Lecture 10: Actor-Critic Methods

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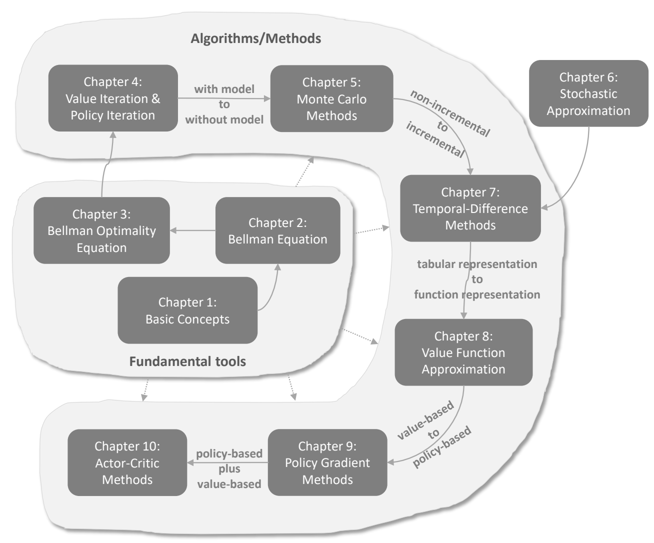
Outline



Algorithms/Methods
Chapter 4: Value Iteration & Policy Iteration
with model to without model
Chapter 5: Monte Carlo Methods
non-incremental to incremental
Chapter 6: Stochastic Approximation
Chapter 7: Temporal-Difference Methods
tabular representation to function representation
Chapter 8: Value Function Approximation
value-based to policy-based
Chapter 9: Policy Gradient Methods
policy-based plus value-based
Chapter 10: Actor-Critic Methods
Chapter 3: Bellman Optimality Equation
Chapter 2: Bellman Equation
Chapter 1: Basic Concepts
Fundamental tools

Actor-critic methods are still policy gradient methods.

- They emphasize the structure that incorporates the policy gradient and value-based methods.

What are "actor" and "critic"?

- Here, "actor" refers to policy update. It is called *actor* is because the policies will be applied to take actions.
- Here, "critic" refers to policy evaluation or value estimation. It is called *critic* because it criticizes the policy by evaluating it.

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The simplest actor-critic

Revisit the idea of policy gradient introduced in the last lecture.

1) A scalar metric $J(\theta)$, which can be $\bar{v}_\pi$ or $\bar{r}_\pi$.
2) The gradient-ascent algorithm maximizing $J(\theta)$ is

$$\begin{aligned}\theta_{t+1} &= \theta_t + \alpha \nabla_\theta J(\theta_t) \\ &= \theta_t + \alpha \mathbb{E}_{S\sim\eta, A\sim\pi}\Big[\nabla_\theta \ln \pi(A|S,\theta_t) q_\pi(S,A)\Big]\end{aligned}$$

3) The stochastic gradient-ascent algorithm is

$$\boxed{\theta_{t+1} = \theta_t + \alpha \nabla_\theta \ln \pi(a_t|s_t,\theta_t) q_t(s_t,a_t)}$$

This expression is very important! We can directly see "actor" and "critic" from it:

- This expression corresponds to actor!
- The algorithm estimating $q_t(s,a)$ corresponds to critic!

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How to get $q_t(s_t, a_t)$?

So far, we have studied two ways to estimate action values:

- **Monte Carlo learning:** If MC is used, the corresponding algorithm is called REINFORCE or Monte Carlo policy gradient.
  - We introduced in the last lecture.
- **Temporal-difference learning:** If TD is used, such kind of algorithms are usually called actor-critic.
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The simplest actor-critic algorithm (QAC)

Initialization: A policy function $\pi(a|s,\theta_0)$ where θ_0 is the initial parameter. A value function $q(s,a,w_0)$ where w_0 is the initial parameter. $\alpha_w, \alpha_\theta > 0$.
Goal: Learn an optimal policy to maximize $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\pi(a|s_t,\theta_t)$, observe r_{t+1}, s_{t+1}, and then generate a_{t+1} following $\pi(a|s_{t+1},\theta_t)$.

