Equations and Symmetry

Bohan Yao

Math Lectures Handout

§1 Introduction

We have all seen the word problems from school that require you to set up a system of equations to solve them. In this lecture, we will be covering similar problems to those, except they are from the AMC 8 and higher competitions instead, so they will require more thinking to solve. However, most of the questions will require nothing but deep thinking, so there is not much "theory" to go over.

§2 Common Techniques for Problems

Since this lesson is actually a combination of the lesson I planned before, or the symmetry lesson, and the equations lesson, there are some cool techniques that you can use in some situations to solve equations faster. We will first be going over the first type of symmetry in a problem, or the type that allows us to exploit either cyclic or symmetric symmetry to our advantage, saving us precious time in a competition.

Theorem 2.1 (Symmetric Equations and Cyclic Equations)

We define the Symmetric Equation to be a set of equations such that by swapping any two variables, the set of equations remains constant. We also define a Cyclic Equation to be a set of equation such that when you "cycle" the variables, or rotate them, the set of equations remains constant.

Note that in particular, all Symmetric Equations are Cyclic.

The above is not really a theorem, but more a shortcut in problems that satisfy one of the symmetry types above. In that case, then you can use this symmetry to solve the problems faster. This is because in problems that use Symmetric or Cyclic Equations, if we find one of the variables, since everything is symmetric, everything must be the same. You can show this by simply switching variables around or "cycling" the variables.

§3 Rate problems

Rate problems are mostly all the same. The general method of solving them involves assigning variables to all of the values that are mentioned in the problem, and using an equation commonly known as the rate equation.

Theorem 3.1 (Rate Equation)

In any distance problem, the average speed you move at times the time you moved equals the distance traveled. In particular, this is more recognized as the equation $d = rt$.

Note that we do not have to just use this for distance problems, we can likewise use this equation for the volume of water left in a leaky bathtub, by simply making a direct correlation between the "distance" and the "volume". All rate equations can be solved using this technique. However, we will also present a few techniques that can help at times.

The first technique is by considering a person to do x of a "job" in a unit amount of time. This can be used to solve the problem "If A finishes the job in 3 hours, and B finishes in 4 hours, how much time does it take them working together to finish the job?".

The second technique is by considering the different "motion" all as one motion. For example, if you have two cars moving towards each other at 30 mph each, that is the same as one car moving towards a wall at 60 mph.

§4 Advanced Symmetry

In this section, we will be going over problems that require the use of a special technique known as "Symmetric Polynomials". Essentially, symmetric polynomials are expressions that are symmetric, and use some number of variables.

Definition 4.1. The kth Symmetric polynomial using the variables $a_1, a_2, \cdots, a_n$ is the sum of this expression: $a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_k$ over all cycles of the variables.

Theorem 4.2

Any symmetric equation in k variables can be written as the sum of the symmetric polynomials in k variables.

The above theorem is actually not very important, so we will omit the proof here. However, this gives us some insight on how to solve the following classic problem:

$x + y = a$, $y + z = b$, $x + z = c$, solve for x, y, z.

How you solve this problem is by adding everything up and dividing by 2 to get an expression for $x + y + z$, then subtract the other equations from this. This illustrates the technique of summing everything or taking the product of everything to get symmetric equations. In general, the more symmetry, the better.

Exercise 4.3 (Classic?)**.** Find a and b if $a^2 + ab = 108$, and $b^2 + ab = 148$.

§5 Problems

Example 5.1 (2020 AIME I P1)

In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. Let $\angle A = x$, then angle chase in terms of x. You only need to use the fact that the angles in a triangle sum to 180, that a line is 180 degrees, and that isoceles triangles have their base angles equal. □

Example 5.2 (2020 AMC 10A P4)

A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid \$0.50 per mile, and her only expense is gasoline at \$2.00 per gallon. What is her net rate of pay, in dollars per hour, after this expense?

Proof. Convert the numbers into how many gallons she drives, and then solve for both her expenses and her pay by using the rates given. □

Example 5.3 (2013 AIME I P1)

The AIME Triathlon consists of a half-mile swim, a 30-mile bicycle, and an eight-mile run. Tom swims, bicycles, and runs at constant rates. He runs five times as fast as he swims, and he bicycles twice as fast as he runs. Tom completes the AIME Triathlon in four and a quarter hours. How many minutes does he spend bicycling?

