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The stable index of 0-1 matrices

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ABSTRACT

We introduce the concept of stable index for 0-1 matrices. Let A be a 0-1 square matrix. If A^k is a 0-1 matrix for every positive integer k, then the stable index of A is defined to be infinity; otherwise, the stable index of A is defined to be the smallest positive integer k such that A^{k+1} is not a 0-1 matrix. We determine the maximum finite stable index of all 0-1 matrices of order n as well as the matrices attaining the maximum finite stable index.

1. Introduction

Properties on the power of nonnegative matrices attract a lot of attentions in combinatorial matrix theory. Let A be a nonnegative square matrix. The Perron-Frobenius

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Theorem states that its spectral radius $\rho(A)$ is an eigenvalue of A. If A has no other eigenvalue of modulus $\rho(A)$, then it is said to be *primitive*. Frobenius proved that A is primitive if and only if there is a positive integer k such that A^k is a positive matrix (see [14], p.134). Given a primitive matrix A, the smallest integer k such that A^k is positive is called the *exponent* of A. Wielandt [13] shows that the exponent of an $n \times n$ primitive matrix is bounded by $(n-1)^2+1$. Dulmage and Mendelsohn [2] revealed that the exponent set of $n \times n$ primitive matrices is not the set $[1,(n-1)^2+1]$, i.e., there are gaps. Lewin, Vitek, Shao and Zhang [8,10,15] determined all the possible exponents of primitive matrices of order n. Brualdi and Ross [1,9], Holladay and Varga [5], Lewin [7], Shao [11,12] investigated exponents of special primitive matrices. Heap and Lynn [3,4] studied *periods* and the *indices of convergence* of nonnegative matrices, which are defined based on combinatorial properties of powers of nonnegative matrices and have connections with stochastic theory.

Denote by $M_n\{0,1\}$ the set of 0-1 matrices of order n. Similar with the exponent of primitive matrices, we introduce a new parameter on 0-1 matrices as follows, which is called the *stable index*.

Definition. Let $A \in M_n\{0,1\}$. If $A^k \in M_n\{0,1\}$ for every positive integer k, then the stable index of A is defined to be ∞; otherwise, the stable index of A is defined to be the smallest positive integer k such that $A^{k+1} \notin M_n\{0,1\}$. We denote the stable index of A by $\theta(A)$.

Note that if $\theta(A)$ is finite, then it is the largest integer k such that $A, A^2, \ldots, A^k$ are all 0-1 matrices.

It is obvious that the stable index can provide an upper bound on the spectral radius of a 0-1 matrix. Recall that the spectral radius of a 0-1 square matrix does not exceed its maximum row sum (see [14], p.126). Given a matrix $A \in M_n\{0,1\}$ with stable index k, since $A^k \in M_n\{0,1\}$, we have $\rho^k(A) = \rho(A^k) \le n$, which leads to

$$\rho(A) \le n^{1/k}.$$

In some sense, both the exponent of primitive matrices and the stable index of 0-1 matrices represent the density of nonzero entries in these matrices. Generally, a matrix with large exponent or stable index can not have many nonzero entries.

Huang and Zhan [6] studied 0-1 matrices with infinite stable index. They determined the maximum number of nonzero entries in matrices from $M_n\{0,1\}$ with infinite stable index as well as the matrices attaining this maximum number. We solve the following problem in this paper.

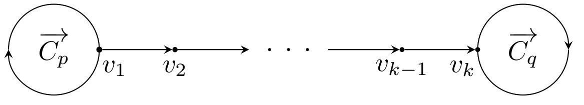
Problem 1. Given a positive integer n. Determine the maximum finite stable index of all 0-1 matrices of order n as well as the 0-1 matrices attaining the maximum finite stable index.

2. Main results

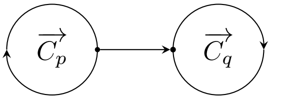
We need some terminology on digraphs. For a digraph D, we denote by $\mathcal{V}(D)$ its vertex set, $\mathcal{A}(D)$ its arc set, (u, v) or uv the arc from u to v. A sequence of consecutive arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{t-1}, v_t)$ is called a *directed walk* (or *walk*) from v_1 to v_t, which is also written as $v_1 v_2 \cdots v_t$. A *directed cycle* (or *cycle*) is a closed walk $v_1 v_2 \cdots v_t v_1$ with $v_1, v_2, \ldots, v_t$ being distinct. A *directed path* (or *path*) is a walk in which all the vertices are distinct. The *length* of a walk (path) is the number of arcs in the walk. A walk (path) of length k is called a *k-walk (k-path)*. In a digraph D, if there is a walk from u to v for all $u, v \in \mathcal{V}(D)$, then D is said to be *strongly connected*.

