Let us start with $k = 1$ and the inequality $p_k \leq \mathbb{1}_{\{k=1\}} + (k-1)p_{k-1} - \frac{kp_k}{2^{d+1}}$. Then, we shall get by substituting $k = 1$, and recalling $p_0 = 0$,

$$p_1 \leq 1 - \frac{p_1}{2^{d+1}} \implies p_1 \cdot \frac{2^{d+1}+1}{2^{d+1}} \leq 1 \implies p_1 \leq \frac{2^{d+1}}{2^{d+1}+1}.$$

Now, from the same inequality, we inductively get for $k \geq 2$,

$$\begin{aligned}
& p_k \leq \mathbb{1}_{\{k=1\}} + (k-1)p_{k-1} - \frac{kp_k}{2^{d+1}} \\
\implies & p_k \cdot \frac{2^{d+1}+k}{2^{d+1}} \leq (k-1)p_{k-1} \\
\implies & p_k \leq \frac{2^{d+1}(k-1)}{2^{d+1}+k} p_{k-1} \\
\implies & p_k \leq \frac{2^{(k-1)(d+1)}(k-1)(k-2)\ldots 1}{(2^{d+1}+k)(2^{d+1}+k-1)\ldots(2^{d+1}+2)} p_1 \\
\implies & p_k \leq \frac{2^{k(d+1)}\Gamma(k)\Gamma(2^{d+1}+1)}{\Gamma(2^{d+1}+k+1)}.
\end{aligned}$$