Numerical PDE's final Project

Peter Solfest Code available at `https://github.com/solter/python-FEM`

1 Problem 1

The equation $u_t + \left(\frac{u^2}{2}\right)_x = \varepsilon u_{xx}$ was solved on the interval $x \in [-1, 1]$, $\varepsilon = 0.01$ until $T = 0.1$.

With the boundary conditions $u(-1, t) = u(-1, 0)$ and $u(1, t) = U(1, 0)$, and initial conditions $u(x, 0) = -.5x + .5$ and $u(x, 0) = 1 - x^2$.

$[-1, 1]$ was meshed into N equi-length intervals with $N = 40, 80$.

Both a standard FEM method and a streamline diffusion method were used to solve the system, using forward euler for the time integration, with 1 and 3 point quadrature schemes used for the nonlinear term.

The figures in appendix A display the solutions.

The results indicate that both streamline and standard FEM methods suffer from oscillations in the solution near the non-0 edge, and mesh refinement exacerbates the oscillations.

2 Problem 2

The equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ was solved on the interval $x \in [-1, 1]$, until $T = 0.05, 0.1$ and 0.2.

A periodic boundary condition was imposed, with the initial condition $u(x, 0) = 0.5(1 + \sin(\pi t))$.

$[-1, 1]$ was meshed into 160 equi-length intervals.

Finite volume methods were used to solve this. An ENO scheme (both 3^{rd} and 1^{st} order) were used for the interface value reconstructions. Both forward euler and a TVD RK3 solver were used for the time integration, and the numerical fluxes were reconstructed using Godunov and Global Lax-Friedrichs (GLF) schemes.

The figures in appendix B display the solutions.

The results indicate that for first order solutions (in time and reconstruction) the Godunov experiences an oscillation as it approaches a shock, this is especially noticeable after letting time run to 1. Whereas GLF's variation seems to be bounded by the original function, even as it approaches the shock.

When third order reconstructions and time stepping are used, the bounded variation displayed by the GLF scheme disappears.

The explosion in the Godunov solution is surprising, as ENO reconstructions are supposed to display TVB behavior.

A Problem 1 Figures

Note that above each figure represents different times during the solution, with the initial solution plotted for reference. The figures are labelled via the number of intervals (N), whether a standard or streamline method was used, and the initial condition.

