MATH 238 Homework #2

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Homework for Section 2.1

1. Use computer software to obtain a direction field for the given differential equation. Also sketch an approximate solution curve passing through each of the given points

$$\frac{dy}{dx} = e^x \sin(y)$$

   (a) $y(-1) = 4.5$
   (b) $y(-1) = -4.5$
   (c) $y(-1) = 1$
   (d) $y(-1) = -1$

   Give the equilibrium solutions as solution functions.

   SOLUTION:

$$(a) \quad \frac{dy}{dx} = e^x \sin(y)$$

$$e^x \sin(y) = 0$$

$$e^x \neq 0,\ \sin(y) = 0$$

$$\Longrightarrow \boxed{y = 0,\ y = \pi n,\ n \in \mathbb{Z}} \text{ are the equilibrium solutions.}$$

Equilibrium equations occur when $\frac{dy}{dx} = 0$ allowing us to the find the $y(x)$.

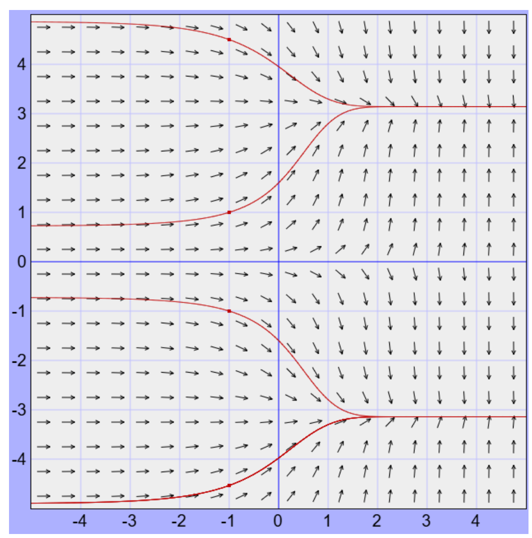
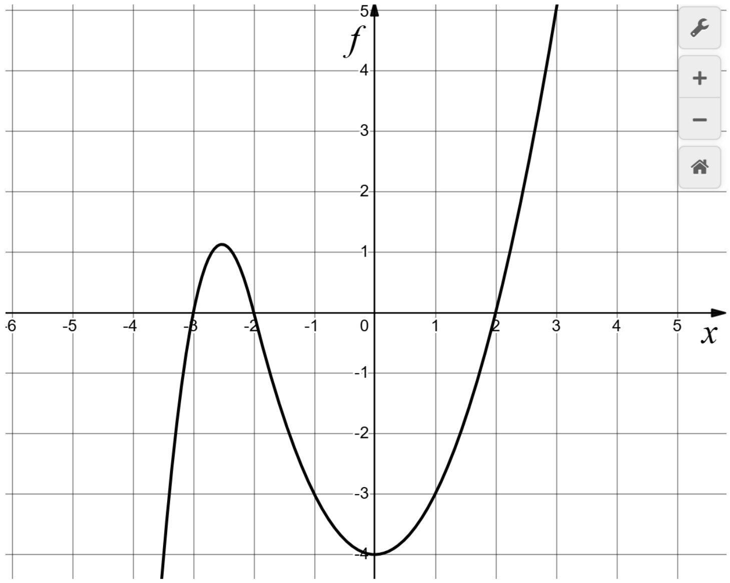


Figure 1: Direction field of $\frac{dy}{dx} = e^x \sin(y)$

2. The given figure represents the graph of $f(x)$ when $dy/dx = f(x)$. By hand, sketch a direction field over an appropriate grid and include it as an image.



SOLUTION: It is pretty easy to guess the function and we can graph the direction field as the form of

