Implementation of HLLD Scheme with Contrained Transport like Algorithm based on Vector Potential Formulation

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Outline

Preface

SINGLE-BLOCK FIXED-GRID

- Solves $\frac{\partial U}{\partial t} + \sum_1^3 \frac{\partial F_i}{\partial x_i} = S$
- Domain decomposition can be both automatic or user-defined
- Currently can solve both HD and MHD
- Will be modified for orthogonal systems

MULTIBLOCK-AMR

- Solves $\frac{\partial U}{\partial t} + \sum_1^3 \frac{\partial F_i}{\partial x_i} = S$
- Domain decomposition is automatic only(hilbert SFC)
- Can only solve HD now
- Intended to be Cartesian only with embebbed boundary functionality

Common Modules

- Riemann Solvers and Interpolation functions
- Physics Specifications (TBD)

Current Presentation

- We implemented HLLD solvers for MHD equation
- We are using Vector potential formulation
- The main goal is to prepare and test a MHD solver which can be shared by both fixed and AMR solvers

Time Integration for AMR (single step)

Calculate the hyperbolic flux for all blocks

↓

Synchronize the hyperbolic flux over all blocks at different level

↓

Iterate over all meshblocks →
- get $Hyp(u)$
- get source terms $S(u)$
- $u + dt(Hyp(u) + S(u))$

↓

Update the primitive variables

↓

Update the boundary values

↓

Update embedded shape

Outline

MHD-SYSTEM : $\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = 0$

ISOTHERMAL

$$\vec{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p_{total} - {B_x}^2 \\ \rho vu - B_x B_y \\ \rho wu - B_x B_z \\ 0 \\ B_y u - B_x v \\ B_z u - B_x w \end{bmatrix}$$

Where $p_{total} = p + \frac{|B|^2}{2}$ with $p = {a_s}^2 \rho$

ADIABATIC/POLYTROPIC

$$\vec{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p_{total} - {B_x}^2 \\ \rho vu - B_x B_y \\ \rho wu - B_x B_z \\ (e + p_{total})u - B_x(\boldsymbol{v} \cdot \boldsymbol{B}) \\ 0 \\ B_y u - B_x v \\ B_z u - B_x w \end{bmatrix}$$

Where $p_{total} = p + \frac{|B|^2}{2}$ with
$p = (\gamma - 1)\left(e - \frac{1}{2}\rho|\boldsymbol{v}|^2 - \frac{1}{2}|\boldsymbol{B}|^2\right)$

Outline

Time Integration (single step)

Calculate the hyperbolic flux

↓

Get the hyperbolic increment terms $Hyp(u)$

↓

Get source terms $S(u)$

↓

$$u_1 = u + dt(Hyp(u) + S(u))$$

Calculation of Hyperbolic Flux

- Rearrange Primitive Variables :

$$q_x = [\rho, v_x, v_y, v_z, P, B_x, B_y, B_z]^T$$
$$q_y = [\rho, v_y, v_z, v_x, P, B_y, B_z, B_x]^T$$
$$q_z = [\rho, v_z, v_x, v_y, P, B_z, B_x, B_y]^T$$

$$For \begin{cases} \text{X-direction} \rightarrow \text{xyz} \\ \text{Y-direction} \rightarrow \text{yzx} \\ \text{Z-direction} \rightarrow \text{zxy} \end{cases}$$

- Perform 1D interpolation to get interface values for primitive variables

$$q_{x\,i} \rightarrow q_{x\,i\pm\frac{1}{2}}$$

- Call the 1D Riemann solver to get the hyperbolic(riemann) flux $f_{x\,i\pm\frac{1}{2}}$

Update step for Vector Potential

- First we calculate the cell centered Electric field from up-winded value of the flux :

$$[E_z]_{i,j,k} = \frac{1}{4}\left(-\left[f_{x\,i-\frac{1}{2}}\right]^* - \left[f_{x\,i+\frac{1}{2}}\right]^* + \left[f_{y\,j-\frac{1}{2}}\right]^* + \left[f_{y\,j+\frac{1}{2}}\right]^*\right)$$

