Lecture 6:
Stochastic Approximation
and
Stochastic Gradient Descent

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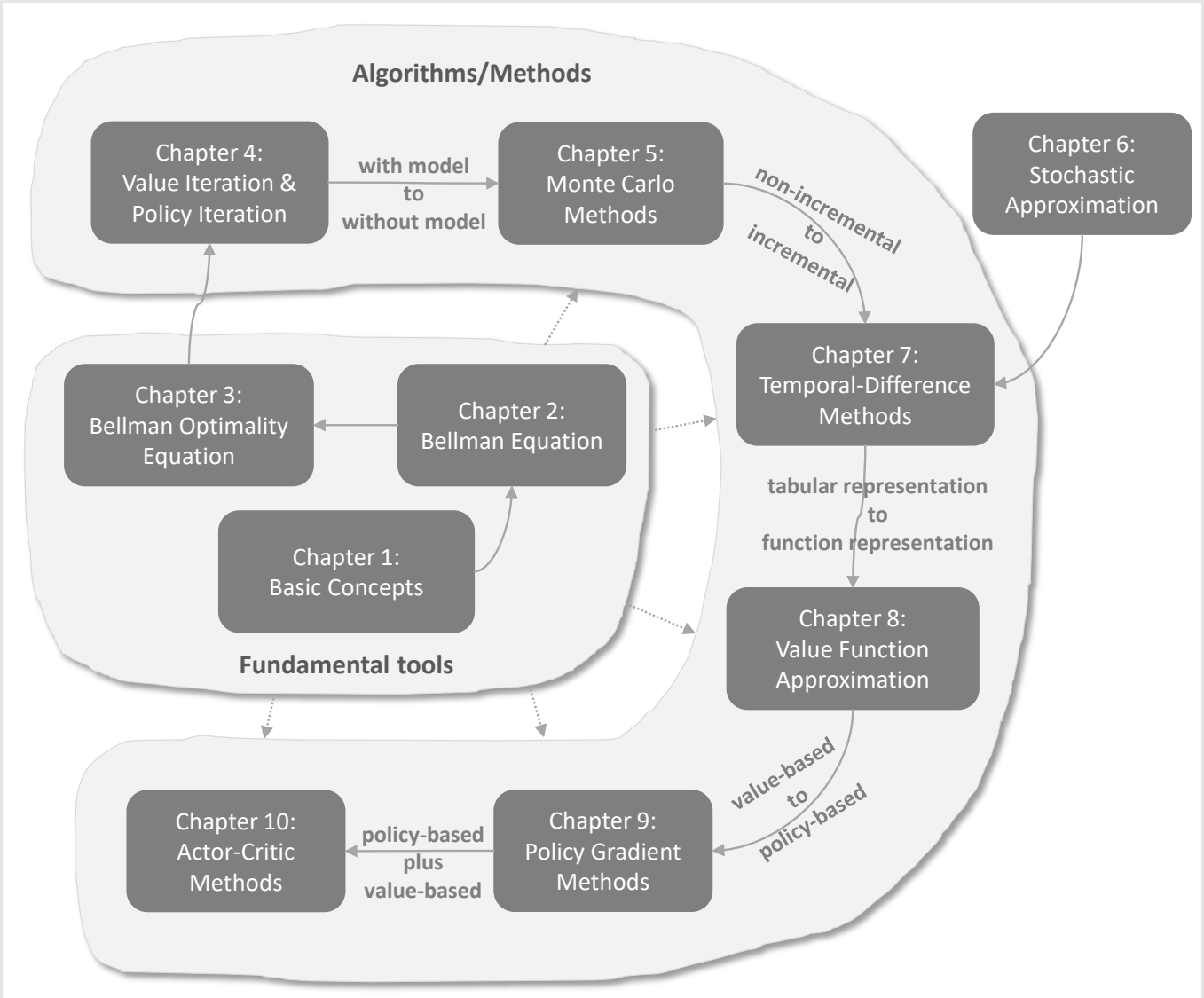
Outline


Algorithms/Methods
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Introduction

- In the last lecture, we introduced Monte-Carlo learning.
- In the next lecture, we will introduce temporal-difference (TD) learning.
- In this lecture, we press the pause button to get us better prepared.
  Why?
  - The ideas and expressions of TD algorithms are very different from the algorithms we studied so far.
  - Many students who see the TD algorithms the first time many wonder why these algorithms were designed in the first place and why they work effectively.
  - There is a knowledge gap!

Introduction

In this lecture,

- We fill the knowledge gap between the previous and upcoming lectures by introducing basic stochastic approximation (SA) algorithms.
- We will see in the next lecture that the temporal-difference algorithms are special SA algorithms. As a result, it will be much easier to understand these algorithms.
- We will also understand the important algorithm of stochastic gradient descent (SGD).

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Motivating example: mean estimation, again

Revisit the mean estimation problem:

- Consider a random variable X.
- Suppose that we collected a sequence of iid samples $\{x_i\}_{i=1}^N$.
- Our aim is to estimate $\mathbb{E}[X]$.
- The expectation of X can be approximated by

$$\mathbb{E}[X] \approx \bar{x} := \frac{1}{N} \sum_{i=1}^{N} x_i.$$

- This approximation is the basic idea of Monte Carlo estimation.
- We know that $\bar{x} \to \mathbb{E}[X]$ as $N \to \infty$.

Why do we care about mean estimation so much?

- Many quantities in RL such as action values and gradients are defined as expectations!

Motivating example: mean estimation

New question: how to calculate the mean $\bar{x}$?

$$\mathbb{E}[X] \approx \bar{x} := \frac{1}{N} \sum_{i=1}^{N} x_i.$$

We have two ways.