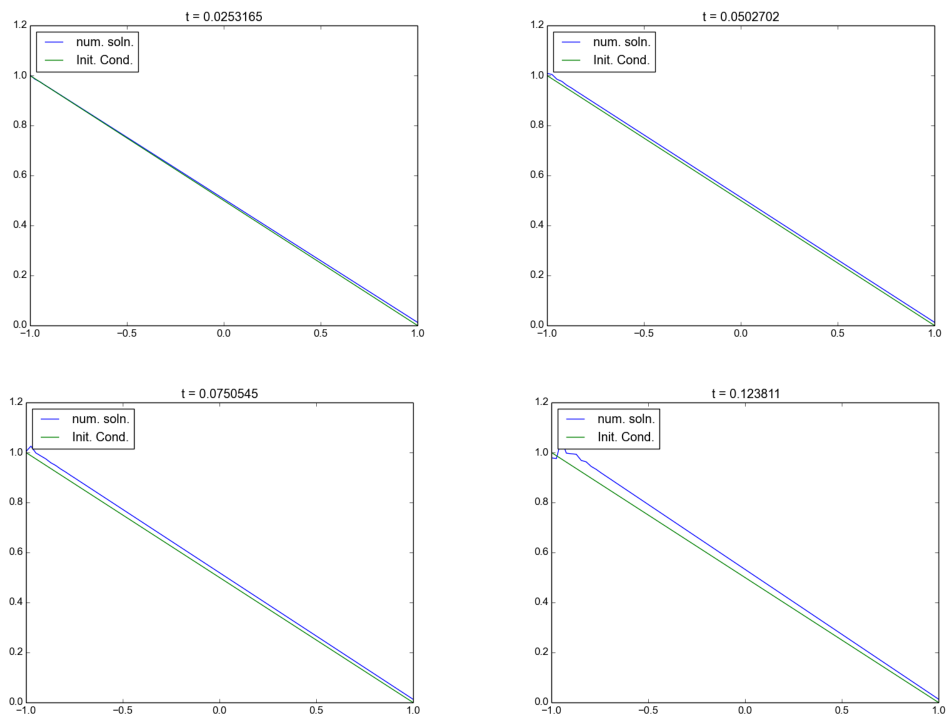
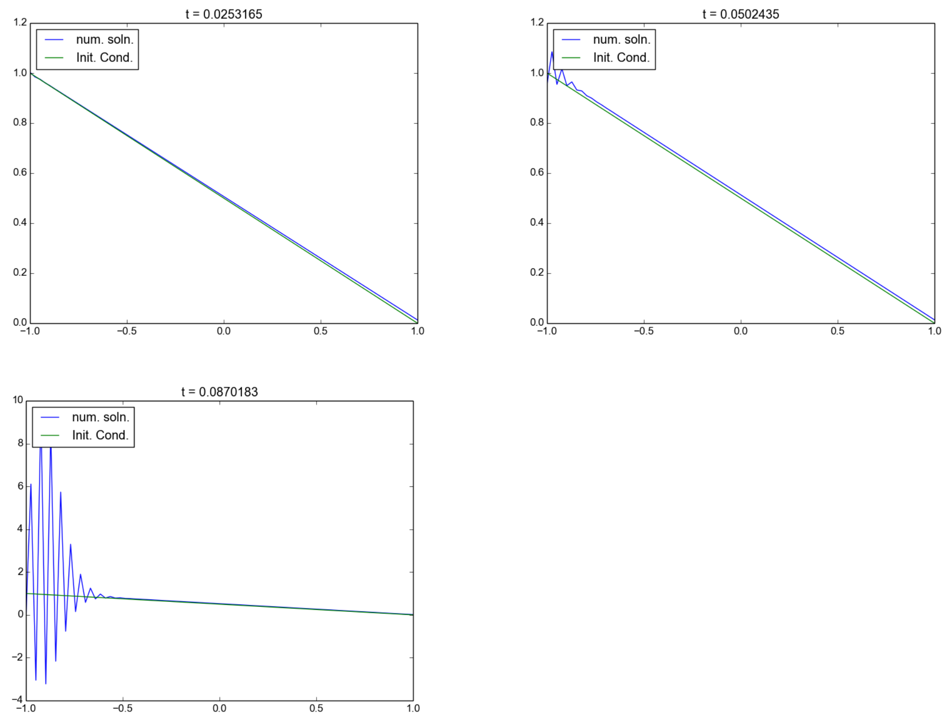


