Lecture 2: Bellman Equation

In this lecture:

- A core concept: state value
- A fundamental tool: the Bellman equation

Outline

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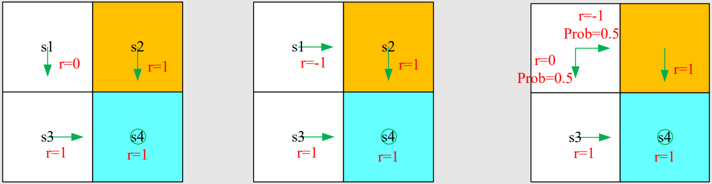
Motivating example 1: Why return is important?

- What is return? The (discounted) sum of the rewards obtained along a trajectory.
- Why return is important? See the following examples.



- Question: From the starting point s_1, which policy is the "best"? Which is the "worst"?
  Intuition: the first is the best and the second is the worst, because of the forbidden area.
- Question: can we use mathematics to describe such an intuition?
  Answer: Return could be used to evaluate policies. See the following.

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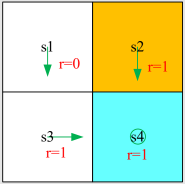

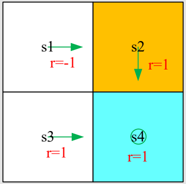

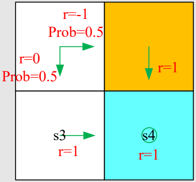

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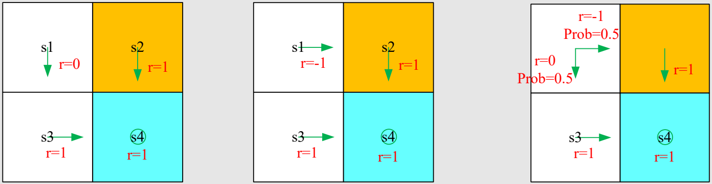

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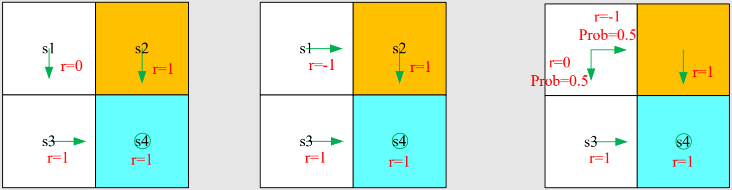

Based on policy 1 (left figure), starting from s_1, the discounted return is

$$\begin{aligned}\text{return}_1 &= 0 + \gamma 1 + \gamma^2 1 + \dots,\\ &= \gamma(1 + \gamma + \gamma^2 + \dots),\\ &= \frac{\gamma}{1-\gamma}.\end{aligned}$$

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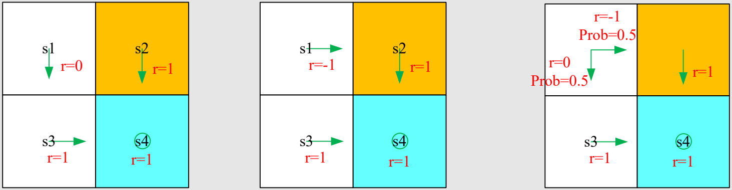

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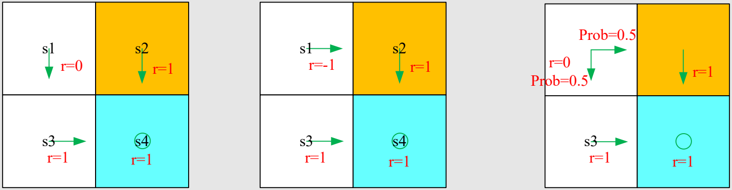

Exercise: Based on policy 2 (middle figure), starting from s_1, what is the discounted return?

