INTRODUCTION TO LINEAR ALGEBRA

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Welcome Message and First Homework Assignment

The origin of the word "algebra" is Arabic. The Arabic word "al-jabr" means "the reunion of broken parts" and the study of algebra is the study of relationships between parts and pieces. The word "linear" means "line-like." To study linear algebra is to study line-like relationships. The purpose of this book is to arm a student with introductory linear algebra knowledge. My sincere hope is that it inspires a student to make connections between pieces of knowledge in order to walk away from the course feeling as though things made sense, that everything is connected, that, in brief, a story can be told.

This book is written with you, the student in mind, as you are growing up in a rapidly changing world, where data science and machine learning are opening up a very interesting and exciting time for human beings, but also frightening to many who have knowledge of what a new technology is in terms of its parts. Due to such change, and also growth in many ways of the subject of linear algebra itself, I fully support a brand new approach to the teaching and learning of introductory linear algebra: that **the subject is contained in five factorizations of a matrix**, listed below, that will be explained once you have reached a middle point in this book. (Reaching that point has proved to me to be the best time to explain them as many students, including ones who have taken the course more than once, have learned the skills that all students learn in any beginning linear algebra course.

$$
\begin{aligned}
A &= CR \\
A &= LU \\
A &= QR \\
S &= Q\Lambda Q^T \\
A &= U\Sigma V^T
\end{aligned}
$$

A brief justification for this is easy to find: factoring decomposes something so we can understand and perhaps rearrange a complex system. We are skilled at giving students an algorithmic procedure, a recipe to follow, to solve the equation $x^2 - 3x = 10$, one of which would be (1) get zero on the right side, (2) factor the polynomial, and (3) notice the information the factored polynomial gives you. Why not go a little further, speak it using language from science: We have a system $x^2 - 3x$, and a desired result of 10. So we would want the difference between the system and the desired result to be as small as possible; that is, we want our error to be zero. That's better than commanding an entire classroom to get a zero on one side: give them a *reason why*. Then factor. Factoring makes it easy to analyze when the system will fail or succeed.

If you are enrolled in this course, then you are familiar with functions of a single variable, and have used the word *linear* in a geometric way: a function whose graph is a line is often referred to as linear. In this course, however, we use this word much more carefully: *Linear* is used in an algebraic way: A function $f(x)$ is **linear** if, for any two elements in its domain x_1 and x_2, and for any scalar (real number) k, the following two conditions hold:

$$
\begin{aligned}
f(x_1 + x_2) &= f(x_1) + f(x_2) \\
f(kx_1) &= kf(x_1)
\end{aligned}
$$

These two conditions have names: *additivity* and *homogeneity.* Together, these properties are often called the principle of superposition, which enables us to write the two properties as one: For any two elements in its domain x_1 and x_2, and for any scalars (real numbers) k_1 and k_2,

$$f(k_1x_1 + k_2x_2) = k_1f(x_1) + k_2f(x_2).$$

This is what characterizes linear functions, also known as linear operators. You are familiar with two linear operations: *differentiation* and *integration.* Notice how the constants migrate outside of the operation, while the operation can happen term by term:

$$\frac{d}{dx}(3x_2 + 4x) = 3\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x)$$
$$\int (3x^2 + 4x)\,dx = 3\int x^2\,dx + 4\int x\,dx$$

From where you are at this moment, perhaps meeting this algebraic definition of *linear* for the first time, $f(x) = 3x$ is linear, and $g(x) = 3x + 2$ is not linear. (Reason why by noticing $g(1) = 5$ and $g(2) = 8$, so $g(1) + g(2) = 13$, but $g(1 + 2) = g(3) = 11$; that is, the first property does not hold. Try the second on your own.) It turns out that in the familiar xy-plane, all functions of the form $f(x) = ax$, where a is any real number, are the true linear functions. We can say ax is "linear" and study the algebra of linear equations when we ask the following question: For what values of x is $ax = b$? Three possible cases arise when considering the solution to $ax = b$: one and only one solution $x = a^{-1}b$, if a has an inverse (all real numbers have an inverse except for zero), no solution if $a = 0$ and $b \neq 0$, and infinitely many solutions if both a and b are zero.

$3x = 6$	one solution $x = 2$
$0x = 0$	infinite solutions $x = \mathbb{R}$
$0x = 4$	no solution

Consider $f(x) = ax$ and $g(y) = by$. The function ax is linear in x, and the function by is linear in y, and we can combine linear functions by writing $ax + by$, which is called a *linear combination* of x and y. The expression $ax + by + cz$ is a linear combination of x, y, and z. The letters a, b, and c are taken to be any real numbers. The set of all linear combinations of x and y is $ax + by$ where you are thinking of a and b as taking on every possible value. The first ten problems on the following page are *systems of linear equations*, and all of you should be familiar with solving systems of linear equations. Therefore, welcome to an introductory course in linear algebra. We begin with action: solve the following ten systems of equations using prerequisite knowledge acquired from your study of high school algebra. Use any method you wish. Solving systems is where this subject begins. This is the central problem of linear algebra. Welcome to the course.

Problems 1-10: Prerequisite Knowledge

1.

$$\begin{aligned} x + 3y &= 2 \\ x + 4y &= 4 \end{aligned}$$

2.

$$\begin{aligned} x - 2y &= 1 \\ 3x - 4y &= 1 \end{aligned}$$

3.

$$\begin{aligned} x + 2y &= 3 \\ x - y &= -3 \end{aligned}$$

4.

$$\begin{aligned} x - y &= 2 \\ x + y &= 4 \end{aligned}$$

5.

$$\begin{aligned} x + y &= 6 \\ x - y &= 2 \end{aligned}$$

6.

$$\begin{aligned} x + 3y &= 2 \\ 2x + 6y &= 1 \end{aligned}$$

7.

$$\begin{aligned} x - 2y &= -5 \\ -3x + 6y &= 15 \end{aligned}$$

8.

$$\begin{aligned} x - y + z &= 1 \\ x - z &= 0 \\ x - y + 2z &= 2 \end{aligned}$$

9.

$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 3 \\ x + 2y - z &= 1 \end{aligned}$$

10.

$$\begin{aligned} x - y + z &= 1 \\ x + y - z &= 1 \\ 2x - y + z &= 2 \end{aligned}$$

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Chapter 1

THINGS TO KNOW

1.1 WHY IS THIS COURSE DIFFERENT FROM OTHER MATHEMATICS COURSES?

Welcome to a new level of mathematical work. You are no longer in a course focused solely on computational work, and you have reached a level of study that invites you to evolve and become more intellectually curious, inventive, creative, careful, and thoughtful, not simply precise. Being precise characterizes mathematics, as usual, but answering the question "*why does this work?*" makes a mathematician.

This course requires you to think and to communicate, in both writing and speaking, beyond the level of computational work, challenges you to make connections among concepts, and encourages you to understand that mathematics is about the art of generalization.

If you were to return to any of the mathematics courses which you have already studied, you might consider studying them with more critical thought. Here are a few random questions from your prior mathematical work to think about, which I hope might steer you in the direction of more profound thought regarding *this* course:

(a) Why are the Hindu-Arabic numerals 1, 2, 3, ... written in the way they are written? How did these numbers come to take their form and shape?

(a) Do you know the origin stories of any of the symbols you use, such as the "is equal to" symbol as used in $2 + 3 = 5$, or "the square root" symbol as used in $\sqrt{4} = 2$, or any other symbol you have met in mathematics?

(c) Think about the operation of addition and how you have generalized this concept: Addition is repeated counting. Multiplication is repeated addition: $2+2+2 = 3 \cdot 2$. Exponentiation is repeated multiplication: $2 \cdot 2 \cdot 2 = 2^3$. Why stop there? Tetration is repeated exponentiation: $2^{2^2} = {}^3 2$. You can keep going, too: there is pentation, hexation, heptation, octation, enneation, decation, vigintation, trigintation, centation, docentation, myriation, circulation... I'm not kidding. Is this not fun to know? These are called **hyper-operators** in mathematics.

(d) Speaking of exponentiation, I assume all of you are comfortable with seeing $2^2, 2^3$, and 2^4 and understanding what these powers of a number mean. So why on earth is $2^{-1} = \frac{1}{2}, 2^{-2} = \frac{1}{4}$, and $2^{-3} = \frac{1}{8}$?

(e) Why is 0!=1?

(f) You began learning calculus as a child: Calculus without limits is called *arithmetic*. arithmetic with limits is called *calculus*. What do I mean by this?

(g) Speaking of calculus, integration is addition with limits, and differentiation is subtraction with limits. Are you aware that you learn introductory calculus kind of backwards in that, historically speaking, integration came way before differentiation? What is so special about Newton and Leibniz?

(h If a line that touches a curve in one and only one point is called *tangent* and a line that cuts through a circle in two points is called *secant*, why are these words used in trigonometry as function names like $\tan\theta$ and $\sec\theta$?

(i) Linear algebra will generalize two statements you made as a child: Sentences such as "*Two points determine a line*" and "*Three non-collinear points determine a plane*" turn into "*One vector determines a line*" and "*Two linearly independent vectors determine a plane.*" The big generalization of these concepts is summed up in a single sentence: *A set of n linearly independent vectors determines an n-dimensional subspace.*

There are a lot more things that could be added to this list, all of which enhance a deeper thinking of the subject under which such questions could arise. These often delight students and bring a greater humanity to the subject of mathematics. Why is it that we leave out these stories when we teach and learn mathematics? (That's a question for any instructor of mathematics, but also one for any of you who might happen to make the courageous decision to one day teach it yourself!)

I dare you to abandon being obedient, only after thinking about when it is necessary to do so, and attempt to establish within yourself the art of play. Ask why when you can. I dare you to seek joy, or some sense of joy, as you explore this subject. Education, in my opinion, can sometimes chase the joy and fun out of learning mathematics: Our exam culture, with its system of rewards and punishments, ruins the party. I invite you to consider the idea that to truly learn and master anything, the art of play must be present.

1.2 WHAT ARE SCALARS?

The term *scalars* refers to numbers, and these numbers are initially limited to *real numbers*. Some of you have probably encountered *complex numbers*, and these are also scalars in some settings. Scalars will be denoted by lowercase, italicized letters and have the following properties:

A: To every pair of scalars a and b, there corresponds another scalar $a+b$ called the **sum** of a and b such that

1: $a+b=b+a$,

2: $a+(b+c)=(a+b)+c$,

3: there is a unique scalar called *zero*, denoted by 0 such that $a+0=a$ for every scalar, and

4: to every scalar a there corresponds a unique scalar $-a$ such that $a+(-a)=0$.

B: To every pair of scalars a and b, there corresponds another scalar ab called the **product** of a and b such that

1: $ab = ba$,

2: $a(bc) = (ab)c$,

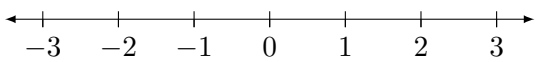
3: there exists a unique scalar 1 called *one* such that $1a = a$ for every scalar a, and

4: to every scalar that is not zero, there corresponds a unique scalar a^{-1} such that $aa^{-1} = 1$.

C: Multiplication is *distributive* with respect to addition: $a(b + c) = ab + ac$.

Given a set of scalars, if addition and multiplication are defined in such a way that all of the conditions above are satisfied, then that set is called a **field**. This is why you often hear someone mention they have a *field of scalars* in mathematics.

The set of all real numbers is denoted by $\mathbb{R}$. This is the field of scalars for this introductory course, meaning that anytime we refer to quantities known as scalars, they are real numbers. The geometric model of $\mathbb{R}$ is the real number line you developed long ago in elementary school: a line drawn horizontally, with a point known as the origin. To the left of the origin are the negative real numbers. Located to the right of the origin are the positive real numbers.



Hopefully you recognize the properties of a field as ones which you have encountered in a beginning algebra course: Both operations are commutative, associative, have an identity element, and an inverse element, and multiplication is distributive with respect to addition.

1.3 WHAT ARE VECTORS?

The term *vectors* refers to things that can be added together. Most of you have used *Euclidean vectors* in physics and mathematics: these are the directed line segments with a magnitude (length) and direction. These vectors indicate a displacement: something is carried from one location to another location. These are not the only kinds of vectors: Polynomials are vectors. Matrices are vectors. Examples abound! Vectors are elements of a space known as a **vector space**. This is the main concept of the course, one in which we do not refine until after several introductory chapters. Vectors are denoted by bold non-italicized letters. A **vector space** is a set V of elements called vectors satisfying the following axioms:

1: To every pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V, the sum $\mathbf{x} + \mathbf{y}$ is also a vector in V,

2: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$,

3: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$,

4: there is a unique vector called the *zero vector* $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$,

5: to every vector $\mathbf{x}$ there corresponds a vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$,

6: for any scalar a and any vector $\mathbf{y}$ in V, the product $a\mathbf{y}$ is also a vector in V,

7: $a(b\mathbf{x}) = (ab)\mathbf{x}$ for any scalars a and b,

8: $1\mathbf{x} = \mathbf{x}$,

9: $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$, and

10: $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

This course will focus on **Euclidean vector spaces**, ones in which you can imagine vectors to be directed line segments from one point to another. The field of scalars $\mathbb{R}$ is itself a vector space, but we will treat it separate from the vector spaces we will study. You are no doubt familiar with the xy-plane, also known as $\mathbb{R}^2$, formed by two real lines (axes), one orthogonal to the other, both meeting at the origin. $\mathbb{R}^2$ is the set of all vectors with two components. Ordered pairs (x, y) have coordinates x and y. Ordered pairs establish a location. The ordered pair $(2, 3)$ indicates a location at which you arrive if you move two units in the positive x-direction and three units in the positive y-direction. The vector from the origin $(0, 0)$ to the location $(2, 3)$ can be thought of as the directed line segment drawn from the origin to $(2, 3)$. If you took calculus, you distinguish between location and displacement in your written work: The location is $(2, 3)$ and the vector is $\langle 2, 3\rangle$. In this class, the two are the same thing; that is, we refer to (x, y) as a vector and do not use the calculus and physics notation. Here are the definitions of Euclidean vector spaces:

$\mathbb{R}^2$: the set of all vectors with two real number components (x_1, x_2).

$\mathbb{R}^3$: the set of all vectors with three real number components (x_1, x_2, x_3).

$\mathbb{R}^4$: the set of all vectors with four real number components (x_1, x_2, x_3, x_4).

... etc....

$\mathbb{R}^n$: the set of all vectors with n real number components $(x_1, x_2, ..., x_n)$.

When n is finite, you have a finite vector space, but it is possible to have a vector space with an infinite number of components. This is denoted by $\mathbb{R}^\infty$ and seems confusing to think about at first, but here is something that helps: if you studied sequences and series, it was often the case that you had an infinite sequence. A sequence is an ordered list of numbers. The catch: **A sequence is a vector!**

VECTOR SPACE	DESCRIPTION
$\mathbb{P}_1$	all polynomials of the form $a_0 + a_1x$
$\mathbb{P}_2$	all polynomials of the form $a_0 + a_1x + a_2x^2$
$\mathbb{P}_3$	all polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$
$\mathbb{M}_{2\times 2}$	all matrices with two rows and two columns.
$\mathbb{P}_{\not\mathbb{R}}$	hey

Table 1.1: A Few Non-Euclidean Vector Spaces

Vectors are defined by two operations: **vector addition** and **scalar multiplication**:

DEFINITION 1: Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors in $\mathbb{R}^n$ and let k be any scalar. **Vector addition** is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n),$$

while **scalar multiplication** is

$$k\mathbf{v} = (kv_1, kv_2, ..., kv_n).$$

It is this definition that truly characterizes a vector. **Table 1.1** lists some examples of vectors from vector spaces that you may not have ever thought of as vectors; however, note the familiar mathematical moves you make and how these moves relate to this first definition in the following examples:

(1) In $\mathbb{R}^2$, $(1,3) + (2,4) = (1+2, 3+4) = (3,7)$ and $4(1,2) = (4\cdot 1, 4\cdot 2) = (4,8)$. If you have taken calculus and physics, you often see and denote an addition like this using the basis vector notation: If $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} + 4\mathbf{j}$, then $\mathbf{v} + \mathbf{w} = 3\mathbf{i} + 7\mathbf{j}$. If $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$, then $4\mathbf{u} = 4\mathbf{i} + 8\mathbf{j}$.

(2) In $\mathbb{R}^3$, $(1,5,-1) + (2,-7,3) = (3,-2,2)$ and $-3(1,2,-3) = (-3,-6,9)$.

(3) In $\mathbb{R}^4$, $(1,-1,0,3) + (3,2,1,7) = (4,1,1,10)$ and $5(1,2,3,2) = (5,10,15,10)$.

(4) The polynomials $1 + 3x$ and $2 + 4x$ are from the vector space $\mathbb{P}_{\not\mathbb{R}}$. Notice that $(1+3x) + (2+4x) = (1+2) + (3x+4x) = 3 + 7x$ and $4(1+2x) = 4 + 8x$. In beginning algebra, you say you "add like terms" or "add similar terms" when adding polynomials. When multiplying a polynomial by a real number, you use the distributive property. This is vector addition and scalar multiplication. Note the same mathematics is executed in (1) above.

(5) Add $1 + 5x - x^2$ to $2 - 7x + 3x^2$ and find $-3(1 + 2x - 3x^2)$. Again, look at the mathematics done in (2) and compare the computations to the ones here. What is the difference?

(6) Matrices are rectangular arrays of numbers that you will see arise naturally from solving systems of equations. You will see them originate from the work we will do beginning in the next several chapters. The vector space of matrices $\mathbb{M}_{2\times 2}$ are all matrices with two rows and two columns, ones of the following form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In many books, there is a special notation for each element that appears in a matrix, a subscript notation ij where the first number i denotes the row the element appears and the second number j denotes the column in which the element appears. In this text, square brackets will be used to denote matrices, and when the matrix itself is referred to, an italicized capital letter is used to denote the matrix. Here are two matrices A and B using these notations:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Matrices are vectors because you can add them and scalar multiply them. If k is any scalar (any real number), then vector addition and scalar multiplication for matrices are as follows:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

1.4 HOW DO YOU WRITE AND TYPE SCALARS AND VECTORS?

In courses where students are using vector quantities for the first time, I advise using proper notation to distinguish between scalars and vectors. It is astonishing to see a growing number of students who dismiss this, ignore it, and could care less. What this often results in, especially in calculus, is complete confusion within problems over *location* and *displacement*. While mathematics can be a challenging subject to master, one of the reasons why students find it confusing at times is because little attention is given to how they communicate what they do in a problem. Accurate answers are important to reach in many problems you encounter in mathematics, which is a goal, of course; however, what many of you will find makes you valuable in a number of future jobs in the mathematical sciences is what a lot of people look for and treasure: GOOD COMMUNICATION SKILLS. In recent years, I have seen a ***lack of attention to the detail involved in writing mathematical notation.*** WRITING USING PROPER NOTATION THAT IS CLEAR AND EASY TO READ IS A CHOICE YOU MAKE AS A STUDENT OF THIS SUBJECT. I cannot express to you how many times I have sat down with a student who is lost in the subject, only to realize that once the student pays attention to how it is written, they begin to understand it much better. If you become confused in your work, consider the thought and care you put into your written communication of the mathematics you write. Have a high bar for yourself. Consider adding this question when you sit down to do homework: *What kind of experience will another human being have who reads my work?* Write as if you are communicating to someone else who is at your level or just beneath your level of mathematics, so your writing teaches them the mathematics. Write in such a way that you do not cause that reader of your work to do work in order to understand you. This can be challenging in this subject, and it is worth making an effort to be as clear as you can in your written communication. It will only benefit you and most likely help you understand the subject better. An answer to a problem is often an

essay using the notation which acts as a kind of shorthand for how you speak it. You will find it important to mentally distinguish between vector quantities and scalar quantities. And if you write mathematics or speak it, I believe you should have others in mind who must read what you write or hear what you speak. Clarity goes a very long way, and you will either be appreciated for it, or you will frustrate and confuse others.

(1) Vectors, when typed in this text, are denoted by bold, non-italicized, lowercase letters:

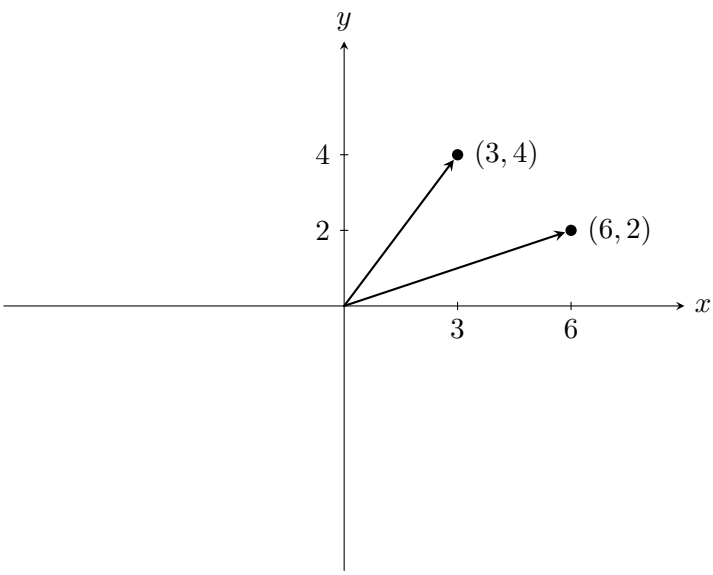
$$\mathbf{v}$$

(2) Vectors, when handwritten by your instructor in the lecture, are denoted by non-italicized lowercase letters with an arrow above them:

$$\vec{\mathrm{v}}$$

(3) When scalars are represented by letters, they are represented by italicized lowercase letters when both typed and handwritten: $k\mathbf{v}$ when typed, or $k\vec{\mathrm{v}}$ when handwritten, where k is the scalar. (You can *see* it is a scalar.)

(4) In this class, Euclidean vectors will be written in a number of different ways. Most of the time vectors will be written primarily as matrices. You will hear "column vector" spoken often. Below are two vectors in $\mathbb{R}^2$.



Both are drawn in what is called **standard position**, meaning from the origin to whatever point they each would carry a particle if the initial point was the origin. If you studied calculus, you denoted these vectors, most likely as $3\mathbf{i}+4\mathbf{j}$ and $6\mathbf{i}+2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the positive x and y directions; that is, $3\mathbf{i}+4\mathbf{j}$ means move three units to the right and four units up. You also denoted these

vectors as $\langle 3,4\rangle$ and $\langle 6,2\rangle$. This latter notation distinguishes these vectors from the locations $(3,4)$ and $(6,2)$. In *this* class, all of these notations are considered synonymous and we do not use the notation $\langle 3,4\rangle$. In fact, we consider $(3,4)$ to be the **space-saver notation** for a vector because we will write vectors in matrix form as one-column matrices. In other words, vectors will be written in the first two of the four forms you see below: For the vector from the origin to the point $(3,4)$, we have the column vector and the space-saver notation for that column vector, which equates to the second two forms with which many of you are familiar from calculus.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = (3,4) = \langle 3,4\rangle = 3\mathbf{i} + 4\mathbf{j}$$

To be clear about the names of each notation, the above vector, written four times from left to right, is first written as a single column matrix, then in the space-saver notation, then in the component vector notation, and finally in the standard basis vector notation.

It will probably be useful for you to think of all vectors in Euclidean vector spaces as column vectors, based on the mathematics you have yet to learn in this class, which involves how these vectors are seen and used in matrices. Below are column vector notations for some general vector $\mathbf{x}$ in some of the Euclidean vector spaces:

If $\mathbf{x}$ is from $\mathbb{R}^2$, then it is of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

If $\mathbf{x}$ is from $\mathbb{R}^3$, then it is of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

If $\mathbf{x}$ is from $\mathbb{R}^4$, then it is of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

In general, if $\mathbf{x}$ is from $\mathbb{R}^n$, then it is of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

(5) It is nice to have some kind of an outline or a "big picture" that kind of summarizes everything you will be learning in a course. This is sometimes challenging to do in mathematics courses, but this has helped some students in the past with clarity, especially when things get confusing. You all know how to solve equations of the form $ax = b$, where a and b are known real numbers and x is the unknown variable. For example, $3x = 6$ has the unique solution $x = 2$. It is fairly easy to come to terms with the three possible solutions for all equations of the form $ax = b$: There is one and only one solution if a has an inverse a^{-1}: The solution is $x = a^{-1}b$. There are an infinite number of solutions if both $a = 0$ and $b = 0$. There is no solution if $a = 0$ and $b \neq 0$. This is a first part of understanding the nature of all equations of the form $ax = b$: finding all values of x that satisfy the equation. A second part of understanding equations of this form is to look at the left side ax as a **function** $f(x) = ax$. As you will see in the last section of this chapter, all functions of the form $f(x) = ax$ are **linear transformations**. Keep this simple idea in mind, pretending that there is a course called BEGINNING BEGINNING LINEAR ALGEBRA out there somewhere that is broken up into these two parts:

BEGINNING BEGINNING LINEAR ALGEBRA

1. PART 1: All solutions to $ax = b$
2. PART 2: All functions $f(x) = ax$

Indeed, this may provide many of you with a simple idea for how to get back on track if you happen to become overwhelmed or confused. Any introductory course in linear algebra can be broken up into two similar parts:

INTRODUCTORY LINEAR ALGEBRA (MATH 208)

1. PART 1: All solutions to $A\mathbf{x} = \mathbf{b}$
2. PART 2: All functions $T(\mathbf{x}) = A\mathbf{x}$

What you most likely do not yet know, however, is that $A\mathbf{x} = \mathbf{b}$ is a matrix equation that represents a system of linear equations, and that $T(\mathbf{x}) = A\mathbf{x}$ represents what are called linear transformations.

1.5 WHAT IS FUNCTION, DOMAIN, CODOMAIN, IMAGE, RANGE?

The concept of *function* is so central to understanding mathematics beyond computational work. A function has a set of inputs, and produces a set of outputs with the following condition: *For each input, there is one and only one output.* Functions you have seen are denoted by a name, followed by a set of parentheses inside of which is a symbol representing any input, followed by an "is equal to" symbol, and followed by a symbol representing any output or the rule for how the inputs are transformed:

$$f(x) = y \tag{1.1}$$
$$f(x) = x^2 \tag{1.2}$$
$$f(x, y) = \sqrt{x - 3y} \tag{1.3}$$
$$f(x_1, x_2, x_3, x_4) = (x_2 + 2x_4, 6x_1 - x_2 + x_3) \tag{1.4}$$

The **domain** of a function is the set of all inputs. The **range** of a function is the set of all outputs. The **codomain** of a function is the space in which its outputs are located. For the function $f(x) = x^2$, its domain is $\mathbb{R}$, its codomain is $\mathbb{R}$, and its range is all non-negative real numbers, sometimes more formally written in the set-builder notation as $\{x \in \mathbb{R} : x \geq 0\}$ or the interval notation $[0, \infty)$. The **image** of an element is the element to which it is mapped, so 4 is the image of 2 since $f(2) = 4$, 9 is the image of 3 since $f(3) = 9$, etc... Later in the course, we refer to functions as *mappings,* or *transformations*, as in f maps 2 to 4, or f transforms 3 to 9. The domain and the codomain are often presented as important spaces when a function is introduced. This function $f(x) = x^2$ maps the set of real numbers into the set of real numbers, and this is often written $f : \mathbb{R} \to \mathbb{R}$.

This is a class of functions with which most of you are familiar, real-valued functions of one variable. Those of you who have studied mutlivariable functions are familiar with the example in (1.3), a function mapping points in the plane to real numbers. What is its domain? range? What is the image of $(2, -1)$? What is its codomain? The function in (1.4) is a function we will study in this course: it is a linear transformation. Later in the course, we will introduce a linear transformation like this in a formal way: Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x_1, x_2, x_3, x_4) = (x_2 + 2x_4, 6x_1 - x_2 + x_3)$. The domain of this function is $\mathbb{R}^4$. The codomain is $\mathbb{R}^2$. These are two important spaces to identify for linear transformations. We have quite a lot of material to cover before we study linear transformations, but it certainly does not hurt to let you know the stronger meaning of the word *linear* based on the functions you all know well from your study of algebra and beginning calculus, which is the focus of the next section.

1.6 WHAT FUNCTIONS $f(x)$ ARE LINEAR?

In this class, you must abandon a careless use of the the word *linear* and recognize that it is used in two ways in mathematics: you often refer to functions whose graphs are lines in the plane as linear functions, but this is a *geometric* use of the word. This class introduces an *algebraic* use of the word. The function $f(x) = 2x+6$ is not linear. The function $f(x) = 2x$ is linear. Here is the definition of the word when the class of functions is restricted to real-valued functions of a single variable: A function $f(x)$ is **linear** if it satisfies two properties: For any two points x_1 and x_2 in the domain of f, and for any scalar k, **(a)** $f(x_1+x_2) = f(x_1)+f(x_2)$ and **(b)** $f(kx_1) = kf(x_1)$. Property **(a)** is known as *additivity* and property **(b)** is known as *homogeneity.* These are usually spelled out together in other courses as *the principle of superposition*, which is $f(k_1x_1 + k_2x_2) = k_1f(x_1) + k_2f(x_2)$. You have all taken calculus and are familiar with the operations of differentiation and integration. Notice that these are operations that obey the principle of superposition. Are they linear operations? What do you think? (Yes, they are...) In the case of functions of one variable, all linear functions are of the form $f(x) = ax$, where a is any real number. The functions $f(x) = ax$ and $g(y) = by$ are linear functions. Combining them together means adding them, so why not call it a linear combination?

1.7 WHAT ARE LINEAR COMBINATIONS?

A linear combination of x and y is $ax+by$, where a and b are scalars. A linear combination of $\mathbf{u}$ and $\mathbf{v}$ is $a\mathbf{u} + b\mathbf{v}$. A linear combination of ω and ϵ is $a\omega + b\epsilon$. The ten systems you solved in problems 1-10, the opening exercises in this text, are systems involving linear combinations of x and y or linear combinations of x, y, and z, so we refer to them as systems of linear equations.

Chapter 2

Euclidean Vectors in the Plane

Over the course of the next several weeks, you will learn that in mathematics, the word "vector" does not always mean "directed line segment with magnitude and direction," despite that being the most popular visual to associate with a vector. Even so, we begin by looking at directed line segments in the xy-plane. It is my understanding that a majority of students enrolled in this course have some experience with these vectors. When vectors are represented by directed line segments, they are usually referred to as **Euclidean vectors**. If you have experience in multivariable calculus, the algebra of Euclidean vectors is the pre-calculus of multivariable calculus. Vectors are the tools that help extend and redevelop the concepts of differential calculus in the plane. What are vectors, then, in linear algebra? They are elements of what we will soon learn is a **vector space**, a concept we will develop carefully. There is a level of abstraction required, but we always have the usual Euclidean vectors, the directed line segments, in mind. This chapter serves as a place to take care of the basics of vectors. We consider only two-dimensional vectors in the plane; that is, vectors with two components. To those of you from calculus, it is important to notate vectors carefully in that subject, and distinguish vectors (directed line segments) from ordered pairs (locations). In this class, we can relax a bit with notation. The notation (x, y) reads as an ordered pair. Ordered pairs have **coordinates** while vectors have **components**. Typically, we know in linear algebra when we are referring to vectors, and when we write (x, y) we most likely will call this a vector. *What is important* is that we will notate vectors by the single letter names they often have: You are reading something that is typed, and you will recognize "the vector vee" as $\mathbf{v}$, a non-italicized, bold, lowercase letter. If you see v, then you can most likely assume it represents a real number, or a **scalar**. Vectors are really characterized by two important operations: **vector addition** and **scalar multiplication**. These operations are defined in the following bulleted list of important things you need to know about vectors in the plane.

- The word "vector" is from the Latin word meaning "carrier" which gives meaning to Euclidean vectors, as they can be thought of as carrying something from one point to another. Vectors have no location. They can be moved anywhere in the plane as long as their magnitude and direction remain the same. Think of all vectors as things that can be housed in what is called **standard position**: the position in which the tail of a vector is at the origin and its tip (the arrow) landing on the ordered pair used to name the vector. In this class, we will use several notations, but two popular notations: the **space-saver notation** $\mathbf{v} = (a, b)$ to represent the vector we can imagine is in standard position, the directed line segment from the origin to the ordered pair (a, b). (In calculus, where it is important to distinguish a vector from a location in the plane, you use the notation $\langle a, b\rangle$ to denote the vector carrying a particle from $(0, 0)$ to (a, b). Ordered pairs have **coordinates**. Vectors have **components**: In the plane, vectors have an x-component and a y-component. The favorite notation for vectors in this course will be **column vector notation**:

The vector $\mathbf{v} = (a, b)$ can be written as a column vector:

$$\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- To clarify, when you see the following typed letters, you can tell which of the six letters represent real numbers (scalars) and which ones represent vectors: $\mathbf{v}, v, x, \mathbf{x}, y.\mathbf{y}$; that is, the first, fourth, and sixth item in that list are vectors and the rest are scalars. Clear notation goes a long way in terms of understanding the mathematics.

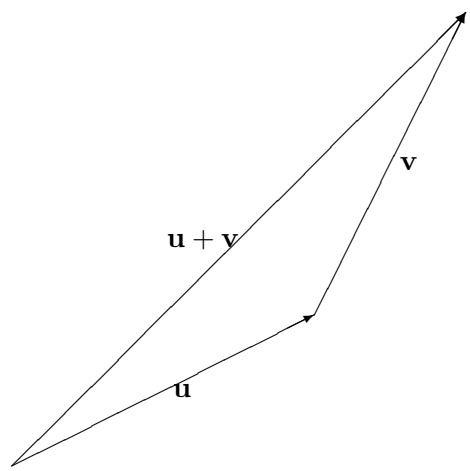
- In a strict mathematical sense, the only operations vectors are required to possess are addition and scalar multiplication. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be any two vectors in $\mathbb{R}^2$. Let k be any scalar (any real number). **Vector addition** is defined in the following way:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

  Scalar multiplication is defined in the following way:

$$k\mathbf{u} = (ku_1, ku_2)$$

- These two operations can be defined graphically: to add $\mathbf{u}$ and $\mathbf{v}$, sketch $\mathbf{u}$ in standard position, thinking that $\mathbf{u}$ carries a particle from the origin to the point (u_1, u_2), and then *from this point*, the vector $\mathbf{v}$ will carry the particle to its final location. This is much easier done, of course, in pictures.



11. Let $\mathbf{u} = (3, 4)$ and let $\mathbf{v} = (6, 2)$. Sketch both vectors in standard position in the xy-plane.

12. For the two vectors in the previous problem, find $\mathbf{u} + \mathbf{v}$. Then use your sketch to demonstrate this vector addition graphically.

13. Illustrate the following vectors, given that $\mathbf{u} = (2, 1)$: $2\mathbf{u}$, $-\mathbf{u}$, $4\mathbf{u}$, and $-4\mathbf{u}$. Then articulate what it means to multiply a vector by a real number. (*Why do we call them scalars? If something is called that name, does it scale something?*)

14. Let $\mathbf{u} = (-1, 3)$ and let $\mathbf{v} = (2, 5)$. Sketch the vectors in standard position on the same coordinate plane. Sketch the parallelogram determined by the two vectors. Show that $\mathbf{u} + \mathbf{v}$ snd $\mathbf{u} - \mathbf{v}$ can both represent the diagonals of the parallelogram.

15. Suppose, instead of a directed line segment, you sketched $\mathbf{u} = (3, 2)$ as a point. Suppose you did this with all possible scalar multiples of the vector $\mathbf{u}$, which can be represented by $k\mathbf{u}$. where k is any real number. The dots you make will "cover a space" in the plane. What space will all of the dots cover?[1]

16. **True** or **false**? All scalar multiples of the vector $\mathbf{u} = (2, 4)$ represents the set of all vectors on the line $y = 2x$.

17. **True** or **false**? All scalar multiples of $\mathbf{u} = (-3, -5)$ is a description of the set of all vectors of the form $(x, 5x/3)$.

18. **True** or **false**? All scalar multiples of $\mathbf{u} = (-3, -5)$ is a description of the set of all vectors of the form $(3a, 5a)$.

19. Let $\mathbf{u} = (4, -3)$ and $\mathbf{v} = (-2, -1)$. Find $2\mathbf{u} + 3\mathbf{v}$.

20. Let $\mathbf{u} = (4, -3)$ and $\mathbf{v} = (-2, -1)$. Find $3\mathbf{u} - 6\mathbf{v}$.

Perhaps you are wondering why this book began with a homework assignment containing ten systems of linear equations, with no solutions or answers provided, and then proceeded quickly into a chapter devoted to Euclidean vectors in the plane. This last part of the chapter builds a strong connection that some find surprising. The fundamental problem of linear algebra is to solve a system of linear equations. And vectors are fundamental to this work. Your job is to do something many of us find challenging at times, and that is to look at things with a different perspective. This last section is one I find crucial for students to understand if they are to understand the basics of linear algebra. You will be required to distinguish between a **row** perspective and a **column** perspective when examining a system of linear equations, such as first ten problems of the course, which we will refer to often in the nest several weeks of work.

If $\mathbf{v}$ is any nonzero vector in the plane, and k is any scalar, then $k\mathbf{v}$, all scalar multiples of the vector, is always a line through the origin. Extending this idea, suppose we take all scalar multiples of two vectors that do not point in the same direction, and combine them by vector addition: Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in the plane. If k_1 and k_2 are any two real numbers, then $k_1\mathbf{u} + k_2\mathbf{v}$ is called a **linear combination** of $\mathbf{u}$ and $\mathbf{v}$. Notice that in homework problems 19 and 20, you were asked to work with two linear combinations of the two vectors introduced in those problems. Perhaps the most popular problem across all the mathematical sciences is solving a mystery surrounding linear combinations of vectors.

We now make our first major connection, and something all students should see on a first day of a course in linear algebra: there is another way to look at a system of linear equations. Consider two ways of looking at the kind of system in which every student has experienced in algebra: two equations and two unknown variables; that is, two lines in the plane. This is the **row** perspective, one in which you are familiar: you view one equation on a first line of writing, followed by another equation on a second line. Consider those rows 1 and 2.[2]

[1] Look up the definition of the word **span** in the dictionary. This is where the concept of *span* in linear algebra comes from.

[2] I claim that you don't really *know* a concept until *you understand it from at least two perspectives*, and I would like to call that *empathy*.

The **column** perspective is another way of viewing a system. Consider the following system:

$$x + 3y = 5$$
$$x - 2y = 0$$

The column perspective is to see the system as a linear combination of two vectors. Instead of the problem being about where two lines intersect, the problem becomes what two scalars x and y are required in order to scale the two vectors so that they add to the vector on the right:

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Imagine going back in time and learning vectors in grade school, then meeting systems of linear equations such as this one, and solving it by using the instructions given to you in the next homework problem.

21. On a piece of graph paper, sketch the two vectors $(1, 1)$ and $(3, -2)$ in standard position. Sketch the vector $(5, 0)$ in standard position, but in a different color or using a dotted line. Next, and try this by trial and error, or by thinking carefully with close inspection: is there a way to scale the two vectors so they add to produce $(5, 0)$?

22. Note the difference in the interpretation of the system in the previous problem: The row perspective is the story of two lines that intersect at the point $(2, 1)$. The column perspective is the story of how to scale a vector by 2, and add it to one other vector in order to produce a third. The column perspective is the story of a linear combination of vectors. Interpret the system below as a linear combination of vectors, and solve the system by inspection. Make sketches of the vectors involved so you can articulate the linear combination. (By the way, those of you who regularly balance chemical equations are solving a system of equations.)

$$x + 3y = 8$$
$$x + 2y = 6$$

23. Let $\mathbf{u} = (2, 3)$, $\mathbf{v} = (1, 1)$, and $\mathbf{w} = (3, 5)$. Show that $\mathbf{w}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. (*Hint*: Solve the linear combination $x\mathbf{u} + y\mathbf{v} = \mathbf{w}$.)

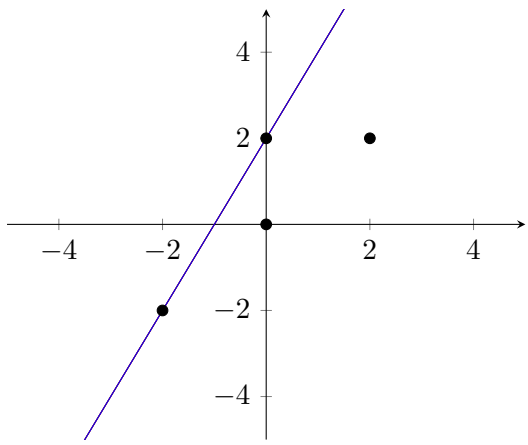
24. Restate the system of equations in homework problem 6 as a problem involving linear combinations of vectors. Make sketches of the vectors on the same set of axes. What is the condition about the *vectors* that enable you to confidently say there cannot possibly be a solution here?

25. Restate the system of equations in homework problem 7 as a problem involving linear combinations of vectors. Make sketches of the vectors on the same set of axes. What is the condition about the *vectors* that enable you to confidently say there are an infinite number of solutions?[3]

[3] Most of you come from a background in which you were trained in algebra to write "infinite number of solutions" as your answer to a system such as this one. In linear algebra, we want to be more precise here. We want to think of describing the set of all points-or *vectors* that satisfy all of the equations. What would be that description here?

Chapter 3

Systems and Augmented Matrices



3.1 Solving Systems of 2 Equations and 2 Unknowns

Of all the methods learned to solve systems, it is *elimination* to which we devote close attention. A majority of you, when solving the first ten problems on your own, most likely used some form of elimination, which involves altering one or both equations, then adding them so that one of the variables disappears. We will refine this process, and learn a solution method known as *Gauss-Jordan elimination*. This gives birth to matrices, and many of the important ideas of linear algebra. It is crucial that you pay close attention to the content in the course, working to understand it as best as you can. We take the following system as our first example of the process of solution by elimination, the first one you solved in the course:

$$\begin{aligned} x + 3y &= 2 \\ x + 4y &= 4 \end{aligned}$$

Whatever solution method is used, the outcome will be the same: a value is found for x and a value is found for y. If we go about our business solving the problem, and keeping things orderly, we can write that the solution to the system is

$$\begin{aligned} x &= -4 \\ y &= 2 \end{aligned}$$

This corresponds to the point of intersection $(-4, 2)$ of the two lines in the system of equations. Here is the important thing to grasp: *From the system as it was given to the result above, we wish to use elimination to rewrite the original system to somehow change it to the form above.* Let me show you how this is done a first time. This is a meticulous process, but important to understand. Before beginning a detailed description of the solution method, one step at a time, notice there are four equations above with

something in common: *each is a sentence containing a* ***linear combination*** *of x and y on the left side of the equation, and a single constant term on the right side.* Additionally, when the system is reinterpreted using the column perspective, it is a linear combination of two vectors. If we were to restate the specific linear combinations at the start, we have this:

$$\begin{aligned} 1x + 3y &= 2 \\ 1x + 4y &= 4 \end{aligned}$$

At the end, we have:

$$\begin{aligned} 1x + 0y &= -4 \\ 0x + 1y &= 2 \end{aligned}$$

In other words, we wish to work with the method of elimination in such a way as to produce this ending set of linear combinations. This brings an important emphasis on the way you should see the columns in a special way.

STEP 1: Acknowledge the system as it is given:

$$\begin{aligned} 1x + 3y &= 2 \\ 1x + 4y &= 4 \end{aligned}$$

STEP 2: We want a 1 by the x in the first equation, first column. We have that. Beneath it, we want a zero in front of the x; that is, we want to eliminate x in the second equation. Therefore, *we must use the first equation to change the second.* We do this mentally by **multiplying the first equation by -1, and adding it to the second equation**. This action produces the following result:

$$\begin{aligned} 1x + 3y &= 2 \\ 0x + 1y &= 2 \end{aligned}$$

STEP 3: We want a 1 by the y in the second equation, second column. We have that. We want a zero in front of the y in the first equation, second column. We don't have that, so we use the second equation to change the first one; that is, we eliminate y in the first equation by **multiplying the second equation by -3 and adding it to the first**, which produces the following result:

$$\begin{aligned} 1x + 0y &= -4 \\ 0x + 1y &= 2 \end{aligned}$$

Notice the algorithmic nature here: we eliminated variables in a strategic way, something we will soon be articulating in more detail. What we actually have done is sometimes surprising to know: matrix multiplication! Most likely, that means absolutely nothing to you unless you have had experience multiplying matrices. What on earth is matrix multiplication? We have some ground to cover. Let's first of all witness the birth of a matrix, nothing other than a rectangular array of numbers.

3.2 Matrices and Augmented Matrices

One of the most important observations you can make regarding this process of elimination is this: *The only elements involved in the mathematics performed were the coefficients and constants that appear in the system.* In fact, the process involved two specific steps spelled out in words as the action taken between steps that changed these numbers from one step to the next. If we were to look at the process again, using the same system as before, but writing it in a special way in which we view the coefficients and the constants as being in certain *columns* and form the following array of numbers, we can say that we have a system of two linear equations and two unknowns x and y, and it is represented by the following **matrix** with two rows and three columns:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix}$$

A **matrix** is a rectangular array of numbers. The plural of *matrix* is **matrices**. A matrix is described by what we call its **size**, which is specified by the numbers of rows and columns it has in the following way: An $m \times n$ matrix is a matrix with m rows and n columns. The matrix we just produced from the system of two linear equations is a 2×3 matrix. Observe the same process of solution we used, which involved two operations:

Multiply the first equation by -1, and add it to the second equation.

Multiply the second equation by -3 and add it to the first equation.

Replace the words "the first equation" with "row 1" and "the second equation" with "row 2" and then use R1 for "row 1" and R2 for "row 2." These two operations can be described symbolically: (-1)R1+R2 and (-3)R2+R1, or just $-$R1+R2 and -3R2+R1. Furthermore, the process itself can be written out in the following way:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$$

Written this way, you can interpret the steps detailed on the previous page concisely, as well as understand the intermediate actions taken to justify the changes. It is crucial that you pick up an important skill as quickly as possible: the ability to read a matrix when you are aware that the matrix represents a system of linear equations. We will learn more technical details about matrices as mathematical objects later, but systems of equations is where things originate. The 2×3 matrix representing the original system (bottom of page 6) is called an **augmented matrix**.

26. Write the augmented matrix for each system in the first homework assignment. Give the size of each matrix.

27. Give an example of a 3×3 matrix, selecting at random any numbers you wish. Then write the system of linear equations this matrix could represent.

28. **True** or **False**: A $m \times n$ augmented matrix represents a system of m equations and $n - 1$ unknown variables.

29. Study carefully the solution to the system given in the three steps on page 6. Between steps 1 and 2, notice that the first equation was *used* to change the second equation; that is, the first equation *did not change* between steps 1 and 2. Notice that, between steps 2 and 3, the second equation was *used* in order to change the first equation. Take the system below and follow the instructions given to make changes to the equations, rewriting the result after each action is taken. (Avoid writing any matrices.)

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 3 \end{aligned}$$

Operation 1: Multiply the first equation by -2 and add it to the second equation in order to change the second equation.

Operation 2: Divide the second equation by -3.

Operation 3: Multiply the second equation by -1 and add it to the first equation in order to change the first equation.

30. Check the following description (in matrix form) of the work in the previous problem. Does it represent the four systems of equations you wrote as your answer?

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 3 \end{bmatrix} \xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{R2/(-3)} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R2+R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

31. Observe the matrix notation in the previous problem in order to appreciate what information is conveyed (especially how the operations are described above the arrows). Translate the following into a system of equations, and explain how the system was changed one step at a time in order to produce this information. At the end of your explanation, state the solution to the system.

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 2 & 12 \end{bmatrix} \xrightarrow{-3R1+R2} \begin{bmatrix} 1 & 2 & 8 \\ 0 & -4 & -12 \end{bmatrix} \xrightarrow{R2/(-4)} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

32. How would you interpret the matrix $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 3 \end{bmatrix}$?

33. How would you interpret the matrix $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$?

34. Use the descriptions of what to do between each step to replace all of the question marks with the correct numbers. (Do not worry about interpreting this work yet, but know that we will return to the results of this problem in the near future.)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 0 \\ ? & ? \end{bmatrix} \xrightarrow{R2/(-3)} \begin{bmatrix} 1 & 0 \\ ? & ? \end{bmatrix} \xrightarrow{-R2+R1} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

35. Repeat the same kind of work for the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R1+R2} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \xrightarrow{R2/(-4)} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

3.3 Clarification of Notation

Besides being described by its size, a matrix is often notated by an italicized capital letter. Below are several matrices that you will use in the next five problems.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 0 & 4 \\ 2 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$$

36. Give the sizes of each matrix.

37. What system of linear equations does C represent?

38. What system of linear equations does D represent?

39. Use the following operations to change the matrix A. How do you interpret A as a system of linear equations and the matrix that results from these operations? **Operation 1:** $-4R1 + R2$; **Operation 2:** $R2/(-3)$; **Operation 3:** $-2R2 + R1$

40. Perform the same three operations on B.

Chapter 4

Gauss-Jordan Elimination: Identifying Free Variables

From this point forward, we will abandon the use of x, y and z and use subscript notation that you will likely find convenient. All of the systems of linear equations that appear in this chapter are solved using Gauss-Jordan elimination, which means every time we are introduced a system, and are asked to find its set of solutions, we will be writing the system as an augmented matrix and employing elementary row operations to obtain the reduced row echelon form of the matrix. The big skill to master in this chapter is the ability to take the matrix in rref and write the solution set of the system. The important things are pointed out throughout the chapter through a succession of examples.

Example 4: Solve the system

$$\begin{aligned} 2x_1 - 5x_2 &= 8 \\ 3x_1 + 9x_2 &= -12 \end{aligned}$$

solution:

$$\begin{bmatrix} 2 & -5 & 8 \\ 3 & 9 & -12 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R2]{R2/3} \begin{bmatrix} 1 & 3 & -4 \\ 2 & -5 & 8 \end{bmatrix} \xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 3 & -4 \\ 0 & -11 & 16 \end{bmatrix}$$
$$\xrightarrow{R2/(-11)} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & \frac{-16}{11} \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & \frac{4}{11} \\ 0 & 1 & \frac{-16}{11} \end{bmatrix}$$

We had not yet met an example in which switching the equations mattered. That translates into one of the elementary row operations here. I try avoiding fractions wherever possible, so I observed that by dividing $R2$ by 3, I would obtain a 1 without introducing fractions if I, next, switched the two rows. Why is this permissible? Switching two rows is indicated by the symbol $\leftrightarrow$. Following this, it is the usual exercise of row reducing the augmented matrix, obtaining the solution $x_1 = \frac{4}{11}$ and $x_2 = \frac{-16}{11}$. Sometimes you cannot avoid fractions. In this section, we are going to view the solution sets as **column vectors**, the brackets indicating that the vector is a 2×1 matrix. Here, we let

$$\mathbf{x} = (x_1, x_2),$$

which, as a column vector, is written this way:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solution to the system is $\mathbf{x} = \begin{bmatrix} \frac{4}{11} \\ -\frac{16}{11} \end{bmatrix}$.

