Similar Triangles

Bohan Yao

Math Lectures Handout

§1 Introduction

Similar triangles. This is yet another one of those techniques that make up at least 50 percent of their subject area, in this case, geometry. Although similar triangles are easy to grasp but so hard to master. Similar triangle problems appear in a very wide range of problems, from the easiest AMC 8 problems to the hardest olympiad problems. Alongside similar triangles, we are also going to go over angle chasing, to supplement our similar triangle knowledge.

§2 Similar Triangle rules

Theorem 2.1 (Similarity)

Say we have two triangles ABC and XYZ. Then if they are directly similar, then this means that $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$. Also, it means that $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$, where MN denotes the length of the line segment with endpoints M and N in this scenario.

This theorem allows us to retrieve information from a pair of similar triangles. How do we find similar triangles? The following few theorems give us ways to find similar triangles.

Theorem 2.2 (SSS Similarity)

If two triangles ABC and XYZ satisfy that $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$, then they are directly similar.

Theorem 2.3 (AA Similarity)

If two triangles ABC and XYZ satisfy $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$, then they are directly similar.

Note that since the angles of a triangle sum to 180°, we actually only need that two of the angles of the triangles are the same, then the third angle is uniquely determined.

Theorem 2.4 (SAS Similarity)

If two triangles ABC and XYZ satisfy $\frac{AB}{XY} = \frac{BC}{YZ}$ and $\angle B = \angle Y$, then they are directly similar.

Note that we can cycle this condition, so we still have SAS Similarity if the triangles become BAC and YXZ instead.

Theorem 2.5 (HL Congruence)

Let ABC and XYZ be right triangles such that B and Y are right angles. Then if $\frac{AC}{XZ} = \frac{AB}{XY}$ or $\frac{AC}{XZ} = \frac{BC}{YZ}$, then ABC and XYZ are directly similar.

Now you may have noticed that we are carefully add the word "directly" to the beginning of "similar". The reason for this is because without it, we do not know if ABC is directly similar to XYZ or YZX or some variant of that. Another pitfall that you must remember is that there is no SSA congruence. The only case where SSA works is HL Congruence, which works because of the special right angle.

Exercise 2.6. Prove all of the similarity methods.

Now that we have all of the similarity methods, we must review basic angle chasing rules, so that we can find similar triangles in diagrams.

§3 Angle Chasing

Most of this is probably a review, but it is important nonetheless.

Theorem 3.1 (Opposite and Adjacent angles)

Adjacent angles on an intersection of two lines always sum to 180°, and opposite angles are always equal.

Theorem 3.2 (Parallel Lines)

Let l and m be parallel lines in the plane. Then let n be a third line not parallel to these two. Then the angles formed by the intersections of n and the two parallel lines are either x or $180 - x$.

From these theorems, prove the following:

Example 3.3

Prove that the sum of the angles in a n - gon is always $180(n - 2)$

§4 Exercises

Now that we have the power of similar triangles and angle chasing on our side, lets see how these two techniques easily solve some problems.

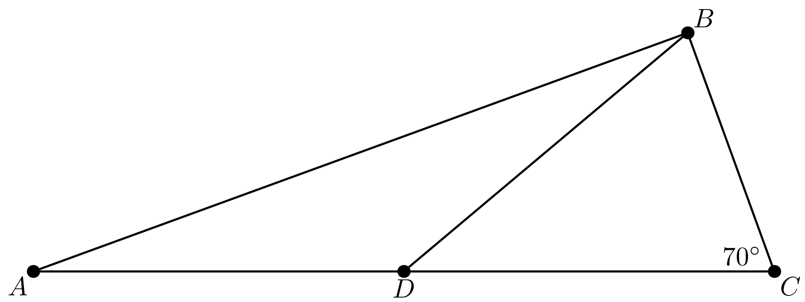
Example 4.1 (2020 AOIME P7)

Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies inside both cones. The maximum possible value for r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. Reduce the problem down to a 2D cross section. Then use similar triangles and the Pythagorean theorem. □

Example 4.2 (2014 AMC 8 P9)

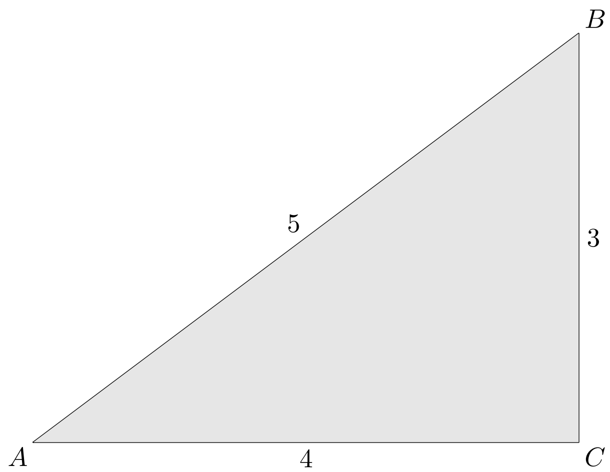
In $\triangle ABC$, D is a point on side $\overline{AC}$ such that $BD = DC$ and $\angle BCD$ measures 70°. What is the degree measure of $\angle ADB$?



