A Simple Auerbach-Kotlikoff Model

Accompanying model description for the `solveOLG_xxx` codes.

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This document briefly describes the model solved in the `solveOLG_xxx` codes.

Demography

The law of motion for population is

$$N_t^a = N_{t-1}^{a-1}\gamma_{t-1}^{a-1},\ \forall a \in [1, \overline{a}], \quad \text{and} \quad N_t^0 = NB_t.$$

N_t^a: mass of persons with age a at time t, γ: conditional survival probability, NB: newborns.

Superscript a is age. t is the period index. Alternatively, it is often useful to switch to cohort view, tracking variables indexed by year of birth z, with $z = t - a$, i.e. $N_z^a = N_{t=z+a}^a$. In cohort view the law of motion of the population is

$$N_z^a = N_z^{a-1}\gamma_z^{a-1},\ \forall a \in [1, \overline{a}], \quad \text{and} \quad N_z^0 = NB_z.$$

Persons with $a < \underline{a}$ are children who do not make any economic decisions.

Household problem

A representative household of cohort z solves the following problem[1])

$$U^a = \max_{C^a, \ell^a} \ u(C^a, \ell^a) + \beta\gamma^a U^{a+1}, \quad \forall a \in [\underline{a}, \overline{a}] \quad \text{s.t.}$$

$$A^{a+1} = (1 + r^a)\left[A^a - p^{C,a}C^a + y^a + iv^a + ab^a\right],$$

[1] I drop the cohort index z in this section.

$$y^a = \phi^a(1-\tau^{W,a})\ell^a\theta^a w^a + (1-\phi^a)(1-\tau^{W,a})p^a - \tau^{l,a}.$$

C: consumption, ℓ: hours worked, β: discount factor, A: household assets, r: real interest rate, $p^C = (1+\tau^C)$: after-tax price of consumption, y: net labor and pension income, iv: net inter-vivo transfers received, ab: accidental bequests received, ϕ share of non-retired households, θ: productivity, w: wage rate per efficiency unit, $\tau^{W,a}$: tax on labor and pension income, p: pension income,τ^l: lump-sum tax. Children do not work, pay no taxes and receive no transfers or bequests.[2] They start adulthood with zero assets, i.e. $A^{\underline{a}} = 0$.

Envelope and optimality conditions

Define the shadow price of assets as $\lambda^a = \frac{\partial U^a}{\partial A^a}$.

$$\begin{aligned}
A^a: &\quad \lambda^a = \gamma^a\beta(1+r^a)\lambda^{a+1}, \\
C^a: &\quad u_{C^a} = \gamma^a\beta(1+r^a)\lambda^{a+1}p^{C,a}, \\
\ell^a: &\quad -u_{\ell^a} = \gamma^a\beta(1+r^a)\lambda^{a+1}\frac{\partial y^a}{\partial \ell^a},
\end{aligned}$$

This gives the Euler equation and the labor supply function:

$$\frac{u_{C^a}}{p^{C,a}} = \lambda^a, \quad \Rightarrow \quad u_{C^a} = \left[\frac{p^{C,a}}{p^{C,a+1}}\beta(1+r^a)\gamma^a\right]u_{C^{a+1}},$$

$$-\frac{u_{\ell^a}}{u_{C^a}} = 1/p^{C,a}\frac{\partial y^a}{\partial \ell^a} = 1/p^{C,a}\phi^a(1-\tau^{W,a})\theta^a w^a.$$

Functional form utility function

Sub-disutility of labor[3] is $\varphi(\ell^a) = \varphi_0\frac{\ell^{1+1/\sigma^L}}{1+1/\sigma^L} + \varphi_1^a$. I assume two different options for $u()$:

1. With income effects:
$$u(C^a, l^a) = \frac{(C^a)^{1-1/\sigma}}{1-1/\sigma} - \varphi(\ell^a).$$

   This just gives a life-time profile of consumption. To pin down the consumption values one has to numerically find the initial consumption level ($C^{\underline{a}}$) that exhausts the budget constraint, i.e. assets at end of life are zero: $A^{\overline{a}} - p^{C,\overline{a}}C^{\overline{a}} + y^{\overline{a}} + iv^{\overline{a}} + ab^{\overline{a}} = 0$.