Example 5: Solve the system:

$$\begin{aligned} x_1 + 2x_2 + \ \ x_3 &= 1 \\ x_1 + \ \ x_2 + \ \ x_3 &= 0 \\ 2x_1 + \ \ x_2 + 2x_3 &= 1 \end{aligned}$$

solution not fully shown: I'll let you do the work here as a homework problem:

61. Write the augmented matrix for the system in example 5. Show that the reduced row echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

This system is *inconsistent*, meaning there is no solution. The third row reads $0 = 1$.

Example 6: Suppose row operations produced the following matrix (rref):

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the solution set for the system of linear equations represented by this matrix?

solution: When you do not obtain a leading 1 where you expect it to show up, this means that the variable associated with the column with no leading 1 is a **free variable**, meaning the variable can be any real number. The third column in the above matrix does not have a leading 1, and so x_3 is the free variable, so let $x_3 = a$, where a is any real number. The above matrix can be translated literally as

$$\begin{aligned} 1x_1 + 0x_2 + 1x_3 &= 0 \\ 0x_1 + 1x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

but with $x_3 = a$, we can solve the first equation for x_1, the second equation for x_2, and form the vector $\mathbf{x}$:

$$\begin{aligned} x_1 &= -a \\ x_2 &= 0 \\ x_3 &= a \end{aligned}$$

Thus the solution to the system is the set of all vectors of the form $(-a, 0, a)$. By scalar multiplication, $(-a, 0, a) = a(-1, 0, 1)$, so the solution to the system is the set of all scalar multiples of the vector $(-1, 0, 1)$, which is a line through the origin in $\mathbb{R}^3$. Of course, I am using the space-saver notation for this set of vectors. As a column vector, $\mathbf{x} = \begin{bmatrix} -a \\ 0 \\ a \end{bmatrix}$.

Systems do not always have to have the same number of equations as unknowns, as the next few examples show.

Example 7: Solve the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + 3x_2 + 2x_3 &= 1 \end{aligned}$$

solution: $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

There is no leading 1 for column 3 in this case, so x_3 is a free variable here. We read the following from this last matrix:

$$\begin{aligned} x_1 + 5x_3 &= 1 \\ x_2 - \quad x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$$

Solving for the variables, letting $x_3 = a$, where a is any real number, we have that the solution to this system is the set of all vectors of the form $(1-5a, a, a)$. By vector addition, $(1-5a, a, a) = (1, 0, 0) + (-5a, a, a)$, and by scalar multiplication, $(1, 0, 0) + (-5a, a, a) = (1, 0, 0) + a(-5, 1, 1)$. So the solution set consists of all vectors on the line in $\mathbb{R}^3$ through the point $(1, 0, 0)$ and parallel to the vector $(-5, 1, 1)$. (If you are a student who has studied multi-variable calculus, you should recognize the vector equation of a line here.)

Example 8: Solve the system:

$$\begin{aligned} x_1 + 2x_2 + \quad\quad - \ x_4 &= -1 \\ -x_1 - 3x_2 + \ x_3 + 2x_4 &= \ \ 3 \\ x_1 - \ x_2 + 3x_3 + \ x_4 &= \ \ 1 \\ 2x_1 - 3x_2 + 7x_3 + 3x_4 &= \ \ 4 \end{aligned}$$

solution not fully shown: As in example 5, I'll let you do the work here as a homework problem:

62. Write the augmented matrix that represents the system, and then use row operations to show that the reduced row echelon form is $\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The third column has no leading 1, and so x_3 is a free variable. Let $x_3 = a$ and translate the matrix to read $x_1 + 2a = -1, x_2 - a = 2, x_3 = a$, and $x_4 = 4$. Solving for each variable, we conclude that the solution set is the set of all vectors of the form $(-2a - 1, a + 2, a, 4)$.[1]

63. Show by using vector addition and scalar multiplication that the set of all vectors of the form $(-2a - 1, a + 2, a, 4)$ is all scalar multiples of the vector $(-2, 1, 1, 0)$, plus the vector $(-1, 2, 0, 4)$.

Example 9: Solve the system

$$\begin{aligned} 2x_1 - 5x_2 - x_3 + 4x_4 &= 0 \\ 3x_1 - 5x_2 - 9x_3 + 11x_4 &= 0 \\ 3x_1 - 7x_2 - 3x_3 + 7x_4 &= 0 \end{aligned}$$

solution:

$\begin{bmatrix} 2 & -5 & -1 & 4 & 0 \\ 3 & -5 & -9 & 11 & 0 \\ 3 & -7 & -3 & 7 & 0 \end{bmatrix}$ is the augmented matrix for this system of linear equations.

It is worth noting the constants zero in each equation. Such a system is known as **homogeneous**. Homogeneous systems of linear equations are an important part of understanding the nature of the solution set of any system. (We explore this in the future.) I thought I would use this example to show you my favorite matrix calculator, free online and always available. (Your graphing calculator is also capable of doing this as well.) Go to www.desmos.com and, under *math tools* click *matrix calculator*. Figure out how to create a matrix of three rows and five columns, and then use the *rref* button to see that the reduced row echelon form of the above matrix is:

$$\begin{bmatrix} 1 & 0 & -8 & 7 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 3 and 4 have no leading 1's, so for the system here, there are *two* free variables x_3 and x_4. Let $x_3 = a$ and $x_4 = b$, where a and b are any real numbers. Written directly from the above matrix, we have:

$$\begin{aligned} x_1 - 8x_3 + 7x_4 &= 0 \\ x_2 - 3x_3 + 2x_4 &= 0 \\ x_3 &= a \\ x_4 &= b \end{aligned}$$

Solving for each variable, we see that $x_1 = 8a - 7b$, $x_2 = 3a - 2b$, $x_3 = a$ and $x_4 = b$. The solution to the system is the set of all vectors of the form $(8a - 7b, 3a - 2b, a, b)$. You

[1] This is a line in $\mathbb{R}^4$ passing through the point $(-1, 2, 0, 4)$ and parallel to the vector $(-2, 1, 1, 0)$.

might be wondering why attention is paid to these solution sets when we utilize vector addition to separate elements, and scalar multiplication to factor. We are setting ourselves up to study important spaces. Solutions to homogeneous equations are always line-like things that contain the origin. These are known as *subspaces*, the kinds of spaces we encounter everywhere we work in linear algebra. They are all possible linear combinations of a set of vectors.

$$\begin{aligned}(8a-7b, 3a-2b, a, b) &= (8a, 3a, a, 0) + (-7b, -2b, 0, b)\\ &= a(8,3,1,0) + b(-7,-2,0,1)\end{aligned}$$

The solution set to this system is the set of all linear combinations of the vectors $(8,3,1,0)$ and $(-7,-2,0,1)$, which is a plane in $\mathbb{R}^4$ that passes through the origin. Before we begin discussing these special spaces, we must understand that we have been doing matrix multiplication in an interesting way. There is much more to see about solving systems. Be sure you practice this section of problems so well that you are comfortable and polished with all of the concepts established thus far.

Solve the following systems of equations, identifying any free variables, and state the solution set as was done in the examples. Show the row operations in your work and use the matrix calculator to check that you have correctly reduced each matrix.

64.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14\\ 2x_1 + 5x_2 + 8x_3 &= 36\\ x_1 - x_2 \qquad &= -4 \end{aligned}$$

65.

$$\begin{aligned} x_1 - x_2 - x_3 &= -1\\ -2x_1 + 6x_2 + 10x_3 &= 14\\ 2x_1 + x_2 + 6x_3 &= 9 \end{aligned}$$

66.

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 15\\ 2x_1 + 4x_2 + 9x_3 &= 33\\ x_1 + 3x_2 + 5x_3 &= 20 \end{aligned}$$

67.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7\\ 2x_1 + 3x_2 + x_3 &= 18\\ -x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

68.

$$\begin{aligned} 3x_1 + 6x_2 - 3x_3 &= 6\\ -2x_1 - 4x_2 - 3x_3 &= -1\\ 3x_1 + 6x_2 - 2x_3 &= 10 \end{aligned}$$

69.

$$\begin{aligned} x_1 + 4x_2 + x_3 \qquad &= 2\\ x_1 + 2x_2 - x_3 \qquad &= 0\\ 2x_1 + 6x_2 \qquad &= 3 \end{aligned}$$

70.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ +2x_2 + 2x_3 &= 11 \\ 2x_1 + 4x_2 + 3x_3 &= 18 \end{aligned}$$

71.

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 10 \\ -3x_1 - 2x_2 + 4x_3 &= -24 \end{aligned}$$

72.

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0 \\ x_1 + 2x_2 \phantom{{}+2x_3} + x_4 &= 4 \\ -x_1 + 2x_2 + 2x_3 + 4x_4 &= 5 \end{aligned}$$

73.

$$\begin{aligned} x_2 + 2x_3 &= 7 \\ x_1 - 2x_2 - 6x_3 &= -18 \\ -x_1 - x_2 - 2x_3 &= -5 \\ 2x_1 - 5x_2 - 15x_3 &= -46 \end{aligned}$$

74.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 3x_2 &= 3 \\ x_1 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

75. Show that the set of all vectors of the form (x, y) is the set of all linear combinations of $(1, 0)$ and $(0, 1)$.

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Chapter 5

Gauss-Jordan Elimination and Reduced Row Echelon Form

5.1 Solving by Elimination (A Second Example)

Perhaps the following example will shed more light on the strategy used to solve the system of equations in the previous chapter. (This is the system from problem 9.)

$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 3 \\ x + 2y - z &= 1 \end{aligned}$$

The solution to this system is

$$\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 2 \end{aligned}$$

Remember the goal here: The goal is to perform steps in which we eliminate variables so that we obtain the form of the answer, which in the strictest of forms can be written in the following way:

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 1 \\ 0x + 0y + 1z &= 2 \end{aligned}$$

This is a special form from which you can easily read the solution to the system. We will go through some meticulous steps before introducing the matrix language and notation for what is known as *Gauss-Jordan elimination.* It is important to understand what you are doing with the equations in the system in order to dispel any strange mysteries that arise when all of this work is spoken and written in matrix form. In order to reach our goal, we take the original system of equations, work from the left to the right, column by column, obtaining a 1 where we want it, and zeros everywhere else above and below that 1. Follow carefully:

STEP 1: Acknowledging the system in the form it is given, be sure everything is lined up properly, column-wise. Starting with the first equation, we want $1x$ here, and no x variables in the other two equations. We have the x, so we use this first equation to eliminate x in the other two. This requires two operations, both of which can be executed in a single step:

$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 3 \\ x + 2y - z &= 1 \end{aligned}$$

Using the first equation, **Operation 1:** multiply it by -2 and add it to the second, and **Operation 2:** multiply it by -1 and add it to the third.
The first two operations produce the following:

$$\begin{aligned} 1x + 1y - 1z &= 0 \\ 0x - 3y + 3z &= 3 \\ 0x + 1y + 0z &= 1 \end{aligned}$$

STEP 2: Next, we want a $1y$ in the second equation. We don't have that, but we can see how easy it is to "divide" by negative three:[1]**Operation 3:** Multiply the second equation by $-1/3$:

$$\begin{aligned} 1x + 1y - 1z &= 0 \\ 0x + 1y - 1z &= -1 \\ 0x + 1y + 0z &= 1 \end{aligned}$$

STEP 3: Next, we use the second equation to eliminate y from the other two. **Operation 4:** Multiply it by -1 and add it to the first, and **Operation 5:** multiply it by -1 and add it to the third:

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y - 1z &= -1 \\ 0x + 0y + 1z &= 2 \end{aligned}$$

STEP 4: Finally, we want a $1z$ in third equation, and we see we already have it. So, using the third equation to eliminate z from the other two (though here, z is already gone from the first equation, so only one operation remains: Using the third equation, **Operation 6:** add it to the second to obtain the result:

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 1 \\ 0x + 0y + 1z &= 2 \end{aligned}$$

Moving from left to right, taking one column at a time, establishing a coefficient of 1 where we need it, and then *using it* to get coefficients of 0 in the other places, we see we are eliminating the variables x, y, and z, in that order. This produces the ideal set of equations that give the solution to the equation. When we do the same work in matrix form, we use different language, sometimes to the point of forgetting what the matrices represent! It is worth repeating several times throughout the course: the fundamental problem of linear algebra is solving a system of linear equations. The above procedure is known as **Gauss-Jordan elimination**, but it is almost always carried out by way of augmented matrices. The next sections of this chapter emphasize the new vocabulary you must learn, while restricting the use of this method to n equations and n unknown variables.

[1] Another option would be to switch the position of the second and third equations, a legal move which does not disturb anything.

5.2 Row Operations

Each of the operations carried out in the previous examples are operations on the equations, and since we are really only dealing with the coefficients and constants, rewriting this procedure using matrices is convenient. The equations become rows, and so we refer to these operations as **row operations**. A question arises: *What are all possible types of row operations?* To answer this question, consider all of the moves you are allowed to perform with a set of equations comprising a system. It is permissible to...

(1) multiply any equation by a non-zero real number and add this to another equation, (such as Operations 1, 2, 4, 5, and 6 in the previous example),

(2) multiply an equation by a nonzero constant (such as Operation 3 in the previous example), and

(3) interchange two equations.

These moves are all legal mathematical moves that do not upset the system in terms of the solutions it possesses. No solutions are added or altered to the solution set, and these operations produce a succession of increasingly simpler systems. These three kinds of moves translate to what we call **elementary row operations**. Since the rows of an augmented matrix correspond to the equations, we can describe an elementary row operation as being an operation in which you

(1) add a constant multiple of one row to another,

(2) multiply a row by a nonzero constant, and

(3) interchange any two rows.

Gauss-Jordan elimination is the process by which you reach the ideal simple system from which you directly read the solution to the system. We borrow the word *echelon* from the French to describe the staircase formation of 1's we see on the main diagonal, and the matrix that is produced from all of these operations is known as a matrix in **reduced row echelon form**. This is how the work is shown in solving a system of linear equations by Gauss-Jordan elimination: the system is written as an augmented matrix, and then altered carefully by row operations, which you are required to notate:

Example 1: Solve the system:

$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 3 \\ x + 2y - z &= 1 \end{aligned}$$

solution: Begin with the augmented matrix, and proceed with identifying row operations between each of the matrices:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \end{bmatrix} \xrightarrow[-R1+R3]{-2R1+R2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R2/(-3)} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-R2+R3]{-R2+R1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R3+R2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to the system can be read from the last matrix. For now, we can write $x = 1, y = 1, z = 2$ or $(1, 1, 2)$. (We will soon be writing these as vectors in column form.)

Return to the very first homework assignment to exercise the method of Gauss-Jordan elimination, using **Example 1** as a guide. Begin by writing the system as an augmented matrix and be sure to include the row operations between matrices.

41. Problem 2
42. Problem 3
43. Problem 4
44. Problem 5
45. Problem 8

5.3 Reduced Row Echelon Form

Think carefully about the form of the augmented matrix that is reached if a system of n equations and n unknowns has a solution: There are 1's along the main diagonal of the matrix, with 0's above and below them. In the last column, there is a column of constants. Consider the next two examples (from the first homework assignment):

Example 2: Solve the system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 6y &= 1 \end{aligned}$$

.

solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 1 \end{bmatrix} \xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

After a single row operation, you ought to pause and *interpret* the augmented matrix: what system of equations does it describe?

$$\begin{aligned} x + 3y &= 2 \\ 0x + 0y &= -3 \end{aligned}$$

The second equation makes no sense mathematically: $0 = -3$. This means the system has no solution, otherwise known as *inconsistent*.

Example 3: Solve the system

$$\begin{aligned} x - 2y &= -5 \\ -3x + 6y &= 15 \end{aligned}$$

solution:

$$\begin{bmatrix} 1 & -2 & -5 \\ -3 & 6 & 15 \end{bmatrix} \xrightarrow{3R1+R2} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

After a single operation, note the 0's in the bottom row. Unlike the previous example, this augmented matrix *does* make sense in that, mathematically speaking, $0 = 0$. As you already know, there are an infinite number of solutions to this system, but writing as your answer "*infinite number of solutions*" does not describe *what the solutions are*.

46. The row perspective of the system of equations in example 3 tells the story of two lines which intersect at an infinite number of points. Interpret the system using the column perspective. How do you make sense of the solution to the system from this perspective?

47. What are all possible types of outcomes in solving a system of two equations and two unknowns?

In both of the previous examples, we were unable to reach the ideal form of an augmented matrix in which successive 1's appear to the right of the 1 in the row above it.Because the row operations produce simpler versions of the system, we say that the form we are approaching is some kind of *reduced* form, characterized by the staircase appearance of 1's along the main diagonal. The ultimate form we wish to reach is known as **reduced row echelon form**, which will be abbreviated as **rref** throughout this text.

DEFINITION 1: A matrix is said to be in **reduced row echelon form** if it has the following four properties:

(1) If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **leading one**.

(2) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(3) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

(4) Each column that contains a leading 1 has zeros everywhere else in that column.[2]

[2] A matrix satisfying the first three properties is said to be in *row echelon form*.

Another popular name for a leading 1 is **pivot**. This is due to the fact that, in the elimination process, going from left to right, one column at a time, a leading 1 is created and then used to annihilate the other numbers in the column, as if the 1 is strapped to a chair with some kind of number gun, and chair pivots around to face up and then face down, while the 1 makes everything 0 above it and below it. It will be important to be able to identify in which columns there are no pivots (no leading 1's). The columns without pivots located in them tell you that the variable whose coefficients appear initially in the column is a **free variable**, a detail we cover in the next chapter.

48. Is the matrix $\begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ in reduced row echelon form? Why or why not?

49. Is the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ in reduced row echelon form? Why or why not?

50. Solve the system by writing the augmented matrix of the system in **rref**.

$$\begin{aligned} x + 4y &= 2 \\ -x + 5y &= 7 \end{aligned}$$

Use elementary row operations to obtain the reduced row echelon form of each matrix.

51. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & 1 & 1 \end{bmatrix}$

52. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 4 & 6 \end{bmatrix}$

53. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \end{bmatrix}$

54. $\begin{bmatrix} 0 & 3 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

55. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

56. $\begin{bmatrix} 2 & 4 & 6 & 6 \\ 2 & 5 & 7 & 1 \\ 3 & 6 & 9 & 3 \end{bmatrix}$

57. $\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

58. $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix}$

59. $\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$

60. $\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$

Chapter 6

The Dot Product and Matrix Multiplication

6.1 Vectors in Euclidean Spaces and Systems as Matrix Equations

The fundamental problem of linear algebra is solving a system of linear equations. The previous chapters introduce you to a systematic way of obtaining the set of all solutions to any system. The ideal system of equations consists of n equations and n variables, and this course develops an important theorem called *the invertible matrix theorem* based on these ideal kinds of systems. As you have seen, systems can have m equations and n unknowns, where $m \neq n$. The solution of any linear system in n unknown variables is a sequence of n numbers. The familiar two equations and two unknowns often has an ordered pair (x, y) as its solution, and the geometric interpretation you are initially taught is that the two lines intersect at this point. Three equations and three unknowns will often have an ordered triple (x, y, z) as its solution. As you have seen, four unknowns might result in a string of four numbers, etc... This brings to mind how to think of what vectors really are in mathematics: *vectors are sequences of numbers*, an ordered list of numbers. We return to *Euclidean vectors* briefly in this section. The set of all real numbers can be viewed geometrically as numbers on a line called the *real line* denoted by $\mathbb{R}$ or sometimes by $\mathbb{R}^1$, where the superscript reinforces the intuitive idea that a line is *one dimensional*. The set of all ordered pairs (also called 2-tuples) of real numbers makes up the xy-plane, and is denoted by $\mathbb{R}^2$. The set of all ordered triples (also called 3-tuples) of real numbers is denoted by $\mathbb{R}^3$. These spaces have dimensions 2 and 3, of course. The extension of this idea is what is known as the n-dimensional space $\mathbb{R}^n$. If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers denoted by $\mathbb{R}^n$. and has dimension n. We can define **vector addition** and **scalar multiplication** for vectors in general:

DEFINITION 2: Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors in $\mathbb{R}^n$ and let k be any scalar. **Vector addition** is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n),$$

while **scalar multiplication** is

$$k\mathbf{v} = (kv_1, kv_2, ..., kv_n).$$

We have already been using these two definitions, such as in the following example:

Example 10: Assume the matrix represents a system of linear equations, and check to see that it is in reduced row echelon form. State the solution set of the system.

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 & 4 \end{bmatrix}$$

solution: A quick run-through of the four criteria for rref proves that the matrix is in rref. The augmented matrix is a 2×5 matrix, so there are four unknown variables x_1, x_2, x_3, and x_4. Columns 2 and 4 do not have any leading 1's and so the variables x_2 and x_4 are free variables. From the matrix, we can write:

$$\begin{aligned} x_1 &= 3 - 3x_2 - 2x_4 \\ x_2 &= x_2 \\ x_3 &= 4 - 2x_4 \\ x_4 &= x_4 \end{aligned}$$

Note how things are lined up in columns. Let $x_2 = a$ and $x_4 = b$, where a and b are any real numbers.

$$\begin{aligned} x_1 &= 3 - 3a - 2b \\ x_2 &= a \\ x_3 &= 4 - 2b \\ x_4 &= b \end{aligned}$$

. In the previous chapter, we used the space-saver notation to express the set of vectors: the solution is the set of all vectors in $\mathbb{R}^4$ of the form $(3 - 3a - 2b, a, 4 - 2b, b)$. Then we used vector addition and scalar multiplication to write that

$$\begin{aligned} (3 - 3a - 2b, a, 4 - 2b, b) &= (3, 0, 4, 0) + (-3a, a, 0, 0) + (-2b, 0, -2b, b) \\ &= (3, 0, 4, 0) + a(-3, 1, 0, 0) + b(-2, 0, -2, 1). \end{aligned}$$

In column vector notation, we can write the same thing by simply looking at the top of the page and observing how things are lined up:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + a \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

This set of vectors consists of all linear combinations of the two vectors $(-3, 1, 0, 0)$ and $(-2, 0, -2, 1)$, which is a plane through the origin $(0, 0, 0, 0)$ in $\mathbb{R}^4$, plus a *particular* vector $(3, 0, 4, 0)$ that moves that plane away from the origin.

When we write these vectors in column form using the brackets, we are writing the vectors as matrices. We will stick with systems of two equations and two unknowns for simplicity in the next couple of examples, but we want to eventually return to this example. Forthcoming examples, by the way, make use of some of the systems from problems 1-10 from the first homework assignment, though the unknown variables are notated using subscripts.

Example 11: Solve the system by Gauss-Jordan elimination:

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ x_1 + 4x_2 &= 4 \end{aligned}$$

solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$$

We interpret the solution to the system as the column vector $\mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. For Gauss-Jordan elimination, we represented the system by a 2×3 matrix called an *augmented matrix*. We now wish to represent the system as a *matrix equation* in the form $A\mathbf{x} = \mathbf{b}$, where A is called **the coefficients matrix**, $\mathbf{x}$ is the column vector whose components are the unknowns, and $\mathbf{b}$ is the column vector containing the constants on the right side of each equation. For the previous example, then, we can identify these:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

As a *matrix equation*, we can write the system in the following way:

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

This choice to write the system in this way defines matrix multiplication. Notice that the left side of the equation represents a product $A\mathbf{x}$, which can be described as the product of a 2×2 matrix and a 2×1 matrix, which produces a 2×1 matrix. The solution to the system is the vector $\mathbf{x} = (-4, 2)$, written in *matrix* notation as a "column vector." If we say that $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ must *satisfy* the equation, meaning

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

The left side of each equation is a linear combination of x_1 and x_2. Notice the row by column result for each combination, meaning that the linear combination

$$x_1 + 3x_2 = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and the linear combination

$$x_1 + 4x_2 = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Some of you might recognize these last two items as the **dot product** of two vectors in $\mathbb{R}^2$. This brings us to the next section. Before proceeding to the next section, let's get acquainted with writing systems as matrix equations:

76. Return to the first ten problems at the beginning of this book. Rewrite each system as a matrix equation.

Write each system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

77.

$$\begin{aligned} x_1 + \ \ x_2 - 3x_3 &= \ \ 10 \\ -3x_1 - 2x_2 + 4x_3 &= -24 \end{aligned}$$

78.

$$\begin{aligned} x_1 + 2x_2 - \ \ x_3 - \ \ x_4 &= 0 \\ x_1 + 2x_2 \qquad\quad + \ \ x_4 &= 4 \\ -x_1 + 2x_2 + 2x_3 + 4x_4 &= 5 \end{aligned}$$

79.

$$\begin{aligned} x_2 + 2x_3 &= 7 \\ x_1 - 2x_2 - 6x_3 &= -18 \\ -x_1 - x_2 - 2x_3 &= -5 \\ 2x_1 - 5x_2 - 15x_3 &= -46 \end{aligned}$$

80.

$$\begin{aligned} x_1 + \ \ x_2 &= 2 \\ 2x_1 + 3x_2 &= 3 \\ x_1 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

81. For each of the previous four systems, note the sizes of the two matrices in the product $A\mathbf{x}$. Is there a pattern you can pick up in terms of these sizes in terms of what makes it possible to multiply two matrices?

6.2 The Dot Product

Here is the general mathematical definition of the dot product of any two vectors in $\mathbb{R}^n$. (In the lecture, we can discuss the origin of the dot product, if there is interest.)

DEFINITION 3: Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors from the same space $\mathbb{R}^n$. The **dot product**, also known as the **inner product** or the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined as the real number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 12: Find the dot product of the vectors $(1, 3)$ and $(-4, 2)$ in $\mathbb{R}^2$.

solution: By definition,

$$\begin{aligned} (1,3) \cdot (-4,2) &= (1)(-4) + (3)(2) \\ &= -4 + 6 \\ &= 2. \end{aligned}$$

Example 13: Find the dot product of the vectors $(1, 4)$ and $(-4, 2)$ in $\mathbb{R}^2$.

solution: By definition,

$$\begin{aligned}(1,4)\cdot(-4,2) &= (1)(-4)+(4)(2)\\ &= -4+8\\ &= 4.\end{aligned}$$

The two dot products in the above examples is what defines the multiplication in the matrix equation

$$\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-4\\ 2\end{bmatrix}=\begin{bmatrix}2\\ 4\end{bmatrix}.$$

Example 14: Let $\mathbf{u}=(2,1,-1,2), \mathbf{v}=(0,1,-2,-2)$. Find $\mathbf{u}\cdot\mathbf{v}$.

solution: By definition,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (2,1,-1,2)\cdot(0,1,-2,-2)\\ &= (2)(0)+(1)(1)+(-1)(-2)+(2)(-2)\\ &= 0+1+2-4\\ &= -1.\end{aligned}$$

6.3 Matrix Multiplication

Using the matrices that arise from example 11, coupled with the definition of the dot product, we see that matrix product $A\mathbf{x}$ is nothing more than the dot products of the vectors in examples 12 and 13, where the matrix A contains components of vectors in its rows, while the matrix on the right is really a vector with the same number of components, but is written in column form, making matrix multiplication the art of computing the dot products of vectors. I have sometimes said to classes, when introducing a matrix product AB, "take the dot product of a row of A with a column of B." You can see how these dot products are carried out in the previous, albeit simple, examples 12 and 13:

$$\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-4\\ 2\end{bmatrix}=\begin{bmatrix}2\\ \\ \end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-4\\ 2\end{bmatrix}=\begin{bmatrix}2\\ 4\end{bmatrix}$$

If this is how matrix multiplication is to be executed, then it must be the case that *the number of columns of the matrix on the left must equal the number of rows of the matrix on the right.* Why? The dot product between two vectors is defined only for vectors with the same number of components. This gives us a *matrix notation* for the dot product of two vectors. For example. the two dot products in the matrix product above can be written in vector form or matrix form:

$$(1,3)\cdot(-4,2)=2$$

can be written

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2.$$

But... don't forget the column perspective of the system of linear equations! That perspective gives us another way to view matrix multiplication. All of this starts with this system of equations:

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ x_1 + 4x_2 &= 4 \end{aligned}$$

Remember that the column perspective means that the system can be interpreted as a linear combination of two vectors, where x_1 and x_2, the unknown variables, are the scalars needed to be found so that

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Notice that this means we can take the matrix product we were looking at, which was

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix},$$

and we can carry out the product by understanding that the rightmost matrix tells us what to do with the columns of the leftmost matrix; that is, the above product can be interpreted as "multiply the first column of A by -4, and add that to twice the second column of A." This gives a stronger meaning to the fact that, in order to multiply two matrices such as AB, the number of columns of A must equal the number of rows of B.

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

I think knowing and using *both* are extremely useful. There are actually *five* different ways to compute a matrix product, and each method has its own use. At the moment, the two methods discussed serve us well in our work with systems of linear equations.

DEFINITION 4: If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **matrix product** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B and, considering these as vectors, compute the dot product of row i and column j.

Use these matrices to study the example and work exercises 82-88.

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} -6 \\ -2 \end{bmatrix}, C = \begin{bmatrix} -3 \\ 7 \end{bmatrix}, D = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, F = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Example 15: Find AF.

solution:

$$AF = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + (-3)(0) & 1 \cdot 3 + (-3)(1) \\ 1 \cdot 1 + (-2)(0) & 1 \cdot 3 + (-2)(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

82. AB

83. DA

84. EA

85. DC

86. FE

87. EF

88. FEA

89. Solve the following system by first writing the system as an augmented matrix, followed by Gauss-Jordan elimination. Be sure to record the row operations between changing matrices. There ought to be only two elementary row operations performed.

$$\begin{aligned} x_1 - 3x_2 &= -3 \\ x_1 - 2x_2 &= 7 \end{aligned}$$

90. You might not be aware of the fact that as you solved the system in the previous problem, you did matrix multiplication. Observe how the augmented matrix changed after the first row operation. Which of the exercises in 82-88 do you see something similar?

91. Write the system as a *matrix equation* $A\mathbf{x} = \mathbf{b}$.

92. Take the matrix equation in 91, $A\mathbf{x} = \mathbf{b}$, and multiply both sides of the equation *on the left* by the matrix D. In other words, find $DA\mathbf{x} = D\mathbf{b}$. What is the result of doing this? (Convert the matrix equation back to a system of equations.)

The following exercises seem a bit unusual, but you will be setting yourself up for what I hope will capture your interest in matrices.

93. Differentiate the function $f(x) = 7e^{2x} \cos 3x - 4e^{2x} \sin 3x$.

94. Find the second derivative of the function in 93.

95. Differentiate the function $p(x) = 3 - x + 2x^2 - x^3$.

Chapter 7

Matrix Operations

7.1 A Matrix That Differentiates a Function

At the start of some of the chapters from here and to the end of this book, you will meet matrices that do things you might not expect matrices to do. Once we get more familiar with the concept of *linear transformations* it will be easier to understand how and why these matrices you meet work in the way that they do. For now, meet the following matrix D, which differentiates any function of the form $ae^{2x}\cos 3x + be^{2x}\sin 3x$, where a and b are any real numbers:

$$D = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

It turns out that any function of the form $ae^{2x}\cos 3x + be^{2x}\sin 3x$ can be expressed as a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Remember that this column vector is *matrix notation*, a 2×1 matrix. The function you were asked to differentiate in problems 93 and 94 can be represented by $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

96. Find $D\mathbf{x}$, and then show that the result matches the result you got for an answer to problem 93.

97. It is possible to find D^2, which is DD. Find this 2×2 matrix D^2.

98. D^2 will take any function of the form above and compute its second derivative. Find the matrix product $D^2\mathbf{x}$ and show that the result matches the result you got for an answer to problem 94.

99. Compute, as any student in a calculus course would, the first and second derivatives of the function $f(x) = ae^{2x}\cos 3x + be^{2x}\sin 3x$.

100. Let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ and find $D^2\mathbf{x}$. Show that this agrees with your work in 99.

7.2 Clarification of Matrix Notation and Terminology

As the above examples suggest, matrices arise in other contexts other than representations for systems of linear equations. A matrix is nothing other than a rectangular array of numbers called **entries** of the matrix. The **size** of a matrix is a description of its numbers of rows and columns: if a matrix has m rows and n columns, then its size is denoted by $m \times n$. The column vectors $\mathbf{x}$ in the examples above are 2×1 matrices. It might be the case that you want to refer to a particular entry of a matrix, and there is a notation for

that. Remember that we use italicized capital letters to denote a matrix. If A represents a matrix, then we use the lowercase a_{ij} to denote the entry that is in the ith row and jth column.

Example 16: Let $A = \begin{bmatrix} 1 & 8 & 0 & 2 \\ 3 & 9 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 7 \\ 5 & 3 \\ 1 & 4 \\ 6 & 2 \end{bmatrix}$. What are the sizes of A and B? Identify a_{23}, b_{22}, and b_{41}.

solution: A is 2×4 and B is 4×2. The entry a_{23} is the number in the second row and third column of A, so $a_{23} = 4$. Similarly, $b_{22} = 3$ and $b_{41} = 6$.

Square matrices are an important class of matrices. A matrix A that has n rows and n columns is a **square matrix of order** n. Square matrices have **main diagonals**. The entries that comprise the **main diagonal** of an $n \times n$ matrix A are all entries a_{kk}, where $k = 1, 2, 3, ..., n$. The main diagonal of a square matrix is always the set of numbers that appear along the diagonal from the first row, first column to the last row, last column.

7.3 Matrix Addition and Scalar Multiplication

If two matrices are equal to each other, then they must have the same size and their corresponding entries must be the same. Matrix addition is defined only for matrices of the same size: simply add corresponding entries. A scalar multiplied by a matrix results in each entry being multiplied by the same scalar. Despite the fact that you have seen a matrix store vectors, a matrix itself *is a vector*. What is really being defined here is vector addition and scalar multiplication!

Example 17:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & 3 & 3 \\ 3 & 5 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 0 \\ 7 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 2 & 6 \end{bmatrix}$$

Find **(a)** $A + C$, **(b)** $A + D$, **(c)** $-2D$, **(d)** $C - A$, and **(e)** $B - C$

solution:

(a) $A + C = \begin{bmatrix} 3+1 & 2+2 & 1+0 \\ 1+7 & 2+3 & 3+2 \\ 0+1 & 4+1 & 1+2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 8 & 5 & 5 \\ 1 & 5 & 3 \end{bmatrix}$

(b) $A + D$ is not defined because the matrices are not the same size. A is 3×3 and D is 3×2.

(c) Multiply each entry of D by -2: $-2D = \begin{bmatrix} -6 & -2 \\ -2 & -2 \\ -4 & -12 \end{bmatrix}$

(d) $C - A = \begin{bmatrix} 1-3 & 2-2 & 0-1 \\ 7-1 & 3-2 & 2-3 \\ 1-0 & 1-4 & 2-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 6 & 1 & -1 \\ 1 & -3 & 1 \end{bmatrix}$

(e) As was the case in part **(b)**, $B - C$ is undefined because the two matrices are of different sizes; B is a 3×4 matrix, while C is s square matrix.

7.4 More Matrix Multiplication

The definition of the product AB of two matrices requires that the number of columns of A be the same as the number of rows of B. If this condition is not met, then the product is undefined. Using the matrices in Example 17, note the sizes of each matrix: A and C are both 3×3 (they are both square matrices of order 3), B is 3×4, and D is 3×2. AB is a possible matrix product, resulting in a 3×4 matrix, but the product BA is not defined. This brings up the important observation: **Matrix multiplication is not commutative**. This is also observed when multiplication *is* defined. Using the matrices given in Example 17, find the following:

101. AC

102. CA

These examples are important to keep in mind as we enter the more challenging theoretical exploration of matrix algebra, a bit of which can now be introduced. Your experience in the study of algebra has been *the algebra of real numbers*, and multiplication of real numbers is commutative; that is, if a and b are real numbers, then $ab = ba$. Another property of real numbers is *the cancellation law*, which you use often in any course: If a, b, and c are real numbers, and $a \neq 0$, then $ab = ac$ implies that $b = c$. This is *not true* with matrices. Another property common to real numbers is that if $ab = 0$, then either $a = 0$ or $b = 0$; that is, at least one of the factors must be zero. This is not true with matrices, as you will demonstrate on your own in the next few exercises.

Use the following matrices for exercises 103-107:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

103. Find AB.

104. Find AC.

105. **True** or **False**: Let A, B and C be matrices. If $AB = AC$, then $B = C$.

106. Find AD.

107. **True** or **False**: Let A and D be matrices. If $AD = O$. where O is a matrix whose entries are all zero, then either $A = O$ or $D = O$.

A matrix whose entries are all zero is called a **zero matrix**, and is denoted by O. In future work, we often denote zero matrices by their size. Anytime you see the notation $O_{m\times n}$, this denotes a matrix with m rows and n columns where all entries are zero.

We conclude this chapter by reviewing the purpose of exercise 92, given towards the end of the previous chapter. By way of Gauss-Jordan elimination, the solution to the system

$$\begin{aligned} x_1 - 3x_2 &= -3 \\ x_1 - 2x_2 &= 7 \end{aligned}$$

can be found using two elementary row operations:

$$\begin{bmatrix} 1 & -3 & -3 \\ 1 & -2 & 7 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 10 \end{bmatrix} \xrightarrow{3R2+R1} \begin{bmatrix} 1 & 0 & 27 \\ 0 & 1 & 10 \end{bmatrix}$$

We now get to a first look at *the algebra of matrix equations* by noting the matrix form of this process: As a matrix equation, the system

$$\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix},$$

and the augmented matrix in reduced row echelon form in which the solution is read, in matrix form, is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 10 \end{bmatrix}.$$

We can write these two matrix equations treating the two unknowns as one unknown, the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and retell the story of solving this system as solving a matrix equation. In problem 92, you were asked to multiply both sides of the original matrix equation by a matrix $D = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$; that is, to solve

$$\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix},$$

multiply both sides by D

$$\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix},$$

which results in

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 27 \\ 10 \end{bmatrix}.$$

Of course, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{x}$. Two notable algebraic items arise here, similar to the algebra of real numbers. When solving the equation $3x = 6$, the algebraic procedure was to "isolate

x." There was a *multiplicative inverse* of 3 which, when multiplied by 3, resulted in the *multiplicative identity element* 1:

$$\begin{aligned} 3x &= 6 \\ 3^{-1} \cdot 3x &= 3^{-1} \cdot 6 \\ 1x &= 2 \\ x &= 2 \end{aligned}$$

The algebra of solving systems in matrix form is similar:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

The notation A^{-1} represents the **inverse** of A, which will be defined in a forthcoming chapter, and the notation I represents the **identity** matrix, which we define here:

DEFINITION 5: A square matrix of order n with 1's on the main diagonal and zeros everywhere else is called an **identity matrix**, and is denoted by I_n or I.

Some examples of some identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given a square matrix A, the identity matrix I behaves like 1 does with real numbers: $AI = IA = A$. If A is *not* a square matrix, there exists some identity matrix I that can be multiplied by A on either side, yielding the result you might imagine. Try these two problems where $A = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 7 & 6 \end{bmatrix}$:

108. Acknowledge the size of A. Multiply A on the right by I_3, and show that the result is A.

109. Multiply A on the left by I_2 and show that the result is also A.

Use the following matrices to work exercises 110-119.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

110. Find $M\mathbf{b}$.

111. Find E_1A.

112. Multiply the resulting matrix in the previous problem *on the left* by E_2, thereby finding E_2E_1A.

113. Write the system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$.

114. Write the augmented matrix that represents the system of equations you wrote, and then solve the system by Gauss-Jordan elimination. (You should record *two* row operations in the process.)

115. Rewrite the system as a matrix equation $A\mathbf{x} = \mathbf{b}$, and, similar to what was done on the previous page, multiply both sides of the matrix equation *on the left* by E_1.

116. Continuing with the result, multiply both sides *on the left* by E_2. Compare the results of these past two problems with the two steps you did in problem 114.

117. Find E_2E_1.

118. Return to the system written as the matrix equation $A\mathbf{x} = \mathbf{b}$, and multiply both sides on the left by M.

119. Compute AM and MA.

120. Return to problem 8 from the very first homework assignment. Write the system as a matrix equation. Multiply both sides of the equation *on the left* by $\begin{bmatrix} -1 & 1 & 1 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$.

Chapter 8

Elementary Matrices and Factoring a Matrix

8.1 A Matrix that Differentiates a Polynomial

If you look back at the last three problems in Chapter 5 (problems 93, 94, and 95), a reason was given for why you met a matrix at the start of Chapter 6 for the first two of those three, but not for the last one, where you had to differentiate $p(x) = 3-x+2x^2-x^3$, perhaps the easiest of functions to be asked to differentiate: $p'(x) = -1+4x-3x^2$. Meet P, the matrix that differentiates cubic polynomials:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

This matrix, like any matrix does, *transforms* the "vector" that gets multiplied *on the left* by P. You are experienced enough in calculus to know that, given any polynomial of degree three, an expression with at most four terms, its first derivative is a polynomial of degree two (an expression with at most three terms). If we interpret the vector (a_0, a_1, a_2, a_3) as representing the polynomial $p(x) = a_0+a_1x+a_2x^2+a_3x^3$, then the polynomial in problem 95 can be represented by the vector $(3, -1, 2, -1)$. Of course, we want to write these as column vectors (matrix notation) because P is a 3×4 matrix, and can only be multiplied by a matrix with four rows (any $4 \times n$ matrix); however, the product of a 3×4 matrix with a 4×1 matrix produces a 3×1 matrix. It is not difficult to see how to interpret this 3×1 matrix output:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

The vector $(-1, 4, -3)$ represents the polynomial $p'(x) = -1+4x-3x^2$.

121. Is it possible to find P^2? If so, how? If not, why not?

122. Rewrite P as a 4×4 matrix by adding a fourth row in which each entry is zero. Find P^2, P^3, and P^4.

123. Find the third derivative of $p(x) = 2-3x+8x^2-7x^3$ like any beginning calculus student would do in a calculus course (no matrices).

124. Find the third derivative of the polynomial in the previous problem by matrix multiplication, demonstrating you can find it in a single computation.

125. What information is gained from $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 2 \\ 1 & 4 & 9 \\ 3 & 0 & 3 \end{bmatrix}$? How might you interpret this product, given the information in this section?

8.2 Converting Row Operations to Matrices

Work exercises 126-136, following the instructions carefully.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

126. Starting with I_2, perform the row operation $-R1 + R2$. Call this new matrix E_1.
127. Again, *starting* with I_2, perform the row operation $-\frac{1}{3}R2$. Call this new matrix E_2.
128. *Starting* with I_2, perform the row operation $-2R2 + R1$. Call this new matrix E_3.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

129. Find E_1M.
130. Multiply the result of the previous exercise on the left by E_2; that is, find E_2E_1M.
131. Multiply the result of the previous problem on the left by E_3; that is, find $E_3E_2E_1M$.
132. Find $E_3E_2E_1$. Call this matrix B.
133. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Find AB.
134. Find BA.
135. Forget matrix multiplication for this exercise, and use elementary row operations to write the matrix M in reduced row echelon form.
136. I once had a mathematics professor who wrote his homework assignments with a purpose or intention in mind. A single set of problems often put together a bigger picture, tied things together, built a foundation for understanding a concept, etc... The idea was that students, rather than blindly going about their work producing answers, would make the necessary connections and/or conclusions about the content on their own. He always ended each assignment with a last problem: *Synthesize this work.* Sometimes this was a very obvious and elementary. Other times it was challenging. No matter how easy or difficult, it made an impression on me, and made the material far more meaningful. I'll ask you a question: What is the purpose of exercises 126-135? Synthesize this work in your own words. What is it, especially if you are able to articulate it, that you have connected or learned? If you would like some help, see the footnote.[1]

DEFINITION 6: Let A be a square matrix of order n (Let A be an $n \times n$ matrix.) If there exists an $n \times n$ matrix B such that $AB = BA = I_n$, then A is said to be an **invertible** matrix or a **nonsingular** matrix, and B is the **inverse** of A. In this event,

[1] What system of linear equations is represented by the matrix M? Solve this system in *at least four* different ways, utilizing the results of your work in problems 126-135.

we write $B = A^{-1}$. In other words, if A is invertible, we denote its inverse by A^{-1}. If no such inverse exists, then A is said to be **singular**.

Example 18: Consider the first problem that appears in this book. Solve the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ x_1 + 4x_2 &= 4 \end{aligned}$$

Solving by Gauss-Jordan elimination requires two elementary row operations:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$$

The system has a *unique* solution. As a matrix equation $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Each elementary row operation corresponds to an **elementary matrix**. In the previous set of exercises, one of the connections I hope you were able to make on your own was that the elementary row operations can be easily converted to matrices for changing both sides of the matrix equation. Notice the first elementary row operation for the system in this example: $-R1 + R2$. This changes the augmented matrix as seen above. If we perform the same row operation on the identity matrix, we get an elementary matrix that we will call E_1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Similarly, we can obtain a second elementary matrix corresponding with the second row operation $-3R2 + R1$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

The stage is now set to solve our matrix equation. The two row operations result in an augmented matrix in which the identity matrix is reached; that is, when we multiply the coefficients matrix A on the left in the same order in which we perform the row operations, we find that $E_2E_1A = I_2$. This means the product $E_2E_1 = A^{-1}$. And, of

course, the solution to the system can be found by $E_2E_1\mathbf{b}$. As you did in beginning algebra in grade school or high school with simple equations involving real numbers, the algebraic steps in our matrix equation involve two steps that correspond to the same two steps in Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

The process of solving the matrix equation is meticulously spelled out (above), and, to be clear, *this is not how we actually want to solve these systems*! Gauss-Jordan elimination is much quicker, and, if we were to solve the matrix equation, we would improve in our efficiency by simply multiplying both sides by the inverse of the matrix A. This means we need to figure out how to find the inverse. It is clear that we can find it by converting the elementary row operations to elementary matrices; however, this is also not the most efficient thing to do. The method of finding the inverse of a matrix ought to be clear from this discussion! *Performing row operations is quick!* Finding the inverse of A is reserved for the next chapter. Before finishing this chapter, we do want to acknowledge something very important about this tedious process outlined in this section regarding elementary matrices. Consider the simple equation $6x = 12$. While you know to "divide both sides by 6" to obtain the solution, suppose you do something similar that is done at the top of this page with the two elementary matrices. The number 6 has two prime factors: 2 and 3. We can mirror the work we did above in solving this equation:

$$\begin{aligned} 6x &= 12 \\ \frac{1}{2} \cdot 6x &= \frac{1}{2} \cdot 12 \\ 3x &= 6 \\ \frac{1}{3} \cdot 3 &= \frac{1}{3} \cdot 6 \\ 1x &= 2 \\ x &= 2 \end{aligned}$$

An alien race might do this, only to discover that *the number can be factored*. The process of finding elementary matrices gives us a way to factor a matrix. **The more modern approach in the study of linear algebra is to direct students towards five different ways of factoring a matrix**. This is the singlemost reason why I have chosen to write a book on introductory linear algebra. It is an extraordinary subject, and it has been evolving, changing, and growing in recent years with the advances that are happening in machine learning and data science!

8.3 Factoring a Matrix

If we look closely at the algebraic nature of the matrices in example 18, we know that

$$E_2E_1A = I_2.$$

This gives us a way to factor the coefficients matrix A. Read each step below very closely, with special attention paid to the side on which matrix multiplication must occur. Note that the row operations can be easily reversed, and so the elementary matrices are invertible (or we can say *nonsingular*).

$$\begin{aligned} E_2E_1A &= I_2 \\ E_2^{-1}E_2E_1A &= E_2^{-1}I_2 \\ E_1A &= E_2^{-1} \\ E_1^{-1}E_1A &= E_1^{-1}E_2^{-1} \\ A &= E_1^{-1}E_2^{-1} \end{aligned}$$

All we need to do is find these inverses, which is a matter of thinking of the row operations in reverse.

Example 19: For the system in the previous example 18, find the inverses of the elementary matrices.

solution: Observe the steps in Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$$

E_1 corresponds with $-R1 + R2$. Think to ask yourself how would you reverse this operation? What row operation would make the following possible?

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{?} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \end{bmatrix}$$

Row 2 is the row that is changing, so you must use row 1 to change row 2. You would have to take row 1 and add it to row 2; that is, $R1 + R2$ does the job, so figure out how

it changes the identity matrix. The resulting matrix will be E_1^{-1}:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

which means that $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. I will leave the other matrix to you:

137. Find the row operation that reverses E_2, explaining this.

138. Show that $E_2^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ by performing the elementary row operation on the identity matrix.

With these two inverses found, and the matrix algebra showing that $A = E_1^{-1}E_2^{-1}$, we have a natural factorization of the matrix:

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Due to the simple nature of this particular system of linear equations, the factorization is an ideal kind of factorization. The matrix on the left, E_1^{-1}, is a **lower triangular matrix**, which is a square matrix where there are zeros above the main diagonal. Usually, lower triangular matrices are given the name L. The matrix on the right,E_2^{-1}, is an **upper triangular matrix**, which is a square matrix with zeros below the main diagonal. These are given the name U. This happens to be what is called an LU-factorization of A. Often, such a factorization comes with a more careful set of guidelines and rules, but we are focusing on the row operations only for the time being, which help us find some kind of breakdown of the coefficients matrix A and rewrite it as a product of other matrices.

Consider the next example after working exercise 139, and know that this is an extra example placed here for a thought experiment. *You are permitted to skip this example if you wish.* Example 20 is a tease for what is to come in a forthcoming chapter in which we give what will be called the CR-factorization of a matrix A.

139. Use row operations to write the matrix $\begin{bmatrix} 1 & 3 & 2 & -4 \\ 2 & 0 & 4 & -8 \\ 1 & 1 & 2 & -4 \end{bmatrix}$ in reduced row echelon form.