- We then add it to the cell centered A_z vector potential $du(iaz,:,:,:)$, note that $A_{z\,i,j,k} = -[E_z]_{i,j,k}$
- During time integration, we update the cell centered value of vector potential A_z as

$$u(iaz,:,:,:) = u(iaz,:,:,:) + dt * du(iaz,:,:,:)$$

Primitive values of Magnetic Field

- We first perform interpolation to get the vector potentials in the edge centers from cell-centered values:

$$[A_x]_{i,j+\frac{1}{2},k+\frac{1}{2}} \quad [A_y]_{i+\frac{1}{2},j,k+\frac{1}{2}} \quad [A_z]_{i+\frac{1}{2},j+\frac{1}{2},k}$$

- We calculate B_x , B_y and B_z and store in the global array qp :

$$[B_x]_{i+\frac{1}{2},j,k} = \frac{[A_z]_{i+\frac{1}{2},j+\frac{1}{2},k} - [A_z]_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} - \frac{[A_y]_{i+\frac{1}{2},j,k+\frac{1}{2}} - [A_z]_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z}$$

$$[B_y]_{i,j+\frac{1}{2},k} = \frac{[A_x]_{i,j+\frac{1}{2},k+\frac{1}{2}} - [A_x]_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z} - \frac{[A_z]_{i+\frac{1}{2},j+\frac{1}{2},k} - [A_z]_{i-\frac{1}{2},j+\frac{1}{2},k}}{\Delta x}$$

$$[B_z]_{i,j,k+\frac{1}{2}} = \frac{[A_y]_{i+\frac{1}{2},j,k+\frac{1}{2}} - [A_y]_{i-\frac{1}{2},j,k+\frac{1}{2}}}{\Delta x} - \frac{[A_x]_{i,j+\frac{1}{2},k+\frac{1}{2}} - [A_x]_{i,j-\frac{1}{2},k+\frac{1}{2}}}{\Delta y}$$

Outline

The Riemann Problem

Basically IVP for a conservation law. The initial state is defined as

$$\mathbf{U}_0 = \begin{cases} \mathbf{U}_l & \text{if} \quad x < x_d, \\ \mathbf{U}_r & \text{if} \quad x > x_d, \end{cases}$$

where $\mathbf{U}_l$ is the initial left state, $\mathbf{U}_r$ is the initial right state, and x_d is the location of the discontinuity.

Linear System

- Consider the system of n linear equations

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \quad A \text{ is an } n \times n \text{ constant coefficient matrix.}$$

- The system is hyperbolic $\rightarrow$ If A has n real eigenvalues, $\lambda_1, \ldots, \lambda_n$, and a corresponding set of n linearly independent right eigenvectors, $[\mathbf{r}^1, \ldots, \mathbf{r}^n]$
- It is called **strictly hyperbolic** system if the eigenvalues are distinct.
- So what if it's a hyperbolic system?

Hyperbolic System

- Let's first define the characteristic variables :

$$W = l^i U \qquad \text{, where } l^i \text{ is the matrix of left eigenvectors}$$

- The hyperbolicity of the system allows us to rewrite it in the following form :

$$\frac{\partial W}{\partial t} + \mathbf{\Lambda}\frac{\partial W}{\partial x} = 0, \qquad \text{, where } \mathbf{\Lambda} \text{ is a diagonal matrix}$$

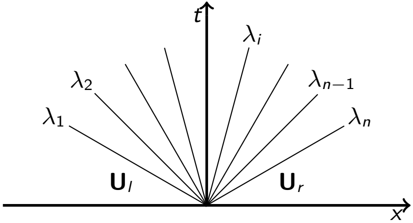
- So, the system can be transformed to n scalar equations of the form

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0,$$

where the eigenvalues (λ_i) are the characteristic speeds.