Two digraphs D and H are *isomorphic*, written $D \cong H$, if there are bijections $\phi : \mathcal{V}(D) \to \mathcal{V}(H)$ and $\varphi : \mathcal{A}(D) \to \mathcal{A}(H)$ such that $(u, v) \in \mathcal{A}(D)$ if and only if $(\phi(u), \phi(v)) \in \mathcal{A}(H)$. In other words, $D \cong H$ if and only if we can get D by relabeling the vertices of H. We say a digraph D contains a *copy* of H if D has a subgraph isomorphic to H.

Denote by $\overrightarrow{C_k}$ the directed cycle with k vertices. Given an integer $k \geq 2$ and two disjoint directed cycles $\overrightarrow{C_p}$ and $\overrightarrow{C_q}$, let $\overrightarrow{g}(p, k, q)$ be the following digraph obtained by adding a $(k-1)$-path from a vertex of $\overrightarrow{C_p}$ to a vertex of $\overrightarrow{C_q}$. Especially, when $k = 2$, we call $\overrightarrow{g}(p, k, q)$ a *glasses digraph* and write it simply as $\overrightarrow{g}(p, q)$. It is clear that $|\mathcal{V}(\overrightarrow{g}(p, k, q))| = p + q + k - 2$.



$\overrightarrow{g}(p, k, q)$



$\overrightarrow{g}(p, q)$

Given a digraph $D = (\mathcal{V}, \mathcal{A})$ with vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and arc set $\mathcal{A}$, its *adjacency matrix* $A_D = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in \mathcal{A}; \\ 0, & \text{otherwise.} \end{cases}$$

We call $\theta(A_D)$ the *stable index* of D, denoted $\theta(D)$.

Let $A = (a_{ij}) \in M_n\{0, 1\}$. We define its digraph as $D(A) = (\mathcal{V}, \mathcal{A})$ with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{A} = \{(v_i, v_j) : a_{ij} = 1, 1 \leq i, j \leq n\}$. For convenience, we will always assume the vertex set of $D(A)$ to be $\{1, 2, \ldots, n\}$.

Denote by $A(i, j)$ the (i, j)-entry of A. Given $A \in M_n\{0, 1\}$, $A^k(i, j) = t$ if and only if $D(A)$ has exactly t distinct walks of length k from i to j. It follows that $D(A)$ contains at most one walk of length k from i to j for all $k \leq \theta(A)$ and $1 \leq i, j \leq n$. If $\theta(A) < \infty$, then there exist vertices i, j such that $D(A)$ contains two distinct walks of length $\theta(A)+1$ from i to j.

Let n be a positive integer. Denote by

$$g(n)=\begin{cases}\frac{n^2-1}{4}, & \text{if } n \text{ is odd},\\ \frac{n^2-4}{4}, & \text{if } n\equiv 0 \pmod 4,\\ \frac{n^2-16}{4}, & \text{if } n\equiv 2 \pmod 4.\end{cases}$$

Our main result states as follows.

Theorem 1. *Let $n \geq 7$ be an integer and $A \in M_n\{0,1\}$ with $\theta(A) < \infty$. Then*

$$\theta(A) \leq g(n) \tag{1}$$

with equality if and only if one of the following holds.

(1) $n \neq 10$ and $D(A) \cong \overrightarrow{g}(p,q)$ with

$$\{p,q\}=\begin{cases}\{\frac{n+1}{2},\frac{n-1}{2}\}, & \textit{if } n \textit{ is odd},\\ \{\frac{n+2}{2},\frac{n-2}{2}\}, & \textit{if } n\equiv 0 \textit{ (mod 4)},\\ \{\frac{n+4}{2},\frac{n-4}{2}\}, & \textit{if } n\equiv 2 \textit{ (mod 4)}.\end{cases} \tag{2}$$

(2) $n = 10$ and $D(A) \cong D$ with

$$D \in \{\overrightarrow{g}(3,7), \overrightarrow{g}(7,3), \overrightarrow{g}(4,3,5), \overrightarrow{g}(5,3,4)\}.$$

Let E_{ij} be the 0-1 matrix with exactly one nonzero entry lying on the (i,j) position, while its size will be clear from the context. Then $C_n = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n1}$ is the basic circulant matrix of order n.