Example 20: Observe *the columns* of the matrix in exercise 139. Think of these columns as vectors. As you view them from left to right, the first column is the vector $(1, 2, 1)$. One by one, left to right, proceed to ask whether or not each column vector is a scalar multiple of this vector, or the next vector. The second column is the vector $(3, 0, 1)$. This is not a scalar multiple of the first column. But the *third column* is twice the first column since, by scalar multiplication, $(2, 4, 2) = 2(1, 2, 1)$, and the fourth column

is -4 times the first column since $(-4,-8,-4) = -4(1,2,1)$. In other words, *three* of the columns, as vectors, are *located on the same line* in the Euclidean space $\mathbb{R}^3$. Vectors that are located on the same line, or vectors that are scalar multiples of each other, are good to locate because you will find a simpler way of working with bits and parts of matrices in which you detail only the vectors that are not scalar multiples of each other. The matrix, you might say, is really constructed from only *two* vectors that are not on the same line: $\mathbf{v_1} = (1,2,1)$ and $\mathbf{v_2} = (3,0,1)$. We will create a matrix to store these two vectors as column vectors and call it C:

$$C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Next, we are going to form a matrix called R by looking at the original matrix, which has four columns, and describe each column of the matrix as **a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$**. Here is how the thinking conversation in your head goes: With your eyes on the matrix in 139, the first column is $1\mathbf{v_1} + 0\mathbf{v_2}$, the second column is $0\mathbf{v_1} + 1mathbfv_2$, the third column is $2\mathbf{v_1} + 0\mathbf{v_2}$, and the fourth column is $-4\mathbf{v_1} + 0\mathbf{v_2}$. These four linear combinations are the columns of R:

$$R = \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It turns out that the matrix in 139 can be factored as the product CR; that is,

$$\begin{bmatrix} 1 & 3 & 2 & -4 \\ 2 & 0 & 4 & -8 \\ 1 & 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(Take a look at your answer to problem 139. Notice anything?)

140. Verify the matrix product above; that is, find $\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

No additional exercises are given as this particular chapter requires careful thought and concentration, but you have ten interesting systems that start this book, if you would like to explore them in similar ways that we discussed the first system. This chapter introduces some knowledge connecting different actions students perform in a linear algebra course, often without understanding what those connections are; hence, the confusion many sometimes express about the subject. Next up: how to find the inverse of a matrix...

Chapter 9

A Method for Finding the Inverse

9.1 Nonsingular and Singular Matrices

When the word *singular* is used in mathematics, it is describing something that is strange, unusual, peculiar, striking, often deviating from the usual or expected. Suppose your initial experience in solving systems of linear equations was something like this: the first time ever an instructor taught you how to solve a system of linear equations, the result was "no solution." You went home, began a homework assignment in which you solved systems of equations and the first several problems also yielded no solution. Suddenly, you worked a problem and there was an answer. You would have most likely had a feeling of "this is unusual" come over you, meaning that the result is a little out of the ordinary. Your experience should have been the opposite, where most problems resulted in some kind of solution. Every once in a while, something unexpected happened that guided you to logically conclude that there could not be a solution. This happens often in the mathematical sciences in different contexts: The point $x = 0$ is referred to as a *singularity* for the function $f(x) = \frac{1}{x}$ because the function is defined for all other values of x with the exception of $x = 0$. The center of a black hole, a point of infinite density and gravity where no object can escape, is referred to as a *singularity* in Albert Einstein's 1915 *Theory of General Relativity.* You are initially taught methods to solve systems of equations and experience the best case scenario of that method working well, producing the ideal result, which means that the system, when written in matrix form, contained a matrix A that was invertible. That feels like the usual. It is somewhat rare to obtain "no solution" or "infinite number of solutions" when using a method to solve the system, which means that the matrix equation of the system contains a matrix A that is a bit peculiar, or *singular.* This is the way to think of the word as it is used in the definition of an invertible matrix, restated from the previous chapter for emphasis. (Ever heard the lyrics to the song "One" from the musical *A Chorus Line*?)

DEFINITION 6: Let A be a square matrix of order n (Let A be an $n \times n$ matrix.) If there exists an $n \times n$ matrix B such that $AB = BA = I_n$, then A is said to be an **invertible** matrix or a **nonsingular** matrix, and B is the **inverse** of A. In this event, we write $B = A^{-1}$. In other words, if A is invertible, we denote its inverse by A^{-1}. If no such inverse exists, then A is said to be **singular**.

9.2 How to Find the Inverse

In matrix form, a system of linear equations can be interpreted as $A\mathbf{x} = \mathbf{b}$, where the goal is to find the unknown $\mathbf{x}$ such that when it is multiplied by A on the left, it results in $\mathbf{b}$. The quickest way to obtain that, in general, by hand, involves employing elementary row operations. The same logic applies to the question of finding an inverse. According to the

definition, the inverse is an unknown matrix X with the property that $AX = I$. It makes sense to create an augmented matrix and use row operations to find X.

Example 21: Assume that $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ is invertible. Find A^{-1}.

solution: Since the definition requires that $AA^{-1} = I$, then, in matrix form,

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Form the augmented matrix and, as you did with solving systems, use elementary row operations to write it in reduced row echelon form:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

The matrix at the end shows that $A^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$. *I strongly encourage you to start paying close attention to* ***definitions*** *in this course.* If we check this answer and honor the definition, we would check and make sure that $AA^{-1} = I$ and $A^{-1}A = I$:

$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4-3 & -3+3 \\ 4-4 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$A^{-1}A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4-3 & 12-12 \\ 1+1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since we have seen that matrix multiplication is *not commutative*, and since we have a definition that states these two products result in the identity, then is it true that the specific product of any matrix with its inverse always commutative? I am glad I pretended you asked. You are a mathematician. This is where real growth in mathematics happens, when you arise from computational mathematical study, and grow into theoretical mathematical study, and start *thinking*.[1]

Example 22: Solve the system by finding the inverse of its coefficients matrix, if it exists:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 11 \\ 2x_1 - x_2 + 3x_3 &= 7 \\ 7x_1 - 3x_2 - 2x_3 &= 2 \end{aligned}$$

solution: As a matrix equation $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$, the solution to the system will be $\mathbf{x} = A^{-1}\mathbf{b}$. We set up the augmented matrix to find the inverse of A, thinking $AA^{-1} = I$:

[1] The secret to doing that is to return yourself to who you were when you were a single-digit age, play, and ask why. You should never be *forced* to think.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 7 & -3 & -2 & 0 & 0 & 1 \end{bmatrix}$$

Next, we apply row operations to obtain the reduced row echelon form of this augmented matrix. This one is a nice exercise in arithmetic:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 7 & -3 & -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-7R1+R3]{-2R1+R2} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 5 & -2 & 1 & 0 \\ 0 & -17 & 5 & -7 & 0 & 1 \end{bmatrix} \tag{9.1}$$

$$\xrightarrow[-\frac{1}{17}R3]{-\frac{1}{5}R2} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & -\frac{5}{17} & \frac{7}{17} & 0 & -\frac{1}{17} \end{bmatrix} \tag{9.2}$$

$$\xrightarrow[-R2+R3]{-2R2+R1} \begin{bmatrix} 1 & 0 & 1 & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -1 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{12}{17} & \frac{1}{85} & \frac{1}{5} & -\frac{1}{17} \end{bmatrix} \tag{9.3}$$

$$\xrightarrow{\frac{17}{12}R2} \begin{bmatrix} 1 & 0 & 1 & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -1 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{60} & \frac{17}{60} & -\frac{1}{12} \end{bmatrix} \tag{9.4}$$

$$\xrightarrow[R3+R2]{-R3+R1} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{60} & \frac{7}{60} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{5}{12} & \frac{1}{12} & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{60} & \frac{17}{60} & -\frac{1}{12} \end{bmatrix} \tag{9.5}$$

Try your best to abandon a common mindset among students in which you learn methods of solution to a problem but do not fully understand why those methods work. Do not forget that the matrix on the left in step (9.1) is an augmented matrix that represents the equation $AX = I$, and the matrix in the last step (9.5) is an augmented matrix that represents $X = A^{-1}$, the *solution* to that equation.[2] Each row operation in the steps between represents an elementary matrix that could be found which can be used to alter the equation. Imagine finding *nine* elementary matrices and then inverting them to give a factorization of the matrix A! The solution to the system is found by acknowledging

$$A^{-1} = \begin{bmatrix} \frac{11}{60} & \frac{7}{60} & \frac{1}{12} \\ \frac{5}{12} & \frac{1}{12} & -\frac{1}{12} \\ \frac{1}{60} & \frac{17}{60} & -\frac{1}{12} \end{bmatrix},$$

[2] Another tip on deepening your level of thought when studying mathematics: *Question the use of specific words that are often given.* Why do you think the word *augmented* is used for these matrices?

and that the solution to the system is obtained by multiplying this inverse by **b**. I'll leave it to you to arrive at the solution $x_1 = 3, x_2 = 5$, and $x_3 = 2$, often written in the space-saver notation as $(3, 5, 2)$, or as the column vector $\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

Though the arithmetic in the previous example is busy, it is worth doing a few of these kinds of problems and then know when it is time to turn to using technology. I chose this example because I thought the inverse of A was worth playing with for a brief moment: Note the denominator of each fraction is either 12 or 60. If we rewrite the matrix such that every denominator is the same, then we can write

$$A^{-1} = \begin{bmatrix} \frac{11}{60} & \frac{7}{60} & \frac{5}{60} \\ \frac{25}{60} & \frac{5}{60} & -\frac{5}{60} \\ \frac{1}{60} & \frac{17}{60} & -\frac{5}{60} \end{bmatrix},$$

and we can factor out the scalar 1/60 from the matrix, and then write

$$A^{-1} = \frac{1}{60} \begin{bmatrix} 11 & 7 & 5 \\ 25 & 5 & -5 \\ 1 & 17 & -5 \end{bmatrix}.$$

The number 60 is a special number related to the original matrix A, a future topic. This is an example to which we will return.

I recall doing work like in the previous example as a student, questioning the use of this method for solving the system. While it is important to appreciate solving a system by Gauss-Jordan elimination, or converting it to a matrix equation, obtaining an inverse to multiply on both sides of that equation, thereby solving it, I noticed that the work done by hand is the same for the two different methods: had we solved the previous system by elimination, we would have performed the same elementary row operations. My instructor at the time convinced me that obtaining the inverse was sometimes an important task. I recall asking "why can't we do both at the same time?" and, every time I had to solve a system of equations, I did my work like I show in the next example:

Example 23: Solve the system:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= -3 \end{aligned}$$

solution by a former smartass:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & -1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & -3 & -6 & -1 & 1 \end{bmatrix} \tag{9.6}$$

$$\xrightarrow{-\frac{1}{3}R2} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \tag{9.7}$$

$$\xrightarrow{-2R2+R1} \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 2 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \tag{9.8}$$

This work shows that the solution to the system is $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and the inverse of $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$. If you want to confirm the solution is $(-1, 2)$, then write the matrix equation of the system, and find the product $A^{-1}\mathbf{b}$.[3]

9.3 A Matrix that Integrates a Function

You've met two interesting matrices that differentiate two kinds of functions:

$$D = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

differentiates all functions of the form $ae^{2x}\cos 3x + be^{2x}\sin 3x$. Suppose we put all functions of this form inside a set called $\mathbb{F}$. All functions in $\mathbb{F}$ can be represented by a vector in the form $\begin{bmatrix} a \\ b \end{bmatrix}$. These functions are vectors because you can add them and scalar multiply them. When the matrix D operates on all vectors of the form (a, b), where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, the result is also a vector in the same family of functions. This is a property[4] that distinguishes important operators: If $\mathbf{x} \in \mathbb{F}$ then $D\mathbf{x} \in \mathbb{F}$; that is the derivative of any constant times $e^{2x}\cos 3x$ plus any constant times $e^{2x}\sin 3x$ is always a constant times $e^{2x}\cos 3x$ plus a constant times $e^{2x}\sin 3x$:

$$D\mathbf{x} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a + 3b \\ -3a + 2b \end{bmatrix}$$

141. Differentiate the function as you did in a first course in calculus to show that the derivative of $ae^{2x}\cos 3x + be^{2x}\sin 3x$ is $(2a + 3b)e^{2x}\cos 3x + (-3a + 2b)e^{2x}\sin 3x$.

[3] In my later years now as a professor of mathematics, I sometimes ask students to solve by elimination, and then find the inverse of the coefficients matrix, and check the solution they found by demonstrating the solution to the equivalent matrix equation.

[4] This is called **closure**, a property we will define and use soon; As an example of something with which you should be familiar, the set of real numbers is *closed under addition* because adding two real numbers results in a real number.

142. Find the inverse of D using the method shown in this chapter. (The method demonstrated in the examples in this chapter is often called the *Matrix Inversion Algorithm* or *Inversion Algorithm.*)

143. Recall the torture in integrating functions of this form. Use integration by parts to evaluate $\int e^{2x} \cos 3x \, dx$.

144. The function you integrated in the previous problem was $f(x) = e^{2x} \cos 3x$. This can be represented by the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $D^{-1}\mathbf{v}$. Does this agree with your answer in 143? What does this inverse matrix *not* do that it ought to do?

Use the following matrices for exercises 145-152.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

145. Find A^{-1}.

146. Find B^{-1}.

147. Let $C = AB$. What is C?

148. Find C^{-1}.

149. Find the matrix product $A^{-1}B^{-1}$

150. Find the matrix product $B^{-1}A^{-1}$.

151. Let $D = BA$. Find D.

152. Find D^{-1}.

153. What do you think is the purpose of exercises 145-152?

154. **True** or **False**: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & -1 & 4 \end{bmatrix}$ is a singular matrix.

155. **True** or **False**: $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -4 \\ 1 & -1 & 7 \end{bmatrix}$ is a singular matrix.

Solve the following systems in 156-158 by Gauss-Jordan elimination. Then check the solution to the system by converting it to a matrix equation, finding the inverse of the coefficients matrix, and using it to justify the solution.

156.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 2 \\ x_2 - 4x_3 &= 5 \\ 2x_1 - x_2 - 2x_3 &= 7 \end{aligned}$$

157.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 - 3x_2 + x_3 &= -1 \\ 5x_1 + 7x_2 + 2x_3 &= -1 \end{aligned}$$

158.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 3x_2 - x_3 &= 0 \end{aligned}$$

159. Derive a formula *on your own* to find the inverse of any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What condition, according to the formula you derived *on your own because it will be blatantly obvious if you use the internet to replace thinking*, must be met in order for an inverse to exist?

160. Test your formula out on the matrices A and B that precede problem 145.

Chapter 10

Recap: What Are We Doing?

10.1 A Matrix That Represents the Square Root of Negative One

Before we address the question that titles this chapter, let's play with the imaginary number i, which belongs to the set of all complex numbers, denoted by $\mathbb{C}$. All numbers in this set are of the form $a + bi$, where a and b belong to the set of real numbers, denoted by $\mathbb{R}$.[1] At some point in your past, you were exposed to complex numbers. All you really paid attention to, aside from refining your algebra skills, was that $i = \sqrt{-1}$, and, of course, $i^2 = -1$. Simplify the following expressions involving complex numbers.

161. $(2 + i)(3 + i)$
162. $(2 + 3i)(4 - 5i)$
163. $2i(1 - i)$
164. $(1 - i)^2$

Meet the matrix C that represents any complex number $a + bi$:

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Expressed in the form of a complex number, $i = 0+1i$, so, as a matrix, i is represented by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

165. Find C^2 and check to see if the result agrees with $i^2 = -1$.
166. Find $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$. Does this agree with your result in 161?
167. Write the problem in 162 as a matrix product, find the product, and compare it to your answer in 162.
168. Repeat this for problem 163.
169. Repeat this one more time for problem 164.
170. Find the inverse of the matrix that represents i. How can you make sense of this result?

[1] By the way, the notation $i \in \mathbb{C}$ means "i belongs to the set $\mathbb{C}$" and the notation $a, b \in \mathbb{R}$ means "a and b belong to the set $\mathbb{R}$."

10.2 Seriously, What *Are* We Doing?

I find it extremely important to take a pause often when studying the content of a course and ask a good question: *What is this course about?* Making connections sets you on a path towards success and alleviates frustration. We move too fast in this culture, treating bits of knowledge as things to see and then quickly pass as if it is a place to one day say "I have been there." In every course, there is a "big picture" that kind of sums up what the course is about, or ties concepts together, bringing some kind of sense to the subject. I believe that this is what people want, and the "exam-grade" culture often forms a barrier.

Mathematics consist of many courses of study. Within each course, there is usually a big truth that sums up its main ideas, something *fundamental* about the course. All of you have heard of a fundamental theorem in calculus. You began studying calculus in elementary school, but it was calculus without limits, and that's called Arithmetic. In Arithmetic, where you can count with numbers and use them to measure things, a fundamental theorem is that addition is the opposite of subtraction. In algebra, where you learn all of the rules and properties of real numbers, do you recall reaching a point of study where you read the Fundamental Theorem of Algebra? After Arithmetic and Algebra, you take calculus with limits, and that's called Calculus. Addition with limits is called integration. Subtraction with limits is called differentiation. The fundamental theorem in calculus is that integration is the opposite of differentiation, similar to the one in Arithmetic. (Linear algebra teaches you that integration and differentiation are actually *pseudoinverses*.)

These are all courses in mathematics that focus on the ways that you can combine real numbers. Crucial in the mastery of this mathematics is understanding the properties of that set of numbers. You learn the properties of the set of real numbers when you study algebra prior to college mathematics. Of those properties, ten stand out in contrast to other sets of objects in a set that you can combine in similar ways:

Let $a, b, c \in \mathbb{R}$.

1. $a + b \in \mathbb{R}$.
2. $ka \in \mathbb{R}$ for any real number k.
3. $a + (b + c) = (a + b) + c$.
4. There exists an *additive identity element* called a **zero element** such that $a + 0 = 0 + a = a$.
5. There is an *additive inverse element*, a negative of a, denoted by $-a$ such that $a + (-a) = (-a) + a = 0$.
6. $a + b = b + a$.
7. $k(a + b) = ka + kb$ for any real number k.
8. $(k + m)a = ka + ma$ for any real numbers k and m.

9. $k(ma) = (km)a$.

10. There is a *multiplicative identity element* 1 such that $1a = a$.

These properties are important because they characterize things in a set that, in the future, we will call vectors. It's worth frequently looking at them, and I triple-dare you to know them well. A way to know them is to understand that there are two *closure* properties, four additive properties, and four multiplicative properties. You will become more and more familiar with these in time.

Here you are in a linear algebra course, however, and what is it that we are doing? I have repeated a sentence several times, both in the lecture and in this book: *The fundamental problem in linear algebra is solving a system of linear equations.* We have reached the part of the course in which we really begin what is known as linear algebra, the study of line-like relationships. This is an *introductory* course in linear algebra, and part of introducing the subject is to (1) explore solving systems of equations, introducing the mathematical objects that arise from systems: *matrices*. There is something extraordinary about matrices. They do things. I hope you have been surprised or entertained or charmed by the ones you have met thus far. They have a lot to do with something always present in this mathematics: **linear combinations**. Systems of linear equations are sentences of linear combinations of unknowns. A matrix is a place to store linear combinations. The column perspective of a system gives us a way to restate a system as a linear combination of vectors. The reason why I decided to open this chapter to meet the matrix C is it gives me a first chance to explain to you how the matrix is formed by helping you *see* things in terms of linear combinations. But remember, this all begins with systems!

$$\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}$$

We line up the x terms and the y terms after noticing the expressions on the left of both sentences in the system are *linear combinations of x and y*. If we store the linear combinations in a matrix, we have

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix},$$

then we can see this is the matrix product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

I firmly believe that if the history of mathematics were taught to you as you learned the subject, then the difficulties and frustrations often encountered would be eased or minimized in some way. (You might think differently about calculus, for example, if you knew that integration came first, then infinite series, and, thanks to a brilliant idea between Newton and Leibniz, differentiation arrived after to make integration possible.) Vectors, for example, came from complex numbers, which are represented by directed line segments in a space called the complex plane. Every complex number can be written in the form $a + bi$, and mathematicians often write these as (a, b), where a is called the real part of the complex number, and b is called the imaginary part of the complex number. You

are accustomed to this notation when you work in the xy-plane. If you see the ordered pair $(3, 5)$, you are not confused that 3 is the x-coordinate and 5 is the y-coordinate. Things have places. To understand how $a + bi$ can be represented by a matrix, let's find the product of $a + bi$ with $x + yi$:

$$(a + bi)(x + yi) = ax + ayi + bxi + byi^2 \tag{10.1}$$
$$= ax + ayi + bxi - by \tag{10.2}$$
$$= (ax - by) + (ay + bx)i \tag{10.3}$$

In (10.1) we employ our algebra to multiply, and in (10.2) we replace i^2 with -1. In (10.3), we group the terms so that we have the terms without i and the terms with i. Take the right side of (10.3) and look carefully. One of my favorite quotations is this: *If you change the way you look at things, the things you look at will often change.* ***Do your eyes see linear combinations on the right side of the last line?***

$$(ax - by) + (ay + bx)i$$

Line up the two linear combinations of x and y:

$$ax - by$$
$$bx + ay$$

Put them in a matrix, and express it as a matrix product:

$$\begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear combinations are at the heart of this subject. Spot them whenever you can. If you are ever lost, try directing your thinking by looking for linear combinations. Notice how the matrix is no longer mysterious. Also, since the real numbers are included in the set of all complex numbers, that you can easily express the number 1 as a matrix. Since $1 = 1 + 0i$, then $a = 1$ and $b = 0$ in the matrix we found that can represent any complex number. What is that matrix? (I am pretending you are writing it down at this very moment and love what you see as the result. There is some sense to this, yes?)

10.3 Recognizing Patterns

If we focus on the ideal system of linear equations, which can result in a unique solution, this is a system of n equations and n unknowns. The goal here is to have you, the student, do some work to see some patterns in certain kinds of equations. Solve all equations and systems of equations first. Then, read the comments that follow these exercises.

1 equation and 1 unknown:

171. $3x = 0$

172. $0x = 0$

173. $0x = 6$

2 equations and 2 unknowns:

174.

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 + 6x_2 &= 0 \end{aligned}$$

175.

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ 6x_1 + 3x_2 &= 0 \end{aligned}$$

176.

$$\begin{aligned} x_1 - x_2 &= 0 \\ -x_1 + x_2 &= 3 \end{aligned}$$

3 equations and 3 unknowns:

177.

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 0 \\ 3x_1 + 6x_2 - 8x_3 &= 0 \\ -2x_1 - x_2 &= 0 \end{aligned}$$

178.

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 0 \\ x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + 8x_3 &= 0 \end{aligned}$$

179.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

180.

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 3 \\ x_2 + 3x_3 &= -1 \\ x_1 + 2x_2 + 8x_3 &= 3 \end{aligned}$$

One broad pattern to notice is that, no matter the size of the system, there are three possible outcomes: the system has one and only one solution, an infinite number of solutions, or no solution. But remember that "infinite number of solutions" is not a good description of a solution set to a system. This is key to doing well in linear algebra and, consequently, key to being able to *use linear algebra in a variety of occupations*, especially in computer science, machine learning, data science, and engineering. If we look closer, specifically with the case when there are an infinite number of solutions, we see a much more interesting pattern:

When there are an infinite number of solutions, notice this:

1. **1 equation, 1 unknown:** all real numbers
2. **2 equations, 2 unknowns:** all scalar multiples of a vector
3. **3 equations, 3 unknowns:** all scalar multiples of a vector, all linear combinations of two vectors
4. **4 equations, 4 unknowns:** all scalar multiples of a vector, all linear combinations of two vectors, all linear combinations of three vectors, etc...

Line-like relationships: Two lines can intersect at a point, or a line. Three planes can intersect each other at a point, a line, or a plane. Four "line-like" things in $\mathbb{R}^4$ can intersect each other at a point, a line, a plane, or a line-like thing in the next dimension... *This is where you start becoming clear about linear algebra.* There are many more patterns to notice! Linear algebra focuses on the things that house linear combinations: ***What do you see when you look at the coefficients matrix of each system?!*** The theory we will study revolves around the invertibility of the square matrix A in each system. In fact, we will develop the largest theorem you will see in mathematics called *the Invertible Matrix Theorem.* But the most important aspect of an introductory course in linear algebra is to learn and understand the language of linear algebra: All linear combinations of a set of vectors, including all scalar multiples of a single vector, is always something line-like that passes through an origin. These are **subspaces**. Here is our agenda, more or less: to master the vocabulary of linear algebra, experience the art of writing mathematical proofs by honoring and appreciating definitions, theorems, and the ten very important axioms mentioned earlier that characterize a set of vectors that form what is known as a **vector space**, experience abstraction of this knowledge, and, perhaps the most important for many of you taking the course, to understand what matrices actually are and have a glimpse at how they are used in applied mathematical work. Linear algebra is a changing and evolving subject, cannot be learned in twelve weeks, is a subject one could spend a lifetime learning, and is perhaps the most important mathematics course to have in mind when studying the subject of mathematics. Welcome to this world. It is my hope that you learn how to play in this subject.

Chapter 11

Subspaces: Line-like Things That Pass Through Origins

11.1 Meet the Zombie Apocalypse Matrix Z

Linear algebra will be a fantastic tool if humans are plagued by a terrible zombie virus. Imagine a college campus has been closed and sealed for safety with the population kept constant inside it at 300 humans. No one is allowed to leave the area and no one is allowed to enter due to a zombie virus. Unfortunately, a zombie outbreak begins inside. Within weeks, algebraists discovered that there was a pattern in the infected humans: Every hour, 20% of humans are converted to zombies due to an infection, but there seems to be something of a cure, and each hour, 10% of zombies are cured and become human. With the outbreak advancing, a count was finally able to be made at a certain hour: there were 240 humans and 60 zombies. The crucial question was whether or not the humans would be able to survive the outbreak while kept in quarantine. Someone with linear algebra powers heard this news, and immediately identified the crucial linear combinations of humans and zombies: Let h represent the number of humans and z represents the number of zombies. If, each hour, 20% of humans are converted to zombies, and 10% of zombies are converted to humans, then *two linear combinations* can be written based on two questions: After each hour, (1) how many humans are there? (2) how many zombies are there? The answers to these are the linear combinations (1) $0.8h + 0.1z$ and (2) $0.2h + 0.9z$. If 20% of humans are converted to zombies, then 80% of humans remain human. If 10% of zombies become human then 90% remain zombies. If, at the start of an hour there are h humans and z zombies, then an hour later, there will be $0.8h + 0.1z$ humans and $0.2h + 0.9z$ zombies. write these two expressions to line up things, place them in a storage container called a matrix.

$$0.8h + 0.1z$$
$$0.2h + 0.9z$$

$$\begin{bmatrix} 0.8h + 0.1z \\ 0.2h + 0.9z \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} h \\ z \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Meet the matrix Z. There is a considerable amount of information held in this matrix. The operation of the matrix on vectors of the form (h, z) will tell you how many humans are present after one hour. Repeated operations of the matrix on that result predict the number of humans after however many hours. The initial condition is given in the above story of the situation is that 240 humans and 60 zombies have been counted. Let $\mathbf{v_0} = \begin{bmatrix} 240 \\ 60 \end{bmatrix}$ represent this initial condition. Then $Z\mathbf{v_o}$ is the number of humans and zombies after one hour, $Z(Z\mathbf{v_0}) = Z^2\mathbf{v_0}$ is the number after two hours, $Z(Z^2\mathbf{v_0})$ after three, etc...

181. Find $\mathbf{v_1} = Z\mathbf{v_0}$, the number of humans and zombies after one hour.

182. Find $\mathbf{v_2} = Z\mathbf{v_1}$.

183. You will want to play on the calculator or some piece of technology to answer this question: if this virus goes on indefinitely, will humans survive the zombie outbreak? Will there eventually be no humans, and only zombies? (Try the desmos matrix calculator.)

We will be returning to this matrix a few times to show you what the matrix is and can do. By the way, any square matrix with nonnegative numbers whose columns add up to 1 is called a **left stochastic** matrix.If both rows and columns add to 1, the matrix is usually called a **doubly stochastic** matrix.

11.2 Linear Combinations of Vectors

DEFINITION 7: A **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is the expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, where $c_1, c_2, ..., c_n$ are scalars.

THEOREM 1: All vectors in $\mathbb{R}^2$ can be written as a linear combination of the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

PROOF: All vectors in $\mathbb{R}^2$ are of the form (a, b), where $a, b \in \mathbb{R}$. By vector addition, $(a, b) = (a, 0) + (0, b)$. By scalar multiplication, $(a, b) = a(1, 0) + b(0, 1)$, which means we can write $(a, b) = a\mathbf{i} + b\mathbf{j}$.

184. Let $\mathbf{u} = (2, 5, -3), \mathbf{v} = (-4, 1, 9), \mathbf{w} = (4, 0, 2)$, and determine the linear combination $2\mathbf{u} - 3\mathbf{v} + \mathbf{w}$.

185. Determine whether the vector $(8, 0, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, -1, 1)$.

11.3 Solutions to Homogeneous Systems

In matrix form, all systems of linear equations is represented by $A\mathbf{x} = \mathbf{b}$. When $\mathbf{b} = \mathbf{O}$, the zero vector, the system is called **homogeneous**. Homogeneous systems always have a solution set. Look at the systems in exercises 174-180. All were homogeneous with the exception of 176 and 180. There is something to notice about the solution sets to systems of the form $A\mathbf{x} = \mathbf{O}$. The zero vector is always a solution. It might be the only solution, but it is always *a* solution. If it is not the only solution, then the solution set is either a scalar multiple of a single vector, or it is the set of all linear combinations of a set of vectors.

Solve the following homogeneous systems of linear equations.

186.

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 2x_1 + x_2 - 5x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 - 8x_3 + \ \ x_4 &= 0 \end{aligned}$$

187.

$$\begin{aligned} x_1 + \ \ x_2 + \ \ x_3 + 3x_4 &= 0 \\ x_1 + 2x_2 + 4x_3 + 5x_4 &= 0 \\ x_1 \qquad\quad - 2x_3 + \ \ x_4 &= 0 \end{aligned}$$

188. Show that the solution set to the system in 186 is the set of all linear combinations of the vectors $(3, -1, 1, 0)$ and $(1, -4, 0, 1)$.[1]

189. Show that the solution set to the system in 187 is the set of all linear combinations of the vectors $(2, -3, 1, 0)$ and $(-1, -2, 0, 1)$.

The phrase "*the set of all linear combinations* of a set of vectors" describes one of the most important spaces in this subject, so much so that we give them a name. In the last section of this chapter, the terms *span* and *subspace* are introduced. You will want to prepare yourself to practice both writing and speaking these terms, and the expectation is that you fully master the use of these terms in both writing and speaking.

11.4 Vocabulary: SPAN and SUBSPACE

Let's do a thought experiment: Take out a piece of graph paper, and consider the vector $\mathbf{u} = (2, 3)$. Some of you think of the directed line segment from the origin to the point $(2, 3)$ when you hear the word "vector" used, but you can also represent $(2, 3)$ as the usual "dot" you make when you plot the point in the xy-plane. Draw a set of axes, and pick one: arrow or dot. Mark this vector on the graph. The task is to plot *all scalar multiples* of $\mathbf{u}$. Suppose this is possible for us to do. What does your picture look like if you were to complete this task, no matter if you used dots or arrows? Once you are done, you would see a line containing the origin and the point $(2, 3)$. There are lots of ways to describe this line: in high school algebra, you wrote its equation as $y = \frac{3x}{2}$. In words, you could say "the line through the origin having slope $\frac{3}{2}$." You are now allowed to describe lines through origins as *all scalar multiples of a single vector.*

Now, perhaps a more challenging thought experiment: take out a second piece of graph paper. Plot the vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (2, 0)$. You can use dots to mark the points on which these vectors land, or arrows. Think this way before being asked to draw other things: when you sketched $\mathbf{u}$, you sketched one of the possible linear combinations of $\mathbf{u}$ and $\mathbf{v}$: $1\mathbf{u} + 0\mathbf{v}$. Then you sketched the other vector: $0\mathbf{u} + 1\mathbf{v}$. Imagine sketching

[1] Some of you may have naturally written your solution in this way, so problems 188 and 189 might be redundant. Those of you who wrote the solutions in the way that they are presented as answers in the appendix of this book will need to do work in 188 and 189.

all possible linear combinations of $\mathbf{u}$ and $\mathbf{v}$. What do you have when you are done? A line-like thing we call a "plane" that contains the origin.

THE SET OF **ALL LINEAR COMBINATIONS OF A SET OF VECTORS IS SOMETHING LINE-LIKE THAT PASSES THROUGH AN ORIGIN.** When the solution set was not a single vector or "no solution" when you solved a system of equations, it was something line-like. The solution set of all homogeneous systems, when not just the zero vector, is something line-like that passes through an origin. If you sit down to draw the set of all linear combinations of a set of vectors, your work will eventually *cover* all vectors that comprise a line-like space through an origin. To cover a space means to **span** a space. And since set of vectors is the genesis of such a space, we say "this set of vectors *spans* the space" or "these vectors span the space." This is the use of the word "span" as a verb. It is also used as an adjective: the vectors that span the space are sometimes known as a **spanning** set. Finally, the word is also used as a noun: The *span* of a set of vectors is the set of all linear combinations of those vectors. Here is the formal definition in most mathematics books:

DEFINITION 8: A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ of vectors in a space V is said to **span** V if every vector $\mathbf{v} \in V$ can be written as a linear combination of the vectors in the set; that is, there exist scalars $c_1, c_2, ..., c_n$ that can be found such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u} + \cdots c_n\mathbf{u}_n$.

Examples: The set $\{\mathbf{i},\mathbf{j}\}$ is a **spanning set** for $\mathbb{R}^2$. These vectors span $\mathbb{R}^2$. The vector in our thought experiment, $(2, 3)$... All scalar multiples of this vector form the line through the origin and containing the vector. The vector $(2, 3)$ spans the line.

Example 24: Do the vectors $(1, 0, 0), (0, 1, 0)$, and $(9p0, 0, 1)$ span $\mathbb{R}^3$?

solution: All vectors in $\mathbb{R}^3$ are of the form (x, y, z), where x, y, and z are any real numbers.You could answer this in a couple of different ways: one way, would be to do what was done in the proof of Theorem 1 a couple of pages ago, and use vector addition and scalar multiplication to show that all vectors of the form (x, y, z) can be written as a linear combination of the three vectors. Another way is to take the set of given vectors and determine all linear combinations; that is, determine the span of this set of vectors: Here, let $a, b, c \in \mathbb{R}$. The set of all linear combinations of the vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is of the form

$$a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = (a, 0, 0) + (0, b, 0) + (0, 0, c) \tag{11.1}$$
$$= (a, b, c). \tag{11.2}$$

Scalar multiplication justifies (11.1) and vector addition justifies step (11.2), and the resulting set of all vectors of the form (a, b, c), where a, b, and c are any real numbers is the description of all vectors in $\mathbb{R}^3$; therefore, YES, the vectors span $\mathbb{R}^3$. This is often written $\text{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} = \mathbb{R}^3$.

Example 25: Describe the space spanned by the vectors $(1, 2, 3, 4)$ and $(1, 0, 1, 0)$.

solution: The span of $(1, 2, 3, 4)$ and $(1, 0, 1, 0)$ is the set of all linear combinations of

$(1, 2, 3, 4)$ and $(1, 0, 1, 0)$. If a and b are any real numbers, then

$$\begin{aligned} span\{(1,2,3,4),(1,0,1,0)\} &= a(1,2,3,4) + b(1,0,1,0) && (11.3)\\ &= (a, 2a, 3a, 4a) + (b, 0, b, 0) && (11.4)\\ &= (a+b, 2a, 3a+b, 4a). && (11.5) \end{aligned}$$

Step (11.3) is using the defintion of *span*, step (11.4) is scalar multiplication, and step (11.5) is addition. A great final answer is a sentence you write, demonstrating your mastery of the new language in which you are learning: *The span of the vectors* $(1, 2, 3, 4)$ *and* $(1, 0, 1, 0)$ *is the set of all vectors of the form* $(a + b, 2a, 3a + b, 4a)$, which is a plane passing through the origin in $\mathbb{R}^4$.

For exercises 190 and 191, use the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

190. Determine the span of the column vectors of A, and mention what space this occurs based on the number of components the vectors have.

191. Determine the span of the row vectors of A.

DEFINITION 9: The **row space** of a matrix A of size $m \times n$ is the span of the row vectors of the matrix; that is, the set of all linear combinations of the rows of A. The row space of A will be denoted by row(A).

DEFINITION 10: The **column space** of a matrix A of size $m \times n$ is the span of the column vectors of the matrix; that is, the set of all linear combinations of the columns of A. The column space of A will be denoted by col(A).

192. Given an $m \times n$ matrix A, Explain why row$(A) \in \mathbb{R}^n$, and why col$(A) \in \mathbb{R}^m$.

193. Avoid being redundant when thinking and executing this problem. Let A be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find row$(A)$ and col(A).

One way to help make sense of this course is to understand the importance of lines in other courses prior to this one, especially in the study of calculus: in the plane, the slope of a line becomes the major building block of the derivative. When you say a function of *one* variable, $f(x)$, is differentiable at a point, what you mean is that if you were to zoom in closer and closer on the curve around where the point is located, then you will eventually see a line segment. (We have technology that shows this. Examine any curve on a touch screen on the graphing calculator on desmos.com, use hour fingers to zoom in until you see a line.) A function of *two* variables, $f(x, y)$ is differentiable at a point if, when you zoom in as close as you wish, the surface looks like a plane, a line-like thing.

Line-like things are what we use to understand things that are not line-like.[2] Why is this? Why are line-like things so important?

The intuitive definition of **subspace** is that it is something line-like that contains an origin. The formal mathematical definition, stated below, is a mathematically precise way of saying what is special about a line-like thing containing an origin: if any two vectors are selected in such a space, then the operations of vector addition and scalar multiplication will always result in a vector that is also in that space. Those two operations will never produce vectors outside of the space, and this is known as the property of **closure** (defined first in DEFINITION 11). And a subspace is always a nonempty **subset** of that space (defined second in DEFINITION 12). This is put into three conditions of what to look for in order to know you can classify a set of vectors as a **subspace** (DEFINITION 13).

DEFINITION 11: Let V be a set of vectors. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be any two vectors in V, and let k be any scalar. If the sum $\mathbf{v}_1 + \mathbf{v}_2$ is also in V, then we say V is **closed under vector addition**. If the product $k\mathbf{v}_1$ is also in V, then we say V is **closed under scalar multiplication**.

DEFINITION 12: A **subset** of V is any collection of vectors in V. If the set U is any collection of elements of a set V, then U is a subset of V, denoted by $U \subseteq V$.

DEFINITION 13: A set U is a **subspace** of a Euclidean vector space $\mathbb{R}^n$ if (1) U is a nonempty subset of $\mathbb{R}^n$, (2) if U is closed under vector addition, and (3) if U is closed under scalar multiplication.

Example 26: Let U be the set of all vectors of the form $(a, 2a)$. Show that U is a subspace of $\mathbb{R}^2$.

solution: Vectors of the form $(a, 2a)$ in which the second component is twice the first component are also contained in $\mathbb{R}^2$, so $U \subseteq \mathbb{R}^2$. Let $\mathbf{v} = (v, 2v)$ and let $\mathbf{w} = (w, 2w)$ be any two vectors in U, and let k be any scalar. [You want to show that adding any two vectors in the set and scalar multiplying any vector in the set will produce a vector in which the form is preserved: do these operations result in a vector whose second component is

[2] One of the most confusing concepts in differential calculus is the *differential* of a function $y = f(x)$, defined to be $dy = f\prime(x)dx$, a concept that leads directly to *linearization*. The differential of $z = f(x, y)$ is defined to be $dz = f_x dx + f_y dy$. These differentials are nothing more than the equations of a line and a plane. When you think of the form in which they are written, the definitions make sense. For example, $y - y_0 = m(x - x_0)$ is the equation of a line with slope m: a difference in y's is equal to a slope times a difference in x's.

twice the first? You must write this out as shown.]

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= (v, 2v) + (w, 2w)\\ &= (v+w, 2v+2w)\\ &= (v+w, 2(v+w)),\end{aligned}$$

so U is closed under vector addition.

$$\begin{aligned}k\mathbf{u} &= k(u, 2u)\\ &= (ku, 2(ku)),\end{aligned}$$

so U is closed under scalar multiplication. U is a subspace of $\mathbb{R}^2$.

The set of all vectors of the form $(a, 2a) = a(1, 2)$, which means that the vector $(1, 2)$ spans U. U is the set of all scalar multiples of this vector, so U is a line through the origin in $\mathbb{R}^2$, specifically the line $y = 2x$.

Example 27: Let W be the set of all vectors of the form (a, a, b), where $a, b \in \mathbb{R}$. Prove, or disprove, that W is a subspace of $\mathbb{R}^3$.

solution: W is a set of vectors with three components in which the first two components are the same, so it is a subset of $\mathbb{R}^3$. Let $\mathbf{u} = (u_1, u_1, u_2), \mathbf{v} = (v_1, v_1, v_2) \in W$, and let k be any scalar.

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (u_1, u_1, u_2) + (v_1, v_1, v_2)\\ &= (u_1+v_1, u_1+v_1, u_2+v_2),\end{aligned}$$

which results in a vector with the first two components equal, so W is closed under vector addition. And

$$\begin{aligned}k\mathbf{u} &= k(u_1, u_1, u_2)\\ &= (ku_1, ku_1, ku_2),\end{aligned}$$

a vector with equal first two components, so W is closed under scalar multiplication. W is a subspace of $\mathbb{R}^3$.

The set of all vectors of the form

$$\begin{aligned}(a, a, b) &= (a, a, 0) + (0, 0, b)\\ &= a(1, 1, 0) + b(0, 0, 1),\end{aligned}$$

so the vectors $(1, 1, 0)$ and $(0, 0, 1)$ span W, which is a plane through the origin in $\mathbb{R}^3$.

Example 28: Let S be the set of all vectors of the form $(a, 2a, a^2)$. where a is any real number. Show that S is *not* a subspace of $\mathbb{R}^3$.

solution: Since a is any real number, then S is nonempty, and the collection of all vectors in S *is* a subset of $\mathbb{R}^3$, but if $\mathbf{u} = (u, 2u, u^2)$ and $\mathbf{v} = (v, 2v, v^2)$ are any two vectors in S,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u, 2u, u^2) + (v, 2v, v^2)\\ &= (u + v, 2u + 2v, u^2 + v^2)\\ &= (u + v, 2(u + v), u^2 + v^2).\end{aligned}$$

If S is closed under vector addition, then the third component of the sum would have to be $u^2 + 2uv + v^2$, which it is not; therefore, S is not a subspace of $\mathbb{R}^3$ because it is not closed under addition.

194. Show that the set S in example 28 is not closed under scalar multiplication.

195. Is the set of points $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ a subset of $\mathbb{R}^2$? Explain.

196. Is the set of points $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ a subspace of $\mathbb{R}^2$? Explain.

197. Let Z be the set containing the zero vector in $\mathbb{R}^2$; that is, let $Z = \{(0, 0)\}$. Is Z a subset of $\mathbb{R}^2$? Why or why not?

198. Is Z a subspace of $\mathbb{R}^2$? Why or why not?

199. Let T be the set of all vectors in $\mathbb{R}^4$ of the form $(a, b, a+b, a-b)$. Prove, or disprove, that T is a subspace of $\mathbb{R}^4$.

200. Find a spanning set for the subspace T in the previous problem, and give a geometric description of T.

201. Let U be the set of all vectors in $\mathbb{R}^3$ where the sum of the first and the third components is always zero. Show that U is a subspace of $\mathbb{R}^3$.

202. Find a spanning set for U in the previous problem, and give a geometric description of U.

203. Let W be the set of all vectors in $\mathbb{R}^3$ such that the first component is zero. Is W a subspace of $\mathbb{R}^3$? If so, prove it, and then find a spanning set for W. If not, explain why.

204. Let X be the set of all vectors in $\mathbb{R}^3$ such that the third component is the product of the first two components. Is X a subspace of $\mathbb{R}^3$? If so, prove it, and then find a spanning set for X and give a geometric description of it. If not, explain why.

205. Let Y be the set of all vectors that are spanned by the $(1, 0, 0)$and $(0, 1, -1)$. Prove, or disprove, that Y is subspace of $\mathbb{R}^3$.

206. Let K be the set of all vectors spanned by $(1, 1, 0, 0)$ and $(0, 1, 1, 1)$. Prove, or disprove, that K is a subspace of $\mathbb{R}^4$.

207. If a vector $\mathbf{u}$ is a solution to a homogeneous system of m linear equations and n unknowns $A\mathbf{x} = \mathbf{O}$, then $A\mathbf{u} = \mathbf{O}$. Let H be the set of all solutions to the homogeneous system of linear equations $A\mathbf{x} = \mathbf{O}$. H is a subset of what Euclidean vector space? Explain.

208. Show that H is a subspace of the Euclidean vector space you found in the previous problem.

209. Let A be an $m \times n$ matrix. The set of all vectors $\{\mathbf{x} : A\mathbf{x} = \mathbf{O}\}$ is called the **null space** of the matrix A, denoted by $\text{nul}(A)$. Explain why the null space of A is a subset of $\mathbb{R}^n$.

210. Explain why the null space of the matrix A in the previous problem is a subspace of $\mathbb{R}^n$.

211. Let A be an $m \times n$ matrix. Prove that $\text{row}(A)$ is a subspace of $\mathbb{R}^n$.

212. Let A be an $m \times n$ matrix. Prove that $\text{col}(A)$ is a subspace of $\mathbb{R}^m$.

213. Try as hard as you can to not get lost in all of the new vocabulary. It is easy to become confused if you are not taking frequent pauses to think carefully. Knowing the definitions and tackling as many problems as you can will help, as will frequent pauses to clarify things: *Remember that linear combinations of vectors is really what we are talking about.* Given any set of vectors from a vector space, the set of all linear combinations of those vectors is the span of the vectors. The set these vectors span is a subset of the vector space from which the vectors are located. The span of any set of vectors from a vector space is always a subspace of that vector space. ***The set of all linear combinations of a set of vectors from a vector space is always a subspace of the vector space***; that is, the set of all linear combinations of a set of vectors in a vector space is always a line-like thing in that space passing through the origin in that vector space. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Let S be the set of all linear combinations of the vectors $\mathbf{u}$ and $\mathbf{v}$. Complete the following sentences, replacing the ? with either S or $\mathbb{R}^n$:
Sentence 1: The span of $\mathbf{u}$ and $\mathbf{v}$ is ?.
Sentence 2: ? is a subset of ?.
Sentence 3: ? is a subspace of ?.

214. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^2$? Why or why not?

215. Let A be an $n \times n$ matrix. Let U be the set of all vectors $\mathbf{u}$ in $\mathbb{R}^n$ such that $A\mathbf{u} = 3\mathbf{u}$. Show that U is a subspace of $\mathbb{R}^n$.

216. Let W be the set of all vectors perpendicular to $\mathbf{w} = (-3, 5, 1)$. Describe this set.

217. Prove that W (the set in the previous problem) is a subspace of $\mathbb{R}^3$.

218. Prove that the set containing the zero vector in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

219. Let U and V be subspaces of $\mathbb{R}^n$. Prove that the intersection $U \cap V$ is a subspace of $\mathbb{R}^n$.

220. Give an example of a subset of $\mathbb{R}^3$ that is not a subspace.

Chapter 12

Linearly, Linear, Linear, Linearity

12.1 Meet Matrices that Compose Functions

In exercises 221-222, follow the instructions given, starting with the point $(2, 3)$. Then complete exercises 223-22? to discover that matrix multiplication is function composition.

221. Write the vector $(2, 3)$ on a piece of paper: Do the following things: (1) Switch the components, and then (2) negate the first component. What is the resulting vector?

222. Starting again with the vector $(2, 3)$, written down on your paper, do the following things: (1) Negate the first component, and then (2) switch the components. What is the resulting vector?

223. We can write the previous two problems using function notation. Suppose f is the function that takes a vector (x, y) and switches the components. Then $f(x, y) = (y, x)$. Suppose g is the function that takes a vector (x, y) and negates the first component. Then $g(x, y) = (-x, y)$. In 221, you took a point and f transformed it first. Then g transformed that result to produce your answer. In algebra, this is written as $g \circ f(x, y)$. In algebra, the function composition, step by step, for f, followed by g, is

$$\begin{aligned} g \circ f(x, y) &= g(f(x, y)) \\ &= g(y, x) \\ &= (-y, x). \end{aligned}$$

Therefore, as a single function, say $h(x, y) = (-y, x)$ will combine both operations. What is $h(2, 3)$ and does this match your answer to 221?

224. Use the previous problem to find a function $j(x, y) = f \circ g(x, y)$. Does this agree with your answer in 222?

225. Meet the matrices F and G:

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, G = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ represent the vector (x, y) and find $F\mathbf{x}$.

226. Find $G\mathbf{x}$.

227. Find H if $H = GF$.

228. Find J if $J = FG$.

229. Find F^{-1}.

230. Find G^{-1}.

231. Find H^{-1}.

232. Find J^{-1}.

233. Find $F^{-1}G^{-1}$.

234. Find $G^{-1}F^{-1}$.

235. **True or False:** $(FG)^{-1} = F^{-1}G^{-1}$.

12.2 The Secret Password Question

Remember what I mentioned once before: If you get lost in linear algebra, in the places outside of the algorithmic mastery, and do not know how find your way to clarity, stop and think and figure out where you see the linear combinations.[1] If you come across someone who you think may hold the key to your success, there is a secret password question that you say out loud: "Where are the linear combinations?" You will get somewhere.

This chapter devotes itself to understanding and using the definitions given for when someone says "these vectors are linearly independent" or "this is a set of linearly dependent vectors." It is often the case in mathematics that students find it challenging and frustrating when reading definitions of concepts. The job is to somehow get at the things that connect the concepts, and maybe why they came about, and perhaps think of how a definition came to be written. Understanding how the definition of a concept was written is one thing. Understanding the reason why the concept is an important thing to define, its origin or its appearance, is a different thing. Intuitive definitions of things sometimes help!

It has been my experience, both as a student and educator, that a majority of students have difficulty understanding the definition of a **linearly independent set** of vectors when they read it, and reread it. The concept is really a description of a set of vectors that lets you know that all of them lie on different line-like things.

