(1) If $H(n) \in O(g(n))$ and $t_2(n) \in (g_2(n))$, then $t_1(n)+t_2(n) \in O(\max \alpha g_1(n); g_2(n)\})$. Prove that assertions.

Sol:- we need to show that $t_1(n)+t_2(n) \in O(\max \{g_1(n); g_2(n)\}$ This means there exists a positive constant c and n_0 such that $t_1(n)+t_2(n) \leq c$.

$$t_1(n) \leq c_1 g_1(n), \text{ for all } n \geq n_1$$

$$t_2(n) \leq c_2 g_2(n), \text{ for all } n \geq n_2$$

let $n_0 = \max \{n_1, n_2\}$ for all $n \geq n_0$

Consider $t_1(n)+t_2(n)$ for all $n \geq n_0$

$$t_1(n)+t_2(n) \leq c_1 g_1(n) + c_2 g_2(n)$$

we need to relate $g_1(n)$ and $g_2(n)$ to $\max \{g_1(n); g_2(n)\}$.

$g_1(n) \leq \max \{g_1(n): g_2(n)\}$ and

$$g_2(n) \leq \max \{g_1(n); g_2(n)\}$$

Thus,

$$c_1 g_1(n) \leq c_1 \max \{g_1(n); g_2(n)\}$$

$$c_2 g_2(n) \leq c_2 \max \{g_1(n); g_2(n)\}$$

$$c_1 g_1(n) + c_2 g_2(n) \leq c_1 \max \{g_1(n); g_2(n)\} + c_2 \max \{g_1(n): g_2(n)\}$$

$$c_1 g_1(n) + c_2 g_2(n) \leq (c_1+c_2) \max (g_1(n); g_2(n)\}$$

$$t_1(n)+t_2(n) \leq (c_1+c_2) \max \{g_1(n); g_2(n)\} \text{ for all } n \geq n_0$$

By the defination of Big-O Notation

$t_1(n)+t_2(n) \in O(\max \{g(n); g_2(n)\}$, then

$t_1(n)+t_2(n) \in O(\max \{g_1(n); g_2(n)\}$

(2) Find the time Complexity of the recurrence equation.

Let us consider such that the recurrence for merge Sort.

$T(n) = 2T(n/2) + n$

By using the master's theorem

$T(n) = aT(n/b) + f(n)$

where, $a \geq 1$; $b \geq 1$ and $f(n)$ is a positive constant function.

Ex:- $T(n) = 2T(n/2) + n$

$a = 2$; $b = 2$; $f(n) = n$

By comparing of $f(n)$ with $n^{\log_b a}$

$\log_b a = \log_2 2 = 1$

compare $f(n)$ with $n^{\log_b a}$

$f(n) = n$

$n^{\log_b a} = n^1 = n$

* $f(n) = O(n^{\log_b a})$ then $t(n) = O(n^{\log_b a} \log n)$

In our case:

$\log_b a = 1$

$T(n) = O(n^1 \log n) = O(n \log n)$

Then time complexity of the recurrance relation is $T(n) = 2T(n/2) + n$ is $O(n \log n$.

(3) $T(n) = \begin{cases} 2T(n/2)+1 & \text{if } n>1 \\ 1 & \text{otherwise} \end{cases}$

By applying of master's theorem

$T(n) = aT(n/b) + f(n)$ where $a \geq 1$, $b \geq 1$

$T(n) = 2T(n/2)+1$

Here

$a=2$; $b=2$; $f(n)=1$

By comparision of $f(n)$ and $n^{\log_b a}$

if $f(n) = O(n^c)$ where $c < \log_b a$, then $T(n) = O(n^{\log_b a})$

if $f(n) = O(n^{\log_b a})$, then $T(n) = O(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^c)$, where $c > \log_b a$ then $T(n) = O(f(n))$

Let us calculate $\log_b a$:

$\log_b a = \log_2 2 = 1$

$f(n) = 1$

$n^{\log_b a} = n^1 = n$

$f(n) = O(n^c)$ with $c \leq \log_b a$ (case 1)

in this case $c=0$ and $\log_b a = 1$

$c<1$, So $T(n) = O(n^{\log_b a}) = O(n^1) = O(n)$

$\therefore$ The time complexity of recurrence relation is

$T(n) = 2T(n/2)+1$ is $O(n)$.

(4)

$$T(n) = \begin{cases} 2T(n-1) & \text{if } n>0 \\ 1 & \text{otherwise} \end{cases}$$

Here,

where $n=0$

$T(0)=1$

recurrance relation analysis

for $n>0$:

$T(n) = 2T(n-1)$

$T(n) = 2T(n-1)$

$T(n-1) = 2T(n-2)$

$T(n-2) = 2T(n-3)$

$\vdots$

$T(1) = 2T(0)$

From this above pattern

$T(n) = 2 \cdot 2 \cdot 2 \cdots\cdots 2 \cdot T(0) = 2^n \cdot T(0)$

Since

$T(0) = 1$; we have

$T(n) = 2^n$

$\therefore$ The recurrance relation is

$T(n) = 2T(n-1)$ for $n>0$ and $T(0)=1$ is

$T(n) = 2^n$

⑤ Big-O-Notation Show that $f(n) = n^2+3n+5$ is $O(n^2)$

$f(n) = O(g(n))$ means $c>0$ and $n_0 \geq 0$

$f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

Given,

$f(n) = n^2+3n+5$

$c>0$, $n_0 \geq 0$ Such that $f(n) \leq c \cdot n^2$

$f(n) = n^2+3n+5$

lets choose $c=2$

$f(n) \leq 2 \cdot n^2$

$f(n) = n^2+3n+5 \leq n^2+3n^2+5n^2$

$= 9n^2$

So, $c=9$; $n_0=1$; $f(n) \leq 9n^2$ for all $n \geq 1$

$f(n) = n^2+3n+5$ is $O(n^2)$