Figure 1: $N = 40$ via standard with $u(x, 0) = -.5x + .5$



Solution diverged

Figure 2: $N = 40$ via streamline with $u(x, 0) = -.5x + .5$

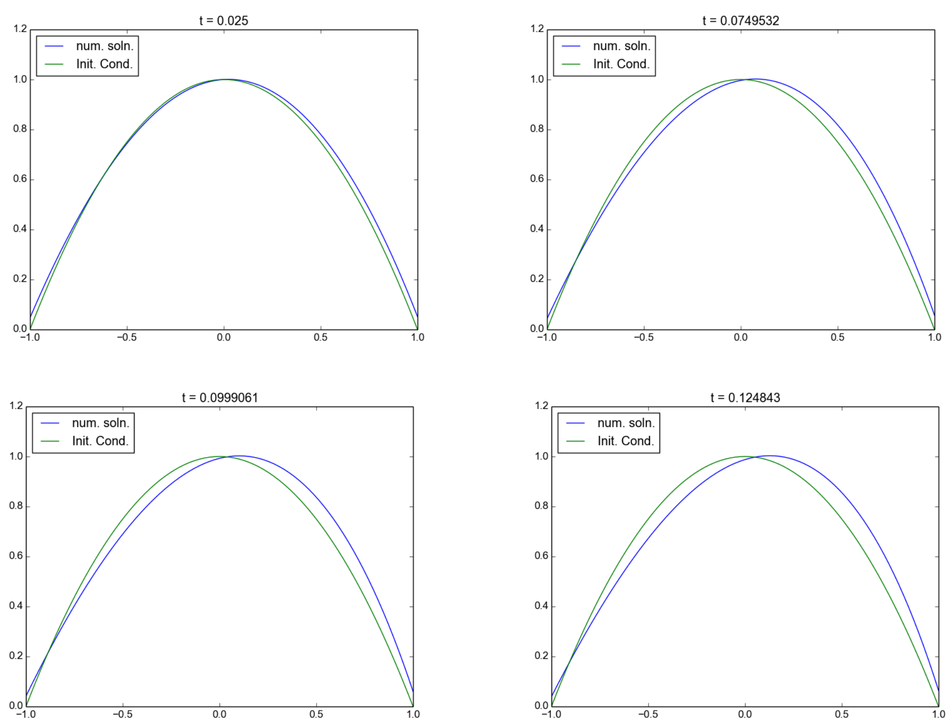


Figure 3: $N = 40$ via standard with $u(x, 0) = 1 - x^2$

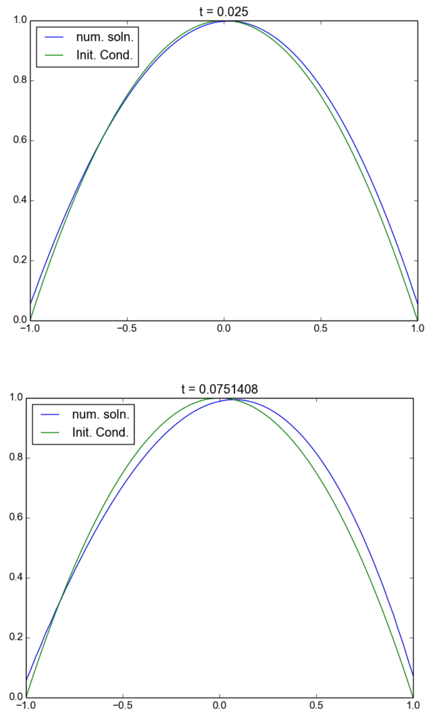


Figure 4: $N = 40$ via streamline with $u(x, 0) = 1 - x^2$

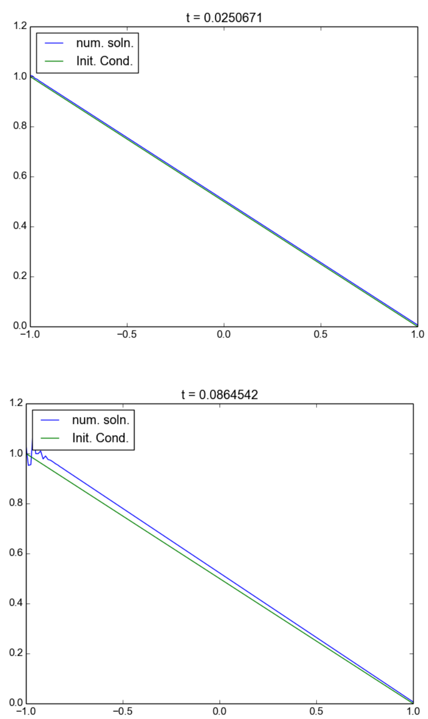
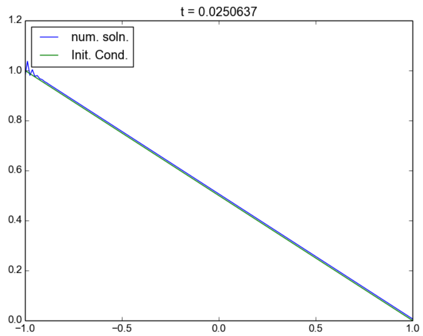


Figure 5: $N = 80$ via standard with $u(x, 0) = -.5x + .5$



Solution diverged

Figure 6: $N = 80$ via standard with $u(x, 0) = -.5x + .5$

B Problem 2 Figures

Note that above each figure represents different times during the solution, with the initial condition plotted for reference. Beneath each group of 4 figures the following code is used

- RO: The ENO reconstruction accuracy used
- OT: The order of accuracy for the time integration method (1 is forward euler, 3 is TVD RK3)