To help you understand why the concept is defined in the way it is, we can use an example and language with which you ought to be familiar. Suppose someone gives you the equation $z = w + 2x - y$, where each variable represents a number. This equation is written in such a way that it makes sense to say that the value of z *depends* on the values of w, x, and y. The equation, however, can be written $w = z - 2x + y$, and we can say that the value of w depends on the values of z, x, and y. Of course, we can see that each of the variables can easily be isolated on one side, which means that the value of each variable

[1] Determine whether or not you are able to answer this question: How were the two matrices F and G formed in the previous discussion?

depends on the values of the others:

$$z = w + 2x - y \tag{12.1}$$
$$w = z - 2x + y \tag{12.2}$$
$$x = \frac{1}{2}z - \frac{1}{2}w + \frac{1}{2}y \tag{12.3}$$
$$y = z - w - 2x \tag{12.4}$$

What is special about this equation? (*Hint: Think of a sentence or four in which you use the words "linear" and "combination."*) When solved for one of the variables, we see that each variable can be written as a linear combination of the other. In such a situation, mathematicians do this: Write the equation in a form in which all of the variables are located on one side of the equation, with zero obtained on the right:

$$w + 2x - y - z = 0$$

Identifying the scalars in blue and the vectors in red, if each variable is dependent on the other by way of a linear combination, then there is a linear combination of the vectors equal to zero:

$$1w + 2x + (-1)y + (-1)z = 0$$

This is what is at the heart of a precise definition of the concept of **linear dependence**, which some books define before linear independence, so think about this for a moment: if each vector in a set of vectors can be written as a linear combination of the other vectors, then there exists scalars that cannot all be zero that you can find to write a linear combination of the vectors that is equal to the zero vector. If *none* of the vectors can be written as a linear combination of the others, then *the only linear combination of the vectors that is equal to the zero vector is one in which all the scalars are zero.* This is the reason why the definition of linear independence is worded the way it is. It's the very next thing to read.

12.3 Linear Dependence and Linear Independence

DEFINITION 14: If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is a set of two or more vectors in a vector space V, then S is said to be a **linearly independent set** if no vector in S can be expressed as a linear combination of the other vectors in the set. A set that is not linearly independent is said to be **linearly dependent**. If S contains one and only one vector, we will agree that the set is linearly independent if and only if that vector is nonzero.

A common problem across disciplines is to take a set of vectors and determine whether one of them can be written as a linear combination of one or more of the others. If one is found, your set is a linearly dependent set; that is, it is not linearly independent.

Example 29: Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$, and $\mathbf{v_2} = \mathbf{v_1} - 3\mathbf{v_3}$, and there is no relationship between $\mathbf{v_4}$ and any of the other three variables. How do you justify that the

set S is a linearly dependent set? Is there a linear combination of all of the vectors in the set equal to the zero vector?

solution: You justify it by using the definition for linear independence. Since one vector can be written as a linear combination of two of the other vectors, then you automatically know, by definition, that S is not linearly independent.

The relationship that was given allows us to write $\mathbf{v_1} - \mathbf{v_2} - 3\mathbf{v_3} = \mathbf{0}$. In order to include the other vector in the set, the scalar zero is appended to the vector:

$$1\mathbf{v_1} + (-1)\mathbf{v_2} + (-3)\mathbf{v_3} + 0\mathbf{v_4} = \mathbf{0}$$

The purpose of this chapter is to, of course, learn these new definitions, which means you want to practice problems in which you might derive some meaning as to *why these concepts might be important.* By far, the important skill to practice is determining whether or not a set of vectors is linearly independent. This ought to come together soon, once we finish this phase of the course in which you are learning the language of linear algebra. The first phase is learning a general method for solving any system of linear equations, introducing matrices in order to have the matrix equation that represents the system $A\mathbf{x} = \mathbf{b}$. The second phase of the course is to learn the vocabulary of linear algebra (and where we, as instructors, can repeatedly point out that matrices are the things to which we eventually return). The third phase of the course is to learn to look at matrices and see them for what they really are: **linear transformations**.

You are often asked to determine whether or not a set of vectors is linearly independent. This means you might have to examine whether or not one vector in the set is a scalar multiple of another in the set, or whether or not the vector is a linear combination of two or more vectors in the set. This takes us right back to the beginning of our course: *solving systems but taking the column perspective!* If you did not pick up on this important detail, let me remind you of it using the following three examples:

Example 30: What is the column perspective of the system below?

$$\begin{aligned} 5x_1 + 3x_2 &= 1 \\ 6x_1 + 2x_2 &= -2 \\ -x_1 + x_2 &= 3 \end{aligned}$$

solution: This was the birthplace of matrices! The column perspective reads the given system in columns, and interprets the unknowns as scalars: Reinterpreted, this system is

$$x_1 \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Is it possible to find scalars x_1 and x_2 so that the vectors $(5, 6, -1)$ and $(3, 2, 1)$ can be scaled in such a way that, when scaled and added together, result in the vector $(1, -2, 3)$? That is one way to restate the problem. Another way is to notice that the left side of the above equation is a linear combination of the vectors $(5, 6, -1)$ and $(3, 2, 1)$ equal to

the vector $(1, -2, 3)$, so the system can be reworded in the exact same style as the next example is worded, so read the next example!

Example 31: Is $\mathbf{v_1} = (1, -2, 3)$ a linear combination of $\mathbf{v_2} = (5, 6, -1)$ and $\mathbf{v_3} = (3, 2, 1)$?

explanation and solution: This question directly translates to this more mathematical question: Are there scalars a and b that can be found such that $\mathbf{v_1} = a\mathbf{v_2} + b\mathbf{v_3}$? This is the column perspective of a system of linear equations. Notice how this translates to what you already know:

$$\begin{aligned}(1, -2, 3) &= a(5, 6, -1) + b(3, 2, 1)\\ (1, -2, 3) &= (5a, 6a, -a) + (3b, 2b, b)\\ (1, -2, 3) &= (5a + 3b, 6a + 2b, -a + b)\end{aligned}$$

The last step above (third step) is the system in example 30, as

$$\begin{aligned}1 &= 5a + 3b\\ -2 &= 6a + 2b\\ 3 &= -a + b\end{aligned}$$

Rewritten to where we solve by Gauss-Jordan elimination (view the system in example 30), we have

$$\begin{bmatrix} 5 & 3 & 1\\ 6 & 2 & -2\\ -1 & 1 & 3\end{bmatrix} \xrightarrow[R1 \leftrightarrow R3]{-R3, \frac{1}{2}R2} \begin{bmatrix} 1 & -1 & -3\\ 3 & 1 & -1\\ 5 & 3 & 1\end{bmatrix} \tag{12.5}$$

$$\xrightarrow[-5R1+R3]{-3R1+R2} \begin{bmatrix} 1 & -1 & -3\\ 0 & 4 & 8\\ 0 & 8 & 16\end{bmatrix} \tag{12.6}$$

$$\xrightarrow[\frac{1}{8}R3]{\frac{1}{4}R2} \begin{bmatrix} 1 & -1 & -3\\ 0 & 1 & 2\\ 0 & 1 & 2\end{bmatrix} \tag{12.7}$$

$$\xrightarrow[-R2+R3]{R2+R1} \begin{bmatrix} 1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix} \tag{12.8}$$

In step (12.5) above, three row operations were carried out. Notice the original augmented matrix is such that if you multiply the third row by -1, you end up with a 1 where you want it if you then switch the first and third rows. At the same time, the second row is divisible by 2, so why not mutliply it by one-half while you can. The remaining steps read as usual, with the last matrix showing $a = -1$ and $b = 2$. Be careful to state a meaningful answer here! YES, the vector can be written as a linear combination of the other two: *If YES is your answer, then you had better state the combination!*

$$(1, -2, 3) = -1(5, 6, -1) + 2(3, 2, 1)$$

Example 32: Determine whether or not the set $S = \{(1, -2, 3), (5, 6, -1), (3, 2, 1)\}$ is linearly independent.

solution: The solution is really the previous example! And if you look closely enough at what you can do when this question is asked, what is a way of working the problem? Notice that the question must be asked: *is it possible to write one vector in the set as a linear combination of the others?* Since we already know the answer is yes, then S is linearly dependent.

Example 33: Determine whether or not the set $W = \{(4, -3), (-2, \frac{3}{2})\}$ is linearly independent.

solution: Since one vector is a scalar multiple of another, $(4, -3) = -2(-2, \frac{3}{2})$, then W is linearly dependent. *Always try and spot whether or not two vectors in a set are scalar multiples of one another. This is often the quickest way to note that a set of vectors is linearly dependent.*

Example 34: Find the reduced row echelon form of the matrix $B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$. Determine whether or not the column vectors of B are linearly independent.

solution: $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

By observation, no column vectors are scalar multiples of one another. Often students assume the set is linearly independent, but when it comes to a set of three or more vectors, you must ask whether one vector can be a linear combination of all, or some, of the other vectors in the set. In this case, since three vectors comprise the columns of B, then ask whether $(1, 1)$ is a linear combination of $(2, 3)$ and $(4, 7)$.[2] If so, then

$$\begin{aligned} (1, 1) &= a(2, 3) + b(4, 7) \\ &= (2a, 3a) + (4b, 7b) \\ &= (2a + 4b, 3a + 7b), \end{aligned}$$

which means that you must solve the system:

$$\begin{aligned} 2a + 4b &= 1 \\ 3a + 7b &= 1 \end{aligned}$$

Solving by Gauss-Jordan elimination

$$\begin{bmatrix} 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R1} \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 3 & 7 & 1 \end{bmatrix} \xrightarrow{-3R1+R2} \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix},$$

[2] Or ask whether the second column is a linear combination of the first and third, or whether the third column is a linear combination of the first and second...

the system has solution $a = \frac{3}{2}$ and $b = -\frac{1}{2}$, which means *yes*, the vector $(1,1)$ is a linear combination of the other two vectors: $(1,1) = \frac{3}{2}(2,3) - \frac{1}{2}(4,7)$. Notice the connection between the two problems here in that it was not necessary to do the work to conclude that the column vectors of B are linearly dependent: That work was done when we wrote the matrix in reduced row echelon form. When B is written in reduced row echelon form, one interpretation of that matrix is that it might possibly represent a system of linear equations whose solution, when interpreted in the column perspective, reveals that the column vectors are related through a linear combination $-2(1,1) + 3(2,3) = (4,7)$. Thus, the columns are linearly dependent.

For a moment, reconsider a factorization of B by looking back at example 20 in Chapter 8, and apply that process to obtain the CR-factorization of B. When examining the columns of B, one by one, and understanding the reduced form interpretation, the first column is $(1,1)$, and when you move to the second column, it is not a scalar multiple of the first, so these two are the columns of C:

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} & & \\ & & \end{bmatrix}$$

The matrix R is found by understanding the column perspective: viewing the three columns of B as linear combinations of the columns of C: The first column of B is $1(1,1) + 0(2,3)$:

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & & \end{bmatrix}$$

The second column of B is $0(1,1) + 1(2,3)$:

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \end{bmatrix}$$

The third column of B is $-2(1,1) + 3(2,3)$:

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

Has anyone caught on to what this CR-factorization might be about? The matrix on the left, C, is a matrix whose column vectors are linearly independent. What is R? (Don't worry; we will return to this repeatedly!)

Example 35: Determine whether or not the columns of $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 2 & 0 & 6 \end{bmatrix}$ are linearly independent.

solution: If you suspect that any column vector *cannot* be written as a linear combination of the other two, then you are probably thinking there is a strong possibility that the vectors are linearly independent: **If a set of vectors is linearly independent, then the only linear combination of the vectors which add to the zero vector is the one in which all the scalars are zero.** In other words, to show that a set is linearly

independent, use row operations to show that $c_1 = c_2 = c_3 = 0$; that is, if you set up the system of equations associated with the question posed, then show that the only solution to the system is the zero vector, which means the vectors are linearly independent:

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.9}$$

$$(c_1 + c_2 + c_3, 2c_1 + 3c_2 + 5c_3, 2c_1 + 6c_3) = (0, 0, 0). \tag{12.10}$$

Notice, in (12.9), this sentence can be written after thinking *whether or not there exists a linear combination of the vectors equal to the zero vector.* This leads directly to (12.10), which is the system below:

$$\begin{aligned} c_1 + \ \ c_2 + \ \ c_3 &= 0 \\ 2c_1 + 3c_2 + 5c_3 &= 0 \\ 2c_1 \qquad\quad + 6c_3 &= 0 \end{aligned}$$

I leave it to you, the student, to show that $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \\ 2 & 0 & 6 & 0 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which means that the only solution to the system is $c_1 = 0, c_2 = 0$, and $c_3 = 0$, and so the columns of M are linearly independent.

236. Determine whether or not the rows of M in example 35 are linearly independent.

237. Is the set $S = \{(2, -6), (-3, 9)\}$ linearly independent?

238. Determine whether or not the set $\{(2, 1), (1, 3), (1, 1)\}$ linearly independent.

239. Is the set $\{(1, 2, 1), (0, 0, 0), (0, 1, -1)\}$ linearly independent?

240. Show that the set $\{(1, 1), (-1, 1)\}$ is linearly independent and spans $\mathbb{R}^2$.

I would like you to make some observations based on the work thus far: Stop, pause, and consider some questions, the kind that lead to a clear understanding of the theory we have yet to put together. To those of you who are intellectually curious, you have most likely, and very naturally, asked or answered some of these:

241. Let A be a square matrix. If its columns are linearly independent, then what is A in reduced row echelon form?

242. Let A be a square matrix. If its rows are linearly independent, then what is A in reduced row echelon form?

243. Basic logic: The sentence "If a then b." is a *conditional* statement. The *converse* statement is "If b then a." Is the converse of the statement in 241 true? Explain.

244. If a matrix A has an inverse, then are its columns linearly independent?

245. If $\mathbf{v} = \mathbf{u} + \mathbf{w}$, prove that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

12.4 The Word *Linear*

What is present in all of the work we are doing is the concept of *linearity.* The word **linear** is used in two ways. For example, the function $f(x) = 2x+3$ is linear. It is also not linear. Remember that the English language contains many words that have the same meaning, and also many words that have multiple different meanings. The word **linear** can be used as an adjective to describe a *geometric* quality, and this is how you have previously used the word: To say $f(x) = 2x + 3$ is linear is to say that its graph is that of a line. In this course, we refine our language to make an important distinction between line-like things through origins (*subspaces*), and all other line-like things that do not contain an origin. It is best to abandon the old use of the word *linear* when talking linear algebra, and redefine it *algebraically.* Mathematicians have two words: *linear* for lines through origins, and *affine* for lines not containing origins. We will formally define the term soon, but here is a first glance at the definition as it pertains to the kind of functions familiar to all of you in this course: The function $f(x)$ is *linear*, or a *linear transformation*, means that $f(x_1 + x_2) = f(x_1) + f(x_2)$, and $f(cx) = cf(x)$, for all x_1, x_2 in the domain of f, and for any real number c.

246. Prove that $f(x) = 2x + 3$ is not linear.

247. Prove that $f(x) = 2x$ is linear.

248. If $g(x)$ is a linear transformation, and $g(4) = 3$, then what is $g(5)$?

249. If $f(x)$ is a linear transformation, then is it always true that $f(0) = 0$?

250. If $h(x)$ is a linear transformation, and $h(6) = a + b$, find $h(3)$.

Chapter 13

Basis and Dimension

13.1 Basis Vectors and Dimension of a Space

Perhaps the most important problem in the text is problem 75: All vectors in $\mathbb{R}^2$ can be written in terms of the vectors $\mathbf{i}$ and $\mathbf{j}$, vectors that are known as *basis vectors*.[1] Though the problem may seem trivial to most (using scalar multiplication and vector addition to show that $(x, y) = x\mathbf{i} + y\mathbf{j}$), it highlights the fact that vector spaces contain an infinite number of vectors, and *all vectors in the space* can be discussed using only two vectors: the basis vectors. Mathematicians clarify what is meant by *dimension* of a space by way of the set of vectors that form a basis for the space. Basis vectors provide a coordinate system that uses numerical coordinates to represent vectors. The concepts of *basis* and *dimension* create a way to understand a rigorous mathematical structure of a vector space. To study the concept of a basis is to generalize the Cartesian coordinate system with which most of us are familiar early on in our mathematics education. *A lot of mathematics is developed by generalizing the concepts you learn in grade school and high school.*

DEFINITION 15: A set of vectors is a **basis** for a space if (1) the vectors span the space, and (2) are linearly independent. The number of vectors in a basis of a space is called the **dimension** of the space, and this number is unique.

This is the section in which you want to make solid the use of all of the new terms in the past several chapters. The examples are carefully written out with this in mind, so please pay close attention to these examples, and work on being able to write and speak precisely.

Example 36: Let W be the set of all vectors in $\mathbb{R}^3$ whose last component is the sum of the first two components; that is, all vectors of the form $(a, b, a + b)$. Show that W is a subspace of $\mathbb{R}^3$ and then find a basis for W. State the dimension of W and give a geometric description of this subspace.

solution: Let $\mathbf{u} = (u_1, u_2, u_1 + u_2), \mathbf{v} = (v_1, v_2, v_1 + v_2) \in W$, and let k be any scalar.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1 + v_1, u_2 + v_2, u_1 + u_2 + v_1 + v_2)\\ &= (u_1 + v_1, u_2 + v_2, (u_1 + v_1) + (u_2 + v_2)),\end{aligned}$$

so W is closed under vector addition.

$$\begin{aligned}k\mathbf{u} &= (ku_1, ku_2, k(u_1 + u_2))\\ &= (ku_1, ku_2, ku_1 + ku_2),\end{aligned}$$

[1] Remember in $\mathbb{R}^2$, $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. In $\mathbb{R}^3$, remember that $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

so W is closed under scalar multiplication. W is a subspace of $\mathbb{R}^3$. Its vectors are of the form

$$\begin{aligned}(a, b, a+b) &= (a, 0, a) + (0, b, b) \\ &= a(1, 0, 1) + b(0, 1, 1),\end{aligned}$$

which shows that the vectors $(1, 0, 1)$ and $(0, 1, 1)$ span W. If these vectors are linearly independent, then since they also span W, they form a basis for W. To show the vectors are linearly independent, show that the only solution to $k_1(1, 0, 1) + k_2(0, 1, 1) = (0, 0, 0)$ is $k_1 = k_2 = 0$: This is left to the student to show! Therefore, a basis for W is the set $B = \{(1, 0, 1), (0, 1, 1)\}$. The dimension of W is 2, so W is a plane.

Example 37: Let V be the set of all vectors in $\mathbb{R}^2$ with a second component that is twice the first component. Prove that V is a one-dimensional subspace of $\mathbb{R}^2$.

solution: The set V consists of all vectors of the form $(a, 2a)$. First, show that this is a subspace of $\mathbb{R}^2$. Let $\mathbf{u} = (u, 2u)$ and $\mathbf{v} = (v, 2v)$ be any vectors in V, and let k be sny scalar.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u.2u) + (v, 2v) \\ &= (u + v, 2u + 2v) \\ &= (u + v, 2(u + v)),\end{aligned}$$

and

$$\begin{aligned}k\mathbf{u} &= k(u, 2u) \\ &= (ku, 2ku),\end{aligned}$$

so V is closed under addition and scalar multiplication, since both adding and scalar multiplying yield vectors whose second component is twice the first component. V is a subspace of $\mathbb{R}^2$. Also, $(a, 2a) = a(1, 2)$, which means the vector $(1, 2)$ spans V, and, because it is a single vector, it is linearly independent. A basis for V is $\{(1, 2)\}$, which has dimension 1.

Example 38: Show that the set $\{(1, 1), (0, 1)\}$ is a basis for $\mathbb{R}^2$.

solution: All vectors in $\mathbb{R}^2$ are of the form (a, b), where $a, b \in \mathbb{R}$. To show the vectors span $\mathbb{R}^2$, we must show that all vectors of the form (a, b) can be written in terms of the two alleged basis vectors. The mathematical way to do this is to solve for the variables c_1 and c_2 in the equation $(a, b) = c_1(1, 1) + c_2(0, 1)$. Note this means that

$$\begin{aligned}a &= c_1 \\ b &= c_1 + c_2,\end{aligned}$$

so $c_1 = a$ and $c_2 = b - a$. Thus, any vector in $\mathbb{R}^2$ $(a, b) = a(1, 1) + (b - a)(0, 1)$. This shows that the vectors span $\mathbb{R}^2$. Since one vector is not a scalar mutliple of the other, then the set $\{(1, 1), (0, 1)\}$ is linearly independent, making it a basis for $\mathbb{R}^2$.

13.2 Tying Concepts Together

In one sense, we are refining knowledge we once expressed in elementary school, and generalizing. Long ago, you established that *two points determine a line*[2] and *three non-collinear points determine a plane.* A line has dimension 1. A plane has dimension 2. As a more sophisticated mathematics student, *two points determine a line* is restated: Two points determine a Euclidean vector from one point to the other, and that single vector *determines a line* because all scalar multiples of that vector comprise the line (the vector *spans* the line). The vector is a basis for the line. Three non-collinear points determine two vectors that are linearly independent; that is, the vectors cannot be placed on the same line (they are not parallel). As a more sophisticated mathematics student, we say that two linearly independent vectors determine a plane. We can come to a better understanding of all of these concepts by addressing past intuitive knowledge we have about Euclidean vector spaces such as $\mathbb{R}, \mathbb{R}^2$, and $\mathbb{R}^3$. These spaces give us a nice geometric interpretation of linear independence. How so? *Two vectors are linearly independent if and only if they do not lie on the same line when in standard position.* Therefore, if one vector is a scalar multiple of another vector, the two vectors are linearly dependent. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors from the same vector space, and $\mathbf{u} = k\mathbf{v}$, where k is a scalar. Then it is true that $\mathbf{u} - k\mathbf{v} = \mathbf{0}$, a linear combination of the two vectors equal to the zero vector in which the scalars are not all zero; hence, the vectors are linearly dependent. *Three vectors are linearly independent if and only if they do not lie in the same plane when in standard position.* Take $\mathbb{R}^2$ and sketch any three vectors while asking these questions: *Is it possible to sketch three vectors on different lines? Is it possible to sketch three vectors so that they lie on different planes?* Take $\mathbb{R}^3$ and do the same. These lead to more interesting questions regarding dimension and basis, as you will experience in some of the exercises.

251. Find a basis for the line $y = 3x$, a subspace of $\mathbb{R}^2$. (What vector spans this subspace?)

252. Let U be the set of all vectors in $\mathbb{R}^3$ that are orthogonal to the vector (1,1,1). Is U a subspace of $\mathbb{R}^3$? If so, find a basis for U, and state its dimension.

253. Let V be the set of all vectors in $\mathbb{R}^4$ of the form $(r, s, r + s, r - s)$. Prove that V is a subspace of $\mathbb{R}^4$, find a basis for V, and state its dimension.

254. The **standard basis** of $\mathbb{R}^2$ is the set $\{(1, 0), (0, 1)\}$. Show that the set of vectors $\{(1, 1), (0, 1)\}$ is also a basis for $\mathbb{R}^2$.

255. Show that the set $\{(2, 3), (1, 1)\}$ is a basis for $\mathbb{R}^2$.

256. Let the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for a vector space V. Prove that the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

257. Why is it true that any set containing the zero vector must be a linearly dependent set?

258. Let W be a set containing exactly two vectors. Prove that if the set is linearly independent, then one vector is not a scalar multiple of the other.

[2] Two *distinct* points, of course, determine a line.

259. Let W be a set containing exactly two vectors. Prove that if one vector is not a scalar multiple of the other, then the set is linearly independent.[3]

260. Show that the set $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{u}\}$ is linearly dependent, for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in a vector space V.

261. Solve the system by Gauss Jordan elimination.

$$\begin{aligned} x_1 + x_2 - 7x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 - 10x_3 + 2x_4 + 2x_5 &= 0 \\ 2x_1 + 3x_2 - 17x_3 + 4x_4 + 3x_5 &= 0 \end{aligned}$$

262. Show that the set of solutions to the system in 261 is a subspace of $\mathbb{R}^5$. Find a basis and the dimension of this subspace.

263. Prove that the solution set of the system is a one-dimensional subspace of $\mathbb{R}^3$.

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 0 \\ 2x_1 + 3x_2 - 13x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

264. Prove that $\{(2,-3,1,0),(-1,-2,0,1)\}$ is a basis for the two-dimensional subspace of all solutions of the following system.

$$\begin{aligned} x_1 + x_2 + x_3 + 3x_4 &= 0 \\ x_1 + 2x_2 + 4x_3 + 5x_4 &= 0 \\ x_1 - 2x_3 + x_4 &= 0 \end{aligned}$$

265. Show that it is possible to have a set of vectors in $\mathbb{R}^3$ that are linearly independent but do not span $\mathbb{R}^3$.

266. Show that it is possible to have a set of vectors that span $\mathbb{R}^3$ but are not linearly independent.

267. Sketch the subspace of $\mathbb{R}^3$ that has as its basis the set $\{(0,1,0)\}$.

268. Sketch the subspace of $\mathbb{R}^3$ that has as its basis the set $\{(0,0,1),(0,1,1)\}$.

269. Why is $\mathbb{R}^2$ **not** a subspace of $\mathbb{R}^3$?

270. Find a basis for the set of all vectors in $\mathbb{R}^3$ that are orthogonal to the vector $(3,2,1)$.

[3] If you combine the solutions to the previous two exercises as a single proof, then what you have proved is the following statement: *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Chapter 14

Linear Transformations

14.1 Meet the Fibonacci Matrix

Mathematicians often describe their work as discovery. Numbers such as π and the natural number e arise in surprising ways, as if they are waiting to be unearthed. Another such number known as the *golden ratio*, denoted by phi φ, is a nice addition to this list of interesting natural numbers. The golden ratio, approximately equal to 1.618, can be found across the universe, is a part of the planet on which you live, and is also part of *you*. A lily (the flower) has three petals, a buttercup has five petals, a chicory has twenty-one petals, a daisy can have thirty-four petals or fifty-five petals. The human body is an astonishing example of the golden ratio. Things that spiral in nature often exhibit the golden ratio (seed heads, pineapples, cauliflowers, sunflowers, the branching of trees, the number of leaves growing on a stem, galaxies, shells of snails, nautilus shells, the cochlea of the inner ear, hurricanes, to name a few...). The number can be a surprising discovery as the answer to some interesting problems. One place is the Fibonacci sequence, a sequence obtained by starting with 0 and 1, adding two terms in succession to obtain the next number in the sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$ If you take the ratio of any term with its predecessor, especially when you consider going out to larger and larger terms, you will find the appearance of φ. The numbers in this sequence are known as the Fibonacci numbers. Notice the limiting nature of taking ratios: $\frac{3}{2} = 1.5, \frac{5}{3} = 1.666..., \frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.61538462, ..., \frac{89}{55} = 1.6181818...$

Meet F, the matrix that generates the Fibonacci sequence: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

271. Let $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $F\mathbf{b}$.

272. Find $F(F\mathbf{b})$ and $F^3\mathbf{b}$.

273. Google "desmos matrix calculator" and keep going. Be convinced that F generates this sequence of numbers. (*We will return to the matrices you have met once we begin studying matrices like we studied real numbers!)*

14.2 Clarification of Function Terms and Notation

The concept of a function is crucial to any work in mathematics at the college level and beyond. All persons present in this course are certainly familiar with *real-valued functions of a single variable*, functions like $f(x) = x^2, f(x) = mx + b, f(x) = \frac{x}{3x-1}$, etc... Some of you have studied functions of two variables, or three, or more, and some of you have studied *vector-valued functions of one or more variables.* A function is a rule that assigns

to each input one and only one output. Think $f(input) = output$. A function *changes* an input, or *transforms* an input. A function is also known as a *transformation* since it does such a thing. Another common word for a function is the word *mapping*. The set of all inputs of a function is called the *domain* of the function. The set of all outputs of a function is called the *range* of the function. In linear algebra, we use the words *transformation* and *mapping* (the verb form of the word: a function maps a space onto another space, for example.) While most of you ought to be comfortable with these words, many of you have not used the word **codomain**. For example, the domain of $f(x) = x^2$ is $\mathbb{R}$. The range of $f(x) = x^2$ is a subset of $\mathbb{R}$, usually given in interval notation: $[0, \infty)$. The codomain of $f(x) = x^2$ is $\mathbb{R}$. The codomain of a function is nothing other than the space in which the outputs can be found. It is the space to which the inputs are mapped. There is a special notation you will see often in the course that specifies the function by its name, its domain, and its codomain:

$$f : \mathbb{R} \to \mathbb{R}$$

This notation is read "f maps the set of real numbers to the set of real numbers." These are, for us, vector spaces. If f maps the vector space V to the vector space W, we write $f : V \to W$. The following examples ought to illuminate these details, so be cautiously observant as you study them.

Example 39: Let $f(x, y) = 2xy + y^2$. This is a function familiar to those of you who have studied multi-variable differential calculus. Function notation reveals in a clear way in what vector space inputs are coming from, and to what vector space they are mapped. Here, $f : \mathbb{R}^2 \to \mathbb{R}$. The function maps vectors in the plane to real numbers. The domain is $\mathbb{R}^2$. The co-domain is $\mathbb{R}$. The range is $\mathbb{R}$. This is a real-valued function that maps the vector space $\mathbb{R}^2$, a plane, to the vector space $\mathbb{R}$, a line.

Example 40: Let $f(x, y) = (3x + y, x - y)$. When you read function notation, you must be able to *see* from what space the inputs and outputs are. Functions have a name: this one is *f*. Following the name is notation indicating the inputs: (x, y) represents familiar ordered pairs one finds in $\mathbb{R}^2$. Following the "is equals to" symbol is the rule for what to do with those inputs. Here, $(3x + y, x - y)$ is a thing with two components; thus, $\mathbb{R}^2$ is the vector space to which the inputs are mapped. This function maps vectors in $\mathbb{R}^2$ (domain) to vectors in $\mathbb{R}^2$ (codomain); that is, $f : \mathbb{R}^2 \to \mathbb{R}^2$. It is common to see this function written

$$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 3x + y \\ x - y \end{bmatrix}.$$

Example 41: Let $f(x_1, x_2, x_3) = (2x_1 - x_3, 3x_1 - x_2 + x_3)$. This is a function that maps vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^2$; that is, $f : \mathbb{R}^3 \to \mathbb{R}^2$. The domain is $\mathbb{R}^3$. The codomain is $\mathbb{R}^2$. Know that we can replace the word *function* with *transformation*, as the next example demonstrates.

Example 42: Let $T(x_1, x_2) = (2x_2 - x_1, 2x_1 - x_2, x_1 + x_2, x_1)$. T is a transformation that maps vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^4$; that is, $T : \mathbb{R}^2 \to \mathbb{R}^4$. The domain is $\mathbb{R}^2$. The codomain is $\mathbb{R}^4$.

Examples 40-42 have something in common: they are linear transformations.

14.3 Linear Transformations

DEFINITION 16: A **linear transformation** is a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \tag{14.1}$$
$$T(k\mathbf{u}) = kT(\mathbf{u}) \tag{14.2}$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars k.

Example 43: Prove that the function in example 40 is a linear transformation.

solution: $f(x, y) = (3x + y, x - y)$ is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be any vectors in the domain $\mathbb{R}^2$, and let k be any scalar.

$$\begin{aligned}
f(\mathbf{u}+\mathbf{v}) &= f((u_1, u_2) + (v_1, v_2)) \\
&= f(u_1 + v_1, u_2 + v_2) \\
&= (3(u_1 + v_1) + (u_2 + v_2), (u_1 + v_1) - (u_2 + v_2)) \\
&= (3u_1 + 3v_1 + u_2 + v_2, u_1 + v_1 - u_2 - v_2) \\
&= (3u_1 + u_2 + 3v_1 + v_2, u_1 - u_2 + v_1 - v_2) \\
&= (3u_1 + u_2, u_1 - u_2) + (3v_1 + v_2, v_1 - v_2) \\
&= f(u_1, u_2) + f(v_1, v_2) \\
&= f(\mathbf{u}) + f(\mathbf{v}),
\end{aligned}$$

and

$$\begin{aligned}
f(k\mathbf{u}) &= f(k(u_1, u_2)) \\
&= f(ku_1, ku_2) \\
&= (3(ku_1) + ku_2, ku_1 - ku_2) \\
&= (k(3u_1 + u_2), k(u_1 - u_2)) \\
&= k(3u_1 + u_2, u_1 - u_2) \\
&= kf(u_1, u_2) \\
&= kf(\mathbf{u}).
\end{aligned}$$

Therefore, f is a linear transformation.

Example 44: Prove that the function in example 39 is *not* a linear transformation.

solution: $f(x, y) = 2xy + y^2$ is a transformation in which $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\mathbf{u} = (u_1, u_2)$

and $\mathbf{v} = (v_1, v_2)$ be any vectors in the domain $\mathbb{R}^2$.

$$\begin{aligned}
f(\mathbf{u}+\mathbf{v}) &= f((u_1,u_2)+(v_1,v_2)) \\
io &= f(u_1+v_1, u_2+v_2) \\
&= 2(u_1+v_1)(u_2+v_2)+(u_2+v_2)^2 \\
&= 2(u_1u_2+u_1v_2+u_2v_1+v_1v_2)+u_2^2+2u_2v_2+v_2^2 \\
&= 2u_1u_2+2u_1v_2+2u_2v_1+2v_1v_2+u_2^2+2u_2v_2+v_2^2 \\
&= 2u_1u_2+u_2^2+2v_1v_2+v_2^2+2u_1v_2+2u_2v_1+2u_2v_2 \\
&= f(\mathbf{u})+f(\mathbf{v})+2u_1v_2+2u_2v_1+2u_2v_2 \\
&\neq f(\mathbf{u})+f(\mathbf{v}),
\end{aligned}$$

so f does not preserve addition; therefore, f is not a linear transformation.

The fancier way to define a linear transformation is to say that it is a function (or mapping) that preserves both addition and scalar multiplication. Preserving addition is (14.1) in the definition. Preserving scalar multiplication is (14.2). Note the extreme detail from one step to the next in the two examples that follow the definition.[1] Also note that when you prove that a function is not linear, it is sufficient to show that only one of the two properties does not hold, since both are required in order for something to be called a linear transformation.[2] You ought to feel a sense of connection with the two operations that define vectors, and with what subspaces are.

274. Show that the function in example 41 is a linear transformation.

275. Show that the function in example 42 is a linear transformation.

276. In example 44, it was shown that the function $f(x,y) = 2xy + y^2$ was not linear by showing that f does not preserve addition. We could have shown it was not linear by demonstrating it does not preserve scalar multiplication. Does this work?

277. If $f(x_1,x_2,x_3) = (2x_1 - x_3, 3x_1 - x_2 + x_3)$, find $f(0,0,0)$.

278. If $T(x_1,x_2) = (2x_2 - x_1, 2x_1 - x_2, x_1 + x_2, x_1)$, find $T(0,0)$.

279. Assume that $T : V \to W$ is a linear transformation. Let $\mathbf{0}_V$ represent the zero vector in V and let $\mathbf{0}_W$ represent the zero vector in W. Prove that $T(\mathbf{0}_V) = \mathbf{0}_W$.

280. A true mastery of logic characterizes the best mathematicians, and it is us instructors who need to improve our work in this area, especially if we wish to train students to do mathematical work: In 279, you proved this statement: *If T is a linear transformation, then T maps the zero vector in the domain to the zero vector in the codomain.* What about the converse? If T maps the zero vector in the domain to the zero vector in the codomain, then T is a linear transformation. *True* or *false*? If it is true, then prove it. If it is false, then provide a counterexample. (*Hint: Use one of the examples.*)

[1] *Advice*: Work exercises 274 and 275 with this in mind, and mock the work you see in the examples.

[2] *Advice*: Work exercise 276 to note an alternative way to show a transformation is not linear, and consider which of the two properties might be less time consuming.

281. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by rotating any vector in the domain by $\frac{\pi}{2}$ radians. Prove that T is a linear transformation.

282. Let $T(x_1, x_2) = x_1 - x_2$. Identify the domain, codomain, and prove, or disprove, that T is a linear transformation.

283. Let $T(x, y, z) = (z, x + y)$. Identify the domain, codomain, and prove, or disprove, that T is a linear transformation.

284. Let $T(\mathbf{x}) = \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$. Prove, or disprove, that T is linear.

285. Let $T(\mathbf{x}) = \mathbf{0}$, where $\mathbf{x}, \mathbf{0} \in \mathbb{R}^n$. Prove, or disprove, that T is linear.

Chapter 15

Every Matrix Defines a Linear Transformation

15.1 A Matrix Can Be Multiplied By What Column Vectors?

Every matrix you meet can be multiplied by a column vector. Let's take a couple of the matrices you have met that actually do something, and figure out what this means: Consider F, the Fibonacci matrix from the previous chapter. F is a 2×2 matrix. The only column vectors you can multiply on the left by F would be 2×1 matrices, matrices of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. For all such products,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix}.$$

The rows of the matrix F contain the coefficients of the linear combinations of x_1 and x_2. This is how you see a matrix define the linear transformation to which it is associated:

$$T(x_1, x_2) = (x_2, x_1 + x_2)$$

286. Prove that $T(x_1, x_2) = (x_2, x_1 + x_2)$ is a linear transformation.

287. For the transformation in 286, find $T(0, 1)$ and then find the composition $T(T(0, 1))$. (Note that the composition corresponds to matrix multiplication; that is, $T \circ T(0, 1)$ is related to the multiplication of F^2 with the column vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.)

288. Repeatedly taking the output of a function and making it an input is called *function iteration*. Another term that describes this process is composing a function with itself. What if you began with the vector $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$? What sequence of numbers, beginning with 2, 8, ... would be generated by this process, if you found $T \circ T \circ T \circ T(2, 8)$? How many terms can you find so that when you find the quotient of last term you have with its predecessor, you get a number that you sense is a limiting value, or does this not happen? Is there a limit of this quotient?

289. What is the domain and codomain of this linear transformation?

Recall the matrix $D = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ from the beginning of Chapter 7 that differentiated any functions of the form $ae^{2x} \cos 3x + be^{2x} \sin 3x$. Since the matrix is of the same size as F, if we restrict ourselves to multiplying it by column vectors, then those are of size 2×1. If we represent all possible column vectors we could multiply on the left by D,

then these would be all vectors with two components; that is, all vectors in $\mathbb{R}^2$, but those are *representations of functions*, functions that we are *differentiating.* The linear transformation associated with this matrix is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by... (you get to answer this!)

290. ... what function?

291. If you were to find $T(1, 8)$, what function are you differentiating, and what is its derivative?

Example 45: Find the function determined by the matrix A below, its domain, and codomain. Take a pause and note the *size* of the matrix and its relationship to the domain and codomain.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

solution: I often joke with students and say "never listen to advice," and then proceed to give them advice. The intent of the joke is this: *Always think for yourself and let yourself be intellectually curious.* You learn best when you *play*, and I even argue that you do not really learn anything well unless there is an element of *play* present, but that's a topic for another book. Here is my advice to students: ***Whenever you meet a matrix for the first time, always note its size.*** The size of a matrix reveals a lot about the information it carries.[1] Here, the matrix A is of size 3×2; A has three rows and two columns. Note that if we refer to the rows as the row vectors of A, and the columns as the column vectors of A, we can see that the row vectors of A are vectors in $\mathbb{R}^2$, and the column vectors of A are in $\mathbb{R}^3$.

Anytime you are asked to find a function matrix determines, ask yourself what column vectors can be multiplied on the right of the matrix. The size of A answers that question. Since A is 3×2, the only column vectors that can be multiplied on the right are of size 2×1. We can make a general statement: If A is $m \times n$, then the vectors $\mathbf{x}$ for which $A\mathbf{x}$ is defined must be $n \times 1$. Given the above A, this means that all vectors in $\mathbb{R}^2$ define the product $A\mathbf{x}$. In other words, the given matrix A **transforms** vectors in a vector space by matrix multiplication. When you view a matrix for the first time, therefore, it is the number of columns that reveal the domain of the function the matrix determines. If you think about the size of $A\mathbf{x}$, and think how the matrix transforms all the vectors in that domain, you get the size of the vectors after they are transformed: A matrix of size $m \times n$, multiplied by a vector of size $n \times 1$ results in a vector of size $m \times 1$. What am I emphasizing here? The output vectors, those in the codomain of the function the matrix determines, are related to the number of rows. For the given matrix here, the product $A\mathbf{x}$ will result in vectors in $\mathbb{R}^3$. What are we really saying?? The transformation T determined by a matrix A is given by $T(\mathbf{x}) = A\mathbf{x}$. If A is of size $m \times n$, then $T : \mathbb{R}^n \to \mathbb{R}^m$. (This once frustrated me as a student, by the way, because I wondered why mathematicians did not change the way they denoted the size of matrices. Would you agree?) Remember this: each row of a

[1] Special note: We are approaching a part of the course in which the content will have serious importance to most of you entering modern computer science. While many of you may be focused on exam performance, and the letters that accompany such horrifying things, you really do need to know this material.

matrix contains the coefficients in the linear combinations of the components of the input vectors. It is not really necessary, if you think about it, to go through an algorithmic procedure to find $A\mathbf{x}$; you can simply read the matrix. Below is a great way to write an answer to the problem posed:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ determines the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2, 5x_1 + 6x_2)$. (The domain of T is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$.)[2]

292. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find the function this matrix determines, its domain, and its codomain?

293. Prove that the function in 290 is a linear transformation.

294. Prove that the function in 292 is a linear transformation.

15.2 The Proof

It is clear we should prove a theorem here: **Every matrix defines a linear transformation.** In our course, when we say "a matrix defines a function" or "a matrix defines a transformation" (both synonymous) we mean this: Given a matrix A, there exists a function $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is a column vector. You may also hear that T is the transformation *induced by* the matrix, language used in the very first sentence of the proof you are about to read. I believe that the heart of understanding this subject is this: Every matrix defines a transformation that is linear, and there is a matrix to be found in any given linear transformation. We began our course with solving systems, quickly learning that these can be written in the form $A\mathbf{x} = \mathbf{b}$. In your own thinking, what does this mean about matrices?

THEOREM 2: Every matrix defines a linear transformation.

PROOF: Let A be a matrix of size $m \times n$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation induced by A, which means $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$ and let k be any scalar.

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= A(\mathbf{u} + \mathbf{v}) \\ &= A\mathbf{u} + A\mathbf{v} \\ &= T(\mathbf{u}) + T(\mathbf{v}), \end{aligned}$$

so T preserves addition.

$$\begin{aligned} T(k\mathbf{u}) &= A(k\mathbf{u}) \\ &= kA\mathbf{u} \\ &= kT(\mathbf{u}), \end{aligned}$$

[2]The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates what the domain and codomain are for the transformation, as that is the point of this notation. T transforms vectors in a domain to vectors in a codomain.

so T preserves scalar multiplication. Since it preserves both addition and scalar multiplication, T is a linear transformation.

15.3 The Standard Matrix of a Linear Transformation

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then it is possible to find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. This matrix is often referred to as the **standard matrix** of the linear transformation T. The following two examples demonstrate how to do this.

Example 46: Find the standard matrix of $f(x_1, x_2, x_3) = (2x_1 - x_3, 3x_1 - x_2 + x_3)$, the linear transformation in exercise 277.

solution: Note that $f : \mathbb{R}^3 \to \mathbb{R}^2$, which means the standard matrix should be a 2×3 matrix. What sheds the most light on these functions is the more popular notation of writing the vectors as column vectors:

$$f(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 2x_1 & & -x_3 \\ 3x_1 & -x_2 & +x_3 \end{bmatrix}$$

Of course, the rows of the matrix house the coefficients of the linear combinations of the input components, and writing these functions in column form reveals the matrix if you line up things properly. From above, it should not be too challenging to see that

$$f(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The standard matrix of f is $\begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & 1 \end{bmatrix}$.

Example 47: Find the standard matrix of the linear transformation $T(x, y) = (y, x, x + y)$.

solution: Here, $T : \mathbb{R}^2 \to \mathbb{R}^3$, so the standard matrix is a 3×2 matrix. Again, written in column form,

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \\ x + y \end{bmatrix}.$$

The output is three rows, three linear combinations of x and y. To make this a bit more clear, note that you could write this explicitly as

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0x + 1y \\ 1x + 0y \\ 1x + 1y \end{bmatrix}.$$

Of course,

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so the standard matrix for this linear transformation is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$.

295. Find the standard matrix for the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that switches the two coordinates. Give the transformation first, specifying the domain and codomain, and explain how you obtain the standard matrix.

296. Find the linear transformation determined by $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, and write it in two different ways. Specify the domain and the codomain.

297. Find the standard matrix for $T(x, y, z) = (x - y + z, z - 2x)$.

298. Find the standard matrix for $S(x, y) = (x - y, 2x - 2y, 2x - y)$.

299. Using T and S from the previous two exercises, find the composition $H = T \circ S$, its domain and codomain. Find $H(4, 5)$. Then find the standard matrix for H.

300. Find $G = S \circ T$ and its standard matrix. What is $G(1, 2, 3)$?

301. Let $T_1 : U \to V$ and $T_2 : V \to U$ be linear transformations. Prove that $T_2 \circ T_1$ is a linear transformation.

The following exercises offer up a challenge.

302. Let T be linear such that $T(1, 1) = (2, 2)$ and $T(2, 0) = (0, 0)$. Find $T(x_1, x_2)$.

303. Let T be linear such that $T(1, 1) = (1, 2)$ and $T(0, -1) = (3, 2)$. Find $T(\mathbf{x})$.

304. Let T be linear such that $T(1, 3, 1) = (4, 4, 0, -2)$ and $T(4, 0, 5) = (4, 5, -1, 5)$. Find $T(-7, 3, -9)$.

305. Let $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2)$. T is a linear transformation (you can prove so if you wish). Determine the inverse transformation T^{-1}, if it exists.

15.4 Student-Teacher Agreement

Going forward, whenever we encounter a linear transformation $T : U \to V$, then we will agree that it is okay to use the following: If T is a linear transformation, then

(1) T preserves vector addition,

(2) T preserves scalar multiplication, which means

(3) T preserves linear combinations, and

(4) T preserves the zero vector.

(1) and (2) can be put together, of course, to mean (3), and it is this quick, one-step move we can now make in forthcoming work.

dfff

Chapter **16**

The Columns of a Matrix, Part 1

16.1 A Momentary Pause for Clarification

Perform the following computations before continuing to read what follows.

306. $\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

307. $2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

308. $3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

309. $-2(3, -1, -2)$

310. $-2(2 - x - 2x^2)$

311. $\int_0^x (5e^x + 6e^{2x})dx$

312. $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

313. $\frac{d}{dx}[2e^x - 3e^{2x}]$

314. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Here are some things to appreciate about the above computations: (1) Each of you is comfortable with the notation you see; that is, very little confusion arose, if any, as you went about computing. (2) If we refer to the computations above as *operations*, we can describe the operations you performed. Those operations were matrix multiplication in 306, 312, and 314, scalar multiplication from 307 to 310, integration in 311, and differentiation in 313. (3) Your eyes see two things in each computation: something on the left and something on the right. It ought to make sense that someone might say that in each computation, an operation was performed on the thing on the right, producing an answer that is of the same species as the thing on the right. If we replace *thing on the right* with *vector*, and rewrite the previous sentence, then we can articulate this: It ought to make sense that someone might say that in each computation, an operation was performed on a vector, producing an answer that is of the same species as that vector. (Are you thinking of another word as you read that sentence? *Closure*, perhaps?) (4) Each operation *transformed* a vector, so it makes sense to say you made some transformations, each of which was a **linear transformation**. Differentiation and integration are often referred to as *linear operations* or *linear operators*. (5) You should note that you did the same computation twice in 309 and 310, then again in 311 and 312, and one more time in 313 and 314. (6) Observe that the answers to 306 and 307 are the same, which means that the matrix in 306 behaves like a scalar: that is, matrix multiplication, in this instance, scaled a vector. When matrices do this, something special is going on. This is the territory we have yet to explore: *eigenvalues* and *eigenvectors* are the things present here, coming soon.

Higher level mathematics requires the ability to see *form*. For example, each of the computations above are of the same form: $A\mathbf{x}$. A function can be defined that corresponds to each computation. The purpose of this chapter is to show you what the column vectors

of a matrix reveal, and then present you with more terms often used when a function, especially a linear transformation, is discussed. Recognizing the forms of things, however, is an important ability, and you acquire this ability over time, repeatedly seeing things and thinking about them on a much deeper level, one beyond that of producing an answer to a computation.

16.2 Systems and Linear Transformations

If $\mathbf{u}$ is a vector from the vector space U, $\mathbf{v}$ is a vector from the vector space V, and T is a linear transformation that maps vectors in U to vectors in V, then a system of linear equations is derived from considering the equality $T(\mathbf{u}) = \mathbf{v}$. Consider the following example.

Example 48: The system

$$\begin{aligned} x + 3y &= 2 \\ x + 4y &= 4 \end{aligned}$$

can be written as a matrix equation $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This is of the form $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$. The matrix A determines the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x+3y, x+4y)$. What is described as a system of equations for which the value on one side ($\mathbf{b}$) is the linear transformation (A) for some other quantity ($\mathbf{x}$). In the next few chapters, you will learn how important the matrix A is in terms understanding this course. I have selected to be more visual in the course, encouraging students to become more and more aware of how much incredible information is locked in a matrix. The remaining part of this chapter gets closer to that conversation, and is the first conversation intended to deepen your understanding of matrices.

315. Describe the linear transformation T in the system in the very first problem in this text. Find $T(1, 0)$ and $T(0, 1)$.

316. Describe the linear transformation T associated with the system in problem 6 of this text. Find $T(1, 0)$ and $T(0, 1)$.

317. Describe the linear transformation T associated with the system in problem 10 of this text. Find $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$.

