Connection of Integration by Parts to Balance of Forces
MATH 255 #25972 Professor Rawlings

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Contents

1 Integration by Parts

If you're ever randomly asked in a calculus class, your best bet is probably integration by parts. We're often taught in school to recognize a specific form and to plug it into a formula. From that point onward, we usually move on and never thing much more.

DEFINITION 1. The formula for Integration by Parts is defined by the following:

$$\int u\,dv \;=\; uv - \int v\,du \tag{1}$$

This is usually derived through the specific form. [2]

$$\begin{aligned}
&\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx \\
&\text{Let } u = u(x),\ du = u'(x)\,dx,\ v = v(x), dv = v'(x)\,dx \\
\implies &\int u\,dv \;=\; uv - \int v\,du
\end{aligned}$$

1.1 Visual Derivation

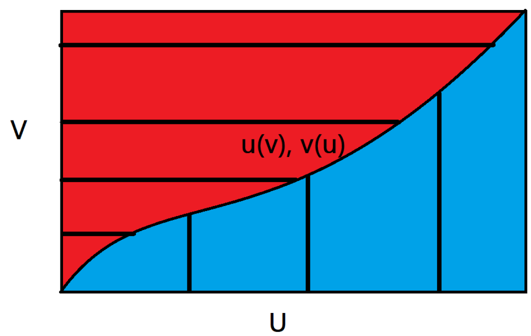

Figure 1: u v visual.

Let the area of the rectangular be calculated through width times height and the curve being in a function of u and v. Additionally, let the axes be u and v. Then, by simply adding up the parts of the whole, we get the area of that same rectangle.

$$\begin{aligned}
&u(v)\,v(u) = \int u(v)\,dv + \int v(u)\,du \\
&\text{Rearranging} \\
&\int u(v)\,dv \;=\; u(v)\,v(u) - \int v(u)\,du \\
\implies &\int u\,dv \;=\; uv - \int v\,du
\end{aligned}$$

1.1.1 Young's Inequality

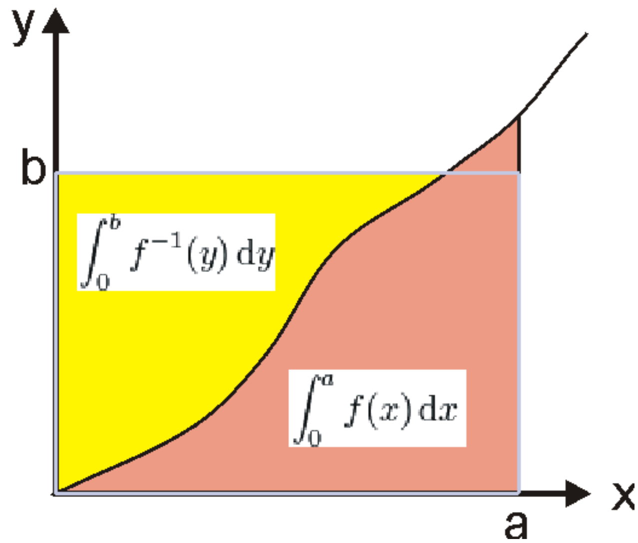

Figure 2: The area of the rectangle a,b can't be larger than sum of the areas under the functions (red) and (yellow). [3]

As a remark, this is process is very similair to Young's inequality for products which states the following:

$$ab \;\leq\; \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \tag{2}$$

1.2 Differential Derivation

Another way (simpler) is simply use the product rule in differential form and integrate. After rearranging, you will arrive at the same solution.

$$\begin{aligned}
&\text{Given: } u(x),\; v(x)\\
&\frac{d}{dx}\left(u(x)\,v(x)\right) = u(x)\,\frac{dv}{dx} + v(x)\,\frac{du}{dx}\\
&d\left(u(x)\,v(x)\right) = u(x)\,dv + v(x)\,du\\
&\int d\left(u(x)\,v(x)\right) = \int u(x)\,dv \;+ \int v(x)\,du\\
&u(x)\,v(x) = \int u(x)\,dv \;+ \int v(x)\,du\\
\Longrightarrow &\int u\,dv \;=\; uv - \int v\,du
\end{aligned}$$

2 Balance of Forces

The balance of forces is all about equal and opposite forces acting on other objects. This is an introductory physics topic that is about adding up parts of forces to a sum. With this idea, we can connect it and show that your simple physics forces is integration by parts in disguise.

