

MATH 238 Homework 5

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Homework for Section 4.2: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, **10**, 11, **12**, 13, **14**, 15, and 16, the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx, \quad y'' + P(x)y' + Q(x)y = 0 \quad (5)$$

Problem 4.2.010

Given: $x^2 y'' + 2xy' - 6y = 0$; $y_1 = x^2$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}x^2 y'' + 2xy' - 6y &= 0; \quad y_1 = x^2 \\y'' + \frac{2}{x}y' - \frac{6}{x^2}y &= 0 \\ \implies P(x) &= \frac{2}{x}\end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\y_2(x) &= (x^2) \int \frac{e^{-\int (\frac{2}{x})dx}}{(x^2)^2} dx \\y_2(x) &= x^2 \int \frac{e^{-2 \ln |x|}}{x^4} dx \\y_2(x) &= x^2 \int \frac{(\frac{1}{x^2})}{x^4} dx \\y_2(x) &= x^2 \int x^{-6} dx\end{aligned}$$

Integrating continued.

$$\begin{aligned}y_2(x) &= x^2 \left(-\frac{x^{-5}}{5} \right) \\y_2(x) &= -\frac{1}{5x^3}\end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \frac{1}{x^3}, \quad y_g = c_1 e^x + \frac{c_2}{x^3}}$$

Problem 4.2.012

Given: $4x^2 y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}4x^2 y'' + y &= 0; \quad y_1 = x^{\frac{1}{2}} \ln x \\4x^2 y'' + 0y' + y &= 0 \\y'' + 0y' + \frac{1}{4x^2}y &= 0 \\ \implies P(x) &= 0\end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^{-\int (0)dx}}{\left(x^{\frac{1}{2}} \ln x\right)^2} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^0}{x(\ln x)^2} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{1}{x(\ln x)^2} dx
 \end{aligned}$$

Using u-substitution.

$$\begin{aligned}
 u = \ln x &\implies du = \frac{1}{x} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int u^{-2} du \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) (-u^{-1}) \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) (-(\ln x)^{-1}) \\
 y_2(x) &= \left(x^{\frac{1}{2}} \cancel{\ln x}\right) \left(\frac{-1}{\cancel{\ln x}}\right) \\
 y_2(x) &= -\sqrt{x}
 \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \sqrt{x}, \quad y_g = c_1 \sqrt{x} \ln x + c_2 \sqrt{x}}$$

Problem 4.2.014

Given: $x^2 y'' - 3xy' + 5y = 0$; $y_1 = x^2 \cos(\ln x)$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}
 x^2 y'' - 3xy' + 5y &= 0; \quad y_1 = x^2 \cos(\ln x) \\
 y'' + \frac{-3}{x} y' + \frac{5}{x^2} y &= 0 \\
 \implies P(x) &= -\frac{3}{x}
 \end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{e^{\int (\frac{3}{x})dx}}{(x^2 \cos(\ln x))^2} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{x^3}{x^4 \cos^2(\ln x)} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{1}{x \cos^2(\ln x)} dx
 \end{aligned}$$

Using u-substitution.

$$\begin{aligned}
 u = \ln x &\implies du = \frac{1}{x} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{1}{\cos^2(u)} du \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \sec^2(u) du \\
 y_2(x) &= (x^2 \cos(\ln x)) (\tan(u)) \\
 y_2(x) &= (x^2 \cos(\ln x)) (\tan(\ln x)) \\
 y_2(x) &= x^2 \sin(\ln x)
 \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = x^2 \sin(\ln(x)), \quad y_g = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x)}$$

Problem 4.2.020

In Problems 17, 18, 19, and **20** the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction (5) of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

Given: $y'' - 4y' + 3y = x; \quad y_1 = e^x$

Find: $y_2(x)$ and $y_p(x)$

SOLUTION: Since this is a nonhomogeneous differential equation, we must use the idea of superposition. We will first use the reduction formula (5), identifying our $P(x)$, plugging in values and **setting the differential equation to zero**, to get our complimentary solution $y_p(x)$ Then, we will use the same process but **set the differential equation to x** .

Solving for $P(x)$.

$$\begin{aligned}
 y'' - 4y' + 3y &= x; \quad y_1 = e^x \\
 \implies P(x) &= -4
 \end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y_2(x) &= (e^x) \int \frac{e^{\int (4)dx}}{(e^x)^2} dx \\
 y_2(x) &= (e^x) \int \frac{e^{4x}}{e^{2x}} dx \\
 y_2(x) &= (e^x) \int e^{2x} dx \\
 y_2(x) &= (e^x) \left(\frac{e^{2x}}{2} \right) \\
 y_2(x) &= \frac{e^{3x}}{2}
 \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = e^{3x}}$$

To solve for our particular solution $y_p(x)$, we have to use the standard reduction of order process.

Assume $y_p = u(x)y_1$

$$\begin{aligned}
 \text{Let } y_p &= ue^x \\
 \text{Then } y_p' &= u'e^x + ue^x \\
 y_p'' &= u''e^x + 2u'e^x + ue^x
 \end{aligned}$$

Substituting these into the nonhomogeneous equation $y'' - 4y' + 3y = x$:

$$\begin{aligned}
 (u''e^x + 2u'e^x + ue^x) - 4(u'e^x + ue^x) + 3(ue^x) &= x \\
 u''e^x - 2u'e^x &= x \\
 u'' - 2u' &= xe^{-x}
 \end{aligned}$$

Let $w = u'$. This gives the first-order linear equation $w' - 2w = xe^{-x}$.

Using the integrating factor $\mu(x) = e^{\int -2dx} = e^{-2x}$:

$$\begin{aligned}
 e^{-2x}(w' - 2w) &= e^{-2x}(xe^{-x}) \\
 \frac{d}{dx}[we^{-2x}] &= xe^{-3x} \\
 we^{-2x} &= \int xe^{-3x} dx \\
 we^{-2x} &= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} \\
 u' = w &= -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x}
 \end{aligned}$$

Integrating to find $u(x)$:

$$\begin{aligned}
 \int u' dx &= \int \left(-\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} \right) dx \\
 u(x) &= \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}
 \end{aligned}$$

Multiplying by $y_1 = e^x$ to get the final particular solution:

$$y_p(x) = \left(\frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x} \right) e^x$$
$$\implies \boxed{y_p(x) = \frac{1}{3}x + \frac{4}{9}}$$

In whole, after combining all components, we arrive here with our general solution.

$$\implies \boxed{y_2(x) = e^{3x}, \quad y_p(x) = \left(\frac{x}{3} + \frac{4}{9} \right), \quad y_g = c_1e^x + c_2e^{3x} + \left(\frac{x}{3} + \frac{4}{9} \right)}$$

Homework for Section 4.3: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and **14** find the general solution $y_g(x)$ of the given second-order differential equation.

Problem 4.3.014

Given: $2y'' - 3y' + 4y = 0$

Find: $y_g(x)$

SOLUTION:

In Problems 15, 16, 17, 18, 19, 20, 21, 22, 23, **24**, 25, 26, 27, and 28 find the general solution of the given higher-order differential equation.

Problem 4.3.024

Given: $y^{(4)} - 2y'' + y = 0$

Find: $y_g(x)$

SOLUTION:

In Problems 29, 30, 31, **32**, 33, 34, 35, and 36 solve the given initial-value problem.

Problem 4.3.032

Given: $4y'' + 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$

Find: $y(x)$

SOLUTION: