

MATH 238 Homework 5

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Contents

Homework for Section 4.2: 50 points	1
Problem 4.2.010	1
Problem 4.2.012	2
Problem 4.2.014	3
Problem 4.2.020	4
 Homework for Section 4.3: 50 points	 6
Problem 4.3.014	6
Problem 4.3.024	7
Problem 4.3.032	7

Homework for Section 4.2: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, **10**, 11, **12**, 13, **14**, 15, and 16, the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx, \quad y'' + P(x)y' + Q(x)y = 0 \quad (5)$$

Problem 4.2.010

Given: $x^2 y'' + 2xy' - 6y = 0$; $y_1 = x^2$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}x^2 y'' + 2xy' - 6y &= 0; \quad y_1 = x^2 \\y'' + \frac{2}{x}y' - \frac{6}{x^2}y &= 0 \\ \implies P(x) &= \frac{2}{x}\end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\y_2(x) &= (x^2) \int \frac{e^{-\int (\frac{2}{x})dx}}{(x^2)^2} dx \\y_2(x) &= x^2 \int \frac{e^{-2 \ln |x|}}{x^4} dx \\y_2(x) &= x^2 \int \frac{(\frac{1}{x^2})}{x^4} dx \\y_2(x) &= x^2 \int x^{-6} dx\end{aligned}$$

Integrating continued.

$$\begin{aligned}y_2(x) &= x^2 \left(-\frac{x^{-5}}{5} \right) \\y_2(x) &= -\frac{1}{5x^3}\end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \frac{1}{x^3}, \quad y_g = c_1 x^2 + \frac{c_2}{x^3}}$$

Problem 4.2.012

Given: $4x^2 y'' + y = 0; \quad y_1 = x^{\frac{1}{2}} \ln x$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}4x^2 y'' + y &= 0; \quad y_1 = x^{\frac{1}{2}} \ln x \\4x^2 y'' + 0y' + y &= 0 \\y'' + 0y' + \frac{1}{4x^2}y &= 0 \\ \implies P(x) &= 0\end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^{-\int (0)dx}}{\left(x^{\frac{1}{2}} \ln x\right)^2} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{e^0}{x(\ln x)^2} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int \frac{1}{x(\ln x)^2} dx
 \end{aligned}$$

Using u-substitution.

$$\begin{aligned}
 u &= \ln x \implies du = \frac{1}{x} dx \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \int u^{-2} du \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) (-u^{-1}) \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) (-(\ln x)^{-1}) \\
 y_2(x) &= \left(x^{\frac{1}{2}} \ln x\right) \left(\frac{-1}{\ln x}\right) \\
 y_2(x) &= -\sqrt{x}
 \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = \sqrt{x}, \quad y_g = c_1 \sqrt{x} \ln x + c_2 \sqrt{x}}$$

Problem 4.2.014

Given: $x^2 y'' - 3xy' + 5y = 0$; $y_1 = x^2 \cos(\ln x)$

Find: $y_2(x)$

SOLUTION: We can simply use the reduction formula (5). All we need to do is identify our $P(x)$ and plug in values, we should arrive at our answer.

Solving for $P(x)$.

$$\begin{aligned}
 x^2 y'' - 3xy' + 5y &= 0; \quad y_1 = x^2 \cos(\ln x) \\
 y'' + \frac{-3}{x} y' + \frac{5}{x^2} y &= 0 \\
 \implies P(x) &= -\frac{3}{x}
 \end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{e^{\int (\frac{3}{x})dx}}{(x^2 \cos(\ln x))^2} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{x^3}{x^4 \cos^2(\ln x)} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{1}{x \cos^2(\ln x)} dx
 \end{aligned}$$

Using u-substitution.

$$\begin{aligned}
 u = \ln x &\implies du = \frac{1}{x} dx \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \frac{1}{\cos^2(u)} du \\
 y_2(x) &= (x^2 \cos(\ln x)) \int \sec^2(u) du \\
 y_2(x) &= (x^2 \cos(\ln x)) (\tan(u)) \\
 y_2(x) &= (x^2 \cos(\ln x)) (\tan(\ln x)) \\
 y_2(x) &= x^2 \sin(\ln x)
 \end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = x^2 \sin(\ln(x)), \quad y_g = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x)}$$

Problem 4.2.020

In Problems 17, 18, 19, and **20** the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction (5) of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

Given: $y'' - 4y' + 3y = x$; $y_1 = e^x$

Find: $y_2(x)$ and $y_p(x)$

SOLUTION: Since this is a nonhomogeneous differential equation, we must use the idea of superposition. We will first use the reduction formula (5), identifying our $P(x)$, plugging in values and **setting the differential equation to zero**, to get our complimentary solution $y_c(x)$ Then, we will use the same process but **set the differential equation to x** .

