1 Solve the following recurrence relation

(a) $x(n) = x(n-1) + 5$ for $n>1$ with $x(1) = 0$

1 write down the first two terms to identify the pattern.

$$x(1) = 0$$
$$x(2) = x(1) + 5 = 5$$
$$x(3) = x(2) + 5 = 10$$
$$x(4) = x(3) + 5 = 15$$

2 Identify the pattern (or) the general term

→ The first term $x(1) = 0$

The common difference $d = 5$

The general formula for the n^{th} term of an AP is

$$x(n) = x(1) + (n-1)d$$

Substituting the given values

$$x(n) = 0 + (n-1)\cdot 5 = 5(n-1)$$

∴ The solution is $x(n) = 5(n-1)$

(b) $x(n) = 3x(n-1)$ for $n>1$ with $x(1) = 4$.

1) write down the first two terms to identify the pattern

$$x(1) = 4$$
$$x(2) = 3x(1) = 3\cdot 4 = 12$$
$$x(3) = 3x(2) = 36$$
$$x(4) = 3x(3) = 108$$

2 Identify the general term

→ The first term $x(1) = 4$

→ The common ratio $r = 3$

The general formula for the n^{th} term of a GP is

$$x(n) = x(1)\cdot 1^{n-1}$$

Substituting the given values

$$x(n) = 4 \cdot 3^{n-1}$$

$\therefore$ The solution is $x(n) = 4 \cdot 3^{n-1}$

(c) $x(n) = x(n/2) + n$ for $n > 1$ with $x(1) = 1$ (solve for $n = 2^k$)

for $n = 2^k$, we can write recurrence in terms of k.

1) Substitute $n = 2^k$ in the recurrence

$$x(2^k) = x(2^{k-1}) + 2^k$$

(2) write down the first few terms to identify the pattern

$$x(1) = 1$$
$$x(2) = x(2^1) = x(1) + 2 = 1 + 2 = 3$$
$$x(4) = x(2^2) = x(2) + 4 = 3 + 4 = 7$$
$$x(8) = x(2^3) = x(4) + 8 = 7 + 8 = 15$$

(3) Identify the general term by finding the pattern

we observe that :-

$$x(2^k) = x(2^{k-1}) + 2^k$$

we sum the series :-

$$x(2^k) = 2^k + 2^{k-1} + 2^{k-2} + \cdots$$

since $x(1) = 1$:

$$x(2^k) \pm 2^k + 2^{k-1} + 2^{k-2} + \cdots -$$

$\therefore$ The geometric series with the term $a = 2$ and the last term 2^k except for the additional term.

The sum of a geometric series S with ratio $r = 2$ is given by $S = \frac{ar^n - 1}{r - 1}$

were, $a = 2$, $r = 2$ and $n = k$

$$S = \frac{2^k - 1}{2 - 1} = 2(2^k - 1) = 2^{k+1}$$

Adding the +1 term.

$$x(2^k) = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

$\therefore$ Solution is

$$x(2^k) = 2^{k+1} - 1$$

d) $x(n) = x(n/3) + 1$ for $n > 1$ with $x(1) = 1$ (solve for $n = 3^k$)

For $n = 3^k$, we can write the recurrence in terms of k.

1) Substitute $n = 3^k$ in recurrence

$$x(3^k) = x(3^{k-1}) + 1$$

2) write down the first few terms to identify the pattern

$$x(1) = 1$$
$$x(3) = x(3^1) = x(1) + 1 = 1 + 1 = 2$$
$$x(9) = (3^2) = x(3) + 1 = 2 + 1 = 3$$
$$x(27) = x(3^3) = x(9) + 1 = 3 + 1 = 4.$$

3) Identify the general term:-

we observe that $x(3^k) = x(3^{k-1}) + 1$

Summing up the series

$$x(3^k) = 1 + 1 + 1 + \dots + 1$$
$$x(3^k) = k + 1$$

$\therefore$ The Solution is $x(3^k) = k + 1$

(2) Evalute the following recurrences complexity.

(i) $T(n) = T(n/2)+1$, where $n=2k$ for all $k \geq 0$.

The recurrence relation can be solved using iteration method.

(i) Substitute $n=2^k$ in the recurrence.

(ii) Iterate the recurrence.

for $k=0$; $T(2^0) = T(1) = T(1)$

$k=1$: $T(2^1) = T(1)+1$

$k=2$: $T(2^2) = T(8) = T(n)+1 = (T(1)+1)+1 = T(1)+2$

$k=3$: $T(2^3) = T(8) = T(n)+1 = (T(1)+2)+1 = T(1)+3$

(3) Generalize the pattern

$T(2^k) = T(1)+k$

since $n=2^k$, $k = \log_2 n$

$T(n) = T(2^k) = T(1)+\log_2 n$

(4) Assume $T(1)$ is a constant c

$T(n) = c+\log_2 n.$

$\therefore$ The solution is $T(n) = O(\log n)$

(ii) $T(n) = T(n/3)+T(2n/3)+T(n)$ where c is constant and 'n' is input }

The recurrence can be solved using the masters theorem for divide-and-conquer recurrence of the form

$T(n) = aT(n/3)+f(n)$

where, $a=2$, $b=3$ and $f(n)=cn$.

let's determine the value of $\log_b a$.