- **The first way**, which is trivial, is to collect all the samples then calculate the average.
  - The drawback of such way is that, if the samples are collected one by one over a period of time, we have to wait until all the samples to be collected.
- **The second way** can avoid this drawback because it calculates the average in an incremental and iterative manner.

Motivating example: mean estimation

In particular, suppose

$$w_{k+1} = \frac{1}{k}\sum_{i=1}^{k} x_i, \quad k = 1, 2, \ldots$$

and hence

$$w_k = \frac{1}{k-1}\sum_{i=1}^{k-1} x_i, \quad k = 2, 3, \ldots$$

Then, w_{k+1} can be expressed in terms of w_k as

$$\begin{aligned} w_{k+1} = \frac{1}{k}\sum_{i=1}^{k} x_i &= \frac{1}{k}\left(\sum_{i=1}^{k-1} x_i + x_k\right) \\ &= \frac{1}{k}((k-1)w_k + x_k) = w_k - \frac{1}{k}(w_k - x_k). \end{aligned}$$

Therefore, we obtain the following iterative algorithm:

$$w_{k+1} = w_k - \frac{1}{k}(w_k - x_k).$$

Motivating example: mean estimation

We can use

$$w_{k+1} = w_k - \frac{1}{k}(w_k - x_k).$$

to calculate the mean $\bar{x}$ incrementally:

$$\begin{aligned}
w_1 &= x_1, \\
w_2 &= w_1 - \frac{1}{1}(w_1 - x_1) = x_1, \\
w_3 &= w_2 - \frac{1}{2}(w_2 - x_2) = x_1 - \frac{1}{2}(x_1 - x_2) = \frac{1}{2}(x_1 + x_2), \\
w_4 &= w_3 - \frac{1}{3}(w_3 - x_3) = \frac{1}{3}(x_1 + x_2 + x_3), \\
&\vdots \\
w_{k+1} &= \frac{1}{k}\sum_{i=1}^{k} x_i.
\end{aligned}$$

Remarks about this algorithm:

$$w_{k+1} = w_k - \frac{1}{k}(w_k - x_k).$$

- An advantage of this algorithm is that it is incremental. A mean estimate can be obtained immediately once a sample is received. Then, the mean estimate can be used for other purposes immediately.
- The mean estimate is not accurate in the beginning due to insufficient samples (that is $w_k \neq \mathbb{E}[X]$). However, it is better than nothing. As more samples are obtained, the estimate can be improved gradually (that is $w_k \to \mathbb{E}[X]$ as $k \to \infty$).

Motivating example: mean estimation

Furthermore, consider an algorithm with a more general expression:

$$w_{k+1} = w_k - \alpha_k(w_k - x_k),$$

where $1/k$ is replaced by $\alpha_k > 0$.

- Does this algorithm still converge to the mean $\mathbb{E}[X]$? We will show that the answer is yes if $\{\alpha_k\}$ satisfy some mild conditions.
- We will also show that this algorithm is a special SA algorithm and also a special stochastic gradient descent algorithm.
- In the next lecture, we will see that the temporal-difference algorithms have similar (but more complex) expressions.

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Robbins-Monro algorithm

Stochastic approximation (SA):

- SA refers to a broad class of stochastic iterative algorithms solving root finding or optimization problems.
- Compared to many other root-finding algorithms such as gradient-based methods, SA is powerful in the sense that it does *not* require to know the expression of the objective function nor its derivative.

Robbins-Monro (RM) algorithm:

- The is a pioneering work in the field of stochastic approximation.
- The famous stochastic gradient descent algorithm is a special form of the RM algorithm.
- It can be used to analyze the mean estimation algorithms introduced in the beginning.

Problem statement: Suppose we would like to find the root of the equation

$$g(w) = 0,$$

where $w \in \mathbb{R}$ is the variable to be solved and $g : \mathbb{R} \to \mathbb{R}$ is a function.

- Many problems can be eventually converted to this root finding problem. For example, suppose $J(w)$ is an objective function to be minimized. Then, the optimization problem can be converged to

$$g(w) = \nabla_w J(w) = 0$$

- Note that an equation like $g(w) = c$ with c as a constant can also be converted to the above equation by rewriting $g(w) - c$ as a new function.

Robbins-Monro algorithm – Problem statement

How to calculate the root of $g(w) = 0$?

- Model-based: If the expression of g is known, there are many numerical algorithms that can solve this problem.
- Model-free: What if the expression of the function g is unknown? For example, the function is represented by an artificial neuron network.

Robbins-Monro algorithm – The algorithm

The Robbins-Monro (RM) algorithm that can solve this problem is as follows:

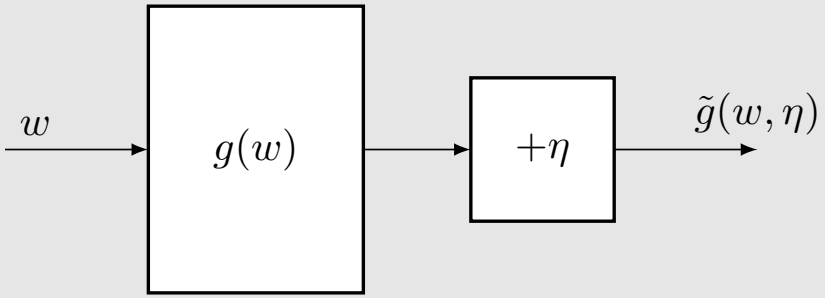
$$w_{k+1} = w_k - a_k \tilde{g}(w_k, \eta_k), \qquad k = 1, 2, 3, \dots$$

where

- w_k is the kth estimate of the root
- $\tilde{g}(w_k, \eta_k) = g(w_k) + \eta_k$ is the kth noisy observation
- a_k is a positive coefficient.