16.3 What Matrices Tell You

We will focus a lot on 2×2 matrices because these matrices are relatively easy to visualize. Any matrix A *is* a linear transformation. The set of all 2×2 matrices consists of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. All matrices of this size determine the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$

given by $T(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$. These matrices transform the plane. The domain and codomain is $\mathbb{R}^2$, but the range (the set of all outputs) is some subspace of $\mathbb{R}^2$. The dimension of the range may not be 2: If $\mathbf{x}$ is any vector in $\mathbb{R}^2$, then a linear transformation might alter the entire plane and produce another plane of vectors (better visuals of this are forthcoming!), it might send every vector to the zero vector like $T(x_1, x_2) = (0, 0)$, which means the range of the transformation has dimension 0. It might collapse a dimension, as in $T(x, y) = (x, 0)$, where all vectors get mapped to the x-axis, where has dimension 1. The possible dimensions of the range are 0, 1, and 2. When we speak dimension, then we are talking about bases of vector spaces. The following discussion is extremely important because it is the first conversation leading to all of the big results. It is worth knowing how to have this conversation. *If you are given any matrix, first notice its size. This tells you what vector space is being transformed. The second thing to notice is how a matrix tells you what happens to basis vectors.* If you know how basis vectors are transformed, then you know a LOT! Here is where it is useful to use the column notation:

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

Take the time to understand each step below, and to understand the point of this section:

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \tag{16.1}$$

$$T(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}) = \begin{bmatrix} ax_1 \\ cx_1 \end{bmatrix} + \begin{bmatrix} bx_2 \\ dx_2 \end{bmatrix} \tag{16.2}$$

$$T(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix} \tag{16.3}$$

$$x_1 T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + x_2 T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix} \tag{16.4}$$

Here is the exact same conversation, but written out in the space-saver notation, worth printing just in case the properties are not as easily seen. The major result of the conversation is how you read the final equation, the fourth step.

$$\begin{aligned} T(x_1, x_2) &= (ax_1 + bx_2, cx_1 + dx_2) \\ T((x_1, 0) + (0, x_2)) &= (ax_1, cx_1) + (bx_2, dx_2) \\ T(x_1(1, 0) + x_2(0, 1)) &= x_1(a, c) + x_2(b, d) \\ x_1 T(1, 0) + x_2 T(0, 1) &= x_1(a, c) + x_2(b, d) \end{aligned}$$

What is the big result here? Vector addition justifies 16.1-16.2. Scalar multiplication justifies 16.2-16.3. Then *both* properties that define a linear transformation justify reaching the last equation, which reveals a huge result: The standard basis vectors of $\mathbb{R}^2$ are $(1, 0)$

and $(0,1)$. And $T(1,0) = (a,c)$, while $T(0,1) = (b,d)$. In other words, *given any* 2×2 *matrix, you are able to see where the standard basis vectors are mapped!* We will be using this soon to visualize things, and we will use it to understand what a basis of a vector space really means.

Example 49: Let T be determined by $\begin{bmatrix} 6 & -1 \\ 5 & -3 \end{bmatrix}$. Find $T(-2,9)$.

solution: While it is certainly permissible to find the formula for T, and then evaluate it at the given vector finding $T(-2,9)$, simply observe the matrix. By observation, $T(1,0) = (6,5)$ and $T(0,1) = (-1,-3)$. Note that $(-2,9) = -2(1,0) + 9(0,1)$. Because T is linear,

$$\begin{aligned} T(-2,9) &= T(-2(1,0) + 9(0,1)) \\ &= -2T(1,0) + 9T(0,1) \\ &= -2(6,5) + 9(-1,-3) \\ &= (-12,-10) + (-9,-27) \\ &= (-21,-37), \end{aligned}$$

so using the properties of a linear transformation gives you another perspective, another way to look at a computation.[1]

Here is a great way to learn to become more and more comfortable with the nature of problem solving in more advanced mathematics courses: cover up the solution to the next problem in example 50, and work the problem on your own. The answer is, in decimal form, $(1.2, 2.4)$, so try and obtain this solution. Then uncover the solution given after the example. The reason why I mention trying this here is due to increasing number of students who do not experience something that the internet (and "googling") has stolen from them, only to experience disaster on exams. This is only advice. (Example 51 forces you to work another problem in a specific kind of way.)

Example 50: Let T be a linear transformation such that $T(2,1) = (4,7)$ and $T(3,-1) = (3,5)$. Find $T(6,0)$.

solution: It depends on what you see, but always know that when you are told that a function is linear, try using the properties of linearity. Here,

$$\begin{aligned} T(2,1) + T(3,-1) &= (4,7) + (3,5) \\ T(5,0) &= (7,12) \\ \frac{6}{5}T(5,0) &= \frac{6}{5}(7,12) \\ T(6,0) &= (\frac{42}{5}, \frac{72}{5}). \end{aligned}$$

[1] The formula is $T(x,y) = (6x - y, 5x - 3y)$, so $T(-2,9) = (-12-9, -10-27) = (-21,-37)$. It's not that one is easier than the other; it's more about stretching your brainwaves in such a way that you see the special nature of linearity through other perspectives.

What if T was a linear transformation mapping $\mathbb{R}^3$ to $\mathbb{R}^2$? Then the standard matrix A of T is a 2×3 matrix. Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, and so $T(x, y, z) = (ax + by + cz, dx + ey + fz)$. Justify each of the changes below:

$$T(x, y, z) = (ax + by + cz, dx + ey + fz) \tag{16.5}$$
$$T((x, 0, 0), (0, y, 0), (0, 0, z)) = (ax, dx) + (by, ey) + (cz, fz) \tag{16.6}$$
$$T(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)) = x(a, d) + y(b, e) + z(c, f) \tag{16.7}$$
$$xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) = x(a, d) + y(b, e) + z(c, f). \tag{16.8}$$

The standard basis vectors of $\mathbb{R}^3$ are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and the final step shows that **the columns of the standard matrix are the vectors in the codomain to which the standard basis vectors are mapped.** Rewriting that last step in column vector notation gives one another reason why that notation is preferred if matrices are the objects used to house the data involved in a linear transformation:

$$xT(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + yT(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + zT(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = x \begin{bmatrix} a \\ d \end{bmatrix} + y \begin{bmatrix} b \\ e \end{bmatrix} + z \begin{bmatrix} c \\ f \end{bmatrix}$$

Example 51: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be linear, where $T(x_1, x_2, x_3) = (2x_1 + x_2, 3x_2 - x_3)$. Find the standard matrix of T and use it to find $T(0, 1, -1)$.

solution: Perhaps you have arrived at a comfortable place in which you can look at the function and write its standard matrix quickly:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

You see that $T(1, 0, 0) = (2, 0)$, $T(0, 1, 0) = (1, 3)$, and $T(0, 0, 1) = (0, -1)$, simply by observing the columns of A. Writing $(0, 1, -1)$ in terms of the standard basis vectors, followed by using linearity,

$$\begin{aligned} (0, 1, -1) &= 1(0, 1, 0) - 1(0, 0, 1) \\ T(0, 1, -1) &= T(1(0, 1, 0) - 1(0, 0, 1)) \\ T(0, 1, -1) &= T(0, 1, 0) - T(0, 0, 1) \\ T(0, 1, -1) &= (1, 3) - (0, -1) \\ T(0, 1, -1) &= (1, 4). \end{aligned}$$

Example 52: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation determined by $\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 8 & 0 \end{bmatrix}$.

Find $T(5, 3)$.

solution: The matrix reveals that $T(1, 0) = (1, 4, 8)$ and $T(0, 1) = (3, 2, 0)$. To those of you who are studying, or have studied multi-variable calculus and/or physics, I thought

I would use the abbreviated notation with which you are more familiar: $T(\mathbf{i}) = (1, 4, 8)$ and $T(\mathbf{j}) = (3, 2, 0)$, so

$$\begin{aligned} T(5,3) &= T(5\mathbf{i} + 3\mathbf{j}) \\ &= 5T(\mathbf{i}) + 3T(\mathbf{j}) \\ &= 5(1,4,8) + 3(3,2,0) \\ &= (5,20,40) + (9,6,0) \\ &= (14,26,40). \end{aligned}$$

16.4 Standard Basis Vector Notation and Generalization

There is something to generalize here: The standard basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$. The standard basis for $\mathbb{R}^3$ is the set $\{(1,0,0),(0,1,0),(0,0,1)\}$. The standard basis for $\mathbb{R}^4$ is... You can see how cumbersome it is to write the basis vectors for Euclidean vector spaces, so there is a notation mathematicians have invented to reduce the agony of writing a vector whose components are all zero except for the single component that is 1. They use $\mathbf{e}$ for "Euclidean" and a subscript indicating which component is 1. For example, $\mathbf{e_1}$ and $\mathbf{e_2}$ can describe the basis vectors of $\mathbb{R}^2$, meaning $\mathbf{e_1} = (1,0)$ and $\mathbf{e_2} = (0,1)$. For $\mathbb{R}^3$, the standard basis vectors are $\mathbf{e_1} = (1,0,0)$, $\mathbf{e_2} = (0,1,0)$, and $\mathbf{e_3} = (0,0,1)$. The basis for any Euclidean space $\mathbb{R}^n$ is written $\{\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}\}$.

THEOREM 3: The columns of a matrix are the vectors to which basis vectors are mapped. More precisely, an $m \times n$ matrix A determines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. The standard basis of $\mathbb{R}^n$ is $\{\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}\}$. If $A = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_n} \end{bmatrix}$, where $\mathbf{c_i}$ is the ith column vector, then $T(\mathbf{e_i}) = \mathbf{c_i}$.

Before the proof, let's be clear on all of this notation. In example 51, the standard matrix of the linear transformation was

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}.$$

If we made use of the notation introduced in this section, then $\mathbf{e_1}, \mathbf{e_2}$, and $\mathbf{e_3}$ are the standard basis vectors of the domain $\mathbb{R}^3$, best thought of as column vectors. The column vectors of A are $\mathbf{c_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. I will leave it to the student to finish connecting this example to the theorem. Here is the proof:

PROOF: Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$. In column vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Given the standard basis of $\mathbb{R}^n$, we know that $\mathbf{x}$ can be written in terms of these vectors:

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}$$

For any vector $\mathbf{x}$, $T(\mathbf{x}) = A\mathbf{x}$, and

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}) \tag{16.9}$$
$$= x_1T(\mathbf{e_1}) + x_2T(\mathbf{e_2}) + \cdots + x_nT(\mathbf{e_n}), \tag{16.10}$$

since a linear transformation preserves linear combinations. We also have that

$$T(\mathbf{x}) = A\mathbf{x} \tag{16.11}$$
$$= \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{16.12}$$
$$= \mathbf{c_1}x_1 + \mathbf{c_2}x_2 + \cdots + \mathbf{c_n}x_n \tag{16.13}$$
$$= x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n}. \tag{16.14}$$

By 16.10 and 16.14,

$$x_1T(\mathbf{e_1}) + x_2T(\mathbf{e_2}) + \cdots + x_nT(\mathbf{e_n}),$$

so $T(\mathbf{e_1}) = \mathbf{c_1}, T(\mathbf{e_2}) = \mathbf{c_2}$, ..., and $T(\mathbf{e_n}) = \mathbf{c_n}$.

We will soon make it okay to say that a matrix transforms a space, since a matrix *is* a linear transformation. The column vectors of any given matrix tell you to what vector every basis vector will be mapped. It helps to change your perception of what a matrix is; or, more specifically, what a matrix *does*.

318. Find the standard matrix for $T(x_1, x_2) = (x_2 - x_1, x_2)$ and use it to compute $T(-1, 4)$. Then check that the result agrees when you use the given formula.

319. Repeat the same for $T(x, y, z) = (2x - z, x + y + 2z)$ and $T(3, 2, 1)$.

320. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $T(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $T(\mathbf{e_2}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $T(\mathbf{e_3}) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$. Find $T(1, 2, 3)$.

321. Let T be a linear transformation with domain $\mathbb{R}^2$. Suppose $T(\mathbf{e_1}) = (2, 0, 1, 1)$ and $T(\mathbf{e_2}) = (1, 4, 5, 5)$. Find $T(3, 5)$.

322. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ define the linear transformation T. Find $T(\mathbf{e_1} - 3\mathbf{e_2})$.

Return to the first section of Chapter 12, where you met the matrices F and G in problem 225. Use these matrices to answer problems 323, 324, and 325. (Congratulate yourself, too, for making it through one hundred problems since you met these two matrices.)

323. Let T_F and T_G be the linear transformations associated with these two matrices. Find formulas for these two functions.

324. Find the standard matrix for $T_F \circ T_G$.

325. Find the standard matrix for $T_G \circ T_F$.

Chapter 17

The Columns of a Matrix, Part 2

17.1 Image and Kernel

If T maps a vector $\mathbf{v}$ to a vector $\mathbf{w}$; that is, if $T(\mathbf{v}) = \mathbf{w}$, then $\mathbf{w}$ is called the **image** of $\mathbf{v}$. The word *image* is used interchangeably with the word *range*. The images of all the vectors in the domain *is* the range, so sometimes people refer to the image of the vector space V is the range of the function. Think simple: if $f(x) = x^2$, then 9 is the image of 3, or the image of 3 is 9. Notice that for this function, the domain is $\mathbb{R}$, the codomain is $\mathbb{R}$, but the range is *not* $\mathbb{R}$. The range is a subset of $\mathbb{R}$, all real numbers greater than, or equal to, zero, and this set is the image of $\mathbb{R}$.

DEFINITION 17: Let $T : V \to W$ be a linear transformation. The **kernel** of T is the set of all vectors in the domain V that are mapped to the zero vector in the codomain W. The kernel of T is denoted by $\ker(T)$. Described in formal set notation, $\ker(T)=\{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$, where $\mathbf{0}_W$ represents the zero vector in the codomain W.

It happens too frequently in mathematics that students are given definitions of things that are so technical in nature that they are cumbersome to remember. It takes a while to get to the stage of life where you might notice you use these mathematical concepts, but your language when using them is meaningful to you because you see how it is used. The kernel of a linear transformation is the collection of all the vectors that get mapped to the zero vector. Let's do a thought experiment and speak as though we are in the future and you have a job in data science. My guess is you might use the following language, knowing that important data is housed in vectors, and you have some kind of linear function T that will take this data and put it somewhere. The kernel of T might be all the data that is lost once the transformation is applied to a set of all data you might have. Does this make sense, and can you imagine why losing data might be something to care about in any kind of occupation?

As a student of mathematics, I often pondered over the choice some historical figure made when giving an official name to a well-defined concept. For example, why would someone use the word *kernel* to describe the set of all vectors that get mapped an origin? or, by its other meaning, all vectors that get mapped to nothing? that go away? get lost? be mapped to a beginning point? to the center of another space? You may have heard the word *kernel* used in relationship to corn: a "kernel of corn." In that context, a kernel is something from which the rest grows. You are familiar with the "origin" in the $xy-$plane, the place from which all other places can be reached by obeying exactly two directions via the (x, y) coordinates. The nature of the word is that it can be used to also describe the "core" of something. If you are a botanist, a core is where seeds are found, and those are "kernels." It can also mean, in another context of what "origin" or "core" are in usage, that it is something that must be kept small. If $\mathbf{v}$ contains important data, and a linear transformation T re-organizes that data and $T(\mathbf{v}) = \mathbf{0}$, then what does that mean? Should the number of vectors mapped to zero be expected to be small?

Think about this as you study this subject. You have a rare opportunity to do so in a linear algebra course: Data science, AI, and most computer science areas are increasing in importance. (This course is a vital part of sustaining progress in those areas, in both humanity and intelligence.) In one area of computer science, the kernel is the first part of an operating system to load into memory before booting, it can be the main component of an operating system, or the central and most important part of anything. *It is my strong belief*, however, that the word was chosen due to the fact that ***the solution set of any system of linear equations*** $A\mathbf{x} = \mathbf{b}$ ***consists of all of the solutions to*** $A\mathbf{x} = \mathbf{0}$ ***and a single solution found somewhere else.*** All solutions are built from the kernel. If you have solved nonhomogeneous, linear, differential equations, your solutions were always the solution to a corresponding equation set equal to zero, added to a particular solution. This is part of the big picture of linear algebra: the structure of all solutions to linear systems.[1]

Example 53: Find the kernel of $T(x, y, z) = (y, z)$, a linear transformation that takes all of three dimensional space, and squashes it flat so that the space is transformed into something infinitely flat: the yz-plane.

solution: Mechanically speaking, find all solutions (x, y, z) where $T(x, y, z) = (0, 0)$. This is fairly easy to accomplish in some cases. Here, for example,

$$\begin{aligned} T(x, y, z) &= (0, 0) \\ (y, z) &= (0, 0), \end{aligned}$$

which means x can be any real number (it's a *free variable*) but the other two variables are zero; that is, the set of all vectors (x, y, z) such that $T(x, y, z) = (0, 0)$ are all vectors of the form (a,0,0), a *subspace* of $\mathbb{R}^3$, a line containing its origin, the $x-$axis. That should make sense to you if you squash in the way described to attain a flat desired result called the $yz-$plane: crush the space in the $x-$direction: loose all the vectors associated with the x-axis, but preserve the origin.

Notice the descriptions in the solutions to the examples are sneaking in connections. The purpose is for you to begin doing so, a sincere recommendation. Notice the visual connection made in the previous solution. In terms of work accomplished in the course thus far, allow the previous example's work to flow to the following example:

Example 54: Find the standard matrix A of the transformation in the previous example, and then find the null space of A.

solution: If we think deeply about this, we know that the null space of A is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and if the kernel of T are all vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$, and $T(\mathbf{x}) = A\mathbf{x}$, then it becomes obvious that the null space of a matrix A is the kernel of the linear transformation determined by A, so the null space of A is the set of all vectors of the form $(a, 0, 0)$; that is, the $x-$axis. The standard matrix of $T(x, y, z) = (y, z)$ is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The matrix equation for $A\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The augmented

[1] Be aware that *kernel* is also used in other contexts and confusing ways in mathematics.

matrix for this system of equations is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and notice its form. The augmented matrix is already in reduced row echelon form in which you can see there is no leading 1 in the first column, so x is a free variable, while both rows explicitly are read as $y = 0$ and $z = 0$; hence, the null space of A is the set of all vectors of the form $(a, 0, 0)$.

17.2 Range

DEFINITION 18: Let $T : V \to W$ be a linear transformation. The set of vectors in W that are the images of vectors in V is called the **range** of T. The range will be denoted by range(T).

THEOREM 4: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$. The range of T is the span of the column vectors of A.

PROOF: Let $\mathbf{w}$ be any vector in the range of T, which means there exists a vector $\mathbf{v}$ in the domain of T such that $T(\mathbf{v}) = \mathbf{w}$. Since $\mathbf{v} \in \mathbb{R}^n$, then it can be written in terms of the standard basis vectors of $\mathbb{R}^n$, so that $T(\mathbf{v}) = \mathbf{w}$ can be written as

$$T(c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n}) = \mathbf{w}.$$

The linearity of T allows us to write this as

$$c_1T(\mathbf{e_1}) + c_2T(\mathbf{e_2}) + \cdots + c_nT(\mathbf{e_n}) = \mathbf{w},$$

where we see that the vector in the range $\mathbf{w}$ is written as a linear combination of the images of the basis vectors of the domain; that is, $\mathbf{w}$ is a linear combination of the column vectors of the matrix A. By definition of *span*, the range of T is the span of the column vectors of A.

Is there anyone reading this last theorem, and thinking how nice it is to be able to find the range of a function simply by finding the span of the set of column vectors?

Example 55:[2] Find the domain, codomain, kernel and range of the linear transformation $T(x, y, z) = (x, 0, z)$ by first finding the standard matrix A associated with T.

solution: $T : \mathbb{R}^3 \to \mathbb{R}^3$, so the domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^3$. The standard matrix A is the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[2]This example is here to be used for visual purposes in a forthcoming chapter.

The kernel of the transformation is all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{0}$, and, by way of quick Gauss-Jordan elimination, the augmented matrix for the system $A\mathbf{x} = \mathbf{0}$ in reduced row echelon form would be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and read as $x = 0, y = a$, and $z = 0$, where a is any scalar since y is a free variable.

The range of the transformation is the span of the set $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$, which is the set of all linear combinations of the vectors in this set. If k_1 and k_2 are any scalars, the span of this set of vectors is the set of all vectors of the form $k_1(1, 0, 0) + k_2(0, 0, 1)$, better articulated as the set of all vectors of the form $(k_1, 0, k_2)$, a plane in $\mathbb{R}^3$, and, more specifically, the $xz-$plane.

Here is a summary of our findings, where a, r, and s are any scalars. Please notice an added dimension to the table, no pun intended.

Space	Description (symbol)	Description (words)	Dimension
domain	$\mathbb{R}^3$	"3D" space	3
codomain	$\mathbb{R}^3$	also, xyz-space	3
kernel	all vectors $(0, a, 0)$	the y-axis	1
range	all vectors $(r, 0, s)$	the xz-plane	2

Example 56: Find the domain, codomain, kernel, and range of the linear transformation $T(x, y, z) = (2x + 6y - 4z, x + 3y - 2z)$ by first finding the standard matrix A associated with T.

solution: $T : \mathbb{R}^3 \to \mathbb{R}^2$, so the domain is "3D" space and the codomain is the xy-plane, and A is a 2×3 matrix:

$$A = \begin{bmatrix} 2 & 6 & -4 \\ 1 & 3 & -2 \end{bmatrix}$$

The kernel of the transformation is all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, which corresponds to a system of equations represented by the augmented matrix

$$\begin{bmatrix} 2 & 6 & -4 & 0 \\ 1 & 3 & -2 & 0 \end{bmatrix},$$

which can be written in reduced row echelon form:

$$\begin{bmatrix} 2 & 6 & -4 & 0 \\ 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -4 & 0 \end{bmatrix}$$
$$\xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables y and z are free variables, and $x + 3y - 2z = 0$ is the information from the matrix. Written in terms of the free variables, $x = 2z - 3y$, so the kernel of T is the set

of all vectors of the form $(2b - 3a, a, b)$, where a and b are any scalars. This is a plane in three-dimensional space.

The range of T is the span of the set $\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}, \begin{bmatrix}-4\\-2\end{bmatrix}\}$. The set of all linear combinations of these vectors leads to a scalar multiple of a single vector, no matter what mathematics you use to examine the span. The range is, therefore, all vectors of the form $(2k, k)$, where k is any scalar. This is a line in a two-dimensional space.

Summarizing, we have the following:

Space	Description (symbol)	Description (words)	Dimension
domain	$\mathbb{R}^3$	"3D" space	3
codomain	$\mathbb{R}^2$	the plane	2
kernel	all vectors $(2b - 3a, a, b)$	a plane	2
range	all vectors $(2k, k)$	the line $y = x/2$	1

All of the transformations T in the following exercises are linear.

326. Determine the kernel and range of $T(x, y) = (2x + 2y, x - y)$.

327. Determine the kernel and range of $T(x, y) = (y - x, 2x - 2y)$.

328. Determine the kernel and range of $T(x_1, x_2, x_3, x_4) = (x_1 + x_3 - x_4, x_2 + x_3)$.

329. Let $D = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$. Find the null space of this matrix.

330. For the matrix D in the previous problem, write the linear transformation T and determine its domain, codomain, range, and kernel.

331. Prove that the kernel of a linear transformation is a subspace of the domain of the transformation. Here is some help to begin writing: Let $T : V \to W$ be a linear transformation. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be any vectors in $\ker(T)$, and let k be any scalar...

332. Prove that the range of a linear transformation is a subspace of the codomain of the transformation.

333. A function is said to be **onto** if the range is equal to the codomain. A function is said to be **one-to-one** if no two vectors in the domain have the same image. Discuss the familiar functions $f(x) = x^2$ and $g(x) = x - 2$. Determine whether they are onto and/or one-to-one, and be able to give a solid reason to justify your answer (without the use of the internet). *Hint: Draw pictures to help.*

334. Determine the kernel and range of the linear transformation defined by the matrix $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$. Is it one-to-one? Is it onto?

335. Determine the kernel and range of the linear transformation defined by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Is it one-to-one? Is it onto?

336. Let $T : V \to \mathbb{R}^3$ be a linear transformation. Geometrically, what are the possibilities for the range of T?

337. Let $T : \mathbb{R}^3 \to W$ be a linear transformation. Geometrically, what are the possibilities for the kernel of T?

For problems 338, 339, and 340, let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$.

338. Recall that the row space of a matrix is the span of the row vectors. Show that A and B have the same row space.

339. Show that A and B have the same null space.

340. Is the column space of A the same as the column space of B?[3]

[3] What do you believe is the point of these last three exercises?

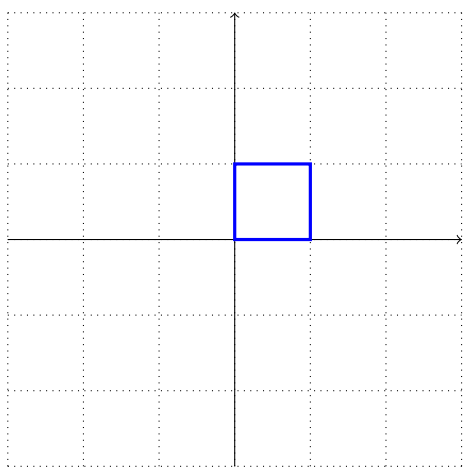
Chapter 18

Visualizing Matrices

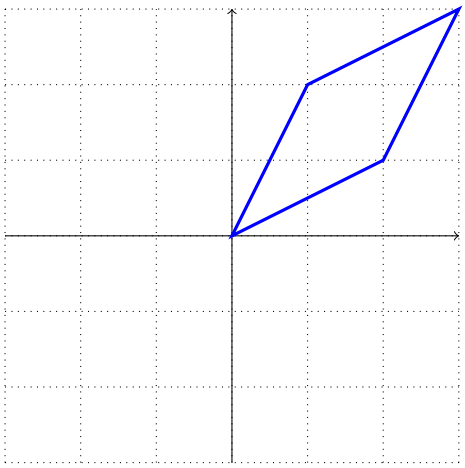
18.1 The Unit Square

In $\mathbb{R}^2$, the square with vertices (0,0), (1,0), (1,1), and (0,1) will be referred to as the **unit square**. When the adjective *unit* describes a mathematical object, it usually means that object has some measure of 1 associated with it: *unit vectors* are vectors of length 1, the *unit circle* $x^2 + y^2 = 1$ has a radius of 1, and, for our purposes here, the *unit square* has an area of 1. Euclidean vector spaces are measured by a quantity associated with their dimensions: $\mathbb{R}$, the real line, can be cut into pieces in which *length* is a measure; the plane, $\mathbb{R}^2$, *area*; $\mathbb{R}^3$, *volume*, etc... There is something about having spaces with a flatness associated with them, along with orthogonality (perpendicularity).

You often hear that two nonparallel vectors in the plane (in standard position) determine a parallelogram, so we can say that the unit square is determined by the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ defines transformation $T(x, y) = (2x + y, x + 2y)$, and we know by looking at the columns that $T(1, 0) = (2, 1)$ and $T(0, 1) = (1, 2)$. If we sketch the two column vectors in the xy-plane, we see that these vectors determine the parallelogram with vertices the origin, (1,2), (3,3), and (2,1):



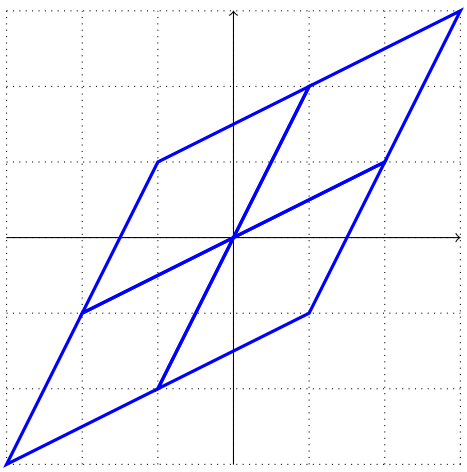
If you know to what vectors the basis vectors are mapped, *then you know how the entire space is altered*!

341. Find $T(1,1)$ by the formula $T(x,y) = (2x+y, x+2y)$.

342. Repeat finding $T(1,1)$ by using linearity, meaning $(1,1) = (1,0) + (0,1)$ and apply the transformation using only the knowledge given by the columns of the matrix.

343. What you see in the two figures preceding these first few exercises is how unit area is altered. This means *all of the unit squares are altered in the same way.* There are 36 unit squares in the first figure (dotted lines). Determine whether or not the four unit squares that surround the origin will be transformed as the picture below depicts.

344. Notice in the figure above, that the parallelogram sits inside the area of a larger rectangle (that happens to be a square). Find the area of the parallelogram using grade school mathematics: Think "area of whole thing minus area of part will equal area of parallelogram."

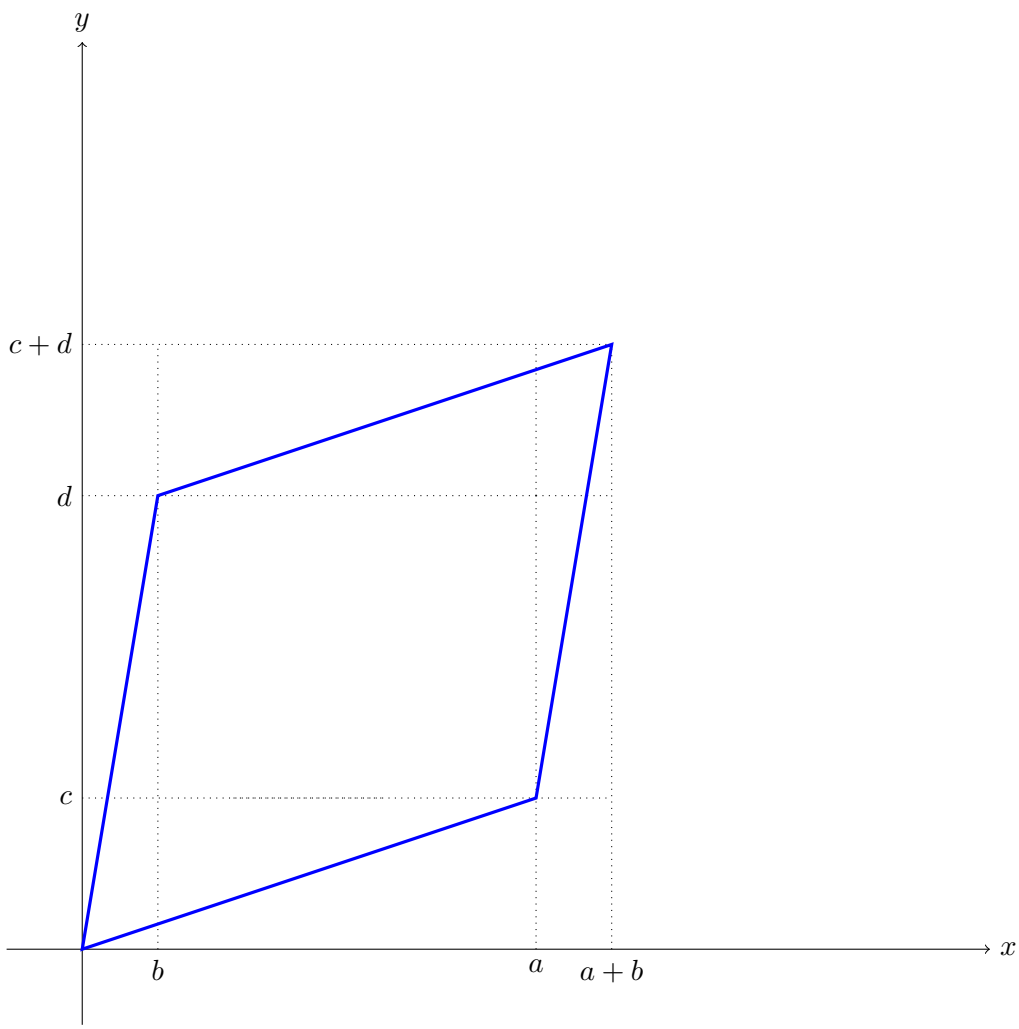


If you use quadrille graph paper, ideally you imagine all the squares you see that make the graph paper what it is as unit squares. It represents the Euclidean vector space $\mathbb{R}^2$,

the xy-plane, and it is built by the standard basis vectors. You can think of the matrix as transforming the coordinate system for the space, creating a new kind of graph paper in which the two major axes point diagonally and are not perpendicular to each other.

18.2 An Origin Story

There is a joke I repeatedly tell students, when the occasion arises: *Did you know that if you clean a vacuum cleaner, then you are a vacuum cleaner?* Another thought that I hope makes you chuckle, even if it is a little laugh, is this: *The other day, I went to the supermarket and bought a trash can that I put inside a bag. After I got home, I took the trash can out of the bag, and then put the bag inside the trash can.* Let's determine something called the determinant.



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Assume we are given a matrix A that transforms the unit square in a similar manner as the matrix in the previous discussion. We can see that the matrix maps the vector $(1, 0)$ to the vector (a, c) and maps the vector $(0, 1)$ to the vector (b, d). The unit square is transformed to a parallelogram.

345. Find the area of the parallelogram using the same method used in 344: whole area minus part equals parallelogram area.

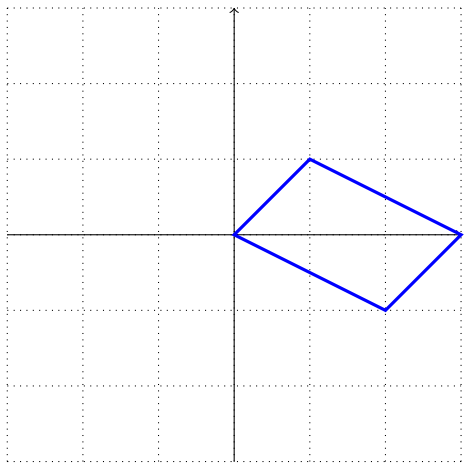
The area you found required some algebra, of course: The rectangle of area $(a + b)(c + d)$ minus the area of six other basic shapes (two triangles, each with area $\frac{ac}{2}$, two triangles, each with area $\frac{bd}{2}$, and two rectangles, each with area bc) results in the area of the parallelogram:

$$\begin{aligned}(a + b)(c + d) - ac - bd - 2bc &= ac + ad + bc + bd - ac - bd - 2bc \\ &= ad - bc\end{aligned}$$

Many of you were taught how to find what is called the determinant of a 2×2 matrix: Subtract the products of its diagonal elements: product of the main diagonal elements a and b subtracted by the product of the elements on the other diagonal b and c, a quick way to obtain the number $ad - bc$. How many of you were taught where the determinant comes from?! We will officially define the *determinant* of a matrix soon. Such a thing ought to *determine* something.

Example 57: Sketch the unit square in the xy-plane. Then sketch the image of the unit square when the matrix $M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ transforms the plane, and find the area of this image. Give the formula for the linear transformation using $T_M : \mathbb{R}^2 \to \mathbb{R}^2$.

partially written solution: I'll let you, the student, sketch the unit square. The columns of the matrix inform us that the image of the basis vector $(1, 0)$ is $(2, -1)$, and the image of the basis vector $(0, 1)$ is $(1, 1)$. Simply sketch these two vectors in the plane, and then sketch the parallelogram they determine. Your parallelogram should have vertices the origin, $(1, 1), (3, 0)$, and $(2, -1)$. Based on the formula we computed, the area of this parallelogram is $2(1) - (-1)(1) = 2 + 1 = 3$, so unit area is tripled under the matrix transformation.



The formula for the transformation $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ is $T_M(x, y) = (2x + y, y - x)$.

Example 58:[1] Using the same matrix M as in the previous example, find the image of the vector $(1, 3)$, and sketch both vectors in the same coordinate plane.

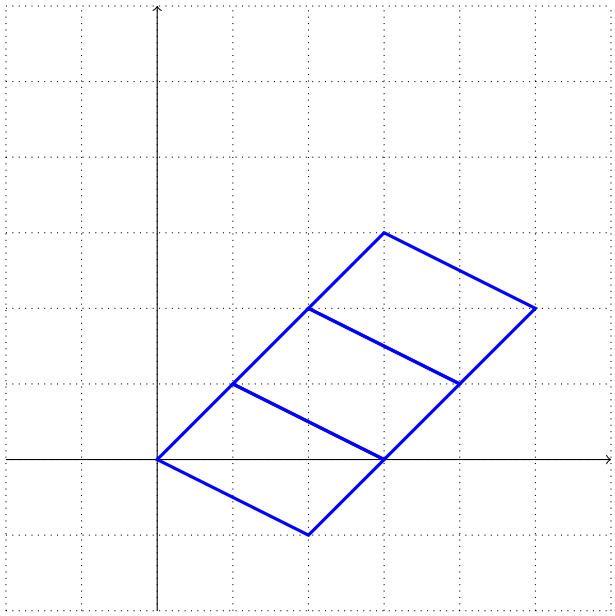
solution and some things to notice: The image of a vector $\mathbf{x}$ can be found by finding $T_M(\mathbf{x})$ or by multiplying the matrix by the vector: $M\mathbf{x}$:

$$T_M(1, 3) = (2(1) + 3, 3 - 1) = (5, 2)$$

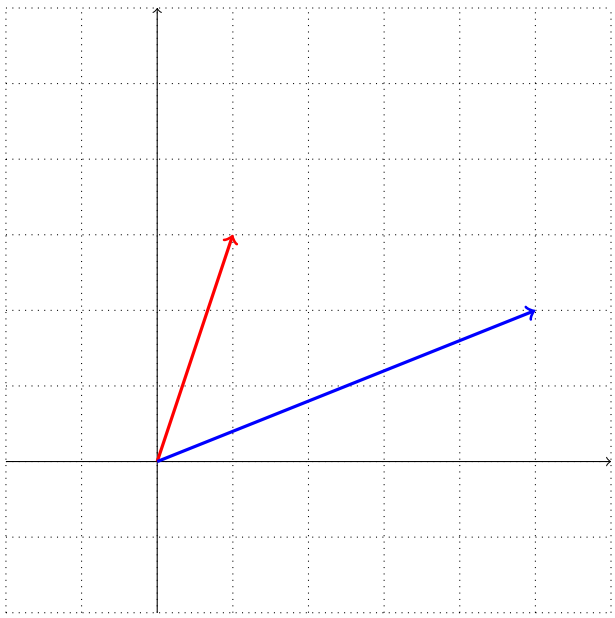
$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Notice something here: The parallelogram in the previous example can serve as the basic building block of the plane in the same way that the unit square makes up graph paper. All scalar multiples of $(1, 0)$ form the x-axis, and all scalar multiples of $(0, 1)$ form the y-axis; You can say that the matrix M transforms the x-axis into a new axis parallel to the longer side of the parallelogram, while the y-axis becomes an axis parallel to the shorter side. To find $T_M(1, 3)$, from the origin, go "right-ish one" long length on the parallelogram and then "up-ish three" short lengths, and notice that you land on $(5, 2)$:

[1] The discussion in this example is *important*.



The example asks us to sketch the vector and its image in the same plane, which opens up another discussion that is vital to understanding some of the big things in the course.



The red vector is transformed to the blue vector by the matrix. It ought to make sense to say that the matrix does two things in the transformation: it *rotates* the red vector, and then *stretches* the vector. In fact, one of the biggest, and most powerful, results in linear algebra is the fact that any matrix can be written as a product of matrices that rotate and stretch. There are two questions about what matrices do to vectors in a vector space, and these questions are central to the subject: *What vectors remain pointing in the same direction after a matrix transforms the space? If there are vectors that are perpendicular to each other before the transformation, are there vectors perpendicular to each other after*

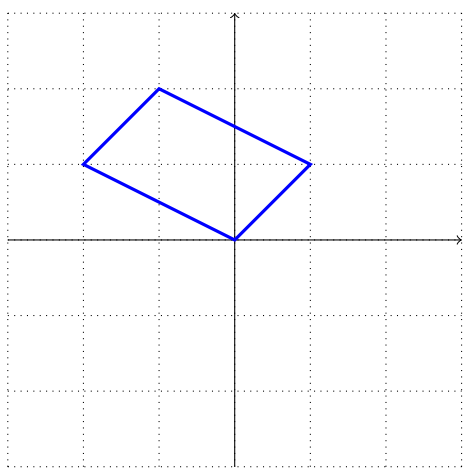
the transformation? We will come back to these questions as they lead to some of the most important mathematical tools in machine learning and data analysis.

Example 59: Sketch the unit square in the xy-plane. Then sketch the image of the unit square when the matrix $P = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ acts on the plane, and find the area of this image. Give the formula for the linear transformation $T_P : \mathbb{R}^2 \to \mathbb{R}^2$. Find the kernel of the transformation, which is the null space of the matrix P, and find the range of the transformation.

partial solution: I will leave it up to the student to show the work details here, and to realize that the image of unit square is a *line segment* from the origin to the point $(3, 6)$; in other words, unit area gets squashed. Notice that this agrees with the area formula we computed, that $1(4) - 2(2) = 0$. The formula for the linear transformation is $T_P(x, y) = (x + 2y, 2x + 4y)$. The null space of the matrix (the kernel of the transformation) is the set of all vectors of the form $(-2a, a)$, the line $x = -2y$. The range of the transformation is the span of the column vectors of P, which is all scalar multiples of $(1, 2)$, otherwise known as the line $y = 2x$. The matrix takes the plane and squashes it to a line.

Example 60: Let $Q = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$. Sketch the image of the unit square under the matrix transformation. What is the area of this image? Write the formula T_Q for this transformation.

solution: Again, the column vectors of Q are the images of the standard basis vectors, and the image of the unit square is the following parallelogram:



It appears as though our area formula collapses here, since $(-2)(1) - 1(1) = -3$. It is *not* because the image of the unit square appears in the second quadrant. The reason for this is due to the fact that the unit square was "flipped" somehow. The area is actually 3, the magnitude of the number computed by our formula. We will get into this later, but that is what a "negative determinant" means geometrically. $T_Q(x, y) = (y - 2x, x + y)$.[2]

[2]Note that $T(1, 3) = (1, 4)$. How would you find the image of the vector in this case using the parallelogram and the idea that it forms a new kind of graph paper?

We are using $\mathbb{R}^2$ to give you exposure to what matrices do to a space, as it is easy to sketch things and visualize. Square matrices are, therefore, the big focus in an introductory course. As we continue, we will see some more matrix transformations. The last two sections of this chapter give you general matrix information for transformations of the plane.

18.3 Rotation Operators

Matrices that rotate vectors along arcs of circles centered at the origin are called *rotation operators*. Your background in trigonometry gives you the power to find the standard matrix for the rotation operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that moves sets of points counterclockwise about the origin by θ, a positive angle.

346. Sketch the unit square in the plane. *In the same plane*, sketch the image of the unit square under the matrix transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

347. Using the same matrix, sketch the vector $(2, 3)$ and its image under the matrix transformation.

348. Sketch the image of the unit square under the transformation by $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

349. Show that the standard matrix for rotating all points in the plane counterclockwise by an angle θ is the following matrix, which we will give the name R_θ:
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

350. Find the matrix that defines a rotation of the plane about the origin through each of the following angles. Determine the image of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under each transformation: **(a)** π. **(b)** $\frac{\pi}{6}$, and **(c)** $-\frac{\pi}{3}$.

351. Find the equation of the image of the ellipse $9x^2 + 4y^2 = 36$ under a rotation through an angle of $\frac{\pi}{2}$.

352. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What does this matrix do to vectors in $\mathbb{R}^2$? (Your job here is to determine an intelligent way to answer this with justification.)

18.4 Dilation and Contraction

Use the following matrices for the next four problems:
$$A = \begin{bmatrix} -7 & 5 \\ -3 & 4 \\ -1 & 9 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, D = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

353. Find AB and AC.

354. Find BC and CB.

355. Find BD and CD.

356. What do the matrices B and C do to unit area in $\mathbb{R}^2$?

Consider the transformation $T(\begin{bmatrix} x \\ y \end{bmatrix}) = k \begin{bmatrix} x \\ y \end{bmatrix}$, one that maps every point in the plane k times as far from the origin. If $k > 1$, points are moved further away from the origin and the transformation is known as a dilation. If $0 < k < 1$, points are moved closer to the origin and the transformation is known as a contraction. The standard matrix of the transformation is a *diagonal matrix*:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Most of us use the word "stretched by a factor of k" rather than *dilation* or *contraction*.

357. Determine a single matrix that defines both rotation about the origin through an angle θ and a dilation of factor k.

358. Let T_1 be a rotation and let T_2 be a reflection in the x-axis. Are these transformations commutative? Discuss this geometrically first, and then algebraically.

359. Find and sketch the image of the unit square under the transformation $T(\mathbf{u}) = A\mathbf{u}+\mathbf{v}$ if A is a diagonal matrix with $k = 2$ and $\mathbf{v} = (4, 4)$.

360. Prove that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (0, y, 0)$ is linear. This transformation is known as a *projection*. Explain why this term is appropriate.

Chapter 19

The Transpose of a Matrix

19.1 Meet Some Interesting Matrix Products

Don't forget the rules of matrix multiplication as you find the products in the following exercises.

361. What is the result of the matrix product ABC if A is size 1×3, B is size 3×3, and C is size 3×1?

362. Find the matrix product $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$.

363. Find the matrix product $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$.

364. Find the matrix product $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

365. Find the products $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

366. Find the products $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$.

367. Think about the previous six exercises for a moment. Without googling, use your own brain to come up with some thoughts or ideas about these products you found. What thoughts come to mind?

19.2 Inner Product and Outer Product of Vectors

When we wish to use a vector $\mathbf{v}$ as a matrix, the context usually makes it clear whether we are using it as a row matrix or column matrix. Using matrix notation, the vector (x, y) is written as a 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. In the exercises above, you see such a vector written as a 1×2 matrix. Note that this allows for two possible matrix products:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$

The product on the left is the familiar *dot product*, a number. This is also known as the *inner product* of two vectors. The product on the right is a matrix known as the *outer product*. You encounter both products if you study quantum physics, but most of you are familiar with the dot product as it is used in circuits, mechanics, and economics. To define the dot product of two vectors in matrix notation, we need to define what is known as the **transpose** of a matrix. Before we do this, here is some important notation: The transpose of a matrix A will be denoted by A^T and involves switching rows and columns (defined in the next section). The transpose of $\mathbf{v}$ is $\mathbf{v}^T$, and if $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $\mathbf{v}^T = \begin{bmatrix} x & y \end{bmatrix}$. As you can see, $\mathbf{v}^T\mathbf{v} = x^2 + y^2$, the dot product, and $\mathbf{v}\mathbf{v}^T$ is the 2×2 matrix. For exercises 368-370, use

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

368. Find $\mathbf{x}\mathbf{x}^T$ and $\mathbf{x}^T\mathbf{x}$.

369. Find $\mathbf{y}\mathbf{y}^T$ and $\mathbf{y}^T\mathbf{y}$.

370. Find the products $\mathbf{x}\mathbf{y}^T$ and $\mathbf{y}\mathbf{x}^T$.

19.3 Switching Rows And Columns

Let A be an $m \times n$ matrix. The **transpose** of A, denoted by A^T, is the $n \times m$ matrix A^T obtained by writing all the rows of A as the columns of A^T; that is, row 1 becomes column 1, row 2 becomes column 2, etc... The following matrices are used in the examples and exercises that follow:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & -3 \\ 4 & -1 & 5 \end{bmatrix}, D = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Example 61: Write the transpose of each matrix.

solution:

$$A^T = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}, C^T = \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ -3 & 5 \end{bmatrix}, D^T = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 5 \end{bmatrix}$$

Compute the following products:

371. AB and then find its transpose, denoted $(AB)^T$

372. $A^T B^T$

373. $B^T A^T$

374. Let $C\mathbf{x} = \mathbf{b}$, where $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $\mathbf{b}$. Then write $\mathbf{b}^T$.

375. Find $\mathbf{x}^T C^T$. Is this equal to $\mathbf{b}^T$?

376. Prove the following statement: If the product AB is defined, then $(AB)^T = B^T A^T$.

19.4 Properties of the Transpose

It is difficult to give introductory students a good intuitive feel for the transpose of a matrix, other than the hope that many students sense the interesting products given at the start of this chapter, namely that matrices can encode a quadratic polynomial, and the dot product can be rewritten using another notation. I came across the following example while reading another book on linear algebra years ago that is related to what these concepts are in more advanced linear algebra coursework, but one that introductory students should be able to follow well, and, perhaps, appreciate:

Example 62: Consider the expression $x_2y_1+x_3y_2-x_1y_1-x_2y_2$. Rewrite the expression as a linear combination of x_1, x_2 and x_3. Then, separately, rewrite the expression as a linear combination of y_1 and y_2. Then read what follows.

solution: It is simply a matter of algebra to move things around in such a way to have these linear combinations: As a linear combination of x_1, x_2 and x_3,

$$x_2y_1 + x_3y_2 - x_1y_1 - x_2y_2 = (-x_1y_1) + (x_2y_1 - x_2y_2) + (x_3y_2) \tag{19.1}$$

$$= (-y_1)x_1 + (y_1 - y_2)x_2 + (y_2)x_3, \tag{19.2}$$

which is the linear combination of the x's we seek. Similarly,

$$x_2y_1 + x_3y_2 - x_1y_1 - x_2y_2 = (x_2y_1 - x_1y_1) + (x_3y_2 - x_2y_2) \tag{19.3}$$

$$= (x_2 - x_1)y_1 + (x_3 - x_2)y_2, \tag{19.4}$$

which is the combination of the y's we seek.

Here comes the interesting part: Remember that a matrix can house the coefficients of linear combinations. If we let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2)$, and interpret both of these vectors as column vectors, then we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and we have another exercise to perform: Rewrite the expressions in (19.2) and (19.4) as matrix inner products:

$$(-y_1)x_1 + (y_1 - y_2)x_2 + (y_2)x_3 = \begin{bmatrix} (-y_1) & (y_1 - y_2) & (y_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{19.5}$$

$$(x_2 - x_1)y_1 + (x_3 - x_2)y_2 = \begin{bmatrix} (x_2 - x_1) & (x_3 - x_2) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{19.6}$$

Keep in mind that (19.5) and (19.6) represent the same expression, so to keep ourselves on track with all of this rewriting, note that

$$\begin{bmatrix}(-y_1) & (y_1-y_2) & (y_2)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}(x_2-x_1) & (x_3-x_2)\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}.$$

The interesting part is the two row matrices on both sides of the equation: We can rewrite them. The row vector on the left side can be rewritten as

$$\begin{bmatrix}(-y_1) & (y_1-y_2) & (y_2)\end{bmatrix}=\begin{bmatrix}y_1 & y_2\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix},$$

while the row vector on the right side can be rewritten as

$$\begin{bmatrix}(x_2-x_1) & (x_3-x_2)\end{bmatrix}=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}.$$

This means that the equation

$$\begin{bmatrix}(-y_1) & (y_1-y_2) & (y_2)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}(x_2-x_1) & (x_3-x_2)\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

can be written in the form $\mathbf{y}^TA^T\mathbf{x}=\mathbf{x}^TA\mathbf{y}$, where A is the 3×2 matrix. This may be a bit of a long and involved example, but it is worth thinking about if you are curious about the transpose of a matrix, and what it actually is. This is also an example that gives introductory linear algebra students something to play with in order to further cement the properties of the transpose. Feel free to spend as much time as you would like playing around with it, as I have found it either captivates intellectually curious students who learn to see the deeper layers of a transpose, or provides others with an annoying example! (In other words, do not spend too much time on this one example; I have included it for those who delight in thinking about these strange definitions that arise in this subject.)