Solving for $P(x)$.

$$\begin{aligned}
 y'' - 4y' + 3y &= x; \quad y_1 = e^x \\
 \implies P(x) &= -4
 \end{aligned}$$

Plugging into (5) and evaluating.

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
y_2(x) &= (e^x) \int \frac{e^{\int (4)dx}}{(e^x)^2} dx \\
y_2(x) &= (e^x) \int \frac{e^{4x}}{e^{2x}} dx \\
y_2(x) &= (e^x) \int e^{2x} dx \\
y_2(x) &= (e^x) \left(\frac{e^{2x}}{2} \right) \\
y_2(x) &= \frac{e^{3x}}{2}
\end{aligned}$$

Because the differential equation is homogeneous and linear, we can drop the constants.

$$\implies \boxed{y_2(x) = e^{3x}}$$

To solve for our particular solution $y_p(x)$, we have to use the standard reduction of order process.

Assume $y_p = u(x)y_1$

$$\begin{aligned}
\text{Let } y_p &= ue^x \\
\text{Then } y_p' &= u'e^x + ue^x \\
y_p'' &= u''e^x + 2u'e^x + ue^x
\end{aligned}$$

Substituting these into the nonhomogeneous equation $y'' - 4y' + 3y = x$:

$$\begin{aligned}
(u''e^x + 2u'e^x + ue^x) - 4(u'e^x + ue^x) + 3(ue^x) &= x \\
u''e^x - 2u'e^x &= x \\
u'' - 2u' &= xe^{-x}
\end{aligned}$$

Let $w = u'$. This gives the first-order linear equation $w' - 2w = xe^{-x}$.

Using the integrating factor $\mu(x) = e^{\int -2dx} = e^{-2x}$:

$$\begin{aligned}
e^{-2x}(w' - 2w) &= e^{-2x}(xe^{-x}) \\
\frac{d}{dx}[we^{-2x}] &= xe^{-3x} \\
we^{-2x} &= \int xe^{-3x} dx \\
we^{-2x} &= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} \\
u' = w &= -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x}
\end{aligned}$$

Integrating to find $u(x)$:

$$\begin{aligned}
\int u' dx &= \int \left(-\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} \right) dx \\
u(x) &= \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}
\end{aligned}$$

Multiplying by $y_1 = e^x$ to get the final particular solution:

$$\begin{aligned} y_p(x) &= u(x)y_1(x) \\ y_p(x) &= \left(\frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x}\right)e^x \\ \Rightarrow y_p(x) &= \frac{1}{3}x + \frac{4}{9} \end{aligned}$$

In whole, after combining all components, we arrive here with our general solution.

$$\Rightarrow y_2(x) = e^{3x}, \quad y_p(x) = \left(\frac{x}{3} + \frac{4}{9}\right), \quad y_g = c_1e^x + c_2e^{3x} + \left(\frac{x}{3} + \frac{4}{9}\right)$$

Homework for Section 4.3: 50 points

In Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and **14** find the general solution $y_g(x)$ of the given second-order differential equation.

For auxiliary differential equations represented by $ay'' + by' + cy = 0$, we know that there exists solutions of $y = e^{mx}$, $m \in \mathbb{C}$. We can substitute the derivatives and the auxiliary differential equation now becomes $e^{mx}(am^2 + bm + c)$ where we can solve for when $m = 0$. With induction, we can apply this to any n -order linear differential equations.

$$c_n y^n + c_{n-1} y^{n-1} + \dots + c_1 y^1 + c_0 y^0 = 0, \quad y = e^{mx} \quad (1)$$

$$e^{mx} (c_n y^n + c_{n-1} y^{n-1} + \dots + c_1 y^1 + c_0 y^0) = 0 \quad (2)$$

Problem 4.3.014

Given: $2y'' - 3y' + 4y = 0$

Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$\begin{aligned} 2y'' - 3y' + 4y &= 0 \\ e^{mx} (2m^2 - 3m + 4) &= 0, e^{mx} \neq 0 \\ 2m^2 - 3m + 4 &= 0 \\ m &= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(4)}}{2(2)} \\ m &= \frac{3 \pm \sqrt{-23}}{4} \\ m &= \frac{3}{4} + i\frac{\sqrt{23}}{4}, \quad \frac{3}{4} - i\frac{\sqrt{23}}{4} \end{aligned}$$

Using distinct conjugate case.

$$\begin{aligned}
 m &= \alpha \pm i\beta \\
 y &= e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)) \\
 \rightarrow y &= e^{\frac{3x}{4}} \left(C_1 \cos\left(\frac{\sqrt{23}}{4}x\right) + C_2 \sin\left(\frac{\sqrt{23}}{4}x\right) \right)
 \end{aligned}$$

In Problems 15, 16, 17, 18, 19, 20, 21, 22, 23, **24**, 25, 26, 27, and 28 find the general solution of the given higher-order differential equation.