Proof. Use the fact that the angles in a triangle sum to 180°. Also use angle chasing techniques to finish off the problem. □

Example 4.3 (2018 AMC 10A P13)

A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



Proof. Draw the crease, then use HL Congruence, and similar triangles finishes. □

Example 4.4 (2017 AMC 10A P21)

A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

Proof. Similar triangles, along with the fact that a square is also a parallelogram. □

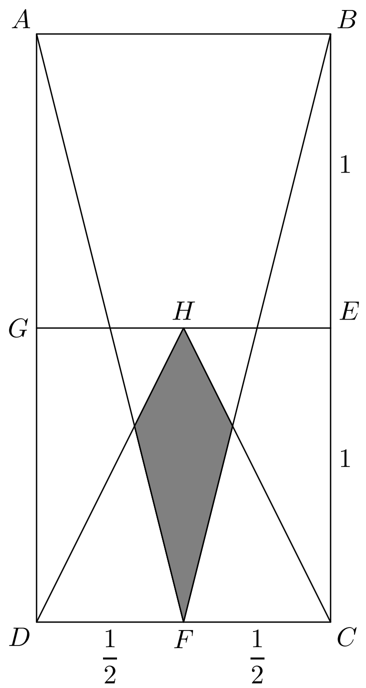
Example 4.5 (2017 AMC 10 B P15)

Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle ADE$?

Proof. HL Congruence on ADE and ABC, then similarity areal ratios finishes. □

Example 4.6 (2014 AMC 10A P16)

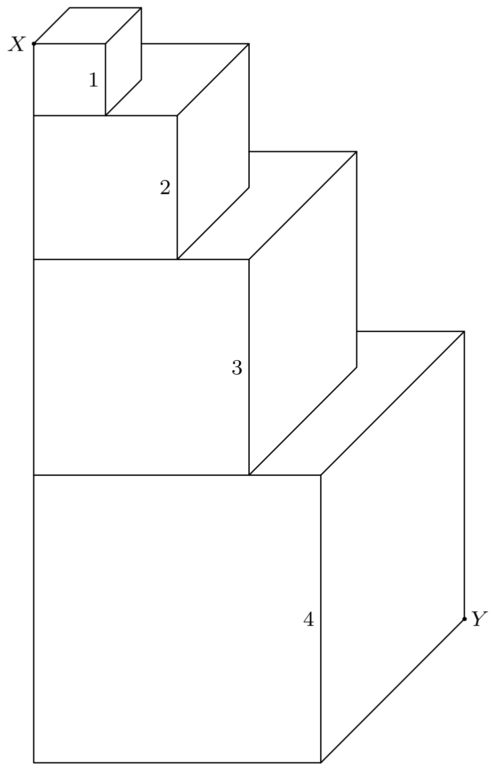
In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region?



Proof. Let the intersections of lines be X and Y. Coordinate bash to find the ratio of AX to FX and similar for other lengths, then use similar triangles. □

Example 4.7 (2014 AMC 10A P19)

Four cubes with edge lengths 1, 2, 3, and 4 are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length 3?

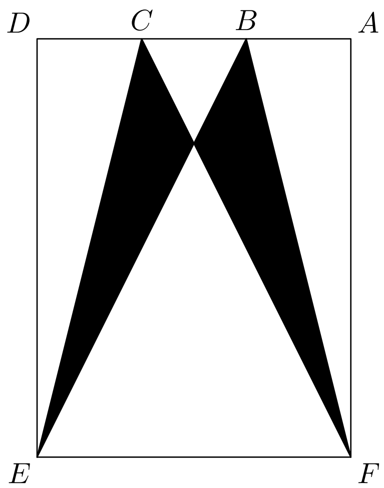


Proof. Find the length of XY, then use the similar triangles in the figure formed by the portions of the segments contained in each of the four cubes. □