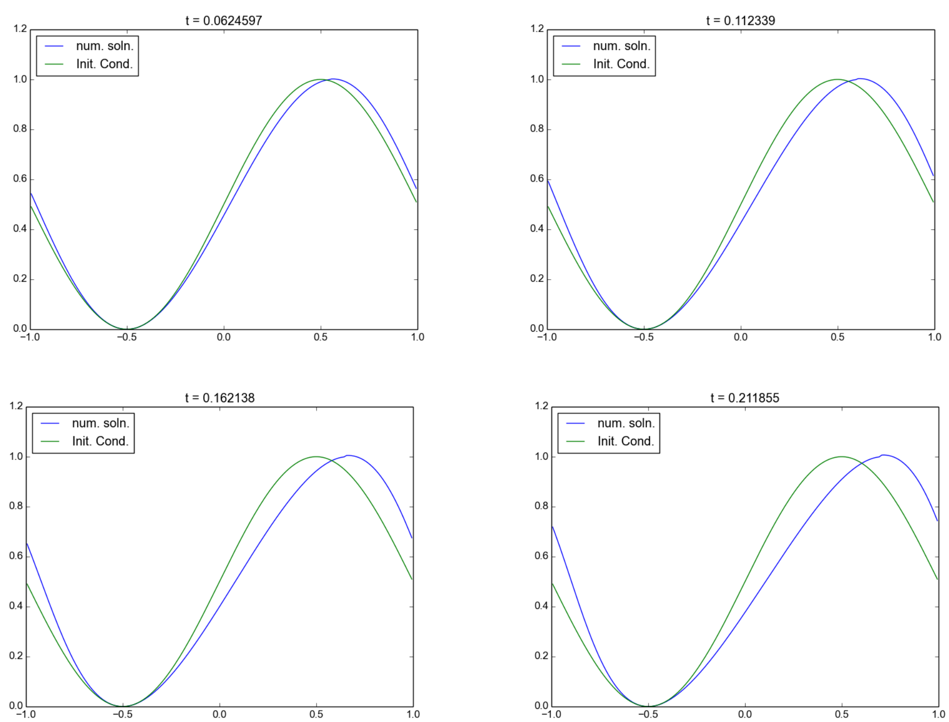


Figure 7: RO = 1, OT = 1, Godunov

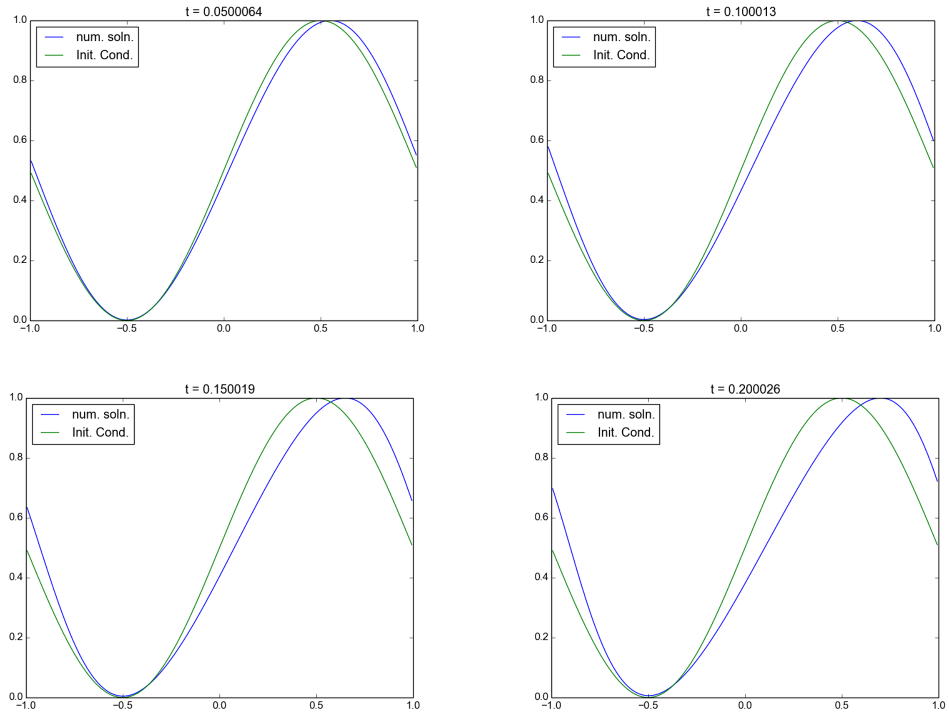


Figure 8: RO = 1, OT = 1, GLF

