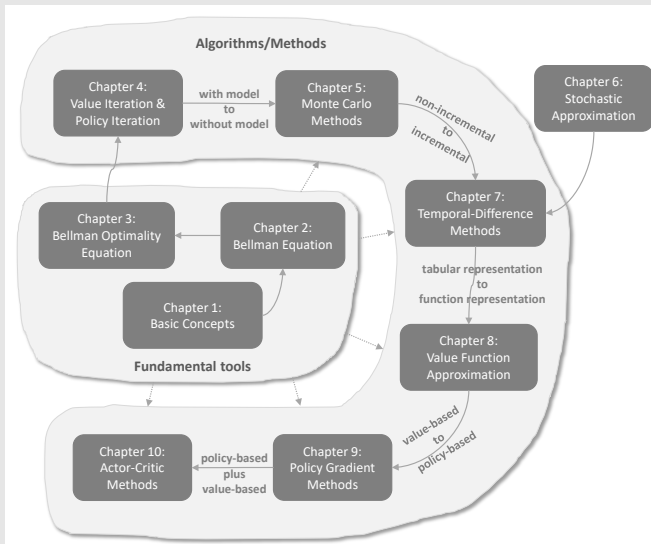


## Lecture 3: Optimal Policy and Bellman Optimality Equation

Shiyu Zhao



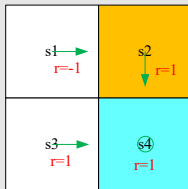
In this lecture:

- Core concepts: optimal state value and optimal policy
- A fundamental tool: Bellman optimality equation (BOE)

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

# Motivating examples



Exercise: write out the Bellman equation and solve the state values (set  $\gamma = 0.9$ )

Bellman equations:

$$v_{\pi}(s_1) = -1 + \gamma v_{\pi}(s_2),$$

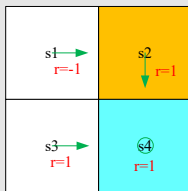
$$v_{\pi}(s_2) = +1 + \gamma v_{\pi}(s_4),$$

$$v_{\pi}(s_3) = +1 + \gamma v_{\pi}(s_4),$$

$$v_{\pi}(s_4) = +1 + \gamma v_{\pi}(s_4).$$

State values:  $v_{\pi}(s_4) = v_{\pi}(s_3) = v_{\pi}(s_2) = 10, v_{\pi}(s_1) = 8$

# Motivating examples



Exercise: calculate the action values of the five actions for  $s_1$

Action values:

$$q_{\pi}(s_1, a_1) = -1 + \gamma v_{\pi}(s_1) = 6.2,$$

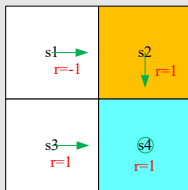
$$q_{\pi}(s_1, a_2) = -1 + \gamma v_{\pi}(s_2) = 8,$$

$$q_{\pi}(s_1, a_3) = 0 + \gamma v_{\pi}(s_3) = 9,$$

$$q_{\pi}(s_1, a_4) = -1 + \gamma v_{\pi}(s_1) = 6.2,$$

$$q_{\pi}(s_1, a_5) = 0 + \gamma v_{\pi}(s_1) = 7.2.$$

# Motivating examples



Question: While the policy is not good, how can we improve it?

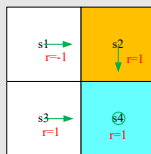
Answer: We can improve the policy based on action values.