It is clear that $D(A) \cong \overrightarrow{g}(p,q)$ if and only if A is permutation similar to

$$\begin{pmatrix} C_p & E_{ij} \\ 0 & C_q \end{pmatrix},$$

where E_{ij} is an arbitrary $p \times q$ 0-1 matrix with exactly one nonzero entry. $D(A) \cong \overrightarrow{g}(p,3,q)$ if and only if A is permutation similar to

$$\begin{pmatrix} C_p & u & 0 \\ 0 & 0 & v^T \\ 0 & 0 & C_q \end{pmatrix},$$

where u, v are 0-1 column vectors with exactly one nonzero entry.

Remark. Denote by $s(n)$ the maximum finite stable index of all matrices in $M_n\{0,1\}$, i.e.,

$$s(n) = max\{\theta(A) : A \in M_n\{0,1\}, \theta(A) < \infty\}.$$

Theorem 1 shows

$$s(n) = g(n) \quad \text{for} \quad n \geq 7.$$

When $n \leq 6$, $s(n) = g(n)$ may not hold. In fact, by using MATLAB we can get all $s(n)$ for $n \leq 6$ as follows.

n	2	3	4	5	6
$s(n)$	1	3	4	6	7

3. Proof of Theorem 1

In this section we present the proof of Theorem 1. We write $B \leq A$ or $A \geq B$ to mean that $A - B$ is a nonnegative matrix. We need the following lemmas.

Lemma 2. *Let A, B be square 0-1 matrices. If B is a principal submatrix of A or $B \leq A$, then*

$$\theta(B) \geq \theta(A).$$

Proof. It is obvious. $\square$

Lemma 3. *Let $A \in M_n\{0,1\}$ be irreducible with $n \geq 2$. If $D(A) \cong \overrightarrow{C}_n$, then $\theta(A) = \infty$; otherwise, we have $\theta(A) \leq n$.*

Proof. Since A is irreducible, $D(A)$ is strongly connected. If $D(A) \cong \overrightarrow{C}_n$, then we have $\theta(A) = \infty$. Otherwise, $D(A)$ has two directed cycles $\overrightarrow{C}_k$ and $\overrightarrow{C}_t$ with nonempty intersection, which can be written as $\overrightarrow{C}_k = v_1v_2 \cdots v_sv_{s+1} \cdots v_kv_1$ and $\overrightarrow{C}_t = v_1v_2 \cdots v_su_{s+1} \cdots u_tv_1$ with $k \geq t \geq s$ and

$$\{v_{s+1}, \ldots, v_k\} \cap \{u_{s+1}, \ldots, u_t\} = \emptyset.$$

It follows that $D(A)$ has the following distinct walks of length $k + (t - s) + 1$ from v_s to v_1:

$$\begin{cases} v_sv_{s+1} \cdots v_kv_1 \cdots v_su_{s+1} \cdots u_tv_1, \\ v_su_{s+1} \cdots u_tv_1 \cdots v_sv_{s+1} \cdots v_kv_1. \end{cases}$$

Hence, we have $\theta(A) \leq k + (t - s) \leq n$. $\square$

Denote by $J_{m\times n}$ the $m \times n$ matrix with all entries equal to 1. We have

Lemma 4. *Let m and n be relatively prime positive integers. Then for any $1 \le i \le m$ and $1 \le j \le n$, we have*

$$\sum_{k=0}^{mn-1} {C_m}^k E_{ij} {C_n}^{mn-k-1} = J_{m\times n}.$$

Proof. Denote by $P = C_m$, $Q = C_n$. Note that $P^k E_{ij} Q^{mn-k-1}$ has exactly one nonzero entry. It suffices to prove

$$P^k E_{ij} Q^{mn-k-1} \ne P^t E_{ij} Q^{mn-t-1} \quad \text{for} \quad k, t \in \{0, 1, \dots, mn-1\} \text{ such that } k \ne t. \tag{3}$$