Proof. Standard problem using $d = rt$, just rewrite all of the conditions in terms of a variable x, which is the speed that he bicycles at. □

Example 5.4 (2017 AMC 10B P12)

Elmer's new car gives 50% percent better fuel efficiency, measured in kilometers per liter, than his old car. However, his new car uses diesel fuel, which is 20% more expensive per liter than the gasoline his old car used. By what percent will Elmer save money if he uses his new car instead of his old car for a long trip?

Proof. Another simple rates problem. This time, all you need to do is to assume that he travels 1000 kilometers, and solve for the cost directly. □

Problem 5.5 (2020 MBMT P28)**.** Consider the system of equations $a+2b+3c+\ldots+26z = 2020b+2c+3d+\ldots+26a = 2019\cdots y+2z+3a+\ldots+26x = 1996z+2a+3b+\ldots+26y = 1995$ where each equation is a rearrangement of the first equation with the variables cycling and the coefficients staying in place. Find the value of $z + 2y + 3x + \ + 26a$.

Proof. This is a classic symmetry problem. Add up all of the equations and divide by a value to get the sum of all of a, b, c, etc. Then it suffices to subtract 27 times this sum from the first equation. □

§6 Homework

Again, please write up your solutions and submit them to me at *solver1104@gmail.com*. For this week, please solve at least 10♣ worth of problems.

Problem 6.1 (2018 AMC 8 P17, 2♣)**.** Bella begins to walk from her house toward her friend Ella's house. At the same time, Ella begins to ride her bicycle toward Bella's house. They each maintain a constant speed, and Ella rides 5 times as fast as Bella walks. The distance between their houses is 2 miles, which is 10,560 feet, and Bella covers $2\frac{1}{2}$ feet with each step. How many steps will Bella take by the time she meets Ella?

Problem 6.2 (2018 AMC 8 P6, 2♣)**.** On a trip to the beach, Anh traveled 50 miles on the highway and 10 miles on a coastal access road. He drove three times as fast on the highway as on the coastal road. If Anh spent 30 minutes driving on the coastal road, how many minutes did his entire trip take?

Problem 6.3 (2011 AMC 10A P15, 2♣)**.** Roy bought a new battery-gasoline hybrid car. On a trip the car ran exclusively on its battery for the first 40 miles, then ran exclusively on gasoline for the rest of the trip, using gasoline at a rate of 0.02 gallons per mile. On the whole trip he averaged 55 miles per gallon. How long was the trip in miles?

Problem 6.4 (2011 AIME I P1, 3♣)**.** Jar A contains four liters of a solution that is 45% acid. Jar B contains five liters of a solution that is 48% acid. Jar C contains one liter of a solution that is k% acid. From jar C, $\frac{m}{n}$ liters of the solution is added to jar A, and the remainder of the solution in jar C is added to jar B. At the end, both jar A and jar B contain solutions that are 50% acid. Given that m and n are relatively prime positive integers, find $k + m + n$.

Problem 6.5 (2017 AMC 10A P9, 3♣)**.** Minnie rides on a flat road at 20 kilometers per hour (kph), downhill at 30 kph, and uphill at 5 kph. Penny rides on a flat road at 30 kph, downhill at 40 kph, and uphill at 10 kph. Minnie goes from town A to town B, a distance of 10 km all uphill, then from town B, a distance of 15 km all downhill, and then back to town A, a distance of 20 km on the flat. Penny goes the other way around using the same route. How many more minutes does it take Minnie to complete the 45-km ride than it takes Penny?

Problem 6.6 (2020 MBMT P35, 5♣)**.** Tim has a multiset of positive integers. Let c_i be the number of occurrences of numbers that are at least i in the multiset. Let m be the maximum element of the multiset. Tim calls a multiset spicy if $c_1, \ldots, c_m$ is a sequence of strictly decreasing powers of 3. Tim calls the hotness of a spicy multiset the sum of its elements. Find the sum of the hotness of all spicy multisets that satisfy $c_1 = 3^{2020}$. Give your answer (mod 1000). (Note: a multiset is an unordered set of numbers that can have repeats)

Problem 6.7 (2002 AMC 10A P17)**.** Sarah pours four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then transfers half the coffee from the first cup to the second and, after stirring thoroughly, transfers half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?