In brief, and this is really a theorem, if A is an $m\times n$ matrix, then A^T is an $n\times m$ matrix such that

(1) $(A^T)^T=A$,

(2) $(A+B)^T=A^T+B^T$, for any matrix B that can add with A,

(3) $(AB)^T=B^TA^T$, for any matrix B that can be multiplied, and

(4) $(kA)^T=kA^T$.

377. Prove that if A is an invertible matrix, then its transpose is also invertible, and $(A^T)^{-1}=(A^{-1})^T$.

378. What is $(ABC)^T$?

379. Find the transpose of both sides of the equation $\mathbf{y}^TA^T\mathbf{x}=\mathbf{x}^TA\mathbf{y}$.

380. Assume that all matrices in the equation are square matrices that are invertible, and solve the equation for D:
$$ABC^TDBA^TC=AB^T$$

Chapter 20

The Complete Solution to $A\mathbf{x} = \mathbf{b}$

20.1 Linear, First-Order, Differential Equations

Not all of you have taken a course on differential equations, but you should be able to easily understand example 63, and somewhat follow example 64. You are not responsible for solving differential equations in this course, but I think you would enjoy learning how to do so.

Example 63: Find all solutions to $\frac{dy}{dx} = 3x - 2$.

solution: All of you can solve this equation by direct integration since the derivative of a function $y(x)$ is given as an explicit function of x:

$$\begin{aligned} \int \frac{dy}{dx} dx &= \int (3x-2)dx \\ y(x) &= \frac{3x^2}{2} - 2x + C \end{aligned}$$

Example 64: Find all solutions to $\frac{dy}{dx} - 2y = 5$.

solution: In a course in which you study the methods for solving differential equations, you will learn that, for an equation of this type (linear, first-order), you must multiply both sides of this equation by the function e^{-2x}, which makes the equation solvable (specifically, in the second step, the left side of the equation becomes the derivative of a product):

$$\begin{aligned} e^{-2x}(\frac{dy}{dx} - 2y) &= 5e^{-2x} \\ e^{-2x}\frac{dy}{dx} - 2e^{-2x}y &= 5e^{-2x} \\ \frac{d}{dx}(ye^{-2x}) &= 5e^{-2x} \\ \int \frac{d}{dx}(ye^{-2x})dx &= \int 5e^{-2x}dx \\ ye^{-2x} &= -\frac{5}{2}e^{-2x} + C \\ y(x) &= -\frac{5}{2} + Ce^{2x} \end{aligned}$$

The point of these two examples is not the methods used to solve these equations; rather, it is to observe *the set of all solutions to the differential equation.* These equations are both linear equations, and it is the **form** of the solution set that you should notice. In example 63, the *homogeneous* equation corresponding to $\frac{dy}{dx} = 3x - 2$ is $\frac{dy}{dx} = 0$, to which the set of all solutions is any constant function C. Notice that this solution is a part of

the set of all solutions to the given equation. Similarly, in example 64, the *homogeneous* equation corresponding to $\frac{dy}{dx} - 2y = 5$ is $\frac{dy}{dx} - 2y = 0$, which has solution Ce^{2x}, and you should observe that this solution is included in the solution to the given equation. Allow the odd lines of changing notation below attempt to shed light on the linear algebra here:

$ay' + by = c$ is a linear equation.
$ay' + by = 0$ is its corresponding homogeneous equation with solution y_h.
y_p is a particular solution to $ay' + by = c$.
All solutions to $ay' + by = c$ are $y = y_p + y_h$.

Example 63: $y' = 3x - 2$ is linear.
$y' = 0$ has solution $y_h = C$.
$y_p = \frac{3x^2}{2} - 2x$ solves $y' = 3x - 2$.
All solutions to $y' = 3x - 2$ are $y = y_p + y_h$.

$y' = 3x - 2$ can be written in operator notation $Dy = 3x - 2$,
which can be written in vector form $D\mathbf{y} = \mathbf{b}$.

This is probably overkill on my part. Agree?
All solutions to $A\mathbf{x} = \mathbf{b}$ are
$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

20.2 Particular Added to the Homogeneous

381. Solve

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ 2x_1 - 4x_2 + x_3 &= 0 \end{aligned}$$

382. Solve

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 2x_1 - 4x_2 + x_3 &= 2 \end{aligned}$$

383. Reinterpret the system in 382 in matrix form $A\mathbf{x} = \mathbf{b}$. Identify $A, \mathbf{b}, \mathbf{x}_h, \mathbf{x}_p$, and what is meant by $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

***. To those of you studying multi-variable calculus, note the **form** of the set of all solutions to these linear systems: a particular solution added to a constant multiple of another solution (or a linear combination of other solutions). Make this connection with equations of lines and planes: The vector equation of a line in $\mathbb{R}^3$ is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$; that is, a vector from the origin to a particular point on the line, plus any scalar multiple of a vector parallel to the line. The vector equation of a plane in $\mathbb{R}^3$ is $\mathbf{r}(s,t) = \mathbf{r_0} + s\mathbf{v_1} + t\mathbf{v_2}$; that is, a vector corresponding to a point on the plane, plus a linear combination of two *linearly independent* vectors in the plane. To all, $y = mx + b$, or should we say $y = b + mx$, a particular point on the line, plus all multiples of something telling you how the line tilts in the plane? Linear algebra has been with you for a very long time, ready to awaken you.

384. Find the row space of the matrix A.

385. Find the null space of the matrix A.

386. Find the linear transformation for which A is its standard matrix. Identify the domain, codomain, the kernel, and the range. For extra thinking on your part, what would you say that the matrix does to a space in this case?

Consider a system of m linear equations and n unknowns. In matrix form, the system is written $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix, $\mathbf{x}$ has n components, and $\mathbf{b}$ has m components. The set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ is the solution set to the system.

The corresponding homogeneous system is $A\mathbf{x} = \mathbf{0}$. The set of all vectors in this solution set form a subspace (you have proved this in a previous homework problem, but let's repeat this here): To prove this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two vectors in this set, which means $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. Therefore,

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0}, \end{aligned}$$

which means $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution, so the set of solutions to $A\mathbf{x} = \mathbf{0}$ is closed under vector addition. If k be any scalar, then

$$\begin{aligned} A(k\mathbf{x}_1) &= k(A\mathbf{x}_1) \\ &= k(\mathbf{0}) \\ &= \mathbf{0}, \end{aligned}$$

which means $k\mathbf{x}_1$ is also a solution, so the set of solutions to $A\mathbf{x} = \mathbf{0}$ is closed under scalar multiplication. If X represents the set of all solutions to $A\mathbf{x} = \mathbf{0}$, then X is a subspace of $\mathbb{R}^n$, a line-like thing passing through the origin in this space. X is also the null space of the matrix A, since it is the set of all vectors that A maps to the zero vector in $\mathbb{R}^m$. The kernel of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$ is X. Remember the brief discussion of the word *kernel*. This set generates the set of all solutions to the non-homogeneous system of equations $A\mathbf{x} = \mathbf{b}$. When solving any linear equation, it is often a requirement that you first obtain the solution to the corresponding homogeneous equation: the *subspace* known as the *null space* of the matrix of coefficients, the *kernel* of the linear transformation. The set of all solutions to the non-homogeneous equation is *parallel* to this line-like thing; that is, the subspace is moved away from the origin.

The challenge in linear algebra is the ability to hold securely among your thoughts the many different ways of saying the same thing. The non-homogeneous system of linear equations after this paragraph, as easy as it is to state its solution set simply by observation, is the system of equations pertaining to the set of exercises that follow, problems that might encourage some of you how much more important it is to truly think about synthesis and connection. The work you will produce in this subset of problems might be something worthy of being a poster you produce, something in picture form that increases the likelihood of you connecting many of the concepts you have studied since day one. Think of these problems as an opportunity to become more acquainted with those myriad pieces of knowledge you have been exposed to, and that, for the moment, you are playing

around with these ideas with the goal of figuring out how they combine to form a kind of overall big picture of what this subject is really about. The graph work you produce ought to be as precise as possible, so use graph paper. And, I am aging and becoming "an old dude" and wish to tell you that doing things by hand without the aid of anything electronic is always an enriching experience. This is simply what I would *encourage* you to do. You can submit it for a serious grade in the course, especially if you become interested or excited in case you feel like you might be seeing some nice things. Use the following system of equations for exercises 387-390:

$$\begin{aligned} x_1 - 3x_2 &= 6 \\ 2x_1 - 6x_2 &= 12 \end{aligned}$$

387. Write the system in $A\mathbf{x} = \mathbf{b}$ form, and then verify that the solution $\mathbf{x}$ to the system is $\mathbf{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + k \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, where k is any scalar. Geometrically, this is a line in $\mathbb{R}^2$. (To Calculus IV students, the vector equation of the line is $\mathbf{r}(t) = (6, 0) + t(3, 1)$. To Calculus III students, the parametric equations of the line are $x = 3t + 6, y = t$. To algebra students, this is the line $y = \frac{1}{3}x - 2$. Ever thought to connect those, by chance?)

388. From the system in matrix form, take the matrix A on its own, as if the only thing you were given to work with is A. Find the row space of A and the null space of A, and sketch these spaces in $\mathbb{R}^2$. Label them clearly as row(A) and null(A). Label this sketch DOMAIN.

389. On the paper you usually write out solutions to problems, write the linear transformation T determined by A as a formula $T(x_1, x_2)$, identifying its domain and codomain. Beneath this work, find the following: ker(T), range(T), and sketch the image of the unit square under the transformation.

390. Return to the sketch labeled DOMAIN, and acknowledge that the kernel of the transformation is the same as the null space of the matrix. Label that on your sketch so that someone else viewing it immediately knows the equivalence here.

391. *On a separate piece of paper*, to be labeled RANGE, find the column space of A, sketch it on a new set of axes sized the same as the coordinate system you used on the other sketch. Label the column space as col(A).

392. Physically take these two pieces of paper, and place them on a flat surface, the sketch DOMAIN to the left of the sketch RANGE. Explain why this makes sense to do, including as much mathematics that you can include *without help of any kind.* This will be a strong indication of your level of understanding things, and the most valuable thing to share with me, or anyone to whom you feel like sharing.

393. Add one final thing to the sketch DOMAIN, and make this sketch stand out from the other sketches on this picture: draw the solution set to the non-homogeneous system given for problem 387.

394. Keep these sketches within your eyesight for a few days, perhaps thinking to hang them on the wall as you would a piece of artwork. If you become curious and interested, take as much time thinking about these pictures. (Consider that there are things present on them that you might not be able to see yet.)

395. Write the transpose of the matrix A. Find its null space, its row space, and its column space. Sketch them on what places you believe would be the most appropriate places to sketch them, and label them $\text{null}(A^T)$, $\text{row}(A^T)$, and $\text{col}(A^T)$.

396. A student hears an instructor say that the system $A\mathbf{x} = \mathbf{b}$ has a solution as long as the vector $\mathbf{b}$ is in the column space of A. If that vector is *not* in the column space, then $A\mathbf{x} = \mathbf{b}$ has no solution. Explain this, or criticize it.

397. Is A an invertible matrix? Why or why not?

These remaining three problems are not related to the ones prior:

398. Let A be an $m \times n$ matrix. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two solutions to the set S of all solutions to the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is not equal to the zero vector, and let k be any scalar. Prove, or disprove, that S is a subspace of $\mathbb{R}^n$.

399. The augmented matrix, in reduced row-echelon form, for a system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Determine the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$.

400. For the system in 399, find $\text{null}(A)$ and $\text{row}(A)$.

Chapter 21

The Four Fundamental Subspaces of a Matrix

21.1 A Quick Self-Check

Cover up Let A be a 2×3 matrix. Let T be the linear transformation determined by this matrix.

401. How does A transform vectors? In terms of T, the function T takes vectors in ??? and maps them to vectors in ???
402. The null space of A is a subspace of?
403. The column space of A is a subspace of?
404. The row space of A is a subspace of?
405. How do you find the range of the transformation?
406. The null space of A^T is a subspace of?
407. If the basis of the null space of A is a single vector, then how many components does it have? What is the dimension of the null space of the matrix in this case?

21.2 Subspaces of the Domain

If A is any $m \times n$ matrix, then the domain of the linear transformation determined by A is $\mathbb{R}^n$. This domain can be split into two important subspaces: the null space of A and the row space of A. We have previously defined these two subspaces (one in definition 9, the other in a prior homework problem). Below, definition 9 is repeated, and we formally add the definition of null space to our list of definitions (hence the strange numbering):

DEFINITION 9: The **row space** of a matrix A of size $m \times n$ is the span of the row vectors of the matrix; that is, the set of all linear combinations of the rows of A. The row space of A will be denoted by $\text{row}(A)$.

DEFINITION 19: The **null space** of a matrix A of size $m \times n$ is the set of all vectors in $\mathbb{R}^n$ that are mapped to the zero vector in $\mathbb{R}^m$. The null space of A will be denoted by $\text{null}(A)$.

In more symbolic language, $\text{null}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. The following two problems are worth repeating:

408. Prove that the row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

409. Prove that the null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

The definition of the null space of a matrix A says something powerful about its relationship with the row space: If $\mathbf{u}$ is any vector in the null space of A, then the product $A\mathbf{u}$ is equal to the zero vector in $\mathbb{R}^m$. What does this product actually mean? Of the many ways to multiply matrices, the most popular is to take the *rows* of A and "dot them" with the vector $\mathbf{u}$, which means that each one of those dot products is 0; hence, each row vector is *orthogonal* to $\mathbf{u}$. Since the row space of A is the span of the rows of A (the set of all linear combinations of the rows of A), then every vector in the row space of A (which can be written in terms of these row vectors), dotted with $\mathbf{u}$ is zero; thus, every vector in the row space is orthogonal to every vector in the null space. This means that the row space is orthogonal to the null space. Moreover, these two subspaces split the entire domain $\mathbb{R}^n$ into these two orthogonal subspaces. Think of what you have done for years in mathematics when you have sketched a curve in the xy-plane, or when you have sketched a surface in the xyz-coordinate system. These two spaces, $\mathbb{R}^2$ and $\mathbb{R}^3$, are familiar spaces to you in that you know a way to visualize the axes to see where the origin is, but let's appreciate the orthogonal nature of our viewpoint: The vertical y-axis is perpendicular to the horizontal x-axis. The vertical z-axis is perpendicular to the horizontal xy-plane. Mechanically work out these two problems carefully, using the fact that, in any Euclidean vector space, $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal:

410. Consider $\mathbb{R}^2$. Find all vectors (a, b) that are orthogonal to $(0, 1)$.

411. Consider $\mathbb{R}^3$. Find all vectors (a, b, c) that are orthogonal to $(0, 0, 1)$.

There is a stronger statement we can make here about subspaces, but let's review a concept many of you have seen appear somewhere in your past, most likely with Venn diagrams or perhaps in high school geometry: the *complement* space. Let $\mathbb{Z}^+$ be the set of all nonzero positive integers. We can split this set up into two distinct sets: the set P of all positive even integers and the set O of all positive odd integers. Within this set $\mathbb{Z}^+$ of only positive integers, the complement of P is O. In 410, the vector (0,1) spans the y-axis, a subspace of $\mathbb{R}^2$. All vectors orthogonal to this one vector will also be orthogonal to *any* vector in this subspace. $\mathbb{R}^2$ is the set of all vectors of the form (x, y). Does it make sense to you if someone were to ask you *what is the orthogonal complement* of the y-axis? Are you able to form, on your own, a conversation about the z-axis and its relationship to the xy-plane in $\mathbb{R}^3$? Spend some time thinking about this. This brings us to some new definitions that will be the focus of the next chapter. I thought I would plant a small seed in this one, if only to give a bit of guidance to where we are headed, and to elicit deeper thinking, if I am able.

SPECIAL NOTATION: For any matrix A, the notation R_0 will represent the reduced row echelon form of A. (I will use this notation throughout the remainder of this book to denote the reduced row echelon form of a matrix.)

Example 65: Let A be the following 2×3 matrix:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

Find the row space of A based on the definition of row space, and the row space of R_0, the reduced row echelon form of A. Show that these row spaces describe the same subspace.

solution: By definition, we can write that row(A) is the set of all linear combinations of the vectors $(1, 3, 4)$ and $(-1, 2, 1)$, so we could say row$(A) = k_1(1, 3, 4) + k_2(-1, 2, 1)$, where k_1 and k_2 are any scalars. That's following the definition. Let's row reduce the matrix:

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow{R2/5} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-3R2+R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The row$(R_0) = a(1, 0, 1) + b(0, 1, 1)$, where a and b are any scalars. If we summarize our two row spaces, we can write them in a more succinct way, describing each one as "the set of all vectors of the form..." as was done in prior work. Using both scalar multiplication and vector addition, we have the following:

(1) The row space of A is the set of all vectors of the form $(k_1 - k_2, 3k_1 + 2k_2, 4k_1 + k_2)$.

(2) The row space of R_0 is the set of all vectors of the form $(a, b, a + b)$.

Notice how nice (2) reads: it is the set of all vectors in $\mathbb{R}^3$ whose third component is the sum of the first two components. You want to appreciate the simplicity of this description in words that is read directly from the mathematical description. Now look at (1). If these row spaces are the same, then (1) should also be a description of the same space in words. Indeed, if you add the first two components, you get the form of the third component:

$$(k_1 - k_2) + (3k_1 + 2k_2) = k_1 + 3k_1 + 2k_2 - k_2 = 4k_1 + k_2.$$

This is the same description as the row space of the reduced row echelon form of A, but it is clearly not as easy to read! The intention of this example is an effort on my part to convince students of two important things: that these row spaces are the same (which we will prove in the next theorem) and that the row space of a matrix ought to be found by using the matrix in its reduced row echelon form. (It might mean you keeping your job one day!) Another way to convince students of this is found in the following two examples:

Example 66: Find the null space of A in example 65.

solution: This requires solving the system $A\mathbf{x} = \mathbf{0}$, but reducing the matrix, as was done above, speeds up our work here in finding the null space. I'll let you, the student, reason why it is quick to write that the solution to the homogeneous system can be found by noting the following:

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$$

If x_3 is any scalar, say k, then the above information can be converted to the set of all vectors of the form $(-k, -k, k)$, or all scalar multiples of the vector $(-1, -1, 1)$. So, null(A) $= k(-1, -1, 1)$, where k is any scalar.

If the null space of A is all scalar multiples of the vector $(-1, -1, 1)$, then the row space of A is the set of all vectors in $\mathbb{R}^3$ that are orthogonal to the vector $(-1, -1, 1)$, leading to this next example, which simply verifies the work we have already done above.

Example 67: Find all vectors in $\mathbb{R}^3$ that are orthogonal to $(-1, -1, 1)$.

solution: This would be the set of all vectors (a, b, c) such that $(a, b, c)\cdot(-1, -1, 1) = 0$, which means

$$\begin{aligned}(a, b, c) \cdot (-1, -1, 1) &= 0\\ -a - b + c &= 0\\ c &= a + b.\end{aligned}$$

Geometrically, it makes sense to expect a plane, given the dimension of this space of vectors. (The bigger picture of how these two subspaces split the vector space is a major result in linear algebra.) Since c can be written in terms of a and b, replace c in (a, b, c) with $a + b$; thus, the set of all vectors orthogonal to $(-1, -1, 1)$ is the set of all vectors of the form $(a, b, a+b)$. Using vector addition to split the variables and scalar multiplication, $(a, b, a + b) = a(1, 0, 1) + b(0, 1, 1)$. A basis for the subspace called the row space is the set $\{(1, 0, 1), (0, 1, 1)\}$. Note this connection to R_0.

The row space of a matrix is the same as the row space of the matrix in its reduced row echelon form. This is because two row equivalent matrices have the same row space. Reduced *row* echelon form of a matrix is fundamentally defined in terms of its rows. Elementary row operations by design do not change the row space because each new row is a linear combination of the older rows. What is impressive about R_0 is that its form gives you linearly independent vectors, which means a basis is easily acquired for the row space. As you perform row operations, changing A to R_0, each new matrix is equivalent to all the others (think of solving a system of equations), so the next theorem takes care of the fact that row(A) = row (R_0) since A and R_0 are equivalent. You have to be careful with column spaces, as an earlier example showed: col(A) is not always equal to col(R_0), an issue we will explore in the next section. You should read this next theorem with the following advice: *consult the definitions!* The proof is one in which the vocabulary of linear algebra allows for some quick logical argumentation, but as a person learning this subject for the first time, you might want to fill in some further explanation as to how one sentence flows logically to the next. That is the popular job of mathematics students, especially at these higher levels of study.

THEOREM 5: Two row equivalent matrices have the same row space.

PROOF: If B is row equivalent to A, then B can be reached by performing a finite number of row operations on A. Each row operation can be represented by an invertible elementary matrix, so in terms of matrix multiplication, suppose $E_1, E_2, ..., E_k$ represent the row operations that change A to B; that is, $E_k...E_2E_1A = B$. The row vectors of B are, thus, linear combinations of the row vectors of A. Consequently, the row space of B must be a subspace of the row space of A. Since the row operations are invertible, we

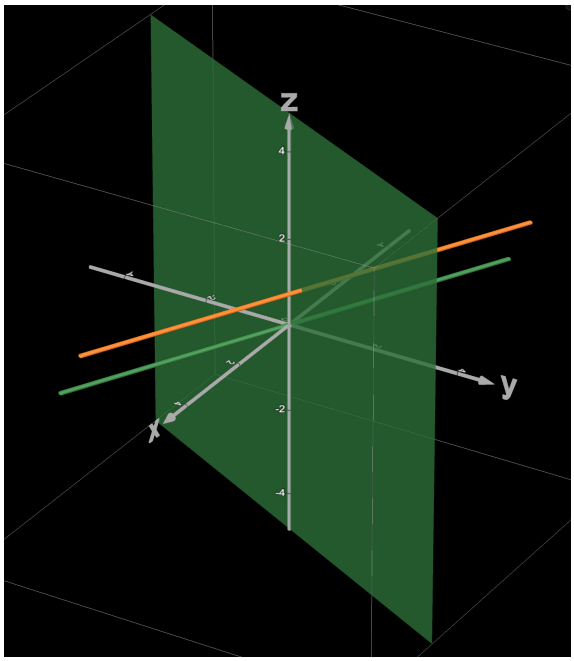


Fig. 21.1: Visual for Exercises 412-418

have, in matrix multiplication form, that $A = E_1^{-1}E_2^{-1}...E_k^{-1}B$, and so A is equivalent to B, and, using the same reasoning, the row space of A must be a subspace of the row space of B. Therefore, row space of A is the same as the row space of B.

Use the following system of equations for exercises 412 to 418, and also later in 419 and 420.

$$x + 2y + 3z = 4$$
$$x + 2y + 4z = 5$$

412. Solve the system by Gauss-Jordan elimination.

413. From your solution in 412, state the solution to the corresponding homogeneous system of equations.

414. Write the system of equations above in the matrix form $A\mathbf{x} = \mathbf{b}$. Find the null space of A.

415. Find the row space of A.

416. Identify the row space, the null space, and the solution set of the system in the above figure, Fig. 21.1.

417. Give the formula for the linear transformation T determined by A. Identify the domain, the codomain, and kernel of the transformation.

418. What is the range of the transformation?

21.3 Subspaces of the Codomain

There are actually six important vector spaces (subspaces) associated with a given matrix A. There are three for the matrix A, and three for its transpose A^T: null(A), row(A), col(A), null(A^T), row(A^T), and col(A^T). Since transposing a matrix is nothing more than switching its rows with its columns, then row(A) = col(A^T) and col(A) = row (A^T). This means that only four of these subspaces are distinct: the row space of A, the column space of A, the null space of A, and the null space of the transpose A^T. The null space of the transpose, null(A^T), is the set of all vectors $\mathbf{v}$ such that $A^T\mathbf{v} = \mathbf{0}$. If we transpose this equation, we get the equation $\mathbf{v}^T A = \mathbf{0}^T$, where the unknown vector is to the left of the matrix A. This is the reason why the null(A^T) is called the *left null space of A*.

Recall the definition of the column space, rewritten below, but this time coupled with the definition for left null space:

DEFINITION 10: The **column space** of a matrix A of size $m \times n$ is the span of the column vectors of the matrix; that is, the set of all linear combinations of the columns of A. The column space of A will be denoted by col(A).

DEFINITION 20: The **left null space** of a matrix A of size $m \times n$ is the null space of the $n \times m$ matrix A^T. The left null space of A will be denoted by null(A^T).

In the next chapter, we will explore the geometric links between these fundamental subspaces of a matrix, which will help us reach a goal in this course: the "big picture" of linear algebra, and a fundamental theorem. This chapter closes with the following two problems, referring to the matrix A in the system of equations used in exercises 412-418:

419. Find the col(A).

420. Find the null(A^T).

Chapter 22

Finding Bases for Subspaces of a Matrix

22.1 Finding the Best Basis for Each Subspace

This first section serves as a clarification for finding and giving the best bases for the four fundamental subspaces of a matrix. A good procedure for doing this is described in detail in examples 68 and 69. It ought to be stated that reducing A is enough, but I am recommending row reducing both the matrix and its transpose (examples 68 and 69) before demonstrating how to use the reduced row echelon form of A only (example 70).

Example 68: Find bases for the four fundamental subspaces of the matrix $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & 0 \\ 3 & 5 & 2 \end{bmatrix}$.

solution: First, let A be this matrix, and find the reduced row echelon form of A to get the row space and the null space:

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & 0 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[-3R1+R3]{-2R1+R2} \begin{bmatrix} 1 & 4 & 3 \\ 0 & -6 & -6 \\ 0 & -7 & -7 \end{bmatrix} \xrightarrow[R3/(-7)]{R2/(-6)} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[-R2+R3]{-4R2+R1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this matrix, you can obtain a basis for the row space by taking the linearly independent rows that appear after the reduction. The null space can be read by interpreting the matrix understanding that $[A|\mathbf{0}]$ would be the augmented matrix leading to the same three columns plus the fourth column of zeros; that is, the null space would be found from knowing $x_1 = x_3, x_2 = -x_3, x_3 = x_3$, so the null space is all vectors $\mathbf{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, where x_3 is the free variable. (If your multivariable calculus skills are in good form, this means the row space is the plane $x - y + z = 0$. Why?) The row space is the span of the set $\{(1, 0, -1), (0, 1, 1)\}$.

Second, find the reduced row echelon form of the matrix A^T, the transpose of A to get the column space and the left null space.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 5 \\ 3 & 0 & 2 \end{bmatrix} \xrightarrow[-3R1+R3]{-4R1+R2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -7 \\ 0 & -6 & -7 \end{bmatrix} \xrightarrow[R3/(-6)]{R2/(-6)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{7}{6} \\ 0 & 1 & \frac{7}{6} \end{bmatrix} \xrightarrow[-R2+R3]{-2R2+R1} \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix}$$

The row space of this matrix is the column space of A, so the column space of A is spanned by the set $\{(1, 0, 2/3), (0, 1, 7/6)\}$. The left null space is when $x_1 = -\frac{2}{3}x_3, x_2 =$

$-\frac{7}{6}x_3$, and $x_3 = x_3$, so the left null space is spanned by the vector $(-\frac{2}{3}, -\frac{7}{6}, 1)$, or, if you do not like fractions, you can choose to say the left null space is spanned by $(-4, -7, 6)$.

The following table is a summary of the fundamental subspaces of the given matrix, where a basis is given for each subspace:

Subspace	Basis	Dimension
null(A)	$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$	1
row(A)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$	2
null(A^T)	$\left\{ \begin{bmatrix} -4 \\ -7 \\ 6 \end{bmatrix} \right\}$	1
col(A)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{7}{6} \end{bmatrix} \right\}$	2

Example 69: Find bases for the four fundamental subspaces of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ -3 & -6 & -1 \end{bmatrix}$.

solution: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -6 & -1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & -3 \\ 2 & -6 \\ -1 & -1 \end{bmatrix}$. Row reduce each matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -6 & -1 \end{bmatrix} \xrightarrow{3R1+R2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R2/(-4)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 2 & -6 \\ -1 & -1 \end{bmatrix} \xrightarrow[R1+R3]{-2R1+R2} \begin{bmatrix} 1 & -3 \\ 0 & 0 \\ 0 & -4 \end{bmatrix} \xrightarrow[R2\leftrightarrow R3]{R3/(-4)} \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{3R2+R1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

From the RREF of A, a basis for the row space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The null space comes from reading the matrix, recognizing that x_2 is a free variable:

$$\begin{aligned} x_1 &= -2x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}$$

The null space is the line spanned by the vector in its basis, $\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\}$. Repeat the same reasoning with the transpose, where its row space is the column space of A, and the left null space: A measure of where you are at in terms of understanding this material, allow me to make a rather annoying comment: you should be comfortable reading the matrix, easily, and without work, write that col(A) is $\mathbb{R}^2$, and the left null space is the zero vector in $\mathbb{R}^2$.

Summary of these bases, including dimension:

Subspace	Basis	Dimension
null(A)	$\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\}$	1
row(A)	$\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$	2
null(A^T)	zero vector	0
col(A)	$\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$	2

Example 70: Find the four fundamental subspaces of the matrix $B = \begin{bmatrix}1 & 1 & -1\\-2 & -1 & 3\\-1 & -1 & 1\end{bmatrix}$.

solution by way of using only the reduced row echelon form of B: Since this matrix is square, and the row operations we will be doing are the same as when finding the inverse of a matrix by keeping track of the elementary operations used, ultimately revealing a basis for the left null space! (How? Why?) While row-reducing the matrix, attempt to find its inverse, if possible:

$$\left[\begin{array}{rrr|rrr}1 & 1 & -1 & 1 & 0 & 0\\-2 & -1 & 3 & 0 & 1 & 0\\-1 & -1 & 1 & 0 & 0 & 1\end{array}\right] \xrightarrow[R1+R3]{2R1+R2} \left[\begin{array}{rrr|rrr}1 & 1 & -1 & 1 & 0 & 0\\0 & 1 & 1 & 2 & 1 & 0\\0 & 0 & 0 & 1 & 0 & 1\end{array}\right]$$

$$\xrightarrow{-R2+R1} \left[\begin{array}{rrr|rrr}1 & 0 & -2 & -1 & -1 & 0\\0 & 1 & 1 & 2 & 1 & 0\\0 & 0 & 0 & 1 & 0 & 1\end{array}\right]$$

This results in a magnificent way to state bases for all four subspaces, all from the results of this one augmented matrix. The first three columns represent the RREF of B: the row space is spanned by $(1, 0, -2)$ and $(0, 1, 1)$, and I leave it to the student to be able to find the null space is spanned by the vector $(2, -1, 1)$. The columns in which there are leading 1's indicate that the first two columns of the original matrix B are a basis for the column space. (The CR-factorization explains this well as understanding that requires you to think in the column perspective.) We can choose the first two columns of B as a basis for the column space of B, though a better basis is found by the RREF of the

transpose. (Why?) And the left null space is spanned by whatever rows have zeros in the first three columns. (Why?! This is something *you*, the student with the brilliant ability to think, should figure out on your own.) Here, the left null space is spanned by the vector $(1, 0, 1)$. A summary of our results:

Subspace	Basis	Dimension
null(B)	$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$	1
row(B)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$	2
null(B^T)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right.$	1
col(B)	$\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$	2

I recommend working the following problems after reading and working through this example, in order to understand something often confusing: how to reconcile that there is a better basis for the column space, but the vectors in both bases span the same space.

421. Write down the transpose B^T and find its reduced row echelon form.

422. Use the result in the previous problem to find a basis for the left null space, and state the basis for its row space, which is a basis for the column space of B.

423. The basis for the col(B) stated in the summary table is $\{(1, -2, -1), (1, -1, -1)\}$. The basis you should have stated in the previous problem: $\{(1, 0, -1), (0, 1, 0)\}$. This mathematics produces a form in which the column space of B can be described as the set of all vectors in $\mathbb{R}^3$ whose first and third components are always opposites of each other. This means the middle component can be any real number. Figure out how to show and clearly explain how both of these bases lead to that description.

Problems 424-430: Find bases for the four fundamental subspaces for each matrix. As you do this work, here is the reason why a course at this level is challenging: *there is no way to understand linear algebra without engaging in deep thought about what information is gained from a matrix regarding the transformation it defines.* This is new to most students of this subject, whose minds have been trained in computational work in mathematics. While accuracy still characterizes any mathematics course, upper level work in mathematics ought to advance your thinking and promote your growth. It is not easy, but it is well worth the effort to ask the questions about how all of this work is connected. I believe it really requires your interest and intellectual curiosity first, and sometimes seeing how things connect can be gratifying and even pleasurable. By your very own human nature, your brain seeks and finds patterns: you are a natural mathematician.

424. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

425. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

426. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

427. $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$

428. $\begin{bmatrix} 1 & 7 & 4 \\ 2 & 8 & 2 \end{bmatrix}$

429. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

430. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ -3 & 7 & 4 & 8 \\ 0 & 3 & 3 & 6 \end{bmatrix}$

22.2 The Pictures of the Input and Output Spaces

The page following this one is one that I think needs no words. On this page are the geometric depictions of these subspaces for the matrices in examples 68 and 69. These pictures help us to eventually build an actual picture of the Fundamental Theorem of Linear Algebra, the focus of the next chapter. A matrix of size $m \times n$ is a linear transformation of vectors in a domain (input space) to vectors in a codomain (output space), and the fundamental subspaces divide the input space and the output space into two subspaces that are orthogonal to each other: the kernel, domain, and range of the transformation are revealed in this work. What sense are you able to make out of these pictures and the matrices associated with them? Consider sketching the subspaces in a similar way for a few of the matrices in the exercises in which you found the fundamental subspaces. What are some observations you can take away from this?

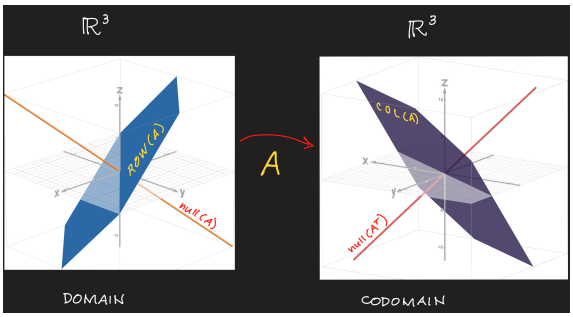


Fig. 22.1: Subspaces of A in Example 68

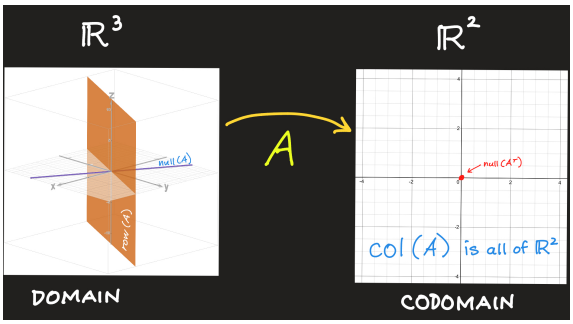


Fig. 22.2: Subspaces of A in Example 69

Chapter 23

Orthogonality of the Subspaces

This chapter focuses on the concept of *orthogonality* by going back to the basics of the dot product of two vectors. Two vectors are orthogonal to each other if their dot product is equal to the scalar zero. A return to vectors in the plane and understanding how the dot product is an extension of regular multiplication you learned in your early elementary school days is covered in order to connect and clarify this concept. As can be seen in the pictures at the end of the previous chapter, the two subspaces in the input space, the row space and null space of a matrix are orthogonal to each other. The two subspaces in the output space, the column space and the left null space of the matrix, are also orthogonal to each other. One theorem is enough to prove this relationship between each pair: all we need to do is to prove that, for any matrix, its row space is orthogonal to its null space. We will not need to prove a second theorem for the column space and the left null space, but we could do so if we wished. It is all about the perspective we take! In order to find the four fundamental subspaces of a matrix A, you do two things: (1) Row reduce A to find its row space and its null space, and (2) row reduce A^T to find its row space and its null space; that is, $\text{row}(A^T) = \text{col}(A)$, and the left null space of A is $\text{null}(A^T)$. This proof is part of the Fundamental Theorem of Linear Algebra, fully stated in the next chapter. We finish this preliminary work and define *rank* and *nullity* of a matrix, which allows us to move into the next chapter and put together what many people call the "big picture" of linear algebra.

23.1 Understanding the Dot Product

Understanding the dot product begins with regular multiplication of real numbers. As a scholar in arithmetic, you learned a fancy way to add numbers quickly, and spent a few months memorizing your multiplication tables. You know that the product $2 \cdot 3$ is equal to 6. If you were to think of the numbers 2 and 3 as vectors on the real number line, sketched

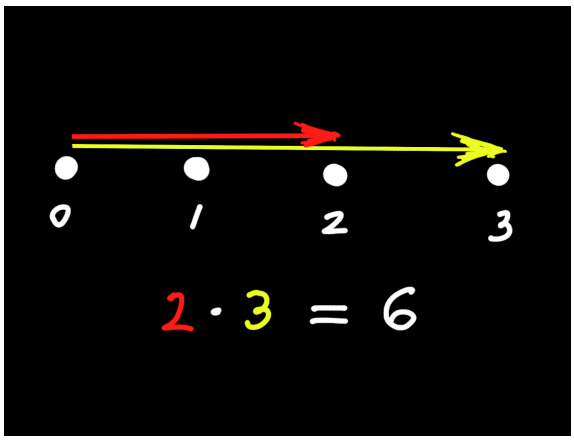


Fig. 23.1: Multiplication in One Direction

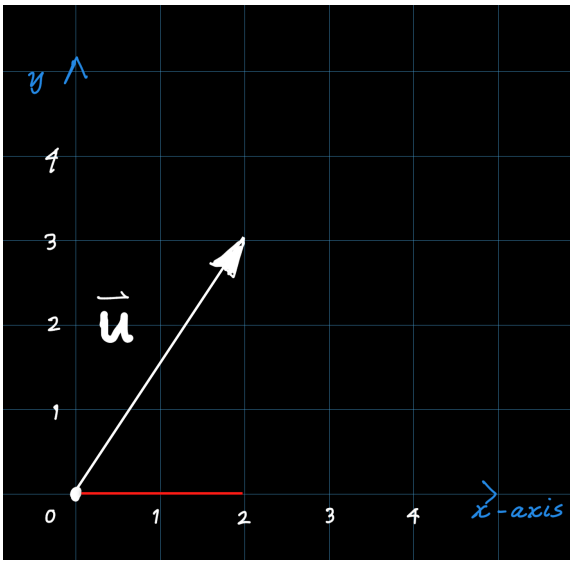


Fig. 23.2: The Length of **u** in a Direction

together in standard position, then the length of 2 in the same direction as the vector 3 is obviously 2. See Fig. 23.1, where the red vector represents 2, the yellow represents 3, and the dot product of 2 and 3 is the product of their lengths. The red vector casts a kind of shadow onto the yellow vector. The dot product of two vectors in the plane can be understood from this picture as it is defined using this same geometry: If two vectors are sketched in standard position, their dot product is the length of the component of one vector in the direction of the other vector, multiplied by the length of the other vector. Figure 23.1 shows this: the length of 2 in the direction of 3 is 2. The length of 3 is 3, so $2 \cdot 3 = 6$. Consider the vector $\mathbf{u} = (2, 3)$ in Fig. 23.2. It is easy to see that the length of **u** in the direction of any vector pointing in the x-direction is 2, the red line segment sketched in this figure. This is the component of **u** in the x-direction. The dot product of **u** with any vector on the x-axis is then the product of the first components of these two vectors. Now tilt this picture as you see in Fig. 23.3. While we are not supposed to change the direction a vector is going, the point here is to understand that the length of the vector going in the direction of the x-axis is the *perpendicular projection of the vector in that direction*; that is, the *orthogonal projection* of **u** in the direction of the axis. We will draw a yellow vector going in the direction where you see the numbers increasing (see Fig. 23.4), and redefine **u**. So start with what you see in Fig. 23.4. On the left, you see two vectors **u** and **v** both sketched in standard position. The red line segment in the direction of **v** is the orthogonal projection of **u** onto **v**. This projection can be found geometrically by dropping a line segment from the tip of **u** to the point on the line on which **v** sits so that the line segment from the tip to the point is perpendicular to the vector **v**. This is indicated as a dotted line in Fig. 23.5. In physics, you often say "the component of **u** in the direction of **v**" when you refer to the length of this projection, which is labeled as L in Figure 23.5. Assume that both vectors are originally given as they are in Fig. 23.4, where you can see the x-axis indicated in this figure. Figures 23.4 and 23.5 contain all of the elements you will put together in the problems that follow these figures to help you refine your understanding of the dot product.

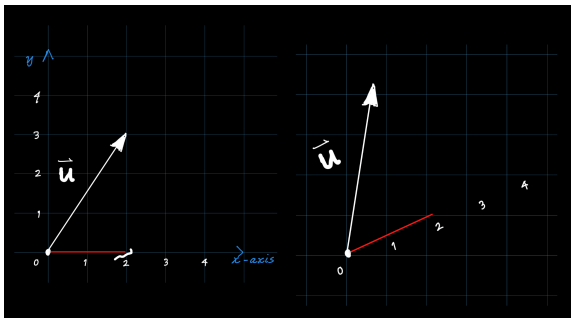


Fig. 23.3: Tilt the Picture

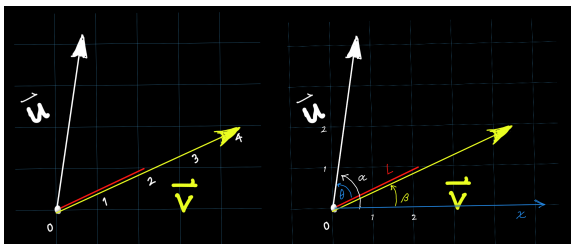


Fig. 23.4: The Dot Product

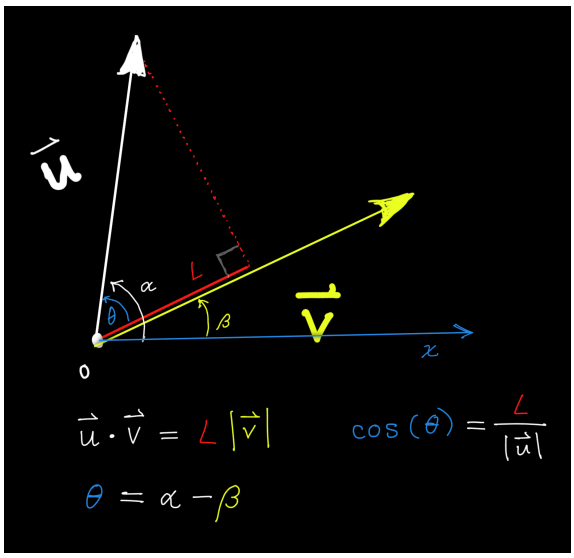


Fig. 23.5: Geometric Elements of the Dot Product

These problems are intended for students who might appreciate a return to some fundamentals of Euclidean vectors. Figures 23.4 and 23.5 are necessary to view in order to complete the work here:

431. View Fig. 23.4: Let $\mathbf{u} = (u_1, u_2)$ and let $\mathbf{v} = (v_1, v_2)$. Let α be the angle $\mathbf{u}$ makes with the positive x-axis, and let β be the angle $\mathbf{v}$ makes with the positive x-axis. Find the following: $\cos\alpha, \sin\alpha, \cos\beta$, and $\sin\beta$ in terms of $u_1, u_2, v_1, v_2, |\mathbf{u}|$, and $|\mathbf{v}|$; that is, in terms of the components of both vectors and the lengths of each vector.

432. View Fig. 23.5. The length L is the component of $\mathbf{u}$ in the direction of $\mathbf{v}$. **The dot product of these two vectors can be defined this way:**

$$\mathbf{u} \cdot \mathbf{v} = L|\mathbf{v}|$$

Use the right triangle in Fig. 23.5 to rewrite this definition in terms of the angle between the two vectors and the lengths of the vectors, thereby deriving the *physics definition of dot product*:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

433. Prove that if the two vectors were perpendicular to each other, then $\mathbf{u} \cdot \mathbf{v} = 0$.

434. Note that $\theta = \alpha - \beta$. Look up the trigonometric identity for the cosine of a difference to rewrite $\cos\theta = \cos(\alpha - \beta)$, epanding the cosine of the dif

435. Use the work from 431 and 434 to prove that $\cos\theta = \frac{u_1v_1 + u_2v_2}{|\mathbf{u}||\mathbf{v}|}$.

436. Show that the physics definition of the dot product in 432 is the same as the mathematics definition of dot product; that is, show that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = u_1v_1 + u_2v_2.$$

23.2 Orthogonal Subspaces

These definitions for dot product apply to Euclidean vectors in any Euclidean vector space, so we can generalize here: For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. More generally:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_iv_i$$

The other big generalization: If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, meaning *perpendicular to*, then their dot product is zero. If the dot product is zero, then the vectors are orthogonal to each other. (Want to prove that?) This is at the heart of understanding *orthogonal subspaces.*

DEFINITION 21: Let W be a subspace of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$.

Some examples might help make this more solid:

Example 71: Consider $\mathbb{R}^2$. The set of all vectors orthogonal to a line W through the origin is the line through the origin perpendicular to W. The lines $y = x$ and $y = -x$ are perpendicular to each other. In vector terminology, thinking of W as the line $y = x$, W is the set of all vectors in $\mathbb{R}^2$ spanned by the vector $(1, 1)$. Find the orthogonal complement of W.

solution: $W^\perp$, the orthogonal complement of W, is the set of all vectors in $\mathbb{R}^2$ that are perpendicular to $(1, 1)$. Let (a, b) be any vector in $W^\perp$. Then $(a, b) \cdot (1, 1) = 0$, which means $a + b = 0$; that is, $b = -a$, so this is the set of all vectors of the form $(a, -a)$, and, by scalar multiplication, $(a, -a) = a(1, -1)$, so $(1, -1)$ spans this subspace. $W^\perp$ is all scalar multiples of the vector $(1, -1)$, or the line $y = -x$.

The following problems force you to produce a few more examples:

437. Let W be the set of all vectors in $\mathbb{R}^3$ spanned by the vector $(3, 2, 1)$. Find $W^\perp$.

438. Let U be the set of all vectors in $\mathbb{R}^3$ of the form $(a, 2a, a)$. Find $U^\perp$.

439. Let V be the set of all vectors in $\mathbb{R}^3$ of the form $(a, b, 2a)$. Find $V^\perp$.

440. Let W be the set of all vectors in $\mathbb{R}^3$ orthogonal to the vector $(1, 1, 1)$. Find a basis for W and then find the orthogonal complement of W.

441. Pick one of the tables in the previous chapter that summarizes the information for the four fundamental subspaces of a matrix. Note how the table is designed to pair up the subspaces. Show that the row space is the orthogonal complement of the null space, and the column space is the orthogonal complement of the left null space.

442. Is the z-axis the orthogonal complement of the xy-plane in $\mathbb{R}^3$? Justify your answer mathematically.

THEOREM 6: Let M be an $m \times n$ matrix. The row space of M and the null space of M are orthogonal complements. This is sometimes written as $\text{row}(M) \perp \text{null}(M)$.

PROOF: Let $\mathbf{x}$ be any vector in the null space of M. By the definition of null space, this means $M\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in the codomain $\mathbb{R}^m$. The matrix product $M\mathbf{x}$ can be carried out by dotting every row vector of the matrix M with the null space vector. Each component of the zero vector is a scalar zero; that is, every row dotted with any null space vector is zero, which means every row vector is orthogonal to any null space vector. Since the rows span the row space, then all vectors in the row space are orthogonal to all vectors in the null space; therefore, $\text{row}(M)$ and $\text{null}(M)$ are orthogonal complements.

443. Find the null space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$. Then, find the orthogonal complement of this subspace as if you were not given the matrix. Show that a basis for the subspace you found matches the basis you get for the row space from the matrix in row reduced echelon form.

444. Suppose the null space of a 2×3 matrix A is spanned by the vector $(2, -1, 1)$ and the left null space contains only the vector $(0, 0)$. Find bases for all four fundamental subspaces of A. State their dimensions.

445. If a 5×5 matrix has a two-dimensional null space, what are the dimensions of all four fundamental subspaces of this matrix?

446. Let B be a 4×2 matrix. Is it possible that B has a one-dimensional null space? Explain.

447. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. Determine the four fundamental subspaces of A. Then explain the linear transformation determined by the matrix by defining it as a function T, and identify its domain, codomain, kernel, and range.

23.3 Rank and Nullity

DEFINITION 22: Let A be any matrix. The **rank** of A is the number of linearly independent columns of A; that is, the rank is the dimension of the column space of A.
DEFINITION 23: Let A be any matrix. The **nullity** of A is the dimension of the null space of A.

448. Determine the rank and nullity of the matrix A in 443.

449. Determine the rank and nullity of the matrix A in 447.

450. What are all possible values for the rank and nullity of an $m \times n$ matrix? Justify your work in answering this question.

Chapter 24

The Fundamental Theorem of Linear Algebra

We arrive at what many call the "big picture" of linear algebra, a fundamental theorem that connects everything we have studied in an interesting manner. The fundamental theorem is really several theorems combined into one. The reason for this is due to the fact that these theorems can be grouped in two ways to show an *algebraic* relationship between the size of a matrix and its fundamental spaces, and a *geometric* relationship between the fundamental spaces themselves. Some of these truths have already been explored, but the purpose of this chapter is to make some connections with almost everything we have studied, with a strong focus on matrices.

THEOREM 7: THE FUNDAMENTAL THEOREM OF LINEAR ALGEBRA
Let A be an $m \times n$ matrix. Then A determines a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$. The domain of the transformation (the input space) $\mathbb{R}^n$ can be split into two subspaces: row(A) and null(A), and the codomain of the transformation (the output space) $\mathbb{R}^m$ can be split into two subspaces: col(A) and null(A^T). Let r be the rank of the matrix. Then the following statements are true:

- The col(A) and row(A) both have dimension r.
- The null(A) has dimension $n - r$.
- The null(A^T) has dimension $m - r$.
- The null(A) is the orthogonal complement of the row(A).
- The null(A^T) is the orthogonal complement of the col(A).

