MATH 238 Homework 5

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Contents

Homework for Section 4.2: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, **10**, 11, **12**, 13, **14**, 15, and 16, the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx, \quad y'' + P(x)y' + Q(x)y = 0 \tag{5}$$

Problem 4.2.010

Given: $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned} & x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2 \\ & y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0 \\ \Longrightarrow & P(x) = \frac{2}{x} \end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx \\ y_2(x) &= \left(x^2\right)\int \frac{e^{-\int\left(\frac{2}{x}\right)dx}}{\left(x^2\right)^2}dx \\ y_2(x) &= x^2\int \frac{e^{-2\ln|x|}}{x^4}dx \\ y_2(x) &= x^2\int \frac{\left(\frac{1}{x^2}\right)}{x^4}dx \\ y_2(x) &= x^2\int x^{-6}dx \end{aligned}$$

Integrating continued.

$$\begin{aligned} y_2(x) &= x^2\left(-\frac{x^{-5}}{5}\right) \\ y_2(x) &= -\frac{1}{5x^3} \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\Longrightarrow \boxed{y_2(x) = \frac{1}{x^3}, \quad y_g = c_1e^x + \frac{c_2}{x^3}}$$

Problem 4.2.012

Given: $4x^2y'' + y = 0; \quad y_1 = x^{\frac{1}{2}}\ln x$

Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned} & 4x^2y'' + y = 0; \quad y_1 = x^{\frac{1}{2}}\ln x \\ & 4x^2y'' + 0y' + y = 0 \\ & y'' + 0y' + \frac{1}{4x^2}y = 0 \\ \Longrightarrow & P(x) = 0 \end{aligned}$$

Plugging into (5) and evaluating.

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^{-\int (0)dx}}{\left(x^{\frac{1}{2}} \ln x\right)^2} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^0}{x(\ln x)^2} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{1}{x(\ln x)^2} dx
\end{aligned}
$$

Using u-substitution.

$$
\begin{aligned}
u &= \ln x \Longrightarrow du = \frac{1}{x} dx \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int u^{-2} du \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \left(-u^{-1}\right) \\
y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \left(-(\ln x)^{-1}\right) \\
y_2(x) &= \left(x^{\frac{1}{2}} \cancel{\ln x}\right) \left(\frac{-1}{\cancel{\ln x}}\right) \\
y_2(x) &= -\sqrt{x}
\end{aligned}
$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$
\Longrightarrow \boxed{y_2(x) = \sqrt{x}, \quad y_g = c_1 \sqrt{x} \ln x + c_2 \sqrt{x}}
$$

Problem 4.2.014

Given: $x^2 y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$
\begin{aligned}
&x^2 y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x) \\
&y'' + \frac{-3}{x} y' + \frac{5}{x^2} y = 0 \\
\Longrightarrow &P(x) = -\frac{3}{x}
\end{aligned}
$$

Plugging into (5) and evaluating.

$$
\begin{aligned}
y_2(x) &= y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx \\
y_2(x) &= \left(x^2\cos(\ln x)\right)\int \frac{e^{\int\left(\frac{3}{x}\right)dx}}{\left(x^2\cos(\ln x)\right)^2}dx \\
y_2(x) &= \left(x^2\cos(\ln x)\right)\int \frac{x^3}{x^4\cos^2(\ln x)}dx \\
y_2(x) &= \left(x^2\cos(\ln x)\right)\int \frac{1}{x\cos^2(\ln x)}dx
\end{aligned}
$$

Using u-substitution.

$$
\begin{aligned}
&u = \ln x \Longrightarrow du = \frac{1}{x}dx \\
&y_2(x) = \left(x^2\cos(\ln x)\right)\int \frac{1}{\cos^2(u)}du \\
&y_2(x) = \left(x^2\cos(\ln x)\right)\int \sec^2(u)du \\
&y_2(x) = \left(x^2\cos(\ln x)\right)(\tan(u)) \\
&y_2(x) = \left(x^2\cos(\ln x)\right)(\tan(\ln x)) \\
&y_2(x) = x^2\sin(\ln x)
\end{aligned}
$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$
\Longrightarrow \boxed{y_2(x) = x^2\sin(\ln(x)), \quad y_g = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)}
$$

Problem 4.2.020

In Problems 17, 18, 19, and **20** the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction (5) of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

Given: $y'' - 4y' + 3y = x; \quad y_1 = e^x$

Find: $y_2(x)$ and $y_p(x)$

SOLUTION: Since this is a nonhomogeneous differential equation, we must use the idea of superposition. We will first use the reduction formula (5), identifying our $P(x)$, plugging in values and **setting the differential equation to zero**, to get our complimentary solution $y_p(x)$ Then, we will use the same process but **set the differential equation to x**.