This algorithm relies on data instead of model:

- Input sequence: $\{w_k\}$
- Noisy output sequence: $\{\tilde{g}(w_k, \eta_k)\}$



Philosophy: without model, we need data!

- The function $g(w)$ is viewed as a black box.
- Here, the model refers to the expression of the function.

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Robbins-Monro algorithm – Illustrative examples

Toy example: manually solve $g(w) = w - 10$ using the RM algorithm.

Set: $w_1 = 20$, $a_k \equiv 0.5$, $\eta_k = 0$ (i.e., no observation error)

$$w_1 = 20 \Longrightarrow g(w_1) = 10$$

$$w_2 = w_1 - a_1 g(w_1) = 20 - 0.5 * 10 = 15 \Longrightarrow g(w_2) = 5$$

$$w_3 = w_2 - a_2 g(w_2) = 15 - 0.5 * 5 = 12.5 \Longrightarrow g(w_3) = 2.5$$

$$\vdots$$

$$w_k \to 10$$

Robbins-Monro algorithm – Illustrative examples

Another example: solve $g(w) = w^3 - 5$ using the RM algorithm.

- The true root is $5^{1/3} \approx 1.71$.
- We only know is $\tilde{g}(w) = g(w) + \eta$.
- Suppose η_k is iid and obeys a standard normal distribution with a mean of zero and standard deviation of 1.
- The initial guess is $w_1 = 0$ and a_k is selected to be $a_k = 1/k$.

The evolution of w_k is shown in the figure. As can be seen, the estimate w_k can converge to the true root.

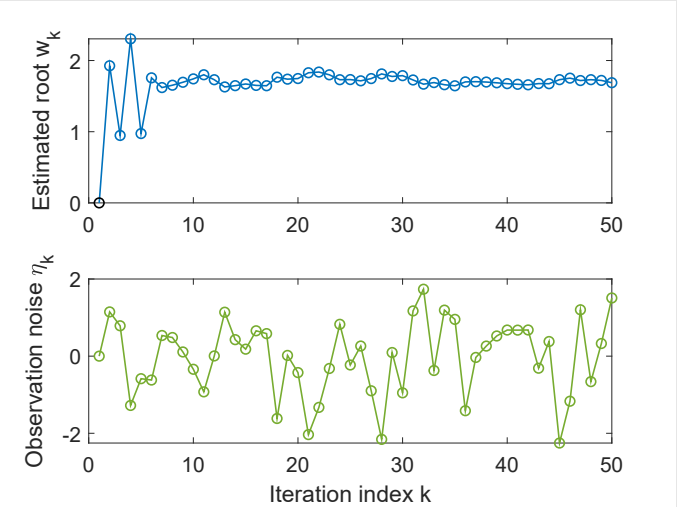


Figure: An illustrative example of the RM algorithm.

Outline

Robbins-Monro algorithm – Convergence properties

Why can the RM algorithm find the root of $g(w) = 0$?

- First present an illustrative example.
- Second give the rigorous convergence analysis.

Robbins-Monro algorithm – Convergence properties

An illustrative example:

- $g(w) = \tanh(w - 1)$
- The true root of $g(w) = 0$ is $w^* = 1$.
- Parameters: $w_1 = 3$, $a_k = 1/k$, $\eta_k \equiv 0$ (no noise for the sake of simplicity)

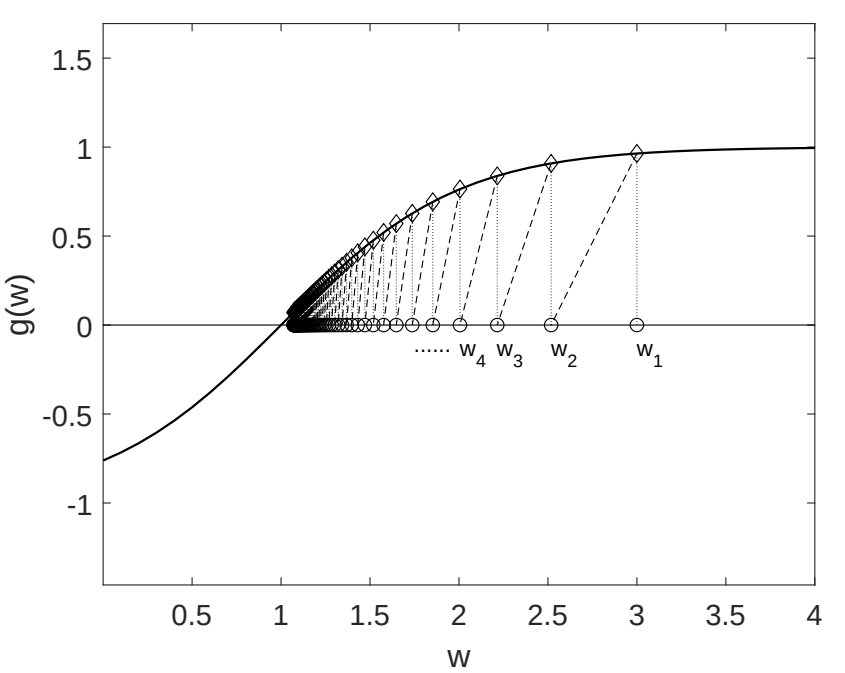
The RM algorithm in this case is

$$w_{k+1} = w_k - a_k g(w_k)$$

since $\tilde{g}(w_k, \eta_k) = g(w_k)$ when $\eta_k = 0$.

