2022 SSMO Team Round

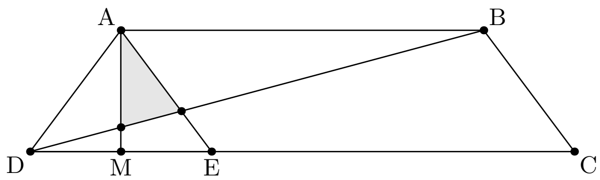
SMO Team

TR 1: In triangle ABC, circumcircle ω is drawn. Let I be the incenter of $\triangle ABC$. Let H_A be the intersection of the A-altitude and ω. Given that $AB = 13, AC = 15$, and $BC = 14$, the area of triangle AIH_A can be expressed as $\frac{m}{n}$, for relatively prime positive integers m and n. Find $m + n$.

TR 2: Consider 8 marbles in a line, where the color of each marble is either black or white and is randomly chosen. Define the period of a lineup of 8 marbles to be the length of the smallest lineup of marbles such that if we consider the infinite repeating sequence of marbles formed by repeating that lineup, the original lineup of 8 marbles can be found within that sequence.

A good ordering of these marbles is defined to be an ordering such that the period of the ordering is at most 6. For example, $bwwbbbww$ is a good ordering because we may consider the lineup $bwwbb$, which has a length equal to 5. If the probability that the marbles form a good ordering can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

TR 3: Let $ABCD$ be an isosceles trapezoid such that $AB \parallel CD$. Let E be a point on CD such that $AB = CE$. Let the midpoint of DE be M such that BD intersects AM at G and AE at F. If $DC = 36, AB = 24$, and $AD = 10$, then $[AGF]$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

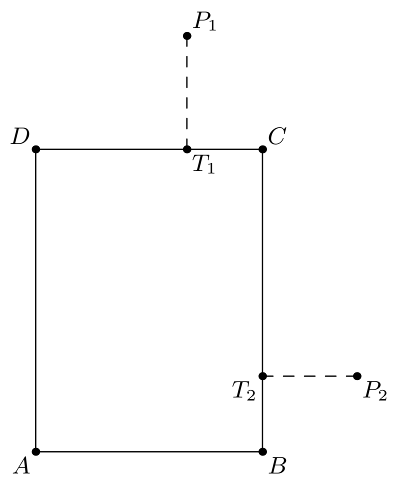


TR 4: Let $a_1 = 2, b_0 = 3, a_n = (a_{n-1})^2$, and $b_n = (b_{n-1})^3$. If $c_n = a_n + b_n$, find the last two digits of $c_1 + c_2 + \cdots + c_{2022}$.

TR 5: Consider the following rectangle $ABCD$ where $BC = 8$. If

$$CD = CT_2, 4T_2P_2 = 2DP_1 = AC, [ADP_1C] = 33, \text{ and } [ABP_2C] = 34,$$

find the value of $[P_1CP_2A]$. (Note that $[ABC]$ is the area of ABC.)



TR 6: Let n be a positive integer, and let x be some variable. Define $P_{x,n}$ as the maximum fraction of elements in the set of the first x natural numbers that may be contained in a subset S such that if k is an element of S, then nk is not. For example, $P_{3,3} = \frac{2}{3}$, since we take the set $\{1, 2\}$. As x approaches infinity, $P_{x,n}$ approaches a value P_n. Given that $\prod_{n=2}^{100} P_n = \frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

TR 7: Let $\cos(2A)$, $\cos(2B)$, and $\cos(2C)$ be the not necessarily distinct roots of a monic cubic f. Given that $f(1) = \frac{17}{30}$, the value of $\sin(A)\sin(B)\sin(C)$ can be expressed as $\frac{\sqrt{m}}{n}$, with m squarefree. Find $m + n$.

TR 8: A frog is at 0 on a number line and wants to go to 9. On each turn, if the frog is at n, the frog hops to one of the numbers from n to 9, inclusive, with equal probability (staying in place counts as a hop). It is then teleported to the largest multiple of 3 that is less than or equal to the frog's position. The expected number of hops it takes for the frog to reach 9 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

TR 9: Given real numbers a, b, x, y such that

$$a^2 + b^2 = 1,$$
$$x^2 + y^2 = 1,$$
$$abxy - \frac{1}{8} = b^2y^2,$$

find the sum of all distinct values of $(a + b + x + y)^2$.

TR 10: If p, q, r are the roots of the polynomial $x^3 - 2x^2 - 4$, find

$$(p^3 + qr)(q^3 + pr)(r^3 + pq).$$

TR 11: Find the number of solutions to $a^3 + 36ab + 64 = 27b^3$, where a is an integer, b is a real number, and $|b| \leq 50$.

TR 12: Regular pentagon $ABCDE$ is inscribed in circle ω_1 with radius $5\sqrt{5}$. Circle ω_2 is the reflection of ω_1 across $\overline{AB}$. Let I be the intersection of $\overline{AD}$ and $\overline{BE}$, let P be an intersection of $\overline{DO}$ and ω_2, and let line ℓ be the tangent to ω_2 at P. The sum of the possible distances from point I to line ℓ can be expressed as $m\sqrt{n}$, where n is a squarefree positive integer. Find $m + n$.

TR 13: In regular hexagon $ABCDEF$ with side length 1, an electron starts at point A. When the electron hits an edge, it reflects off of it, with the angle of reflection equal to the angle of incidence. The electron first travels in a straight line to a point on edge CD. The electron bounces off of a total of 5 edges before hitting a vertex. The electron stops and its total distance traveled is measured. If the shortest possible distance the electron could have traveled can be expressed as $\sqrt{m}$, find m.

TR 14: On a hot summer day, three little piggies decide to play with water balloons. The three piggies travel to a 200-floor parking garage each armed with exactly one water balloon. The game works as follows:

1. If a piggie drops a water balloon from any floor of the building, it will either break, or it will survive the fall.
2. If the water balloon breaks, then any greater fall would have broken it as well.
3. If the water balloon survives, then it would have survived any lesser fall.
4. Every water balloon is identical and interchangeable.

The goal for the piggies is to find the lowest floor that will break a water balloon. Assuming they play optimally, what is the minimum number of tries in which they are guaranteed to find the lowest balloon-breaking floor?

TR 15: Consider two externally tangent circles ω_1 and ω_2 with centers O_1 and O_2. Suppose that ω_1 and ω_2 have radii of 1 and 3 respectively. There exist points A, B on ω_1 and points C, D on ω_2 such that AC and BD are the external tangents of ω_1 and ω_2. The circumcircle of $\triangle BO_2D$ intersects AC at two points X and Y such that $AX < AY$. If CX can be expressed as $\frac{\sqrt{m}}{n}$, where m and n are relatively prime positive integers, find $m + n$.