   With this specification the left-hand side of the labor supply function is given as
$$-\frac{u_{\ell^a}}{u_{C^a}} = \varphi_0(\ell^a)^{1/\sigma^L}(C^a)^{1/\sigma}.$$

[2] We keep track of the number of children as they affect government consumption.

[3] φ_1^a is an additive age-depedent shifter.

2. Without income effects:

$$u(C^a, l^a) = \frac{\left[C^a - \varphi(\ell^a)\right]^{1-1/\sigma}}{1-1/\sigma}.$$

The left-hand side of the labor supply function is now independent of C^a

$$-\frac{u_{\ell^a}}{u_{C^a}} = \varphi_0(\ell^a)^{1/\sigma^L}.$$

This enables us to explicitly solve for the consumption function[4]:

$$\begin{aligned}
\Lambda^a &= (p^{C,a})^{1-\sigma} + (\beta\gamma^a)^{\sigma}(1+r^a)^{\sigma-1}\Lambda^{a+1}, \\
H^a &= y^a - \varphi(\ell^a)p^{C,a} + iv^a + ab^a + H^{a+1}/(1+r^a), \\
\Omega^a &= \Lambda^a(p^{C,a})^{\sigma-1}, \\
C^a &= (A^a + H^a)/(p^{C,a}\Omega^a) + \varphi(\ell^a).
\end{aligned}$$

H: human wealth (discounted future labor and pension income), Ω: marginal propensity to consume out of life-time wealth.

Aggregation

To aggregate household variables one must first convert them to period view and then compute weighted sums. E.g. aggregate consumption is $C_t = \sum_a C_t^a N_t^a$. Total effective labor supply is $L_t^S = \sum_a \phi_t^a \ell_t^a \theta_t^a N_t^a$ and so on.

Firm problem

The representative firm solves the following optimization problem

$$V_t = \max_{I_t, L_t^D} \; \chi_t + V_{t+1}/(1+r_t), \quad s.t.$$

$$\begin{aligned}
\chi_t &= Y(K_t, L_t^D) - (1+\tau_t^F)w_t L_t^D - I_t - T_t^F, \\
T_t^F &= \tau_t^{prof}\left[Y(K_t, L_t^D) - (1+\tau_t^F)w_t L_t^D - \delta K_t\right], \\
K_{t+1} &= (1-\delta)K_t + I_t.
\end{aligned}$$

V: discounted value of the firm, L: aggregate labor in efficiency units, I: investment, χ: per-period dividends, τ^F: pay-roll tax rate, T^F: profit taxes, K: capital stock, δ: depreciation rate.

[4] See e.g. Schuster (2021): "The FISK OLG model - A numerical overlapping generations model for Austria", Working paper of the Office of the Austrian Fiscal Advisory Council 7, for a more general proofs, also for other results in this note.

Envelope and optimality conditions

Define the shadow price of capital as $q_t = \frac{\partial V_t}{\partial K_t}$.

$$K_t: \quad q_t = (1-\tau_t^{prof})Y_{K_t} + \tau_t^{prof}\delta + \frac{q_{t+1}}{1+r_t}(1-\delta),$$

$$I_t: \quad 1 = \frac{q_{t+1}}{1+r_t},$$

$$L_t^D: \quad Y_{L_t^D} = (1+\tau_t^F)w_t.$$

This implies

$$Y_{K_t} = \frac{r_{t-1} + \delta(1-\tau_t^{prof})}{1-\tau_t^{prof}}.$$

The right-hand side is user cost of capital uck_t. We can think of $Y_{K_t}()$ as the inverse capital demand function that returns the user cost of capital for a given capital stock consistent with optimal behavior of the firm. I assume $Y()$ to be linearly homogenoeous. Therefore, Hayashi's theorem applies: $V_t = q_t K_t$. Inserting this in the capital demand function gives an implicit relationship of r_t and V_{t+1} that can be used to find the asset market clearing interest rate (all other things constant).

$$Y_{K_{t+1}}(V_{t+1}/(1+r_t)) = \frac{r_t + \delta(1-\tau_{t+1}^{prof})}{1-\tau_{t+1}^{prof}}.$$

Functional form production function

Here I assume: $Y(K,L) = TFP \cdot K^{\alpha}L^{1-\alpha}$. Therefore, $Y_K = TFP \cdot (K/L)^{\alpha-1}$ and $Y_L = TFP \cdot (K/L)^{\alpha}$.

