MATH 238 Homework 3

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Homework for Section 2.3: 50 points

It is usually easier to do the homework on paper and then type the solutions in the latex document compiling frequently to catch the errors early! Each of the ordinary differential equation in first-order linear and will be referred to as a equation.

1. (2 points) On what rectangular regions does the equation below possess a unique solution?

$$a_1(x)\frac{dy}{dx} + a_0(x)y = q(x)$$

SOLUTION: A unique solution exists when both $\frac{a_0(x)}{a_1(x)}$ and $\frac{q(x)}{a_1(x)}$ are both continuous. In a rectangular region, it would be $\boxed{R = [a,b] \times [c,d] \text{ where } a_1(x) \neq 0 \text{ and } a_1(x),\ a_0(x),\ q(x) \text{ are continuous.}}$[1]

2. (2 points) On what rectangular regions does the equation below possess a unique solution?

$$\frac{dy}{dx} + p(x)y = q(x)$$

SOLUTION: A unique solution exists in a standard first order linear equation when $p(x)$ and $q(x)$ are both continuous. In a rectangular region, it would be $\boxed{R = [a,b] \times [c,d] \text{ where both } p(x),\ q(x) \text{ are continuous.}}$

[1] If $a_1(x) \neq 0$ and $a_1(x),\ a_0(x),\ q(x)$ are continous, then $\frac{a_0(x)}{a_1(x)}$ and $\frac{q(x)}{a_1(x)}$ are also continous.

3. (6 points) Solve the equation below

$$\frac{dP}{dt} + 5tP = P + 2t - 2$$

SOLUTION: To approach this problem, we must first put it into the standard first order linear equation $y' + P(x)y = H(x)$. After that, we will find the integrating factor $\mu(x)$.

$$\frac{dP}{dt} + 5tP = P + 2t - 2$$
$$\frac{dP}{dt} + (5t-1)P = 2t - 2$$
$$P(t) = 5t - 1, \quad H(t) = 2t - 2$$
$$\mu(t) = e^{\int P(t)dt}$$
$$\mu(t) = e^{\int 5t - 1 dt}$$
$$\mu(t) = e^{\frac{5t^2}{2} - t}$$
$$\mu(t) = e^{\frac{5}{2}\left(t^2 - \frac{2}{5}t + \frac{1}{25}\right) - \frac{1}{10}}$$

We don't care about the constant $e^{-\frac{1}{10}}$

$$\Longrightarrow \mu(t) = e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}$$
$$\mu(t)\left(\frac{dP}{dt} + (5t-1)P = 2t - 2\right)$$
$$\Longrightarrow \frac{d}{dt}(\mu(t)P) = \mu(t)(2t-2)$$
$$\int \frac{d}{dt}(\mu(t)P)dt = \int \mu(t)(2t-2)dt$$
$$\mu(t)P = \int e^{\frac{5t^2}{2} - t}(2t-2)dt$$

$$\mu(t)P = \int e^{\frac{5t^2}{2} - t}(2t-2)dt$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}(2t-2)dt$$
$$u = t - \frac{1}{5} \Longrightarrow t = u + \frac{1}{5}$$
$$dt = du, \quad (2t-2) = (2u - \frac{8}{5})$$

$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int \left(2u - \frac{8}{5}\right) e^{\frac{5}{2}u^2} du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \left(\int 2ue^{\frac{5}{2}u^2} du - \frac{8}{5}\int e^{\frac{5}{2}u^2} du\right)$$
$$v = u^2 \Longrightarrow dv = 2u\, du$$

$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \int e^{\frac{5}{2}v} dv - \frac{8}{5}\int e^{\frac{5}{2}u^2} du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{8}{5}\int e^{\left(\sqrt{\frac{5}{2}}u\right)^2} du$$
$$m = \sqrt{\frac{5}{2}}u \Longrightarrow \sqrt{\frac{2}{5}}dm = du$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{8}{5}\int e^{m^2}\sqrt{\frac{2}{5}}dm$$

$$\text{Remark: } \operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt$$
$$C \cdot \operatorname{erfi}(x) = C\frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt = \int e^{m^2} du$$
$$C = \frac{\sqrt{\pi}}{2}$$

