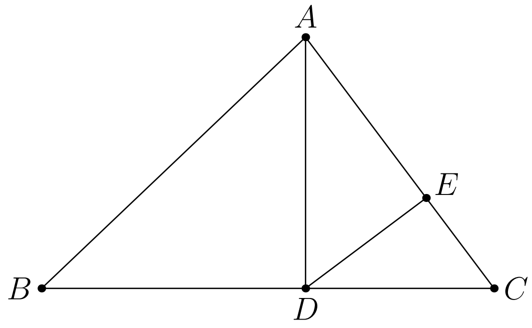
2024 SSMO Relay Round 1 Solutions

SMO Team

RR 1 Part 1: Let AD be an altitude of triangle ABC and let DE be an altitude of triangle ACD. If $AB = 29$, $CE = 9$, and $DE = 12$, what is the area of triangle ABC?



Answer: $\boxed{360}$

Solution: From the Pythagorean Theorem, we have $DC = 15$. From similar triangles, $\frac{DC}{AD} = \frac{EC}{DE} = \frac{3}{4} \implies AD = 20$. So, $BD = \sqrt{29^2 - 20^2} = 21$. In conclusion,

$$[ABC] = \frac{1}{2}(BC)AD = \frac{1}{2}(BD + DC)(AD) = \frac{1}{2}(21 + 15)(20) = \boxed{360}$$

RR 1 Part 2: Let $T = TNYWR$. A circular necklace is called *interesting* if it has T black beads and T white beads. A move consists of cutting out a segment of consecutive beads and reattaching it in reverse. It is possible to change any *interesting* necklace into any other *interesting* necklace using at most x moves. Find x. (Note: Rotations and reflections of a necklace are considered the same necklace).

Answer: $\boxed{719}$

Solution: Every time we reverse a section, we can "fix" the position of at least one bead. When only two beads are out of place, we can "fix" their positions with only one move. So, the answer is $2T - 1$. Since $T = 360$, we have $2T - 1 = \boxed{719}$.

RR 1 Part 3: Let $T = TNYWR$. In a circle, there are T people. $T - 2$ of them have red shoes, and two of them have blue shoes. First, they will randomly eliminate somebody from the circle. Then, they will randomly eliminate somebody with red shoes from the circle, and the cycle repeats until there is only one person left. The probability this person has blue shoes can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{181}$

Solution: Of the 719 people, 717 have red shoes. Note that we are guaranteed to remove 359 of those shoes, on the $2, 4, 6, \ldots$ cycles. Therefore, the question is equivalent to "what is the probability a randomly choosen shoe from $717 - 359$ red shoes and 2 blue shoes is blue?" This answer is simply $\frac{2}{2+358} = \frac{1}{180} \implies 1 + 180 = \boxed{181}$.

2024 SSMO Relay Round 2 Solutions

SMO Team

RR 2 Part 1: In a regular hexagon $ABCDEF$, let X be a point inside the hexagon such that $XA = XB = 3$. If the area of the hexagon is $6\sqrt{3}$, then XE^2 can be expressed as $a + b\sqrt{c}$, where a, b, c are positive integers with c squarefree. Find $a + b + c$.

Answer: $\boxed{19}$

Solution: Let s be the sidelength of the hexagon. We have $\frac{6s^2\sqrt{3}}{4} = 6\sqrt{3} \implies s = 2$. Let h be the distance from X to AB. Thus, we have $\sqrt{h^2 + \left(\frac{s}{2}\right)^2} = 3 \implies h^2 = 8 \implies h = 2\sqrt{2}$. Since the distance from AB to DE is $2\sqrt{3}$,

$$XE = \sqrt{(2\sqrt{3} - 2\sqrt{2})^2 + \left(\frac{s}{2}\right)^2} \implies XE^2 = 21 - 8\sqrt{6} \implies 21 - 8 + 6 = \boxed{19}.$$

RR 2 Part 2: Let $T = TNYWR$. If

$$a = \sum_{n=1}^{N} n(n+1)(n+2),$$

find the last three digits of a.

Answer: $\boxed{890}$

Solution: We seek to find the last three digits of $\sum_{n=1}^{19} n(n+1)(n+2)$. Note that

$$\begin{aligned}
\sum_{n=1}^{19} n(n+1)(n+2) &= \sum_{n=1}^{19}(n^3 + 3n^2 + 2n) \\
&= \sum_{n=1}^{19} n^3 + 3\sum_{n=1}^{19} n^2 + 2\sum_{n=1}^{19} n \\
&= \left(\frac{19 \cdot 20}{2}\right)^2 + 3\left(\frac{19 \cdot 20 \cdot 39}{6}\right) + 2\left(\frac{19 \cdot 20}{2}\right) \\
&= 190^2 + (19 \cdot 39) \cdot 10 + 380 \\
&= 36100 + 7410 + 380 \\
&\equiv 100 + 410 + 380 \\
&\equiv \boxed{890} \pmod{1000}.
\end{aligned}$$

RR 2 Part 3: Let $T = TNYWR$. A point P is randomly chosen inside the square with vertices $A = (0,0), B = (0,T), C = (T,T)$, and $D = (T,0)$. Find the perimeter of the set S containing all points P for which $AP + CP \geq BP + DP$.