The following examples require you working several exercises within them in order to become more familiar with the way this theorem brings so many pieces of our course together. *Special note*: The space-saver notation is used in these examples to, of course, save space. You may be in the habit by now of writing all of these vectors as column matrices.

Example 72: Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$.

451. Show that the four fundamental subspaces of A have the following bases: $\{(-3, 1)\}$ for the null space, $\{(1, 3)\}$ for the row space, $\{(-1, 1)\}$ for the left null space, and $\{(1, 1)\}$ for the column space.

452. Show that the linear transformation determined by A is s function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + 3y, x + 3y)$. Find the kernel of the transformation and the range of the transformation, recognizing (if you have not already) that these are two of the subspaces: the null space and column space.

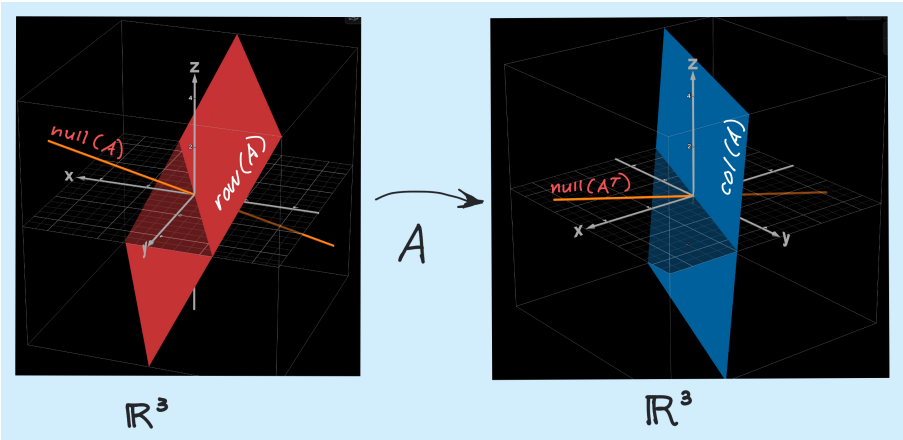


Fig. 24.1: Fundamental Subspaces of A in Problem 454

453. Sketch a picture of the four fundamental subspaces, two in the domain and then, to the right of that, the two in the codomain. State the rank of the matrix and the nullity. Check the theorem and verify that everything holds true.

Example 73: Consider the matrix $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & -7 \\ -2 & 2 & 2 \end{bmatrix}$.

454. Show that bases for the four fundamental subspaces of the matrix are the following: The null(A) has basis $\{(2,1,1)\}$, and the row(A) has basis $\{(1,0,-2),(0,1,-1)\}$. The left null space, null(A^T), has basis $\{(10,-4,1)\}$ and the col(A) has basis $\{(1,0,-10),(0,1,4)\}$. Take a moment to pause and think about the linear transformation A defines and view Fig. 24.1, accurate pictures of the four subspaces of this matrix.

455. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$. Write the linear transformation determined by A, identifying the domain and the codomain.

456. Find bases for the four fundamental subspaces of the matrix in the previous problem. Then state the geometric description of each subspace. What is the kernel and the range of the linear transformation?

457. Sketch a picture (similar to Fig. 24.1) of the four fundamental subspaces of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$.

458. Plot the point $(2,3,0)$ in the output space in the picture you sketched. Notice that this point is not a point in the column space of the matrix.

459. (Challenge problem:) What is the *shortest distance* from this point $(2,3,0)$ to the column space of A? (*Hint: What is the distance between a point and a plane?*)

460. Suppose the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}$ is an augmented matrix representing a system of linear equations. Write the system of equations it represents, and then show the system is *inconsistent* (has no solution). Bring the work you have accomplished in the last five problems, including this one, into the next chapter, where we jump into the real world to not only see how linear algebra is applied, but where you can start to see some clear meaning in the four fundamental subspaces of a matrix.

Chapter 25

The Method of Least Squares

In practice, systems of linear equations are often inconsistent; that is, they have no solution. While mathematics can precisely say so, we can also use mathematics to *find a solution that is good enough* to work in the context of whatever it is we are doing. The system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 \qquad &= 3 \\ x_1 + x_2 &= 0 \end{aligned}$$

has no solution. When written in matrix form $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, the vector $\mathbf{b}$ is not in the column space of A, which means the system will not have a solution. You should be more and more familiar and comfortable with knowing that fact. The span of the columns of A give you the range of the transformation. The rank of A is its dimension. Solving the system is really answering the question "is $\mathbf{b}$ in the column space of A?" But the more real-world approach to this is to look for a solution that is close enough to being a solution, approximating one. This section is a wonderful way to see a meaningful use of the four fundamental subspaces of a matrix. Data obtained in practice might be produced in picture form revealing a possible relationship between variables in such a way as to be closely modeled by a well known polynomial. Data appearing somewhat close to a line might result in finding a *line of best fit.* Other data may show it is necessary to find a *parabola of best fit.* This chapter introduces an example detailing *how* one would go about finding a line of best fit, then asks you to exercise some similar problems to become acquainted with the procedure, which is called the method of least squares. Finally, the reason why the method works is a direct and logical application of the fundamental subspaces. Students are often gratified to see this reason, as it helps to truly understand why these four subspaces, and the transpose of a matrix, are important.

25.1 The Procedure for Finding a Polynomial of Best Fit

Example 74: Find the line of best fit for the points $((1, 2), (0, 3)$, and $(1, 0)$.

solution: The problem tells us to seek an equation of a line (a degree one polynomial approximation), which means that we want to find an equation $y = a_0 + a_1 x$ that best fits these points. Therefore, we must assume the given points satisfy this equation: Each

point is an ordered pair (x, y) such that $y = a_0 + a_1 x$. Three equations can be formed:

$$\begin{aligned} 2 &= a_0 + a_1(1) \\ 3 &= a_0 + a_1(0) \\ 0 &= a_0 + a_1(1) \end{aligned}$$

We seek to solve the system:

$$\begin{aligned} a_0 + a_1 &= 2 \\ a_0 \qquad &= 3 \\ a_0 + a_1 &= 0 \end{aligned}$$

In matrix form $A\mathbf{x} = \mathbf{b}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. The mechanical approach of solving the system means obtaining the reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}$, which is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which reads $a_0 = 0, a_1 = 0$, and $0 = 1$, which cannot be true. The system has no solution: The vector $\mathbf{b}$ is not in the column space of the matrix A. This is the key to finding an approximate solution: since $\mathbf{b}$ is not in the column space, then *what vector in the column space is closest to this vector?* We explore this in the next section, which will also explain why a closest solution is found by doing the following:

1: Multiply both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^T.

2: Solve for $\mathbf{x}$ by finding the inverse of $A^T A$.

The algebra can be summarized by the following steps:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^T A\mathbf{x} &= A^T\mathbf{b} \\ \mathbf{x} &= (A^T A)^{-1} A^T \mathbf{b} \end{aligned}$$

This amounts to some busy matrix multiplication problems for this system, where $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$: $A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$, $A^T\mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, which means $(A^T A)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$, and

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

The line of best fit for these data points is $y = 3 - 2x$. See Fig. 25.1. This procedure of finding the polynomial of best fit can be summarized in the following recipe of instructions: Given the set of points in the plane, (1) What degree of polynomial do you seek? This must be decided first. A line of best fit begins with $y = a_o + a_1 x$, a parabola of best fit begins with $y = a_0 + a_1 x + a_2 x^2$, and so on... (2) Form the system of equations $A\mathbf{x} = \mathbf{b}$. (3) Find the products $A^T A$ and $A^T\mathbf{b}$. (4) Find the inverse $(A^T A)^{-1}$. (5) Multiply the inverse on the left of $A^T\mathbf{b}$ to obtain the coefficients of the polynomial. Finding this polynomial is a matter of obeying this procedure. Knowing why the procedure works is linear algebra.

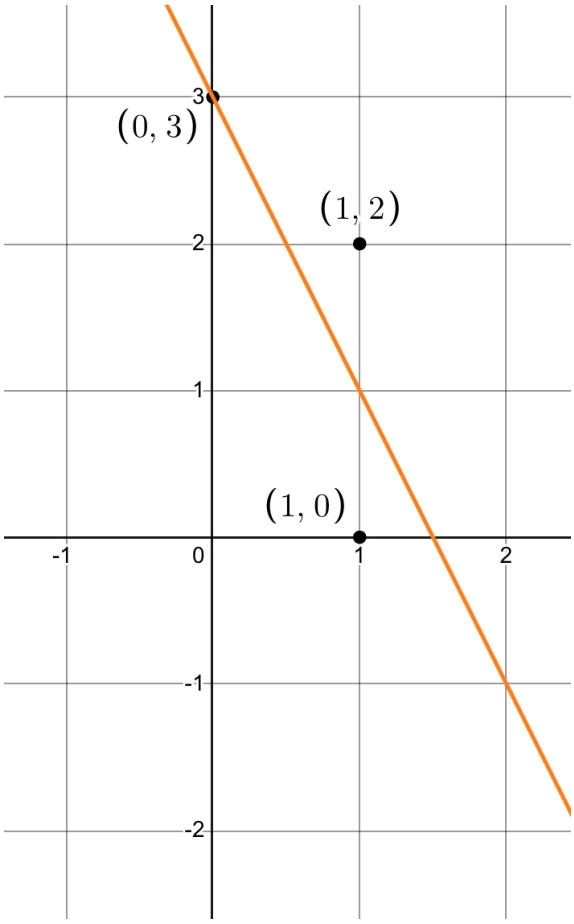


Fig. 25.1: Example 74: Line of Best Fit

461. Determine the line of best fit for $(1,1),(2,2)$, and $(3,1)$.

462. Determine the line of best fit for $(1,3),(3,4)$, and $(4,5)$.

463. Determine the line of best fit for $(0,2),(2,2)$, and $(2,1)$.

464. Determine the line of best fit for $(-1,1),(1,2),(2,3)$, and $(3,3)$.

465. Determine the parabola of best fit for $(-2,2),(-1,0),(0,-2)$, and $(1,1)$.

25.2 Why the Procedure Works

The method being used here is known as **the method of least squares**. In practice, it is often the case that a system $A\mathbf{x} = \mathbf{b}$ has no solution; that is $\mathbf{b}$ is not in the column space of A. The job is to find a solution that is "close enough" and there is a wonderful connection to the fundamental subspaces of the matrix A that shed light on what they can mean. For this method to work, the columns of A must be independent, and the matrix A^TA must be invertible. To help understand this method, and why it works, let's use the system in example 74, and focus on the four fundamental subspaces of the matrix A, work you completed at the end of the previous chapter. The reduced row echelon form of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since A is 3×2, it determines a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$. We can see that the row space is all of $\mathbb{R}^2$ and the null space is the zero vector $(0,0)$. It is the output space we wish to focus on in this particular situation: The column space of A is a plane, and the null space is a line. Before continuing, let's examine some really subtle details that I hope clarify a few things about the bases for these two subspaces of this matrix:

466. One description of the column space of this matrix is this: The col(A) is the set of all vectors in $\mathbb{R}^3$ of the form (a,b,a), where a and b are any scalars. If you had been given this information (without knowing the matrix), what would be a basis for this subspace?

467. The basis you extracted from the description of the subspace in the previous problem is not unique. Show that the columns of A are a basis for the same subspace.

468. (Challenge problem) If you took multivariable calculus, show that a Cartesian equation of the plane is $z = x$; that is, the column space of A can be described as the plane $z = x$ in $\mathbb{R}^3$.

The left null space is all scalar multiples of the vector $(-1,0,1)$. These two subspaces of the output space are pictured together in Fig. 25.2. The vector $\mathbf{b} = (2,3,0)$ can be sketched in the output space as the directed line segment from the origin to the point $(2,3,0)$. In the picture, you see the ordered triple as a dot in the xy-plane; that is, you have a *visual* of how the vector $\mathbf{b}$ is not in the column space. The best solution, then, would be *the closest vector in column space*. Calculus minimizes the square of the distance from $\mathbf{b}$ to the column space, and proves that it has to be the perpendicular distance, the length of

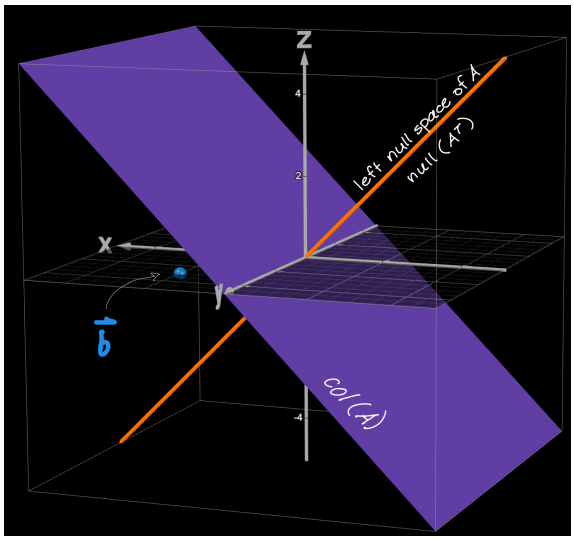


Fig. 25.2: The Output Space

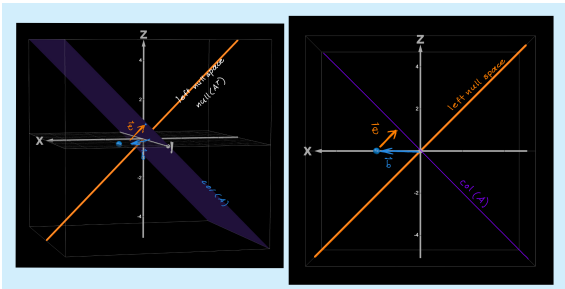


Fig. 25.3: **e**, a Left Null Space Vector

an orthogonal projection of the point onto the plane; hence the name *least squares*. Call this vector of minimum length from $(2, 3, 0)$ to the col(A) $\mathbf{e}$ for *error*. We want minimum error, of course. It should be easy to spot the vector addition in Fig. 25.3: $A\mathbf{x}$ needs to be equal to this closest column vector $\mathbf{b} + \mathbf{e}$; that is,

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} + \mathbf{e} \\ A\mathbf{x} - \mathbf{b} &= \mathbf{e} \end{aligned}$$

The most gratifying thing to see here is the use of the fact that $\mathbf{e}$ is a left null space vector, since the left null space is orthogonal to the column space; that is, the Fundamental Theorem of Linear Algebra at work! Any left null space vector $\mathbf{v}$ such that $A^T\mathbf{v} = \mathbf{0}$. So $A\mathbf{x} - \mathbf{b}$, being a left null space vector, is such that

$$\begin{aligned} A^T(A\mathbf{x} - \mathbf{b}) &= \mathbf{0} \\ A^TA\mathbf{x} - A^T\mathbf{b} &= \mathbf{0} \\ A^TA\mathbf{x} &= A^T\mathbf{b} \\ \mathbf{x} &= (A^TA)^{-1}A^T\mathbf{b} \end{aligned}$$

I believe it is worth taking time to appreciate the role the fundamental subspaces play in clarifying a best solution to inconsistent systems. This section touches lightly on the

method of least squares, a comprehensive study of which would have a clear theoretical framework that begins with first defining the *least squares problem*, defined here:

DEFINITION 24: The Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations and n unknowns, a vector $\mathbf{x}$ in $\mathbb{R}^n$ that minimizes $|A\mathbf{x} - \mathbf{b}|$ with respect to the dot product (Euclidean inner product) on $\mathbb{R}^m$, if it exists, is known as a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

469. Set up the system to find the parabola of best fit for the points $(-2, 2), (0, 0), (2, 1)$, and $(4, 3)$. Identify the 4×3 matrix A and find bases for its four fundamental subspaces, and describe them geometrically. (Identify the vector $\mathbf{b}$, too, in preparation for the next problem.

470. Find the parabola of best fit for the points in the previous problem. How difficult would it be to picture the location of $\mathbf{b}$ and visualize the direction of the left null space in order to find the solution found?

471. Find the least squares solution of the system:

$$\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

472. What is the least squares solution to a system of linear equations that is *consistent*? Find, for example, the least squares solution of the very first problem in this book, a system with the solution $(-4, 2)$.

473. Not to be redundant, but, in light of the previous problem, find the line of best fit for the points $(1, 1)$ and $(3, 3)$.

474. Prove that if the columns of A are linearly independent, then $A^T A$ is invertible.

475. Prove that if $A^T A$ is invertible, then the columns of A are linearly independent.

Chapter 26

Graphs and Incidence Matrices

The purpose of this chapter is to show you another area in which linear algebra is applied to further highlight the four fundamental subspaces of a matrix, and get better acquainted with them. I have chosen a topic that introduces you to graphs, usually studied in discrete mathematics courses. This mathematics is applied to electrical circuits, but you do not have to have studied this in physics. If you have studied it in another course, I hope you experience a joyful moment that is rare in a mathematics course. I hope you enjoy what you read.

26.1 Graphs

A *graph* is a collection of *nodes* (points) that are joined by *edges* (usually line segments between nodes, but generally a path connecting two nodes). Graphs are useful in displaying some kind of connection between places (users on websites or social media platforms, traffic flow between points connecting roads) or objects (typically represented by points). A graph is a picture of the specific connections between nodes. Take a look at Figure 26.1 showing a graph with four nodes and five edges. Notice the arrows on the edges. This graph might possibly represent traffic flow (traffic network) along four one-way streets (at node 1, drivers have three choices of roads along which to leave; at node 2, traffic arrives from node 1, and leaves towards node 3, etc...). We will use this graph, however, to represent and electrical network known as a circuit. The arrows represent the direction of the flow of current. In circuit analysis, the word *branch* replaces the word *edge*. So, in the figure, mathematically speaking, the graph has $m = 5$ edges (branches) and $n = 4$ nodes. An *incidence matrix* is a matrix representing a relationship between two classes of objects (here, that would be the nodes and the edges).

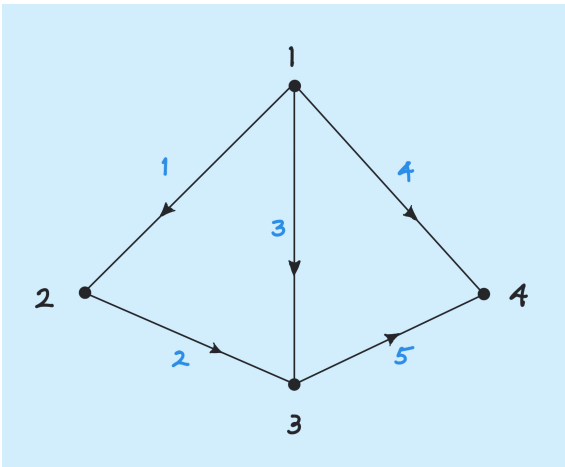


Fig. 26.1: Graph: 4 Nodes, 5 Edges

26.2 Subspaces of the Incidence Matrix

Let A be the 5×4 incidence matrix representing which nodes are connected with which edges:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

This matrix can be written directly from viewing the graph in the following way: Let the rows represent the edges and the columns represent the nodes. For each edge, you want to indicate the connection between nodes by one of three entries in the matrix $(-1, 0, 1)$: Edge 1 in the graph goes from node 1 to node 2, so under node 1 (column 1), enter a -1 and under node 2 (column 2), enter a 1. Enter a 0 for all other nodes. The first row of the matrix indicates this. Read the second row: this represents edge 2: it leaves node 2 and arrives at node 3. And so on... Since this is a mathematics course, we will focus on the mathematics of this matrix first, and then I will simply tell you the relationship between the mathematics and the physics, and leave it to the physics instructors to explain the electrical circuit analysis details. To set yourself up for that latter part, work the following pure linear algebra problems (good for a self-check as some of the answers will be discussed in the next section):

476. There is a nice way to say what the four fundamental subspaces do, given any matrix, and it is this: *The four fundamental subspaces of a matrix describe the kernels and ranges of the linear transformations determined by the matrix and its transpose.* Explain this for any $m \times n$ matrix.

477. For the incidence matrix A for the graph in Fig. 26.1, A maps vectors in ? to vectors in ? A^T maps vectors in ? to vectors in ?

478. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$. Write the system of linear equations $A\mathbf{v} = \mathbf{0}$.

479. Let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$. Write the system of linear equations $A^T\mathbf{y} = \mathbf{0}$.

480. Find bases for the four fundamental subspaces of the incidence matrix A.

481. What is the rank of A?

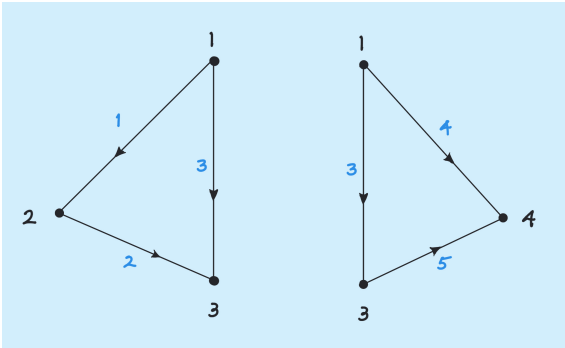


Fig. 26.2: Two Loops

26.3 Connection to Electrical Networks

26.3.0.1 The Null Spaces

The components of the vector $\mathbf{v}$ represent the electric potential (potential energy per charge) at each node, measured in volts. The null space of A is the solution to the system you wrote in problem 478. $A\mathbf{v}$ measures the potential difference across each edge. All five voltage differences are zero, which means all four voltages are the same. Every vector in the null space is a constant vector, spanned by $(1, 1, 1, 1)$, making the null space a line in $\mathbb{R}^4$, which means the rank of the matrix is 3. A *loop* is formed when the number of edges is the same as the number of nodes; that is, when $m = n$. In the graph, there are two loops: one from edges 1 to 2 to 3, and one from edges 1 to 3 to 4. (See Fig. 26.2). *Each loop in the graph gives a vector in the left null space!* In a loop, the currents must balance themselves, and the easiest way to do that is to flow around a loop. Notice the first loop on the left, where a current goes around the triangle: in the positive direction (Think counterclockwise) from node 1 to node 2 along edge 1, from node 2 to node 3 along edge 2, and in the negative direction (think clockwise) from node 1 to node 3. This is the vector $(1, 1, -1, 0, 0)$. (Think of writing the incidence matrix for each loop.) The loop on the right: positively from node 1 to node 3 along edge 3, from node 3 to node 4 along edge 5, and negatively from node 4 to node 1 along edge 4. This is the vector $(0, 0, 1, -1, 1)$. The left null space, the solutions to $A^T\mathbf{y} = \mathbf{0}$, is spanned by the vectors $(-1, -1, 1, 0, 0)$ and $(1, 1, 0, -1, 1)$; that is, the basis for the left null space is the set $\{(-1, -1, 1, 0, 0), (1, 1, 0, -1, 1)\}$. Notice that the two vectors found in the two loops are combinations of these vectors: the first, a scalar multiple of the first basis vector, the second, the sum of the two basis vectors. There is a third loop here that you should notice.

482. Consider the two left null space vectors found from the two loops pictured in Fig. 26.2. Do these vectors form a basis for the left null space? Prove it.

483. Consider the third loop not mentioned explicitly. Show that it is a left null space vector.

If you are familiar with the physics, then you might recognize **Kirchhoff's Current Law**, that flow in equals flow out at each node, and is given by $A^T\mathbf{y} = \mathbf{0}$. I kind of hope

you see more here, if you are familiar with electrical circuits, and enjoy seeing pure linear algebra applied!

26.3.1 The Row and Column Spaces

A basis for the row space of A for our example is $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$. All vectors in the row space represent potential voltage differences across nodes in an electrical network. Every row adds to zero, which means all combinations of the rows add to zero. This is, of course, really saying that the null space, which contains all vectors of the form (a,a,a,a), where a is any real number, is orthogonal to the row space. The column space is the interesting one because when $\mathbf{b}$ is in the column space of A then $A\mathbf{v}=\mathbf{b}$, and this vector has to obey **Kirchhoff's Voltage Law**. You found a basis for the column space as either the first three columns of A or the set $\{(1,0,1,0,-1),(0,1,1,0,-1),(0,0,0,1,1)\}$. There is a lot to unpack, collect, organize, and think about here, and I must inform anyone reading this particular chapter to be cautious of its contents: I have not spelled out in detail how one should go about studying electrical networks. I leave it to the student who is interested to figure out other details independently with some intellectual curiosity. What *should* be clear is that any student reading it should be able to successfully see how the matrix A was formed from the graph, in addition to finding bases for its four fundamental subspaces. The last section does give some kind of summary of the linear algebra in connection with the physics for anyone inspired to do the work in understanding the connection between the two, hopefully discovering some more details that are not mentioned. To those who have studied circuits, my advice would be to take the time to study this material to see how powerful linear algebra can be.

26.4 Summary

The incidence matrix A written from our graph in Fig. 26.1 (reproduced in Fig. 26.3 but including the variables that were defined in the exposition) is an example of one of the most important types of models in discrete applied mathematics: graphs. Graphs are everywhere: roads, traffic networks, pipelines, fluid flow, blood flow, the human brain, economies, the internet, etc... In the example of an electrical circuit, which comes with Kirchhoff's laws from physics, it is linear algebra that connects these laws to the four fundamental subspaces of the matrix A:

- **The null space**

Chapter **27**

When a Matrix Behaves Like a Scalar

Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Work the following exercises.

486. Sketch the unit square and the image of the unit square under the transformation determined by A *on the same set of axes.*

487. Find the area of the parallelogram you sketched.

488. View your sketch of the unit square and the parallelogram. Note that the vector from the origin to the vertex $(1, 1)$ of the unit square was stretched only by the transformation, but its direction did not get transformed; that is, the vector remained on the line it had been on prior to the transformation by A. By viewing the picture, can you tell by how much this vector was scaled?

489. Find $A\mathbf{u}$. What is $5\mathbf{u}$?

490. On the same set of axes, sketch the vector $\mathbf{v}$ and the image of the vector under the transformation by A. Under the sketch, find $2\mathbf{v}$ and find $A\mathbf{v}$.

491. If k is any nonzero scalar, is it true that $A(k\mathbf{u}) = 5k\mathbf{u}$?

492. Find A^2. What is $A^2\mathbf{u}$?

493. Find A^3. What is $A^3\mathbf{u}$? Are you able to make a conjecture on what $A^6\mathbf{u}$ is without doing the calculations?

A linear transformation that is a mapping from n-dimensional space to m-dimensional space is always associated with an $m \times n$ matrix. In this chapter, we restrict ourselves to mappings within the same spaces. A linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$ is associated with a square matrix (matrix of order n). Since we are familiar with $\mathbb{R}^2$ then the focus of the work is with 2×2 matrices. The matrix A in the exercises above defines a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (4x + y, 2x + 3y)$. In the lecture, we will see a nice animation of the transformation, taking the grid formed by the standard basis vectors and noting how the matrix changes the space, preserving vector addition and scalar multiplication. Fig. 27.1 should be a nice way to check some of the work you did above (you should see the parallelogram you sketched, for example), and includes another interesting observation: note the two lines that are emphasized in the sketch. All vectors along these lines are vectors that remain on the same line once the matrix has transformed the space. Vectors on these lines are scaled under the transformation, but their direction is remains unchanged. In other words, the matrix, when multiplied by any of the vectors along these lines, does the same thing that scalar multiplication does. In the exercises the matrix scales the vector $\mathbf{u}$ by a factor of 5 and scales the vector $\mathbf{v}$ by a factor of 2. The lines $y = x$ and $y = -2x$ are *invariant under the matrix transformation*, meaning the lines remain unchanged. These lines ought to provide you with a stronger way of seeing how

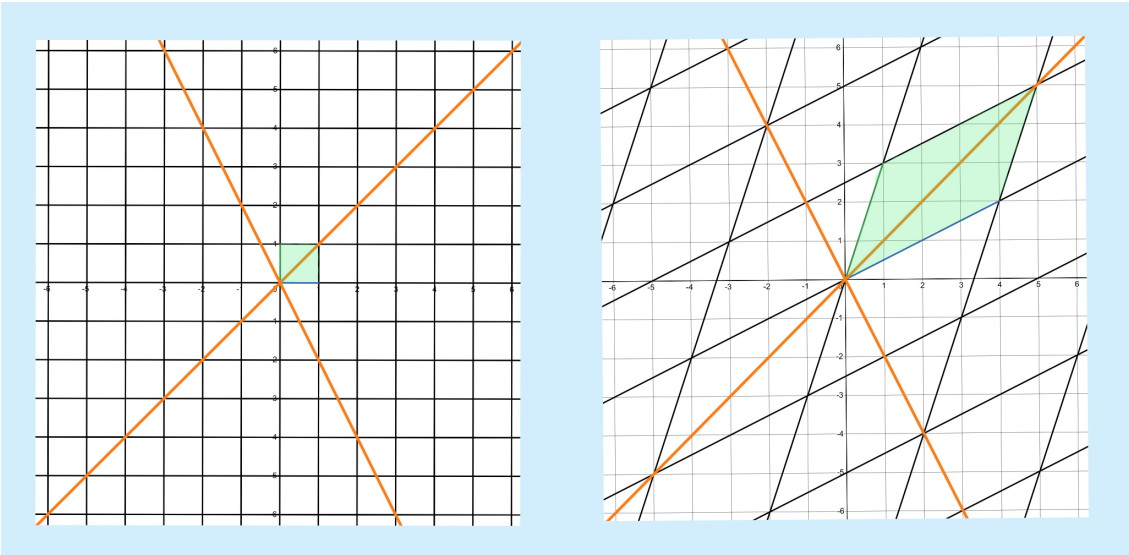


Fig. 27.1: How A Transforms the Plane

the matrix A changes the space. What do you think of the following sentence? *These lines are the axes on which the matrix transformation hinges.* This is a geometric description of what are known as *eigenspaces*, additional subspaces (line-like things through origins) that characterize square matrices. Vectors within these spaces are called *eigenvectors*, and the scalars by which they are stretched are the *eigenvalues* of the matrix. An eigenvector comes with a corresponding eigenvalue. Here, the matrix A has two eigenvalues: 5 and 2: 5 is the eigenvalue corresponding to the eigenvector $(1, 1)$, and 2 is the eigenvalue corresponding to the eigenvector $(-1, 2)$. The computational focus of this chapter is how to find these eigenvalues and corresponding eigenvectors. The conceptual focus is geometrical at first, but many of you have likely come across eigenvalues or eigenvectors in other courses, and know that they are important things. Eigenvalues and eigenvectors are instrumental in understanding electrical circuits, mechanical systems, Google's PageRank algorithm, and a long list of other items in applied mathematics. So the greater goal is to expose you to reasons why these are important, but a grasp of the geometric property of eigenvalues and eigenvectors is essential!

27.1 Determinants

The unit square was transformed to a parallelogram of area 10 square units. In other words, unit area is increased by a factor of ten under the transformation by the matrix A. This number ten is known as the **determinant** of the matrix A, which we will denote by $\det(A)$ or $|A|$. You may have seen determinants denoted by the bars replacing the brackets for the matrix. This is a common notation as well, and you want to make sure you know the difference between a square matrix and the determinant of a square matrix.

$$\text{If } A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \text{ then } \det(A) = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10.$$

We restrict ourselves to square matrices of order 2 in this chapter. If A is any 2×2

matrix, then the determinant is quick to find, of course:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Another way to arrive at the determinant being an important thing that determines something is through the solution of a system of n equations and n unknowns. In this restrictive case, when $n = 2$, the solution to any system of the form

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

is, by algebra, $x = \frac{pd-qb}{ad-bc}$ and $y = \frac{aq-cp}{ad-bc}$.[1] Notice that the denominator of both fractions in the solution is the determinant of the coefficients matrix. If the system has a unique solution, then the determinant of this matrix cannot be zero. What happens if the determinant is zero? Take a moment and look back at the first few pages of this book: Go to the systems $A\mathbf{x} = \mathbf{b}$ in problems 5, 6, and 7.

494. Write each system in matrix form $A\mathbf{x} = \mathbf{b}$. Under each, compute $\det(A)$.

495. Consider the corresponding homogeneous systems for all three systems of equations. What are the solutions?

496. Make a conjecture about the relationship between the determinant of the matrix A and the solution set of $A\mathbf{x} = \mathbf{b}$. (If $\det(A) \neq 0$, then? If $\det(A) = 0$, then?)

497. Make a conjecture about the relationship between the determinant of the matrix A and the solution set of $A\mathbf{x} = \mathbf{0}$.

The word *trivial* in mathematics often means something very obvious, something easy to see, simple, and usually something requiring little to no proof at all, and also met with a hundred percent agreement by all parties involved. For example, $\mathbf{x} = \mathbf{0}$ is always a solution to the system $A\mathbf{x} = \mathbf{0}$. The zero vector, in this context, is known as the *trivial* solution. All other solutions, if any exist, are called *nontrivial* solutions. The takeaway from this section, and what you really want to be clear about, is the following: For any system $A\mathbf{x} = \mathbf{b}$, the system has a solution if the vector $\mathbf{b}$ is in the column space of A. The system has no solution if $\mathbf{b}$ is not in the column space. Every subspace contains the zero vector, so the zero vector is always in the column space of any matrix. Here is where the determinant comes in: If $\det(A) \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$. If $\det(A) = 0$, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

27.2 Eigenvalues and Eigenvectors

Consider the zero vector a trivial vector. When we want to find all of the vectors that remain on the same line under the transformation by a matrix, we are asking the following

[1] There is a marvelous theorem known as Cramer's Rule showing that solutions to square systems can be found by determinants.

question: when is matrix multiplication the same as scalar multiplication? Given a 2×2 matrix A, we want to find *nontrivial* vectors $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, where λ is a scalar. This leads to the following definitions:

DEFINITION 25: If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is, if $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an **eigenvalue** of A, and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

To find eigenvalues and corresponding eigenvectors of a matrix is to understand this definition using $A\mathbf{x} = \lambda\mathbf{x}$. The identity matrix I is inserted in the second step below in order to factor the vector $\mathbf{x}$ in the final step. (If you write this out in full matrix form, then you will see why that is necessary! You have to hold sizes of matrices into account.)

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} &= \lambda I\mathbf{x} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

The zero vector is certainly a solution to the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$; however, this will lead us nowhere. Why? In order for λ to be an eigenvalue of A this system must have a nontrivial solution for $\mathbf{x}$, which means the determinant of $A - I\lambda$ has to be zero. This leads to what is known as the *characteristic equation* of the matrix A: $\det(A - \lambda I) = 0$, or, as it is sometimes written,

$$|A - \lambda I| = 0.$$

If we had set out to find the eigenvalues and corresponding eigenvectors of the matrix given at the start of this chapter, this is where we would begin. Example 76 shows the work typically written down on paper when the command is to find all eigenvalues of a matrix, but it is worth summarizing what we are doing by reminding ourselves of the simple thing we are seeking: under what conditions is a matrix behaving like a scalar? Given the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, when is it the case that $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\mathbf{x} = \lambda\mathbf{x}$? Ideally, this would be the place to begin mathematically, but you would easily arrive at the place where the example begins.

Example 76: Find all eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

solution: It is the question prior to the example where the work truly begins, but theoretically, we have done that work. We seek all scalars λ such that $\det(A - \lambda I) = 0$, so we begin this work with this characteristic equation which, when written out for this matrix, is:

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} &= 0 \\
(4-\lambda)(3-\lambda) - (2)(1) &= 0 \\
12 - 7\lambda + \lambda^2 - 2 &= 0 \\
\lambda^2 - 7\lambda + 10 &= 0 \\
(\lambda - 2)(\lambda - 5) &= 0
\end{aligned}$$

There are two eigenvalues of the matrix: $\lambda_1 = 2$ and $\lambda_2 = 5$. To find the eigenvectors corresponding to each eigenvalue, we then solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for each eigenvalue. The solution to each system is a subspace (the null space of the matrix $A - \lambda I$), an *eigenspace*, from which we select a single vector to denote its basis:

For $\lambda_1 = 2$, solve $\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which has an augmented matrix $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. This reduces to the augmented matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where we find $\mathbf{x} = \begin{bmatrix} -y/2 \\ y \end{bmatrix}$. If we let $y = 2$, we get the vector $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. (You get to choose one vector from the infinitely many.)

For $\lambda_2 = 5$, solve $\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which has an augmented matrix $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$. This reduces to the augmented matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where we find $\mathbf{x} = \begin{bmatrix} y \\ y \end{bmatrix}$. If we let $y = 1$, we get the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For clarity, always summarize your work to demonstrate you understand that each eigenvalue comes with an eigenvector: This matrix has the following eigenvalues and corresponding eigenvectors $\lambda_1 = 2$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\lambda_2 = 5$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Determine all eigenvalues and corresponding eigenvectors of the following matrices.

498. $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

499. $\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

500. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where a and b are any scalars

Chapter 28

More on Eigenvalues and Eigenvectors

28.1 The Characteristic Polynomial

DEFINITION 26: The **trace** of a square matrix A, denoted by $\text{tr}(A)$, is the sum of the entries on the main diagonal of the matrix.

The trace of the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ is 7, and the determinant of the matrix A is 10. Notice in example 76 how these two numbers appeared in the characteristic equation through which we found the eigenvalues. We can make an interesting generalization here for any 2×2 matrix. I think you should do the work here:

501. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\text{tr}(A) = a + d$ and $\det(A) = ad - bc$. Show that, if one were seeking the eigenvalues of the matrix, the characteristic equation for the matrix is always of the form $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$.

502. Determine all eigenvalues and corresponding eigenvectors (basis of the eigenspace) of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

503. Determine all eigenvalues and corresponding eigenvectors of $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

504. Show that $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ has no real eigenvalues. Think of a geometric reason why this is the case, and see if it is possible to figure out a solid geometric reason for this particular matrix.

505. Find all eigenvalues and corresponding eigenvectors of the linear transformation $T(x, y) = (x + 4y, 2x + 3y)$.

506. Find equations for all lines that are invariant under the matrix $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

507. Prove that if λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is its corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.

508. Prove that if λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is its corresponding eigenvector, and k is a scalar, then $\lambda - k$ is an eigenvalue of $A - kI$ and $\mathbf{x}$ is its corresponding eigenvector.

509. Prove that if A is a 2×2 matrix, then A and A^T have the same eigenvalues.

510. Prove that if 0 is an eigenvalue of a 2×2 matrix A, then the columns of A are linearly dependent.

28.2 Polynomials Are Amazing

If you have studied calculus in which sequences and series are the main topic of discussion, then you are aware of how extraordinary polynomials are in terms of the information they carry about a function and its derivatives. There are a lot more properties to discover about polynomials. In our work on finding the eigenvalues of a matrix, we have a *characteristic equation*, which, in problem 501, can be written as $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$. What's known as the *characteristic polynomial* is, of course, the expression on the left side of that equation:

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$$

Your prerequisite knowledge arms you with the ability to find the *roots* of a polynomial. The roots of a polynomial are the values of the unknown variable that make the polynomial equal to zero. In the plane, the geometric importance of finding the roots of a polynomial enables you to know where the graph of the polynomial intersects the x-axis. Knowing where the polynomial touches this axis gives you a rough idea of what the graph of the polynomial looks like; that is, identifying the roots of a polynomial limit the possible shapes it can take graphically. The roots of the characteristic polynomial we are working with are the eigenvalues of the matrix from which the polynomial was written. If a matrix is of order two, then the polynomial has degree two. Assume its roots are λ_1 and λ_2. The *Fundamental Theorem of Algebra* is helpful here: all polynomials of degree n have n roots over the set of complex numbers. If we know our polynomial is of degree two, and we know the two roots, then we can think (and write) backwards: Finding them meant you had solved a quadratic equation, perhaps factored it in order to write that $\lambda = \lambda_1$ and $\lambda = \lambda_2$, and the step just prior to stating these two roots would have come from the factorization:

$$\begin{aligned}(\lambda - \lambda_1)(\lambda - \lambda_2) &= 0\\ \lambda^2 - \lambda\lambda_1 - \lambda\lambda_2 + \lambda_1\lambda_2 &= 0\\ \lambda^2 - (\lambda_1 + \lambda_2)\lambda + (\lambda_1\lambda_2) &= 0\end{aligned}$$

There are two extraordinary results to notice here:

- *The sum of the eigenvalues is the trace of the matrix.*
- *The product of the eigenvalues is the determinant of the matrix.*

Is this true for any square matrix? If we return to high school algebra and work with the roots of a general polynomial in the variable x, we see a pattern. In fact, we see something more in terms of the coefficients of the polynomial in terms of the roots of the polynomial. Go through these exercises to discover something about the roots of a polynomial and its coefficients, and then we can return to matrices:

511. If r_1 and r_2 are the roots of a degree two polynomial in x, then show that the polynomial is $x^2 - r_1r_2x + r_1r_2$. (Yes, this is really a repeat of the work we just did...)

512. If r_1, r_2, and r_3 are the roots of a degree three polynomial in x, what is the polynomial?

513. If r_1, r_2, r_3, and r_4 are the roots of a degree four polynomial in x, what is the polynomial?

514. Are you able to pick up a pattern yet? If so, write the degree eight polynomial in x if its roots are elements of the set $\{r_1, r_2, r_3, ..., r_8\}$ without doing any algebra.

515. Make a strong attempt to precisely describe the pattern you see. Make a conjecture about a polynomial of degree n whose roots are in the set $\{r_1, r_2, ..., r_n\}$. Discuss your description with your peers in order to judge whether or not what you wrote is precise enough. What do you learn from this?

28.3 Determinants and Eigenvalues of Matrices of Order Three

There is a general form for the determinant of any $n \times n$ matrix, a topic that deserves its own book, and one not included in this course (though it can easily be added as many study determinants in introductory courses). There are a couple of ways to find the determinant of a 3×3 matrix A. If we repeated what we did with 2×2 matrices, which makes the most sense to do, we imagine the linear transformation determined by the matrix. If A is the 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

then the matrix maps $\mathbb{R}^3$ into $\mathbb{R}^3$, a transformation of three dimensional space. Unit measure of this space would be in volume, so it is natural to ask what happens with the volume of the unit cube under the transformation. It is possible to calculate volume of the resulting transformation, assuming it is a parallelopiped, and show that the magnitude of the determinant is the volume. The determinant would have the following value:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

There is a nice way to remember this product that is analogous to what happened with 2×2 matrices, but it is a bit more involved: copy the determinant's columns, and then copy the first two columns to the right of those three:

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Start with the main diagonal of the matrix that goes from a to e to i, and you can see the first product in the formula for the determinant. Move one diagonal to the right, from b to f to g, etc... I think you see the picture, perhaps relating it in a way to the mnemonic that happened with the 2×2 matrices? (For those of you who have studied the cross product

of vectors in $\mathbb{R}^3$, you know another technique used in computing determinants known as *expansion by minors*, which is not discussed in this problem, but briefly mentioned in the forthcoming example.)

Example 77: Find all eigenvalues and corresponding eigenvectors of $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$.

solution (partially shown): The eigenvalues of this matrix would be found by solving the characteristic equation $\det(A - I\lambda) = 0$:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix} = 0 \tag{28.1}$$

$$(2-\lambda)(2-\lambda)(-2-\lambda) - 3(-1)(2-\lambda) = 0 \tag{28.2}$$

$$(2-\lambda)[(2-\lambda)(-2-\lambda) - 3(-1)] = 0 \tag{28.3}$$

$$(2-\lambda)(-4+\lambda^2+3) = 0 \tag{28.4}$$

$$-8 + 2\lambda^2 + 6 + 4\lambda - \lambda^3 - 3\lambda = 0 \tag{28.5}$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \tag{28.6}$$

The two zeros in the first row certainly help with this computation. I am certain you can sense how tedious the calculation might be, which is why you don't have students finding eigenvalues of larger matrices in an introductory course. To those of you who know how to find the determinant by expanding by the first row, notice that step 28.3 is the first element of the first row, multiplied by what's inside the brackets, the *minor* 2×2 determinant that is left when you cross out the first row and first column where the first element of the first row sits. The cubic equation can be factored by grouping the first two terms, grouping the last two terms:

$$\begin{aligned} \lambda^2(\lambda-2) - (\lambda-2) &= 0 \\ (\lambda-2)(\lambda^2-1) &= 0 \\ (\lambda-2)(\lambda-1)(\lambda+1) &= 0 \end{aligned}$$

This leads to the three eigenvalues $\lambda_1 = -1, \lambda_2 = 1$, and $\lambda_3 = 2$. For each eigenvalue λ, its corresponding eigenvectors are the solutions to the the system $A - \lambda I = \mathbf{0}$, work I leave to you, the student: For problems 516, 517, and 518, verify that the eigenvalue is correctly paired with an eigenvector.

516. $\lambda_1 = -1, \mathbf{x_1} = (0, 1, 3)$

517. $\lambda_2 = 1, \mathbf{x_2} = (0, 1, 1)$

518. $\lambda_3 = 2, \mathbf{x_3} = (1, 1, 1)$

519. Is the sum of the eigenvalues equal to the trace of the matrix?

520. Is the product of the eigenvalues equal to the determinant of the matrix?

Chapter 29

Eigenspaces

It has been said that the famous Tacoma Narrows Bridge collapse... When you find the eigenvalues of a matrix and then the corresponding eigenvectors, ultimately what you are doing with that information is you are giving an eigenvalue and a basis for an **eigenspace**. In the case of 2×2 matrices, it is nice to see a visual of these important subspaces, especially with some meaningful matrices you have met in previous chapters. This section sheds some more light on why eigenvalues and eigenvectors are crucial things to figure out.

DEFINITION 27: If A is an $n \times n$ matrix and λ is an eigenvalue of A corresponding to the eigenvector $\mathbf{x}$, then the set of all solution vectors $\mathbf{x}$ such that $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ form a vector space called the **eigenspace of A corresponding to** λ.k

521. Let S be the eigenspace of A corresponding to eigenvalue λ and eigenvector $\mathbf{x}$. Prove that S is a subspace of A.

522. Find all eigenvalues and give a basis for the corresponding eigenspaces for the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Make a sketch of these eigenspaces while exploring how the matrix transforms the plane. (For those who are motivated, find and sketch all the fundamental subspaces of A, and put together some narrative with pictures about this matrix. Tell its story, in a sense.)

523. Repeat the same for the matrix $B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$.

524. Let $C = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$. Determine all eigenspaces associated with C.

525. Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix}$. Determine all eigenspaces associated with D.

526. Find all eigenspaces of $E = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

29.1 The Fibonacci Matrix

Recall the matrix that generates the famous Fibonacci sequence in which the sum of two consecutive terms everywhere is equal to the third consecutive term:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

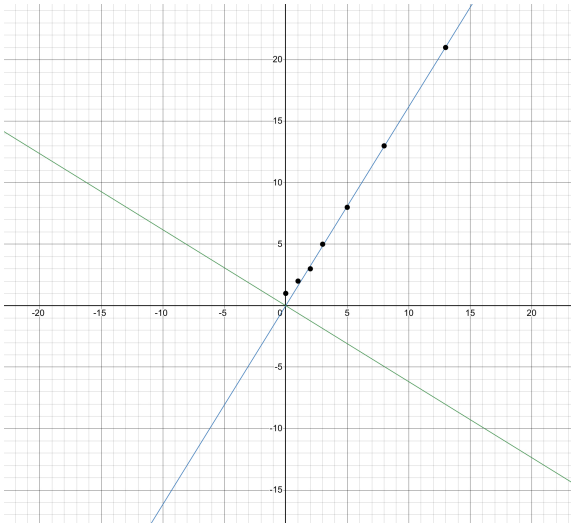


Fig. 29.1: Eigenspaces of the Fibonacci Matrix

Chapter 30

Change of Basis: Leave, Do It There, and Return

This chapter focuses on an important mathematical move you have seen before in many different settings: a problem you encounter cannot be solved in its current form, but if you change its form by rewriting it in another way that does not alter what it truly is, you may find it easy to solve in the other form, ultimately converting it back to the form in which it was originally written. The substitution method, for example, for integrals in calculus (often referred to as "u-substitution") is one such case: The function $e^{\tan(x)}\sec^2(x)$ might appear intimidating at first when seen as an integrand:

$$\int e^{\tan(x)}\sec^2(x)\,dx$$

But a simple substitution of $u = \tan(x)$ reveals that the integral is handled easily, since the differential $du = \sec^2(x)\,dx$:

$$\int e^u\,du$$

The integral in terms of x is obtained by first finding it in terms of u. Think of the procedure as a combination of three steps. In order to obtain the answer in x, you must first go from x to u, then obtain the answer in u, and then go from u to x. To solve the differential equation $y''(t) - y'(t) - 6y(t) = 0$, one could intelligently come to the conclusion that the only family of functions that could be combined in such a way as to sum to zero would be exponential functions of the form $y(t) = e^{rt}$, a substitution that changes the differential equation to the equation $e^{rt}(r^2 - r - 6) = 0$, which is solved by understanding that e^{rt} is never zero, and so obtaining the solution to the original equation involves solving a quadratic equation whose roots are $r_1 = -2$ and $r_2 = 3$. This information is returned to that pertaining to the differential equation's general solution, which is ultimately all linear combinations of two linearly independent functions e^{-2t} and e^{3t}, written as $y(t) = C_1e^{-2t} + C_2e^{3t}$. Some of you have met the Laplace transform, a powerful tool that converts complex differential equations into simpler algebraic equations. Each of these mathematical procedures can be thought of in three steps in order to achieve a desired result. Plenty of examples in many branches of mathematics abound. I like the following example for our purposes, however, because it gives any person a way to understand the nature of what is going on with matrices without having to revisit some complex mathematics from the past: Suppose you have a box at your home, but it is locked. You wish to have it unlocked at home, but in order to unlock it, you must take it to a friend's house first, where you are able to unlock it. Let U_H represent the box unlocked at home, U_F represent the box unlocked at your friend's house, and let J represent the journey from your house to your friend's house. J^{-1} will be the journey from your friend's house back to your house. Having the box unlocked at home is equal to the journey from your house to your friend's house, followed by having the box unlocked at your friend's house, followed by the journey from your friend's house to your house:

$$U_H = JU_FJ^{-1}$$

I hope you can read that and agree this U_H and U_F are not the same, but are *similar*.

30.1 Similar Matrices

DEFINITION 28: Let A and B both be $n \times n$ matrices. A is said to be **similar** to B if there exists an invertible (nonsingular) $n \times n$ matrix S such that $A = SBS^{-1}$.

