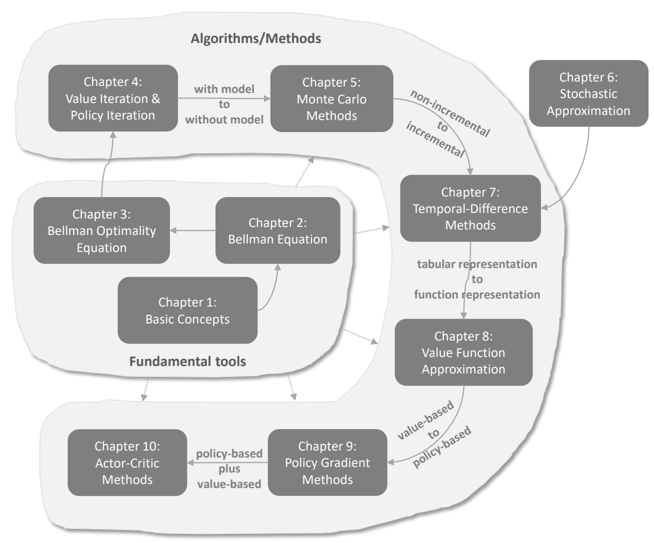
# Lecture 9: Policy Gradient Methods

Shiyu Zhao

# Outline


Algorithms/Methods
Chapter 4: Value Iteration & Policy Iteration
with model to without model
Chapter 5: Monte Carlo Methods
non-incremental to incremental
Chapter 6: Stochastic Approximation
Chapter 7: Temporal-Difference Methods
Chapter 3: Bellman Optimality Equation
Chapter 2: Bellman Equation
Chapter 1: Basic Concepts
Fundamental tools
tabular representation to function representation
Chapter 8: Value Function Approximation
value-based to policy-based
Chapter 10: Actor-Critic Methods
policy-based plus value-based
Chapter 9: Policy Gradient Methods

In this lecture, we will move

- from value-based methods to policy-based methods
- from value function approximation to policy function approximation

# Outline

# Basic idea of policy gradient

Previously, policies have been represented by tables:

- The action probabilities of all states are stored in a table $\pi(a|s)$. Each entry of the table is indexed by a state and an action.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $s_1$ | $\pi(a_1\|s_1)$ | $\pi(a_2\|s_1)$ | $\pi(a_3\|s_1)$ | $\pi(a_4\|s_1)$ | $\pi(a_5\|s_1)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_9$ | $\pi(a_1\|s_9)$ | $\pi(a_2\|s_9)$ | $\pi(a_3\|s_9)$ | $\pi(a_4\|s_9)$ | $\pi(a_5\|s_9)$ |

# Basic idea of policy gradient

Now, policies can be represented by parameterized functions:

$$\pi(a|s,\theta)$$

where $\theta \in \mathbb{R}^m$ is a parameter vector.

- The function can be, for example, a neural network, whose input is $s$, output is the probability to take each action, and parameter is $\theta$.
- **Advantage:** when the state space is large, the tabular representation will be of low efficiency in terms of storage and generalization.
- The function representation is also sometimes written as $\pi(a, s, \theta)$, $\pi_\theta(a|s)$, or $\pi_\theta(a, s)$.

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**Differences between tabular and function representations:**

- First, how to define optimal policies?
  - In the tabular case, a policy $\pi$ is optimal if it can maximize *every state value*.
  - In the function case, a policy $\pi$ is optimal if it can maximize certain *scalar metrics*.

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**Differences between tabular and function representations:**

- Second, how to access the probability of an action?
  - In the tabular case, the probability of taking $a$ at $s$ can be directly accessed by looking up the tabular policy.
  - In the function case, we need to calculate the value of $\pi(a|s,\theta)$ given the function structure and the parameter.

**Differences between tabular and function representations:**

- Third, how to update policies?
  - In the tabular case, a policy $\pi$ can be updated by directly changing the entries in the table.
  - In the function case, a policy $\pi$ cannot be updated in this way anymore. Instead, it can only be updated by changing *the parameter* $\theta$.

