De-Preferential Attachment Random Graphs

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- It is natural to believe that given a choice, a new predator will like to choose its food from the existing species which are not eaten by many.
- In other words, a new predator will have *less incentive* or *less preference* to choose its prey from the existing species which have many predators.

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- This is opposite of the usual "*rich get richer model*", also known as, *preferential attachment model* [Barabási and Albert (1999)].
- We will call any such model a *de-preferential attachment model*.
- Our goal will be to study such a model rigorously and compare its properties with the preferential attachment model.

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- At every (discrete) time $n + 1 \geq 2$, we will add one new vertex, say v_{n+1} to the existing graph, say G_n, by letting it to join to the existing vertices $\{v_1, v_2, \ldots, v_n\}$.

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  - In this case we can have *multiple edges* and *self-loops* depending on the mechanism in which the m new links will be formed. Also there can be formation of cycles.
  - None of these are good for a food-chain network, as `A multiply eats B` or `A eats itself` or even `A eats B which eats C but C eats A` are not suitable for such a network.

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- Let $\left\{ \mathcal{F}_{n,k} \,\middle|\, 0 \leq k \leq m-1, n \geq 1 \right\}$ be the natural filtration of the random attachments.
- If $m = 1$ then we will simply write the natural filtration as $\{\mathcal{F}_n\}_{n\geq 1}$.

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where $C_n^{-1} =: D_n = \sum_{i=1}^{n} \frac{1}{d_i(n)}$.

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- This prevents the formation of the *self-loops*.
- We still have the possibility of having *multiple edges* between two vertices.

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- **Inverse De-Preferential Model:**

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where $C^{-1}_{n+1,k} =: D_{n+1,k} = \sum_{j=1}^{n} \frac{1}{d_j(n+1,k)}$.

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- They obtain by some intuitive arguments (not quite rigorous) the asymptotic degree distribution and validated their claims by simulation results.
- Our results support their observations.

Main Results: Linear Case with $m = 1$

Theorem 1 (WLLN for fixed vertex degree)

Fix a vertex $i \geq 1$ then

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$$\frac{d_i(n) - \log n}{\sqrt{\log n}} \xrightarrow{d} \text{Normal}\,(0,1)\,.$$

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Theorem 3 (Asymptotic degree distribution)

Let $P_k(n)$ be the proportion of vertices in G_n with degree $k \geq 1$. Then for any $k \geq 1$,

$$P_k(n) \longrightarrow \frac{1}{2^k} \text{ a.s.}$$

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Remark: The *asymptotic degree distribution* of G_n is Geometric $\left(\frac{1}{2}\right)$ which has mean 2, mode 1 and exponential tail.

Main Results: Linear Case with $m = 1$

Theorem 4 (Asymptotic degree distribution of the chosen vertex)

Let U_{n+1} be the (random) selected vertex from $\{v_1, v_2, \ldots, v_n\}$ where the new vertex v_{n+1} connects. Then for any $k \geq 1$,

$$\mathbf{P}\left(\text{degree}_{G_n}\left(U_{n+1}\right) = k\right) \longrightarrow \frac{1}{2^k}.$$

Main Results: Linear Case with $m \geq 1$

Theorem 5 (WLLN for fixed vertex degree)

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Main Results: Inverse Case with $m = 1$

Theorem 7 (SLLN for fixed vertex degree)

Fix a vertex $i \geq 1$ then

$$\frac{d_i(n)}{\sqrt{\log n}} \longrightarrow \sqrt{\frac{2}{\lambda^*}} \text{ a.s.},$$

where $\lambda^* > 0$ is the unique positive solution of the equation

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{1}{1 + i\lambda^*} = 1.$$

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Theorem 8 (Asymptotic degree distribution)

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Theorem 9 (Asymptotic degree distribution of the chosen vertex)

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Main Results: Inverse Case with $m > 1$

Theorem 10 ("WLLN" for fixed vertex degree)

$\exists$ constants $0 < C_1 < C_2 < \infty$ such that for any fixed vertex i,

$$\mathbf{P}\left(C_1 \leq \frac{d_i(n)}{m\sqrt{\log n}} \leq C_2 \right) \longrightarrow 1,$$

as $n \to \infty$.