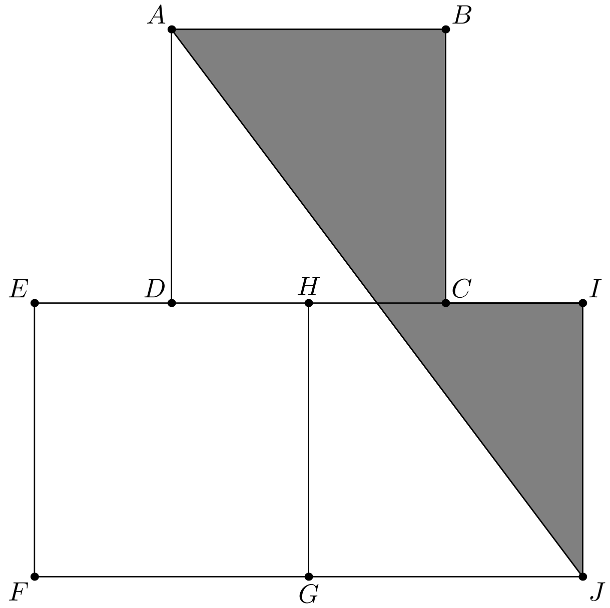
§5 Homework

Same as every week, please write up your solutions and send them to me at *solver1104@gmail.com*. Please try to solve at least 10♣ worth of problems.

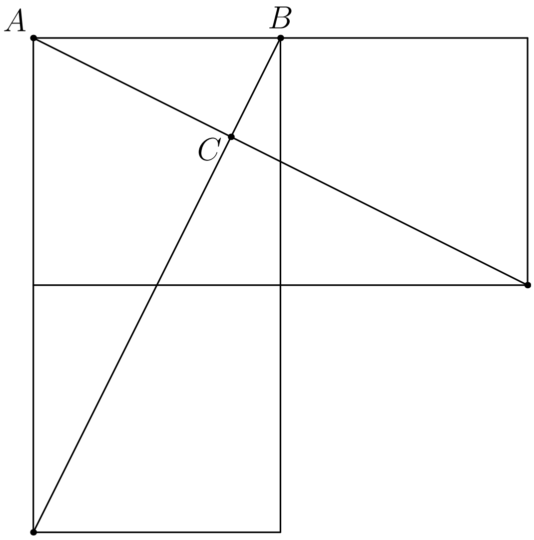
Problem 5.1 (2016 AMC 8 P 22, 2♣)**.** Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA$. The area of the "bat wings" is



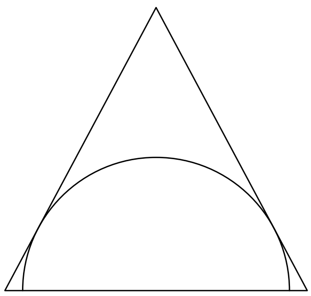
Problem 5.2 (2013 AMC 8 P24)**.** Squares $ABCD$, $EFGH$, and $GHIJ$ are equal in area. Points C and D are the midpoints of sides IH ad HE, respectively. What is the ratio of the area of the shaded pentagon $AJICB$ to the sum of the areas of the three squares?



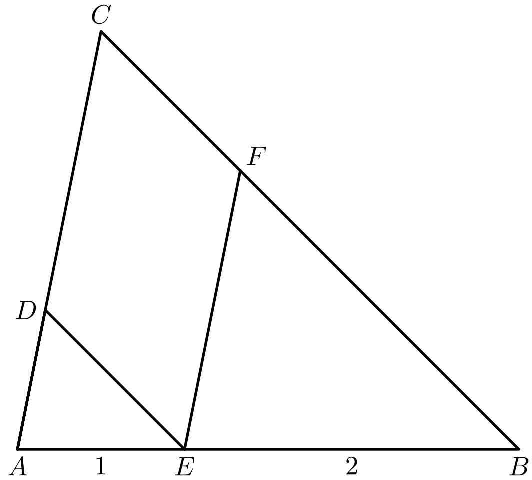
Problem 5.3 (2012 AMC 10 A, 3♣). Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?



Problem 5.4 (2016 AMC 8 P25, 3♣). A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



Problem 5.5 (2018 AMC 8 P20, 3♣). In $\triangle ABC$, a point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$?



Problem 5.6 (2019 AIME II P7, 5♣). Triangle ABC has side lengths $AB = 120$, $BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, such that the intersection of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of length 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on ℓ_A, ℓ_B, and ℓ_C.

Problem 5.7 (2019 AIME I P6, 5♣). In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.