# Basic idea of policy gradient

The basic idea of the policy gradient is simple:

- First, metrics (or objective functions) to define optimal policies: $J(\theta)$, which can define optimal policies.
- Second, gradient-based optimization algorithms to search for optimal policies:

$$\theta_{t+1} = \theta_t + \alpha \nabla_\theta J(\theta_t)$$

Although the idea is simple, the complication emerges when we try to answer the following questions.

- What appropriate metrics should be used?
- How to calculate the gradients of the metrics?

These questions will be answered in detail in this lecture.

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# Metric 1: average value

The first metric is the average state value or simply called average value:

$$\bar{v}_\pi = \sum_{s\in\mathcal{S}} d(s) v_\pi(s)$$

- $\bar{v}_\pi$ is a weighted average of the state values.
- $d(s) \geq 0$ is the weight for state $s$.

Since $\sum_{s\in\mathcal{S}} d(s) = 1$, we can interpret $d(s)$ as a probability distribution. Then, the metric can be written as

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**An important equivalent expression:**

You will see the following metric often in the literature:

$$J(\theta) = \lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^{n} \gamma^t R_{t+1}\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1}\right].$$

Question: What is its relationship to the metric we introduced just now?

Answer: They are the same. That is because

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# Metric 1: average value

**How to select the distribution $d$? There are two cases.**

Case 1: $d$ is **independent** of the policy $\pi$.

- This case is relatively simple because the gradient of the metric is easier to calculate.
- In this case, we specifically denote $d$ as $d_0$ and $\bar{v}_\pi$ as $\bar{v}_\pi^0$.

How to select $d_0$?

- One trivial way is to treat all the states equally important and hence select $d_0(s) = 1/|\mathcal{S}|$.
- Another important case is that we are only interested in a specific state $s_0$. For example, the episodes in some tasks always start from the same state $s_0$. Then, we only care about the long-term return starting from $s_0$. In this case,

$$d_0(s_0) = 1, \quad d_0(s \neq s_0) = 0$$

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**How to select the distribution $d$? There are two cases.**

Case 2: $d$ **depends** on the policy $\pi$.

- A common way is to select $d$ as $d_\pi(s)$, which is the stationary distribution under $\pi$.

  Details of stationary distribution can be found in the last lecture and the book.
- The interpretation of selecting $d_\pi$ is as follows.
  - $d_\pi$ reflects the long-run behavior of the Markov decision process under a given policy $\pi$.
  - If one state is frequently visited in the long run, it is more important and deserves more weight.
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# Metric 2: average reward

The second metric is average one-step reward or simply average reward:

$$\bar{r}_\pi \doteq \sum_{s\in\mathcal{S}} d_\pi(s) r_\pi(s) = \mathbb{E}[r_\pi(S)],$$

where $S \sim d_\pi$,

$$r_\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s) r(s,a)$$

$$r(s,a) = \mathbb{E}[R|s,a] = \sum_r r p(r|s,a)$$

Remarks:

- $r_\pi(s)$ is the average immediate reward that can be obtained starting from state $s$.
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**An important equivalent expression:**

- Suppose an agent follows a given policy and generate a trajectory with the rewards as $(R_1, R_2, \dots)$.
- The average single-step reward along this trajectory is

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\Big[R_1+R_2+\cdots+R_n|S_0=s_0\Big]\\=&\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{t=0}^{n-1}R_{t+1}|S_0=s_0\right]\end{aligned}$$

where $s_0$ is the starting state of the trajectory.

# Metrics to define optimal policies - Remarks

An important fact is that

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{t=0}^{n-1}R_{t+1}|S_0=s_0\right]=\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{t=0}^{n-1}R_{t+1}\right]$$

Remarks:

- The derivation of the above equation is nontrivial and can be found in my book.
- Highlight: the starting state $s_0$ does not matter.

