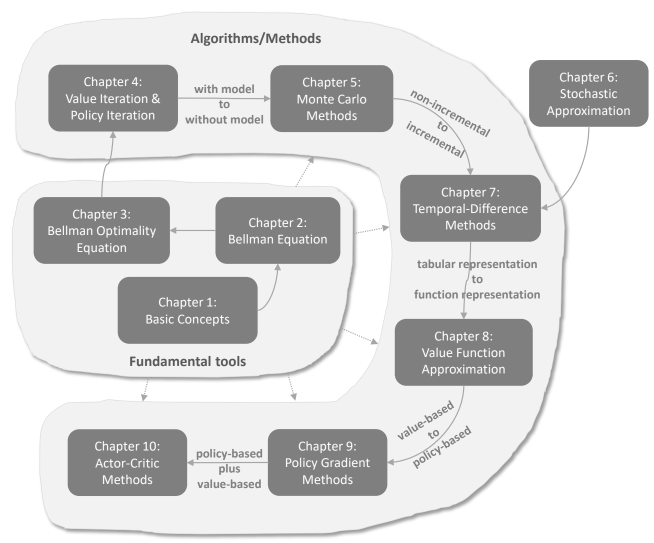
# Lecture 4: Value Iteration and Policy Iteration

Shiyu Zhao

# Outline



Algorithms/Methods
Chapter 4: Value Iteration & Policy Iteration
with model to without model
Chapter 5: Monte Carlo Methods
non-incremental to incremental
Chapter 6: Stochastic Approximation
Chapter 7: Temporal-Difference Methods
Chapter 3: Bellman Optimality Equation
Chapter 2: Bellman Equation
Chapter 1: Basic Concepts
Fundamental tools
tabular representation to function representation
Chapter 8: Value Function Approximation
value-based to policy-based
Chapter 10: Actor-Critic Methods
policy-based plus value-based
Chapter 9: Policy Gradient Methods

# Outline

1. Value iteration algorithm
2. Policy iteration algorithm
3. Truncated policy iteration algorithm

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3. Truncated policy iteration algorithm

# Value iteration algorithm

▷ How to solve the Bellman optimality equation?

$$v = f(v) = \max_{\pi}(r_\pi + \gamma P_\pi v)$$

▷ The **contraction mapping theorem** suggests an iterative algorithm:

$$v_{k+1} = f(v_k) = \max_{\pi}(r_\pi + \gamma P_\pi v_k), \quad k = 1, 2, 3 \ldots$$

where $v_0$ can be arbitrary. This algorithm can eventually find the optimal state value and an optimal policy.

▷ This algorithm is called value iteration!

▷ We next study the **implementation** of this algorithm.

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# Value iteration algorithm

The algorithm (matrix-vector form)

$$v_{k+1} = f(v_k) = \max_{\pi}(r_\pi + \gamma P_\pi v_k), \quad k = 1, 2, 3 \dots$$

can be decomposed to two steps.

- Step 1: policy update. This step is to solve

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

  where $v_k$ is given.

- Step 2: value update.

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

Question: is $v_k$ a state value? No, because it is not ensured that $v_k$ satisfies a Bellman equation.

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# Value iteration algorithm

$\triangleright$ Next, we need to study the elementwise form in order to implement the algorithm.

- Matrix-vector form is useful for **theoretical analysis**.
- Elementwise form is useful for **implementation**.

▷ Step 1: Policy update

The elementwise form of

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

is

$$\pi_{k+1}(s) = \arg\max_{\pi} \sum_a \pi(a|s) \underbrace{\left(\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_k(s')\right)}_{q_k(s,a)}, \quad s \in \mathcal{S}$$

The optimal policy solving the above optimization problem is

$$\pi_{k+1}(a|s) = \begin{cases} 1 & a = a_k^*(s) \\ 0 & a \neq a_k^*(s) \end{cases}$$

where $a_k^*(s) = \arg\max_a q_k(a,s)$. $\pi_{k+1}$ is called a **greedy policy**, since it simply selects the greatest q-value.

