1) Big omega Notation: prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

Sol)

$$g(n) \geq c \cdot n^3$$

$$g(n) = n^3 + 2n^2 + 4n$$

For finding constants c and n_0

$$n^3 + 2n^2 + 4n \geq c \cdot n^3$$

Divide both sides with n^3

$$1 + \frac{2n^2}{n^3} + \frac{4n}{n^3} \geq c$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq c$$

Here $\frac{2}{n}$ and $\frac{4}{n^2}$ approaches 0

$$1 + \frac{2}{n} + \frac{4}{n^2} \approx 1$$

Example $c = \frac{1}{2}$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2} \qquad (1 \geq \frac{1}{2},\ n \geq 1)$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq 1 \qquad (n \geq 1,\ n_0 = 1)$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$$

Thus, $g(n) = n^3 + 2n^2 + 4n$ is indeeded $\Omega(n^3)$

(2) Big Theta Notation : Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not

$$c_1 n^2 \leq h(n) \leq c_2 n^2$$

In upper bound $h(n)$ is $O(n^2)$

In Lower bound $h(n)$ is $\Omega(n^2)$

Upper Bound $(O(n^2))$:

$h(n) = 4n^2 + 3n$

$h(n) \leq c_2 n^2$

$4n^2 + 3n \leq c_2 n^2 \Rightarrow 4n^2 + 3n \leq 5n^2$

let's $c_2 = 5$

Divide both sides by n^2

$4 + \frac{3}{n} \leq 5$

$h(n) = 4n^2 + 3n$ is $O(n^2)$ $(c_2 = 5, n_0 = 1)$

Lower bound :

$h(n) = 4n^2 + 3n$

$h(n) \geq c_1 n^2$

$4n^2 + 3n \geq c_1 n^2$

let's $c_1 = 4 \Rightarrow 4n^2 + 3n \geq 4n^2$

Divide both sides by n^2

$4 + \frac{3}{n} \geq 4$

$h(n) = 4n^2 + 3n$ $(c_1 = 4, n_0 = 1)$

$h(n) = 4n^2 + 3n$ is $\Theta(n^2)$

(3) Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = -n^2$ Show whether $f(n) = \Omega(g(n))$ is true or false and Justify your answer.

$$f(n) \geq c.g(n)$$

Substuting $f(n)$ and $g(n)$ into this inequality we get

$$n^3 - 2n^2 + n \geq c.(-n^2)$$

Find c and n_0 holds $n \geq n_0$

$$n^3 - 2n^2 + n \geq -cn^2$$

$$n^3 - 2n^2 + n + cn^2 \geq 0$$

$$n^3 + (c-2)n^2 + n \geq 0$$

$$n^3 + (c-2)n^2 + n \geq 0. \quad (n^3 \geq 0)$$

$$n^3 + (1-2)n^2 + n = n^3 - n^2 + n \geq 0 \quad (c=2)$$

$$f(n) = n^3 - 2n^2 + n \text{ is } \Omega(g(n)) = \Omega(-n^2)$$

Therfore the Statement $f(n) = \Omega(g(n))$ is True.

(4) Determine whether $h(n) = n\log n + n$ is $\Theta(n\log n)$ prove a rigorous proof for your conclusion

$$c_1 n\log n \leq h(n) \leq c_2 n\log n$$

upper Bound:

$$h(n) \leq c_2 n\log n$$

$$h(n) = n\log n + n$$

$$n\log n + n \leq c_2 n\log n$$

Divide both sides by $n\log n$

$$1 + \frac{n}{n\log n} \le 2$$

$$1 + \frac{1}{\log n} \le C_2 \quad \text{(simplify)}$$

$$(C_2 = 2)$$

$$1 + \frac{1}{\log n} \le 2$$

Then $h(n)$ is $O(n \log n)$

Lower bound:-

$h(n) \ge C_1 n \log n$

$h(n) = n\log n + n$

$n\log n + n \ge C_1 n \log n$

Divide both sides by $n\log n$

$$1 + \frac{n}{n\log n} \ge C_1$$

$$1 + \frac{1}{\log n} \ge C_1 \quad \text{(simplify)}$$

$$(C_1 = 1)$$

$$1 + \frac{1}{\log n} \ge 1$$

$$\frac{1}{\log n} \ge 0$$

$h(n)$ is $\Omega(n\log n)$ $\quad (C_1 = 1, n_0 = 1)$

$h(n) = n\log n + n$ is $\Theta(n\log n)$

5. Solve the following recurrence relations and find the order of growth of solutions

$$T(n) = 4T(n/2) + n^2, \quad T(1) = 1$$

$T(n) = 4T(n/2) + n^2, \quad T(1) = 1$

$T(n) = aT(n/b) + f(n)$

$a = 4, \; b = 2 \quad f(n) = n^2$

Applying master theorem

$T(n) = aT(n/b) + f(n)$

$f(n) = O(n^{\log_b a - E}) \quad \left(\begin{array}{l} E > 0 \\ T(n) = \Theta(n^{\log_b a}) \end{array} \right.$

$f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

$f(n) = \Omega(n^{\log_b a + E})$, then $T(n) = \Theta(n^{\log_b a} \log n)$ (or) $F(n)$

Calculating $\log_b a$:

$\log_b a = \log_2 4 = 2$

$f(n) = n^2 = \Theta(n^2)$ (comparing $f(n)$ with $n^{\log_b a}$)

$f(n) = \Theta(n^2) = \Theta(n^{\log_b a})$, (case 2)

$T(n) = 4T(n/2) + n^2$

$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^2 \log n)$

<u>Order of growth.</u>

$T(n) = 4T(n/2) + n^2$ with $T(1) = 1$

is $\Theta(n^2 \log n)$