(20a+5) + \sum_{i=1}^{4a+2}(20a+8) + \sum_{i=1}^{4a+3}(20a+13) + \sum_{i=1}^{4a+4}(20a+20)\right) \\
&= \sum_{a=0}^{24}(80a+46) \\
&= 80 \cdot \left(\frac{24 \cdot 25}{2}\right) + 46 \cdot 25 \\
&= \boxed{25150}.
\end{aligned}$$

AR 8: 20 unit spheres are stacked in a triangular pyramid formation, such that the first layer has 1 sphere, the second layer has 3 spheres, the third layer has 6 spheres, and the fourth layer has 10 spheres. The radius of the smallest sphere that fully contains all of these spheres is $\frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a+b+c+d$.

Answer: $\boxed{13}$

Solution: The centers of the four outermost spheres form a regular tetrahedron of side-length 6. Adding the radius of a unit sphere gives a total radius of

$$6 \cdot \frac{\sqrt{6}}{4} + 1 = \frac{3\sqrt{6}+2}{2} \implies 3+6+2+2 = \boxed{13}.$$

AR 9: Let $x = 1 + \frac{5}{2+\frac{3}{2+\frac{3}{2+\dots}}}$. The value of $\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}$ can be written as $\frac{a+\sqrt{b}}{c}$, where b is a squarefree positive integer and a and c are relatively prime positive integers. Find $a+b+c$.

Answer: $\boxed{114}$

Solution: We first compute the value of

$$S = 2 + \frac{3}{2+\frac{3}{2+\dots}}.$$

Note that

$$S = 2 + \frac{3}{S} \implies S^2 - 2S - 3 = 0 \implies S = 3.$$

So, $x = 1 + \frac{5}{3} = \frac{8}{3}$. Now, let

$$t = \sqrt{x + \sqrt{x + \dots}}.$$

We have

$$t = \sqrt{x + t} \implies t^2 - t - x = 0.$$

Substituting $x = \frac{8}{3}$ and applying the quadratic formula, we have

$$t = \frac{1 + \sqrt{1 + 4 \cdot \frac{8}{3}}}{2} = \frac{3 + \sqrt{105}}{6} \implies 3 + 105 + 6 = \boxed{114}.$$

AR 10: The largest value of x that satisfies the equation $5x^2 - 7\lfloor x \rfloor \{x\} = \frac{26\lfloor x \rfloor^2}{5}$ can be expressed as $\frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a + b + c + d$.

Answer: $\boxed{82}$

Solution: Let $a = \lfloor x \rfloor$ and $b = \{x\}$. The equation is equivalent to

$$5(a + b)^2 - 7ab = \frac{26a^2}{5} \implies a^2 - 15ab - 25b^2 = 0 \implies \frac{a}{b} = \frac{15 + 5\sqrt{13}}{2}.$$

Since $b < 1$, x is maximized when

$$a = \left\lfloor \frac{15 + 5\sqrt{13}}{2} \right\rfloor = 16 \implies b = \frac{32}{15 + 5\sqrt{13}}.$$

So,

$$x = a = b = 16 + \frac{32}{15 + 5\sqrt{13}} = \frac{56 + 8\sqrt{13}}{5} \implies 56 + 8 + 13 + 5 = \boxed{82}.$$