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# Value iteration algorithm - Elementwise form

▷ Step 2: Value update

The elementwise form of

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

is

$$v_{k+1}(s) = \sum_a \pi_{k+1}(a|s) \underbrace{\left( \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k(s') \right)}_{q_k(s,a)}, \quad s \in \mathcal{S}$$

Since $\pi_{k+1}$ is greedy, the above equation is simply

$$v_{k+1}(s) = \max_a q_k(a,s)$$

# Value iteration algorithm - Elementwise form

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# Value iteration algorithm - Pseudocode

▷ Procedure summary:

$$v_k(s) \to q_k(s,a) \to \text{greedy policy } \pi_{k+1}(a|s) \to \text{new value } v_{k+1} = \max_a q_k(s,a)$$

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$$v_k(s) \rightarrow q_k(s,a) \rightarrow \text{greedy policy } \pi_{k+1}(a|s) \rightarrow \text{new value } v_{k+1} = \max_a q_k(s,a)$$

**Pseudocode: Value iteration algorithm**

**Initialization:** The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all $(s,a)$ are known. Initial guess $v_0$.

**Aim:** Search the optimal state value and an optimal policy solving the Bellman optimality equation.

While $v_k$ has not converged in the sense that $\|v_k - v_{k-1}\|$ is greater than a predefined small threshold, for the $k$th iteration, do

  For every state $s \in \mathcal{S}$, do

    For every action $a \in \mathcal{A}(s)$, do

      q-value: $q_k(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a)v_k(s')$

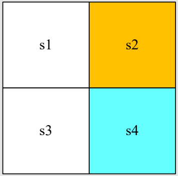
    Maximum action value: $a_k^*(s) = \arg\max_a q_k(a,s)$

    *Policy update:* $\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

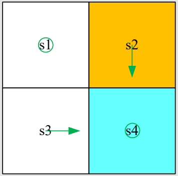
    *Value update:* $v_{k+1}(s) = \max_a q_k(a,s)$

# Value iteration algorithm - Example

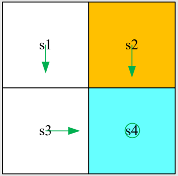
▷ The reward setting is $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$.



(a)



(b)



(c)

q-table: The expression of $q(s, a)$.

| q-value | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $s_1$ | $-1+\gamma v(s_1)$ | $-1+\gamma v(s_2)$ | $0+\gamma v(s_3)$ | $-1+\gamma v(s_1)$ | $0+\gamma v(s_1)$ |
| $s_2$ | $-1+\gamma v(s_2)$ | $-1+\gamma v(s_2)$ | $1+\gamma v(s_4)$ | $0+\gamma v(s_1)$ | $-1+\gamma v(s_2)$ |
| $s_3$ | $0+\gamma v(s_1)$ | $1+\gamma v(s_4)$ | $-1+\gamma v(s_3)$ | $-1+\gamma v(s_3)$ | $0+\gamma v(s_3)$ |
| $s_4$ | $-1+\gamma v(s_2)$ | $-1+\gamma v(s_4)$ | $-1+\gamma v(s_4)$ | $0+\gamma v(s_3)$ | $1+\gamma v(s_4)$ |

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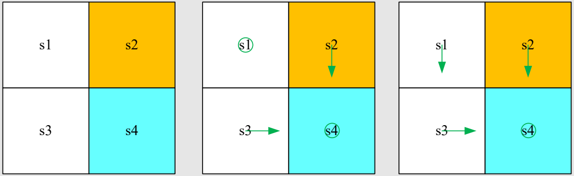


(a) (b) (c)

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Step 1: Policy update:

$$\pi_1(a_5|s_1) = 1,\ \pi_1(a_3|s_2) = 1,\ \pi_1(a_2|s_3) = 1,\ \pi_1(a_5|s_4) = 1$$

Step 2: Value update:

$$v_1(s_1) = 0,\ v_1(s_2) = 1,\ v_1(s_3) = 1,\ v_1(s_4) = 1.$$

This policy is visualized in Figure (b).