Strictly Hyperbolic System

- For strictly hyperbolic systems , we can rearrange the eigenvalues such that $\lambda_1 < \ldots < \lambda_i < \ldots < \lambda_n$. The solutions in x-t plane forms gives us :



For $\lambda_m < x/t < \lambda_{m+1}$

$$U = U_l + \sum_{i=1}^{m} \alpha_i r^i,$$

or

$$U = U_r - \sum_{i=m+1}^{n} \alpha_i r^i$$

$$\alpha_i = l_i \cdot (U_l - U_r)$$

Extension to Nonlinear System

- We extend the concept to nonlinear hyperbolic systems of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0, \begin{cases} \mathbf{U} & = \text{vector of conserved variables} \\ \mathbf{F}(\mathbf{U}) & = \text{flux vector} \end{cases}$$

- The extension can be done by considering $A \equiv \mathbf{J}(\mathbf{U}) = \partial \mathbf{F}/\partial \mathbf{U}$
- Here, for each of the n eigenvalues there is an associated wave with speed S_i
- These speeds are functions of the conservative variables (unlike linear case) and thus the structure of the wave is not always a jump discontinuity

Waves in Ideal Magnetohydrodynamics

- The Jacobian matrix, $\mathbf{J}(\mathbf{U}) = \partial \mathbf{F} / \partial \mathbf{U}$, has real but not necessarily distinct eigenvalues in ideal MHD. The ideal MHD system is called non strictly hyperbolic because it can have degenerate eigenvalues.
- Each eigenvalue is associated with a wave that travels at the characteristic speed

$$\begin{aligned}
v_n \quad &: \text{contact or tangential discontinuity (entropy)},\\
v_n \pm c_s &: \text{slow rarefaction or shock},\\
v_n \pm c_a &: \text{rotational discontinuity (Alfvén), and}\\
v_n \pm c_f &: \text{fast rarefaction or shock},
\end{aligned}$$

where c_s, c_a, c_f are the slow, Alfvén, and fast wave propagation speeds, respectively

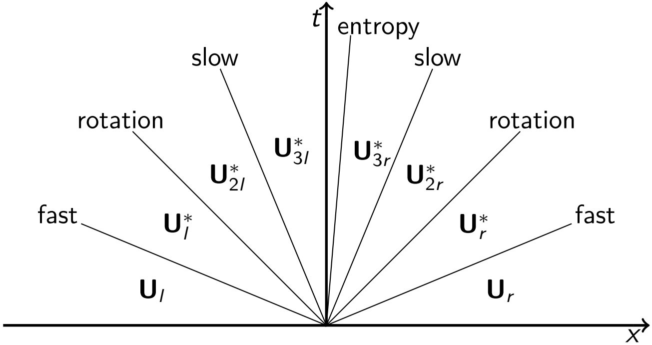
Waves in Ideal Magnetohydrodynamics : Propagation Speeds

The propagation speeds are

$$
\begin{aligned}
c_a^2 &= \frac{B_n^2}{\rho} \\
c_f^2 &= \frac{1}{2}\left[a^2 + c_a^2 + c_t^2 + \sqrt{\left(a^2 + c_a^2 + c_t^2\right)^2 - 4a^2c_a^2}\right] \\
c_s^2 &= \frac{1}{2}\left[a^2 + c_a^2 + c_t^2 + \sqrt{\left(a^2 + c_a^2 + c_t^2\right)^2 - 4a^2c_a^2}\right]
\end{aligned}
$$

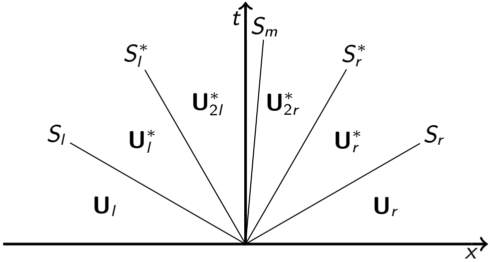
$$
where \begin{cases}
c_t^2 & = B_t^2/\rho \\
a^2 & = \gamma p_g/\rho \\
c_{a,t} & = \text{the Alfvén speed normal /tangential to the wave front} \\
a & = \text{the speed of sound}
\end{cases}
$$

7-Waves states



Originally proposed by Dai and Woodward (JCP,1994) and improved by Ryu and Jones(ApJ,1995)

HLLD Scheme



- It's basically an extension of HLLC
- Two nonlinear waves, i.e., shocks and rarefactions, separate the left and right initial states from the intermediate states in the Riemann fan, i.e., $\mathbf{U}_l^*$, $\mathbf{U}_{2l}^*$, $\mathbf{U}_{2r}^*$, and $\mathbf{U}_r^*$.The four intermediate states are separated by two rotational discontinuities with velocities, S_l^*, S_r^*, and one contact discontinuity with velocity, S_m.

