A Subdivision Scheme for Approximating Circular Helix with NURBS Curve

Xiongyin Pu[1,2],Weijun Liu[1]
1. Modern Equipment Research Design Center
Shenyang Institute of Automation, Chinese Academy of Sciences
Shenyang, Liaoning Province 110016, China
2. Graduate University of Chinese Academy of Sciences
Beijing,100049,China
puxy@sia.cn, wjliu@sia.cn

Abstract

An initial approximation circular helix with quadratic NURBS curve is represented by combining the algorithm for the planar NURBS circular arc and the height function defined by using the knot insertion and the degree elevation for linear NURBS. For given tolerance, a subdivision scheme based on de Boor algorithm for approximating NURBS helix curve is proposed. The modification subdivision scheme is easy and stabilized for keeping on improvement the approximation value and eliminates the blending algorithm for Bézier segments. Numerical examples are given to illustrate the efficiency of the method.

Keywords: Approximation, Helix, NURBS curves, Subdivision

1. Introduction

In the fields of computer aided geometric design, computer aided design and computer graphics, helices can be used for the end-milling cutter geometry, the tool path description or the simulation of kinematic motion, etc. A circular helix is a unique spatial curve with constant curvature and constant torsion, and the circle and the straight line are the only two plane curves with this property. Since helix is an irrational curve, it cannot be represented polynomials or rational polynomials in explicit form, the problem of approximating the helix by rational Bézier curves have been discussed in many papers.

S. Mick and O. Röschel[1] have presented a kinematic method to interpolate the helix by rational cubic Bézier curves. Juhász[2]has proposed an improved algorithm which ensures 2nd-order continuity at the junction of two approximating curves, but it is impossible join more than two segments to form a C^2 curve. Sun[3] has presented a degree elevation method for the helix approximated by NURBS curves. Seemann has studied algorithms to approximate a helix segment with rational Bézier curves of order 4 to 6，which for larger angles the approximations become worse. Yang[4] also gave the method for helix approximation using quintic rational Bézier curves.

Without an optimal error bound, the approximation will inevitably contain more data than that actually needs for the approximation to be within the given tolerance. In classical splines approximation[5], the optimal approximation order and optimal smoothness are $O(h^{n+1})$ and C^{n-1}. Ahn[6, 7]has presented the error bound analysis for the approximation of a cylindrical helix by conic, bi-conic, quadratic and bi-quadratic Bézier curves and indicated that the error bound has the approximation order three and monotone increase as the length of the helix increases. In this paper we present the concept of the relative error bound to obtain a suitable level of subdivision.

Even though NURBS have been described as the ‘geometry standard’ for curve and surface modeling, it is nevertheless worthwhile investigating how well one can approximate helix with quadrics NURBS since a helix also is not represented exactly by NURBS. In this paper, we present a method to construct an initial approximation helix with NRUBS curve and a subdivision scheme to approximate helix with quadratic NURBS according to the futures of the helix.

The paper is organized as follows. In Section 2, a circular helix is approximated by an initial NURBS curve with one to four segments. The definition of the upper error bound is given and the factors influencing the error are discussed. The subdivision scheme for the approximation helix with NURBS curve subdivided at the mid point is given is described formally in Section 3. The concept of the relative error bound is presented to predetermine the subdivision level. Examples for the approximation helix with NURBS curves are presented in Section 4. In the last section we conclude this paper.

2. Helix approximation with NURBS

Using rotation and translation any cylindrical helix could be represented by

$$h(\theta) = (r\cos(\theta), r\sin(\theta), p\theta), \qquad \theta \in [-\alpha, \alpha] \tag{1}$$

for some positive real numbers α, r, p, where θ is the center angle of the corresponding circular arc on xy-plane.