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- For the CLTs we use martingale CLT.

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- One type of embedding for $m = 1$ and a different embedding for $m > 1$.

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- Each vertex reproduces independently according to identical copies of a age dependent pure birth process $(\xi(t))_{t\geq 0}$ such that $\mathbf{P}(\xi(0) = 1) = 1$ and

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- This process is an example of a *Crump-Mode-Jagers (CMJ) branching process* [Crump and Mode (1968) and Jagers (1969)].

Techniques Used for the Inverse De-Preferential with $m = 1$

Embedding Theorem for $m = 1$

Starting with $\tau_1 = 0$ consider the following sequence of stopping times

$$\tau_n := \inf\left\{ t \geq \tau_{n-1} \,\middle|\, |\Upsilon(t)| = n \right\}.$$

For $m = 1$, the sequence of random graphs $\{G_n\}_{n=1}^{\infty}$ have the same distribution as the sequence of random trees $\{\Upsilon(\tau_n)\}_{n=1}^{\infty}$.

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(ii) For studying preferential attachment model with non-linear weights a similar observation was made by Rudas and Tóth (2007).

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- $\lambda^* > 0$ is called the *Malthusian parameter* for the (supercritical) CMJ process.

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Theorem A of Nerman (1961)

Suppose $\{\Upsilon(t) : t \geq 0\}$ is a (supercritical) CMJ process with Multhusian parameter λ^* and let $\phi : \mathcal{G} \to \mathbb{R}$ be bounded function. Then the following limit holds almost surely

$$\lim_{t\to\infty} \frac{1}{|\Upsilon(t)|} \sum_{x\in\Upsilon(t)} \phi(\Upsilon(t)_{\downarrow x}) = \lambda^* \int_0^\infty \exp\{-\lambda^* t\}\mathbf{E}(\phi(\Upsilon(t)))dt,$$

where for a tree $\mathcal{T} \in \mathcal{G}$ and a vertex $x \in \mathcal{T}$ we define $\mathcal{T}_{\downarrow x}$ as the sub-tree rooted at x consisting of all the descendants of x.

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Remark: This proves the SLLN for the degree of a fixed vertex and also the asymptotic degree distribution in the inverse de-preferential case.

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- We recursively define the following stopping times starting with $\tau_1 = 0$,

$$
\begin{aligned}
\tau_2 &:= \inf \left\{ t \geq 0 \,\middle|\, Z_1(t) - m = m \right\} \\
\tau_3 &:= \inf \left(t \geq \tau_2 \,\middle|\, Z_1(t) + Z_2(t - \tau_2) - 2m = m \right\} \\
\vdots \quad & \vdots \quad \vdots \\
\tau_{n+1} &:= \inf \left(t \geq \tau_n \,\middle|\, Z_1(t) + Z_2(t - \tau_2) + \cdots + Z_n(t - \tau_n) - n\,m = m \right\}
\end{aligned}
$$

Techniques Used for the Inverse De-Preferential with $m > 1$

Embedding Theorem for $m > 1$

For $m \geq 1$, the two sequence of random variables, namely, $\left\{ (d_i(n))_{i=1}^n \,\middle|\, n \geq 1 \right\}$ and $\left\{ (Z_i(\tau_n - \tau_i))_{i=1}^n \,\middle|\, n \geq 1 \right\}$ has the same distribution.

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WLLN for the Pure Birth Process

Let $\{Z(t) : t \geq 0\}$ be a pure birth process with $\mathbf{P}\left(Z(0) = m\right) = 1$ and birth rates $\lambda_i = \frac{1}{i}$. Then

$$\frac{Z(t)}{\sqrt{t}} \xrightarrow{P} \sqrt{2}.$$

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- But it is necessary assumption for the results on inverse case which we prove using the embedding to CMJ branching process.

Open Problems

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- For $m > 1$ case it seems that the Athreya-Karlin Embedding technique is fairly unsatisfactory for the inverse de-preferential case. Proofs of a complete WLLN and CLT remain open for the degree of a fixed vertex.
- For $m = 1$ case one should remove the dependency on the initial configuration but it seems it is a technically very difficult problem!

Thank You