Robbins-Monro algorithm – Convergence properties

Simulation result: w_k converges to the true root $w^* = 1$.



Intuition: w_{k+1} is closer to w^* than w_k.

- When $w_k > w^*$, we have $g(w_k) > 0$. Then, $w_{k+1} = w_k - a_k g(w_k) < w_k$ and hence w_{k+1} is closer to w^* than w_k.
- When $w_k < w^*$, we have $g(w_k) < 0$. Then, $w_{k+1} = w_k - a_k g(w_k) > w_k$ and w_{k+1} is closer to w^* than w_k.

Robbins-Monro algorithm – Convergence properties

The above analysis is intuitive, but not rigorous. A rigorous convergence result is given below.

Theorem (Robbins-Monro Theorem)

In the Robbins-Monro algorithm, if

1) $0 < c_1 \leq \nabla_w g(w) \leq c_2$ *for all* w*;*

2) $\sum_{k=1}^{\infty} a_k = \infty$ *and* $\sum_{k=1}^{\infty} a_k^2 < \infty$*;*

3) $\mathbb{E}[\eta_k | \mathcal{H}_k] = 0$ *and* $\mathbb{E}[\eta_k^2 | \mathcal{H}_k] < \infty$*;*

where $\mathcal{H}_k = \{w_k, w_{k-1}, \dots\}$*, then* w_k *converges with probability 1 (w.p.1) to the root* w^* *satisfying* $g(w^*) = 0$*.*

Robbins-Monro algorithm – Convergence properties

Explanation of the three conditions:

- **Condition 1:** $0 < c_1 \leq \nabla_w g(w) \leq c_2$ for all w
  - g should be monotonically increasing,which ensures that the root of $g(w) = 0$ exists and is unique
  - The gradient is bounded from the above.
  - This condition is not strict. Consider the example $g(w) = \nabla_w J(w) = 0$. This condition requires that $g(w)$ is convex.

- **Condition 2:** $\sum_{k=1}^{\infty} a_k = \infty$ and $\sum_{k=1}^{\infty} a_k^2 < \infty$
  - $\sum_{k=1}^{\infty} a_k^2 < \infty$ ensures that a_k converges to zero as $k \to \infty$.
  - $\sum_{k=1}^{\infty} a_k = \infty$ ensures that a_k do not converge to zero too fast.

- **Condition 3:** $\mathbb{E}[\eta_k | \mathcal{H}_k] = 0$ and $\mathbb{E}[\eta_k^2 | \mathcal{H}_k] < \infty$
  - A special yet common case is that $\{\eta_k\}$ is an iid stochastic sequence satisfying $\mathbb{E}[\eta_k] = 0$ and $\mathbb{E}[\eta_k^2] < \infty$. The observation error η_k is not required to be Gaussian.

Examine the second condition more closely:

$$\sum_{k=1}^{\infty} a_k^2 < \infty \qquad \sum_{k=1}^{\infty} a_k = \infty$$

- First, $\sum_{k=1}^{\infty} a_k^2 < \infty$ indicates that $a_k \to 0$ as $k \to \infty$.
- **Why is this condition important?**

  Since

  $$w_{k+1} - w_k = -a_k \tilde{g}(w_k, \eta_k),$$

  - If $a_k \to 0$, then $a_k \tilde{g}(w_k, \eta_k) \to 0$ and hence $w_{k+1} - w_k \to 0$.
  - We need the fact that $w_{k+1} - w_k \to 0$ if w_k converges eventually.
  - If $w_k \to w^*$, $g(w_k) \to 0$ and $\tilde{g}(w_k, \eta_k)$ is dominant by η_k.

Examine the second condition more closely:

$$\sum_{k=1}^{\infty} a_k^2 < \infty \qquad \sum_{k=1}^{\infty} a_k = \infty$$

- Second, $\sum_{k=1}^{\infty} a_k = \infty$ indicates that a_k should not converge to zero too fast.
- **Why is this condition important?**
  Summarizing $w_2 = w_1 - a_1 \tilde{g}(w_1, \eta_1)$, $w_3 = w_2 - a_2 \tilde{g}(w_2, \eta_2)$, $\ldots$, $w_{k+1} = w_k - a_k \tilde{g}(w_k, \eta_k)$ leads to

$$w_1 - w_\infty = \sum_{k=1}^{\infty} a_k \tilde{g}(w_k, \eta_k).$$

  Suppose $w_\infty = w^*$. If $\sum_{k=1}^{\infty} a_k < \infty$, then $\sum_{k=1}^{\infty} a_k \tilde{g}(w_k, \eta_k)$ may be bounded. Then, if the initial guess w_1 is chosen arbitrarily far away from w^*, then the above equality would be invalid.

What $\{a_k\}$ satisfies the two conditions? $\sum_{k=1}^{\infty} a_k^2 < \infty, \sum_{k=1}^{\infty} a_k = \infty$

One typical sequence is

$$a_k = \frac{1}{k}$$

- It holds that

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = \kappa,$$

  where $\kappa \approx 0.577$ is called the Euler-Mascheroni constant (also called Euler's constant).

- It is notable that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty.$$

  The limit $\sum_{k=1}^{\infty} 1/k^2$ also has a specific name in the number theory: Basel problem.

Robbins-Monro algorithm – Convergence properties

If the three conditions are not satisfied, the algorithm may not work.