2.1. Helix approximation with one element NURBS

For 0<α<=π/4, we assume that the approximation helix is represented by quadratic NURBS curve with only one element that is defined as the span between two distinct knot values, as

$$C(\xi) = \frac{\sum_{i=0}^{n} N_{i,q}(\xi) w_i B_i}{\sum_{i=0}^{n} N_{i,q}(\xi) w_i} \tag{2}$$

where the degree q = 2, the number of control points is 3, $\xi \in [0,1]$

$$N_{i,q}(\xi)=\frac{\xi-\xi_i}{\xi_{i+q}-\xi_i}N_{i,q-1}(\xi)+\frac{\xi_{i+q+1}-\xi}{\xi_{i+q+1}-\xi_{i+1}}N_{i+1,q-1}(\xi)$$

$$N_{i,0}(\xi)=\begin{cases}1 & \text{if } \xi_i\le\xi\le\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}$$

having the control vertexes,

$$B_0=(x_0,y_0,z_0,w_0)=(r\cos\alpha,-r\sin\alpha,-p\alpha,1)$$

$$B_1=(x_1,y_1,z_1,w_1)=(r/\cos\alpha,0,0,\cos\alpha)$$

$$B_2=(x_2,y_2,z_2,w_2)=(r\cos\alpha,r\sin\alpha,p\alpha,1)$$

and a knot vector $E=\{0,0,0,1,1,1\}$, We can express as following

$$w(\xi)=\sum_{i=0}^{n}N_{i,q}(\xi)w_i$$

$$X^w(\xi)=r\sum_{i=0}^{n}N_{i,q}(\xi)w_i\bar{x}_i=r\bar{x}(\xi),$$

$$Y^w(\xi)=r\sum_{i=0}^{n}N_{i,q}(\xi)w_i\bar{y}_i=r\bar{y}(\xi),$$

$$Z^w(\xi)=p\sum_{i=0}^{n}N_{i,q}(\xi)w_i\bar{\theta}_i=p\bar{z}(\xi),$$

$$C(\xi)=(X^w(\xi),Y^w(\xi),Z^w(\xi))/w(\xi)$$

(3)

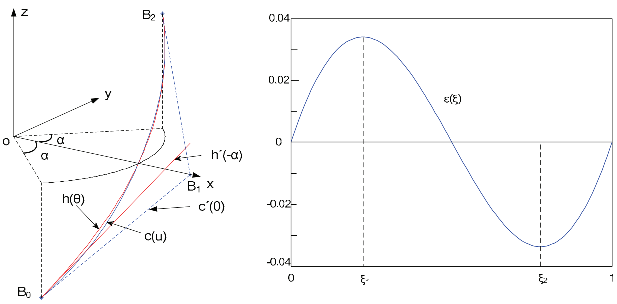
The helix segment and the approximated curve with NURBS curve do not have the same tangent direction at both end-points. See Fig. 1, where $\alpha=\pi/4$, $r=1$, $p=1$, the red curve is $h(\theta)$, the red line is tangent of $h(\theta)$ at point $h(-\alpha)$, the error bound is 3.31×10^{-2}.

$$h'(-\alpha)=(r\sin\alpha,r\cos\alpha,p),$$

$$h'(\alpha)=(-r\sin\alpha,r\cos\alpha,p),$$

$$C'(0)=(2r\sin^2\alpha,r\sin2\alpha,2p\alpha\cos\alpha),$$

$$C'(1)=(-2r\sin^2\alpha,r\sin2\alpha,2p\alpha\cos\alpha),$$



(a) (b)

Figure. 1 the approximation helix with quadratic NURBS and the error

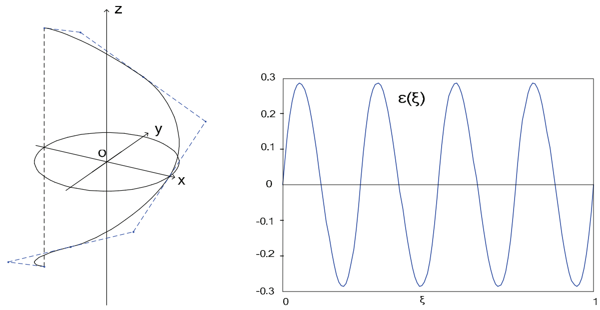
2.2. Helix approximation with multi elements NURBS

For π/4<α<=π, an initial approximation helix is respected by quadratic NURBS curve with two to four elements connected with C^1 continuity.

First to construct a planar NURBS arc using Piegl Algorithm (A7.1)[8]. There are two elements and five control points for π/4<α<=π/2, three elements and seven control points for π/2<α<=3π/2, four elements and nine control points for 3π/2<α<=π.

