We start with $k = 1$ and the inequality $p_k \geq \mathbb{1}_{\{k=1\}} + \frac{(k-1)p_{k-1}}{2^{d+1}} - kp_k$. Then, we shall get by substituting $k = 1$, and recalling $p_0 = 0$,

$$p_1 \geq 1 - p_1 \implies p_1 \geq 1/2.$$

Now, from the same inequality, we inductively get for $k \geq 2$,

$$\begin{aligned}
p_k &\geq \frac{(k-1)p_{k-1}}{2^{d+1}} - kp_k \\
\implies (k+1)p_k &\geq \frac{k-1}{2^{d+1}} p_{k-1} \\
\implies p_k &\geq \frac{(k-1)\ldots 1}{2^{(k-1)(d+1)}(k+1)\ldots 3} p_1 \\
\implies p_k &\geq \frac{\Gamma(k)}{2^{(k-1)(d+1)}} \\
\implies p_k &\geq \frac{\Gamma(k)}{2^{(k-1)(d+1)}\Gamma(k+2)}.
\end{aligned}$$