Suppose $P^k E_{ij} Q^{mn-k-1} = P^t E_{ij} Q^{mn-t-1}$. Then we have

$$P^{k-t} E_{ij} = E_{ij} Q^{k-t},$$

which implies $k-t = um = vn$ for some integers u, v. Since m and n are relatively prime, we obtain $k - t = rmn$ for some integer r. Now $k - t \in [0, mn - 1]$ leads to $k - t = 0$. Therefore, we have (3). □

Given two positive integers p and q, we write $(p, q) = 1$ if they are relatively prime. Denote by $lcm(p, q)$ the least common multiple of p and q. Since we can find two distinct walks of length $lcm(p, q) + k - 1$ with the same initial and terminal vertices in $\overrightarrow{g}(p, k, q)$, we have

$$\theta(\overrightarrow{g}(p, k, q)) \le lcm(p, q) + k - 2. \tag{4}$$

Note that

$$\begin{cases} (\frac{n+1}{2}, \frac{n-1}{2}) = 1, & \text{if } n \text{ is odd}, \\ (\frac{n+2}{2}, \frac{n-2}{2}) = 1, & \text{if } n \equiv 0 \pmod 4, \\ (\frac{n+4}{2}, \frac{n-4}{2}) = 1, & \text{if } n \equiv 2 \pmod 4. \end{cases} \tag{5}$$

We have

$$g(n) = \max\{pq : p + q = n \text{ and } p, q \text{ are relatively prime}\}.$$

Lemma 5. *Let $n \ge 7$ be an integer. Then for any matrix $B \in M_n\{0, 1\}$ with the form*

$$B = \begin{pmatrix} C_p & X \\ 0 & C_q \end{pmatrix}, \quad X \ne 0,$$

we have

$$\theta(B) \leq g(n).$$

Equality holds if and only if $D(B) \cong \overrightarrow{g}(p,q)$ *with* p, q *satisfying* (2).

Proof. Note that $D(B)$ contains a copy of $\overrightarrow{g}(p,q)$. By (4) and Lemma 2 we have

$$\theta(B) \leq \theta(\overrightarrow{g}(p,q)) \leq lcm(p,q) \leq g(n). \tag{6}$$

Now suppose $\theta(B) = g(n)$. Then (6) implies that p, q satisfy (2) and they are relatively prime. By direct computation we have

$$B^m = \begin{pmatrix} {C_p}^m & \sum_{k=0}^{m-1} {C_p}^k X {C_q}^{m-1-k} \\ 0 & {C_q}^m \end{pmatrix}.$$

Next we distinguish two cases.

Case 1. X has at least two nonzero entries. Applying Lemma 4 we have

$$\sum_{k=0}^{pq-1} {C_p}^k X {C_q}^{pq-1-k} \geq 2J_{p\times q},$$

which leads to $B^{pq} \notin M_n\{0,1\}$ and

$$\theta(B) < pq = g(n).$$

Case 2. X has exactly one nonzero entry. By Lemma 4 and its proof, we have

$$\sum_{k=0}^{m-1} {C_p}^k X {C_q}^{m-1-k} \leq J_{p\times q} \quad \text{for} \quad m \leq pq,$$

where equality holds if and only if $m = pq$. On the other hand, it is easy to check that $B^{pq+1} \notin M_n\{0,1\}$. Therefore, we have

$$\theta(B) = pq = g(n).$$

Thus we get $D(B) \cong \overrightarrow{g}(p,q)$ and the second part of the lemma holds. □

Lemma 6. *Let* $\phi(t) = g(t) - t$. *Then*

$$\phi(r) \geq \phi(s) \text{ for all positive integers } r \geq 7 \text{ and } s < r.$$

Equality holds if and only if $(r,s) = (10,9)$.