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This policy is visualized in Figure (b).

# Value iteration algorithm - Example

- $k = 1$: since $v_1(s_1) = 0, v_1(s_2) = 1, v_1(s_3) = 1, v_1(s_4) = 1$, we have

| q-table | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
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  Step 1: Policy update:

$$\pi_2(a_3|s_1) = 1,\ \pi_2(a_3|s_2) = 1,\ \pi_2(a_2|s_3) = 1,\ \pi_2(a_5|s_4) = 1.$$

  Step 2: Value update:

$$v_2(s_1) = \gamma 1,\ v_2(s_2) = 1+\gamma 1,\ v_2(s_3) = 1+\gamma 1,\ v_2(s_4) = 1+\gamma 1.$$

  This policy is visualized in Figure (c).

  The policy is already optimal!!

- $k = 2, 3, \ldots$. Stop when $\|v_k - v_{k+1}\|$ is smaller than a predefined threshold.

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# Outline

# Policy iteration algorithm

▷ Algorithm description:

Given a random **initial policy** $\pi_0$,

- Step 1: policy evaluation (PE)

  This step is to calculate the state value of $\pi_k$:

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

  Note that $v_{\pi_k}$ is a state value function.

- Step 2: policy improvement (PI)

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_{\pi_k})$$

  The maximization is componentwise!

Similar to the value iteration algorithm? Be patient. We will compare them later.

# Policy iteration algorithm

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  This step is to calculate the state value of $\pi_k$:

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  Note that $v_{\pi_k}$ is a state value function.

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  The maximization is componentwise!

Similar to the value iteration algorithm? Be patient. We will compare them later.

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PE=policy evaluation, PI=policy improvement

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- Q1: In the policy evaluation step, how to get the state value $v_{\pi_k}$ by solving the Bellman equation?
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$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

- Closed-form solution:

$$v_{\pi_k} = (I - \gamma P_{\pi_k})^{-1} r_{\pi_k}$$

- Iterative solution:

$$v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, \ldots$$

Already studied in the lecture about Bellman equation.

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▷ **Q2: In the policy improvement step, why is the new policy $\pi_{k+1}$ better than $\pi_k$?**

See the proof in the book.

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**Lemma (Policy Improvement)**

*If* $\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$, *then* $v_{\pi_{k+1}} \geq v_{\pi_k}$ *for any* $k$.

See the proof in the book.

## Policy iteration algorithm

▷ **Q3: Why can such an iterative algorithm finally reach an optimal policy?**

Since every iteration would improve the policy, we know

$$v_{\pi_0} \le v_{\pi_1} \le v_{\pi_2} \le \cdots \le v_{\pi_k} \le \cdots \le v^*.$$

As a result, $v_{\pi_k}$ keeps **increasing** and will converge.

Still need to prove what value it converges to.

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**Theorem (Convergence of Policy Iteration)**

*The state value sequence $\{v_{\pi_k}\}_{k=0}^{\infty}$ generated by the policy iteration algorithm converges to the optimal state value $v^*$. As a result, the policy sequence $\{\pi_k\}_{k=0}^{\infty}$ converges to an optimal policy.*

The proof is given in my book.

# Policy iteration algorithm

▷ **Q4: What is the relationship between policy iteration and value iteration?**

Will be explained in detail later.

# Policy iteration algorithm - Elementwise form

Step 1: Policy evaluation

▷ Matrix-vector form: $v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, \ldots$

▷ Elementwise form:

$$v_{\pi_k}^{(j+1)}(s) = \sum_a \pi_k(a|s) \left( \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}^{(j)}(s') \right), \quad s \in \mathcal{S}$$

Stop when $j$ is sufficiently large or $\|v_{\pi_k}^{(j+1)} - v_{\pi_k}^{(j)}\|$ is sufficiently small.