Answer:

$$\begin{aligned}
\text{return}_2 &= -1 + \gamma 1 + \gamma^2 1 + \dots, \\
&= -1 + \gamma(1 + \gamma + \gamma^2 + \dots), \\
&= -1 + \frac{\gamma}{1-\gamma}.
\end{aligned}$$

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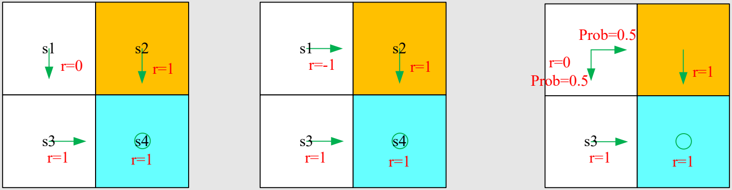

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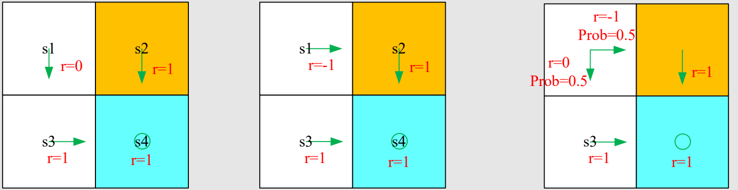

Policy 3 is stochastic!

Exercise: Based on policy 3 (right figure), starting from s_1, the discounted return is

Answer:

$$\begin{aligned}\text{return}_3 &= 0.5\left(-1+\frac{\gamma}{1-\gamma}\right)+0.5\left(\frac{\gamma}{1-\gamma}\right),\\ &= -0.5+\frac{\gamma}{1-\gamma}.\end{aligned}$$

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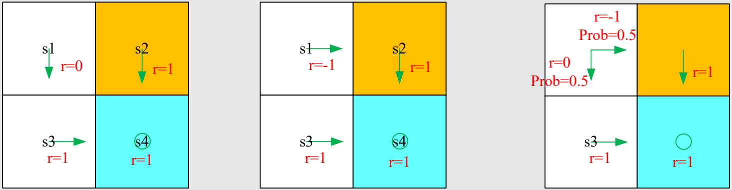

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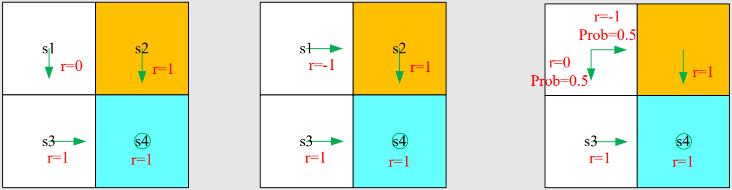


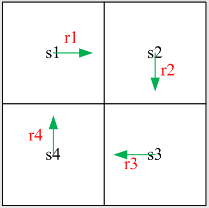
In summary, starting from s_1,

$$\text{return}_1 > \text{return}_3 > \text{return}_2$$

The above inequality suggests that the first policy is the best and the second policy is the worst, which is exactly the same as our intuition. Calculating return is important to evaluate a policy.

Motivating example 2: How to calculate return?

While return is important, how to calculate it?



Method 1: by definition

Let v_i denote the return obtained starting from s_i $(i = 1, 2, 3, 4)$

$$v_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \dots$$

$$v_2 = r_2 + \gamma r_3 + \gamma^2 r_4 + \dots$$

$$v_3 = r_3 + \gamma r_4 + \gamma^2 r_1 + \dots$$

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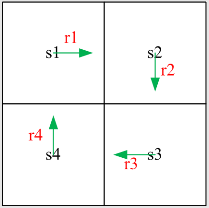

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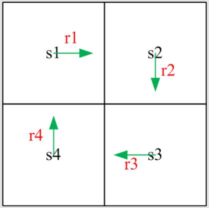

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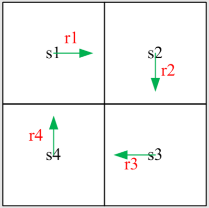

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$$v_1 = r_1 + \gamma(r_2 + \gamma r_3 + \dots) = r_1 + \gamma v_2$$

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- The returns rely on each other. *Bootstrapping!*

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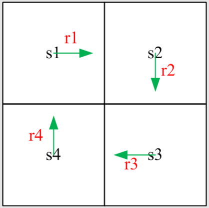

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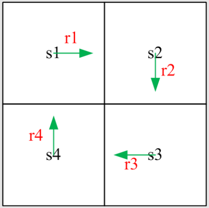

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- The returns rely on each other. *Bootstrapping!*

Motivating example 2: How to calculate return?