Second to build a high function $z=p\theta$ with linear NURBS. The two end control points are respectively $z_1=-p\alpha$ and $z_2=p\alpha$. Using the knot insertion and the degree elevation algorithm makes the high function have the same knot vector and degree with the planar NURBS arc.

Last to combine the control points of the planar NURBS arc and linear NURBS. The initial NURBS initial approximation helix is represented. Fig. 2(a) show an approximation helices $r_2=15, \beta=\pi/3, \theta\in[-\pi,\pi]$, are consist of four segments and nine control points, the error shows in Fig. 2 (b).



(a) (b)

Figure. 2 an initial approximation NURBS helix with four elements and the error

2.3. Error Analysis

Proposition 1. For each α, *p* and *r*, the helix approximations with NURBS curve $C(\xi), \xi\in[0,1]$, have the error bounds

$$h(\theta)-C(\xi)\le\varepsilon(\xi_1)=\left|p\left(\arctan\frac{\bar{y}(\xi_1)}{\bar{x}(\xi_1)}-\frac{\bar{z}(\xi_1)}{w(\xi_1)}\right)\right| \quad (4)$$

where ξ_1 indicates the first value to maximize the error *ε*.

Proof. The upper error bound of the Hausdorff distance betwween the helix $h(\theta)$ and approximation curve $C(\xi)$ is presented, where the Hausdorff distance [6, 7, 9-11] is defined

$$d_H(h,C)=\max\left\{\max_{\theta\in[-\alpha,\alpha]}\min_{\xi\in[0,1]}\left|h(\theta)-C(\xi)\right|,\max_{\xi\in[0,1]}\min_{\theta\in[-\alpha,\alpha]}\left(\left|h(\theta)-C(\xi)\right|\right)\right\}.$$

It is well known

$$d_H(h,C)\le\max\left|h(\theta(\xi))-C(\xi)\right|$$

for a reparametrisation

$$\theta=\theta(\xi)=\arctan(Y^w(\xi)/X^w(\xi)),$$

where $0\leqslant\xi\leqslant1$. Since $\left(X^w(\xi)\right)^2+\left(Y^w(\xi)\right)^2=r^2w^2(\xi)$, the error between the helix and approximation NURBS curves $C(\xi)$ is expressed

$$h(\theta(\xi))-C(\xi)=\left(0,0,p\arctan\left(\frac{Y^w(\xi)}{X^w(\xi)}\right)-\frac{Z^w(\xi)}{w(\xi)}\right),$$

let

$$\varepsilon=p\arctan\left(\frac{Y^w(\xi)}{X^w(\xi)}\right)-\frac{Z^w(\xi)}{w(\xi)} \quad (5)$$

Its derivative is

$$\varepsilon'(\xi)=p\frac{Y^{w\prime}(\xi)X^w(\xi)-Y^w(\xi)X^{w\prime}(\xi)}{\left(X^w(\xi)\right)^2+\left(Y^w(\xi)\right)^2}-\frac{Z^{w\prime}(\xi)w(\xi)-Z^w(\xi)w'(\xi)}{w^2(\xi)}$$

$$=p\frac{\bar{y}'(\xi)\bar{x}(\xi)-\bar{y}(\xi)\bar{x}'(\xi)-\bar{z}'(\xi)w(\xi)+\bar{z}(\xi)w'(\xi)}{w^2(\xi)}$$

Let the numerator of the derivative ε'(ξ) equal to zeros. The solution to this quadratic equation is

$$\xi_1 = \frac{1-\sqrt{1-2E(\gamma)}}{2m}$$

where $E(\gamma) = \dfrac{\sin\gamma - \gamma\cos\gamma}{\sin\gamma - \gamma\cos\gamma - \sin\gamma\cos\gamma + \gamma}$, $\gamma = \alpha / m$, m is the number of elements.

As shown in Fig. 1(b), $\varepsilon(\xi)$ has the local maximum $\varepsilon(\xi_1)$ and the local minimum $\varepsilon(\xi_2)$, $\varepsilon(\xi_1) = -\varepsilon(\xi_2)$. With the reparametrisation $\theta = \arctan(\overline{y}(\xi)/\overline{x}(\xi))$,

$$\left|h(\theta) - C(\xi)\right| \le \varepsilon(\xi_1) = \left|p\left(\arctan\frac{\overline{y}(\xi_1)}{\overline{x}(\xi_1)} - \frac{\overline{z}(\xi_1)}{w(\xi_1)}\right)\right|$$

Thus we obtain the upper bound of the Hausdorff distance $d_H(h,C)$.