- For example, $g(w) = w^3 - 5$ does not satisfy the first condition on gradient boundedness. If the initial guess is good, the algorithm can converge (locally). Otherwise, it will diverge.

We will see that a_k is often selected as a sufficiently small constant in many RL algorithms. Although the second condition is not satisfied in this case, the algorithm can still work effectively.

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Robbins-Monro algorithm – Apply to mean estimation

Recall that

$$w_{k+1} = w_k + \alpha_k(x_k - w_k).$$

is the mean estimation algorithm.

We know that

- If $\alpha_k = 1/k$, then $w_{k+1} = 1/k \sum_{i=1}^{k} x_i$.
- If α_k is not $1/k$, the convergence was not analyzed.

Next, we show that this algorithm is a special case of the RM algorithm. Then, its convergence naturally follows.

Robbins-Monro algorithm – Apply to mean estimation

1) Consider a function:

$$g(w) \doteq w - \mathbb{E}[X].$$

Our aim is to solve $g(w) = 0$. If we can do that, then we can obtain $\mathbb{E}[X]$.

2) The observation we can get is

$$\tilde{g}(w, x) \doteq w - x,$$

because we can only obtain samples of X. Note that

$$\begin{aligned}\tilde{g}(w, \eta) = w - x &= w - x + \mathbb{E}[X] - \mathbb{E}[X] \\ &= (w - \mathbb{E}[X]) + (\mathbb{E}[X] - x) \doteq g(w) + \eta,\end{aligned}$$

3) The RM algorithm for solving $g(x) = 0$ is

$$w_{k+1} = w_k - \alpha_k \tilde{g}(w_k, \eta_k) = w_k - \alpha_k(w_k - x_k),$$

which is exactly the mean estimation algorithm.

The convergence naturally follows.

Dvoretzkys convergence theorem (optional)

Theorem (Dvoretzky's Theorem)

Consider a stochastic process

$$w_{k+1} = (1 - \alpha_k)w_k + \beta_k \eta_k,$$

where $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}, \{\eta_k\}_{k=1}^{\infty}$ *are stochastic sequences. Here* $\alpha_k \geq 0, \beta_k \geq 0$ *for all* k*. Then,* w_k *would converge to zero with probability 1 if the following conditions are satisfied:*

1) $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$; $\sum_{k=1}^{\infty} \beta_k^2 < \infty$ *uniformly w.p.1;*

2) $\mathbb{E}[\eta_k | \mathcal{H}_k] = 0$ *and* $\mathbb{E}[\eta_k^2 | \mathcal{H}_k] \leq C$ *w.p.1;*

where $\mathcal{H}_k = \{w_k, w_{k-1}, \ldots, \eta_{k-1}, \ldots, \alpha_{k-1}, \ldots, \beta_{k-1}, \ldots\}$.

- A more general result than the RM theorem. It can be used to prove the RM theorem
- It can also directly analyze the mean estimation problem.
- An extension of it can be used to analyze Q-learning and TD learning algorithms.

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Stochastic gradient descent

Next, we introduce stochastic gradient descent (SGD) algorithms:

- SGD is widely used in the field of machine learning and also in RL.
- SGD is a special RM algorithm.
- The mean estimation algorithm is a special SGD algorithm.

Problem setup: Suppose we aim to solve the following optimization problem:

$$\min_{w} \quad J(w) = \mathbb{E}[f(w, X)]$$

- w is the parameter to be optimized.
- X is a random variable. The expectation is with respect to X.
- w and X can be either scalars or vectors. The function $f(\cdot)$ is a scalar.

Method 1: gradient descent (GD)

$$w_{k+1} = w_k - \alpha_k \nabla_w \mathbb{E}[f(w_k, X)] = w_k - \alpha_k \mathbb{E}[\nabla_w f(w_k, X)]$$

How to calculate the expectation? Model-based or model-free

We focus on model-free here.

Method 2: batch gradient descent (BGD)

$$\mathbb{E}[\nabla_w f(w_k, X)] \approx \frac{1}{n} \sum_{i=1}^{n} \nabla_w f(w_k, x_i).$$

$$w_{k+1} = w_k - \alpha_k \frac{1}{n} \sum_{i=1}^{n} \nabla_w f(w_k, x_i).$$

Drawback: it requires many samples in each iteration for each w_k.

Method 3: stochastic gradient descent (SGD)

$$w_{k+1} = w_k - \alpha_k \nabla_w f(w_k, x_k),$$

- Compared to the gradient descent method: Replace the true gradient $\mathbb{E}[\nabla_w f(w_k, X)]$ by the stochastic gradient $\nabla_w f(w_k, x_k)$.
- Compared to the batch gradient descent method: let $n = 1$.

Outline

We next consider an example:

$$\min_{w} \quad J(w) = \mathbb{E}[f(w, X)] = \mathbb{E}\left[\frac{1}{2}\|w - X\|^2\right],$$

where

$$f(w, X) = \|w - X\|^2/2 \qquad \nabla_w f(w, X) = w - X$$

Excises:

- Excise 1: Show that the optimal solution is $w^* = \mathbb{E}[X]$.
- Excise 2: Write out the GD algorithm for solving this problem.
- Excise 3: Write out the SGD algorithm for solving this problem.

Answer:

- The GD algorithm for solving the above problem is

$$\begin{aligned} w_{k+1} &= w_k - \alpha_k \nabla_w J(w_k) \\ &= w_k - \alpha_k \mathbb{E}[\nabla_w f(w_k, X)] \\ &= w_k - \alpha_k \mathbb{E}[w_k - X]. \end{aligned}$$

- The SGD algorithm for solving the above problem is

$$w_{k+1} = w_k - \alpha_k \nabla_w f(w_k, x_k) = w_k - \alpha_k (w_k - x_k)$$

- Note:
  - It is the same as the mean estimation algorithm we presented before.
  - That mean estimation algorithm is a special SGD algorithm.