How to solve these equations? Write in the following matrix-vector form:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} \gamma v_2 \\ \gamma v_3 \\ \gamma v_4 \\ \gamma v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{\mathbf{r}} + \gamma \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}$$

which can be rewritten as

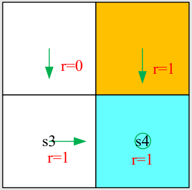
$$\mathbf{v} = \mathbf{r} + \gamma \mathbf{P} \mathbf{v}$$

This is the Bellman equation (for this specific deterministic problem)!!

- Though simple, it demonstrates the core idea: the value of one state relies on the values of other states.
- A matrix-vector form is more clear to see how to solve the state values.

Motivating example 2: How to calculate return?

Exercise: Consider the policy shown in the figure. Please write out the relation among the returns (that is to write out the Bellman equation)



Answer:

$$v_1 = 0 + \gamma v_3$$

$$v_2 = 1 + \gamma v_4$$

$$v_3 = 1 + \gamma v_4$$

$$v_4 = 1 + \gamma v_4$$

Exercise: How to solve them? We can first calculate v_4, and then v_3, v_2, v_1.

Outline

Some notations

Consider the following single-step process:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1}$$

- $t, t+1$: discrete time instances
- S_t: state at time t
- A_t: the action taken at state S_t
- R_{t+1}: the reward obtained after taking A_t
- S_{t+1}: the state transited to after taking A_t

Note that S_t, A_t, R_{t+1} are all *random variables*.

This step is governed by the following probability distributions:

- $S_t \rightarrow A_t$ is governed by $\pi(A_t = a|S_t = s)$
- $S_t, A_t \rightarrow R_{t+1}$ is governed by $p(R_{t+1} = r|S_t = s, A_t = a)$
- $S_t, A_t \rightarrow S_{t+1}$ is governed by $p(S_{t+1} = s'|S_t = s, A_t = a)$

At this moment, we assume we know the model (i.e., the probability distributions)!

Some notations

Consider the following multi-step trajectory:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1} \xrightarrow{A_{t+1}} R_{t+2}, S_{t+2} \xrightarrow{A_{t+2}} R_{t+3}, \ldots$$

The discounted return is

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$$

- $\gamma \in [0, 1)$ is a discount rate.
- G_t is also a random variable since $R_{t+1}, R_{t+2}, \ldots$ are random variables.

State value

The expectation (or called expected value or mean) of G_t is defined as the *state-value function* or simply *state value*:

$$v_\pi(s) = \mathbb{E}[G_t|S_t = s]$$

Remarks:

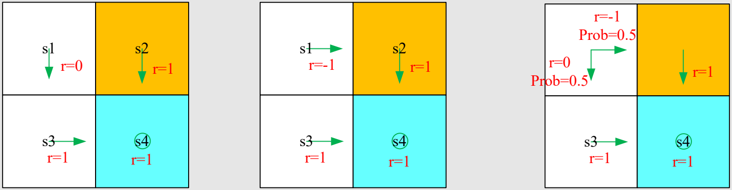
- It is a function of s. It is a conditional expectation with the condition that the state starts from s.
- It is based on the policy π. For a different policy, the state value may be different.
- It represents the "value" of a state. If the state value is greater, then the policy is better because greater cumulative rewards can be obtained.

Q: What is the relationship between return and state value?

A: The state value is the mean of all possible returns that can be obtained starting from a state. If everything - $\pi(a|s)$, $p(r|s,a)$, $p(s'|s,a)$ - is deterministic, then state value is the same as return.

