Full resolution photo on my Instagram @feenafoto

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ECOR1043: Circuits

Introduction to AC, Complex Numbers and Reactance

Sinusoids, complex numbers, capacitance and Inductance

Link to Fourier

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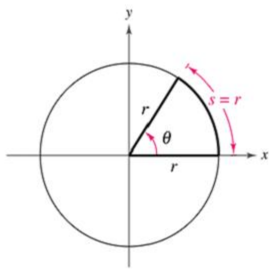
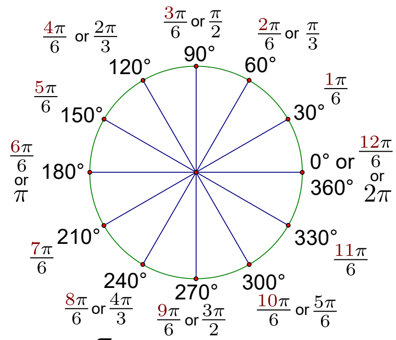
Radians and Degrees

- Radians
  - One radian is the measure of a central angle θ that intercepts arc *s* equal in length to the radius *r* of the circle.



Arc length = radius when θ = 1 radian.

$2\pi = 360$



  - To convert degrees to radians, multiply degrees by $\dfrac{\pi}{180°}$
  - To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi}$

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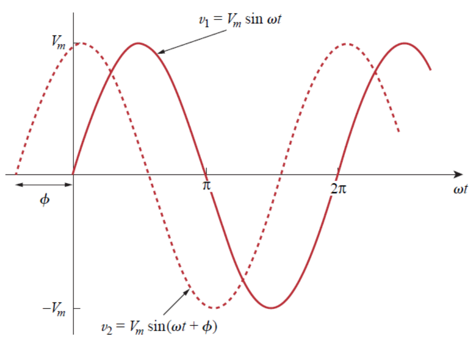
Sinusoids

- A sinusoid is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \varphi)$$



V_m = Amplitude (measure in V or A)

f = Signal frequency, measured in Hertz (Hz)

$\omega = 2\pi f$ = Angular frequency, units of radians/second (rad/s)

$T = \frac{1}{f}$ = Signal period (length of time for 1 'cycle')

ϕ = Phase (degrees or radians)

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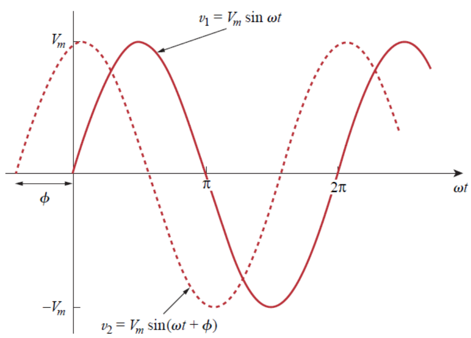
Sinusoids

- Ex. 1:
  - Given a sinusoid $5\sin(4\pi t - 60°)$, calculate its amplitude, phase, angular frequency, period, and frequency.

- Solution

$v(t) = V_m \sin(\omega t + \varphi)$ $\omega = 2\pi f$

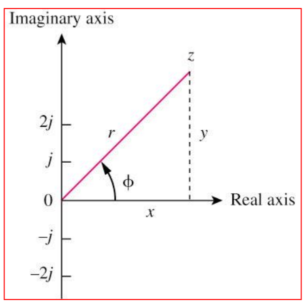
  - Amplitude V_m =
  - Phase φ =
  - Angular frequency ω=
  - Frequency f=
  - Period T=



$V_m = 5, \phi = -60^o, \omega = 4\pi \frac{rad}{s}, f = 2Hz, T = 0.5s$

Complex Numbers

- Complex numbers
  - It is more convenient to work with complex numbers in our sinusoidal analysis
  - Dealt with similar to vectors in mechanics (two components)



- **Rectangular** representation

$$\boxed{z = x + jy} = r(\cos\varphi + j\sin\varphi)$$

where $\boxed{j = \sqrt{-1}}$

- **Polar** representation

$$z = r\angle\varphi$$

Rectangular to Polar conversion

Magnitude → $r = \sqrt{x^2 + y^2}$

where

Phase angle → $\phi = \tan^{-1}\frac{y}{x}$

Complex Numbers

- Mathematic operation of complex number:

