2.4 Notes

someonecantcode

February 5, 2026

Contents

1 Normal Exact ODES

DEFINITION 1. An exact equation is defined as the following:

$$M(x,y)dx + N(x,y)dy = 0, \text{where } M_y = N_x \tag{1}$$

It should be analogous to

$$f_x dx + f_y dy = dz = 0 \Longrightarrow z = f(x,y) = c \tag{2}$$

c is a constant.

1.1 solving process

Here is the formal general process:

$$\begin{aligned}
&f_x = M, \quad f_y = N \\
&\int \frac{\partial f}{\partial x} dx = \int M(x,y)dx \\
&f(x,y) = \int M(x,y)dx + h(y), \quad \text{this is your general solution, now solve for } h(y) \\
\\
&\frac{\partial}{\partial y}\left(f(x,y) = \int M(x,y)dx + h(y)\right) \\
&\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + h'(y) \\
&\text{Recall that } f_y = N \\
\Longrightarrow &N(x,y) = \frac{\partial}{\partial y}\int M(x,y)dx + h'(y) \\
&N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx = h'(y) \\
\\
&\text{substitue it back into the solution} \\
&\int\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right)dy = \int h'(y)dy = h(y) \\
&f(x,y) = \int M(x,y)dx + h(y) \\
\Longrightarrow &\boxed{f(x,y) = \int M(x,y)dx + \int\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right)dy = C}
\end{aligned}$$

This is not a formula but this is the general solving practice. Can be applied for in the direction of y.

1.2 Luke Rawling shortcut method

Simply just take the derivatives instead of that weird definition method. You use pattern recognition to see which one is a function of both x and y which is our $f(x,y)$, the functions of x are $h(x)$, and the functions of y are $g(y)$. This is the process and assume $M_y = N_x$ or the ODE is exact in English. You may need to seperate out terms to isolate them.

$$
\begin{aligned}
&M(x,y)dx + N(x,y)dy = 0 \\
&\int M(x,y)dx + \int N(x,y)dy = 0 \\
&f(x,y) + g(y) + f(x,y) + h(x) = 0 \\
&2f(x,y) + g(y) + h(x) = 0 \\
\implies &\boxed{f(x,y) + g(y) + h(x) = C}
\end{aligned}
$$

Example:

$$
\begin{aligned}
&(y-1)dx + (x+1)dy = 0 \\
&\frac{\partial}{\partial y}(y-1) = 1, \quad \frac{\partial}{\partial x}(x+1) = 1 \\
&M_y = N_x \\
&\int (y-1)dx + \int (x+1)dy = 0 \\
&xy - y + xy + x = 0 \\
&2xy - y + x = 0 \\
&xy \text{ is our } f(x,y)\text{, having both } x \text{ and } y, \quad g(y) = -y, h(x) = x \\
&2f(x,y) + g(y) + h(x) = 0 \\
\implies &\boxed{xy - y + x = C}
\end{aligned}
$$

Yes test it by yourself and you should get the same answer.

2 Near-exact ODEs

This method is simply multiplying the ODE by an integrating factor usually denoted as ($\mu(x,y)$ to be formal) either a function of $\mu(x)$ or $\mu(y)$ to make the ODE exact. Afterwards, you just solve exactly the same as an exact problem.

Find your $P(x)$ or $P(y)$ and use the following formula.

DEFINITION 2. The integrating factor is the following:

$$
\mu(x) = e^{\int P(x)dx}, \quad \mu(y) = e^{\int P(y)dy} \tag{3}
$$

I will not derive, solve, and prove the formulas for $P(x)$ and $P(y)$. An easy way to memorize this is that these are the exact same partial derivatives as when you are testing for exactness. KEY THING, THESE SHOULD BE FUNCTIONS OF EITHER x or y. YOU MUST TEST EITHER.

DEFINITION 3. Formulas for $P(x)$ and $P(y)$.

$$
P(x) = \frac{M_y - N_x}{N}, \quad \mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \tag{4}
$$

$$
P(y) = \frac{N_x - M_y}{M}, \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy} \tag{5}
$$

2.1 Example

Example (video walkthrough https://www.youtube.com/watch?v=6eNgoXoVTWM) :

$$
\begin{aligned}
&(y^2+2xy)dx - x^2dy = 0\\
&\frac{\partial}{\partial y}(y^2+2xy) = 2y+2x, \quad \frac{\partial}{\partial x}(-x^2) = -2x, \text{DO NOT MAKE THE MISTAKE OF } x^2\\
&2y+2x \neq -2x\\
&P(x) = \frac{M_y - N_x}{N} = \frac{2y+2x-(-2x)}{x^2} = \frac{2y+4x}{x^2}, \quad \text{NOT FUNCTION OF X}\\
&P(y) = \frac{N_x - M_y}{M} = \frac{(-2x)-(2y+2x)}{y^2+2xy} = \frac{-2y-4x}{y(y+2x)} = \frac{-2(y+2x)}{y(y+2x)}\\
&\frac{N_x - M_y}{M} = -\frac{2}{y} = P(Y), \text{IS A FUNCTION OF Y}\\
&\mu(y) = e^{\int P(y)dy} = e^{\int -2y^{-1}dy} = e^{\ln|y^{-2}|} = y^{-2}, \text{we ignore sign due to } x^2.\\
\\
&\mu(y)\big((y^2+2xy)dx - x^2dy = 0\big)\\
&\frac{1}{y^2}\big((y^2+2xy)dx - x^2dy = 0\big)\\
&(1+2xy^{-1})dx - x^2y^{-2}dy = 0\\
&\int(1+2xy^{-1})dx - \int x^2y^{-2}dy = 0\\
&x + \frac{x^2}{y} + \frac{x^2}{y} = 0\\
&2f(x,y) = \frac{x^2}{y}, g(x) = x\\
\implies &\boxed{f(x,y) = \frac{x^2}{y} + x = C}
\end{aligned}
$$