In particular, the current policy  $\pi(a|s_1)$  is

$$\pi(a|s_1) = \begin{cases} 1 & a = a_2 \\ 0 & a \neq a_2 \end{cases}$$



# Motivating examples



Observe the action values that we obtained just now:

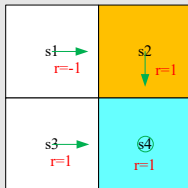
$$q_{\pi}(s_1, a_1) = 6.2, \quad q_{\pi}(s_1, a_2) = 8, \quad q_{\pi}(s_1, a_3) = 9,$$

$$q_{\pi}(s_1, a_4) = 6.2, \quad q_{\pi}(s_1, a_5) = 7.2.$$

What if we select the **greatest action value**? Then, the **new policy** is

$$\pi_{\text{new}}(a|s_1) = \begin{cases} 1 & a = a_3 \\ 0 & a \neq a_3 \end{cases}$$

# Motivating examples



Question: why doing this can improve the policy?

- Intuition: easy! Actions with greater values are better.
- Math: nontrivial! Will be introduced in this and next lectures!

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

The state value could be used to evaluate if a policy is good or not: if

$$v_{\pi_1}(s) \geq v_{\pi_2}(s) \quad \text{for all } s \in \mathcal{S}$$

then  $\pi_1$  is “better” than  $\pi_2$ .

## Definition

A policy  $\pi^*$  is optimal if  $v_{\pi^*}(s) \geq v_{\pi}(s)$  for all  $s$  and for any other policy  $\pi$ .

The definition leads to many questions:

- Does the optimal policy exist?
- Is the optimal policy unique?
- Is the optimal policy stochastic or deterministic?
- How to obtain the optimal policy?

To answer these questions, we study the *Bellman optimality equation*.

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction**
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

**Bellman optimality equation (elementwise form):**

$$\begin{aligned} v(s) &= \max_{\pi} \sum_a \pi(a|s) \left( \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v(s') \right), \quad s \in \mathcal{S} \\ &= \max_{\pi} \sum_a \pi(a|s) q(s, a), \quad s \in \mathcal{S} \end{aligned}$$

Remarks:

- $p(r|s, a), p(s'|s, a), r, \gamma$  are known.
- $v(s), v(s')$  are unknown and to be calculated.
- Is  $\pi(s)$  known or unknown?

**Bellman optimality equation (matrix-vector form):**

$$v = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v)$$

where the elements corresponding to  $s$  or  $s'$  are

$$\begin{aligned} [r_{\pi}]_s &\triangleq \sum_a \pi(a|s) \sum_r p(r|s, a) r, \\ [P_{\pi}]_{s, s'} &= p(s'|s) \triangleq \sum_a \pi(a|s) \sum_{s'} p(s'|s, a) \end{aligned}$$

Here  $\max_{\pi}$  is performed elementwise.

## Bellman optimality equation (matrix-vector form):

$$v = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v)$$

- BOE is **tricky** yet **elegant**!
  - Why elegant? It describes the optimal policy and optimal state value in an elegant way.
  - Why tricky? There is a maximization on the right-hand side, which may not be straightforward to see how to compute.
- This lecture will answer all the following questions:
  - Algorithm: how to solve this equation?
  - Existence: does this equation have solutions?
  - Uniqueness: is the solution to this equation unique?
  - Optimality: how is it related to optimal policy?



- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries**
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries**
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

# Maximization on the right-hand side of BOE

BOE: elementwise form

$$v(s) = \max_{\pi} \sum_a \pi(a|s) \left( \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v(s') \right), \quad \forall s \in \mathcal{S}$$

BOE: matrix-vector form  $v = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v)$

Example (How to solve two unknowns from one equation)

Solve two unknown variables  $x, a \in \mathbb{R}$  from the following equation:

$$x = \max_a (2x - 1 - a^2).$$

To solve them, first consider the right hand side. Regardless the value of  $x$ ,  $\max_a (2x - 1 - a^2) = 2x - 1$  where the maximization is achieved when  $a = 0$ . Second, when  $a = 0$ , the equation becomes  $x = 2x - 1$ , which leads to  $x = 1$ . Therefore,  $a = 0$  and  $x = 1$  are the solution of the equation.