Proof. For $r \geq 11$ and $s < r$, we have

$$\phi(r) - \phi(s) \geq \frac{r^2 - 16}{4} - r - \frac{s^2 - 1}{4} + s = \frac{(r+s-4)(r-s)}{4} - \frac{15}{4} > 0.$$

Note that

$$\begin{array}{lllll} \phi(1) = -1, & \phi(2) = -5, & \phi(3) = -1, & \phi(4) = -1, & \phi(5) = 1, \\ \phi(6) = -1, & \phi(7) = 5, & \phi(8) = 7, & \phi(9) = 11, & \phi(10) = 11. \end{array}$$

The conclusion follows clearly. □

Lemma 7. *Let $n \geq 7$ be an integer and*

$$\begin{aligned} \mathcal{B} = \big\{ & B \in M_n\{0,1\} : \\ & D(B) \text{ contains a copy of } \overrightarrow{g}(p,k,q) \text{ for some positive integers } p,q,k \big\}. \end{aligned}$$

Then

$$\theta(B) \leq g(n) \quad \text{for all} \quad B \in \mathcal{B}.$$

Moreover, $\theta(B) = g(n)$ if and only if $D(B) \cong \overrightarrow{g}(p,q)$ with p, q satisfying (2) for $n \neq 10$ and

$$D(B) \cong D \quad \text{with} \quad D \in \{\overrightarrow{g}(3,7), \overrightarrow{g}(7,3), \overrightarrow{g}(4,3,5), \overrightarrow{g}(5,3,4)\} \quad \text{for} \quad n = 10.$$

Proof. Let $B \in \mathcal{B}$. Suppose $D(B)$ contains a copy of $\overrightarrow{g}(p,k,q)$ with $p + q = m$. Then by Lemma 2 and Lemma 6 we have

$$\theta(B) \leq \theta(\overrightarrow{g}(p,k,q)) \leq lcm(p,q) + k - 2 \leq g(m) + k - 2 \leq g(m) + n - m \leq g(n). \tag{7}$$

Now suppose $\theta(B) = g(n)$. We distinguish two cases.

Case 1. $m = n$. The condition $p + q = m$ implies that B is permutation similar with

$$H = \begin{pmatrix} C_p + B_{11} & X \\ Y & C_q + B_{22} \end{pmatrix},$$

where $X \neq 0$. Note that if two 0-1 matrices are permutation similar, then they have the same stable index. Applying Lemma 3 we get $Y = 0$, $B_{11} = 0$ and $B_{22} = 0$, since otherwise we would have $\theta(B) = \theta(H) \leq n < g(n)$. By Lemma 5 we see that $\theta(B) = g(n)$ if and only if $D(B) \cong \overrightarrow{g}(p,q)$ with p, q satisfying (2).

Case 2. $m < n$. By (7) we get

$$g(m) - m = g(n) - n.$$

Applying Lemma 6 we have

$$n = 10, \quad m = 9, \quad k = 3 \quad \text{and} \quad lcm(p,q) = g(9),$$

which leads to $\{p, q\} = \{4, 5\}$. Moreover, $D(B)$ contains no copy of $\overrightarrow{g}(4,5)$ or $\overrightarrow{g}(5,4)$. Otherwise we have

$$\theta(B) \leq g(9) < g(10).$$

Therefore, $D(B)$ contains a copy of $\overrightarrow{g}(4,3,5)$ or $\overrightarrow{g}(5,3,4)$.

Next we prove that if $D(B)$ has a copy of $\overrightarrow{g}(4,3,5)$, then $D(B) \cong \overrightarrow{g}(4,3,5)$. Note that B is permutation similar with

$$H = \begin{pmatrix} C_4 + B_{11} & u & B_{13} \\ x^T & \alpha & v^T \\ B_{31} & y & C_5 + B_{33} \end{pmatrix},$$

where both u and v are nonzero column vectors. Applying Lemma 3 we obtain

$$B_{11} = 0, \quad B_{33} = 0, \quad x = 0 \quad \text{and} \quad y = 0,$$

since otherwise we would have $\theta(B) = \theta(H) \leq 10$. Since $D(B)$ has no copy of $\overrightarrow{g}(4,5)$ or $\overrightarrow{g}(5,4)$, we have $B_{13} = B_{31}^T = 0$. Moreover, it is easy to check that if $\alpha \neq 0$, then $H^5 \notin M_{10}\{0,1\}$, which leads to $\theta(B) = \theta(H) \leq 4$. Therefore,

$$H = \begin{pmatrix} C_4 & u & 0 \\ 0 & 0 & v^T \\ 0 & 0 & C_5 \end{pmatrix}. \tag{8}$$

By direct computation, we have

$$H^m = \begin{pmatrix} {C_4}^m & {C_4}^{m-1}u & \sum_{i=0}^{m-2} {C_4}^i(uv^T){C_5}^{m-2-i} \\ 0 & 0 & v^T{C_5}^{m-1} \\ 0 & 0 & {C_5}^m \end{pmatrix}.$$