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# Policy iteration algorithm - Elementwise form

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▷ Matrix-vector form: $\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$

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$$\pi_{k+1}(s) = \arg\max_\pi \sum_a \pi(a|s) \underbrace{\left( \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}(s') \right)}_{q_{\pi_k}(s,a)}, \quad s \in \mathcal{S}.$$

Here, $q_{\pi_k}(s,a)$ is the action value under policy $\pi_k$. Let

$$a_k^*(s) = \arg\max_a q_{\pi_k}(a,s)$$

Then, the greedy policy is

$$\pi_{k+1}(a|s) = \begin{cases} 1 & a = a_k^*(s), \\ 0 & a \neq a_k^*(s). \end{cases}$$

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# Policy iteration algorithm - Implementation

**Pseudocode: Policy iteration algorithm**

**Initialization:** The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all $(s,a)$ are known. Initial guess $\pi_0$.

**Aim:** Search for the optimal state value and an optimal policy.

While $v_{\pi_k}$ has not converged, for the $k$th iteration, do

*Policy evaluation:*

Initialization: an arbitrary initial guess $v_{\pi_k}^{(0)}$

While $v_{\pi_k}^{(j)}$ has not converged, for the $j$th iteration, do

For every state $s \in \mathcal{S}$, do

$v_{\pi_k}^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}^{(j)}(s')\right]$

*Policy improvement:*

For every state $s \in \mathcal{S}$, do

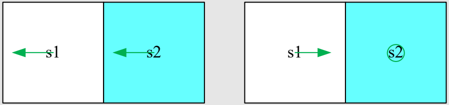
For every action $a \in \mathcal{A}$, do

$q_{\pi_k}(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}(s')$

$a_k^*(s) = \arg\max_a q_{\pi_k}(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

# Policy iteration algorithm - Simple example




(a) initial policy (b) optimal policy

- The reward setting is $r_{\text{boundary}} = -1$ and $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$.
- Actions: $a_\ell, a_0, a_r$ represent go left, stay unchanged, and go right.
- Aim: use policy iteration to find out the optimal policy.

# Policy iteration algorithm - Simple example

▷ Iteration $k = 0$: Step 1: policy evaluation

$\pi_0$ is selected as the policy in Figure (a). The Bellman equation is

$$v_{\pi_0}(s_1) = -1 + \gamma v_{\pi_0}(s_1),$$
$$v_{\pi_0}(s_2) = 0 + \gamma v_{\pi_0}(s_1).$$

- Solve the equations directly:

$$v_{\pi_0}(s_1) = -10, \quad v_{\pi_0}(s_2) = -9.$$

- Solve the equations iteratively. Select the initial guess as $v_{\pi_0}^{(0)}(s_1) = v_{\pi_0}^{(0)}(s_2) = 0$:

$$\begin{cases} v_{\pi_0}^{(1)}(s_1) = -1 + \gamma v_{\pi_0}^{(0)}(s_1) = -1, \\ v_{\pi_0}^{(1)}(s_2) = 0 + \gamma v_{\pi_0}^{(0)}(s_1) = 0, \end{cases}$$

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# Policy iteration algorithm - Simple example

▷ Iteration $k = 0$: Step 2: policy improvement

The expression of $q_{\pi_k}(s, a)$:

| $q_{\pi_k}(s,a)$ | $a_\ell$ | $a_0$ | $a_r$ |
|---|---|---|---|
| $s_1$ | $-1 + \gamma v_{\pi_k}(s_1)$ | $0 + \gamma v_{\pi_k}(s_1)$ | $1 + \gamma v_{\pi_k}(s_2)$ |
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By seeking the greatest value of $q_{\pi_0}$, the improved policy is:

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This policy is optimal after one iteration! In your programming, should continue until the stopping criterion is satisfied.

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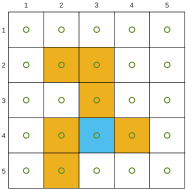
- Setting: $r_{\text{boundary}} = -1$, $r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\gamma = 0.9$.
- Let's check out the intermediate policies and state values.