Proposition 2. The error bounds of $d_H(h,C)$ is linear with r and is monotone decreasing as the helix angle β increases for the given α; and the error bounds of $d_H(h,C)$ is monotone increasing as α increases to any definite value p.

Proof. Since the ratio of the curvature of the circular helix to the torsion of it is constant which means $k/\tau = \tan\beta$, where β is the helix angle, we can obtain $p = r/\tan\beta$ and insert in (5), the upper error bound is the function with respect to r, α, β, as shown below,

$$\begin{aligned}\varepsilon(r,\beta,\alpha) &= p\arctan\frac{Y^w(\xi_1(\alpha))}{X^w(\xi_1(\alpha))} - \frac{Z^w(\xi_1(\alpha))}{w(\xi_1(\alpha))} \\ &= \frac{r}{\tan\beta}\left(\arctan\frac{\overline{y}(\xi_1(\alpha))}{\overline{x}(\xi_1(\alpha))} - \frac{\overline{z}(\xi_1(\alpha))}{w(\xi_1(\alpha))}\right)\end{aligned} \tag{6}$$

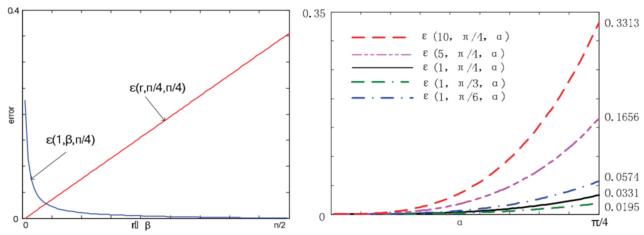
Using the chain rule for the multi-variables function we have

$$\varepsilon_r{}' = \frac{1}{\tan\beta}\left(\arctan\frac{\overline{y}(\xi_1(\alpha))}{\overline{x}(\xi_1(\alpha))} - \frac{\overline{z}(\xi_1(\alpha))}{w(\xi_1(\alpha))}\right)$$

$$\varepsilon_\beta{}' = r\left(\arctan\frac{\overline{y}(\xi_1(\alpha))}{\overline{x}(\xi_1(\alpha))} - \frac{\overline{z}(\xi_1(\alpha))}{w(\xi_1(\alpha))}\right)\csc^2\beta$$

$$\varepsilon_\alpha{}' = \varepsilon_\xi{}' \cdot \xi_\alpha{}'$$

As shown in Fig. 2(a) and 2(b), $\varepsilon_r{}' > 0$, $\varepsilon_\beta{}' < 0$ and $\varepsilon_\alpha{}' > 0$.



(a) (b)

Figure. 3 The relationship among the upper error bound and the parameters

It is clear that $\varepsilon(r,\beta,\alpha)$ is increasing as r increases and is decreasing as the helix angle β increases, as Fig. 3(a) shown. The red line is the value of the error varied as radius increases and the blue curve is the curve of the error variation as the helix angle increases, where r is from 1 to 90 mm and β is from 1 to 90 degree. In Fig. 3(b), the curve of the error variation is increasing as α increase to any definite value r and β.

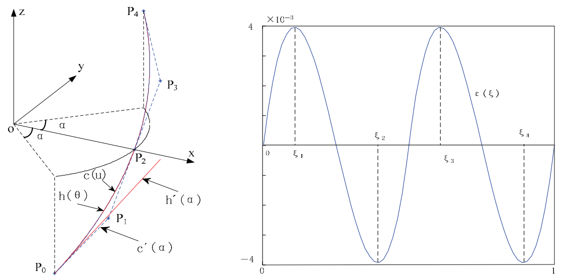
3.1. The first subdivision of the approximation NURBS helix with one element