Answer: $\boxed{3560}$

Solution: Let $P = (x, y)$. We have

$$\begin{aligned}
&\sqrt{x^2+y^2}+\sqrt{(X-T)^2+(y-T)^2} \geq \sqrt{x^2+(y-T)^2}+\sqrt{(x-T)^2+y^2} \implies \\
&\quad x^2+y^2+(x-T)^2+(y-T^2)+2\sqrt{x^2+y^2}\cdot\sqrt{(X-T)^2+(y-T)^2} \geq \\
&\qquad x^2+(y-T)^2+(x-T)^2+y^2+2\sqrt{x^2+(y-T)^2}\cdot\sqrt{(x-T)^2+y^2} \implies \\
&\quad (x^2+y^2)((x-T)^2+(y-T)^2) \geq (x^2+(y-T)^2)(y^2+(x-T)^2) \implies \\
&\quad y^2(x-T)^2+(x^2)(y-T)^2 \geq x^2y^2+(x-T)^2(y-T)^2 \implies \\
&\quad (x^2-(x-T)^2)(y^2-(y-T)^2) \leq 0 \implies \\
&\quad (T^2-2xT)(T^2-2yT) \leq 0 \implies \\
&\quad \left(x-\frac{T}{2}\right)\left(y-\frac{T}{2}\right) \leq 0.
\end{aligned}$$

So, if we split the square into four smaller squares by drawing lines $x = \frac{T}{2}, y = \frac{T}{2}$, the set S is equivalent to the bottom left and top right squares. Thus, the perimeter of the set S is equivalent to $4T = 4 \cdot 890 = \boxed{3560}$.

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2024 SSMO Relay Round 3 Solutions

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RR 3 Part 1: Alice and Bob on the second floor of the building right next to the escalator moving upwards. Alice and Bob each run at 200 steps per minute and the escalator is a total of 225 steps. Alice chooses to use the escalator moving upwards, while Bob chooses to run to the escalator moving downwards, 300 steps away, and then ride the escalator down. If Alice and Bob hit the bottom floor at the exact same time, find the speed of the escalators in terms of steps per minute.

Answer: $\boxed{100}$

Solution: Let the speed of the escalators be s. We have the following Distance/Speed/Time chart.

Distance	Speed	Time
225	$200 - s$	a_1
300	200	b_1
225	$200 + s$	b_2

Since Alice and Bob reach the bottom floor at the same time, we have $a_1 = b_1 + b_2$. Since distance is the product of speed and time, we have

$$\begin{aligned}\frac{225}{200-s} &= \frac{3}{2} + \frac{225}{200+s} \implies \\ \frac{450s}{200^2 - s^2} &= \frac{3}{2} \implies \\ s^2 + 300s - 200^2 &= 0 \implies \\ (s+400)(s-100) &= 0 \implies \\ s &= \boxed{100}.\end{aligned}$$

RR 3 Part 2: Let $T = TNYWR$. Find the greatest odd integer n for which $n^2 + (T-1)n$ is a perfect square.

Answer: $\boxed{2401}$

Solution: We have $n^2 + 99n = s^2$, for some integer s. Factoring, we have

$$\left(\left(n + \frac{99}{2}\right) + s\right)\left(\left(n + \frac{99}{2}\right) - s\right) = \frac{99^2}{4} \implies$$

$$(2n + 99 - 2s)(2n + 99 + 2s) = 99^2.$$

To maximize n, we let $2n + 99 - 2s = 1$ and $2n + 99 + 2s = 99^2 = 9801$. This gives $n = \frac{\frac{9801+1}{2} - 99}{2} = \boxed{2401}$.

RR 3 Part 3: Let $T = TNYWR$. Riley and Boris are playing a game on a $(T-1)\times(T-1)$ grid of dots. The game alternates turns and starts with Riley. Each turn, a player draws a line connected two different random dots, exactly 1 unit apart. The first person to complete the first unit square loses the game. Given that Riley plays optimally and Boris plays randomly, the probability that Riley wins can be expressed as P. Find the least positive integer a such that aP is an integer.

Solution: Let the "score" of an edge be the number of unit squares it is part of. Note the $4T-4$ edges on the edge of the square are part of one unit square and the other $2(T-1)^2 - 2(T-1)$ edges all lie on two unit squares. Now, note that the first player that places an edge such that the total score of all edges placed exceeds $3(T-1)^2$ loses. Note that if Riley places all the edges, then after $4(T-1)$ turns, there will be a score of $12(T-1)$. Afterwards, there will be an odd score after Boris's turn each time and an even score after Riley's turn each time. Since $3(T-1)^2$ is even, this means that Boris places the edge that increases the total score to $3(T-1)^2+1$, in other words, Boris completes the first unit square. So, $P = 1$, meaning $a = \boxed{1}$.