# Maximization on the right-hand side of BOE

Fix  $v'(s)$  first and solve  $\pi$ :

$$\begin{aligned} v(s) &= \max_{\pi} \sum_a \pi(a|s) \left( \sum_r p(r|s, a) r + \gamma \sum_{s'} p(s'|s, a) v(s') \right), \quad \forall s \in \mathcal{S} \\ &= \max_{\pi} \sum_a \pi(a|s) q(s, a) = \max_{\pi} [\pi(a_1|s) q(s, a_1) + \cdots + \pi(a_5|s) q(s, a_5)] \\ &\doteq \max_{c_1, \dots, c_5} [c_1 q(s, a_1) + \cdots + c_5 q(s, a_5)], \quad c_1 + \cdots + c_5 = 1 \end{aligned}$$

Example (How to solve  $\max_{\pi} \sum_a \pi(a|s) q(s, a)$ )

Suppose  $q_1, q_2, q_3 \in \mathbb{R}$  are given. Find  $c_1^*, c_2^*, c_3^*$  solving

$$\max_{c_1, c_2, c_3} c_1 q_1 + c_2 q_2 + c_3 q_3.$$

where  $c_1 + c_2 + c_3 = 1$  and  $c_1, c_2, c_3 \geq 0$ .

Answer: Suppose  $q_3 \geq q_1, q_2$ . Then, the optimal solution is  $c_3^* = 1$  and  $c_1^* = c_2^* = 0$ . That is because for any  $c_1, c_2, c_3$

$$q_3 = (c_1 + c_2 + c_3) q_3 = c_1 q_3 + c_2 q_3 + c_3 q_3 \geq c_1 q_1 + c_2 q_2 + c_3 q_3.$$

Inspired by the above example, considering that  $\sum_a \pi(a|s) = 1$ , we have

$$\begin{aligned} v(s) &= \max_{\pi} \sum_a \pi(a|s) \left( \sum_r p(r|s, a) r + \gamma \sum_{s'} p(s'|s, a) v(s') \right), \quad \forall s \in \mathcal{S} \\ &= \max_{\pi} \sum_a \pi(a|s) q(s, a) \\ &= \max_{a \in \mathcal{A}(s)} q(s, a) \end{aligned}$$

where the optimality is achieved when

$$\pi(a|s) = \begin{cases} 1 & a = a^* \\ 0 & a \neq a^* \end{cases}$$

where  $a^* = \arg \max_a q(s, a)$ .

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries**
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies

# Solve the Bellman optimality equation

The BOE is  $v = \max_{\pi}(r_{\pi} + \gamma P_{\pi}v)$ . Let

$$f(v) := \max_{\pi}(r_{\pi} + \gamma P_{\pi}v)$$

Then, the Bellman optimality equation becomes

$$v = f(v)$$

where

$$[f(v)]_s = \max_{\pi} \sum_a \pi(a|s)q(s, a), \quad s \in \mathcal{S}$$

This equation looks very simple. How to solve it?

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries**
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - **Contraction mapping theorem**
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies



Some concepts:

- **Fixed point:**  $x \in X$  is a fixed point of  $f : X \rightarrow X$  if

$$f(x) = x$$

- **Contraction mapping (or contractive function):**  $f$  is a contraction mapping if

$$\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|$$

where  $\gamma \in (0, 1)$ .

- $\gamma$  must be strictly less than 1 so that many limits such as  $\gamma^k \rightarrow 0$  as  $k \rightarrow \infty$  hold.
- Here  $\|\cdot\|$  can be any vector norm.

Examples to demonstrate the concepts.

## Example

- $x = f(x) = 0.5x, x \in \mathbb{R}$ .

It is easy to verify that  $x = 0$  is a fixed point since  $0 = 0.5 \times 0$ . Moreover,  $f(x) = 0.5x$  is a contraction mapping because

$$\|0.5x_1 - 0.5x_2\| = 0.5\|x_1 - x_2\| \leq \gamma\|x_1 - x_2\| \text{ for any } \gamma \in [0.5, 1).$$

- $x = f(x) = Ax$ , where  $x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$  and  $\|A\| \leq \gamma < 1$ .