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \nabla_\theta \ln \pi(a_t|s_t,\theta_t) q(s_t,a_t,w_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \big[r_{t+1} + \gamma q(s_{t+1},a_{t+1},w_t) - q(s_t,a_t,w_t)\big] \nabla_w q(s_t,a_t,w_t)$$

Remarks:

- The *critic* corresponds to "SARSA+value function approximation".
- The *actor* corresponds to the policy update algorithm.
- This particular actor-citric algorithm is sometimes referred to as Q Actor-Critic (QAC).
- Though simple, this algorithm reveals the core idea of actor-critic methods. It can be extended to generate many other algorithms as shown later.

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Outline

Introduction

Next, we extend QAC to advantage actor-critic (A2C)

- The core idea is to introduce a baseline to reduce variance.

Baseline invariance

Property: the policy gradient is invariant to an additional baseline:

$$\nabla_\theta J(\theta) = \mathbb{E}_{S\sim\eta, A\sim\pi}\Big[\nabla_\theta \ln \pi(A|S,\theta_t) q_\pi(S,A)\Big]$$

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Here, the additional baseline $b(S)$ is a scalar function of S.

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Next, we answer two questions:

- Why is it valid?
- Why is it useful?

First, why is it valid?

That is because

$$\mathbb{E}_{S\sim\eta,A\sim\pi}\Big[\nabla_\theta \ln \pi(A|S,\theta_t)b(S)\Big]=0$$

The details:

$$\mathbb{E}_{S\sim\eta,A\sim\pi}\Big[\nabla_\theta \ln \pi(A|S,\theta_t)b(S)\Big]=\sum_{s\in\mathcal{S}}\eta(s)\sum_{a\in\mathcal{A}}\pi(a|s,\theta_t)\nabla_\theta \ln \pi(a|s,\theta_t)b(s)$$

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Baseline invariance

Second, why is the baseline useful?

The gradient is $\nabla_\theta J(\theta) = \mathbb{E}[X]$ where

$$X(S, A) \doteq \nabla_\theta \ln \pi(A|S, \theta_t)[q_\pi(S, A) - b(S)]$$

We have

- $\mathbb{E}[X]$ is invariant to $b(S)$.
- $\mathrm{var}(X)$ is NOT invariant to $b(S)$.
  - Why? Because $\mathrm{tr}[\mathrm{var}(X)] = \mathbb{E}[X^T X] - \bar{x}^T \bar{x}$ and

$$\begin{aligned}\mathbb{E}[X^T X] &= \mathbb{E}\left[(\nabla_\theta \ln \pi)^T (\nabla_\theta \ln \pi)(q_\pi(S, A) - b(S))^2\right] \\ &= \mathbb{E}\left[\|\nabla_\theta \ln \pi\|^2 (q_\pi(S, A) - b(S))^2\right]\end{aligned}$$

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- Benefit: when we use a random sample to approximate $\mathbb{E}[X]$, the estimation variance would also be small.

In the algorithms of REINFORCE and QAC,

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- Although this baseline is optimal, it is complex.
- We can remove the weight $\|\nabla_\theta \ln \pi(A|s,\theta_t)\|^2$ and select the suboptimal baseline:

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The algorithm of advantage actor-critic

When $b(s) = v_\pi(s)$:

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  where

$$\delta_\pi(S,A) \doteq q_\pi(S,A) - v_\pi(S)$$

  is called the advantage function (why called advantage?).

