Direct Counting

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Math Lectures Handout

"So we have two ways of approaching this problem. The first approach is by using constructive counting, and the second way is, wait there is no second way." - Bohan Yao

§1 Introduction

This handout focuses on the combinatorical questions that can be approached in a "direct" manner. These are the problems that can be solved in a "direct" manner, meaning that you do not have to make modifications to the problem to make it easier to solve, instead, the approaches to these problems will generally be straightforward, which does not mean that they are easy in any sense, but means that the solution is straightforward. However, many of the problems here are generally easier, as there are rarely problems on difficult contests that are of the "direct" type. In this handout, most things will be conveyed in the text, and there will be no theorems.

§2 Constructive Counting

Constructive counting is a technique about counting the number of ways to make several independent choices together. For example, if we have to make a first choice out of n objects, then we have to make a second choice out of x objects, then we have to make a final choice out of y objects, then if all of the choices are *independent* of each other, then the number of choices we can make is nxy. The important part of this is that the choices are independent of each other, which means that you can choose the same objects in a later step, even if your choices in earlier steps are different. If this does not hold true, then this technique doesn't work.

§3 Easy Problems

Example 3.1 (2016 AMC 8 P17)
An ATM password at Fred's Bank is composed of four digits from 0 to 9, with repeated digits allowable. If no password may begin with the sequence $9, 1, 1$, then how many passwords are possible?

Proof. Use Constructive Counting to find the total amount of combinations without restrictions, then subtract off the number of combinations that are restricted. □

Example 3.2 (2015 AMC 8 P11)

In the small country of Mathland, all automobile license plates have four symbols. The first must be a vowel (A, E, I, O, or U), the second and third must be two different letters among the 21 non-vowels, and the fourth must be a digit (0 through 9). If the symbols are chosen at random subject to these conditions, what is the probability that the plate will read "AMC8"?

Proof. Use Constructive counting to find the total number of combinations that are allowed under the restrictions for the digits given, then divide the number of ways to get a license plate reading "AMC8" by the total amount. □

Example 3.3 (2011 AMC 8 P23)

How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

Proof. Begin by considering cases based on whether the last digit is 0 or not. If it is not zero, then the last digit is 5. Then use constructive counting, but make sure you do not overcount cases. □

§4 Permutations and Combinations

We now define two useful pieces of notation that will be used extensively in combinatorics mainly, but will also be used in number theoretical problems and algebraic problems. First of all, I am assuming that you know what a factorial is. ($n! = n \cdot (n-1) \cdot (n-2) \cdots 1$, where the exclamation mark represents a factorial). With this information, we will define *permutations and combinations.*

Definition 4.1. Define $P(x, y)$ to be $\frac{x!}{y!}$. Then this function represents the number of ways to choose $x - y$ objects from a list of x objects, if order matters.

Permutations may be represented differently in different places, but the main idea is there.

Definition 4.2. Define $\binom{x}{y}$ to mean $\frac{x!}{y!(x-y)!}$. Then this function represents the number of ways to choose y objects from a list of x objects, if order does not matter.

Using these definitions, we can solve many harder problems. However, it is just a matter of applying these techniques carefully.

§5 Casework

Casework is the technique where you break up a problem into several different cases. It is very straightforward, but you need to make sure that you do not overcount any cases. One of the most common mistakes contestants make is called an OBOE, or an Off By One Error.

§6 Harder Exercises

Example 6.1 (2015 AIME II P2)

The nine delegates to the Economic Cooperation Conference include 2 officials from Mexico, 3 officials from Canada, and 4 officials from the United States. During the opening session, three of the delegates fall asleep. Assuming that the three sleepers were determined randomly, the probability that exactly two of the sleepers are from the same country is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. Casework on which country's officials fell asleep, then divide by the total number of ways to choose 3 officials. □

Example 6.2 (2017 AIME II P2)

Teams T_1, T_2, T_3, and T_4 are in the playoffs. In the semifinal matches, T_1 plays T_4 and T_2 plays T_3. The winners of those two matches will play each other in the final match to determine the champion. When T_i plays T_j, the probability that T_i wins is $\frac{i}{i+j}$, and the outcomes of all the matches are independent. The probability that T_4 will be the champion is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Proof. Casework on the different combinations of previous matches such that T_4 is the champion. □

Problem 6.3 (2016 AIME I P2)**.** Two dice appear to be standard dice with their faces numbered from 1 to 6, but each die is weighted so that the probability of rolling the number k is directly proportional to k. The probability of rolling a 7 with this pair of dice is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. First, find the probability of each number being rolled. Then, casework on what numbers you roll from the dice. □

Problem 6.4 (2012 AIME II P3)**.** At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible comittees that can be formed subject to these requirements.

Proof. Casework on the distribution of genders coming from each of the departments. □

§7 Homework

As always, submit your solutions to me. *solver1104@gmail.com* is my email. Try to solve at least 10♣

Problem 7.1 (2016 AIME II P2, 2♣)**.** There is a 40% chance of rain on Saturday and a 30% of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Problem 7.2 (2008 AMC 8 P24, 2♣)**.** Ten tiles numbered 1 through 10 are turned face down. One tile is turned up at random, and a die is rolled. What is the probability that the product of the numbers on the tile and the die will be a square?

Problem 7.3 (2011 AMC 8 P18, 2♣)**.** A fair 6-sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number?

Problem 7.4 (2014 AMC 10A P17)**.** Three fair six-sided dice are rolled. What is the probability that the values shown on two of the dice sum to the value shown on the remaining die?

Problem 7.5 (2007 AIME I P6, 3♣)**.** A frog is placed at the origin on a number line, and moves according to the following rule: in a given move, the frog advanced to either the closest integer point with a greater integer coordinate that is a multiple of 3, or to the closest integer point with a greater integer coordinate that is a multiple of 13. A move sequence is a sequence of coordinates which correspond to valid moves, beginning with 0, and ending with 39. For example, 0, 3, 6, 13, 15, 26, 39 is a move sequence. How many move sequences are possible for the frog?

Problem 7.6 (2015 AIME II P5, 5♣)**.** Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

Problem 7.7 (2015 AIME I P5, 5♣)**.** In a drawer Sandy has 5 pairs of socks, each pair a different color. On Monday Sandy selects two individual socks at random from the 10 socks in the drawer. On Tuesday Sandy selects 2 of the remaining 8 socks at random and on Wednesday two of the reaining 6 socks at random. The probability that Wednesday is the first day Sandy selects matching socks is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 7.8 (2020 AOIME P9, 7♣)**.** While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Problem 7.9 (2017 AIME II P9, 7♣)**.** A special deck of cards contains 49 cards, each labeled with a number from 1 to 7 and colored with one of seven colors. Each number-color combination appears on exactly one card. Sharon will select a set of eight cards from the deck at random. Given that she gets at least one card of each color and at least one card with each number, the probability that Sharon can discard one of her cards and still have at least one card of each color and at least one card with each number is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 7.10 (2020 AIME I P11, 10♣)**.** For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.