MATH 238 Homework 5

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Contents

Homework for Section 4.2: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, **10**, 11, **12**, 13, **14**, 15, and 16, the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx, \quad y'' + P(x)y' + Q(x)y = 0 \tag{5}$$

Problem 4.2.010

Given: $x^2 y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$x^2 y'' + 2xy' - 6y = 0; \quad y_1 = x^2$$

$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$$

$$\implies P(x) = \frac{2}{x}$$

Plugging into (5) and evaluating.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$y_2(x) = (x^2) \int \frac{e^{-\int \left(\frac{2}{x}\right)dx}}{(x^2)^2} dx$$

$$y_2(x) = x^2 \int \frac{e^{-2\ln|x|}}{x^4} dx$$

$$y_2(x) = x^2 \int \frac{\left(\frac{1}{x^2}\right)}{x^4} dx$$

$$y_2(x) = x^2 \int x^{-6} dx$$

Integrating continued.

$$y_2(x) = x^2 \left(-\frac{x^{-5}}{5}\right)$$

$$y_2(x) = -\frac{1}{5x^3}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \frac{1}{x^3}, \quad y_g = c_1 x^2 + \frac{c_2}{x^3}}$$

Problem 4.2.012

Given: $4x^2 y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$4x^2 y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x$$

$$4x^2 y'' + 0y' + y = 0$$

$$y'' + 0y' + \frac{1}{4x^2}y = 0$$

$$\implies P(x) = 0$$

Plugging into (5) and evaluating.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \int \frac{e^{-\int (0)dx}}{\left(x^{\frac{1}{2}} \ln x \right)^2} dx$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \int \frac{e^0}{x(\ln x)^2} dx$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \int \frac{1}{x(\ln x)^2} dx$$

Using u-substitution.

$$u = \ln x \implies du = \frac{1}{x} dx$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \int u^{-2} du$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \left(-u^{-1} \right)$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \left(-(\ln x)^{-1} \right)$$

$$y_2(x) = \left(x^{\frac{1}{2}} \ln x \right) \left(\frac{-1}{\ln x} \right)$$

$$y_2(x) = -\sqrt{x}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \sqrt{x}, \quad y_g = c_1 \sqrt{x} \ln x + c_2 \sqrt{x}}$$

Problem 4.2.014

Given: $x^2 y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$
Find: $y_2(x)$

SOLUTION: We can simply the use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$x^2 y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$$

$$y'' + \frac{-3}{x} y' + \frac{5}{x^2} y = 0$$

$$\implies P(x) = -\frac{3}{x}$$

Plugging into (5) and evaluating.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \int \frac{e^{\int \left(\frac{3}{x}\right)dx}}{\left(x^2 \cos\left(\ln x\right)\right)^2} dx$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \int \frac{x^3}{x^4 \cos^2\left(\ln x\right)} dx$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \int \frac{1}{x \cos^2\left(\ln x\right)} dx$$

Using u-substitution.

$$u = \ln x \Longrightarrow du = \frac{1}{x}dx$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \int \frac{1}{\cos^2\left(u\right)} du$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \int \sec^2(u) du$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \left(\tan(u)\right)$$

$$y_2(x) = \left(x^2 \cos\left(\ln x\right)\right) \left(\tan(\ln x)\right)$$

$$y_2(x) = x^2 \sin(\ln x)$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\Longrightarrow \boxed{y_2(x) = x^2 \sin\left(\ln\left(x\right)\right), \quad y_g = c_1 x^2 \cos\left(\ln x\right) + c_2 x^2 \sin\left(\ln x\right)}$$

Problem 4.2.020

In Problems 17, 18, 19, and **20** the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction (5) of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

Given: $y'' - 4y' + 3y = x; \quad y_1 = e^x$
Find: $y_2(x)$ and $y_p(x)$

SOLUTION: Since this is a nonhomogeneous differential equation, we must use the idea of superposition. We will first use the reduction formula (5), identifying our $P(x)$, plugging in values and **setting the differential equation to zero**, to get our complimentary solution $y_p(x)$ Then, we will use the same process but **set the differential equation to x**.