- The stochastic version is

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The algorithm of advantage actor-critic

Furthermore, the advantage function is approximated by the TD error:

$$\delta_t = q_t(s_t, a_t) - v_t(s_t) \to r_{t+1} + \gamma v_t(s_{t+1}) - v_t(s_t)$$

- This approximation is reasonable because

$$\mathbb{E}[q_\pi(S, A) - v_\pi(S)|S = s_t, A = a_t] = \mathbb{E}\Big[R + \gamma v_\pi(S') - v_\pi(S)|S = s_t, A = a_t\Big]$$

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Interpretation of the A2C algorithm:

$$\theta_{t+1} = \theta_t + \alpha \nabla_\theta \ln \pi(a_t|s_t, \theta_t)\delta_t(s_t, a_t)$$

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Advantage actor-critic (A2C) or TD actor-critic

Initialization: A policy function $\pi(a|s,\theta_0)$ where θ_0 is the initial parameter. A value function $v(s,w_0)$ where w_0 is the initial parameter. $\alpha_w, \alpha_\theta > 0$.
Goal: Learn an optimal policy to maximize $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\pi(a|s_t,\theta_t)$ and then observe r_{t+1}, s_{t+1}.

Advantage (TD error):

$$\delta_t = r_{t+1} + \gamma v(s_{t+1}, w_t) - v(s_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \delta_t \nabla_\theta \ln \pi(a_t|s_t, \theta_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \delta_t \nabla_w v(s_t, w_t)$$

It is on-policy.

Since the policy $\pi(\theta_t)$ is stochastic, no need to use techniques like ε-greedy.

Outline

- Policy gradient is on-policy.
  - Why? because the gradient is $\nabla_\theta J(\theta) = \mathbb{E}_{S\sim\eta, A\sim\pi}[*]$
- Can we convert it to off-policy?
  - Yes, by importance sampling
  - The importance sampling technique is not limited to AC, but also to any algorithm that aims to estimate an expectation.

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Illustrative examples

Consider a random variable $X \in \mathcal{X} = \{+1, -1\}$.

If the probability distribution of X is p_0:

$$p_0(X = +1) = 0.5, \quad p_0(X = -1) = 0.5$$

then the expectation of X is

$$\mathbb{E}_{X \sim p_0}[X] = (+1) \cdot 0.5 + (-1) \cdot 0.5 = 0.$$

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Then, the average value can converge to the expectation:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}[X], \quad \text{as } n \to \infty$$

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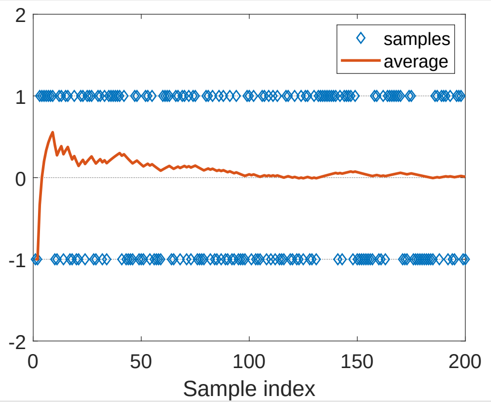
Figure: Samples and $\bar{x} \to \mathbb{E}[X]$



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  We may want to estimate $\mathbb{E}_{A\sim\pi}[*]$ where π is the *target policy* based on the samples of a *behavior policy* β.

- **How to do that?**

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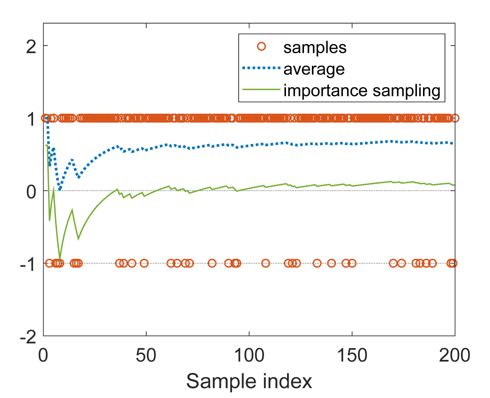

Outline

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- How to estimate $\mathbb{E}_{X\sim p_1}[f(X)]$? Easy. Let

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  - If $p_1(x_i) = p_0(x_i)$, the importance weight is one and $\bar{f}$ becomes $\bar{x}$.
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You may ask: While $\bar{f} = \frac{1}{n}\sum_{i=1}^{n} \frac{p_0(x_i)}{p_1(x_i)} x_i$ requires $p_0(x)$, if I know $p_0(x)$, why not directly calculate the expectation?