$$\Longrightarrow e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}u^2} - \frac{\sqrt{\pi}}{2}\frac{8}{5}\sqrt{\frac{2}{5}}\operatorname{erfi}(m) + C$$
$$e^{\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} P = \frac{2}{5}e^{\frac{5}{2}\left(t - \frac{1}{5}\right)^2} - \frac{4}{5}\sqrt{\frac{2\pi}{5}}\operatorname{erfi}\left(\sqrt{\frac{5}{2}}\left(t^2 - \frac{1}{5}\right)\right) + C$$
$$\boxed{P = \frac{2}{5} - \frac{4}{5}\sqrt{\frac{2\pi}{5}}\operatorname{erfi}\left(\sqrt{\frac{5}{2}}\left(t^2 - \frac{1}{5}\right)\right)e^{-\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2} + Ce^{-\frac{5}{2}\left(t^2 - \frac{1}{5}\right)^2}}$$

4. (7 points) Solve the equation below

$$2L\frac{di}{dt} + 3Ri = E, i(0) = i_0$$

where L, R and E are constants.

SOLUTION:

$$2L\frac{di}{dt} + 3Ri = E, i(0) = i_0$$
$$\frac{di}{dt} + \frac{3R}{2L}i = \frac{E}{2L}$$
$$\mu(t) = e^{\frac{3R}{2L}t}$$

$$\mu(t)\left(\frac{di}{dt} + \frac{3R}{2L}i = \frac{E}{2L}\right)$$
$$\Longrightarrow \frac{d}{dt}(e^{\frac{3R}{2L}t}i) = e^{\frac{3R}{2L}t}\frac{E}{2L}$$
$$e^{\frac{3R}{2L}t}i = \int e^{\frac{3R}{2L}t}\frac{E}{2L}dt$$
$$e^{\frac{3R}{2L}t}i = \frac{\cancel{2L}}{3R}e^{\frac{3R}{2L}t}\frac{E}{\cancel{2L}} + C$$
$$i(t) = \frac{E}{3R} + Ce^{-\frac{3R}{2L}t}$$

$$i(0) = i_0 = \frac{E}{3R} + C$$
$$C = i_0 - \frac{E}{3R}$$
$$\Longrightarrow \boxed{i(t) = \frac{E}{3R} + \left(i_0 - \frac{E}{3R}\right)e^{-\frac{3R}{2L}t}}$$

5. (7 points) Solve the equation below

$$\cos^2 x \sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 4$$

SOLUTION:

$$\cos^2 x \sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 4$$
$$\frac{dy}{dx} + (\cot x)y = 4\sec^2 x \csc x$$
$$\mu(x) = e^{\ln|\sin x|} = \sin x$$

$$\mu(x)\left(\frac{dy}{dx} + (\cot x)y = 4\sec^2 x \csc x\right)$$
$$\Longrightarrow \frac{d}{dx}(e^{\sin x}y) = \cancel{\sin x}4\sec^2 x\cancel{\csc x}$$
$$y\sin x = \int \sec^2 x dx$$
$$y\sin x = \tan x + C$$
$$\Longrightarrow \boxed{y(x) = 4\sec x + C\csc x}$$

6. (10 points) Solve the equation below

$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$$

SOLUTION:

$$
\begin{aligned}
&(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x} \\
&\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1} \\
&\mu(x) = e^{\int \frac{x+2}{x+1}\,dx} \\
&\mu(x) = e^{\int \frac{x+1+1}{x+1}\,dx} \\
&\mu(x) = e^{\int 1+\frac{1}{x+1}\,dx} \\
&\mu(x) = e^{x+\ln|x+1|} = (x+1)e^{x}
\end{aligned}
$$

$$
\begin{aligned}
&\mu(x)\left(\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}\right) \\
&((x+1)e^{x})\left(\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}\right) \\
\Longrightarrow &\frac{d}{dx}\left((x+1)e^{x}y\right) = \cancel{(x+1)e^{x}}\frac{2x\cancel{e^{-x}}}{\cancel{x+1}} \\
&(x+1)e^{x}y = \int 2x dx \\
\Longrightarrow &\boxed{y = \frac{x^2+C}{(x+1)e^{x}}}
\end{aligned}
$$

7. (16 points) Solve the equation below

$$\frac{dy}{dx} + 6xy = f(x)$$

where

$$f(x) = \begin{cases} x^2 & x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$$

Graph the right side and one of the the solutions on separate graphs.

