MATH 238 Homework 3

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Homework for Section 2.3: 50 points

It is usually easier to do the homework on paper and then type the solutions in the latex document compiling frequently to catch the errors early! Each of the q ordinary differential equation in first-order linear and will be referred to as a equation.

1. (2 points) On what rectangular regions does the equation below possess a unique solution?

$$a_1(x)\frac{dy}{dx} + a_0(x)y = q(x)$$

SOLUTION: A unique solution exists when both $\frac{a_0(x)}{a_1(x)}$ and $\frac{q(x)}{a_1(x)}$ are both continuous. In a rectangular region, it would be $\boxed{R = [a, b] \times [c, d] \text{ where } a_1(x) \neq 0 \text{ and } a_1(x),\ a_0(x),\ q(x) \text{ are continuous.}}$[1]

2. (2 points) On what rectangular regions does the equation below possess a unique solution?

$$\frac{dy}{dx} + p(x)y = q(x)$$

SOLUTION: A unique solution exists in a standard first order linear equation when $p(x)$ and $q(x)$ are both continuous. In a rectangular region, it would be $\boxed{R = [a, b] \times [c, d] \text{ where both } p(x),\ q(x) \text{ are continuous.}}$

[1] If $a_1(x) \neq 0$ and $a_1(x),\ a_0(x),\ q(x)$ are continous, then $\frac{a_0(x)}{a_1(x)}$ and $\frac{q(x)}{a_1(x)}$ are also continous.

3. (6 points) Solve the equation below

$$\frac{dP}{dt} + 5tP = P + 2t - 2$$

SOLUTION: To approach this problem, we must first put it into the standard first order linear equation $y' + P(x)y = H(x)$. After that, we will find the integrating factor $\mu(x)$.

$$\frac{dP}{dt} + 5tP = P + 2t - 2$$
$$\frac{dP}{dt} + (5t-1)P = 2t - 2$$
$$P(t) = 5t - 1, \quad H(t) = 2t - 2$$
$$\mu(t) = e^{\int P(t)dt}$$
$$\mu(t) = e^{\int 5t - 1 dt}$$
$$\mu(t) = e^{\frac{5t^2}{2} - t}$$
$$\mu(t) = e^{\frac{5}{2}\left(t^2 - \frac{2}{5}t + \frac{1}{25}\right) - \frac{1}{10}}$$

We don't care about the constant $e^{-\frac{1}{10}}$

$$\Longrightarrow \mu(t) = e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}$$
$$\mu(t)\left(\frac{dP}{dt} + (5t-1)P = 2t - 2\right)$$
$$\Longrightarrow \frac{d}{dt}(\mu(t)P) = \mu(t)(2t-2)$$
$$\int \frac{d}{dt}(\mu(t)P)dt = \int \mu(t)(2t-2)dt$$
$$\mu(t)P = \int e^{\frac{5t^2}{2} - t}(2t-2)dt$$

$$\mu(t)P = \int e^{\frac{5t^2}{2} - t}(2t-2)dt$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}(2t-2)dt$$
$$u = t - \frac{1}{5} \Longrightarrow t = u + \frac{1}{5}$$
$$dt = du, \quad (2t-2) = (2u - \frac{8}{5})$$

$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int \left(2u - \frac{8}{5}\right) e^{\frac{5}{2}u^2} du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \left(\int 2ue^{\frac{5}{2}u^2} du - \frac{8}{5}\int e^{\frac{5}{2}u^2} du\right)$$
$$v = u^2 \Longrightarrow dv = 2u\ du$$

$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int e^{\frac{5}{2}v} dv - \frac{8}{5}\int e^{\frac{5}{2}u^2} du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{8}{5}\int e^{\left(\sqrt{\frac{5}{2}}u\right)^2} du$$
$$m = \sqrt{\frac{5}{2}}u \Longrightarrow \sqrt{\frac{2}{5}}dm = du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{8}{5}\int e^{m^2}\sqrt{\frac{2}{5}}dm$$

$$\text{Remark: } \operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt$$
$$C \cdot \operatorname{erfi}(x) = C\frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt = \int e^{m^2} du$$
$$C = \frac{\sqrt{\pi}}{2}$$

$$\Longrightarrow e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{\sqrt{\pi}}{2}\frac{8}{5}\sqrt{\frac{2}{5}}\operatorname{erfi}(m) + C$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}\left(t - \frac{1}{5}\right)^2} - \frac{4}{5}\sqrt{\frac{2\pi}{5}}\operatorname{erfi}\left(\sqrt{\frac{5}{2}}\left(t^2 - \frac{1}{5}\right)\right) + C$$
$$\boxed{P = \frac{2}{5} - \frac{4}{5}\sqrt{\frac{2\pi}{5}}\operatorname{erfi}\left(\sqrt{\frac{5}{2}}\left(t^2 - \frac{1}{5}\right)\right)e^{-\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} + Ce^{-\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}}$$