If u or v has more than one nonzero entry, then applying Lemma 4 we have

$$F(m) \equiv \sum_{i=0}^{m-2} {C_4}^i(uv^T){C_5}^{m-2-i} \geq 2J_{4\times 5} \quad \text{when} \quad m = 21,$$

which implies $H^{21} \notin M_{10}\{0,1\}$ and

$$\theta(B) = \theta(H) \leq 20 < g(10).$$

Hence, each of u and v has exactly one nonzero entry and $D(B) \cong \overrightarrow{g}(4,3,5)$.

Similarly, if $D(B)$ has a copy of $\overrightarrow{g}(5,3,4)$, then $D(B) \cong \overrightarrow{g}(5,3,4)$.

On the other hand, if $D(B) = \overrightarrow{g}(4,3,5)$ or $\overrightarrow{g}(5,3,4)$, then B or B^T is permutation similar to a matrix H with form (8), where each of u and v has exactly one nonzero entry. It follows that

$$F(m) \leq J_{4\times 5} \quad \text{for all} \quad 1 \leq m \leq 21$$

and $F(22) \notin M_{10}\{0,1\}$, which means $\theta(B) = \theta(H) = 21 = g(10)$.

This completes the proof. $\square$

Now we are ready to present the proof of Theorem 1.

Proof of Theorem 1. Let $A \in M_n\{0,1\}$ with $\theta(A) < \infty$. Without loss of generality, we may assume

$$A = \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix} \tag{9}$$

with each A_i being an $n_i \times n_i$ irreducible square matrix for $i = 1, 2, \ldots, k$. We denote by A_{ij} the (i,j)-block of A in (9).

If there exists $i \in \{1, 2, \ldots, k\}$ such that $n_i \geq 2$ and $D(A_i)$ is not isomorphic to C_{n_i}, then by Lemma 3 we have

$$\theta(A) \leq n < g(n). \tag{10}$$

If there exist $i, j \in \{1, 2, \ldots, k\}$ such that $n_i \geq 2, n_j \geq 2$ and $A_{ij} \neq 0$, then $D(A)$ contains a copy of $\overrightarrow{g}(n_i, n_j)$. Applying Lemma 5 we have

$$\theta(A) \leq g(n_i + n_j) \leq g(n).$$

Moreover, $\theta(A) = g(n)$ if and only if $D(A) \cong \overrightarrow{g}(p,q)$ with $\{p,q\} = \{n_i, n_j\}$ satisfying (2).

Now we are left to verify the case

$$A_i = C_{n_i} \quad \text{for all} \quad n_i \geq 2 \quad \text{and} \quad A_{i_1 i_2} = 0 \quad \text{for all} \quad n_{i_1} \geq 2, n_{i_2} \geq 2.$$

We consider the digraph $D(A)$. By the definition of stable index, $D(A)$ has two distinct walks $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ with length $\theta(A) + 1$ from x to y for some vertices x, y. Denote by $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ the union of $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$, i.e., the digraph with vertex set $\mathcal{V}(\overrightarrow{w}_1) \cup \mathcal{V}(\overrightarrow{w}_2)$ and arc set $\mathcal{A}(\overrightarrow{w}_1) \cup \mathcal{A}(\overrightarrow{w}_2)$. We distinguish three cases.

Case 1. $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ is acyclic. Then both $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ are directed path and $\theta(A) \leq n-1 < g(n)$.

Case 2. $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ has exactly one cycle $\overrightarrow{C}_p$. If only one of the two walks, say $\overrightarrow{w}_1$, contains copies of $\overrightarrow{C}_p$, then $\overrightarrow{w}_2$ is a directed path with length less than $n-1$, which implies $\theta(A) \leq n-1$. If both $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ contain copies of $\overrightarrow{C}_p$, then by deleting a copy of $\overrightarrow{C}_p$ in each of $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ we get two distinct walks of length $\theta(A)+1-p$ from x to y, which contradicts the definition of $\theta(A)$. Hence, we always have $\theta(A) < g(n)$ in this case.

Case 3. $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ has at least two cycles. We distinguish two subcases.