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | -100.0 | -100.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | -100.0 | 0.0 | 0.0 |
| 4 | 0.0 | -100.0 | 10.0 | -100.0 | 0.0 |
| 5 | 0.0 | -100.0 | 0.0 | 0.0 | 0.0 |

$\pi_0$ and $v_{\pi_0}$

# Policy iteration algorithm - Complicated example

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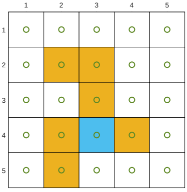


| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
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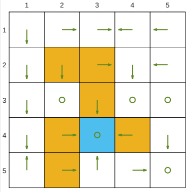
$\pi_0$ and $v_{\pi_0}$

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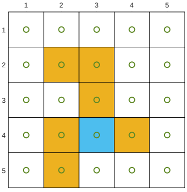
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 0.0 |
| 5 | 0.0 | 9.0 | 10.0 | 0.0 | 0.0 |

$\pi_1$ and $v_{\pi_1}$

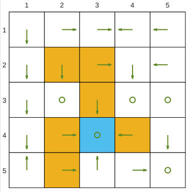
# Policy iteration algorithm - Complicated example

- Setting: $r_{\text{boundary}} = -1$, $r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\gamma = 0.9$.
- Let's check out the intermediate policies and state values.



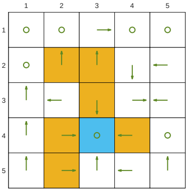
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | -100.0 | -100.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | -100.0 | 0.0 | 0.0 |
| 4 | 0.0 | -100.0 | 10.0 | -100.0 | 0.0 |
| 5 | 0.0 | -100.0 | 0.0 | 0.0 | 0.0 |

$\pi_0$ and $v_{\pi_0}$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 0.0 |
| 5 | 0.0 | 9.0 | 10.0 | 0.0 | 0.0 |

$\pi_1$ and $v_{\pi_1}$

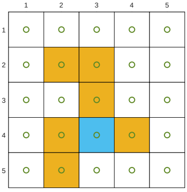


| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 0.0 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 0.0 |

$\pi_2$ and $v_{\pi_2}$

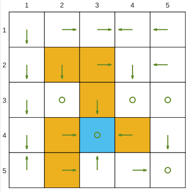
# Policy iteration algorithm - Complicated example

- ▷ Setting: $r_{\text{boundary}} = -1$, $r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\gamma = 0.9$.
- ▷ Let's check out the intermediate policies and state values.



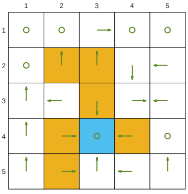
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | -100.0 | -100.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | -100.0 | 0.0 | 0.0 |
| 4 | 0.0 | -100.0 | 10.0 | -100.0 | 0.0 |
| 5 | 0.0 | -100.0 | 0.0 | 0.0 | 0.0 |

$\pi_0$ and $v_{\pi_0}$



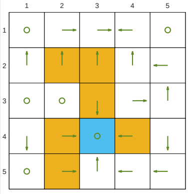
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 0.0 |
| 5 | 0.0 | 9.0 | 10.0 | 0.0 | 0.0 |

$\pi_1$ and $v_{\pi_1}$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 0.0 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 0.0 |

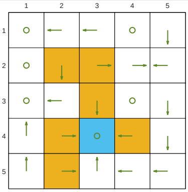
$\pi_2$ and $v_{\pi_2}$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 0.0 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_3$ and $v_{\pi_3}$

- ▷ Interesting pattern of the policies and state values

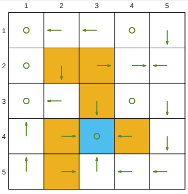



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_4$ and $v_{\pi_4}$

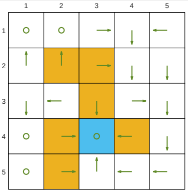
## Policy iteration algorithm - Complicated example

▷ Interesting pattern of the policies and state values




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_4$ and $v_{\pi_4}$

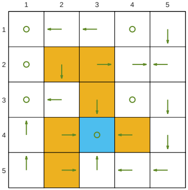



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 4.3 | 4.8 | 4.3 |
| 2 | 0.0 | 0.0 | 4.8 | 5.3 | 5.9 |
| 3 | 0.0 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_5$ and $v_{\pi_5}$