4. (7 points) Solve the equation below

$$2L\frac{di}{dt} + 3Ri = E, i(0) = i_0$$

where L, R and E are constants.

SOLUTION:

$$\begin{aligned}
&2L\frac{di}{dt} + 3Ri = E, i(0) = i_0\\
&\frac{di}{dt} + \frac{3R}{2L}i = \frac{E}{2L}\\
&\mu(t) = e^{\frac{3R}{2L}t}
\end{aligned}$$

$$\begin{aligned}
&\mu(t)\left(\frac{di}{dt} + \frac{3R}{2L}i = \frac{E}{2L}\right)\\
\Longrightarrow &\frac{d}{dt}(e^{\frac{3R}{2L}t}i) = e^{\frac{3R}{2L}t}\frac{E}{2L}\\
&e^{\frac{3R}{2L}t}i = \int e^{\frac{3R}{2L}t}\frac{E}{2L}dt\\
&e^{\frac{3R}{2L}t}i = \frac{\cancel{2L}}{3R}e^{\frac{3R}{2L}t}\frac{E}{\cancel{2L}} + C\\
&i(t) = \frac{E}{3R} + Ce^{-\frac{3R}{2L}t}
\end{aligned}$$

$$\begin{aligned}
&i(0) = i_0 = \frac{E}{3R} + C\\
&C = i_0 - \frac{E}{3R}\\
\Longrightarrow &\boxed{i(t) = \frac{E}{3R} + \left(i_0 - \frac{E}{3R}\right)e^{-\frac{3R}{2L}t}}
\end{aligned}$$

5. (7 points) Solve the equation below

$$\cos^2 x \sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 4$$

SOLUTION:

$$\begin{aligned}
&\cos^2 x \sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 4\\
&\frac{dy}{dx} + (\cot x)y = 4\sec^2 x\csc x\\
&\mu(x) = e^{\ln|\sin x|} = \sin x
\end{aligned}$$

$$\begin{aligned}
&\mu(x)\left(\frac{dy}{dx} + (\cot x)y = 4\sec^2 x\csc x\right)\\
\Longrightarrow &\frac{d}{dx}(e^{\sin x}y) = \cancel{\sin x}4\sec^2 x\cancel{\csc x}\\
&y\sin x = \int \sec^2 x dx\\
&y\sin x = \tan x + C\\
\Longrightarrow &\boxed{y(x) = 4\sec x + C\csc x}
\end{aligned}$$

6. (10 points) Solve the equation below

$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$$

SOLUTION:

$$\begin{aligned}
&(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}\\
&\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}\\
&\mu(x) = e^{\int \frac{x+2}{x+1}dx}\\
&\mu(x) = e^{\int \frac{x+1+1}{x+1}dx}\\
&\mu(x) = e^{\int 1+\frac{1}{x+1}dx}\\
&\mu(x) = e^{x+\ln|x+1|} = (x+1)e^{x}
\end{aligned}$$

$$\begin{aligned}
&\mu(x)\left(\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}\right)\\
&((x+1)e^{x})\left(\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}\right)\\
\Longrightarrow &\frac{d}{dx}\left((x+1)e^{x}y\right) = \cancel{(x+1)e^{x}}\frac{2x\cancel{e^{-x}}}{\cancel{x+1}}\\
&(x+1)e^{x}y = \int 2x dx\\
\Longrightarrow &\boxed{y = \frac{x^2+C}{(x+1)e^{x}}}
\end{aligned}$$

7. (16 points) Solve the equation below

$$\frac{dy}{dx} + 6xy = f(x)$$

where

$$f(x) = \begin{cases} x^2 & x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$$

Graph the right side and one of the the solutions on separate graphs.