First of all, we consider the first subdivision of $C(\xi)$, the new control points can be obtained using following method

$$P_{0,1} = B_0$$
$$P_{1,1}^{x,y} = (B_0^{x,y} + w_1 B_1^{x,y})/(1+w_1)$$
$$P_{1,1}^{z} = (B_0^{z} + B_1^{z})/2$$
$$P_{1,1}^{w} = \sqrt{(1+w_1)/2};$$
$$P_{2,1}^{x,y} = (B_0^{x,y} + 2w_1 B_1^{x,y} + B_2^{x,y})/2(1+w_1)$$
$$P_{2,1}^{z} = (B_0^{z} + 2B_1^{z} + B_2^{z})/4$$
$$P_{2,1}^{w} = 1;$$
$$P_{3,1}^{x,y} = (B_2^{x,y} + w_1 B_1^{x,y})/(1+w_1)$$
$$P_{3,1}^{z} = (B_2^{z} + B_1^{z})/2$$
$$P_{3,1}^{w} = \sqrt{(1+w_1)/2};$$
$$P_{4,1} = B_2$$

where $\alpha=\pi/4, r=1, p=1$, the knot vector $E = \{0,0,0,0.5,0.5,1,1,1\}$. There are two segments. In Fig. 4, the number of the control points become five, the red curve is $h(\theta)$, the red line is tangent at point $h(-\alpha)$, the upper error bound is 3.95×10^{-3}. After splitting once, the tangent at the end point of the curve is more approximately than the initial NURBS curve. The error between the helix and the approximate curve with NURBS curves is reduced rapidly.

Remark: The z-coordinate of new control points by splitting are independent to the weight, in other words, when we compute the z-coordinate of new control points, set w_1=1.



(a) (b)

Fig. 4 Subdividing once the helix and the error

3.2. Subdivision scheme for the approximation NURBS helix with multi elements

We build the following subdivision scheme. For the iteration $k = 1,\ldots$, and the number of segments $i = 1,\ldots 2^k m$, m is the number of the initial elements,

$$P_{0,i,k} = P_{2i-2,k-1}$$
$$P_{1,i,k}^{x,y} = (P_{2i-2,k-1}^{x,y} + P_{2i-1,k-1}^{w} \cdot P_{2i-1,k-1}^{x,y})/(1+P_{2i-1,k-1}^{w})$$
$$P_{1,i,k}^{z} = (P_{2i-2,k-1}^{z} + P_{2i-1,k-1}^{z})/2$$
$$P_{1,i,k}^{w} = \sqrt{(1+P_{2i-1,k-1}^{w})/2}$$

$$P_{2,i,k}^{x,y} = (P_{2i-2,k-1}^{x,y} + 2P_{2i-1,k-1}^{w} \cdot P_{2i-1,k-1}^{x,y} + P_{2i,k-1}^{x,y}) / 2(1 + P_{2i-1,k-1}^{w})$$

$$P_{2,i,k}^{z} = (P_{2i-2,k-1}^{z} + 2P_{2i-1,k-1}^{z} + P_{2i,k-1}^{z}) / 4$$

$$P_{2,i,k}^{w} = 1$$

$$P_{3,i,k}^{x,y} = (P_{2i,k-1}^{x,y} + P_{2i-1,k-1}^{w} \cdot P_{2i-1,k-1}^{x,y}) / (1 + P_{2i-1,k-1}^{w})$$

$$P_{3,i,k}^{z} = (P_{2i,k-1}^{z} + P_{2i-1,k-1}^{z}) / 2$$

$$P_{3,i,k}^{w} = \sqrt{(1 + P_{2i-1,k-1}^{w}) / 2}$$

$$P_{4,i,k} = P_{2i,k-1}.$$

The k-th approximation $C_k(\xi)$ is represented with quadratic NURBS curve with $2^k m$ elements, For each time subdivision, each segment will produce five control points, and the end control point of this segment is the start control point of the next segment, that is to say $P_{4,i,k} = P_{0,i+1,k}$. The control points of the whole curve are

$$P_{0,1,k}, P_{1,1,k}, \cdots, P_{4,i,k}, P_{1,i+1,k}, P_{2,i+1,k}, \cdots, P_{4,2^k m,k}$$

where $i = 1, \cdots, 2^k m$, the number of control points is $2^{k+1} m + 1$ and the knot vector is

$E = \{0,0,0,\ldots, j/2^k m, j/2^k m, \ldots, 1,1,1\}$, $j = 1, \cdots, 2^k m - 1$.