$$\frac{dy}{dx} = \frac{(x+3)(x+2)(x-2)}{3}$$

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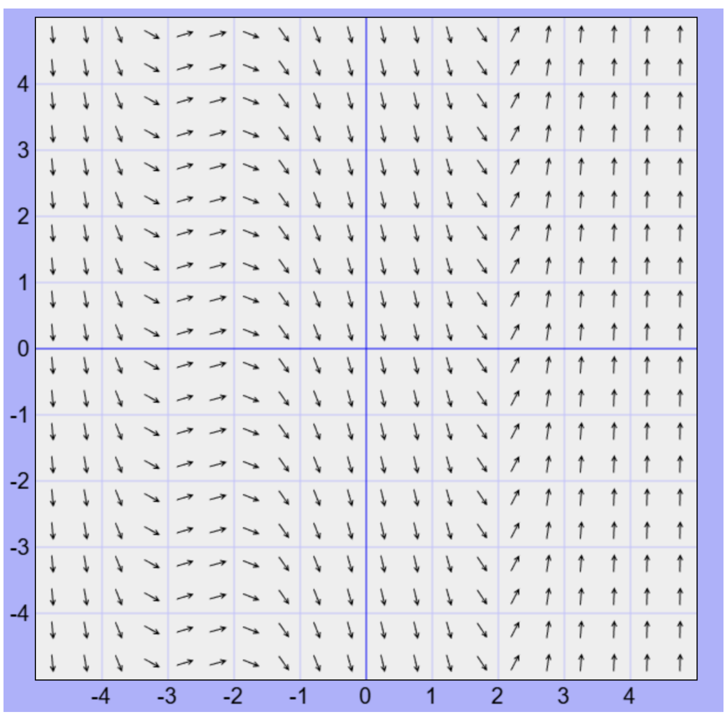
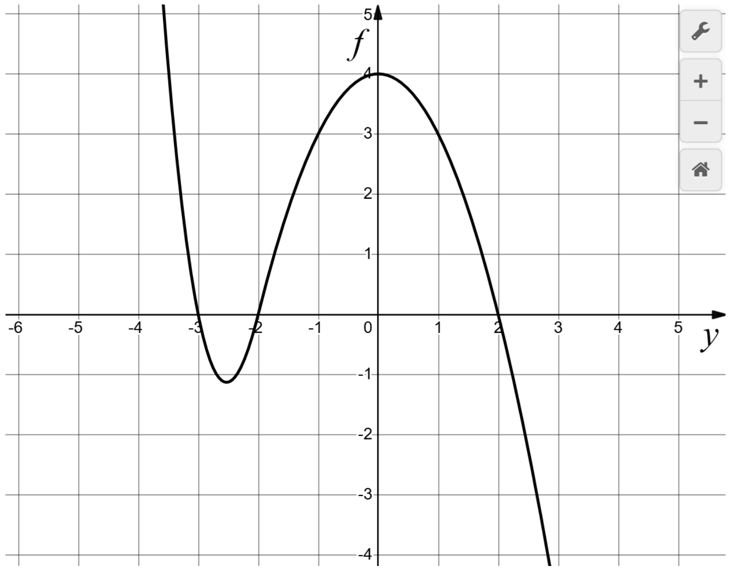
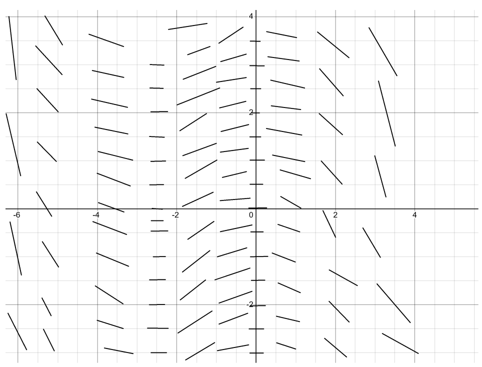


3. The given figure represents the graph of $f(y)$ when $dy/dx = f(y)$. By hand, sketch a direction field over an appropriate grid and include it as an image.



SOLUTION:



4. Consider the autonomous first-order differential equation $dy/dx = 2y - y^2$ and the initial condition $y(0) = y_0$. By hand, sketch the graph of a typical solution $y(x)$ when y_0 has the given valuesand include them as images.

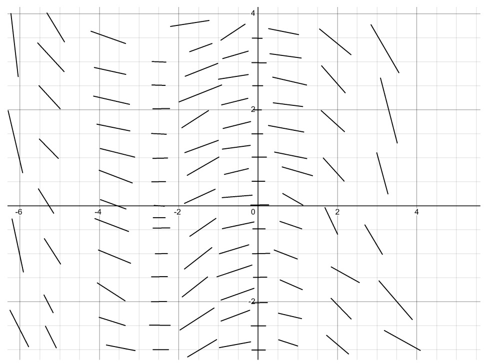
(a) $y_0 < 0$

(b) $0 < y_0 < 2$

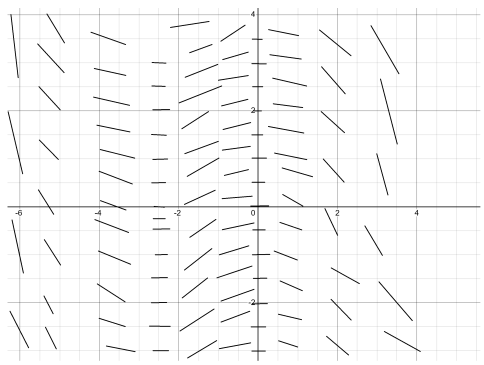
(c) $y_0 > 2$

SOLUTION: We can simply graph the direction field and pick our starting point $(0, y_0)$ on the direction field and follow the slopes to create our solution curve.