  $z_1 = x_1 + jy_1$ $\quad$ $z_2 = x_2 + jy_2$

  - Addition:

    $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

  - Subtraction:

    $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

  - Multiplication:

    In polar form

    $z_1 z_2 = r_1 r_2 \angle(\varphi_1 + \varphi_2)$

  - Division:

    In polar form

    $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\varphi_1 - \varphi_2)$

Complex Numbers

- Ex. 2: Complex Addition/Subtraction
  - Solve for $C = A + B$ and $D = B - A$
  - $A = 3 + j2$
  - $B = -6 + j9$

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Solve for C=A+B

Solve for D=B-A

$C = -3 + j11$ $\quad$ $D = -9 + j7$

Complex Numbers

- Ex. 3: Complex Multiplication/Division
  - Solve for $G = E \times F$ and $H = \frac{E}{F}$
  - $E = 5\angle 70°$
  - $F = 3\angle -\pi/2$

$z_1 z_2 = r_1 r_2 \angle \varphi_1 + \varphi_2$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \varphi_1 - \varphi_2$

Solve for $G = E \times F$

Solve for $H = E/F$

G= $15\angle -20°$ *or* $15\angle\, 340°$

$H = \frac{5}{3}\angle 160°$ *or* $H = \frac{5}{3}\angle -200°$

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Resistance, Reactance and Impedance

- Resistance
  - The friction against the flow of current
  - It is present in all conductors to some extent most notably in resistors
  - Symbolized with R and measured in ohms Ω. It is frequency independent

- Reactance
  - The inertia against the flow of currents
  - It is present anywhere electric or magnetic fields are developed in proportion to an applied alternating voltage or current
  - It is present in capacitors and inductors, and it is frequency dependent
  - Symbolized with X and measured in ohms Ω.

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Resistance, Reactance and Impedance

- Impedance Z
  - This is a comprehensive expression of any and all forms of opposition to current flow, including both resistance and reactance.
  - It is present in all circuits, and in all components
  - As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX$$

Where R is the real part which is resistance and X is imaginary part and represents reactance

  - It is measured in ohms Ω.
  - The magnitude of this impedance is

$$Z = \sqrt{R^2 + X^2}$$

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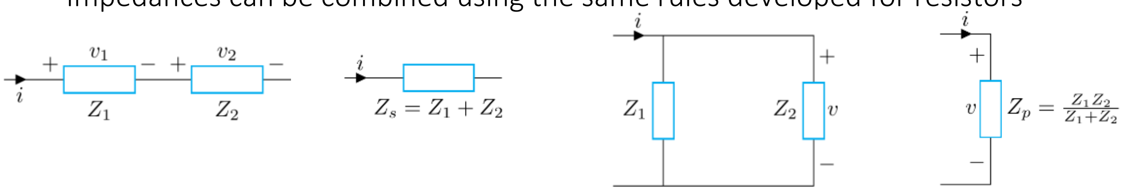
Resistance, Reactance and Impedance

- Impedance Z (cont.)
  - Impedances can be combined using the same rules developed for resistors



$$Z_s = \sum kZ_k$$

$$\frac{1}{Z_p} = \sum k\frac{1}{Z_k}$$

  - Ohm's law holds for impedance

$$V = IZ$$

  - KVL and KCL laws hold for impedance

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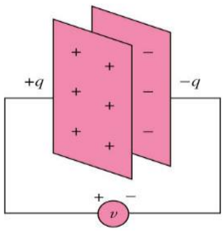
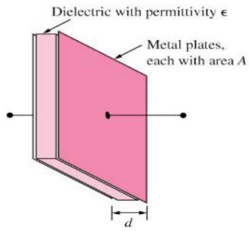
Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- A capacitor consists of two conducting plates separated by an insulator (or dielectric).
- Capacitance C is the ratio of the charge q on a capacitor to the voltage difference v between the two plates, measured in farads (F).