DEFINITION 2. A force is defined by the following.

$$\vec{F} = m\vec{a} \tag{3}$$

DEFINITION 3. When we have a balance of forces, we are at equilibrium with the sum of all total forces in every direction.

$$\sum \vec{F} = \vec{0},\ \vec{F}_{\text{net}} = \vec{0} \tag{4}$$

2.1 Connection to Integration by Parts

To arrive back to integration by parts, we will have to do some algebraic manipulation. After which, we will use our differential derivation product rule trick.

$$\begin{aligned}
\vec{F}_{\text{net}} &= 0 \\
\vec{F}_{\text{net}}\, dx &= 0\, dx \\
\int \vec{F}_{\text{net}}\, dx &= \int 0\, dx \\
\vec{F}_{\text{net}}\, x &= C
\end{aligned}$$

Using differential form from 1.2.

$$\begin{aligned}
d(\vec{F}_{\text{net}}x) &= \vec{F}_{\text{net}}\, dx + x\, d(\vec{F}_{\text{net}}) + \cancelto{0}{d(C)} \\
\int d(\vec{F}_{\text{net}}x) &= \int \vec{F}_{\text{net}}\, dx + \int x\, d(\vec{F}_{\text{net}}) \\
\vec{F}_{\text{net}}x &= \int \vec{F}_{\text{net}}\, dx + \int x\, d(\vec{F}_{\text{net}}) \\
\implies &\boxed{\int \vec{F}_{\text{net}}\, dx = \vec{F}_{\text{net}}x - \int x\, d(\vec{F}_{\text{net}})}
\end{aligned}$$

Just like that, we have the exact same equation. The next question is how should to interpret this.

2.2 Interpretation

Right off the bat, we can recognize the form of work. Now, we're left with this weird $\int x\, d(\vec{F}_{\text{net}})$ term. Let us start back with this equation instead and work from there.

$$\vec{F}_{\text{net}}x = \int \vec{F}_{\text{net}}\, dx + \int x\, d(\vec{F}_{\text{net}})$$

In structurual mechanics, we deal with both strain and complementary strain energy alongside work.

DEFINITION 4. Strain Energy

$$U = \int \vec{F}_{\text{net}}\, dx \tag{5}$$

DEFINITION 5. Complementary Strain Energy

$$C = \int x \, d\vec{F}_{\text{net}} \tag{6}$$

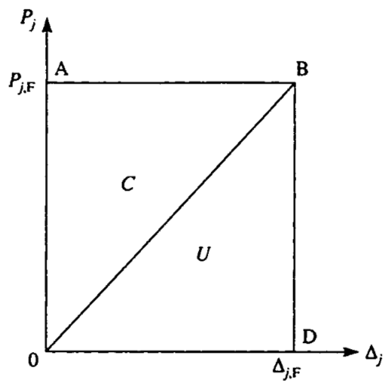

Figure 3: Load-deflection curve for a linearly elastic member.

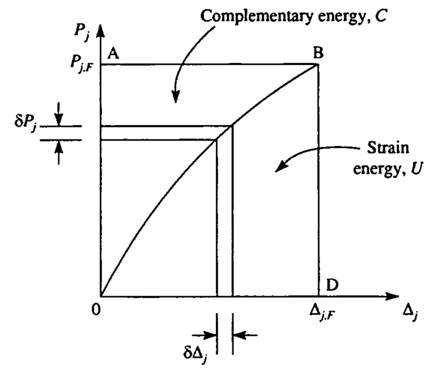

Figure 4: Load-deflection cruve for a non-linearly elastic member

[1]

Let the y axis be in terms of P, a force and a displacement Δ. To calculate the area the of our rectangle, we can use the same process from 1.1.

$$\begin{aligned}
\vec{F}_{\text{net}} x &= \int \vec{F}_{\text{net}} \, dx + \int x \, d(\vec{F}_{\text{net}}) \\
\text{Work} &= U + C
\end{aligned}$$

This shows that integration by parts in the balance of forces is about strain energy within internal structures alongside external structures.

References

[1] T. H. G. Megson. *Structural and Stress Analysis*. 2nd ed. Oxford: Butterworth-Heinemann, 2005. Chap. 15, pp. 443–446. ISBN: 978-0-7506-6221-5. URL: http://freeit.free.fr/Knovel/Structural%20and%20Stress%20Analysis/31961_15.pdf.

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[3] Wikipedia contributors. *Young's inequality for products — Wikipedia, The Free Encyclopedia*. [Online; accessed 18-February-2026]. 2025. URL: https://en.wikipedia.org/w/index.php?title=Young%27s_inequality_for_products&oldid=1316559643.