SOLUTION:

$$\frac{dy}{dx} + 6xy = x^2,\ x < 1$$
$$\mu(x) = e^{3x^2}$$
$$\frac{d}{dx}(e^{3x^2}y) = e^{3x^2}x^2$$
$$e^{3x^2}y = \int e^{3x^2}x^2 dx$$
$$u = x, dv = xe^{3x^2}dx$$
$$du = dx, v = \frac{1}{6}e^{3x^2}$$
$$e^{3x^2}y = \frac{1}{6}xe^{3x^2} - \int \frac{1}{6}e^{3x^2}dx,\ u = \sqrt{3}x,\ \frac{du}{\sqrt{3}} = dx$$
$$e^{3x^2}y = \frac{1}{6}xe^{3x^2} - \frac{1}{12}\sqrt{\frac{\pi}{3}}\text{erfi}(\sqrt{3}x)$$
$$\Longrightarrow \boxed{y(x) = \frac{x}{6} - \frac{\sqrt{\pi}\text{erfi}(\sqrt{3}x)}{12\sqrt{3}e^{3x^2}} + C_2e^{-3x^2}, \quad x < 1}$$

$$\frac{dy}{dx} + 6xy = 2x - 1,\ x \geq 1$$
$$\mu(x) = e^{3x^2}$$
$$\frac{d}{dx}(e^{3x^2}y) = e^{3x^2}(2x-1)$$
$$e^{3x^2}y = \int e^{3x^2}(2x-1)dx$$
$$e^{3x^2}y = \int 2xe^{3x^2} - e^{3x^2}dx$$
$$e^{3x^2}y = \frac{1}{3}e^{3x^2} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right) + C$$
$$\Longrightarrow \boxed{y(x) = \frac{1}{3} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right)e^{-3x^2} + C_1e^{-3x^2}, \quad x \geq 1}$$

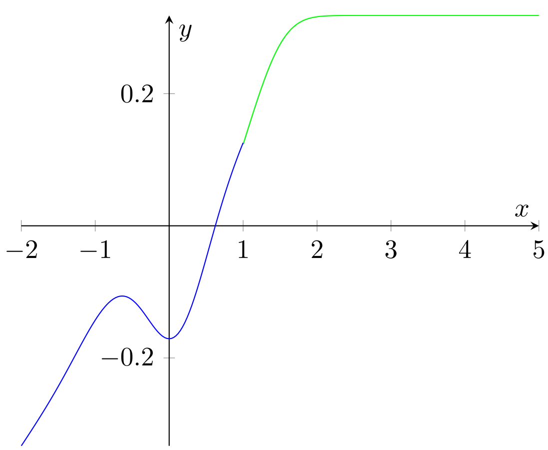

Figure 1: Graph of both solution curves when $C_1 = 0$ $C_2 = -0.170920506876$ using numeric methods.

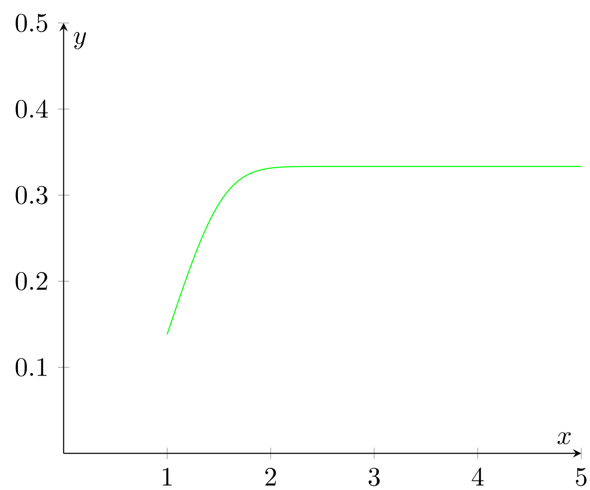

Figure 2: Right Graph of solution curve when $C_1 = 0$. We can use this to find $y(1) \approx 0.12312177618$

$$\boxed{y(x) = \begin{cases} \frac{x}{6} - \frac{\sqrt{\pi}\text{erfi}(\sqrt{3}x)}{12\sqrt{3}e^{3x^2}} + C_2e^{-3x^2} & x < 1 \\ \frac{1}{3} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right)e^{-3x^2} + C_1e^{-3x^2} & x \geq 1 \end{cases}}$$