$$\boldsymbol{q = C \times v}$$





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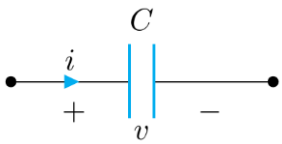
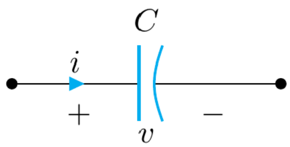
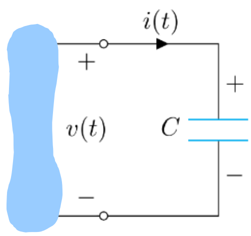
Capacitors

- It acts as an open circuit at DC (0Hz) and a short circuit at very high-frequencies (∞)
- Each plate will hold an electrical charge, one positive and the other negative (similar to a battery)
- Ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns stored energy when delivering power to the circuit.







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Capacitors

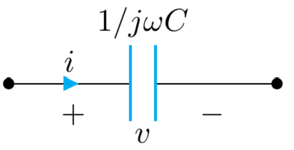
- The impedance is

$$\mathbf{Z_C} = \frac{1}{j\omega C}$$

$$\mathbf{Z_C} = -\frac{j}{\omega C}$$

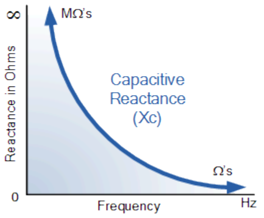
Note that real part=0. Recall $z = x + jy$

$$\mathbf{Z_C} = -jX_c$$



Where $X_c = 1/\omega C$ is reactance of capacitor

ω is angular frequency
$\omega = 2\pi f$



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Capacitors

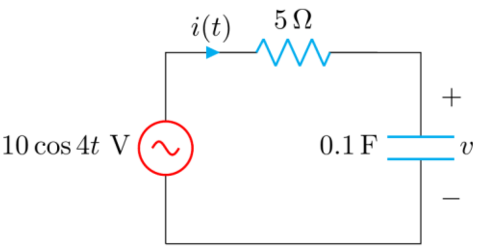
- Ex. 4:
  - What is the input impedance of the circuit?
  - We know that

    $\mathbf{Z} = R + \mathbf{Z}_c$

    $R = 5\Omega, \mathrm{C} = 0.1\mathrm{F}, \quad \omega = 4$ rad/s

    $\mathbf{Z}_c = \frac{1}{j\omega C}$

    $\mathbf{Z}_c = \frac{1}{j4\times 0.1}$
  - So, the input impedance is



$$\mathbf{Z} = R + \mathbf{Z}_c$$

$$\mathbf{Z} = 5 + \frac{1}{j0.4}$$

$$\mathbf{Z} = 5 - j2.5\Omega$$

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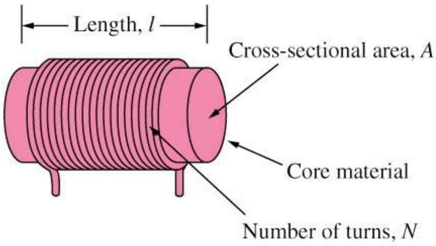
Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor consists of a coil of conducting wire.
- Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it and measured in *Henry (H)*.



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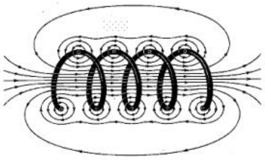
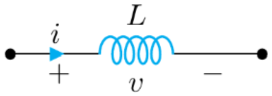
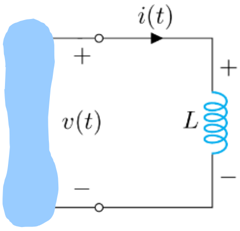
Inductors

- It acts as a short circuit for DC (0Hz), and an open circuit for high frequencies (∞)
- When current flows through a wire, it creates a magnetic field around the wire in circular lines.
- Ideal inductor does not dissipate energy. It takes power from the circuit when storing energy and returns stored energy when delivering power to the circuit.