# Policy iteration algorithm - Complicated example

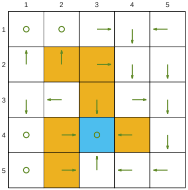
▷ Interesting pattern of the policies and state values



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_4$ and $v_{\pi_4}$

$\vdots$



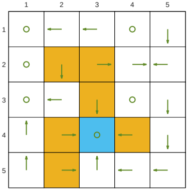
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 4.3 | 4.8 | 4.3 |
| 2 | 0.0 | 0.0 | 4.8 | 5.3 | 5.9 |
| 3 | 0.0 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_5$ and $v_{\pi_5}$

$\vdots$

# Policy iteration algorithm - Complicated example

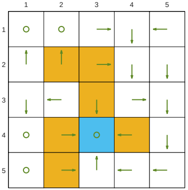
▷ Interesting pattern of the policies and state values




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_4$ and $v_{\pi_4}$

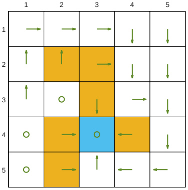
⋮




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 4.3 | 4.8 | 4.3 |
| 2 | 0.0 | 0.0 | 4.8 | 5.3 | 5.9 |
| 3 | 0.0 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_5$ and $v_{\pi_5}$

⋮

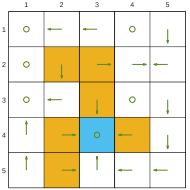



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3.5 | 3.9 | 4.3 | 4.8 | 5.3 |
| 2 | 3.1 | 3.5 | 4.8 | 5.3 | 5.9 |
| 3 | 2.8 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_9$ and $v_{\pi_9}$

# Policy iteration algorithm - Complicated example

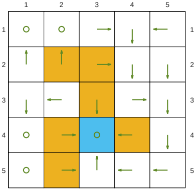
▷ Interesting pattern of the policies and state values



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 10.0 | 0.0 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_4$ and $v_{\pi_4}$

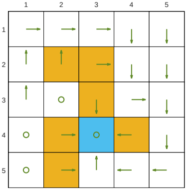
$\vdots$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 4.3 | 4.8 | 4.3 |
| 2 | 0.0 | 0.0 | 4.8 | 5.3 | 5.9 |
| 3 | 0.0 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

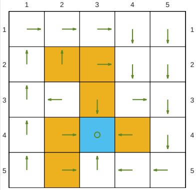
$\pi_5$ and $v_{\pi_5}$

$\vdots$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3.5 | 3.9 | 4.3 | 4.8 | 5.3 |
| 2 | 3.1 | 3.5 | 4.8 | 5.3 | 5.9 |
| 3 | 2.8 | 0.0 | 10.0 | 5.9 | 6.6 |
| 4 | 0.0 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 0.0 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_9$ and $v_{\pi_9}$



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3.5 | 3.9 | 4.3 | 4.8 | 5.3 |
| 2 | 3.1 | 3.5 | 4.8 | 5.3 | 5.9 |
| 3 | 2.8 | 2.5 | 10.0 | 5.9 | 6.6 |
| 4 | 2.5 | 10.0 | 10.0 | 10.0 | 7.3 |
| 5 | 2.3 | 9.0 | 10.0 | 9.0 | 8.1 |

$\pi_{10}$ and $v_{\pi_{10}}$

# Outline

1. Value iteration algorithm
2. Policy iteration algorithm
3. Truncated policy iteration algorithm

# Compare value iteration and policy iteration

**Policy iteration:** start from $\pi_0$

- Policy evaluation (PE):

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

- Policy improvement (PI):

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_{\pi_k})$$

**Value iteration:** start from $v_0$

- Policy update (PU):

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

- Value update (VU):

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

## Compare value iteration and policy iteration

**Policy iteration:** start from $\pi_0$

- Policy evaluation (PE):

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

- Policy improvement (PI):

$$\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$$

**Value iteration:** start from $v_0$

- Policy update (PU):

$$\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_k)$$

- Value update (VU):