It is easy to verify that  $x = 0$  is a fixed point since  $0 = A0$ . To see the contraction property,

$$\|Ax_1 - Ax_2\| = \|A(x_1 - x_2)\| \leq \|A\|\|x_1 - x_2\| \leq \gamma\|x_1 - x_2\|. \text{ Therefore, } f(x) = Ax \text{ is a contraction mapping.}$$

### Theorem (Contraction Mapping Theorem)

*For any equation that has the form of  $x = f(x)$ , if  $f$  is a contraction mapping, then*

- **Existence:** *there exists a fixed point  $x^*$  satisfying  $f(x^*) = x^*$ .*
- **Uniqueness:** *The fixed point  $x^*$  is unique.*
- **Algorithm:** *Consider a sequence  $\{x_k\}$  where  $x_{k+1} = f(x_k)$ , then  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ . Moreover, the convergence rate is exponentially fast.*

For the proof of this theorem, see the book.

Examples:

- $x = 0.5x$ , where  $f(x) = 0.5x$  and  $x \in \mathbb{R}$   
 $x^* = 0$  is the unique fixed point. It can be solved iteratively by

$$x_{k+1} = 0.5x_k$$

- $x = Ax$ , where  $f(x) = Ax$  and  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  and  $\|A\| < 1$   
 $x^* = 0$  is the unique fixed point. It can be solved iteratively by

$$x_{k+1} = Ax_k$$

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution**
- 6 BOE: Optimality
- 7 Analyzing optimal policies

Let's come back to the Bellman optimality equation:

$$v = f(v) = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v)$$

## Theorem (Contraction Property)

*$f(v)$  is a contraction mapping satisfying*

$$\|f(v_1) - f(v_2)\| \leq \gamma \|v_1 - v_2\|$$

*where  $\gamma$  is the discount rate!*

For the proof of this lemma, see our book.

# Solve the Bellman optimality equation

Applying the contraction mapping theorem gives the following results.

## Theorem (Existence, Uniqueness, and Algorithm)

For the BOE  $v = f(v) = \max_{\pi}(r_{\pi} + \gamma P_{\pi}v)$ , there always *exists* a solution  $v^*$  and the solution is *unique*. The solution could be solved iteratively by

$$v_{k+1} = f(v_k) = \max_{\pi}(r_{\pi} + \gamma P_{\pi}v_k) \quad (1)$$

This sequence  $\{v_k\}$  converges to  $v^*$  *exponentially fast* given any initial guess  $v_0$ . The convergence rate is determined by  $\gamma$ .

**Important:** The algorithm in (1) is called the *value iteration algorithm*. We will analyze it in the next lecture! This lecture focuses more on the fundamental properties.

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality**
- 7 Analyzing optimal policies



Suppose  $v^*$  is the solution to the Bellman optimality equation. It satisfies

$$v^* = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v^*)$$

Suppose

$$\pi^* = \arg \max_{\pi} (r_{\pi} + \gamma P_{\pi} v^*)$$

Then

$$v^* = r_{\pi^*} + \gamma P_{\pi^*} v^*$$

Therefore,  $\pi^*$  is a policy and  $v^* = v_{\pi^*}$  is the corresponding state value.

Is  $\pi^*$  the optimal policy? Is  $v^*$  the greatest state value can be achieved?

## Theorem (Policy Optimality)

*Suppose that  $v^*$  is the unique solution to  $v = \max_{\pi}(r_{\pi} + \gamma P_{\pi}v)$ , and  $v_{\pi}$  is the state value function satisfying  $v_{\pi} = r_{\pi} + \gamma P_{\pi}v_{\pi}$  for any given policy  $\pi$ , then*

$$v^* \geq v_{\pi}, \quad \forall \pi$$

For the proof, please see our book.

