Bijections

Bohan Yao

Math Lectures Handout

"Stars and Bars, Mudkips and Blueberries, Sticks and Stones, all the same thing" - Unknown

§1 Introduction

One of the most interesting parts of combinatorics is making a relation between two seemingly different problems and showing that they are equal. This technique is called a bijection. In this class, we will be examining some of the most important bijection for solving problems, including the so called "Stars and Bars" technique and blockwalking arguments, which are two of the most popular/ famous problems in all of competition math.

§2 Proving a Bijection

To prove a bijection, you must prove that both sets are subsets of the other. This means that given two problems, A and B, to establish a bijection between the two, we need to show that any solution to A corresponds to a unique solution in B, and any solution in B corresponds to a unique solution in A. It is pretty obvious that by showing this, we show that the two problems are equivalent under a transformation.

Bijections are usually useful for transforming a harder problem into an easier one, that is easier to find the solution to. In proving a bijection, you are essentially reducing the problem down to another problem, so it makes no sense to establish a bijection from an easy problem to a harder one.

§3 Stars and Bars

We start off our discussion by examining the innnocent looking problem shown below:

Example 3.1

How many ways are there to distribute n objects to k groups, if the objects are indistinguishable but the groups are distinguishable?

This looks like a regular problem that can be done with the techniques we learned in the previous class, but if you try to solve it without bashing (listing out all of the cases in the problem), you will not be able to find a clean solution. This is where we need to consider the following problem:

Example 3.2

How many ways are there to arrange $n + k - 1$ letters where there are n of one letter and $k - 1$ of another letter?

It turns out that we can make a bijection between these two problems, or a connection between these two problems, which allows us to equate the answers of these two problems. This is the so called "Stars and Bars" technique.

The proof of the bijection is as follows:

Let the $k - 1$ letters be "dividers", and let the n letters be the objects. Then each of the arrangements of the digits represents a distinct distribution of objects. The first group receives an amount of objects equal to the amount of letters in the n letter group before the first divider letter, the second group receives an amount of objects equal to the amount of letters in the n letter category between the first and second divider, and so on. In this way, we establish the bijection, as all orderings of the letters are equivalent to a distinct distribution, and clearly, all distributions can be represented by one ordering.

§4 Stars and Bars Problems

Example 4.1 (Modified Stars and Bars)

Find the number of ways to distribute n indistinguishable objects to k distinguishable groups, if each group must receive at least 1 object?

Proof. Give each group one object first, then proceed with normal stars and bars. □

Example 4.2 (Even more modified Stars and Bars)

How many ways are there to distribute 7 objects to 3 groups, if no group may receive more than 3 objects?

Proof. Write instead that group 1 receives $3 - x$ objects, group 2 receives $3 - y$ objects, and group 3 receivves $3 - z$ objects. Then we just need to distribute two objects to the x , y, z, which is just stars and bars in its regular form. □

§5 Blockwalking

Now we will go over the other famous bijection, which is the block walking bijection. Consider the following problem:

Example 5.1

In a $n \times k$ grid, how many ways are there to walk from the lower left corner to the upper right corner?

Again, there should not be an obvious way to solve this problem without using bijections. One of the ways to solve this is to use a recursive technique. How you can solve this problem using recursion is by noticing that the number of ways to get to any location is just the sum of the number of ways to get to the location beneath it, and the number of ways to get to the location to the left of it. However, this way requires lots of addition, which is not very favorable if n and k are large.

So now, consider the following problem. We will show that these two are indeed equal problems, meaning that the solutions to them will be the same.

Example 5.2

How many ways are there to arrange the letters in the word "$RRR\cdots RUU\cdots U$"?

The proof that these two are equivalent is as follows:

To show that any solution of the blockwalk problem is equivalent to a solution in the ordering problem, we just note that each "move" of the blockwalk corresponds to a letter. If the move is right, then the letter will be R, otherwise, the letter is U, and in the blockwalk, you move up. The other direction of showing that a ordering corresponds to a blockwalk is very similar to this proof, so I will omit it here. Now that we know that these two problems are equivalent, their solutions are equivalent.

§6 Blockwalking Problems

Example 6.1

Find the number of ways to go from the bottom left corner of a 7 by 7 grid if you must pass through the point (5,3).

Proof. Find the number of ways to get to (5,3), then multiply by the number of ways to get to the destination. □

Problem 6.2. Find the number of ways to go from the lower left corner of a 7x7 grid to the top right corner, if in every "move" you can move either 1 or 2 moves in a single direction.

Proof. For this problem, you cannot solve it by using the blockwalk argument, because there are moves that move 2 "units". However, you can still use the recursive solution. This is favorable because the numbers are small. □

§7 Other examples

Example 7.1 (2019 AMC 8 P25)

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?

Proof. Stars and Bars in the modified version solves this. Alternatively, casework on how many apples she gives to herself also works. □

Example 7.2 (Unknown)

In a 10x10 grid, we want to walk from the bottom left corner to the top left corner. We can walk any distance in one direction (right or up) per turn. How many turns does it take?

Proof. Use the same recursive approach we took in the blockwalk problems. You need to modify the recursion to be the sum of all of the numbers in the same column, but the proof is virtually the same. □

Example 7.3 (Unknown)
How many positive integer solutions are there to the equation $a \cdot b \cdot c \cdot d = 10^100$? (You can keep your answer in binomial form)

Proof. First, use the prime factorizations of a, b, c, d, e. Then the problem becomes Stars and Bars after breaking the problem into the powers of 2 case and the powers of 5 case. □

§8 Homework

Again, please send your solutions to me at *solver1104@gmail.com*. Also, please fill out the feedback form at: https://forms.gle/3HzpAAnEaKh6kfe5A. Thanks!

Problem 8.1 (Unknown, 2♣)**.** How many ways are there to choose odd integers w, x, y, z, such that $w + x + y + z = 98$?

Problem 8.2 (Unknown, 2♣)**.** How many quadruples of integers a, b, c, d are there such that $a \leq b \leq c \leq d \leq 12$?

Problem 8.3 (Well- known, 2♣)**.** Prove that the number of subsets of a set with n distinct elements is 2^n.

Problem 8.4 (2018 AMC 10A P11, 2♣)**.** When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$ where n is a positive integer. What is n?

Problem 8.5 (Own, 3♣)**.** Find the number of solutions to the equation $a + b - c - d = -9$ if a, b, c, d are nonegative integers that are less than 10.

Problem 8.6 (1984 AIME P11, 3♣)**.** A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

Problem 8.7 (Mandelbrot, 5♣)**.** In a certain lottery, 7 balls are drawn at random from n balls numbered from 1 through n. If the probability that no pair of consecutive numbers is drawn equals the probability of drawing exactly one pair of consecutive numbers, find n.

Problem 8.8 (NIMO, 5♣)**.** Zang is at the point $(3, 3)$ in the coordinate plane. Every second, he can move one unit up or one unit right, but he may never visit points where the x and y coordinates are both composite. In how many ways can he reach the point $(20, 13)$?

Problem 8.9 (2016 AMC 10A P20. 5♣)**.** For some particular value of N, when $(a + b + c + d + 1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?