Outline

Stochastic gradient descent – Convergence

From GD to SGD:

$$w_{k+1} = w_k - \alpha_k \mathbb{E}[\nabla_w f(w_k, X)]$$

$$\Downarrow$$

$$w_{k+1} = w_k - \alpha_k \nabla_w f(w_k, x_k)$$

- $\mathbb{E}[\nabla_w f(w_k, X)]$: true gradient
- $\nabla_w f(w_k, X)$: stochastic gradient

The stochastic gradient is a noisy measurement or an approximation of the true gradient:

$$\nabla_w f(w_k, x_k) = \mathbb{E}[\nabla_w f(w, X)] + \underbrace{\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w, X)]}_{\eta}$$

where η is the noise. Since

$$\nabla_w f(w_k, x_k) \neq \mathbb{E}[\nabla_w f(w, X)]$$

whether $w_k \to w^*$ as $k \to \infty$ by SGD?

We next show that SGD is a special RM algorithm. Then, the convergence naturally follows.

The aim of SGD is to minimize

$$J(w) = \mathbb{E}[f(w, X)]$$

This problem can be converted to a root-finding problem:

$$\nabla_w J(w) = \mathbb{E}[\nabla_w f(w, X)] = 0$$

Let

$$g(w) = \nabla_w J(w) = \mathbb{E}[\nabla_w f(w, X)].$$

Then, the aim of SGD is to find the root of $g(w) = 0$.

What we can measure is

$$\begin{aligned}\tilde{g}(w,\eta) &= \nabla_w f(w,x) \\ &= \underbrace{\mathbb{E}[\nabla_w f(w,X)]}_{g(w)} + \underbrace{\nabla_w f(w,x) - \mathbb{E}[\nabla_w f(w,X)]}_{\eta}.\end{aligned}$$

Then, the RM algorithm for solving $g(w) = 0$ is

$$w_{k+1} = w_k - a_k \tilde{g}(w_k, \eta_k) = w_k - a_k \nabla_w f(w_k, x_k).$$

- It is exactly the SGD algorithm.
- Therefore, SGD is a special RM algorithm.

Since SGD is a special RM algorithm, its convergence naturally follows.

Theorem (Convergence of SGD)

In the SGD algorithm, if

1) $0 < c_1 \leq \nabla_w^2 f(w, X) \leq c_2$*;*

2) $\sum_{k=1}^{\infty} a_k = \infty$ *and* $\sum_{k=1}^{\infty} a_k^2 < \infty$*;*

3) $\{x_k\}_{k=1}^{\infty}$ *is iid;*

then w_k *converges to the root of* $\nabla_w \mathbb{E}[f(w, X)] = 0$ *with probability 1.*

For the proof see the book.

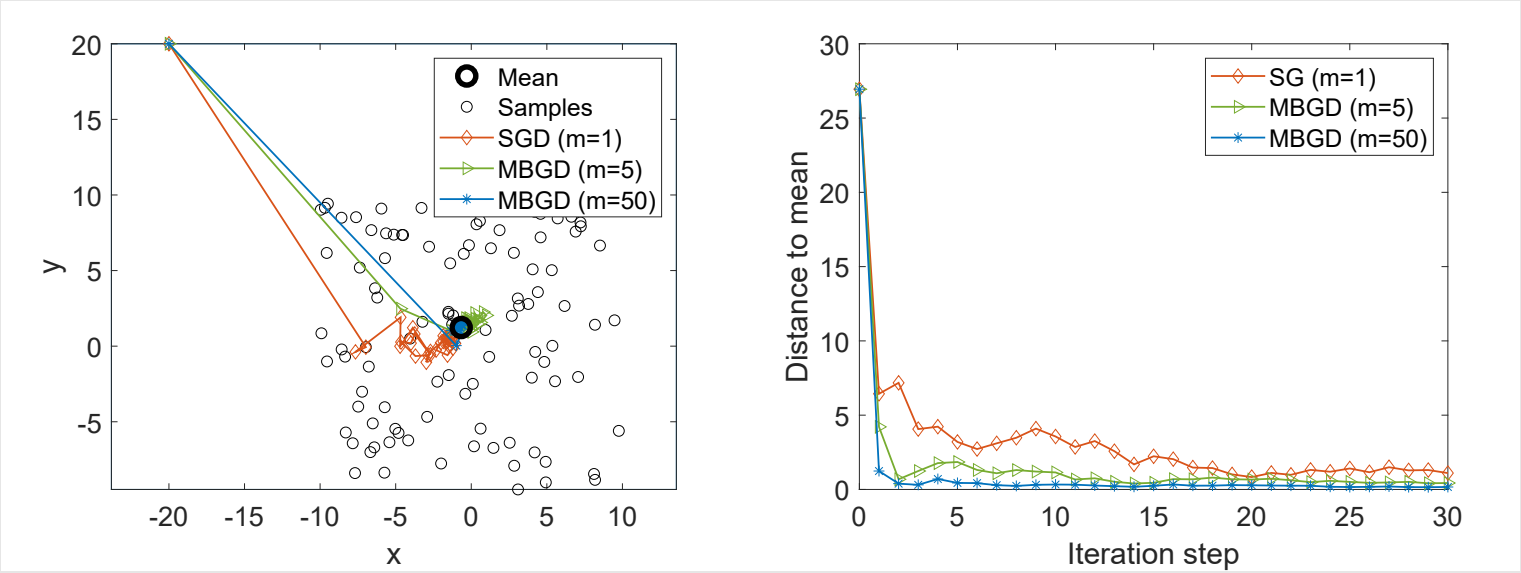
Outline

Stochastic gradient descent – Convergence pattern

Question: Since the stochastic gradient is random and hence the approximation is inaccurate, whether the convergence of SGD is slow or random?

