Modern Statistics	**Lecture 1 - 2/17/2025**

Probability

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1 Sample Space and Event

Ω: Sample space, the set of possible outcomes of experiments.

A: Event, the subset of Ω ($A \subset \Omega$).

Example 1 *The letter "H" is used to denote the event "the coin is heads up", and "T" is used to denote the event "the coin is tails up". Then we can list these situations as follows:*

1) Toss a ***finite*** *number of coins*

- *Toss a coin once:* $A = \{H, T\}$.
- *Toss a coin twice:* $A = \{HH, HT, TH, TT\}$.
- *Add a condition (toss a coin twice, but only list the possible events when the 1st time is heads up):* $A = \{HH, HT\}$.

2) Toss an ***infinite*** *number of coins*

- $\Omega = \{\omega_1, \omega_2, \omega_3, \cdots\}$.

2 Concepts

- $A \cap B = \{\omega | \omega \in A \text{ and } \omega \in B\}$.
- $A \cup B = \{\omega | \omega \in A \text{ or } \omega \in B\}$.
- $A^c = \{\omega | \omega \in \Omega, \omega \notin A\}$.
- $A \setminus B = \{\omega | \omega \in A, \omega \notin B\}$.
- Disjoint: $\{A_i\}_{i=1}^n$ if $\forall i \neq j$, $A_i \cap A_j = \emptyset$. $\bigcup\limits_{i=1}^{n} A_i = \Omega$.
- $A \subseteq B$: $\omega \in A \implies \omega \in B$.
- Monotone increasing: $A_1 \subseteq A_2 \subseteq A_3 \cdots$, $\lim_{n\to\infty} A_n = \bigcup\limits_{i=1}^{\infty} A_i$.

- Monotone decreasing: $A_1 \supseteq A_2 \supseteq A_3 \cdots$, $\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$.

Example 2 *Analyze the set sequences* $A_i = \left[0, \frac{1}{i}\right), i = 1, 2, \cdots$.

Easy to know that $A_1 = [0,1), A_2 = \left[0, \frac{1}{2}\right), \cdots$*,and* $A_1 \supseteq A_2 \supseteq A_3 \cdots$.

Then $\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i = \{0\}$.

3 Probability

The concept “probability” can be understood as a function (or a mapping).

def. P: $A \subseteq \Omega \to [0,1]$ that satisfies 3 axioms:

- $P(A) \geq 0, \forall A \subseteq \Omega$.
- $P(\Omega) = 1$.
- $\{A_i\}_{i=1}^n, A_i \cap A_j = \emptyset, P(\bigcap_{i=1}^{n} A_i) = \sum_{i=1}^n P(A_i)$.

Properties:

- $P(\emptyset) = 0$.
  proof: $P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) = 1$.
- $P(A) + P(A^c) = 1$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  proof:
  $P(A \cup B) = P((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) = P(A \setminus B) + P(A \cap B) + P(B \setminus A) = P((A \setminus B) \cup (B \setminus A)) + P(B \setminus A) + P(A \cap B) - P(A \cap B) = P(A) + P(B) - P(A \cup B)$.
- $P(A) = \frac{|A|}{|\Omega|} \in [0,1]$.

def. Independence: used for hypotheses, but cannot be verified in real-life situations.

- $A \perp B \iff P(A \cap B) = P(A)P(B)$.
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$.
- $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^n P(A_i)$.

Example 3 *Do the coin toss experiment. It is known that A={10 times at least 1 H }. Solve for P(A).*

Let event B_i denotes "the i-th toss is tails up".

Then $P(A) = 1 - P(A^c) = 1 - P(\bigcap_{i=1}^{n} B_i) = 1 - \prod_{i=1}^{n} P(B_i) = 1 - (\frac{1}{2})^{10} = \frac{1023}{1024}$.