$\log_b a = \log_3 2.$

using the properties of algorithms

$$\log_3 2 = \frac{\log 2}{\log 3}$$

Now, we compare $f(n) = cn$ with $n^{\log_3 2}$

$$f(n) = O(n)$$
$$n = n^1$$

Since $\log_3 2$ we are in third case of master's theorem

$$f(n) = O(n^c) \text{ with } c > \log_b a$$

$\therefore$ The solution is $T(n) = O(f(n)) = O(cn) = O(n)$

3. consider the following recurrence algorithm

min[A(0.... n-2)]
if n=1 return A[0]
Else temp=min(1A| 0... n-2])
if temp<=A(n-1) return temp
Else
return A[n-1]

(a) what does this algorithm compute?

The given algorithm, min[A[0,...n-1]] computes the min value in the array 'A' from index '0' for 'n-1'. it does this by recurrsivey finding the minimum value in the sub array A[0... n-2] and then comparing it with the last Element A[n-1] to determine the overall max value.

(b) Setup a recurrence relation for the algorithm basic operation count and Solve it.

TO determine the recurrence relation for the algorithms basic operation count, let's analyse

The Steps involved in the algorithm the basic operation are the comparision and function calls.

Recurrence relation setup

1) Base case when $n=1$, the algorithm performs a single operation to return $A[v]$.

(2) Recursive case. when $n>1$, the algorithm

makes a recursive call to min $(A[0...n-2])$

performs a comparison b/w temp and $A(n-1)$.

let $t(n)$ represent the no.of basic operation the algorithm performs for an array of size 'n'.

(1) Base case :-

$$T(1) = 1$$

(2) Recursive case :-

$$T(n) = T(n-1) + 1$$

Here $T(n-1)$ accounts for the operations performed by the recursive call to min $(A[0...n-2])$ and the +1 accounts for the comparison b/w temp and $A[n-1]$

To Solve this recurrences relation we can use iteration method:

$$\begin{aligned} T(n) &= T(n-1)+1 \\ &= (T(n-2)+1)+1 \\ &= ((T(n-3)+1)+1)+1 \\ &= 1+(n-1) \\ &= n. \end{aligned}$$

∴ The solution is $T(n) = n$.

(4) Analysize the order of growth.

(i) $f(n) = 2n^2+5$ and $g(n) = 7n$ use the $\Omega(g(n))$ notation.

To analyse the order of growth and use the Ω notation, we need to compare the give function $f(n)$ and $g(n)$.

given functions :-

$$f(n) = 2n^2+5$$
$$g(n) = 7n.$$

order of growth using $\Omega(g(n))$ notation.

The notation $\Omega(g(n))$ describes a lower bound on the growth rate that for sufficiently large n, $f(n)$, grows at least as $g(n)$

$$f(n) = c \cdot g(n)$$

lets analyse $f(n) = 2n^2+5$ with respect to $g(n) = 7n$.

1) Identify dominant terms:

⇒ The dominant terms in $f(n)$ is $2n^2$ since it grows faster then constant terms as 'n' increases.

⇒ The dominant term in $g(n)$ is $7n$.

2) Establish the inequality :-

⇒ we want to find constants 'c' and n_0 such that $2n^2+5 \geq c \cdot 7n$ for all $n \geq n_0$

3) simplify the inequality.

⇒ Ignore the lower order term 5 for larger.

$$2n^2 \geq 7cn.$$

⇒ Divide both sides by n.

$$2n \geq 7c.$$

$\Rightarrow$ Solve form :-

$$n \geq 7c/2$$

(4) choose constants :-

let $c=1$

$$n \geq \frac{7 \cdot 1}{2} = 3.5$$

$\therefore$ for $n \geq n$, the inequality holds:

$$2n^2+5 \geq 7n \text{ for all } n \geq n.$$

we have shown that there exist constant $c=1$ and $n_0 = n$ Such that for all $n \geq n_0$:

$$2n^2+5 \geq 7n.$$

Thus, we can conclude that :-

$$f(n) = 2n^2+5 = \Omega(7n)$$

in Ω notation, the dominant term $2n^2$ in $f(n)$ clearly grows faster than $7n$. Hence

$$f(n) = \Omega(n^2)$$

However, for the specific comparision asked $f(n) = \Omega(7n)$ is also correct.

Showing that $f(n)$ grows at least as fast as $7n$.