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# Summary of the two metrics

| Metric | Expression 1 | Expression 2 | Expression 3 |
|---|---|---|---|
| $\bar{v}_\pi$ | $\sum_{s\in\mathcal{S}} d(s)v_\pi(s)$ | $\mathbb{E}_{S\sim d}[v_\pi(S)]$ | $\lim_{n\to\infty}\mathbb{E}\left[\sum_{t=0}^{n}\gamma^t R_{t+1}\right]$ |
| $\bar{r}_\pi$ | $\sum_{s\in\mathcal{S}} d_\pi(s)r_\pi(s)$ | $\mathbb{E}_{S\sim d_\pi}[r_\pi(S)]$ | $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{t=0}^{n-1} R_{t+1}\right]$ |

Table: Summary of the different but equivalent expressions of $\bar{v}_\pi$ and $\bar{r}_\pi$.

**Remark 1 about the metrics:**

- All these metrics are functions of $\pi$.
- Since $\pi$ is parameterized by $\theta$, these metrics are functions of $\theta$.
- In other words, different values of $\theta$ can generate different metric values.

Therefore, we can search for the optimal values of $\theta$ to maximize these metrics.

This is the basic idea of policy gradient methods.

**Remark 2 about the metrics:**

- One complication is that the metrics can be defined in either the discounted case where $\gamma \in (0, 1)$ or the undiscounted case where $\gamma = 1$.
- The undiscounted case is nontrivial.
- We only consider the discounted case so far in this book. For details about the undiscounted case, see the book.

**Remark 3 about the metrics:**

- Intuitively, $\bar{r}_\pi$ is more short-sighted because it merely considers the immediate rewards, whereas $\bar{v}_\pi$ considers the total reward over all steps.
- However, the two metrics are equivalent to each other. Specifically, in the discounted case where $\gamma < 1$, it holds that

$$\bar{r}_\pi = (1-\gamma)\bar{v}_\pi.$$

Therefore, they can be optimized simultaneously. See the proof in the book.

# Outline

# Gradients of the metrics

Given a metric, we next

- derive its gradient
- and then, apply gradient-based methods to optimize the metric.

The gradient calculation is one of the most complicated parts of policy gradient methods! That is because

- first, we need to distinguish different metrics $\bar{v}_\pi$, $\bar{r}_\pi$, $\bar{v}_\pi^0$
- second, we need to distinguish the discounted and undiscounted cases.

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# Gradients of the metrics

I simply give the expression of the gradient without proof:

$$\nabla_\theta J(\theta) = \sum_{s \in \mathcal{S}} \eta(s) \sum_{a \in \mathcal{A}} \nabla_\theta \pi(a|s,\theta) q_\pi(s,a)$$

The above is a unified expression of many cases:

- $J(\theta)$ can be $\bar{v}_\pi$, $\bar{r}_\pi$, or $\bar{v}_\pi^0$.
- "$=$" may denote strict equality, approximation, or proportional to.
- $\eta$ is a distribution or weight of the states.

The derivation of this expression is **very complex**.
Details are not given here. Interested readers can read my book.
For most readers, it is sufficient to know this expression.

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**A compact and useful form of the gradient:**

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**First, why is this expression useful?**

- Because we can use samples to approximate the gradient:

$$\nabla_\theta J \approx \nabla_\theta \ln \pi(a|s,\theta) q_\pi(s,a)$$

where $s, a$ are samples. This is the idea of stochastic gradient descent.

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---

**Second, how to prove the above equation?**

Consider the function $\ln \pi$ where $\ln$ is the natural logarithm. It is easy to see that

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# Gradients of the metrics

**A compact and useful form of the gradient:**

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\nabla_\theta J(\theta) &= \sum_{s\in\mathcal{S}} \eta(s) \sum_{a\in\mathcal{A}} \nabla_\theta \pi(a|s,\theta) q_\pi(s,a) \\
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**Remarks:** It is required by $\ln \pi(a|s,\theta)$ that for any $s, a, \theta$

$$\pi(a|s,\theta) > 0$$

- This can be achieved by using softmax functions that can normalize the entries in a vector from $(-\infty, +\infty)$ to $(0, 1)$.
  - For example, for any vector $x = [x_1, \ldots, x_n]^T$,

$$z_i = \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}$$

    where $z_i \in (0, 1)$ and $\sum_{i=1}^{n} z_i = 1$.