Of course, if $A = SBS^{-1}$, then multiplying both sides on the right by S means that $AS = SB$, a useful equation for a problem such as the following:

Example 80: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then is A similar to B?

solution: Assume, by definition, that $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and set up the equation $AS = SB$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix} = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix}$$

By comparison, $a = -a$ implies $a = 0$, and, by the same reasoning, $d = 0$. Meanwhile, b and c are arbitrary real numbers, but $b \neq 0$ and $c \neq 0$ because S must be invertible (nonsingular), so any matrix of the form below can be used to show that $A = SBS^{-1}$, which means A and B are similar.

$$S = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, b \neq 0, c \neq 0$$

531. Find the inverse of S in example 80.

532. Show that $A = SBS^{-1}$.

533. Let I_2 be the 2×2 identity matrix. Show that the only matrix similar to I_2 is I_2 itself. (*Hint:* Start with the sentence "A is any 2×2 matrix that is similar to I_n" and directly apply the definition. Then see where it leads you.

534. Prove that every $n \times n$ matrix A is similar to itself.

535. Prove that if A is similar to B, then B is similar to A.

536. Prove that if A is similar to B, and B is similar to C, then A is similar to C.

30.2 A Change of Basis Conversation

Before getting to the level of being given a formula to execute problems involving changing of bases, this entire section is a single conversation that meticulously takes you through the idea of the title of this chapter. Follow along with pencil and paper in hand as it is helpful to read it slowly and understand it before connecting this chapter to the previous

couple of chapters on eigenvalues and eigenvectors. This chapter serves as a kind of pivotal moment in linear algebra, as it opens the door to a larger world in which linear algebra is applied, and gets to the heart of why this entire book was written: that factoring matrices in certain ways (decomposition) is an extraordinary skill in applications. The conversation begins with a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$:

$$T(x, y, z) = (x + y, y + z, x + z)$$

Let A be the standard matrix representing this linear transformation; that is, the matrix A for which $T(\mathbf{x}) = A\mathbf{x}$ is the matrix below, which you should verify for yourself:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

You have learned that the columns of this matrix reveal how the standard basis vectors are transformed; that is, you know that $T(1, 0, 0) = (1, 0, 1), T(0, 1, 0) = (1, 1, 0)$, and $T(0, 0, 1) = (0, 1, 1)$. Recall the shorthand notation for these basis vectors:

$$\mathbf{e_1} = (1, 0, 0), \mathbf{e_2} = (0, 1, 0), \mathbf{e_3} = (0, 0, 1)$$

This enables us to write:

$$T(\mathbf{e_1}) = (1, 0, 1), T(\mathbf{e_2}) = (1, 1, 0), T(\mathbf{e_3}) = (0, 1, 1)$$

This is something you *see* when you view A. It makes sense to say that the matrix A is the **matrix representation of T with respect to the standard basis for** $\mathbb{R}^3$. If $E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, then we can say the set E represents a basis for $\mathbb{R}^3$, the standard basis. As you have learned (*notice how I have repeated that phrase in this conversation*), any set of three linearly independent vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. The set $G = \{\mathbf{u_1} = ((1, -1, 0), \mathbf{u_2} = (1, 0, -1), \mathbf{u_3} = (0, 1, 1)\}$ is also a basis for $\mathbb{R}^3$. (You ought to verify this as a review exercise.) If G is a basis for $\mathbb{R}^3$, then we can ask the following question: Is it possible to create a matrix representing the linear transformation with respect to this other basis, rather than the standard basis? If so, we would want the matrix to have columns in which we can see the vectors to which these basis vectors $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$ are mapped but the vectors *must each be written in terms of these basis vectors*; that is, we use T to figure out the following outputs:

$$\begin{aligned} T(\mathbf{u_1}) &= T(1, -1, 0) \\ &= (1 + (-1), -1 + 0, 1 + 0) \\ &= (0, -1, 1) \end{aligned}$$

$$\begin{aligned} T(\mathbf{u_2}) &= T(1, 0, -1) \\ &= (1 + 0, 0 + (-1), 1 + (-1) \\ &= (1, -1, 0) \end{aligned}$$

$$\begin{aligned} T(\mathbf{u_3}) &= T(0, 1, 1) \\ &= (0 + 1, 1 + 1, 0 + 1) \\ &= (1, 2, 1) \end{aligned}$$

These outputs need to be written in terms of the basis vectors $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$. So we must find scalars a_i, b_i, and $c_i, i = 1, 2, 3$ such that:

$$\begin{aligned} T(\mathbf{u_1}) &= a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3} \\ T(\mathbf{u_2}) &= b_1\mathbf{u_1} + b_2\mathbf{u_2} + b_3\mathbf{u_3} \\ T(\mathbf{u_3}) &= c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} \end{aligned}$$

The above can be written as the following:

$$(0, -1, 1) = a_1(1, -1, 0) + a_2(1, 0, -1) + a_3(0, 1, 1) \tag{30.1}$$

$$(1, -1, 0) = b_1(1, -1, 0) + b_2(1, 0, -1) + b_3(0, 1, 1) \tag{30.2}$$

$$(1, 2, 1) = c_1(1, -1, 0) + c_2(1, 0, -1) + c_3(0, 1, 1) \tag{30.3}$$

Each one of these can be converted to a system of linear equations represented by the following augmented matrices:

$$(30.1): \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$$(30.2): \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$(30.3): \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

If you look closely at the above three augmented matrices, notice how they can be put together as a single augmented matrix that can be row reduced:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & -1 & -1 & 2 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{bmatrix} &\xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 3 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{bmatrix} \\ &\xrightarrow[R_2+R_3]{-R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 4 \end{bmatrix} \\ &\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \\ &\xrightarrow[=R_3+R_2]{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

The last three columns reveal the scalars for (30.1), (30.2), and (30.3): We can put them in a matrix, but we want to notice the place where this begins: Let B represent the matrix

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

B is the matrix we are looking for. Its first column is where the basis vector $\mathbf{u}_!$ is mapped, but the vector is written in terms of $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$. Its second column is where the basis vector $\mathbf{u_2}$ is mapped, but the vector is written in terms of $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$. Its third column is where the basis vector $\mathbf{u_3}$ is mapped, but the vector is written in terms of $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$. This seems like a long process, but this conversation is doing nothing more than a direct search for these linear combinations. To find B we really solved three systems *simultaneously*, just like, at the beginning of this course, we were solving three equations *simultaneously*. If we take a closer look at our process, we find a much more efficient way of finding B. To discover how, let's define S to be a matrix whose columns house the basis vectors $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$; that is, for this particular example, S is the matrix:

$$S = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{bmatrix}, \text{ which means } S = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

The problem we formulated to find B involved first finding how each of these basis vectors were transformed by the matrix A, and by defining the matrix S, the matrix product AS accomplishes this task. The first column of this matrix product is $T(\mathbf{u_1})$, the second column is $T(\mathbf{u_2})$, and the third column is $T(\mathbf{u_3})$:

$$AS = \begin{bmatrix} T(\mathbf{u_1}) & T(\mathbf{u_2}) & T(\mathbf{u_3}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

But each column of that matrix product were written in terms of the standard basis vectors, and we wanted to write each of the columns of that matrix in terms of the new basis vectors: Initially, we wrote these three separate systems like so:

$$\begin{aligned} T(\mathbf{u_1}) &= a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3} \\ T(\mathbf{u_2}) &= b_1\mathbf{u_1} + b_2\mathbf{u_2} + b_3\mathbf{u_3} \\ T(\mathbf{u_3}) &= c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} \end{aligned}$$

The right side of each of those equations is a column, therefore, of AS. But I like taking the above set of three equations we initially wrote and write it as we have always written a system of equations, in matrix form:

$$\begin{bmatrix} T(\mathbf{u_1}) \\ T(\mathbf{u_2}) \\ T(\mathbf{u_3}) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \\ \mathbf{u_3} \end{bmatrix}$$

With $AS = \begin{bmatrix} T(\mathbf{u_1}) & T(\mathbf{u_2}) & T(\mathbf{u_3}) \end{bmatrix}, B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, and $S = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{bmatrix}$, that matrix equation is this:

$$(AS)^T = B^T S^T$$

By the property of transposes, we know $B^T S^T = (SB)^T$, so $(AS)^T = (SB)^T$, which means that A and B are similar, because

$$AS = SB,$$

so if B is the matrix we are looking for, then multiplying $AS = SB$ both sides on the left by the inverse of S gives us a way to find B efficiently, the **Change of Basis Formula:**

If A is the standard matrix of a linear transformation, one whose columns map standard basis vectors to vectors in the same basis, then the matrix B that represents the same linear transformation, but with respect to another basis, whose columns map its basis vectors to vectors in terms of its basis vectors, is:

$$B = S^{-1}AS$$

Notice this can be read *right to left* as matrix multiplication is function composition, "In order to have an opened box B at home, first go from your house to friend's house (S), then open the box there (A), and finally go from friend's house to your house (S^{-1})."

30.3 Why? A Meaningful Change of Basis Conversation

Why might this be an extraordinary thing to be able to do? Let's make a strong connection in this last section of this chapter, and enter the doorway into the more exciting areas of linear algebra you will meet following an introductory course in linear algebra, and depending on your area of study: *applied linear algebra, notably principal component analysis and singular value decomposition.* The starting place for that is, in my very humble opinion, connecting the eigenvalues and eigenvectors of a matrix with this idea of rewriting a matrix in terms of another basis other than the standard basis vectors. This conversation provides you with a solid reason, and kind of a logical one I believe most people would agree might be obvious based on *how the matrix transforms a space.* Before the actual conversation, however, I thought I'd present a pure linear algebra problem in an example, but it will pertain to the conversation that follows. The example and its solution is dry and boring. The conversation that follows is the same example, but with meaning, and includes the example's origin story.

Example 81: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (2x+3y, x+4y)$. Find a matrix that represents the same linear transformation, but in terms of the basis $\{(-3, 1), (1, 1)\}$ for $\mathbb{R}^2$.

solution: Let A be the standard matrix for the transformation, and place the new basis vectors as columns of a matrix S. Then find S^{-1}, followed by the change of basis formula:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, S = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}, \text{ so find the inverse of } S:$$

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ -3 & 1 & 1 & 0 \end{bmatrix}$$
$$\xrightarrow{3R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{4}R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$
$$\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Finally, let B be the matrix we are looking for, which is found by the product:

$$B = S^{-1}AS = \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Think about this result. A new basis is given for the xy-plane: $\{(-3,1),(1,1)\}$, and by the given linear transformation, $T(-3,1) = (2(-3)+3(1),(-3)+4(1)) = (-3,1)$ and $T(1,1) = (2(1)+3(1),(1)+4(1)) = (5,5)$. This can also be confirmed by multiplying both of these basis vectors on the left by A. The matrix B can be read this way:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$T(-3,1) = \mathbf{1}(-3,1)+\mathbf{0}(1,1) = (-3,1)$ by the first column of B, and
$T(1,1) = \mathbf{0}(-3,1)+\mathbf{5}(1,1) = (5,5)$ by the second column of B.

Here are some important things to notice about this procedure: Given the linear transformation $T(x,y) = (2x+3y, x+4y)$, when you find the image of $(2,6)$, you directly plug these coordinates in the function and find $T(2,6) = (2\cdot 2+3\cdot 6, 2+4\cdot 6) = (20,26)$. You are reading this in the usual x and y coordinates of the plane; that is, you are using the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$:

$$T(2(1,0)+6(0,1)) = 20(1,0)+26(0,1)$$

The other basis for $\mathbb{R}^2$ in the example is $\{(-3,1),(1,1)\}$. And the matrix you found, B, tells you about the linear transformation *using a different coordinate system*, coordinates (u,v) where the same linear transformation can be written in terms of these new coordinates: $T(u,v) = (u,5v)$. Question: What is the equivalent of $T(2,6) = (20,26)$? You would want to convert $(2,6)$ to the new coordinate system, and that question is easily answered by asking it more precisely: how would you write the ordered pair in terms of the new basis vectors? This is, of course, solving a linear combination, and notice what happens: The problem is to solve

$$(2,6) = a(-3,1)+b(1,1), \text{ which means to solve the system:}$$

$$\begin{aligned} -3a + b &= 2 \\ a + b &= 6 \end{aligned}$$

Notice what this system is in matrix form, and notice how to solve it by way of matrices:

$$\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

is solved by noticing you are using the matrices S and S^{-1}:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

In other words, if $\mathbf{x}$ is a point in terms of the standard basis, then to convert it into the coordinates of the new basis, multiply this vector by S^{-1}. Of course, the solution to the system is $a = 1$ and $b = 5$, which means the point in the other basis is $(1, 5)$. This is not a point on the regular grid in which you are familiar! Suppose that we say $\{(-3, 1), (1, 1)\}$ is a basis for the uv-plane. While $(2, 6)$ is from the xy-plane. Note that they describe the same point since $(1, 5)$ means $1(-3, 1) + 5(1, 1)$, while $(2, 6)$ means $2(1, 0) + 6(0, 1)$. (Check this.) You will see a new grid in a page or two. The matrix B is the way that coordinates in the uv-plane are transformed in the same way that A does with coordinates in the xy-plane:

$$T(2, 6) = (22, 26) = 22(1, 0) + 26(0, 1) \tag{30.4}$$

$$T(1, 5) = (1, 25) = 1(-3, 1) + 25(1, 1) \tag{30.5}$$

Do you see how these result in the same location? (Other books introduce a new notation to indicate the difference in coordinate systems; that is, (30.4) is in (x, y) and (30.5) is in (u, v).) Let's ask the question: *why might this be important to be able to do?*

The 2×2 matrix A in the previous example is the standard matrix that defines the linear transformation $T(x, y) = (2x + 3y, x + 4y)$. We can visualize what this matrix does, how it transforms $\mathbb{R}^2$. See Figure 30.1. The standard basis vectors get mapped to the column vectors you see in the matrix; that is, $(1, 0)$ is mapped to $(2, 1)$, and $0, 1)$ is mapped to $(3, 4)$. The unit square is mapped to a parallelogram, whose area is 5. This is the linear transformation with respect to the standard basis vectors. It is easy for us to work in the usual coordinate system we call the *rectangular* coordinate system, using the x and y directions, but for this transformation, another coordinate system might be more helpful. The eigenspaces are the spaces where the transformation can be seen as being grounded; that is, finding all vectors whose directions are not changed is easier to work with in terms of *seeing* the transformation. The eigenvalues and corresponding eigenvectors of A are $\lambda_1 = 1, \mathbf{x_1} = (-3, 1)$ and $\lambda_2 = 5, \mathbf{x_2} = (1, 1)$. The two eigenspaces associated with this matrix are sketched in Figure 30.2: the lines $x = -3y$ and $y = x$. It is along these lines that vectors are scaled by the matrix (all vectors along the line $x = -3y$ are scaled by 1, meaning they remain the same size, and all vectors along the line $y = x$ are scaled by a factor of 5). If we chose the eigenvectors as a basis for the plane, we can view the linear transformation as something more simple: the matrix can be seen as scaling in the

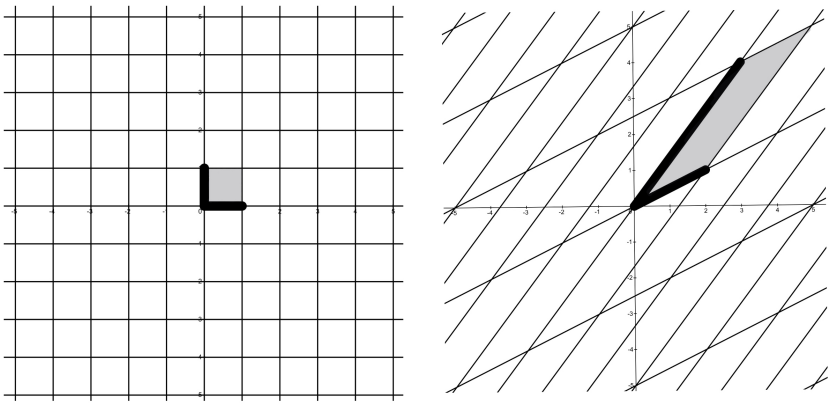


Fig. 30.1: How A Transforms $\mathbb{R}^2$ (Standard Basis)

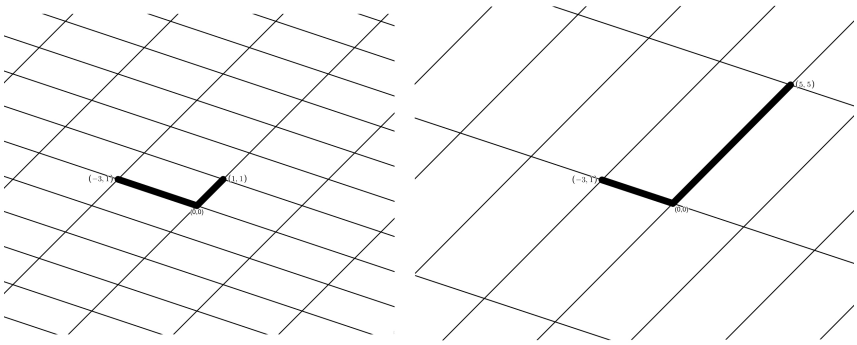


Fig. 30.2: How A Transforms $\mathbb{R}^2$ (Eigenvectors as Basis)

two directions given by the eigenvectors. Once these eigenvectors are found, then working example 81 follows. Notice the matrix that results in the end, that the eigenvalues show up in the columns of B, and that this matrix is a diagonal matrix, one of the easiest of linear transformations to understand. (Why?) It is worth thinking about such matrices, ones with any real numbers on the main diagonal and zeros everywhere else: changing the basis in such a way as to see the transformation in a much easier way is similar to letting $u = x^2$ in the integral $\int 2x\cos(x^2)\,dx$, because it transforms the integral into something easy: $\int \cos(u)\,du$. It is like transforming $y'' - y' - 6y = 0$ with the substitution $y = e^{rt}$ to transform the equation into a high school algebra problem: $r^2 - r - 6 = 0$, which is quick to solve. In other words, this skill might be a computational superpower.

Take a moment to notice that, using Fig. 30.1, under the standard basis, that $T(-4, 0) = (-8, -4)$. This is the same as, in Fig. 30.2, $T(1, -1) = (1, -5)$ since $(1, -5)$ means $1(-3, 1) - 5(1, 1) = (-8, -4)$. Notice that example 81 is a demonstration of decomposing a matrix into a very interesting product:

$$A = SBS^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

If you were to go to wolframalpha.com and input the matrix A in the window, then hit the "return" key, scroll down and notice what that machine returns on your screen: one of the items you will see is known as the *diagonalization* of the matrix. Some matrices can be *diagonalized* and some cannot. Diagonalizing a matrix has its own chapter or section

in most linear algebra texts.

537. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (6x - 2y, 9y - 2x)$. Find the standard matrix A for the transformation. Then find a matrix that represents the transformation with respect to the basis $\{(1, 1), (1, -1)\}$.

538. Using the same transformation $T(x, y) = (6x-2y, 9y-2x)$, find a matrix representing T with respect to the basis $\{(2, 1), (1, -2)\}$.

539. The answer to the previous problem is the diagonal matrix with 5 and 10 on the main diagonal. Are 5 and 10 eigenvalues of the standard matrix A? Find the eigenvalues and corresponding eigenvectors of A, the standard matrix of the linear transformation $T(x, y) = (6x - 2y, 9y - 2x)$.

Use the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ for exercises 540-54?.

540. Determine all eigenvalues and corresponding eigenvectors of A.

541. b

Chapter **31**

Abstract Vector Spaces

Chapter **32**

Diagonalization

Chapter **33**

Matrix Factorizations

Chapter **34**

More on Abstract Stuff

Chapter 35

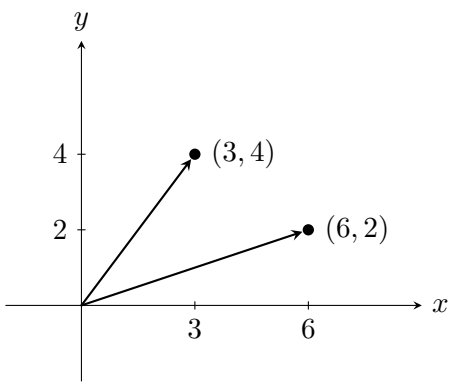
Review

Bibliography

Appendix A

Answers to Selected Problems

1. $(-4, 2)$
2. $(-1, -1)$
3. $(-1, 2)$
4. $(3, 1)$
5. $(4, 2)$
6. no solution
7. the set of all vectors of the form $(2k - 5, k)$, where k is any scalar
8. $(1, 1, 1)$
9. $(1, 1, 2)$
10. the set of all vectors of the form $(1, k, k)$, where k is any scalar.
11. Vectors given in the form (a, b) are directed line segments from the origin to the *ordered pair* (a, b).



12. The sum $\mathbf{u} + \mathbf{v} = (9, 6)$. To explain this graphically, take the vector $(6, 2)$ and slide it along the other vector until its tail meets the arrow of the other vector, and its arrow will land on $(9, 6)$

15. Eventually, the dots would cover the line $y = \frac{2x}{3}$.

16. **True**; It is important to note that all scalar multiples of *any nonzero vector* is another way of describing a line through the origin on which the given vector sits. Thinking of slope $\Delta y/\Delta x$, the vector $(2, 4)$ means $\Delta y = 4$ and $\Delta x = 2$ for the line, and so the line contains the origin and has slope 2; hence, the equation $y = 2x$. (The line can be described as the set of all vectors of the form $(x, 2x)$.)

17. **True** Let a be any real number; $a(-3, -5) = (-3a, -5a)$, so if $x = -3a$, then $a = -\frac{x}{3}$. Replace this a in $y = -5a$ and get $y = -5(-\frac{X}{3}$, which simplifies to $y = 5x/3$, which is the set of all points of the form $(x, 5x/3)$.

18. **True** Justify this on your own.

19. $(2, -9)$

20. $(24, -3)$

21. If you scale $(1, 1)$ by 2, and add it to $(3, -2)$, then you arrive at the vector $(5, 0)$.

22. Think of telling a story in pictures to show that you must double the lengths of both vectors in order to add them to arrive at the resulting vector.

23. This is the same as solving the system involving the two equations $2x + y = 3$ and $3x + y = 5$, which has solution $(2, -1)$. You should be able to show that $2\mathbf{u} - \mathbf{v} = \mathbf{w}$.

24. Stated another way: Find all x and y such that

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Your sketch should show two vectors on the same line, the constant vector elsewhere, in which case you know there is no way to scale and add two vectors together to sum to a vector that is not on that line. (*Be absolutely sure about understanding this!*)

25. Stated another way (which ought to be as was done in 24, written in column vector form) and using the space-saver notation, find all x and y such that $x(1, -3) + y(-2, 6) = (-5, 15)$. Your sketch should show all three vectors on the same line. Why, though, an infinite number of solutions? By the way, this linear combination problem is often stated in the following way: *Is it possible to write* $(-5, 15)$ *as a linear combination of* $(1, -3)$ *and* $(-2, 6)$? Later on, the fancy way of stating the problem is this: *Is* $(-5, 15)$ *in the span of the set* $\{(1, -3), (-2, 6)\}$?

26. I'll assume this one is an easy one, and you wrote 2×3 augmented matrices for 1-7, and 3×4 augmented matrices for 8-10.

32. As an augmented matrix, the matrix describes a system of two equations and two unknowns with no solution because the second row reads as $0x + 0y = 3$, which does not make sense.

33. As an augmented matrix, the matrix describes a system of two equations and two unknowns in which $x = 7$ and y is any real number. You will soon be instructed to write this like so: If $x = 7$ and $y = k$, where k is any real number, then this is the set of all vectors

$$\mathbf{x} = \begin{bmatrix} 7 + 0k \\ 0 + k \end{bmatrix},$$

which is written as $\mathbf{x} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} + k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

34.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R1+R2} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \xrightarrow{R2/(-3)} \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \xrightarrow{-R2+R1} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

35.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R1+R2} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \xrightarrow{R2/(-4)} \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

36. A, B, C, and D are, respectively, 2×3, 2×2, 3×4, and 4×3.

38.

$$\begin{aligned} 3x_1 + 2x_2 &= 2 \\ x_1 \qquad &= 4 \\ 2x_1 + 2x_2 &= 0 \\ x_2 &= 7 \end{aligned}$$

47. The reduced augmented matrix reads as

$$\begin{aligned} x - 2y &= 5 \\ y &= y \end{aligned}$$

and you will be learning to write this in column vector form (these are the "accounting skills" of lining things up properly):

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 + 2y \\ 0 + 1y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The variable y will be designated as a *free* variable.

48. Justify your "yes" by checking all four criteria.

49. No because the first nonzero element of row 2 is not a 1.

50. You should record three specific row operations, resulting in $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$, which means the solution is $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, or $x = -2$ and $y = 1$.

***Check your own answers to problems 51-60 by using the desmos matrix calculator, free online.

64. $(0, 4, 2)$

65. $(1, 1, 1)$

66. $(-1, 2, 3)$

101. $AC = \begin{bmatrix} 18 & 13 & 6 \\ 18 & 11 & 10 \\ 29 & 13 & 10 \end{bmatrix}$

102. $CA = \begin{bmatrix} 5 & 6 & 7 \\ 24 & 28 & 18 \\ 4 & 12 & 6 \end{bmatrix}$

142. $\begin{bmatrix} \frac{2}{13} & -\frac{3}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix}$

153. The inverse of a product obeys the equation $(AB)^{-1} = B^{-1}A^{-1}$.

161. $5 + 5i$

162. $23 + 2i$

163. $2 + 2i$

164. $-2i$

165. $C^2 = \begin{bmatrix} a^2 - b^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix}$, so $i^2 = -1$, and, as a matrix, -1 is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. If $a = 0$ and $b = 1$, the matrix C represents i, and notice that these substitutions lead to the matrix C^2 being the one that represents -1.

166. The matrix product is $\begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix}$, which does agree with 161, since this matrix represents $5 + 5i$.

167. As a matrix product, this would be $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 23 & -2 \\ 2 & 23 \end{bmatrix}$, which agrees with 162.

170. Part of understanding this problem is knowing that $i^{-1} = -i$.

171. 0

172. the set of all real numbers

173. no solution

174. $(0, 0)$

175. all scalar multiples of the vector $(-1, 2)$

176. no solution

177. (0,0,0)

178. all scalar multiples of $(-2, -3, 1)$

179. all linear combinations of the vectors $(3, 1, 0)$ and $(-1, 0, 1)$

180. no solution

183. Though it might be too small, check out the graph that decorates your syllabus and also the Canvas page for the course. It is related to this set of problems involving the humans and zombies. See if you can decipher what information is located in the graph.

184. $(20, 7, -31)$

190. The span of the column vectors of A is $\mathbb{R}^2$.

191. The span of the row vectors of A is the set of all vectors of the form (a, b, b); that is, the row vectors span a plane in $\mathbb{R}^3$, the set of all points where the second and third coordinates are equal.

192. An $m \times n$ matrix has m rows and n columns, so each row has n entries in it; therefore, if each row is a vector, it is a vector with n components and, consequently, is a vector from $\mathbb{R}^n$. Use the same logic to conclude that each column vector must be a vector from $\mathbb{R}^m$.

193. In this case, both the row and column spaces are lines in $\mathbb{R}^2$: both are all scalar multiples of $(1,1)$, or you could say the row space is the line $y = x$, as is the column space.

194. Let k be any scalar and let $\mathbf{u} = (u, 2u, u^2)$ be any vector in S. If S is closed under scalar multiplication, then $k\mathbf{u}$ would have the form in which its second component is twice the first, and its third component is the square of the first; however, $k\mathbf{u} = (ku, 2ku, ku^2)$. Because the third component is not k^2u^2, then we can conclude that the set S is not closed under scalar multiplication.

195. *Yes*, the set of points described here (the unit circle) is a subset of $\mathbb{R}^2$.

196. Intuitively speaking, *no*, it is not a subspace of $\mathbb{R}^2$ because it is not a line through the origin. To show this mathematically, pick any two points in the set and think in terms of vectors! Any two vectors from the origin to any point on the unit circle, when added together, would land on a point that is not on the circle. That's one argument that works because it shows that the set is not closed under vector addition; hence, the set of points cannot be a subspace.

236. The rows are linearly independent. To conclude that, you did something similar to what was done in example 35.

237. The set is dependent because you showed one vector is a scalar multiple of the other.

238. The set is dependent since any vector in the set can be written as a linear combination of all the others (give a solid example of this in your work).

239. The set is dependent due to the fact that the zero vector is in the set.

240. To show the set is linearly independent, show that the only solution to $c_1(1,1) + c_2(-1,1) = (0,0)$ is $c_1 = c_2 = 0$. The vectors span $\mathbb{R}^2$ since all vectors in this space can be written as

$$\begin{aligned}
(x,y) &= (\frac{x}{2} + \frac{x}{2} + \frac{y}{2} - \frac{y}{2}, \frac{y}{2} + \frac{y}{2} + \frac{x}{2} - \frac{x}{2}) \\
&= (\frac{x+y}{2} + \frac{x-y}{2}, \frac{x+y}{2} + \frac{y-x}{2}) \\
&= (\frac{x+y}{2}, \frac{x+y}{2}) + (\frac{x-y}{2}, \frac{y-x}{2}) \\
&= (\frac{x+y}{2}, \frac{x+y}{2}) + ((-1)\frac{y-x}{2}, \frac{y-x}{2}) \\
&= \frac{x+y}{2}(1,1) + \frac{y-x}{2}(-1,1).
\end{aligned}$$

241. the identity matrix

242. the identity matrix

245. Since it is possible to show that $(1)\mathbf{u} + (-1)\mathbf{v} + (1)\mathbf{w} = \mathbf{0}$, then, by definition, the set is linearly dependent.

246. One way to show this is to demonstrate one of the properties fails: $f(cx) = 2(cx)+3$, and $cf(x) = c(2x+3) = 2cx + 3c$; thus, $f(cx) \neq cf(x)$.

248. Since g is linear, $g(5) = g(5 \cdot \frac{4}{4}) = g(\frac{5}{4} \cdot 4) = \frac{5}{4}g(4) = \frac{5}{4} \cdot 3 = \frac{15}{4}$.

249. Yes, if f is linear, then it is always true that $f(0) = 0$. You can prove this by assuming the contrary: Assume $f(0) \neq 0$. See if you can arrive at a contradiction.

250. Since h is linear, then $h(3) = h(3 \cdot \frac{6}{6}) = h(\frac{3}{6} \cdot 6) = h(\frac{1}{2} \cdot 6) = \frac{1}{2}h(6) = \frac{a+b}{2}$.

251. The line $y = 3x$ is the set of all vectors in $\mathbb{R}^2$ that are of the form $(x, 3x) = x(1, 3)$, which means $(1, 3)$ spans the line $y = 3x$. A single vector is independent, and so a basis for $y = 3x$ is the set $\{(1, 3)\}$.

252. The set of all vectors orthogonal to $(1, 1, 1)$ is found knowing that if (a, b, c) is orthogonal to the given vector, then $(a, b, c) \cdot (1, 1, 1) = 0$; that is, $a + b + c = 0$, so $c = -a - b$, and U can be described as the set of all vectors of the form $(a, b, -a - b)$. All vectors of this form can be written $(a, b, -a - b) = (a, 0, -a) + (0, b, -b) = a(1, 0, -1) + b(0, 1, -1)$, so the vectors $(1, 0, -1)$ and $(0, 1, -1)$ span U. Since they are not parallel to each other, the vectors are linearly independent. A basis for U is the set $\{(1, 0, -1), (0, 1, -1)\}$. The dimension of U is 2, which means U is a plane in $\mathbb{R}^3$.

253. Be sure the proof that V is a subspace of $\mathbb{R}^4$ matches the standards and expectations of writing as demonstrated in examples 36 and 37 in the text. To find a basis for V, proceed to show that all vectors in this set are of the form $(r, s, r + s, r - s) = (r, 0, r, r) + (0, s, s, -s) = r(1, 0, 1, 1) + s(0, 1, 1, -1)$, which shows that a spanning set for V is $\{(1, 0, 1, 1), (0, 1, 1, -1)\}$. Show that the only solution to $c_1(1, 0, 1, 1) + c_2(0, 1, 1, -1) = (0, 0, 0, 0)$ is $c_1 = 0$ and $c_2 = 0$, which proves that the spanning set is linearly independent, thus forming a basis for V. The dimension of V is 2, so V is a plane in $\mathbb{R}^4$.

254. Do the necessary work to show that the vectors span $\mathbb{R}^2$ and are linearly independent (see example 38; yes, the writer of this text is very much aware that it is the exact same set of vectors, an intentional move).

255. Use example 38 as a guide to accomplish this.

256. *preliminary thinking: to show the set is linearly independent, use definition 14 (see appendix C) along with what is given.* **Proof:** Since $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for V, then the set is a linearly independent spanning set for V, so no vectors in the set can be written as a linear combination of other vectors in the set; therefore, $\mathbf{v_1}$ cannot be written in terms of $\mathbf{v_2}$, so the removal of a single vector from the set yields a set that is linearly independent.

257. Read definition 14 and consider the linear combination $a\mathbf{0} + b\mathbf{v} = \mathbf{0}$. While $b = 0$ must be true, a can be any real number.

258. If $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set, then it is not possible to find nonzero scalars c_1, c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$; that is, $c_1\mathbf{u} + c_2\mathbf{v} \neq \mathbf{0}$ if c_1 and c_2 are nonzero scalars, and so $\mathbf{u} \neq -\frac{c_2}{c_1}\mathbf{v}$.

260. Let $\mathbf{x} = \mathbf{u} - \mathbf{v}, \mathbf{y} = \mathbf{v} - \mathbf{w}$, and $\mathbf{z} = \mathbf{w} - \mathbf{u}$. By observation, $\mathbf{x} = (-1)\mathbf{y} + (-1)\mathbf{z}$, showing that a vector in the set is a linear combination of the other vectors in the set; therefore, the set is linearly dependent.

261. The matrix in rref after four row operations reveals that x_3, x_4, and x_5 are free variables, and the following information can be read from this matrix:

$$x_1 = 4x_3 - 2x_4$$
$$x_2 = 3x_3 - x_5$$

This leads to the solution $\mathbf{x} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

262. This is an annoying problem by now, yes? You can see the basis vectors in the solution, and the dimension of the subspace is 3. Make sure your writing matches that of the examples.

263. The solution set is all scalar multiples of $(2, 3, 1)$, a one-dimensional subspace of $\mathbb{R}^3$.

265. There are many different answers possible here: The easiest vectors to go to for examples would be the three standard basis vectors for $\mathbb{R}^3$. Because they form a basis for the vector space, they are linearly independent (none can be written as a linear combination of the others), and they span $\mathbb{R}^3$, so remove a single vector from the set, and you will have two linearly independent vectors that do not span $\mathbb{R}^3$.

266. Here is one example: $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 3)\}$. These vectors span $\mathbb{R}^3$ but are not linearly independent.

267. This would be the y-axis.

268. This would be the yz-plane.

269. $\mathbb{R}^2$ is the set of all vectors of the form (a, b); that is, all vectors with two components, whereas $\mathbb{R}^3$ is the set of all vectors with three components. In order for a set of vectors to be a subspace of $\mathbb{R}^3$, the set must consist of vectors with three components; that is, it must be a subset of $\mathbb{R}^3$. The set of all vectors of the form $(a, b, 0)$ *is* a subspace of $\mathbb{R}^3$, however. (In mathematics, it is all about the precision in the definitions!)

270. A basis for this set is $\{(1, 0, -3), (0, 1, -2)\}$.

271.

461. $y = \frac{4}{3}$

462. $y = \frac{16}{7} + \frac{9x}{14}$

463. $y = 2 - \frac{x}{4}$

464. $y = \frac{11}{7} + \frac{19x}{35}$

465. $y = -\frac{5}{4} + \frac{3x}{4} + \frac{5x^2}{4}$

466. $\{(1, 0, 1), (0, 1, 0)\}$

467. The span of the columns of A is all combinations of $(1,1,1)$ and $(1,0,1)$, so consider $r(1,1,1)+s(1,0,1)$, where $r,s \in \mathbb{R}$; that is, all vectors of the form $(r+s,r,r+s)$, which is of the same form as all vectors of the form (a,b,a). (In words, the column space is the set of all vectors in $\mathbb{R}^3$ whose first and third components are the same.)

468. A course in calculus teaches you that a plane passing through the origin in $\mathbb{R}^3$ with normal vector (a,b,c) is $ax+by+cz=0$.

470. $y = 0.3 - 0.3x + 0.25x^2$ is the parabola of best fit. The least squares solution to the system is the vector $\mathbf{v} = (0.3, -0.3, 0.25)$. The product $A\mathbf{v} = (1.9, 0.3, 0.7, 3.1)$. The vector $\mathbf{b} = (2,0,1,3)$. How would picture the distance between the points $(1.9, 0.3, 0.7, 3.1)$ and $(2,0,1,3)$? This distance would be 0.4472135955 (the length of the vector $\mathbf{e}$).

471. $x_1 = \frac{17}{95}, x_2 = \frac{143}{285}$

472. The solution to the system, $(-4,2)$.

473. $y = x$

474. **Proof:** Let A be an $m \times n$ matrix with linearly independent columns. The size of A means that there are n column vectors from $\mathbb{R}^m$, and since these column vectors are linearly independent, then the only linear combination of the columns that results in the zero vector in $\mathbb{R}^m$ is the zero vector in $\mathbb{R}^n$; that is, the system $A\mathbf{x} = \mathbf{0}$ has one and only one solution $\mathbf{x} = \mathbf{0}$; that is, the null space of the matrix A contains one and only one vector: the zero vector in $\mathbb{R}^n$. The matrix A^TA is a square matrix of order n (its size is $n \times n$). The null space of A^TA are all vectors $\mathbf{x}$ such that $A^TA\mathbf{x} = \mathbf{0}$, which means that $A\mathbf{x}$ is a left null space vector; that is, $A\mathbf{x} \in \text{null}(A^T)$. But $A\mathbf{x}$ is also in the column space of A, which means $A\mathbf{x} = \mathbf{0}$, the only element that is common to both the column and left null spaces of A; hence, the only solution to $A^TA\mathbf{x} = \mathbf{0}$ is the trivial solution, which means A^TA is invertible.

476. Use the fact that $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$. The kernel of any matrix M is $\text{null}(M)$ and the range is $\text{col}(M)$, and apply that to both matrices A and A^T.

477. Since A is a 5×4 matrix, then $A : \mathbb{R}^4 \to \mathbb{R}^5$ and $A^T : \mathbb{R}^5 \to \mathbb{R}^4$.

480.

498. $\lambda_1 = -1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

499. $\lambda_1 = -3, \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$ and $\lambda_2 = 2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

500. As long as $a \neq b$, then $\lambda_1 = a, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = b, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; If $a = b$ then what happens?

Appendix B

Descriptions of Common Vector Spaces

- $\mathbb{R}^2$ is the set of all vectors of the form (a, b), where a and b are any scalars. $\mathbb{R}^2$ is also known as *the two-dimensional Euclidean plane* or *the Cartesian plane*, and the xy-plane. All lines through the origin are subspaces of $\mathbb{R}^2$. The x-axis is a subspace of $\mathbb{R}^2$, but $\mathbb{R}$ is not a subspace of $\mathbb{R}^2$.
- $\mathbb{R}^3$ is the set of all vectors of the form (a, b, c), where a, b, and c are any scalars. $\mathbb{R}^3$ is a set of vectors with three components. $\mathbb{R}^2$ is a set of vectors with two components; thus, $\mathbb{R}^2$ is not subset, nor a subspace, of $\mathbb{R}^3$, but you can say the xy-plane is a subspace of $\mathbb{R}^3$ as long as you are clear in your expression of the plane as vectors in $\mathbb{R}^3$; that is, the set of all vectors of the form $(x, y, 0)$, where x and y are any scalars (the set of all vectors in $\mathbb{R}^3$ whose third component is zero) is a subspace of $\mathbb{R}^3$.
- More information coming here...

Appendix C

Definitions

DEFINITION 1: Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors in $\mathbb{R}^n$ and let k be any scalar. **Vector addition** is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n),$$

while **scalar multiplication** is

$$k\mathbf{v} = (kv_1, kv_2, ..., kv_n).$$

DEFINITION 2: A matrix is said to be in **reduced row echelon form** if it has the following four properties:

(1) If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **leading one**.

(2) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(3) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

(4) Each column that contains a leading 1 has zeros everywhere else in that column.[1]

DEFINITION 3: Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors from the same space $\mathbb{R}^n$. The **dot product**, also known as the **inner product** or the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined as the real number

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

DEFINITION 4: If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **matrix product** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B and, considering these as vectors, compute the dot product of row i and column j.

DEFINITION 5: A square matrix of order n with 1's on the main diagonal and zeros everywhere else is called an **identity matrix**, and is denoted by I_n or I.

[1] A matrix satisfying the first three properties is said to be in *row echelon form*.

DEFINITION 6: Let A be a square matrix of order n (Let A be an $n \times n$ matrix.) If there exists an $n \times n$ matrix B such that $AB = BA = I_n$, then A is said to be an **invertible** matrix or a **nonsingular** matrix, and B is the **inverse** of A. In this event, we write $B = A^{-1}$. In other words, if A is invertible, we denote its inverse by A^{-1}. If no such inverse exists, then A is said to be **singular**.

DEFINITION 7: A **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is the expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, where $c_1, c_2, ..., c_n$ are scalars.

DEFINITION 8: A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ of vectors in a space V is said to **span** V if every vector $\mathbf{v} \in V$ can be written as a linear combination of the vectors in the set; that is, there exist scalars $c_1, c_2, ..., c_n$ that can be found such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u} + \cdots c_n\mathbf{u}_n$.

DEFINITION 9: The **row space** of a matrix A of size $m \times n$ is the span of the row vectors of the matrix; that is, the set of all linear combinations of the rows of A. The row space of A will be denoted by $\text{row}(A)$.

DEFINITION 10: The **column space** of a matrix A of size $m \times n$ is the span of the column vectors of the matrix; that is, the set of all linear combinations of the columns of A. The column space of A will be denoted by $\text{col}(A)$.

DEFINITION 11: Let V be a set of vectors. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be any two vectors in V, and let k be any scalar. If the sum $\mathbf{v}_1 + \mathbf{v}_2$ is also in V, then we say V is **closed under vector addition**. If the product $k\mathbf{v}_1$ is also in V, then we say V is **closed under scalar multiplication**.

DEFINITION 12: A **subset** of V is any collection of vectors in V. If the set U is any collection of elements of a set V, then U is a subset of V, denoted by $U \subseteq V$.

DEFINITION 13: A set U is a **subspace** of a Euclidean vector space $\mathbb{R}^n$ if (1) U is a nonempty subset of $\mathbb{R}^n$, (2) if U is closed under vector addition, and (3) if U is closed under scalar multiplication.

DEFINITION 14: If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is a set of two or more vectors in a vector space V, then S is said to be a **linearly independent set** if no vector in S can be expressed as a linear combination of the other vectors in the set. A set that is not linearly independent is said to be **linearly dependent**. If S contains one and only one vector, we will agree that the set is linearly independent if and only if that vector is nonzero.

DEFINITION 15: A set of vectors is a **basis** for a space if (1) the vectors span the space, and (2) are linearly independent. The number of vectors in a basis of a space is called the **dimension** of the space, and this number is unique.

DEFINITION 16: A **linear transformation** is a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(k\mathbf{u}) = kT(\mathbf{u})$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars k.

DEFINITION 17: Let $T : V \to W$ be a linear transformation. The **kernel** of T is the set of all vectors in the domain V that are mapped to the zero vector in the codomain W. The kernel of T is denoted by $\ker(T)$. Described in formal set notation, $\ker(T)=\{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$, where $\mathbf{0}_W$ represents the zero vector in the codomain W.

DEFINITION 18: Let $T : V \to W$ be a linear transformation. The set of vectors in W that are the images of vectors in V is called the **range** of T. The range will be denoted by $\text{range}(T)$.

DEFINITION 19: The **null space** of a matrix A of size $m \times n$ is the set of all vectors in $\mathbb{R}^n$ that are mapped to the zero vector in $\mathbb{R}^m$. The null space of A will be denoted by $\text{null}(A)$.

DEFINITION 20: The **left null space** of a matrix A of size $m \times n$ is the null space of the $n \times m$ matrix A^T. The left null space of A will be denoted by $\text{null}(A^T)$.

DEFINITION 21: Let W be a subspace of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$.

DEFINITION 22: Let A be any matrix. The **rank** of A is the number of linearly independent columns of A; that is, the rank is the dimension of the column space of A.

DEFINITION 23: Let A be any matrix. The **nullity** of A is the dimension of the null space of A.

DEFINITION 24: The Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations and n unknowns, a vector $\mathbf{x}$ in $\mathbb{R}^n$ that minimizes $|A\mathbf{x} - \mathbf{b}|$ with respect to the dot product (Euclidean inner product) on $\mathbb{R}^m$, if it exists, is known as a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

DEFINITION 25: If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is, if $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an **eigenvalue** of A, and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

DEFINITION 26: The **trace** of a square matrix A, denoted by $\text{tr}(A)$, is the sum of the entries on the main diagonal of the matrix.

DEFINITION 27: If A is an $n \times n$ matrix and λ is an eigenvalue of A corresponding to the eigenvector $\mathbf{x}$, then the set of all solution vectors $\mathbf{x}$ such that $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ form a vector space called the **eigenspace of A corresponding to** λ.

Appendix D

Theorems

THEOREM 1: All vectors in $\mathbb{R}^2$ can be written as a linear combination of the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

PROOF: All vectors in $\mathbb{R}^2$ are of the form (a, b), where $a, b \in \mathbb{R}$. By vector addition, $(a, b) = (a, 0) + (0, b)$. By scalar multiplication, $(a, b) = a(1, 0) + b(0, 1)$, which means we can write $(a, b) = a\mathbf{i} + b\mathbf{j}$.

THEOREM 2: Every matrix defines a linear transformation.

PROOF: Let A be a matrix of size $m \times n$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation induced by A, which means $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$ and let k be any scalar.

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= A(\mathbf{u} + \mathbf{v}) \\ &= A\mathbf{u} + A\mathbf{v} \\ &= T(\mathbf{u}) + T(\mathbf{v}), \end{aligned}$$

so T preserves addition.

$$\begin{aligned} T(k\mathbf{u}) &= A(k\mathbf{u}) \\ &= kA\mathbf{u} \\ &= kT(\mathbf{u}), \end{aligned}$$

so T preserves scalar multiplication. Since it preserves both addition and scalar multiplication, T is a linear transformation.

THEOREM 3: The columns of a matrix are the vectors to which basis vectors are mapped. More precisely, an $m \times n$ matrix A determines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. The standard basis of $\mathbb{R}^n$ is $\{\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}\}$. If $A = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_n} \end{bmatrix}$, where $\mathbf{c_i}$ is the ith column vector, then $T(\mathbf{e_i}) = \mathbf{c_i}$. **PROOF:**Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$. In column vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Given the standard basis of $\mathbb{R}^n$, we know that $\mathbf{x}$ can be written in terms of these vectors:

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}$$

For any vector $\mathbf{x}$, $T(\mathbf{x}) = A\mathbf{x}$, and

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}) && \text{(D.1)} \\ &= x_1T(\mathbf{e_1}) + x_2T(\mathbf{e_2}) + \cdots + x_nT(\mathbf{e_n}), && \text{(D.2)} \end{aligned}$$

since a linear transformation preserves linear combinations. We also have that

$$T(\mathbf{x}) = A\mathbf{x} \tag{D.3}$$

$$= \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{D.4}$$

$$= \mathbf{c_1}x_1 + \mathbf{c_2}x_2 + \cdots + \mathbf{c_n}x_n \tag{D.5}$$

$$= x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n}. \tag{D.6}$$

By 16.10 and 16.14,

$$x_1T(\mathbf{e_1}) + x_2T(\mathbf{e_2}) + \cdots + x_nT(\mathbf{e_n}),$$

so $T(\mathbf{e_1}) = \mathbf{c_1}, T(\mathbf{e_2}) = \mathbf{c_2}$, ..., and $T(\mathbf{e_n}) = \mathbf{c_n}$.

THEOREM 4: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$. The range of T is the span of the column vectors of A.

PROOF: Let $\mathbf{w}$ be any vector in the range of T, which means there exists a vector $\mathbf{v}$ in the domain of T such that $T(\mathbf{v}) = \mathbf{w}$. Since $\mathbf{v} \in \mathbb{R}^n$, then it can be written in terms of the standard basis vectors of $\mathbb{R}^n$, so that $T(\mathbf{v}) = \mathbf{w}$ can be written as

$$T(c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n}) = \mathbf{w}.$$

The linearity of T allows us to write this as

$$c_1T(\mathbf{e_1}) + c_2T(\mathbf{e_2}) + \cdots + c_nT(\mathbf{e_n}) = \mathbf{w},$$

where we see that the vector in the range $\mathbf{w}$ is written as a linear combination of the images of the basis vectors of the domain; that is, $\mathbf{w}$ is a linear combination of the column vectors of the matrix A. By definition of *span*, the range of T is the span of the column vectors of A.

THEOREM 5: Two row equivalent matrices have the same row space.

PROOF: If B is row equivalent to A, then B can be reached by performing a finite number of row operations on A. Each row operation can be represented by an invertible elementary matrix, so in terms of matrix multiplication, suppose $E_1, E_2, ..., E_k$ represent the row operations that change A to B; that is, $E_k...E_2E_1A = B$. The row vectors of B are, thus, linear combinations of the row vectors of A. Consequently, the row space of B must be a subspace of the row space of A. Since the row operations are invertible, we have, in matrix multiplication form, that $A = E_1^{-1}E_2^{-1}...E_k^{-1}B$, and so A is equivalent to B, and, using the same reasoning, the row space of A must be a subspace of the row space of B. Therefore, row space of A is the same as the row space of B.

THEOREM 6: Let M be an $m \times n$ matrix. The row space of M and the null space of M are orthogonal complements. This is sometimes written as $\text{row}(M) \perp \text{null}(M)$.

PROOF: Let $\mathbf{x}$ be any vector in the null space of M. By the definition of null space, this means $M\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in the codomain $\mathbb{R}^m$. The matrix product

$M\mathbf{x}$ can be carried out by dotting every row vector of the matrix M with the null space vector. Each component of the zero vector is a scalar zero; that is, every row dotted with any null space vector is zero, which means every row vector is orthogonal to any null space vector. Since the rows span the row space, then all vectors in the row space are orthogonal to all vectors in the null space; therefore, row(M) and null(M) are orthogonal complements .