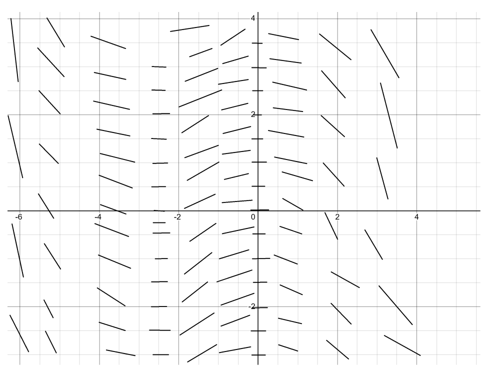
(a) $y_0 < 0$



(b) $0 < y_0 < 2$



(c) $y_0 > 2$



Homework for Section 2.2

1. Solve the initial-value problem

$$\frac{dy}{x^2+e^x} = \frac{ydx}{y^2+1},\ y(0)=1$$

SOLUTION:

$$\begin{aligned}
\frac{dy}{x^2+e^x} &= \frac{ydx}{y^2+1}\\
\frac{y^2+1}{y}dy &= \left(x^2+e^x\right)dx\\
\int \frac{y^2+1}{y}dy &= \int x^2+e^x dx\\
\int y+y^{-1}dy &= \frac{x^3}{3}+e^x+C\\
\implies \frac{y^2}{2}+\ln|y| &= \frac{x^3}{3}+e^x+C
\end{aligned}$$

$$\begin{aligned}
y(0) &= 1\\
\frac{(1)^2}{2}+\ln|(1)| &= \frac{(0)^3}{3}+e^{(0)}+C\\
\frac{1}{2}+0 &= 0+1+C\\
\implies & \boxed{C=-\frac{1}{2}}
\end{aligned}$$

2. Solve the initial-value problem

$$x+3y^2\sqrt{x+1}\frac{dy}{dx}=0,\ y(0)=1$$

SOLUTION:

$$\begin{aligned}
x+3y^2\sqrt{x+1}\frac{dy}{dx} &= 0\\
3y^2dy &= \frac{-x}{\sqrt{x+1}}dx\\
\int 3y^2dy &= \int \frac{-x}{\sqrt{x+1}}dx\\
y^3 &= -\int \frac{(x+1)-1}{\sqrt{x+1}}dx\\
y^3 &= -\int \frac{(x+1)}{\sqrt{x+1}}-\frac{1}{\sqrt{x+1}}dx\\
y^3 &= -\int \sqrt{x+1}-\frac{1}{\sqrt{x+1}}dx\\
\implies y^3 &= -\left(\frac{2}{3}(x+1)^{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}\right)+C
\end{aligned}$$

$$\begin{aligned}
&y(0)=1\\
&(1)^3=-\left(\frac{2}{3}((0)+1)^{\frac{3}{2}}-2((0)+1)^{\frac{1}{2}}\right)+C\\
&\implies \boxed{C=-\frac{1}{3}}
\end{aligned}$$

3. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < \frac{r}{k}$, find limit: $\lim\limits_{t\to\infty} C(t)$ and interpret your answer.

SOLUTION:

(a) Given that $C(0) = C_0$, we can solve the differential equation by integrating and Determine the concentration at any time t.

$$\begin{aligned}
\frac{dC}{dt} &= r - kC \\
\frac{dC}{r - kC} &= dt \\
\int \frac{dC}{r - kC} &= \int dt \\
-\frac{1}{k}\ln|r - kC| &= t + C_1 \\
|r - kC| &= e^{-k(t+C_1)} \\
r - kC &= e^{-k(t)}e^{-kC_1} \\
C(t) &= \frac{r - e^{-k(t)}e^{-kC_1}}{k} \\
C(t) &= \frac{r}{k} - \frac{e^{-kC_1}}{k}e^{-kt} \\
\implies & \boxed{C(t) = \frac{r}{k} + Ce^{-kt}}
\end{aligned}$$

(b) To find $\lim\limits_{t\to\infty} C(t)$, assuming that $C_0 < \frac{r}{k}$ implies that the constant is negative: $C < 0$. We can interpret this as when the concentration is less than $\frac{r}{k}$, the concentration will try to increase towards and approach $\frac{r}{k}$. We can see this happen vice versa when the concentration is larger, it decreases and stabalizes back to $\frac{r}{k}$.

$$\begin{aligned}
\text{Given } C_0 < \frac{r}{k},\ C(0) &= \frac{r}{k} + C < \frac{r}{k} \\
\implies & \ C < 0 \\
C(t) &= \frac{r}{k} - Ce^{-kt} \\
\lim_{t\to\infty} C(t) &= \frac{r}{k} - \lim_{t\to\infty} Ce^{-kt} \\
\lim_{t\to\infty} C(t) &= \frac{r}{k} - 0 \\
\implies & \boxed{\lim_{t\to\infty} C(t) = \frac{r}{k}}
\end{aligned}$$