MATH 238 Homework 5

Brendan Tea

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Contents

Homework for Section 4.2: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, **10**, 11, **12**, 13, **14**, 15, and 16, the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx, \quad y'' + P(x)y' + Q(x)y = 0 \tag{5}$$

Problem 4.2.010

Given: $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
Find: $\;y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned} &x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2 \\ &y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0 \\ \Longrightarrow &P(x) = \frac{2}{x} \end{aligned}$$

Plugging into (5) and evaluating.

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
y_2(x) &= \left(x^2\right) \int \frac{e^{-\int \left(\frac{2}{x}\right)dx}}{\left(x^2\right)^2} dx \\
y_2(x) &= x^2 \int \frac{e^{-2\ln|x|}}{x^4} dx \\
y_2(x) &= x^2 \int \frac{\left(\frac{1}{x^2}\right)}{x^4} dx \\
y_2(x) &= x^2 \int x^{-6} dx
\end{aligned}
$$

Integrating continued.

$$
\begin{aligned}
y_2(x) &= x^2 \left(-\frac{x^{-5}}{5}\right) \\
y_2(x) &= -\frac{1}{5x^3}
\end{aligned}
$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$
\Longrightarrow \boxed{y_2(x) = \frac{1}{x^3}, \quad y_g = c_1 e^x + \frac{c_2}{x^3}}
$$

Problem 4.2.012

Given: $4x^2y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x$

Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$
\begin{aligned}
&4x^2y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x \\
&4x^2y'' + 0y' + y = 0 \\
&y'' + 0y' + \frac{1}{4x^2} y = 0 \\
\Longrightarrow &P(x) = 0
\end{aligned}
$$

Plugging into (5) and evaluating.

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^{-\int (0)dx}}{\left(x^{\frac{1}{2}} \ln x\right)^2} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^0}{x(\ln x)^2} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{1}{x(\ln x)^2} dx
\end{aligned}
$$

Using u-substitution.

$$\begin{aligned}
u &= \ln x \Longrightarrow du = \frac{1}{x}dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int u^{-2} du \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \left(-u^{-1}\right) \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \left(-(\ln x)^{-1}\right) \\
y_2(x) &= \left(x^{\frac{1}{2}} \cancel{\ln x}\right) \left(\frac{-1}{\cancel{\ln x}}\right) \\
y_2(x) &= -\sqrt{x}
\end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\Longrightarrow \boxed{y_2(x) = \sqrt{x}, \quad y_g = c_1\sqrt{x}\ln x + c_2\sqrt{x}}$$

Problem 4.2.014

Given: $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}
& x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x) \\
& y'' + \frac{-3}{x}y' + \frac{5}{x^2}y = 0 \\
\Longrightarrow & P(x) = -\frac{3}{x}
\end{aligned}$$

plug into (5) and evaluating.

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) \int \frac{e^{\int \left(\frac{3}{x}\right)dx}}{\left(x^2 \cos(\ln x)\right)^2} dx \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) \int \frac{x^3}{x^4 \cos^2(\ln x)} dx \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) \int \frac{1}{x \cos^2(\ln x)} dx
\end{aligned}$$

Using u-substitution.

$$
\begin{aligned}
u &= \ln x \Longrightarrow du = \frac{1}{x}dx \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) \int \frac{1}{\cos^2(u)} du \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) \int \sec^2(u) du \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) (\tan(u)) \\
y_2(x) &= \left(x^2 \cos(\ln x)\right) (\tan(\ln x)) \\
y_2(x) &= x^2 \sin(\ln x)
\end{aligned}
$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\Longrightarrow \boxed{y_2(x) = x^2 \sin(\ln(x)), \quad y_g = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln(x))}$$

Problem 4.2.020

In Problems 17, 18, 19, and **20** the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction (5) of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

Given: $y'' - 4y' + 3y = x; \quad y_1 = e^x$
Find: $y_2(x)$ and $y_p(x)$

SOLUTION:

$$\Longrightarrow \boxed{y_2(x) = e^{3x}, \quad y_p(x) = \left(\frac{x}{3} + \frac{4}{9}\right), \quad y_g = c_1 e^x + c_2 e^{3x} + \left(\frac{x}{3} + \frac{4}{9}\right)}$$

Homework for Section 4.3: 50 points