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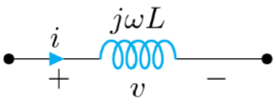
Inductors

- The impedance is

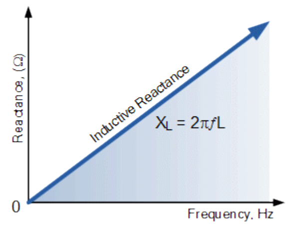
$$\mathbf{Z_L} = j\omega L$$
$$\mathbf{Z_L} = jX_L$$

Here also, real part=0. Recall $z = x + jy$



Where $X_L = \omega L$ is reactance of inductor

ω is angular frequency
$\omega = 2\pi f$

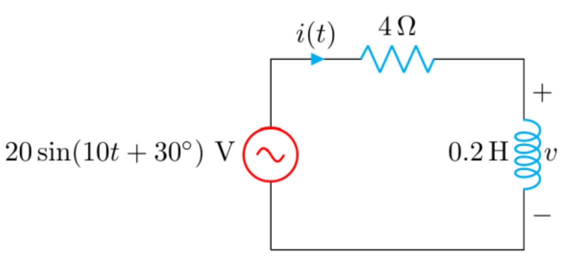


Inductors

- Ex. 5: What is the input impedance of the circuit?
- We know that

$$\mathbf{Z} = R + \mathbf{Z}_L$$
$$R = 4\Omega, \mathrm{L} = 0.2\mathrm{H}, \quad \omega = 10 \text{ rad/s}$$

$$\mathbf{Z}_L = j\omega L$$
$$\mathbf{Z}_L = j \times 10 \times 0.2$$
$$\mathbf{Z}_L = j2$$



- So, the input impedance is

$$\mathbf{Z} = R + \mathbf{Z}_L$$
$$\mathbf{Z} = 4 + j2\ \Omega$$

Practice Problems

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Input Impedance

- Prob 1:
  - Given a sinusoid $50\sin(6\pi t - 50°)$, what is the angular frequency ω and frequency f.
- Solution:
  - We know that the general formula for a sinusoid is

$$v(t) = V_m \sin(\omega t + \varphi)$$

  - Therefore, the angular frequency is

$$\omega = 6\pi \text{ rad/s} \qquad \text{----- A}$$

  - To get the frequency f in Hz:
  - Since $\omega = 2\pi f$, substitute this in eq. A above
  - $2\pi f = 6\pi$
  - $f = \frac{6\pi}{2\pi}$
  - $f = 3\text{Hz}$

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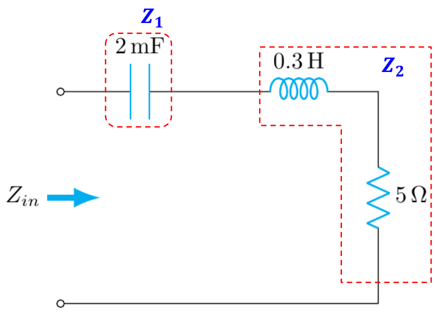
Input Impedance

- Prob 2:
  - Find the input impedance of the circuit. Assume that the circuit operates at $\omega = 50$ rad/s.



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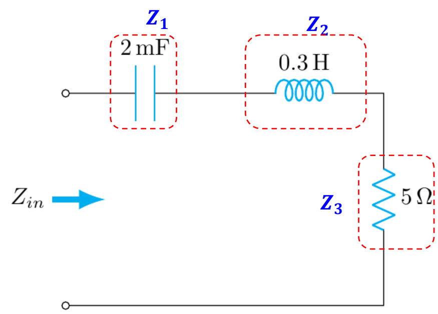
Input Impedance

- Prob 2. (Solution):
  - Find the input impedance of the circuit. Assume that the circuit operates at $\omega = 50$ rad/s.
  - Solution:

$$Z_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\Omega$$
$$Z_2 = j\omega L = j50 \times 0.3 = j15\Omega$$
$$Z_3 = 5\Omega$$

Therefore:

$$Z_{in} = Z_1 + Z_2 + Z_3$$
$$Z_{in} = -j10 + j15\Omega + 5$$
$$Z_{in} = (5 + j5)\Omega$$



  - Note that this impedance is presented by the circuit at frequency ω=50 rad/s and it will change at any other frequency

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