Solving for $P(x)$.

$$
\begin{aligned}
&y'' - 4y' + 3y = x; \quad y_1 = e^x \\
\Longrightarrow &P(x) = -4
\end{aligned}
$$

Plugging into (5) and evaluating.

$$
\begin{aligned}
y_2(x) &= y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx \\
y_2(x) &= (e^x)\int \frac{e^{\int (4)dx}}{(e^x)^2}dx \\
y_2(x) &= (e^x)\int \frac{e^{4x}}{e^{2x}}dx \\
y_2(x) &= (e^x)\int e^{2x}dx \\
y_2(x) &= (e^x)\left(\frac{e^{2x}}{2}\right) \\
y_2(x) &= \frac{e^{3x}}{2}
\end{aligned}
$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\Longrightarrow \boxed{y_2(x) = e^{3x}}$$

To solve for our particular solution $y_p(x)$, we have to use the standard reduction of order process.

Assume $y_p = u(x)y_1$

$$
\begin{aligned}
\text{Let } y_p &= ue^x \\
\text{Then } y_p' &= u'e^x + ue^x \\
y_p'' &= u''e^x + 2u'e^x + ue^x
\end{aligned}
$$

Substituting these into the nonhomogeneous equation $y'' - 4y' + 3y = x$:

$$
\begin{aligned}
(u''e^x + 2u'e^x + ue^x) - 4(u'e^x + ue^x) + 3(ue^x) &= x \\
u''e^x - 2u'e^x &= x \\
u'' - 2u' &= xe^{-x}
\end{aligned}
$$

Let $w = u'$. This gives the first-order linear equation $w' - 2w = xe^{-x}$.
Using the integrating factor $\mu(x) = e^{\int -2dx} = e^{-2x}$:

$$
\begin{aligned}
e^{-2x}(w' - 2w) &= e^{-2x}\left(xe^{-x}\right) \\
\frac{d}{dx}[we^{-2x}] &= xe^{-3x} \\
we^{-2x} &= \int xe^{-3x}dx \\
we^{-2x} &= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} \\
u' = w &= -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x}
\end{aligned}
$$

Integrating to find $u(x)$:

$$
\begin{aligned}
\int u'dx &= \int \left(-\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x}\right)dx \\
u(x) &= \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}
\end{aligned}
$$

Multiplying by $y_1 = e^x$ to get the final particular solution:

$$y_p(x) = \left(\frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}\right)e^x$$
$$\Longrightarrow \boxed{y_p(x) = \frac{1}{3}x + \frac{4}{9}}$$

In whole, after combining all components, we arrive here with our general solution.

$$\Longrightarrow \boxed{y_2(x) = e^{3x}, \quad y_p(x) = \left(\frac{x}{3} + \frac{4}{9}\right), \quad y_g = c_1e^x + c_2e^{3x} + \left(\frac{x}{3} + \frac{4}{9}\right)}$$

Homework for Section 4.3: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and **14** find the general solution $y_g(x)$ of the given second-order differential equation.

For auxiliary differential equations represented by $ay'' + by' + cy = 0$, we know that there exists solutions of $y = e^{mx}, \quad m \in \mathbb{C}$. We can substitute the derivates and the auxiliary differential equation now becomes $e^{mx}\left(am^2 + bm + c\right)$ where we can solve for when $m = 0$. With induction, we can apply this for n amount of derivatives.

$$c_1y^n + c_2y^{n-1} + \cdots + c_ny^1 + c_{n+1}y^0 = 0, \quad y = e^{mx} \tag{1}$$

$$e^{mx}\left(c_1m^n + c_2m^{n-1} + \cdots + c_nm^1 + c_{n+1}m^0\right) = 0 \tag{2}$$

