2023 SSMO Team Round

SMO Team

TR 1: Let (a, b, c, d) be a permutation of $(2, 0, 2, 3)$. Find the largest possible value of $a^b + b^c + c^d + d^a$

TR 2: A plane and a car start both move northward. The car moves northbound at 60 miles per hour. The plane moves northeast and increases in altitude at an angle of 30°. Let s the speed in feet per second that the plane must fly at to move north at the same speed as the car. Find $3s^2$.

TR 3: Let ABC be a triangle such that $AB = 4\sqrt{2}, BC = 5\sqrt{2}$, and $AC = \sqrt{82}$. Let ω be the circumcircle of $\triangle ABC$. Let D be on the circle such that $\overline{BD} \perp \overline{AC}$. Let E be the point diametrically opposite of B. Let F be the point diametrically opposite D. Find the area of the quadrilateral $ADEF$ in terms of a mixed number $a\frac{b}{c}$. Find $a + b + c$.

TR 4: Find the sum of values for prime p such that $p \mid (2023^{p^2} + (p-1)! + 2^{p^4})$.

TR 5: Joshy is playing a game with a dartboard that has two sections. If Joshy hits the first section, he gets 20 points, and if he hits the second section, he gets 23 points. Assume Joshy always hits one of the two sections. Let a be the maximum value that Joshy cannot achieve. Let b be the number of positive integer scores Joshy cannot achieve. Let c be the number of ways for Joshy to achieve 2023 points. Find $(a - b)c$.

TR 6: Suppose that a, b, c are positive reals satisfying

$$(a^3 + 4)(b^3 + 6)(c^3 + 8) = 8(a + b + c)^3.$$

Find the sum of all possible values of $\frac{bc}{a^2}$. If you believe there are no solutions, put 0 as your answer. If you believe the sum is infinity, put 1000 as your answer.

TR 7: Let $S = \{1, 2, 3, 4, \cdots, 23\}$ and let there be randomly chosen sets A, B, C where $A, B, C \subseteq S$. The probability that $|A| + |B| = |C|$ can be expressed as $\frac{m}{n}$. Let 2^a be the largest power of 2 such $2^a \mid n$. Find a.

TR 8: Three rabbits run away from the origin at the same speed and constant velocity such that the angle between any two rabbits' directions is 120°. After 12 seconds, a hunter with a speed $\sqrt{7}$ times that of the rabbits runs from the origin. Let the minimal time in seconds needed for her to meet (and subsequently) catch all three rabbits be $a + b\sqrt{c}$. Find $a + b + c$.

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TR 9: Let B, K, and R be the total number of possible moves for a bishop, knight, or rook from any position of a 9 by 9 grid. Find $B+K+R$.

(A bishop moves along diagonals, a rook moves along rows, and a knight moves in the form of a 2×1 "L" shape)

TR 10: There exists a lane of infinite cars. Each car has a $\frac{1}{3}$ chance of being high quality and a $\frac{2}{3}$ chance of being low quality. John goes down the row of cars buying high-quality cars. However, after John sees 3 low-quality cars, he gives up on buying additional cars. Let the probability that he buys at least 5 cars before giving up as $\frac{m}{n}$. Find $m+n$.

TR 11: Let $ABCD$ be a cyclic quadrilateral such that AC is the diameter. Let P be the orthocenter of ABD. Define $X = AB \cup CD$, and $Y = AD \cup BC$. If $AB = 8$, $BC = 1$, and $CD = 4$, suppose $\frac{[CBPD]}{[AXY]} = \frac{m}{n}$. Find $m+n$.

TR 12: Let T be the set of rationals of the form $\frac{a}{2^b}$ for nonnegative a and b. Define the function $f\colon T \to \mathbb{Z}$ such that, for $t = \frac{a}{2^b}$ such b is minimal, we have that

$$f\left(\frac{a}{2^b}\right) = \begin{cases} 1 & b = 0 \\ f\left(\frac{a-1}{2^b}\right) + f\left(\frac{a+1}{2^b}\right) & b \neq 0 \end{cases}$$

Suppose

$$\sum_{i=0}^{2^{10}-1} \frac{f\left(\frac{i+1}{2^{10}}\right)}{f\left(\frac{i}{2^{10}}\right)}$$

equals $\frac{m}{n}$. Find $m+n$.

TR 13: Let $D(n)$ denote the product of all divisors of n Let $P(i,j)$ denote the set of all integers that are both a multiple of i and a factor of j. Let

$$-F(a) = \sqrt{\left|\log_{10}\left(\frac{D(10^a)}{\prod_{\omega \in P(10^2, 10^{a+2})} \omega}\right)\right|} \text{ and } G(n) = \sqrt[n-1]{\prod_{i=2}^{n} 10^{F(i)}}.$$

Suppose $\sum_{k=2}^{\infty} G(k)$ is $\frac{a+b\sqrt{c}}{d}$. Find the value of $a+b+c+d$.

TR 14: Find the sum of all perfect squares of the form $2p^3 - 5p^2q + q^2$ where p and q are positive integers such p is prime and $p \nmid q$.

TR 15: Consider a piece of paper in the shape of a regular pentagon with sidelength 2. We fold it in half. We then fold it such that the vertices of the longest side become the same side. The area of the folded figure can be expressed as $\frac{1}{a}\sqrt{b+c\sqrt{5}}$ where a, b, c are integers and $\gcd(b,c)$ is squarefree. Find $a+b+c$. (For convenience, note that $\cos(36^\circ) = \frac{1+\sqrt{5}}{4}$)