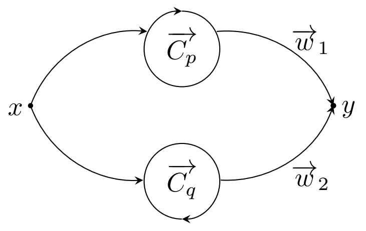
Subcase 3.1. $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ contains a copy of $\overrightarrow{g}(p,k,q)$ for some positive integers p,q,k. Then $(p,q) = (n_i, n_j)$ for some $i,j \in \{1,2,\ldots,k\}$. Since $A_{ij} = 0$ for all $n_i \geq 2, n_j \geq 2$, applying Lemma 7 we have either

$$\theta(A) \leq \theta(\overrightarrow{w}_1 \cup \overrightarrow{w}_2) < g(n)$$

or $g(A) = g(n)$ with

$$D(A) \cong \overrightarrow{g}(p,3,q), \quad \text{where} \quad \{p,q\} = \{4,5\}.$$

Subcase 3.2. $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ does not contain any copy of $\overrightarrow{g}(p,k,q)$. Then $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ contain disjoint cycles $\overrightarrow{C}_p$ and $\overrightarrow{C}_q$, respectively. Moreover, $\overrightarrow{w}_1 \cup \overrightarrow{w}_2$ has the following diagram.



Note that there exists a unique directed path from x to y in each of $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$, denoted $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$, which can be obtained by deleting copies of $\overrightarrow{C}_p$ and $\overrightarrow{C}_q$ in $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$, respectively. Suppose the length of $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$ are r and s, respectively. Then $\overrightarrow{w}_1$ is the union of $\overrightarrow{p}_1$ and $(\theta(A)+1-r)/p$ copies of $\overrightarrow{C}_p$; $\overrightarrow{w}_2$ is the union of $\overrightarrow{p}_2$ and $(\theta(A)+1-s)/q$ copies of $\overrightarrow{C}_q$.

Let u and v be the smallest nonnegative integers such that $r+up = s+vq$. Then

$$u = (\theta(A)+1-r)/p \quad \text{and} \quad v = (\theta(A)+1-s)/q.$$

Otherwise let $\overrightarrow{w}_3$ be the union of $\overrightarrow{p}_1$ and u copies of $\overrightarrow{C}_p$, and let $\overrightarrow{w}_4$ be the union of $\overrightarrow{p}_2$ and v copies of $\overrightarrow{C}_q$. Then $\overrightarrow{w}_3$ and $\overrightarrow{w}_4$ are two distinct walks of length $r+up < \theta(A)+1$ from x to y, which contradicts the definition of $\theta(A)$.

Next we prove that

$$r + up \leq g(n), \tag{11}$$

which implies

$$\theta(A) \leq r + up - 1 < g(n).$$

Note that $r, s \leq n-1$. If $p = 1$, then

$$\begin{aligned} r + up &\leq \min\{s + mq : m \text{ is a nonnegative integer such that } s + mq \geq n-1\} \\ &< n - 1 + q \leq n - 1 + (n-3) \leq g(n). \end{aligned}$$

Similarly, we have (11) when $q = 1$. Suppose $p \geq 2$ and $q \geq 2$. Since the cycle $\overrightarrow{C}_q$ in $\overrightarrow{w}_2$ is disjoint with $\overrightarrow{w}_1$, $\overrightarrow{p}_1$ has at most $n-2$ vertices and we have $r \leq n-3$. Similarly, $s \leq n-3$. Let $z = lcm(p, q)$. If $r + up \geq g(n) + 1$, then

$$r + up - z \geq g(n) + 1 - g(p+q) \geq g(n) + 1 - g(n-2) \geq n - 3 \geq \max\{r, s\}.$$

It follows that

$$r + (u - z/p)p = r + up - z = s + vq - z = s + (v - z/q)q \quad \text{with} \quad u - z/p \geq 0, v - z/q \geq 0,$$

which contradicts the choices of u and v. Thus we obtain (11).

Now we can conclude that $\theta(A) \leq g(n)$ with equality if and only if $D(A) \cong D$, where

$$D \in \{\overrightarrow{g}(p, q) : p, q \text{ satisfies } (2)\} \cup \{\overrightarrow{g}(4, 3, 5), \overrightarrow{g}(5, 3, 4)\}.$$

This completes the proof. □

Declaration of competing interest

The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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