SOLUTION:

$$\frac{dy}{dx} + 6xy = 2x - 1,\ x \geq 1$$

$$\mu(x) = e^{3x^2}$$

$$\frac{d}{dx}(e^{3x^2} y) = e^{3x^2}(2x - 1)$$

$$e^{3x^2} y = \int e^{3x^2}(2x - 1)dx$$

$$e^{3x^2} y = \int 2xe^{3x^2} - e^{3x^2} dx$$

$$e^{3x^2} y = \frac{1}{3}e^{3x^2} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right) + C$$

$$\Longrightarrow \boxed{y(x) = \frac{1}{3} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right)e^{-3x^2} + C_1 e^{-3x^2}, \quad x \geq 1}$$

$$\frac{dy}{dx} + 6xy = x^2,\ x < 1$$

$$\mu(x) = e^{3x^2}$$

$$\frac{d}{dx}(e^{3x^2} y) = e^{3x^2} x^2$$

$$e^{3x^2} y = \int e^{3x^2} x^2 dx$$

$$u = x, dv = xe^{3x^2} dx$$

$$du = dx, v = \frac{1}{6}e^{3x^2}$$

$$e^{3x^2} y = \frac{1}{6}xe^{3x^2} - \int \frac{1}{6}e^{3x^2} dx,\ u = \sqrt{3}x,\ \frac{du}{\sqrt{3}} = dx$$

$$e^{3x^2} y = \frac{1}{6}xe^{3x^2} - \frac{1}{12}\sqrt{\frac{\pi}{3}}\text{erfi}(\sqrt{3}x)$$

$$\Longrightarrow \boxed{y(x) = \frac{x}{6} - \frac{\sqrt{\pi}\text{erfi}(\sqrt{3}x)}{12\sqrt{3}e^{3x^2}} + C_2 e^{-3x^2}, \quad x < 1}$$

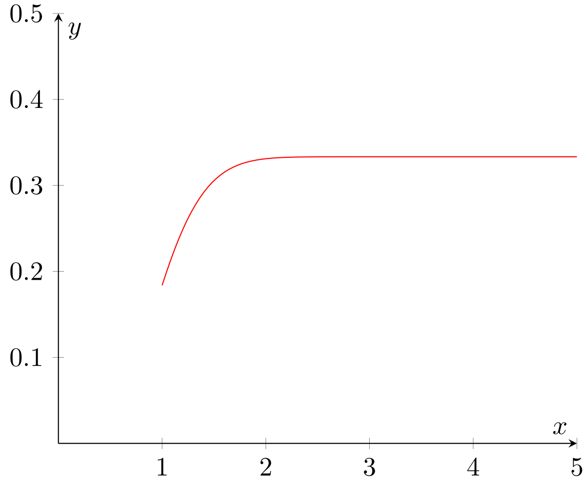

Figure 1: Right Graph of solution curve when $C_1 = 0$

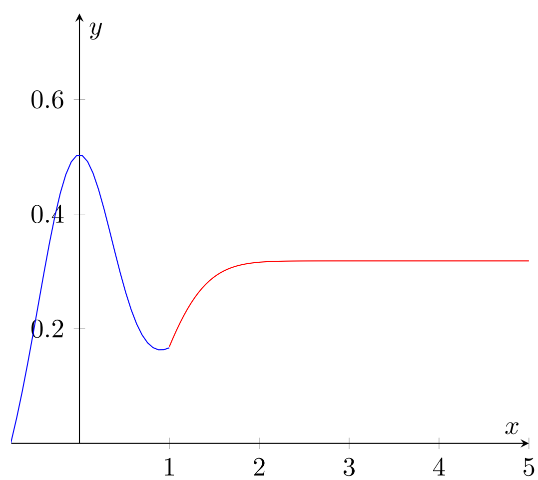

Figure 2: Graph of both solution curves when $C_1 = 0$ $C_2 = 0.503611718084$

$$\boxed{y(x) = \begin{cases} \frac{x}{6} - \frac{\sqrt{\pi}\text{erfi}(\sqrt{3}x)}{12\sqrt{3}e^{3x^2}} + C_2 e^{-3x^2} & x < 1 \\ \frac{1}{3} - \frac{1}{2}\sqrt{\frac{\pi}{3}}\text{erfi}\left(\sqrt{3}x\right)e^{-3x^2} + C_1 e^{-3x^2} & x \geq 1 \end{cases}}$$