Factors

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Math Lectures Handout

§1 Introduction

I think we all know what factors are. They are just equivalent to the divisors of a number. Today in this course, we will be learning some more advanced tools with factors. Although factors at the computational level may seem like a whole new technique, they are actually equivalent to modular arithmetic in a very deep way. Nonetheless, we will still cover factors, as they are also very important for many computational style problems.

§2 Basic Tools

Definition 2.1. Let n be a positive integer, and let k be a factor of n. Then we have that $xk = n$ for some positive integer x.

This is basically what a factor is. Note that sometimes a question may ask you to find negative factors as well, or find factors of negative numbers. The definition for negative numbers is the same as above, just with x being negative. However, most contests assume that factors are positive. Note that 0 is not a factor of anything, but everything is a factor of 0.

Theorem 2.2 (Categorizing Factors)

Let the prime factorization of a positive integer n be $\prod p_i^{a_i}$. Then if a number k is a factor of n, then k only has prime factors that are in the p_i, and all of the exponents of the primes p_i are less than or equal to a_i.

Proof. First show that all numbers of this form are factors, then show that there are no factors that are not of this form. □

This theorem categorizes the factors that a number has. Using this theorem, we can prove two other useful theorems:

Theorem 2.3 (Sigma function and Tau function in Number Theory)

Let a number be n. Then its number of factors is represented by the function τ, and its sum of its divisors as σ. We can also write closed forms for these two functions.

§3 Divisibility Rules

Now we can also use modular arithmetic and a little bit of creativity to prove the following:

Theorem 3.1 (Divisibility)

A number is divisible by 3 if and only if the sum of its digits is. (Same thing for 9). A number is divisible by 5 if and only if its last digit is 0 or 5, and similar for powers of 5. A number is divisible by 2 if and only if its last digit is (Similar things hold for powers of 2).

These are usually only important for computational contests.
However, it is instructive to prove these, so let us do so.

Exercise 3.2. Prove the rules above.

§4 Examples

Example 4.1 (2016 AMC 10A P22)

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

Proof. Consider the prime factorization of 110. This will limit the number of possible values for n, and it suffices to check all of the possible cases. □

Example 4.2 (2014 AMC 10B P12)

The largest divisor of $2,014,000,000$ is itself. What is its fifth largest divisor?

Proof. Count the smallest factors, then divide 2014000000 by the 5th smallest one. □

Example 4.3 (2014 AMC 10B P17)

What is the greatest power of 2 that is a factor of $10^{1002} - 4^{501}$?

Proof. Take out the obvious factors of 2 first, then consider the rest of the expression mod increasing powers of 2. □

Example 4.4 (2017 AMC 10B P23)

Let $N = 123456789101112\dots4344$ be the 79-digit number obtained that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

Proof. Consider the expression $\pmod 9$ and $\pmod 5$, then use divisibility rules and CRT the two $\pmod{}$ expressions together. □

Example 4.5 (2013 AIME I P2)

Find the number of five-digit positive integers, n, that satisfy the following conditions: (a) the number n is divisible by 5, (b) the first and last digits of n are equal, and (c) the sum of the digits of n is divisible by 5.

Proof. Use the 5 divisibility rule along with some clever reasoning. □

§5 Homework

As always, please write up your solutions and send them to me at *solver1104@gmail.com*. Please solve at least 10♣ worth of problems for this week.

Problem 5.1 (2017 AMC 8 P7, 2♣)**.** Let Z be a 6-digit positive integer, such as 247247, whose first three digits are the same as its last three digits taken in the same order. Which of the following numbers must also be a factor of Z?

Problem 5.2 (2018 AMC 8 P7, 2♣)**.** The 5-digit number $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

Problem 5.3 (2018 AMC 8 P18, 2♣)**.** How many positive factors does $23,232$ have?

Problem 5.4 (2017 AMC 8 P19, 3♣)**.** For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?

Problem 5.5 (2016 AMC 8 P15, 3♣)**.** What is the largest power of 2 that is a divisor of $13^4 - 11^4$?

Problem 5.6 (2019 AIME II P4, 5♣)**.** A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 5.7 (2017 AMC 10B P20, 5♣)**.** The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?