State value

Example:



Recall the returns obtained from s_1 for the three examples:

$$v_{\pi_1}(s_1) = 0 + \gamma 1 + \gamma^2 1 + \cdots = \gamma(1 + \gamma + \gamma^2 + \dots) = \frac{\gamma}{1-\gamma}$$

$$v_{\pi_2}(s_1) = -1 + \gamma 1 + \gamma^2 1 + \cdots = -1 + \gamma(1 + \gamma + \gamma^2 + \dots) = -1 + \frac{\gamma}{1-\gamma}$$

$$v_{\pi_3}(s_1) = 0.5\left(-1 + \frac{\gamma}{1-\gamma}\right) + 0.5\left(\frac{\gamma}{1-\gamma}\right) = -0.5 + \frac{\gamma}{1-\gamma}$$

Outline

Deriving the Bellman equation

Consider a random trajectory:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1} \xrightarrow{A_{t+1}} R_{t+2}, S_{t+2} \xrightarrow{A_{t+2}} R_{t+3}, \ldots$$

The return G_t can be written as

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots, \\ &= R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \ldots), \\ &= R_{t+1} + \gamma G_{t+1}, \end{aligned}$$

Then, it follows from the definition of the state value that

$$\begin{aligned} v_\pi(s) &= \mathbb{E}[G_t | S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s] \\ &= \mathbb{E}[R_{t+1} | S_t = s] + \gamma \mathbb{E}[G_{t+1} | S_t = s] \end{aligned}$$

Next, calculate the two terms, respectively.

Deriving the Bellman equation

First, calculate the first term $\mathbb{E}[R_{t+1}|S_t = s]$:

$$\begin{aligned}\mathbb{E}[R_{t+1}|S_t = s] &= \sum_a \pi(a|s)\mathbb{E}[R_{t+1}|S_t = s, A_t = a] \\ &= \sum_a \pi(a|s) \sum_r p(r|s,a)r\end{aligned}$$

Note that

- This is the mean of *immediate rewards*

Deriving the Bellman equation

Second, calculate the second term $\mathbb{E}[G_{t+1}|S_t = s]$:

$$\begin{aligned}\mathbb{E}[G_{t+1}|S_t = s] &= \sum_{s'} \mathbb{E}[G_{t+1}|S_t = s, S_{t+1} = s']p(s'|s) \\ &= \sum_{s'} \mathbb{E}[G_{t+1}|S_{t+1} = s']p(s'|s) \\ &= \sum_{s'} v_\pi(s')p(s'|s) \\ &= \sum_{s'} v_\pi(s') \sum_a p(s'|s,a)\pi(a|s)\end{aligned}$$

Note that

- This is the mean of *future rewards*
- $\mathbb{E}[G_{t+1}|S_t = s, S_{t+1} = s'] = \mathbb{E}[G_{t+1}|S_{t+1} = s']$ due to the memoryless Markov property.

Deriving the Bellman equation

Therefore, we have

$$
\begin{aligned}
v_\pi(s) &= \mathbb{E}[R_{t+1}|S_t = s] + \gamma \mathbb{E}[G_{t+1}|S_t = s], \\
&= \underbrace{\sum_a \pi(a|s) \sum_r p(r|s,a) r}_{\text{mean of immediate rewards}} + \underbrace{\gamma \sum_a \pi(a|s) \sum_{s'} p(s'|s,a) v_\pi(s')}_{\text{mean of future rewards}}, \\
&= \sum_a \pi(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right], \quad \forall s \in \mathcal{S}.
\end{aligned}
$$

Highlights:

- The above equation is called the *Bellman equation*, which characterizes the relationship among the state-value functions of different states.
- It consists of two terms: the immediate reward term and the future reward term.
- A set of equations: every state has an equation like this!!!