Government

Total government expenditures are:

$$Exp_t = C_t^G + P_t, \quad \text{with} \;\; C_t^G = \sum_a c_t^{G,a} N_t^a, \quad P_t = \sum_a (1-\phi_t^a) p_t^a N_t^a,$$

Total government revenues are:

$$Rev_t = T_t^F + (\tau_t^F L_t^D + \tau_t^W L_t^S)w_t + \sum_a \tau_t^{l,a} N_t^a + \tau_t^C C_t + \tau_t^W P_t.$$

One of the government instruments is always set such that the primary balance ($PB_t = Rev_t - Exp_t$) is consistent with an exogenous debt trajectory following the law of motion

$$D_{t+1}^G = (1+r_t)\left[D_t^G - PB_t\right].$$

Equilibrium

Equilibrium in the closed economcy is given by optimal behavior by households (aggregated accordingly) and the representative firm and the following market clearing conditions

$$\zeta_t^Y = I_t + C_t + C_t^G - Y_t = 0,$$

$$\zeta_t^A = D_t^G + V_t - A_t = 0,$$

$$\zeta_t^L = L_t^D - L_t^S = 0,$$

$$\zeta_t^G = Rev_t - Exp_t - PB_t = 0,$$

and the resource constraints for intervivo transfers ($\zeta^{iv} = 0$) and ($\zeta^{ab} = 0$) binding.

It is good practice to always check if Walras' Law holds (even out of equilibrium):

$$\zeta_t^W = \zeta_t^Y + w_t \zeta_t^L + \zeta_t^G + \zeta_t^{iv} + \zeta_t^{ab} + \zeta_t^A - \frac{\zeta_{t+1}^A}{1 + r_t} = 0.$$

Alternative closure: Small open economy

Instead of the closed economy setting from above I now consider the small open economy case. Domestic and foreign goods as well as assets are perfect substitutes, respectively. The international price of the homogeneous good is 1 and the international real rate r_t is exogenously given. While the household problem is unaltered, I have to introduce capital adjustment costs J_t to the firm problem for realistic dynamics. The following equations replace their counterparts from above

$$\chi_t = Y(K_t, L_t^D) - (1 + \tau_t^F) w_t L_t^D - I_t - J_t - T_t^F,$$

$$T_t^F = \tau_t^{prof} \left[Y(K_t, L_t^D) - (1 + \tau_t^F) w_t L_t^D - J_t - \delta K_t \right],$$

with $J_t = \frac{\psi}{2} K_t (I_t/K_t - \delta)^2$. The envelope condition and the first order condition for optimal investment change to

$$K_t : \quad q_t = (1 - \tau_t^{prof})(Y_{K_t} - J_{K_t}) + \tau_t^{prof} \delta + \frac{q_{t+1}}{1 + r_t}(1 - \delta),$$

$$I_t : \quad 1 + J_{I_t} = \frac{q_{t+1}}{1 + r_t},$$

Because of linear homogeneity of $J(\cdot)$, Hayashi's theorem still applies: $V_t = q_t K_t$. In addition we have the law of motion of foreign assets D^F as function of the current account/trade balance TB

$$D_{t+1}^F = (1 + r_t) \left[D_t^F + TB_t \right].$$

Consequently, the excess demands for the final good and assets change to

$$\zeta_t^Y = I_t + C_t + C_t^G + TB_t - Y_t = 0,$$
$$\zeta_t^A = D_t^G + D_t^F + V_t - A_t = 0.$$

Other extensions

Based on this core set-up some extensions are conceptually more challenging, in the sense of requiring alterations deep in the solution procedure. Those are for example:

- aggregate uncertainty
- structural unemployment
- endogenous technology
- explicit bequest motive

Other extensions are conceptually much simpler to implement (in the sense of 'more of the same'), for example:

- moderate cross-sectional household heterogeneity
- imperfect substitution between domestic and foreign goods (Armington assumption)
- migration
- more labor supply margins
- more household states (e.g. stock of accumulated pension rights)
- non-linear income tax
- more fiscal instruments
- monopolistic competition in the goods' market
- more advanced government debt rules
- detrending by productivity growth, population growth, trend inflation