Answer: We may only be able to obtain $p_0(x)$ of a given x, but not all x.

- For example, continuous case, complex expression of p_0, or no expression of p_0 (e.g., p_0 represented by a neural network).

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Importance sampling

Summary: if $\{x_i\} \sim p_1$,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}_{X\sim p_1}[X]$$

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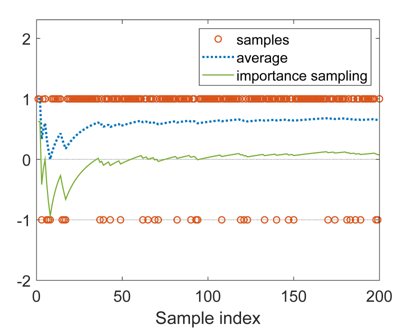

Outline

The theorem of off-policy policy gradient

Like the previous on-policy case, we need to derive the policy gradient in the off-policy case.

- Suppose β is the behavior policy that generates experience samples.
- Our goal is to use these samples to update the target policy $\pi(\theta)$ that can optimize the metric

$$J(\theta) = \sum_{s \in \mathcal{S}} d_\beta(s) v_\pi(s) = \mathbb{E}_{S \sim d_\beta}[v_\pi(S)]$$

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Theorem (Off-policy policy gradient theorem)

In the discounted case where $\gamma \in (0,1)$, the gradient of $J(\theta)$ is

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{S\sim\rho, A\sim\pi}\left[\nabla_\theta \ln \pi(A|S,\theta) q_\pi(S,A)\right] \\ &= \mathbb{E}_{S\sim\rho, A\sim\beta}\left[\frac{\pi(A|S,\theta)}{\beta(A|S)} \nabla_\theta \ln \pi(A|S,\theta) q_\pi(S,A)\right]\end{aligned}$$

where β is the behavior policy and ρ is a state distribution.

See the details and the proof in my book.

The off-policy policy gradient is also invariant to a baseline $b(s)$.

- In particular, we have

$$\nabla_\theta J(\theta) = \mathbb{E}_{S\sim\rho, A\sim\beta}\left[\frac{\pi(A|S,\theta)}{\beta(A|S)}\nabla_\theta \ln \pi(A|S,\theta)\big(q_\pi(S,A) - b(S)\big)\right]$$

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The corresponding stochastic gradient-ascent algorithm is

$$\theta_{t+1} = \theta_t + \alpha_\theta \frac{\pi(a_t|s_t, \theta_t)}{\beta(a_t|s_t)} \nabla_\theta \ln \pi(a_t|s_t, \theta_t)\big(q_t(s_t, a_t) - v_t(s_t)\big)$$

Similar to the on-policy case,

$$q_t(s_t, a_t) - v_t(s_t) \approx r_{t+1} + \gamma v_t(s_{t+1}) - v_t(s_t) \doteq \delta_t(s_t, a_t)$$

Then, the algorithm becomes

$$\theta_{t+1} = \theta_t + \alpha_\theta \frac{\pi(a_t|s_t, \theta_t)}{\beta(a_t|s_t)} \nabla_\theta \ln \pi(a_t|s_t, \theta_t) \delta_t(s_t, a_t)$$

The interpretation can be seen from

$$\theta_{t+1} = \theta_t + \alpha_\theta \left(\frac{\delta_t(s_t, a_t)}{\beta(a_t|s_t)} \right) \nabla_\theta \pi(a_t|s_t, \theta_t)$$

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Off-policy actor-critic based on importance sampling

Initialization: A given behavior policy $\beta(a|s)$. A target policy $\pi(a|s,\theta_0)$ where θ_0 is the initial parameter. A value function $v(s,w_0)$ where w_0 is the initial parameter. $\alpha_w, \alpha_\theta > 0$.
Goal: Learn an optimal policy to maximize $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\beta(s_t)$ and then observe r_{t+1}, s_{t+1}.