- Specifically, the policy function has the form of

$$\pi(a|s,\theta) = \frac{e^{h(s,a,\theta)}}{\sum_{a' \in \mathcal{A}} e^{h(s,a',\theta)}}$$

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- Such a form based on the softmax function can be realized by a neural network whose input is $s$ and parameter is $\theta$. The network has $|\mathcal{A}|$ outputs, each of which corresponds to $\pi(a|s,\theta)$ for an action $a$. The activation function of the output layer should be softmax.
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# Outline

# Gradient-ascent algorithm

Now, we are ready to present the first policy gradient algorithm to find optimal policies!

- The gradient-ascent algorithm maximizing $J(\theta)$ is

$$\begin{aligned}\theta_{t+1} &= \theta_t + \alpha \nabla_\theta J(\theta) \\ &= \theta_t + \alpha \mathbb{E}\Big[\nabla_\theta \ln \pi(A|S, \theta_t) q_\pi(S, A)\Big]\end{aligned}$$

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**Pseudocode: Policy Gradient by Monte Carlo (REINFORCE)**

**Initialization:** Initial parameter $\theta$; $\gamma \in (0,1)$; $\alpha > 0$.
**Goal:** Learn an optimal policy to maximize $J(\theta)$.

For each episode, do
    Generate an episode $\{s_0, a_0, r_1, \ldots, s_{T-1}, a_{T-1}, r_T\}$ following $\pi(\theta)$.
    For $t = 0, 1, \ldots, T-1$:
        *Value update:* $q_t(s_t, a_t) = \sum_{k=t+1}^{T} \gamma^{k-t-1} r_k$
        *Policy update:* $\theta \leftarrow \theta + \alpha \nabla_\theta \ln \pi(a_t|s_t, \theta) q_t(s_t, a_t)$

**Remark 1: How to do sampling?**

$$\mathbb{E}_{S\sim\eta, A\sim\pi}\Big[\nabla_\theta \ln \pi(A|S,\theta_t) q_\pi(S,A)\Big] \longrightarrow \nabla_\theta \ln \pi(a|s,\theta_t) q_\pi(s,a)$$

- How to sample $S$?
  - $S \sim \eta$, where the distribution $\eta$ is a long-run behavior under $\pi$.
  - In practice, people usually do not care about it.
- How to sample $A$?
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Since

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The interpretation of

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**Math:** When $\theta_{t+1} - \theta_t$ is sufficiently small, the definition of differential implies

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**Interpretation:**

- If $\beta_t > 0$, the probability of choosing $(s_t, a_t)$ is increased:

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**Interpretation (continued)**: $\beta_t$ can balance exploration and exploitation.

The reason is as follows.

- First, $\beta_t$ is proportional to $q_t(s_t, a_t)$.

$$\text{greater } q_t(s_t, a_t) \Longrightarrow \text{greater } \beta_t \Longrightarrow \text{greater } \pi(a_t|s_t, \theta_t)$$

  Therefore, the algorithm intends to exploit actions with greater values.

- Second, $\beta_t$ is inversely proportional to $\pi(a_t|s_t, \theta_t)$.

$$\text{smaller } \pi(a_t|s_t, \theta_t)) \Longrightarrow \text{greater } \beta_t \Longrightarrow \text{greater } \pi(a_t|s_t, \theta_t)$$

  Therefore, the algorithm intends to explore actions that have low probabilities.

$$\theta_{t+1} = \theta_t + \alpha \underbrace{\left( \frac{q_t(s_t, a_t)}{\pi(a_t|s_t, \theta_t)} \right)}_{\beta_t} \nabla_\theta \pi(a_t|s_t, \theta_t)$$

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# Gradient-ascent algorithm

$$\theta_{t+1} = \theta_t + \alpha \underbrace{\left( \frac{q_t(s_t, a_t)}{\pi(a_t|s_t, \theta_t)} \right)}_{\beta_t} \nabla_\theta \pi(a_t|s_t, \theta_t)$$

**Interpretation (continued)**: $\beta_t$ can balance exploration and exploitation.

The reason is as follows.

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# Outline

# Summary

Contents of this lecture:

- Metrics for optimality
- Gradients of the metrics
- Gradient-ascent algorithm
- A special case: REINFORCE

Next lecture: Actor-critic