Stochastic gradient descent – Convergence pattern

Example: $X \in \mathbb{R}^2$ represents a random position in the plane. Its distribution is uniform in the square area centered at the origin with the side length as 20. The true mean is $\mathbb{E}[X] = 0$. The mean estimation is based on 100 iid samples $\{x_i\}_{i=1}^{100}$.



- When the estimate (e.g., the initial guess) is far away from the true value, the SGD estimate can approach the neighborhood of the true value fast.
- When the estimate is close to the true value, it exhibits certain randomness but still approaches the true value gradually.

Question: Why such a pattern?

Answer: We answer this question by considering the relative error between the stochastic and batch gradients:

$$\delta_k \doteq \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w f(w_k, X)]|}.$$

Since $\mathbb{E}[\nabla_w f(w^*, X)] = 0$, we further have

$$\delta_k = \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w f(w_k, X)] - \mathbb{E}[\nabla_w f(w^*, X)]|} = \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w^2 f(\tilde{w}_k, X)(w_k - w^*)]|}.$$

where the last equality is due to the mean value theorem and $\tilde{w}_k \in [w_k, w^*]$.

Suppose f is strictly convex such that

$$\nabla_w^2 f \geq c > 0$$

for all w, X, where c is a positive bound.

Then, the denominator of δ_k becomes

$$\begin{aligned}\left|\mathbb{E}[\nabla_w^2 f(\tilde{w}_k, X)(w_k - w^*)]\right| &= \left|\mathbb{E}[\nabla_w^2 f(\tilde{w}_k, X)](w_k - w^*)\right| \\ &= \left|\mathbb{E}[\nabla_w^2 f(\tilde{w}_k, X)]\right|\left|(w_k - w^*)\right| \geq c|w_k - w^*|.\end{aligned}$$

Substituting the above inequality to δ_k gives

$$\delta_k \leq \frac{\left|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]\right|}{c|w_k - w^*|}.$$

Stochastic gradient descent – Convergence pattern

Note that

$$\delta_k \leq \frac{\big|\overbrace{\nabla_w f(w_k, x_k)}^{\text{stochastic gradient}} - \overbrace{\mathbb{E}[\nabla_w f(w_k, X)]}^{\text{true gradient}}\big|}{\underbrace{c|w_k - w^*|}_{\text{distance to the optimal solution}}}.$$

The above equation suggests an interesting convergence pattern of SGD.

- The relative error δ_k is inversely proportional to $|w_k - w^*|$.
- When $|w_k - w^*|$ is large, δ_k is small and SGD behaves like GD.
- When w_k is close to w^*, the relative error may be large and the convergence exhibits more randomness in the neighborhood of w^*.

Outline

BGD, MBGD, and SGD

Suppose we would like to minimize $J(w) = \mathbb{E}[f(w, X)]$ given a set of random samples $\{x_i\}_{i=1}^n$ of X. The BGD, SGD, MBGD algorithms solving this problem are, respectively,

$$w_{k+1} = w_k - \alpha_k \frac{1}{n} \sum_{i=1}^{n} \nabla_w f(w_k, x_i), \qquad \text{(BGD)}$$

$$w_{k+1} = w_k - \alpha_k \frac{1}{m} \sum_{j \in \mathcal{I}_k} \nabla_w f(w_k, x_j), \qquad \text{(MBGD)}$$

$$w_{k+1} = w_k - \alpha_k \nabla_w f(w_k, x_k). \qquad \text{(SGD)}$$

- **In the BGD algorithm**, all the samples are used in every iteration. When n is large, $(1/n) \sum_{i=1}^n \nabla_w f(w_k, x_i)$ is close to the true gradient $\mathbb{E}[\nabla_w f(w_k, X)]$.
- **In the MBGD algorithm**, $\mathcal{I}_k$ is a subset of $\{1, \ldots, n\}$ with the size as $|\mathcal{I}_k| = m$. The set $\mathcal{I}_k$ is obtained by m times idd samplings.
- **In the SGD algorithm**, x_k is randomly sampled from $\{x_i\}_{i=1}^n$ at time k.

BGD, MBGD, and SGD

Compare MBGD with BGD and SGD:

- Compared to SGD, MBGD has less randomness because it uses more samples instead of just one as in SGD.
- Compared to BGD, MBGD does not require to use all the samples in every iteration, making it more flexible and efficient.
- If $m = 1$, MBGD becomes SGD.
- If $m = n$, MBGD does NOT become BGD strictly speaking because MBGD uses randomly fetched n samples whereas BGD uses all n numbers. In particular, MBGD may use a value in $\{x_i\}_{i=1}^n$ multiple times whereas BGD uses each number once.