Advantage (TD error):

$$\delta_t = r_{t+1} + \gamma v(s_{t+1}, w_t) - v(s_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \frac{\pi(a_t|s_t,\theta_t)}{\beta(a_t|s_t)} \delta_t \nabla_\theta \ln \pi(a_t|s_t,\theta_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \frac{\pi(a_t|s_t,\theta_t)}{\beta(a_t|s_t)} \delta_t \nabla_w v(s_t, w_t)$$

Outline

Up to now, the policies used in the policy gradient methods are all stochastic since $\pi(a|s,\theta) > 0$ for every (s,a).

Can we use deterministic policies in the policy gradient methods?

- Benefit: it can handle continuous action.

Introduction

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The ways to represent a policy:

- Up to now, a general policy is denoted as $\pi(a|s,\theta) \in [0,1]$, which can be either stochastic or deterministic.
- Now, the deterministic policy is specifically denoted as

$$a = \mu(s,\theta) \doteq \mu(s)$$

  - μ is a mapping from $\mathcal{S}$ to $\mathcal{A}$.
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Outline

- The policy gradient theorems introduced before are merely valid for stochastic policies.
- If the policy must be deterministic, we must derive a new policy gradient theorem.
- The ideas and procedures are similar.

The theorem of deterministic policy gradient

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The theorem of deterministic policy gradient

Consider the metric of average state value in the discounted case:

$$J(\theta) = \mathbb{E}[v_\mu(s)] = \sum_{s \in \mathcal{S}} d_0(s) v_\mu(s)$$

where $d_0(s)$ is a probability distribution satisfying $\sum_{s \in \mathcal{S}} d_0(s) = 1$.

- d_0 is selected to be independent of μ. The gradient in this case is easier to calculate.
- There are two special yet important cases of selecting d_0.
  - The first special case is that $d_0(s_0) = 1$ and $d_0(s \neq s_0) = 0$, where s_0 is a specific starting state of interest.
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Theorem (Deterministic policy gradient theorem in the discounted case)

In the discounted case where $\gamma \in (0,1)$, the gradient of $J(\theta)$ is

$$\begin{aligned}\nabla_\theta J(\theta) &= \sum_{s\in\mathcal{S}} \rho_\mu(s)\nabla_\theta \mu(s)\big(\nabla_a q_\mu(s,a)\big)|_{a=\mu(s)} \\ &= \mathbb{E}_{S\sim\rho_\mu}\left[\nabla_\theta \mu(S)\big(\nabla_a q_\mu(S,a)\big)|_{a=\mu(S)}\right]\end{aligned}$$

Here, ρ_μ is a state distribution.
See more details and the proof in my book.

One important difference from the stochastic case:

- The gradient does not involve the distribution of the action A (why?).
- As a result, the deterministic policy gradient method is off-policy.

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Outline

The algorithm of deterministic actor-critic

Based on the policy gradient, the gradient-ascent algorithm for maximizing $J(\theta)$ is:

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The algorithm of deterministic actor-critic

Deterministic policy gradient or deterministic actor-critic

Initialization: A given behavior policy $\beta(a|s)$. A deterministic target policy $\mu(s, \theta_0)$ where θ_0 is the initial parameter. A value function $q(s, a, w_0)$ where w_0 is the initial parameter. $\alpha_w, \alpha_\theta > 0$.
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Remarks:

- This is an off-policy implementation where the behavior policy β may be different from μ.
- β can also be replaced by μ+noise.
- How to select the function to represent $q(s, a, w)$?
  - Linear function: $q(s, a, w) = \phi^T(s, a)w$ where $\phi(s, a)$ is the feature vector. Details can be found in the DPG paper.
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Summary

- The simplest actor-critic
- Advantage actor-critic
- Off-policy actor-critic
- Deterministic actor-critic

The end

This is the end of the course, but a start for your journey in the field of reinforcement learning!