Deriving the Bellman equation

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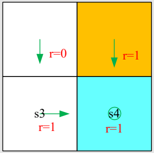
\end{aligned}
$$

Highlights: symbols in this equation

- $v_\pi(s)$ and $v_\pi(s')$ are state values to be calculated. Bootstrapping!
- $\pi(a|s)$ is a given policy. Solving the equation is called policy evaluation.
- $p(r|s,a)$ and $p(s'|s,a)$ represent the dynamic model. What if the model is known or unknown?

An illustrative example



Write out the Bellman equation according to the general expression:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

This example is simple because the policy is deterministic.

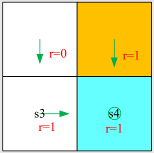
First, consider the state value of s_1:

- $\pi(a = a_3|s_1) = 1$ and $\pi(a \neq a_3|s_1) = 0$.
- $p(s' = s_3|s_1, a_3) = 1$ and $p(s' \neq s_3|s_1, a_3) = 0$.
- $p(r = 0|s_1, a_3) = 1$ and $p(r \neq 0|s_1, a_3) = 0$.

Substituting them into the Bellman equation gives

$$v_\pi(s_1) = 0 + \gamma v_\pi(s_3)$$

An illustrative example



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Similarly, it can be obtained that

$$\begin{aligned} v_\pi(s_1) &= 0 + \gamma v_\pi(s_3), \\ v_\pi(s_2) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_3) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_4) &= 1 + \gamma v_\pi(s_4). \end{aligned}$$

An illustrative example

How to solve them?

$$\begin{aligned}
v_\pi(s_1) &= 0 + \gamma v_\pi(s_3),\\
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\end{aligned}$$

Solve the above equations one by one from the last to the first:

$$\begin{aligned}
v_\pi(s_4) &= \frac{1}{1-\gamma},\\
v_\pi(s_3) &= \frac{1}{1-\gamma},\\
v_\pi(s_2) &= \frac{1}{1-\gamma},\\
v_\pi(s_1) &= \frac{\gamma}{1-\gamma}.
\end{aligned}$$

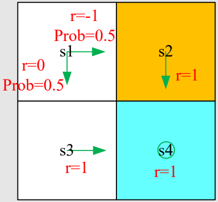
An illustrative example

If $\gamma = 0.9$, then

$$\begin{aligned} v_\pi(s_4) &= \frac{1}{1-0.9} = 10, \\ v_\pi(s_3) &= \frac{1}{1-0.9} = 10, \\ v_\pi(s_2) &= \frac{1}{1-0.9} = 10, \\ v_\pi(s_1) &= \frac{0.9}{1-0.9} = 9. \end{aligned}$$

What to do after we have calculated state values? Be patient (calculating action value and improve policy)

Exercise



Exercise:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

- write out the Bellman equations for each state.
- solve the state values from the Bellman equations.
- compare with the policy in the last example.

Exercise

Answer:

$$\begin{aligned}
v_\pi(s_1) &= 0.5[0 + \gamma v_\pi(s_3)] + 0.5[-1 + \gamma v_\pi(s_2)],\\
v_\pi(s_2) &= 1 + \gamma v_\pi(s_4),\\
v_\pi(s_3) &= 1 + \gamma v_\pi(s_4),\\
v_\pi(s_4) &= 1 + \gamma v_\pi(s_4).
\end{aligned}$$

Solve the above equations one by one from the last to the first.

$$\begin{aligned}
&v_\pi(s_4) = \frac{1}{1-\gamma}, \quad v_\pi(s_3) = \frac{1}{1-\gamma}, \quad v_\pi(s_2) = \frac{1}{1-\gamma},\\
&v_\pi(s_1) = 0.5[0 + \gamma v_\pi(s_3)] + 0.5[-1 + \gamma v_\pi(s_2)],\\
&\qquad\quad = -0.5 + \frac{\gamma}{1-\gamma}.
\end{aligned}$$

Substituting $\gamma = 0.9$ yields

$$v_\pi(s_4) = 10, \quad v_\pi(s_3) = 10, \quad v_\pi(s_2) = 10, \quad v_\pi(s_1) = -0.5 + 9 = 8.5.$$

Compare with the previous policy. This one is worse.