Solving for $P(x)$.

$$y'' - 4y' + 3y = x; \quad y_1 = e^x$$

$$\Longrightarrow P(x) = -4$$

Plugging into (5) and evaluating.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$y_2(x) = (e^x) \int \frac{e^{\int (4)dx}}{(e^x)^2} dx$$

$$y_2(x) = (e^x) \int \frac{e^{4x}}{e^{2x}} dx$$

$$y_2(x) = (e^x) \int e^{2x} dx$$

$$y_2(x) = (e^x) \left(\frac{e^{2x}}{2} \right)$$

$$y_2(x) = \frac{e^{3x}}{2}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = e^{3x}}$$

To solve for our particular solution $y_p(x)$, we have to use the standard reduction of order process.

Assume $y_p = u(x)y_1$

$$\text{Let } y_p = ue^x$$
$$\text{Then } y_p' = u'e^x + ue^x$$
$$y_p'' = u''e^x + 2u'e^x + ue^x$$

Substituting these into the nonhomogeneous equation $y'' - 4y' + 3y = x$:

$$(u''e^x + 2u'e^x + ue^x) - 4(u'e^x + ue^x) + 3(ue^x) = x$$
$$u''e^x - 2u'e^x = x$$
$$u'' - 2u' = xe^{-x}$$

Let $w = u'$. This gives the first-order linear equation $w' - 2w = xe^{-x}$.
Using the integrating factor $\mu(x) = e^{\int -2dx} = e^{-2x}$:

$$e^{-2x}(w' - 2w) = e^{-2x}(xe^{-x})$$
$$\frac{d}{dx}[we^{-2x}] = xe^{-3x}$$
$$we^{-2x} = \int xe^{-3x} dx$$
$$we^{-2x} = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}$$
$$u' = w = -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x}$$

Integrating to find $u(x)$:

$$\int u'dx = \int \left(-\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} \right) dx$$
$$u(x) = \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}$$

Multiplying by $y_1 = e^x$ to get the final particular solution:

$$y_p(x) = u(x)y_1(x)$$

$$y_p(x) = \left(\frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}\right)e^x$$

$$\implies \boxed{y_p(x) = \frac{1}{3}x + \frac{4}{9}}$$

In whole, after combining all components, we arrive here with our general solution.

$$\implies \boxed{y_2(x) = e^{3x}, \quad y_p(x) = \left(\frac{x}{3} + \frac{4}{9}\right), \quad y_g = c_1e^x + c_2e^{3x} + \left(\frac{x}{3} + \frac{4}{9}\right)}$$

Homework for Section 4.3: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and **14** find the general solution $y_g(x)$ of the given second-order differential equation.

For auxiliary differential equations represented by $ay'' + by' + cy = 0$, we know that there exists solutions of $y = e^{mx}, \quad m \in \mathbb{C}$. We can substitute the derivates and the auxiliary differential equation now becomes $e^{mx}\left(am^2 + bm + c\right)$ where we can solve for when $m = 0$. With induction, we can apply this to any n-order linear differential equations.

$$c_ny^n + c_{n-1}y^{n-1} + \cdots + c_1y^1 + c_0y^0 = 0, \quad y = e^{mx} \tag{1}$$

$$e^{mx}\left(c_ny^n + c_{n-1}y^{n-1} + \cdots + c_1y^1 + c_0y^0\right) = 0 \tag{2}$$

Problem 4.3.014

Given: $2y'' - 3y' + 4y = 0$
Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$2y'' - 3y' + 4y = 0$$

$$e^{mx}\left(2m^2 + -3m + 4\right) = 0, e^{mx} \neq 0$$

$$2m^2 + -3m + 4 = 0$$

$$m = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(4)}}{2(2)}$$

$$m = \frac{3 \pm \sqrt{-23}}{4}$$

$$m = \frac{3}{4} + i\frac{\sqrt{23}}{4}, \quad \frac{3}{4} - i\frac{\sqrt{23}}{4}$$