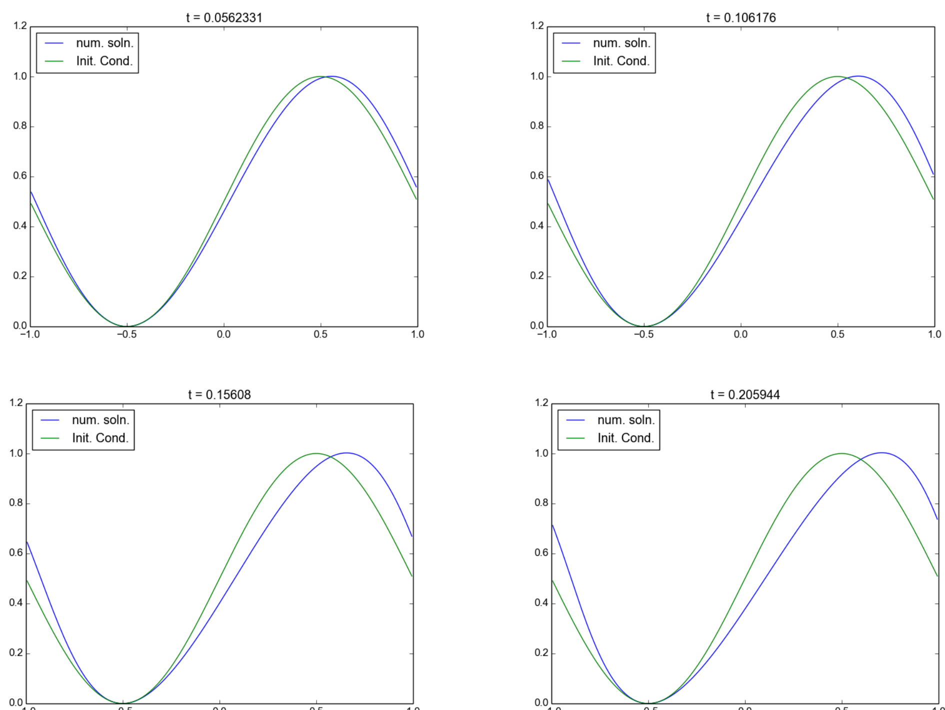


Figure 9: RO = 3, OT = 3, Godunov

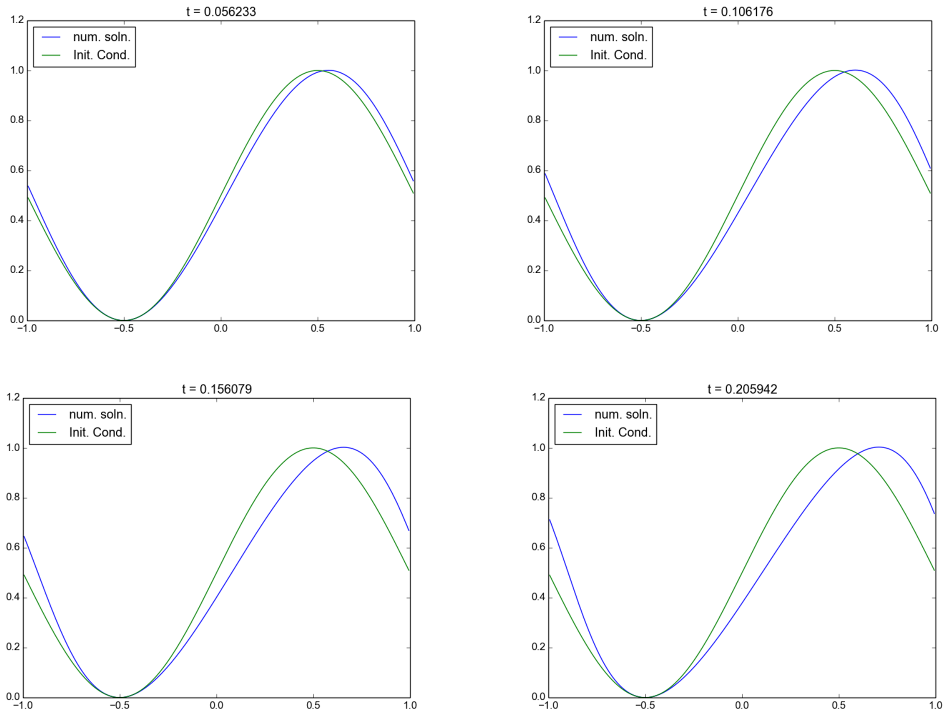


Figure 10: RO = 3, OT = 3, GLF

