

2.4 Notes

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February 5, 2026

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1 Normal Exact ODES

DEFINITION 1. An exact equation is defined as the following:

$$M(x, y)dx + N(x, y)dy = 0, \text{ where } M_y = N_x \quad (1)$$

It should be analogous to

$$f_x dx + f_y dy = dz = 0 \implies z = f(x, y) = c \quad (2)$$

c is a constant.

1.1 solving process

Here is the formal general process:

$$f_x = M, \quad f_y = N$$

$$\int \frac{\partial f}{\partial x} dx = \int M(x, y) dx$$

$$f(x, y) = \int M(x, y) dx + h(y), \quad \text{this is your general solution, now solve for } h(y)$$

$$\frac{\partial}{\partial y} \left(f(x, y) = \int M(x, y) dx + h(y) \right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y) dx + h'(y)$$

Recall that $f_y = N$

$$\implies N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + h'(y)$$

$$N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx = h'(y)$$

substitute it back into the solution

$$\int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right) dy = \int h'(y) dy = h(y)$$

$$f(x, y) = \int M(x, y) dx + h(y)$$

$$\implies \boxed{f(x, y) = \int M(x, y) dx + \int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right) dy = C}$$

This is not a formula but this is the general solving practice. Can be applied for in the direction of y .

1.2 Luke Rawling shortcut method

Simply just take the derivatives instead of that weird definition method. You use pattern recognition to see which one is a function of both x and y which is our $f(x, y)$, the functions of x are $h(x)$, and the functions of y are $g(y)$. This is the process and assume $M_y = N_x$ or the ODE is exact in English. You may need to separate out terms to isolate them.

$$\begin{aligned}
& M(x, y)dx + N(x, y)dy = 0 \\
& \int M(x, y)dx + \int N(x, y)dy = 0 \\
& f(x, y) + g(y) + f(x, y) + h(x) = 0 \\
& 2f(x, y) + g(y) + h(x) = 0 \\
& \implies \boxed{f(x, y) + g(y) + h(x) = C}
\end{aligned}$$

Example:

$$\begin{aligned}
& (y - 1)dx + (x + 1)dy = 0 \\
& \frac{\partial}{\partial y}(y - 1) = 1, \quad \frac{\partial}{\partial x}(x + 1) = 1 \\
& M_y = N_x \\
& \int (y - 1)dx + \int (x + 1)dy = 0 \\
& xy - y + xy + x = 0 \\
& 2xy - y + x = 0 \\
& xy \text{ is our function with both } x \text{ and } y, \quad g(y) = -y, h(x) = x \\
& 2f(x, y) + g(y) + h(x) = 0 \\
& \implies \boxed{xy - y + x = C}
\end{aligned}$$

Yes test it by yourself and you should get the same answer.

2 Near-exact ODEs

This method is simply multiplying the ODE by an integrating factor usually denoted as $(\mu(x, y))$ to be formal either a function of $\mu(x)$ or $\mu(y)$ to make the ODE exact. Afterwards, you just solve exactly the same as an exact problem.

Find your $P(x)$ or $P(y)$ and use the following formula.

DEFINITION 2. The integrating factor is the following:

$$\mu(x) = e^{\int P(x)dx}, \quad \mu(y) = e^{\int P(y)dy} \quad (3)$$

I will not derive, solve, and prove the formulas for $P(x)$ and $P(y)$. An easy way to memorize this is that these are the exact same partial derivatives as when you are testing for exactness. KEY THING, THESE SHOULD BE FUNCTIONS OF EITHER x or y . YOU MUST TEST EITHER.

DEFINITION 3. Formulas for $P(x)$ and $P(y)$.

$$P(x) = \frac{M_y - N_x}{N}, \quad \mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \quad (4)$$

$$P(y) = \frac{N_x - M_y}{M}, \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy} \quad (5)$$

2.1 Example

Example (video walkthrough <https://www.youtube.com/watch?v=6eNgoXoVTWM>) :

$$(y^2 + 2xy)dx - x^2dy = 0$$

$$\frac{\partial}{\partial y}(y^2 + 2xy) = 2y + 2x, \quad \frac{\partial}{\partial x}(-x^2) = -2x, \text{ DO NOT MAKE THE MISTAKE OF } x^2$$

$$2y + 2x \neq -2x$$

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y + 2x - (-2x)}{x^2} = \frac{2y + 4x}{x^2}, \quad \text{NOT FUNCTION OF } x$$

$$P(y) = \frac{N_x - M_y}{M} = \frac{(-2x) - (2y + 2x)}{y^2 + 2xy} = \frac{-2y - 4x}{y(y + 2x)} = \frac{-2(y + 2x)}{y(y + 2x)}$$

$$\frac{N_x - M_y}{M} = -\frac{2}{y} = P(Y), \text{ IS A FUNCTION OF } Y$$

$$\mu(y) = e^{\int P(y)dy} = e^{\int -2y^{-1}dy} = e^{\ln |y^{-2}|} = y^{-2}, \text{ we ignore sign due to } x^2.$$

$$\mu(y) ((y^2 + 2xy)dx - x^2dy = 0)$$

$$\frac{1}{y^2} ((y^2 + 2xy)dx - x^2dy = 0)$$

$$(1 + 2xy^{-1})dx - x^2y^{-2}dy = 0$$

$$\int (1 + 2xy^{-1})dx - \int x^2y^{-2}dy = 0$$

$$x + \frac{x^2}{y} + \frac{x^2}{y} = 0$$

$$2f(x, y) = \frac{x^2}{y}, g(x) = x$$

$$\Rightarrow \boxed{f(x, y) = \frac{x^2}{y} + x = C}$$