Using distinct conjugate case.

$$m = \alpha \pm i\beta$$
$$y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$$
$$\longrightarrow \boxed{y = e^{\frac{3x}{4}}\left(C_1 \cos\left(\frac{\sqrt{23}}{4}x\right) + C_2 \sin\left(\frac{\sqrt{23}}{4}x\right)\right)}$$

In Problems 15, 16, 17, 18, 19, 20, 21, 22, 23, **24**, 25, 26, 27, and 28 find the general solution of the given higher-order differential equation.

Problem 4.3.024

Given: $y^{(4)} - 2y'' + y = 0$
Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$y^{(4)} - 2y'' + y = 0$$
$$e^{mx}\left(m^4 - 2m^2 + 1\right) = 0, e^{mx} \neq 0$$
$$m^4 - 2m^2 + 1 = 0$$
$$(m^2 - 1)^2 = 0$$
$$m = 1, -1, 1, -1$$
$$y_1(x) = e^x + e^{-x}$$

Using reduction formula (5) to obtain $y_2(x)$.

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$
$$y^{(4)} - 2y'' + y = 0, \quad y_1(x) = e^{m_1 x}, \quad P(x) = 2$$

$$y_2(x) = e^{m_1 x}\int \frac{e^{2m_1 x}}{e^{2m_1 x}}dx = e^{m_1}\int dx = xe^{m_1 x}$$
$$y_2(x) = xe^x + xe^{-x}$$

Combining $y_1(x)$ and $y_2(x)$.

$$y_g(x) = y_1(x) + y_2(x)$$
$$\Longrightarrow \boxed{y_g(x) = C_1 e^x + C_2 xe^x + C_3 e^{-x} + C_4 xe^{-x}}$$

In Problems 29, 30, 31, **32**, 33, 34, 35, and 36 solve the given initial-value problem.

Problem 4.3.032

Given: $4y'' + 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$
Find: $y(x)$

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SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation. With the IVP, we will simply need to plug in values and solve for the constants.

Solving for $y_g(x)$.

$$4y'' + 4y' - 3y = 0$$

$$e^{mx} \left(4m^2 + 4m - 3\right) = 0, e^{mx} \neq 0$$

$$4m^2 + 4m - 3 = 0$$

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(-3)}}{2(4)}$$

$$m = \frac{-4 \pm \sqrt{16 + 48}}{8}$$

$$m = -\frac{1}{2} \pm 1$$

$$m = \frac{1}{2}, -\frac{3}{2}$$

$$\Longrightarrow \boxed{y_g(x) = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}}}$$

Solving for $y(x)$ and $y'(x)$

$$y(x) = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}}$$

$$y'(x) = \frac{C_1 e^{\frac{x}{2}}}{2} - \frac{3C_2 e^{-\frac{3x}{2}}}{2}$$

Solving for constants C_1 and C_2

$$y(0) = 1$$

$$1 = C_1 e^{\frac{(0)}{2}} + C_2 e^{-\frac{3(0)}{2}}$$

$$\Longrightarrow 1 = C_1 + C_2$$

$$y'(0) = 5$$

$$5 = \frac{C_1 e^{\frac{(0)}{2}}}{2} - \frac{3C_2 e^{-\frac{3(0)}{2}}}{2}$$

$$5 = \frac{C_1}{2} - \frac{3C_2}{2}$$

$$\Longrightarrow 10 = C_1 - 3C_2$$

$$C_1 + C_2 = 1$$

$$C_1 - 3C_2 = 10$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 \\ -9 \end{bmatrix}}$$

Plugging back constants and combining solutions.

$$\implies \boxed{y(x) = \frac{13e^{\frac{x}{2}}}{4} - \frac{9e^{-\frac{3x}{2}}}{4}}$$