Outline

Matrix-vector form of the Bellman equation

Why consider the matrix-vector form?

- How to solve the Bellman equation?
  One unknown relies on another unknown.

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

- The above *elementwise form* is valid for every state $s \in \mathcal{S}$. That means there are $|\mathcal{S}|$ equations like this!
- If we put all the equations together, we have a set of linear equations, which can be concisely written in a *matrix-vector form*.
- The matrix-vector form is very elegant and important.

Matrix-vector form of the Bellman equation

Recall that:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

Rewrite the Bellman equation as

$$v_\pi(s) = r_\pi(s) + \gamma \sum_{s'} p_\pi(s'|s) v_\pi(s') \tag{1}$$

where

$$r_\pi(s) \triangleq \sum_a \pi(a|s) \sum_r p(r|s,a) r, \qquad p_\pi(s'|s) \triangleq \sum_a \pi(a|s) p(s'|s,a)$$

Matrix-vector form of the Bellman equation

Suppose the states could be indexed as s_i $(i = 1, \ldots, n)$.
For state s_i, the Bellman equation is

$$v_\pi(s_i) = r_\pi(s_i) + \gamma \sum_{s_j} p_\pi(s_j|s_i) v_\pi(s_j)$$

Put all these equations for all the states together and rewrite to a matrix-vector form

$$v_\pi = r_\pi + \gamma P_\pi v_\pi$$

where

- $v_\pi = [v_\pi(s_1), \ldots, v_\pi(s_n)]^T \in \mathbb{R}^n$
- $r_\pi = [r_\pi(s_1), \ldots, r_\pi(s_n)]^T \in \mathbb{R}^n$
- $P_\pi \in \mathbb{R}^{n \times n}$, where $[P_\pi]_{ij} = p_\pi(s_j|s_i)$, is the *state transition matrix*

Illustrative examples

If there are four states, $v_\pi = r_\pi + \gamma P_\pi v_\pi$ can be written out as

$$\underbrace{\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}}_{v_\pi} = \underbrace{\begin{bmatrix} r_\pi(s_1) \\ r_\pi(s_2) \\ r_\pi(s_3) \\ r_\pi(s_4) \end{bmatrix}}_{r_\pi} + \gamma \underbrace{\begin{bmatrix} p_\pi(s_1|s_1) & p_\pi(s_2|s_1) & p_\pi(s_3|s_1) & p_\pi(s_4|s_1) \\ p_\pi(s_1|s_2) & p_\pi(s_2|s_2) & p_\pi(s_3|s_2) & p_\pi(s_4|s_2) \\ p_\pi(s_1|s_3) & p_\pi(s_2|s_3) & p_\pi(s_3|s_3) & p_\pi(s_4|s_3) \\ p_\pi(s_1|s_4) & p_\pi(s_2|s_4) & p_\pi(s_3|s_4) & p_\pi(s_4|s_4) \end{bmatrix}}_{P_\pi} \underbrace{\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}}_{v_\pi}.$$

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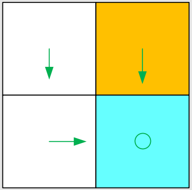

For this specific example:

$$\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}$$

Illustrative examples

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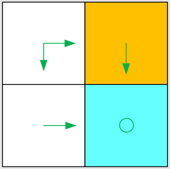


For this specific example:

$$\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix} = \begin{bmatrix} 0.5(0) + 0.5(-1) \\ 1 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}.$$

Outline

Solve state values

Why to solve state values?

- Given a policy, finding out the corresponding state values is called *policy evaluation*! It is a fundamental problem in RL. It is the foundation to find better policies.
- It is important to understand how to solve the Bellman equation.