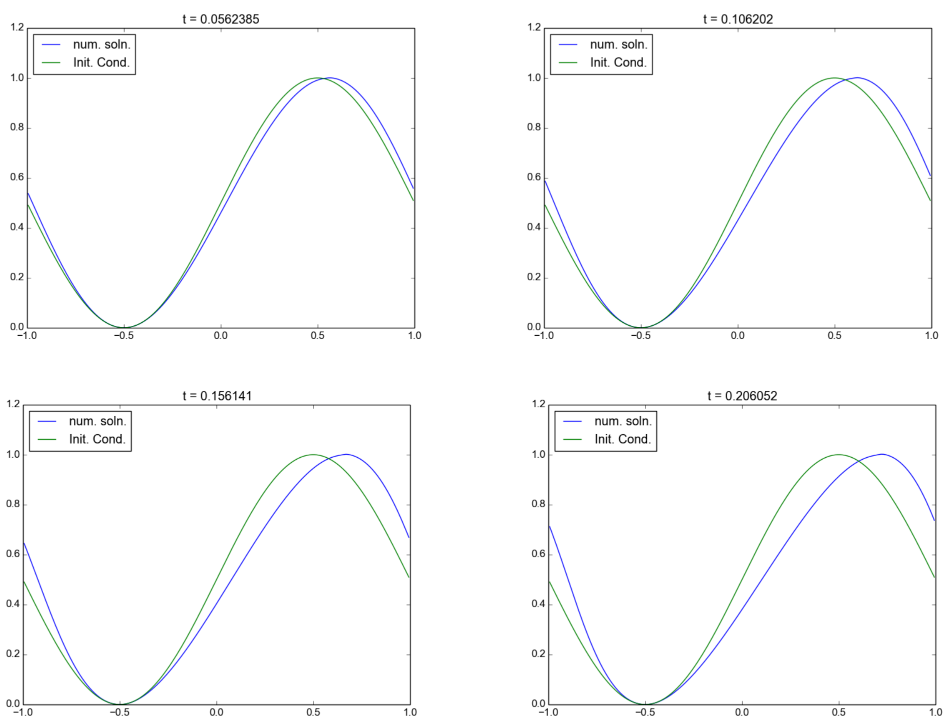


Figure 11: RO = 3, OT = 1, Godunov

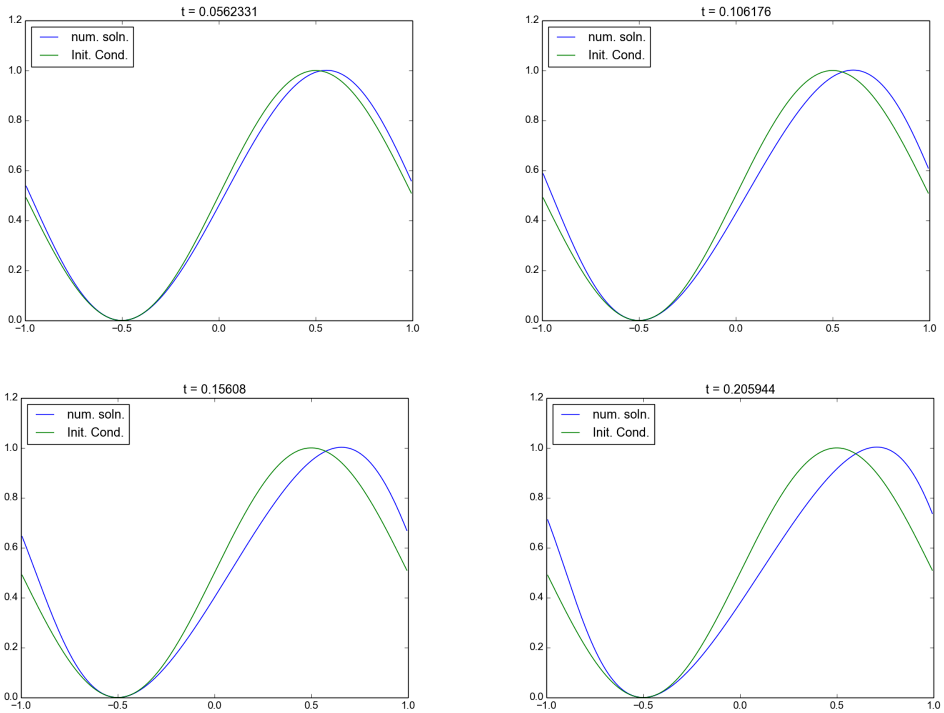


Figure 12: RO = 1, OT = 3, Godunov