3.3. Subdivision level

In classical splines approximation, the optimal approximation order and optimal smoothness are $O(h^{q+1})$ and C^{q-1}; the asymptotic error bounds of the form $O(h^{q+1})$ are approximation order $q+1$, where h denotes the maximum length of parameter interval tending to zero and q is the degree. That is to say, the correct approximation method is chosen, there exists some constant K for which

$$d_H(h, C) \le K h^{q+1} \tag{7}$$

for each time h is halved, the new error is roughly $2^{-(q+1)}$ times the previous one. For the approximation helix with quadratic NURBS, the error bound is $d_H(h, C_k) \le \varepsilon_k \le 2^{-3k} \varepsilon_0$.

Definition: For each α, p and r, the helix approximations with NURBS curve $C(\xi)$, $\xi \in [0,1]$, have the relative approximation error bounds

$$e_r = \varepsilon / p \tag{8}$$

where $p = r / \tan\beta$, β is the helix angle. Let $e_{r,k}$ be the k-th level relative error bound. Then

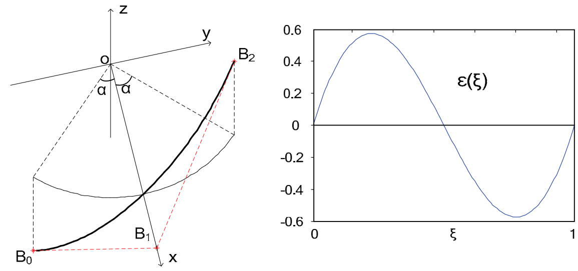
$$e_{r,k} = \varepsilon_k / p \le 2^{-3k} \varepsilon_0 / p = 2^{-3k} e_{r,0}$$
(9)

where $e_{r,0}$ is the initial relative error. So we can obtain the minimum subdivision level for a given tolerance

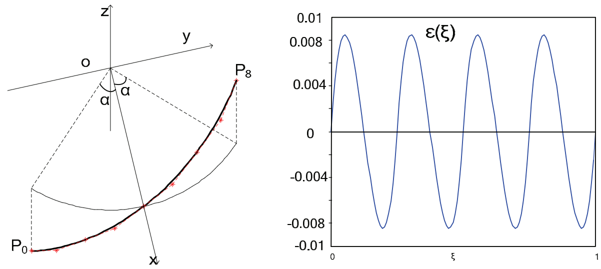
$$k \ge \frac{\ln e_r - \ln e_{r,0}}{-3 \ln 2}$$
(10)

4. Example

In this section, we have applied the new algorithm to the helix is approximated by NURBS curves with center angles, the radius, the helix angle and the given toleration. We just sample a few examples here to show the efficiency of the methods.



(a) Initial approximation and error



(b) Subdividing two times approximation and error

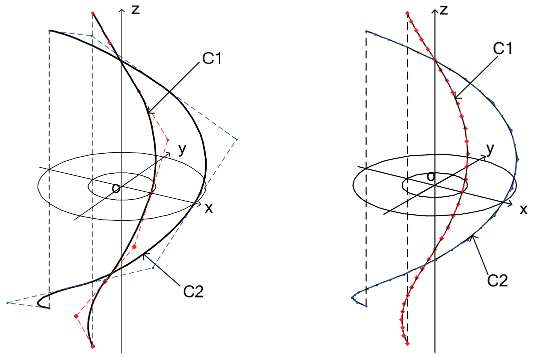
Figure. 5 Approximation helix

In Fig. 5(a), the initial approximation helix is represented one element and three control points, where $r = 10, \beta = \pi/6, \alpha = \pi/4$, and the maximum error is 0.5737. If the given upper error bound is 0.01, we can compute the iteration $k = 2$. Using the subdivision scheme, the approximation helix is represented four elements and nine control points, and the maximum error is 0.0084 less than the given error as shown Fig. 5(b).