Problem 4.3.024

Given: $y^{(4)} - 2y'' + y = 0$

Find: $y_g(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation.

$$\begin{aligned}
 y^{(4)} - 2y'' + y &= 0 \\
 e^{mx} (m^4 - 2m^2 + 1) &= 0, e^{mx} \neq 0 \\
 m^4 - 2m^2 + 1 &= 0 \\
 (m^2 - 1)^2 &= 0 \\
 m &= 1, -1, 1, -1 \\
 y_1(x) &= e^x + e^{-x}
 \end{aligned}$$

Using reduction formula (5) to obtain $y_2(x)$.

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \\
 y^{(4)} - 2y'' + y &= 0, \quad y_1(x) = e^{m_1 x}, \quad P(x) = 2 \\
 y_2(x) &= e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1} \int dx = xe^{m_1 x} \\
 y_2(x) &= xe^x + xe^{-x}
 \end{aligned}$$

Combining $y_1(x)$ and $y_2(x)$.

$$\begin{aligned}
 y_g(x) &= y_1(x) + y_2(x) \\
 \Rightarrow y_g(x) &= C_1 e^x + C_2 x e^x + C_3 e^{-x} + C_4 x e^{-x}
 \end{aligned}$$

In Problems 29, 30, 31, **32**, 33, 34, 35, and 36 solve the given initial-value problem.

Problem 4.3.032

Given: $4y'' + 4y' - 3y = 0$, $y(0) = 1$, $y'(0) = 5$

Find: $y(x)$

SOLUTION: Recognizing the differential equation as an auxiliary equation allows us to solve this very easily and reduce it to a simple quadratic equation. With the IVP, we will simply need to plug in values and solve for the constants.

Solving for $y_g(x)$.

$$\begin{aligned}
 4y'' + 4y' - 3y &= 0 \\
 e^{mx} (4m^2 + 4m - 3) &= 0, e^{mx} \neq 0 \\
 4m^2 + 4m - 3 &= 0 \\
 m &= \frac{-4 \pm \sqrt{(4)^2 - 4(4)(-3)}}{2(4)} \\
 m &= \frac{-4 \pm \sqrt{16 + 48}}{8} \\
 m &= -\frac{1}{2} \pm 1 \\
 m &= \frac{1}{2}, -\frac{3}{2} \\
 \Rightarrow y_g(x) &= C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}}
 \end{aligned}$$

Solving for $y(x)$ and $y'(x)$

$$\begin{aligned}
 y(x) &= C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}} \\
 y'(x) &= \frac{C_1 e^{\frac{x}{2}}}{2} - \frac{3C_2 e^{-\frac{3x}{2}}}{2}
 \end{aligned}$$

Solving for constants C_1 and C_2

$$\begin{aligned}
 y(0) &= 1 \\
 1 &= C_1 e^{\frac{(0)}{2}} + C_2 e^{-\frac{3(0)}{2}} \\
 \Rightarrow 1 &= C_1 + C_2 \\
 \\
 y'(0) &= 5 \\
 5 &= \frac{C_1 e^{\frac{(0)}{2}}}{2} - \frac{3C_2 e^{-\frac{3(0)}{2}}}{2} \\
 5 &= \frac{C_1}{2} - \frac{3C_2}{2} \\
 \Rightarrow 10 &= C_1 - 3C_2
 \end{aligned}$$

$$\begin{aligned}
 C_1 + C_2 &= 1 \\
 C_1 - 3C_2 &= 10 \\
 \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 10 \end{bmatrix} \\
 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 10 \end{bmatrix} \\
 \boxed{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 \\ -9 \end{bmatrix}}
 \end{aligned}$$

Plugging back constants and combining solutions.

$$\Rightarrow y(x) = \frac{13e^{\frac{x}{2}}}{4} - \frac{9e^{-\frac{3x}{2}}}{4}$$