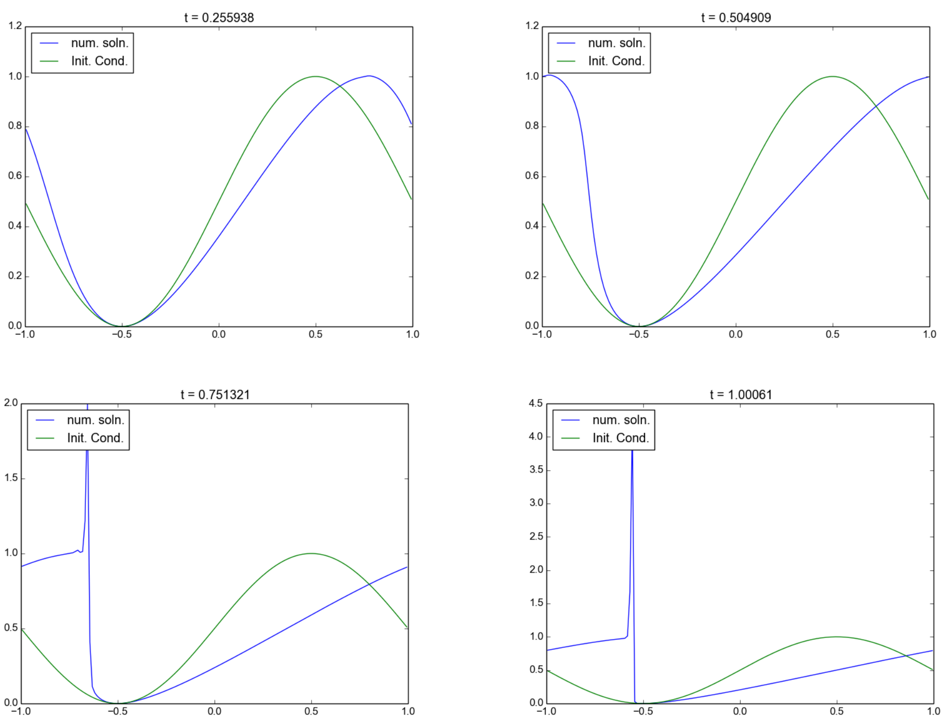


Figure 13: RO = 1, OT = 1, Godunov

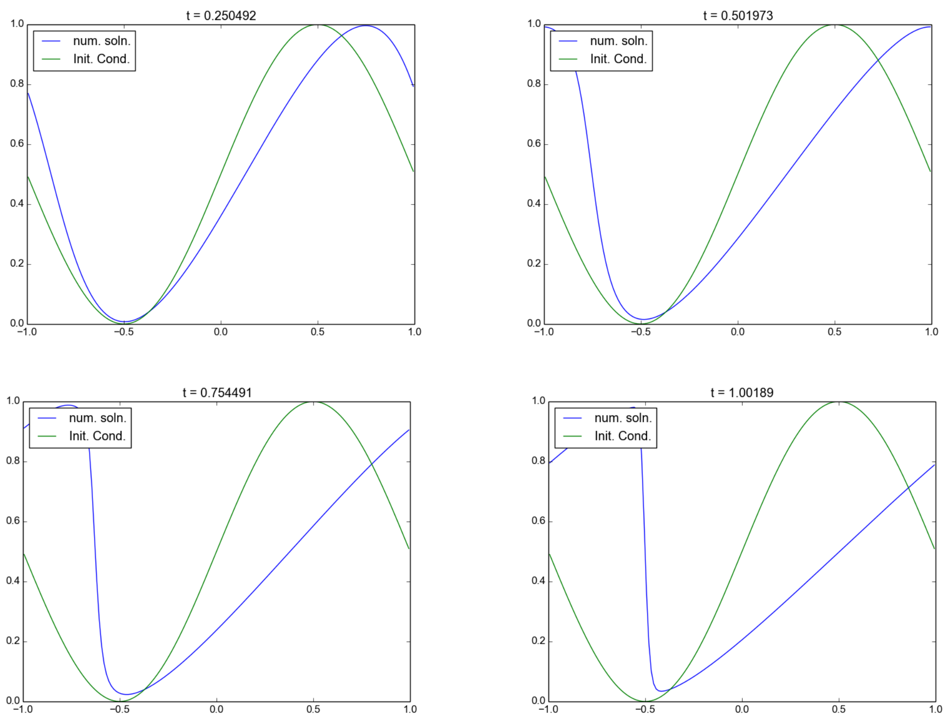


Figure 14: RO = 1, OT = 1, GLF