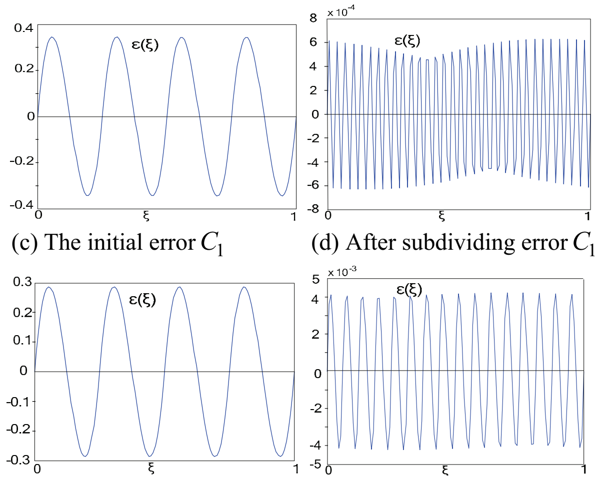
Fig. 6(a) show two initial approximation helices C_1 is of $r_1 = 6, \beta = \pi/6, \theta \in [-\pi, \pi], \varepsilon = 0.005$, C_2 is of $r_2 = 15, \beta = \pi/3, \theta \in [-\pi, \pi], \varepsilon = 0.01$, the initial approximation helices are consist of four segments, and the blue and red dash lines are the control polygon, the blue dots and the red starts are the control points respectively and the errors show in Fig. 6(c) and (e). As shown in Fig. 6(b), the two approximation helices used the subdivision scheme respectively level times and the errors after subdividing show in Fig. 6(d) and (f).The dates of the two approximation helices are obtained as shown in Table 1.

5. Conclusions

In this paper a simple approximation method of the cylindrical helix by quadratic NURBS curves has been presented. The subdivision scheme is different from de-Boor algorithm since the z-coordinate of new produced control points is not affected by weight. The presented subdivision scheme is efficient not only to make the control polygon approximate the helix but also the curve represented by NURBS approximate the helix. The factors influencing the error bound is derived. The error bound varies linearly with the radius, increases as the center angle increases, decreases as the helix angle increase. The relative error is used for determining how many subdivision levels are required to approximate the helix by NURBS curves satisfy a given tolerance.



(a) Initial approximation C_1 and C_2 (b) Subdividing C_1 and C_2



(c) The initial error C_1 (d) After subdividing error C_1

(e) The initial error C_2 (f) After subdividing error C_2

Figure. 6 Approximation helix

Table 1. Two approximation helices within the given toleration

	p	ε_0	$e_{r,0}$	ε	k	$d_H(h,C_k)$	$e_{r,k}$	$2^{-3k}\varepsilon_0$
C_1	10.3929	0.3442	0.0331	0.005	3	6.31×10^{-4}	6.07×10^{-5}	6.46×10^{-5}
C_2	8.6603	0.2869	0.0331	0.010	2	4.2×10^{-3}	4.85×10^{-4}	5.17×10^{-4}

References

[1] Mick, S., and Röschel, O. "Interpolation of helical patches by kinematic rational Bézier patches", *Computers & Graphics*, 1990, 14, (2), pp. 275-280

[2] Juhász, I. "Approximating the helix with rational cubic Bézier curves", *Computer-Aided Design*, 1995, 27, (8), pp. 587-593

[3] Sun, K.H. "Representing the Cylindrical Helix with NURBS Curves", *Mathematic Numerical Sinica* (China), 1998, 20, (4), pp. 393-402

[4] Yang, X. "High accuracy approximation of helices by quintic curves", *Computer Aided Geometric Design*, 2003, 20, (6), pp. 303-317

[5] de Boor, C. "A Practical Guide to Splines" *Springer-Verlag*, 1978.

[6] Ahn, Y.J. "Helix approximations with conic and quadratic Bézier curves", *Computer Aided Geometric Design*, 2005, 22, (6), pp. 551-565

[7] Ahn, Y.J., and Kim, P. "Error Analysis for Approximation of Helix by Bi-conic and Bi-quadratic Bezier Curves", *Comm. Korean Math*. Soc., 2005, 20, (4), pp. 861-873

[8] Piegl, L., and Tiller, W. "The NURBS Book (Monographs in Visual Communication)", Springer-Verlag, 1997, 2nd edn. 1997)

[9] Degen, W.L.F. "High accurate rational approximation of parametric curves", *Computer Aided Geometric Design*, 1993, (10), pp. 293-313

[10] Floater, M. "An analysis of cubic approximation schemes for conic sections", *Advances in Computational Mathematics,* 1996, 5, (1), pp. 361-379

[11] Floater, M. "High-order approximation of conic sections by quadratic splines", *Computer Aided Geometric Design*, 1995, 12, (6), pp. 617-637