Solve state values

The Bellman equation in matrix-vector form is

$$v_\pi = r_\pi + \gamma P_\pi v_\pi$$

- The *closed-form solution* is:

  $$v_\pi = (I - \gamma P_\pi)^{-1} r_\pi$$

  In practice, we still need to use numerical tools to calculate the matrix inverse.

  Can we avoid the matrix inverse operation? Yes, by iterative algorithms.

- *An iterative solution is:*

  $$v_{k+1} = r_\pi + \gamma P_\pi v_k$$

  This algorithm leads to a sequence $\{v_0, v_1, v_2, \dots\}$. We can show that

  $$v_k \to v_\pi = (I - \gamma P_\pi)^{-1} r_\pi, \quad k \to \infty$$

Solve state values (optional)

Proof.

Define the error as $\delta_k = v_k - v_\pi$. We only need to show $\delta_k \to 0$. Substituting $v_{k+1} = \delta_{k+1} + v_\pi$ and $v_k = \delta_k + v_\pi$ into $v_{k+1} = r_\pi + \gamma P_\pi v_k$ gives

$$\delta_{k+1} + v_\pi = r_\pi + \gamma P_\pi(\delta_k + v_\pi),$$

which can be rewritten as

$$\delta_{k+1} = -v_\pi + r_\pi + \gamma P_\pi \delta_k + \gamma P_\pi v_\pi = \gamma P_\pi \delta_k.$$

As a result,

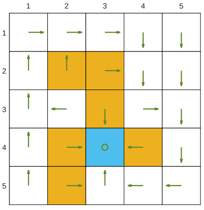
$$\delta_{k+1} = \gamma P_\pi \delta_k = \gamma^2 P_\pi^2 \delta_{k-1} = \cdots = \gamma^{k+1} P_\pi^{k+1} \delta_0.$$

Note that $0 \leq P_\pi^k \leq 1$, which means every entry of P_π^k is no greater than 1 for any $k = 0, 1, 2, \ldots$. That is because $P_\pi^k \mathbf{1} = \mathbf{1}$, where $\mathbf{1} = [1, \ldots, 1]^T$. On the other hand, since $\gamma < 1$, we know $\gamma^k \to 0$ and hence $\delta_{k+1} = \gamma^{k+1} P_\pi^{k+1} \delta_0 \to 0$ as $k \to \infty$. □

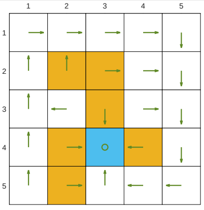
Solve state values

Examples: $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = +1$, $\gamma = 0.9$

The following are two "good" policies and the state values. The two policies are different for the top two states in the forth column.



	1	2	3	4	5
1	3.5	3.9	4.3	4.8	5.3
2	3.1	3.5	4.8	5.3	5.9
3	2.8	2.5	10.0	5.9	6.6
4	2.5	10.0	10.0	10.0	7.3
5	2.3	9.0	10.0	9.0	8.1

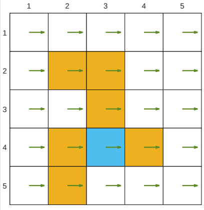


	1	2	3	4	5
1	3.5	3.9	4.3	4.8	5.3
2	3.1	3.5	4.8	5.3	5.9
3	2.8	2.5	10.0	5.9	6.6
4	2.5	10.0	10.0	10.0	7.3
5	2.3	9.0	10.0	9.0	8.1

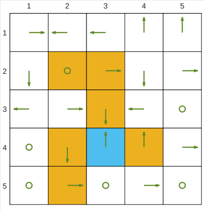
Solve state values

Examples: $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = +1$, $\gamma = 0.9$

The following are two "bad" policies and the state values. The state values are less than those of the good policies.