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

# Compare value iteration and policy iteration

▷ The two algorithms are very similar:

$$\text{Policy iteration: } \pi_0 \xrightarrow{PE} v_{\pi_0} \xrightarrow{PI} \pi_1 \xrightarrow{PE} v_{\pi_1} \xrightarrow{PI} \pi_2 \xrightarrow{PE} v_{\pi_2} \xrightarrow{PI} \dots$$

$$\text{Value iteration: } u_0 \xrightarrow{PU} \pi_1' \xrightarrow{VU} u_1 \xrightarrow{PU} \pi_2' \xrightarrow{VU} u_2 \xrightarrow{PU} \dots$$

PE=policy evaluation. PI=policy improvement.

PU=policy update. VU=value update.

# Compare value iteration and policy iteration

▷ Let's compare the steps carefully:

| | Policy iteration algorithm | Value iteration algorithm | Comments |
|---|---|---|---|
| 1) Policy: | $\pi_0$ | N/A | |
| 2) Value: | $v_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} v_{\pi_0}$ | $v_0 \doteq v_{\pi_0}$ | |
| 3) Policy: | $\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_0})$ | $\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_0)$ | The two policies are the same |
| 4) Value: | $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$ | $v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$ | $v_{\pi_1} \geq v_1$ since $v_{\pi_1} \geq v_{\pi_0}$ |
| 5) Policy: | $\pi_2 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_1})$ | $\pi_2' = \arg\max_\pi (r_\pi + \gamma P_\pi v_1)$ | |
| ⋮ | ⋮ | ⋮ | ⋮ |

- They start from the same initial condition.
- The first three steps are the same.
- The fourth step becomes different:
  - In policy iteration, solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$ requires an iterative algorithm (an infinite number of iterations)
  - In value iteration, $v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$ is a one-step iteration

# Compare value iteration and policy iteration

▷ Let's compare the steps carefully:

| | Policy iteration algorithm | Value iteration algorithm | Comments |
|---|---|---|---|
| 1) Policy: | $\pi_0$ | N/A | |
| 2) Value: | $v_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} v_{\pi_0}$ | $v_0 \doteq v_{\pi_0}$ | |
| 3) Policy: | $\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_0})$ | $\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_0)$ | The two policies are the same |
| 4) Value: | $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$ | $v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$ | $v_{\pi_1} \geq v_1$ since $v_{\pi_1} \geq v_{\pi_0}$ |
| 5) Policy: | $\pi_2 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_1})$ | $\pi_2' = \arg\max_\pi (r_\pi + \gamma P_\pi v_1)$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

- They start from the same initial condition.
- The first three steps are the same.
- The fourth step becomes different:
  - In policy iteration, solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$ requires an iterative algorithm (an infinite number of iterations)
  - In value iteration, $v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$ is a one-step iteration

# Compare value iteration and policy iteration

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
v_{\pi_1}^{(0)} &= v_0 \\
v_{\pi_1}^{(1)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
v_{\pi_1}^{(2)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
&\vdots \\
v_{\pi_1}^{(j)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
&\vdots \\
v_{\pi_1}^{(\infty)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

# Compare value iteration and policy iteration

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
& v_{\pi_1}^{(0)} = v_0 \\
\text{value iteration} \leftarrow v_1 \longleftarrow & v_{\pi_1}^{(1)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
& v_{\pi_1}^{(2)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
& \quad \vdots \\
& v_{\pi_1}^{(j)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
& \quad \vdots \\
& v_{\pi_1}^{(\infty)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

# Compare value iteration and policy iteration

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
& v_{\pi_1}^{(0)} = v_0 \\
\text{value iteration} \leftarrow v_1 \longleftarrow \; & v_{\pi_1}^{(1)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
& v_{\pi_1}^{(2)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
& \vdots \\
& v_{\pi_1}^{(j)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
& \vdots \\
\text{policy iteration} \leftarrow v_{\pi_1} \longleftarrow \; & v_{\pi_1}^{(\infty)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