HLLD Scheme

$$\mathbf{F}(x,t) = \begin{cases} \mathbf{F}_l & \text{if} \quad 0 < S_l \\ \mathbf{F}_l^* = \mathbf{F}_l + S_l(\mathbf{U}_l^* - \mathbf{U}_l) & \text{if} \quad S_l \leq 0 \leq S_l^* \\ \mathbf{F}_{2l}^* = \mathbf{F}_l^* + S_l^*(\mathbf{U}_{2l}^* - \mathbf{U}_l^*) & \text{if} \quad S_l^* \leq 0 \leq S_m \\ \mathbf{F}_{2r}^* = \mathbf{F}_r^* + S_r^*(\mathbf{U}_{2r}^* - \mathbf{U}_r^*) & \text{if} \quad S_m \leq 0 \leq S_r^*, \\ \mathbf{F}_r^* = \mathbf{F}_r + S_r(\mathbf{U}_r^* - \mathbf{U}_r) & \text{if} \quad S_r^* \leq 0 \leq S_r \\ \mathbf{F}_r & \text{if} \quad S_r < 0 \end{cases}$$

Some Notes Regarding the Implementation

- Calculate for each variable :

$$W_L = S_L * U_L - FLUX_L$$
$$W_R = S_R * U_R - FLUX_R$$
$$S_M = \frac{[W_R]_{xmom} - [W_L]_{xmom}}{[W_R]_{density} - [W_L]_{density}}$$
$$S_L{}^* = S_M - \sqrt{\frac{B_x{}^2}{\rho_L}}$$
$$S_R{}^* = S_M + \sqrt{\frac{B_x{}^2}{\rho_R}}$$

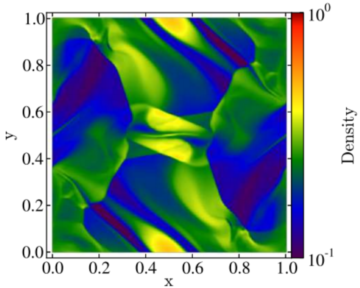
if $S_R{}^* \to S_R$ or $S_L{}^* \to S_L$ revert to HLLC

Outline

Orszag-Tang case

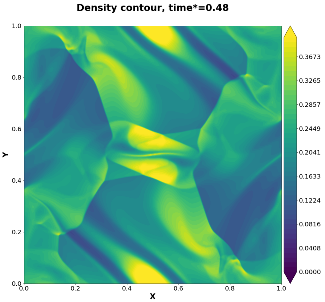
Case : Bryan et al, APJ (2014), Toth (JCP, 2000) etc

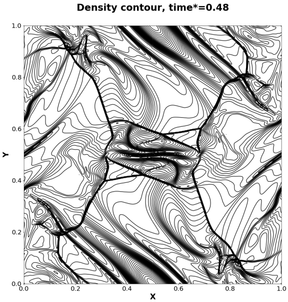
IC : $[0 \leq x \leq 1; 0 \leq y \leq 1.]$; $\rho = 25/(36\pi)$, $p = 5/(12\pi)$ $\gamma = 5/3$;
$u = -sin(2\pi y); v = sin(2\pi x)$;
$A_z = B_0(cos(4\pi x)/(4\pi) + cos(2\pi y)/(2\pi)), with B_0 = 1/(4\pi)^{1/2}$



Orszag-Tang case

Results : HLLD, RK2,WENO3, cfl=0.4, nx=ny=512





Ongoing works

- The python script for post-process is still under development.
- Multigrid for fixed grid is under development

Conclusion and Future Work

- MHD system is working but we still need more tests
- Need to extend to AMR
- Need to run more test cases

Thanks!