Coordinate Bashing

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Math Lectures Handout

"If the only tool you have is a hammer, it makes sense to treat everything as if it was a nail" - Maslow's Hammer

Coordinate Bashing. The technique of choice of many AMC 8/10 contestants when faced with a geometry problem. Although coordinate bashing may not help as much for harder and harder contests, where the problem writers specifically avoid problems killed by coordinates, it is still very useful for many AMC 8 problems, and AMC 10 problems.

§1 Prerequisites

We are assuming that you already know how to solve systems of linear equations.

§2 Basic knowledge

Coordinate bashing is the method of plotting a geometry problem onto the coordinate plane and solving it by using the algebraic representation of the figures in the problem.

Theorem 2.1

Lines in the coordinate plane can be represented as $y = mx + b$. In particular, m is the slope of the line, and b is the y-intercept. Also, lines with slopes that are negative reciprocals of each other are perpendicular.

Theorem 2.2

Circles in the coordinate plane can be represented as $(x - a)^2 + (y - b)^2 = r^2$. In particular, (a, b) is the center of this circle, and r is the radius.

Theorem 2.3

The distance from a point (x_0, y_0) to the line represented by $Ax + By + C = 0$ is $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$.

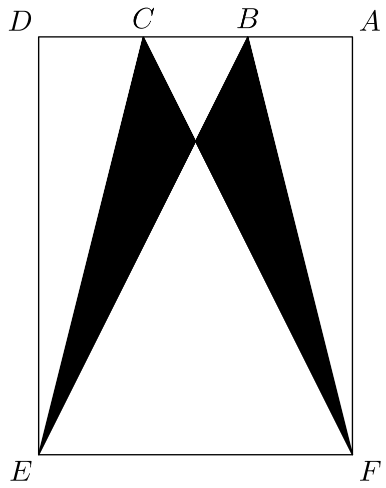
Exercise 2.4. Prove theorem 1.3.

§3 Easy Applications

To get you familiarized with this technique, what better way to do so but to slaughter some AMC problems with this technique!

Example 3.1 (2016 AMC 8 Problem 22)

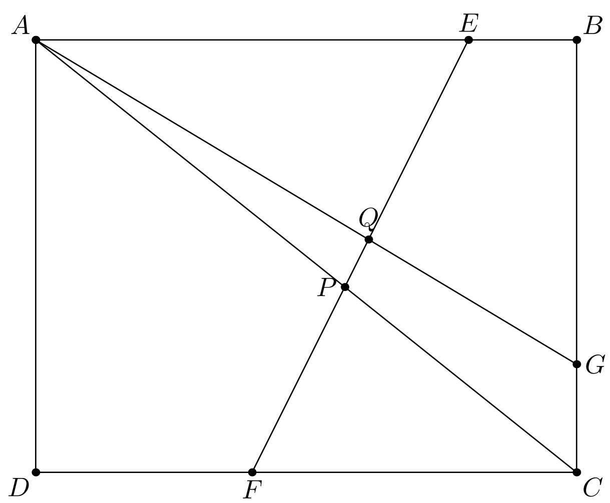
Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA$. The area of the "bat wings" is



Proof. Set E as the center of the coordinate system, and set ED as the y axis, and EF as the x axis. Then just coordinate bash. □

Example 3.2 (2016 AMC 10B P19)

Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$. and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$?



Proof. Set A as the origin, and sides of the rectangle as axes, then coordinate bash. □

One may also notice that both of these problems succumb to similar triangles.

§4 When to employ coordinates

Generally, computational geometry problems, or problems that have you find a numerical answer, are the only problems that coordinate bash solve. However, not all computational geometry problems can be solved using coordinates feasibly.
One of the big warnings that coordinates will not work, or will be very painful to use is the presence of circles. Although circles do have a reasonably compact coordinates definition, as there are more and more circles, it will eventually be too messy. The reason for this is clear: circles are a second degree curve, meaning that you will have to solve many higher degree equations when involving circles in coordinate bashing. One exception to this rule is when the circles are centered at the origin, as many terms get killed off then, and it is slightly less messy.
However, lines are coordinates best friend. Lines are a first degree curve in the coordinate plane (although they are not curved), meaning that when you have a problem involving only lines, it is a lot easier to use coordinates. However, due to the nature of coordinates, it is weaker than some more powerful types of bash, like barycentrics, animation, and complex numbers. This is also the reason why most coordinate problems can be solved using similar triangles, because there is simply not enough power in coordinates. However, in the next section, we will see how to implement more power into your coordinate system.

§5 Advanced Tools

Theorem 5.1 (Shoelace)
Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area of P is

$$\frac{1}{2}|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)|$$

.

Proof. Show that this theorem holds for triangles, then induct on the number of vertices of P $\square$

We will not actually go over this proof, as it is very messy algebraically and provides little to no insight on actual problem solving.
The reason why this theorem is called the Shoelace theorem is because of a certain way you can remember this theorem, which makes the computation look like intersecting shoelaces.
This theorem is not too useful in general, but it has some uses in problems.