Problem 4.3.014

Given: $2y'' - 3y' + 4y = 0$
Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$\begin{aligned}
&2y'' - 3y' + 4y = 0\\
&e^{mx}\left(2m^2 + -3m + 4\right) = 0, e^{mx} \neq 0\\
&2m^2 + -3m + 4 = 0\\
&m = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(4)}}{2(2)}\\
&m = \frac{3 \pm \sqrt{-23}}{2}\\
&m = \frac{3}{2} + i\sqrt{23}, \quad \frac{3}{2} - i\sqrt{23}\\
&y = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)\\
\longrightarrow &\boxed{y = e^{\frac{3x}{2}}\left(C_1\cos(\sqrt{23}x) + C_2\sin(\sqrt{23}x)\right)}
\end{aligned}$$

In Problems 15, 16, 17, 18, 19, 20, 21, 22, 23, **24**, 25, 26, 27, and 28 find the general solution of the given higher-order differential equation.

Problem 4.3.024

Given: $y^{(4)} - 2y'' + y = 0$
Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$\begin{aligned}
&y^{(4)} - 2y'' + y = 0 \\
&e^{mx}\left(m^4 - 2m^2 + 1\right) = 0, e^{mx} \neq 0 \\
&m^4 - 2m^2 + 1 = 0 \\
&(m^2 - 1)^2 = 0 \\
&m = 1, -1, 1, -1 \\
&y_1(x) = e^x + e^{-x}
\end{aligned}$$

Using reduction formula (5) to obtain $y_2(x)$.

$$\begin{aligned}
&y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
&y^{(4)} - 2y'' + y = 0, \quad y_1(x) = e^{m_1 x},\ P(x) = 2 \\
\\
&y_2(x) = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1} \int dx = xe^{m_1 x} \\
&y_2(x) = xe^x + xe^{-x}
\end{aligned}$$

Combining $y_1(x)$ and $y_2(x)$.

$$\begin{aligned}
&y_g(x) = y_1(x) + y_2(x) \\
\Longrightarrow &\boxed{y_g(x) = C_1 e^x + C_2 xe^x + C_3 e^{-x} + C_4 xe^{-x}}
\end{aligned}$$

In Problems 29, 30, 31, **32**, 33, 34, 35, and 36 solve the given initial-value problem.

Problem 4.3.032

Given: $4y'' + 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$
Find: $y(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation. With the IVP, we will simply need to plug in values and solve for the constants.

Solving for $y_g(x)$.

$$\begin{aligned}
&4y'' + 4y' - 3y = 0 \\
&e^{mx}\left(4m^2 + 4m - 3\right) = 0, e^{mx} \neq 0 \\
&4m^2 + 4m - 3 = 0 \\
&m = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(-3)}}{2(4)} \\
&m = \frac{-4 \pm \sqrt{16 + 48}}{8} \\
&m = -\frac{1}{2} \pm 1 \\
&m = \frac{1}{2}, -\frac{3}{2} \\
\Longrightarrow &y_g(x) = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}}
\end{aligned}$$

Solving for $y(x)$ and $y'(x)$

$$\begin{aligned}
y(x) &= C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}} \\
y'(x) &= \frac{C_1 e^{\frac{x}{2}}}{2} - \frac{3C_2 e^{-\frac{3x}{2}}}{2}
\end{aligned}$$

Solving for constants C_1 and C_2

$$\begin{aligned}
&y(0) = 1 \\
&1 = C_1 e^{\frac{(0)}{2}} + C_2 e^{-\frac{3(0)}{2}} \\
\Longrightarrow &1 = C_1 + C_2
\end{aligned}$$

$$\begin{aligned}
&y'(0) = 5 \\
&5 = \frac{C_1 e^{\frac{(0)}{2}}}{2} - \frac{3C_2 e^{-\frac{3(0)}{2}}}{2} \\
&5 = \frac{C_1}{2} - \frac{3C_2}{2} \\
\Longrightarrow &10 = C_1 - 3C_2
\end{aligned}$$

$$\begin{aligned}
&C_1 + C_2 = 1 \\
&C_1 - 3C_2 = 10 \\
&\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \\
&\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 10 \end{bmatrix} \\
&\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 \\ -9 \end{bmatrix}
\end{aligned}$$

Plugging back constants and combining solutions.

$$\Longrightarrow \boxed{y(x) = \frac{13e^{\frac{x}{2}}}{4} - \frac{9e^{-\frac{3x}{2}}}{4}}$$