Now we understand why we study the BOE. That is because it describes the optimal state value and optimal policy.

What does an optimal policy  $\pi^*$  look like?

$$\pi^*(s) = \arg \max_{\pi} \sum_a \pi(a|s) \underbrace{\left( \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v^*(s') \right)}_{q^*(s, a)}$$

## Theorem (Greedy Optimal Policy)

For any  $s \in \mathcal{S}$ , the deterministic greedy policy

$$\pi^*(a|s) = \begin{cases} 1 & a = a^*(s) \\ 0 & a \neq a^*(s) \end{cases}$$

is an optimal policy solving the BOE. Here,

$$a^*(s) = \arg \max_a q^*(a, s),$$

where  $q^*(s, a) \doteq \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v^*(s')$ .

- 1 Motivating examples
- 2 Definition of optimal policy
- 3 BOE: Introduction
- 4 BOE: Preliminaries
  - BOE: Maximization on the right-hand side
  - BOE: Rewrite as  $v = f(v)$
  - Contraction mapping theorem
- 5 BOE: Solution
- 6 BOE: Optimality
- 7 Analyzing optimal policies**

What factors determine the optimal state value and optimal policy?

It can be clearly seen from the BOE

$$v(s) = \max_{\pi} \sum_a \pi(a|s) \left( \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v(s') \right)$$

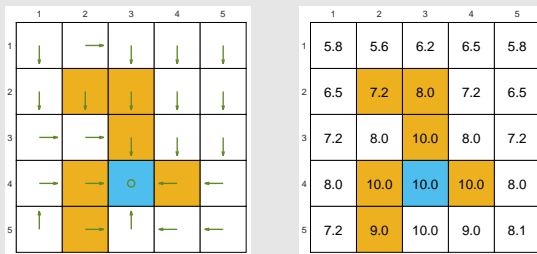
that there are three factors:

- System model:  $p(s'|s, a)$ ,  $p(r|s, a)$
- Reward design:  $r$
- Discount rate:  $\gamma$

We next show how  $r$  and  $\gamma$  can affect the optimal policy.

# Analyzing optimal policies

The optimal policy and the corresponding optimal state value are obtained by solving the BOE.

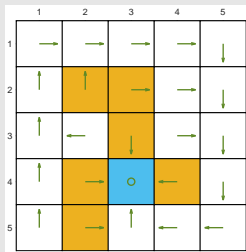


(a)  $r_{\text{boundary}} = r_{\text{forbidden}} = -1$ ,  $r_{\text{target}} = 1$ ,  $\gamma = 0.9$

The optimal policy dares to take risks: entering forbidden areas!!

# Analyzing optimal policies

If we change  $\gamma = 0.9$  to  $\gamma = 0.5$



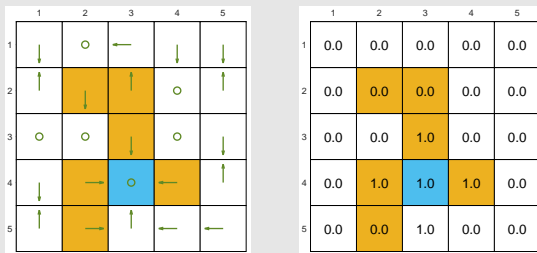
	1	2	3	4	5
1	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.1
3	0.0	0.0	2.0	0.1	0.1
4	0.0	2.0	2.0	2.0	0.2
5	0.0	1.0	2.0	1.0	0.5

(b) The discount rate is  $\gamma = 0.5$ . Others are the same as (a).

The optimal policy becomes shorted-sighted! Avoid all the forbidden areas!

# Analyzing optimal policies

If we change  $\gamma$  to 0



(c) The discount rate is  $\gamma = 0$ . Others are the same as (a).