# Compare value iteration and policy iteration

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
& v_{\pi_1}^{(0)} = v_0 \\
\text{value iteration} \leftarrow v_1 \longleftarrow & v_{\pi_1}^{(1)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
& v_{\pi_1}^{(2)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
& \vdots \\
\text{truncated policy iteration} \leftarrow \bar{v}_1 \longleftarrow & v_{\pi_1}^{(j)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
& \vdots \\
\text{policy iteration} \leftarrow v_{\pi_1} \longleftarrow & v_{\pi_1}^{(\infty)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

# Compare value iteration and policy iteration

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
& & v_{\pi_1}^{(0)} &= v_0 \\
\text{value iteration} \leftarrow v_1 &\longleftarrow & v_{\pi_1}^{(1)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
& & v_{\pi_1}^{(2)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
& & &\vdots \\
\text{truncated policy iteration} \leftarrow \bar{v}_1 &\longleftarrow & v_{\pi_1}^{(j)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
& & &\vdots \\
\text{policy iteration} \leftarrow v_{\pi_1} &\longleftarrow & v_{\pi_1}^{(\infty)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

- The value iteration algorithm computes *once*.
- The policy iteration algorithm computes *an infinite number of iterations*.
- The truncated policy iteration algorithm computes *a finite number of iterations* (say $j$). The rest iterations from $j$ to $\infty$ are truncated.

# Truncated policy iteration - Pseudocode

**Pseudocode: Truncated policy iteration algorithm**

**Initialization:** The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all $(s,a)$ are known. Initial guess $\pi_0$.

**Aim:** Search for the optimal state value and an optimal policy.

While $v_k$ has not converged, for the $k$th iteration, do

*Policy evaluation:*

Initialization: select the initial guess as $v_k^{(0)} = v_{k-1}$. The maximum iteration is set to be $j_{\text{truncate}}$.

While $j < j_{\text{truncate}}$, do

For every state $s \in \mathcal{S}$, do

$v_k^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_k^{(j)}(s')\right]$

Set $v_k = v_k^{(j_{\text{truncate}})}$

*Policy improvement:*

For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

$q_k(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_k(s')$

$a_k^*(s) = \arg\max_a q_k(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

▷ Will the truncation undermine convergence?

**Proposition (Value Improvement)**

*Consider the iterative algorithm for solving the policy evaluation step:*

$$v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, ...$$

*If the initial guess is selected as* $v_{\pi_k}^{(0)} = v_{\pi_{k-1}}$, *it holds that*

$$v_{\pi_k}^{(j+1)} \geq v_{\pi_k}^{(j)}$$

*for every* $j = 0, 1, 2, \ldots$.

For the proof, see the book.

# Truncated policy iteration - Convergence

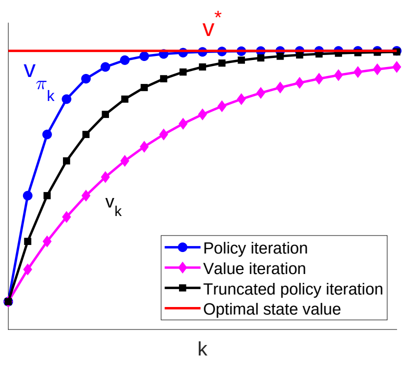



Figure: Illustration of the relationship among value iteration, policy iteration, and truncated policy iteration.

The convergence proof of PI is based on that of VI. Since VI converges, we know PI converges.

# Summary

▷ Value iteration: it is the iterative algorithm solving the Bellman optimality equation: given an initial value $v_0$,

$$v_{k+1} = \max_\pi (r_\pi + \gamma P_\pi v_k)$$

$$\Updownarrow$$

$$\begin{cases} \text{Policy update: } \pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_k) \\ \text{Value update: } v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k \end{cases}$$

▷ Policy iteration: given an initial policy $\pi_0$,

$$\begin{cases} \text{Policy evaluation: } v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k} \\ \text{Policy improvement: } \pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k}) \end{cases}$$

▷ Truncated policy iteration