	1	2	3	4	5
1	-6.6	-7.3	-8.1	-9.0	-10.0
2	-8.5	-8.3	-8.1	-9.0	-10.0
3	-7.5	-8.3	-8.1	-9.0	-10.0
4	-7.5	-7.2	-9.1	-9.0	-10.0
5	-7.6	-7.3	-8.1	-9.0	-10.0



	1	2	3	4	5
1	0.0	0.0	0.0	-10.0	-10.0
2	-9.0	-10.0	-0.4	-0.5	-10.0
3	-10.0	-0.5	0.5	-0.5	0.0
4	0.0	-1.0	-0.5	-0.5	-10.0
5	0.0	0.0	0.0	0.0	0.0

Outline

Action value

From state value to action value:

- State value: the average return the agent can get *starting from a state*.
- Action value: the average return the agent can get *starting from a state* and *taking an action*.

Why do we care action value? Because we want to know which action is better. This point will be clearer in the following lectures.

We will frequently use action values.

Action value

Definition:

$$q_\pi(s,a) = \mathbb{E}[G_t|S_t = s, A_t = a]$$

- $q_\pi(s,a)$ is a function of the state-action pair (s,a)
- $q_\pi(s,a)$ depends on π

It follows from the properties of conditional expectation that

$$\underbrace{\mathbb{E}[G_t|S_t = s]}_{v_\pi(s)} = \sum_a \underbrace{\mathbb{E}[G_t|S_t = s, A_t = a]}_{q_\pi(s,a)} \pi(a|s)$$

Hence,

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a) \tag{2}$$

Action value

Recall that the state value is given by

$$v_\pi(s) = \sum_a \pi(a|s) \underbrace{\Big[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \Big]}_{q_\pi(s,a)} \qquad (3)$$

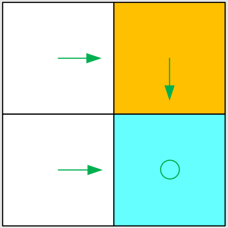
By comparing (2) and (3), we have the **action-value function** as

$$q_\pi(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \qquad (4)$$

(2) and (4) are the two sides of the same coin:

- (2) shows how to obtain state values from action values.
- (4) shows how to obtain action values from state values.

Illustrative example for action value



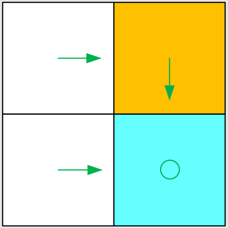
Write out the action values for state s_1.

$$q_\pi(s_1, a_2) = -1 + \gamma v_\pi(s_2),$$

Questions:

- $q_\pi(s_1, a_1), q_\pi(s_1, a_3), q_\pi(s_1, a_4), q_\pi(s_1, a_5) =$? Be careful!

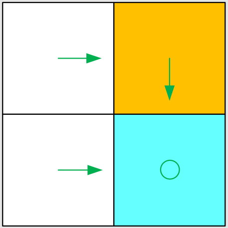
Illustrative example for action value



For the other actions:

$$q_\pi(s_1, a_1) = -1 + \gamma v_\pi(s_1),$$
$$q_\pi(s_1, a_3) = 0 + \gamma v_\pi(s_3),$$
$$q_\pi(s_1, a_4) = -1 + \gamma v_\pi(s_1),$$
$$q_\pi(s_1, a_5) = 0 + \gamma v_\pi(s_1).$$

Illustrative example for action value



Highlights:

- Action value is important since we care about which action to take.
- We can first calculate all the state values and then calculate the action values.
- We can also directly calculate the action values with or without models.

Summary

Key concepts and results:

- State value: $v_\pi(s) = \mathbb{E}[G_t|S_t = s]$
- Action value: $q_\pi(s,a) = \mathbb{E}[G_t|S_t = s, A_t = a]$
- The Bellman equation (elementwise form):

$$\begin{aligned} v_\pi(s) &= \sum_a \pi(a|s) \underbrace{\Big[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \Big]}_{q_\pi(s,a)} \\ &= \sum_a \pi(a|s) q_\pi(s,a) \end{aligned}$$

- The Bellman equation (matrix-vector form):

$$v_\pi = r_\pi + \gamma P_\pi v_\pi$$

- How to solve the Bellman equation: closed-form solution, iterative solution