Theorem 5.2 (Rotating Points)
Let the transformation Ψ be defined as the rotation a degrees clockwise around the coordinate plane. Then Ψ takes any point (x, y) to the point $(\sqrt{x^2+y^2}(\cos(\arctan(\frac{y}{x}) - a)), \sqrt{x^2+y^2}(\sin(\arctan(\frac{y}{x}) - a)))$.

Although this specific formulation is usually way too messy to use in a practical scenario, there is a reformulation of this that is actually useful, which will be covered in the next section.

Exercise 5.3. Prove Theorem 5.2 (Apply trigonometry)

Theorem 5.4 (Weighted Averages of Points)
Let there be two points in the plane, $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Then given any real number α, the point R such that $\frac{|PR|}{|RQ|} = \alpha$ is the point $(x_1 + (x_2 - x_1) \cdot \alpha, y_1 + (y_2 - y_1) \cdot \alpha)$. (Where distances are signed)

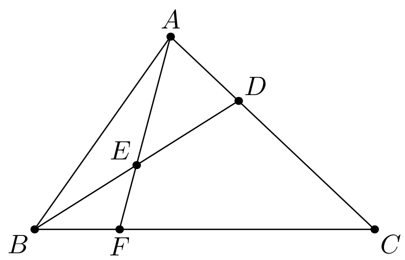
This is actually pretty simple, but is actually very useful. This highlights another one of the strengths of coordinates, ratios along lines.
There are actually quite a few ways to prove this, one of the easiest being just checking that this holds true by using the distance formula. An easier way to do this is to take a transformation ζ taking P to the origin, and from here, it is obvious (remember to shift back!). This highlights yet another important tip about coordinates: Choose the origin and the axes wisely. A good choice of origin can sometimes be all you need to finish a problem. Also, you can choose the unit size of the coordinate plane freely, so if a right triangle has sides of length 90, 120, 150, you can scale the triangle into a $3-4-5$ triangle, just remember to scale back at the end.

§6 Examples

Example 6.1 (Classic, AIME?)
What is the acute angle formed by the two lines $y = \frac{12x}{5}$ and the line $y = \frac{4x}{3}$?

Proof. Use the ideas from theorem 5.2 to do this problem. (You just need to find the angle between the points $(5, 12), (3, 4), (0, 0)$). □

Example 6.2 (2019 AMC 8 P24)
In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



Proof. In this problem, note that the problem does not have any control over what type of triangle ABC is, so let it be a right triangle! Then just set B to be the origin and coordinate bash. □

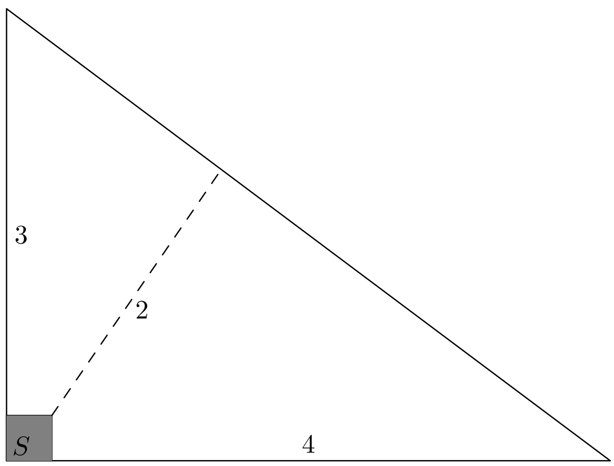
Example 6.3 (2019 AIME I P8)

A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. Set O as the origin, and use the obvious set of axes. Then its simply just trigonometry and infinite geometric series sum. □

Example 6.4 (2018 AMC 10A P23)

Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



Proof. Simply set the right angle as the origin, then let the sides of the right triangle be the axes, then coordinate bashing using point to line formula works. □

§7 Closing Remarks

Although coordinate bashing is fairly simple for most problems, it is important to recognize when to bash and when to try to find a different approach. If the problem involves many circles, it is often better to use a different approach. However, if the problem is very length based and has many lines, it is a good sign that coordinates will work. Good luck on the problem set!

§8 Problems

Try to solve at least 15 ♣. All problems will be related to the lecture. Try to solve it with ideas from the lecture, but this is not required, all approaches will be admitted.