BGD, MBGD, and SGD – Illustrative examples

Given some numbers $\{x_i\}_{i=1}^n$, our aim is to calculate the mean $\bar{x} = \sum_{i=1}^n x_i/n$.
This problem can be equivalently stated as the following optimization problem:

$$\min_w \quad J(w) = \frac{1}{2n}\sum_{i=1}^{n} \|w - x_i\|^2$$

The three algorithms for solving this problem are, respectively,

$$w_{k+1} = w_k - \alpha_k \frac{1}{n}\sum_{i=1}^{n}(w_k - x_i) = w_k - \alpha_k(w_k - \bar{x}), \qquad \text{(BGD)}$$

$$w_{k+1} = w_k - \alpha_k \frac{1}{m}\sum_{j\in\mathcal{I}_k}(w_k - x_j) = w_k - \alpha_k\left(w_k - \bar{x}_k^{(m)}\right), \qquad \text{(MBGD)}$$

$$w_{k+1} = w_k - \alpha_k(w_k - x_k), \qquad \text{(SGD)}$$

where $\bar{x}_k^{(m)} = \sum_{j\in\mathcal{I}_k} x_j/m$.

Furthermore, if $\alpha_k = 1/k$, the above equation can be solved as

$$w_{k+1} = \frac{1}{k}\sum_{j=1}^{k} \bar{x} = \bar{x}, \qquad \text{(BGD)}$$

$$w_{k+1} = \frac{1}{k}\sum_{j=1}^{k} \bar{x}_j^{(m)}, \qquad \text{(MBGD)}$$

$$w_{k+1} = \frac{1}{k}\sum_{j=1}^{k} x_j. \qquad \text{(SGD)}$$

- The estimate of BGD at each step is exactly the optimal solution $w^* = \bar{x}$.
- The estimate of MBGD approaches the mean faster than SGD because $\bar{x}_k^{(m)}$ is already an average.

Let $\alpha_k = 1/k$. Given 100 points, using different mini-batch sizes leads to different convergence speed.

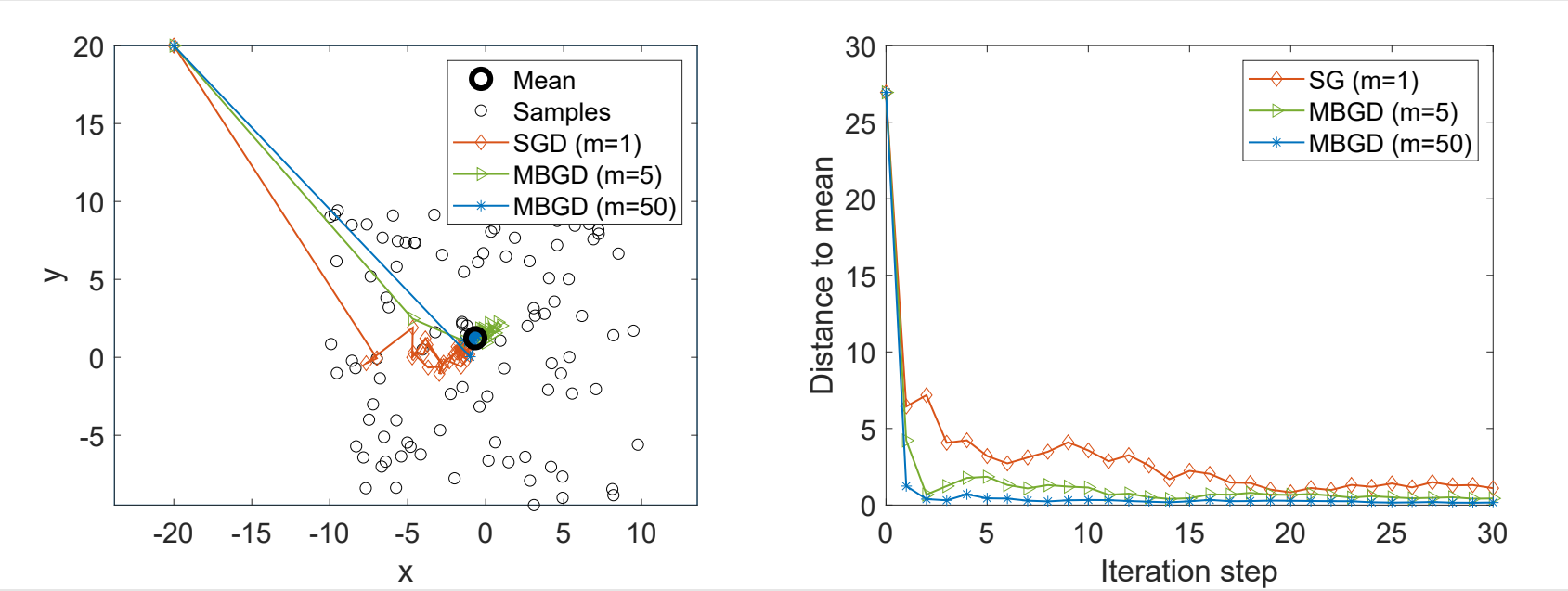


Figure: An illustrative example for mean estimation by different GD algorithms.

Outline

Summary

- Mean estimation: compute $\mathbb{E}[X]$ using $\{x_k\}$

$$w_{k+1} = w_k - \frac{1}{k}(w_k - x_k).$$

- RM algorithm: solve $g(w) = 0$ using $\{\tilde{g}(w_k, \eta_k)\}$

$$w_{k+1} = w_k - a_k \tilde{g}(w_k, \eta_k)$$

- SGD algorithm: minimize $J(w) = \mathbb{E}[f(w, X)]$ using $\{\nabla_w f(w_k, x_k)\}$

$$w_{k+1} = w_k - \alpha_k \nabla_w f(w_k, x_k),$$

These results are useful:

- We will see in the next chapter that the temporal-difference learning algorithms can be viewed as stochastic approximation algorithms and hence have similar expressions.
- They are important optimization techniques that can be applied to many other fields.