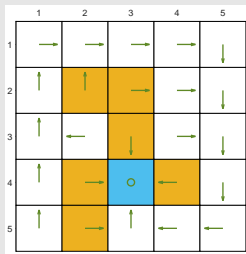
The optimal policy becomes extremely short-sighted! Also, choose the action that has the greatest *immediate reward*! Cannot reach the target!



# Analyzing optimal policies

If we increase the punishment when entering forbidden areas: **change**

$$r_{\text{forbidden}} = -1 \text{ to } r_{\text{forbidden}} = -10$$



	1	2	3	4	5
1	3.5	3.9	4.3	4.8	5.3
2	3.1	3.5	4.8	5.3	5.9
3	2.8	2.5	10.0	5.9	6.6
4	2.5	10.0	10.0	10.0	7.3
5	2.3	9.0	10.0	9.0	8.1

(d)  $r_{\text{forbidden}} = -10$ . Others are the same as (a).

The optimal policy would also avoid the forbidden areas.

What if we change  $r \rightarrow ar + b$ ?

For example,

$$r_{\text{boundary}} = r_{\text{forbidden}} = -1, \quad r_{\text{target}} = 1, \quad r_{\text{otherstep}} = 0$$

becomes

$$r_{\text{boundary}} = r_{\text{forbidden}} = 0, \quad r_{\text{target}} = 2, \quad r_{\text{otherstep}} = 1$$

The optimal policy remains the same!

What matters is not the absolute reward values! It is their relative values!

## Theorem (Optimal Policy Invariance)

*Consider a Markov decision process with  $v^* \in \mathbb{R}^{|S|}$  as the optimal state value satisfying  $v^* = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v^*)$ . If every reward  $r$  is changed by an affine transformation to  $ar + b$ , where  $a, b \in \mathbb{R}$  and  $a > 0$ , then the corresponding optimal state value  $v'$  is also an affine transformation of  $v^*$ :*

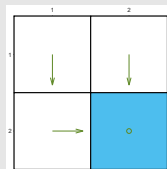
$$v' = av^* + \frac{b}{1 - \gamma} \mathbf{1},$$

*where  $\gamma \in (0, 1)$  is the discount rate and  $\mathbf{1} = [1, \dots, 1]^T$ . Consequently, the optimal policies are invariant to the affine transformation of the reward signals.*

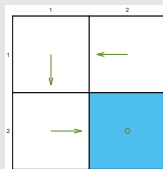
The proof is given in my book.

# Analyzing optimal policies

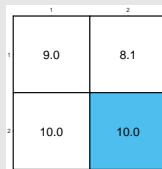
Meaningless detour?



(a) Optimal policy



(b) Not optimal



Question: Why does the optimal policy not take meaningless detours? We don't punish for taking detours because  $r_{\text{otherstep}} = 0$ .

Answer: Wrong! We do punish by using the discount rate!

Policy (a):  $\text{return} = 1 + \gamma 1 + \gamma^2 1 + \dots = 1/(1 - \gamma) = 10$ .

Policy (b):  $\text{return} = 0 + \gamma 0 + \gamma^2 1 + \gamma^3 1 + \dots = \gamma^2/(1 - \gamma) = 8.1$

Bellman optimality equation:

- Elementwise form:

$$v(s) = \max_{\pi} \sum_a \pi(a|s) \underbrace{\left( \sum_r p(r|s, a)r + \gamma \sum_{s'} p(s'|s, a)v(s') \right)}_{q(s, a)}, \quad \forall s \in \mathcal{S}$$

- Matrix-vector form:

$$v = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v)$$

Questions about the Bellman optimality equation:

- Existence: does this equation have solutions?
  - Yes, by the contraction mapping Theorem
- Uniqueness: is the solution to this equation unique?
  - Yes, by the contraction mapping Theorem
- Algorithm: how to solve this equation?
  - Iterative algorithm suggested by the contraction mapping Theorem
- Optimality: why we study this equation
  - Because its solution corresponds to the optimal state value and optimal policy.

Finally, we understand why it is important to study the BOE!