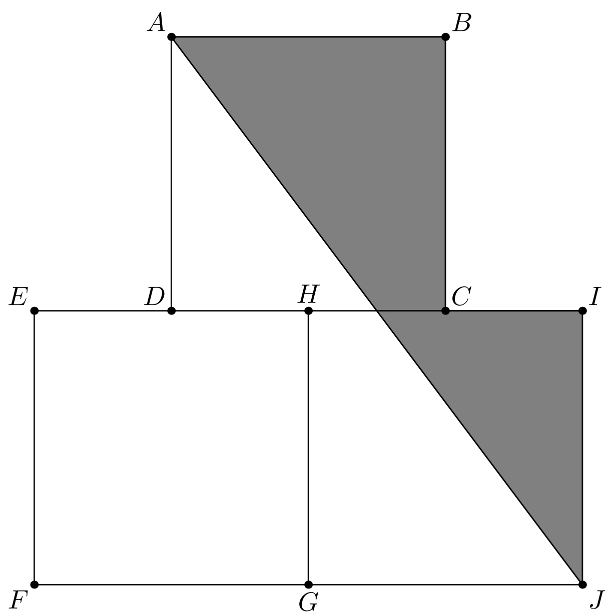
Please submit your solutions (Show most of your steps, but being a little brief is fine) to me. My email is solver1104@gmail.com . I suggest writing up the solutions on paper first, then scanning it or taking a picture of the paper.

Problem (2 ♣, 2019 AMC 10A P7)**.** Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

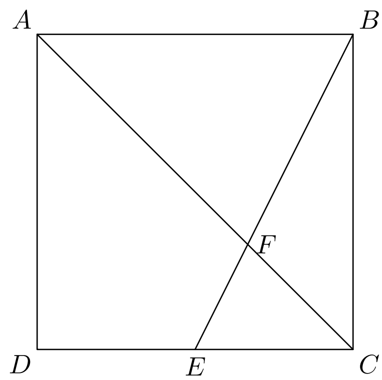
Problem (2 ♣, 2020 MATHCOUNTS School P20)**.** What is the area of the region in the first quadrant that lies between the lines $x + 3y = 12$ and $x + 3y = 18$?

Problem (2 ♣, 2020 MATHCOUNTS School P25)**.** The lines given by the equations $y = 9 - \frac{x}{3}$ and $y = mx + b$ are perpendicular and intersect at a point on the x axis. What is b?

Problem (3 ♣, 2013 AMC8 P 24)**.** Squares $ABCD$, $EFGH$, and $GHIJ$ are equal in area. Points C and D are the midpoints of sides IH ad HE, respectively. What is the ratio of the area of the shaded pentagon $AJICB$ to the sum of the areas of the three squares?



Problem (3 ♣, 2018 AMC8 P22)**.** Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?

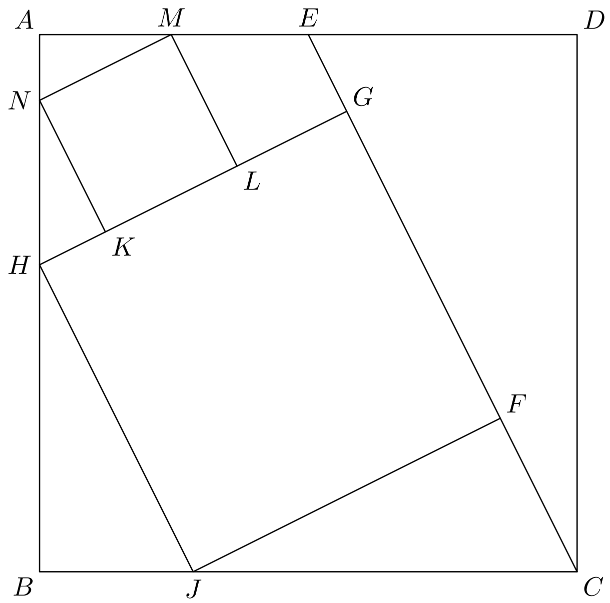


Problem (5 ♣, Own)**.** Let $ABCD$ be an isoceles trapezoid with AB parallel to CD. Then let the intersection of lines AC and BD be X, and let the line through X parallel to AB intersect line BC at point Z. Then find the distance BZ in terms of the side lengths of the trapezoid.

Problem (5 ♣, 2018 AIME II P4)**.** In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, that total area of the six triangular regions. Then $K = \frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

Problem (5 ♣, 2020 AOIME P4)**.** Triangles $\triangle ABC$ and $\triangle A'B'C'$ lie in the coordinate plane with vertices $A(0,0)$, $B(0,12)$, $C(16,0)$, $A'(24,18)$, $B'(36,18)$, and $C'(24,2)$. A rotation of m degrees clockwise around the point (x,y), where $0 < m < 180$, will transform $\triangle ABC$ to $\triangle A'B'C'$. Find $m + x + y$.

Problem (7 ♣, 2015 AIME I P7)**.** In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.



Problem (7 ♣, 2018 AMC 10B P24)**.** Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

Problem (10 ♣, 2020 AIME I P13)**.** Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

Problem (10 ♣, 2014 AIME II P11)**.** In $\triangle RED$, $RD = 1$, $\angle DRE = 75^\circ$ and $\angle RED = 45^\circ$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that

$\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = \frac{a-\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a + b + c$.