2024 SSMO Relay Round 4 Solutions

SMO Team

RR 4 Part 1: Freddy the Frog can jump 1 unit right or up. He is at $(1, 1)$ and wants to get to $(7, 4)$. However, he is scared of points $(3, 1)$ and $(3, 2)$ and will not hop onto those points. How many ways can he reach his destination?

Answer: $\boxed{100}$

Solution: Using the Principle of Inclusion and Exclusion, the answer is the number of ways to get to $(7, 4)$ minus the number of ways to get to $(7, 4)$ passing through $(3, 1)$ minus the number of ways to get to $(7, 4)$ passing through $(3, 2)$ plus the number of ways to get to $(7, 4)$ passing through both $(3, 1)$ r $(3, 2)$. This is equivalent to

$$\binom{11}{4} - \binom{4}{1}\binom{7}{3} - \binom{5}{2}\binom{6}{2} + \binom{4}{1}\binom{6}{2} = 330 - 140 - 150 + 60 = \boxed{100}.$$

RR 4 Part 2: Let $T = TNYWR$. Regular octagon $OLYMPIAD$ is perfectly inscribed within Circle Q. Circle Q has area $T\pi$. If the area of octagon $OLYMPIAD$ is $a\sqrt{b}$, where a and b are positive integers with b squarefree. Find $a + b$.

Answer: $\boxed{202}$

Solution: Since the area of circle Q is 100π, the radius of circle Q is 10. Let the center of circle Q be Q_1. From the Law of Cosines on OQ_1L, we have

$$OL^2 = 10^2 + 10^2 - 2 \cdot \left(\frac{\sqrt{2}}{2}\right)(10)(10) = 200 - 100\sqrt{2}.$$

Now, since the area of an octagon with sidelength s is $s^2(2 + 2\sqrt{2})$, we have

$$[OLYMPIAD] = (200 - 100\sqrt{2})(2 + 2\sqrt{2}) = 200\sqrt{2} \implies 200 + 2 = \boxed{202}.$$

RR 4 Part 3: Let $T = TNYWR$. Real numbers a, b, and c satisfy

$$a + b = -ca^3 - abc \qquad = 4b^3 - abc = T$$

Find the value of $abc - c^3$.

Answer: $\boxed{206}$

Solution: Since $a + b = -c \implies a + b + c = 0$, we have

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0.$$

So, $4 + T - x = 0 \implies x = 4 + T = \boxed{206}$.

2024 SSMO Relay Round 5 Solutions

SMO Team

RR 5 Part 1: Let the super factorial $!(n)$ be defined on positive integers as $\prod_{i=1}^{n} i!$. Find the largest positive integer k such that that there are exactly k positive integers n such that $!(n)$ has fewer than k trailing zeroes.

Answer: $\boxed{12}$

Solution: Let $f(n)$ be the number of trailing zeroes $!(n)$ has and $g(n)$ denote the number of trailing zeroes $n!$ has. Clearly, $f(n) = f(n-1) + g(n)$. Consider the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
g(n)	0	0	0	0	1	1	1	1	1	2	2	2	2	2
f(n)	0	0	0	0	1	2	3	4	5	7	9	11	13	15

It is easy to see that for $k = 12$, exactly 12 integers such that $!(n)$ has fewer than 12 trailing zeroes. Thus, the answer is $\boxed{12}$.

RR 5 Part 2: Let $T = TNYWR$. In the game of high and low, the computer chooses two integers without replacement from the set $\{1, 2, 3, \ldots, T\}$. The computer displays the first integer and asks the player if the second integer is higher or lower. Given that the player always plays optimally, the chances of guessing correctly can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{39}$

Solution: Note that if the computer chooses $k \leq 6$, then there is a $\frac{12-k}{11}$ chance that the player guesses correctly and if $k \geq 6$, then there is a $\frac{k-1}{11}$ chance that the player guesses correctly. Thus, the answer is

$$\frac{2(6 + \cdots + 11)}{11 \cdot 12} = \frac{17}{22} \implies 17 + 22 = \boxed{39}.$$

RR 5 Part 3: Let $T = TNYWR$. Let k be the maximum prime factor that divides T. How many values of x satisfy

$$\sin x^2 + \cos x^2 = \sin^2 x + \cos^2 x \quad \text{and} \quad -k \leq x \leq k?$$

Answer: $\boxed{107}$

Solution: Note that $k = 13$. Now, $\sin^2 x + \cos^2 x = 1$, so we are seeking to find the number of solutions to $\sin x^2 + \cos x^2 = 1$. The only solutions are when $x^2 = 2k\pi, \frac{\pi}{2} + 2k\pi$, for integer

k. Now, for $-13 \le x \le 13$, we have $x^2 \le 169$. Now, note that $52\pi + 0.5 \le 169 < 54\pi$. So, there are 54 possible values for x^2, each giving two unique solutions, except $x^2 = 0$. In conclusion, the answer is $54 \cdot 2 - 1 = \boxed{107}$.