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Title: Limit theorems for a class of critical superprocesses with stable branching
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Summary:

A critical superprocess with a general spatial motion is considered with spatially dependent stable branching mechanism. It is shown that the process 'extinguishes' and the survival probability is regularly varying. Conditioned on survival, the large time scaling limit is obtained too. It turns out that the lowest stable index determines the results quantitatively. The proofs are based on a spine decomposition and the fact that the assumptions imply the ergodicity of the spine.

The article under review can constitute a nice contribution to the long list of Yaglom type theorems for (spatial) branching processes; the main point being that the assumption on the finiteness of the variance is dropped. I am happy to

recommend the publication of this article

in SPA, provided the authors address the points below.

Errors/Suggestions:

(1) (**Important!**) General remark: There are a number of restrictive looking assumptions and thus it is crucial that the authors provide examples which are covered by the setting. It is true that before Assumption 4, there is a reference to [24] 'for a list of examples,' but this is somewhat hidden. At least a brief summary of examples should be placed at a more visible spot that convinces the reader that the scope is not as restricted as it seems.

In [24] there are indeed 12 examples given. I would at least vaguely describe (some of) these, possibly in an appendix and definitely without proofs, just mentioning that details and proofs (of these being examples) are in [24]. If it is relegated to an appendix then somewhere at the beginning at a visible spot I'd refer the reader to the appendix.

(2) p. 2 in the middle: define positively regular

(3) p. l.4: 'They showed' instead 'he showed.'

(4) p. 4 the fourth displayed formula $\psi(x, z) = ...$: Explain how this fits into the general formula with the integral term.

(5) p. 4 (1.16) and (1.17): perhaps mention that the limits do not depend on x resp. μ.

(6) p.5 right after (1.18): Here β has not been defined yet, only later in Assumption 4 it will be.

(7) p. 6 l.-15: Why isn't it $P_t^{\beta}\phi(x) = e^{\lambda t}\phi(x)$ and $P_t^{\beta *}\phi^*(x) = e^{\lambda t}\phi^*(x)$?

(8) p.6 paragraph after Assumption 2: The notation is a bit confusing. So ϕ is the principal eigenfunction for the motion with the potential β but φ is the same without the potential. This should be made more clear.

(9) Same place (**Important!**): It is not clear why φ is well defined. Let's say ξ is a diffusion process. If $\mathcal{L}$ is the elliptic operator corresponding to it and $\lambda(\mathcal{L})$ is its (generalized) principal eigenvalue then $\mathcal{L} - \lambda(\mathcal{L})$ may or may not have a Green's function. In the second case everything is fine and φ is uniquely determined up to constant multiples; it is the ground state. However, in the first case the cone of positive harmonic functions w.r. to $\mathcal{L} - \lambda(\mathcal{L})$ is not necessarily one dimensional. For example, if we know that ξ is recurrent (and so $\lambda(\mathcal{L})$=0), then we are fine. But when it is transient, uniqueness does not follow.

(10) p. 8 l.-6: I would rather say that the family of functions is just $(e^{\lambda t}\phi)_{t\geq 0}$, while $(e^{\lambda t}X_t(\phi))_{t\geq 0}$ is indeed the martingale.

(11) p. 11 Lemma 2.4 and Corollary 2.5: Why do we suddenly use α instead of γ?

(12) p.12 Explain how you get (2.4) from (2.3).

(13) p.13 top: Why not '$\mathcal{N}$ is a Poisson random measure on $\mathbb{W}$?'

(14) p.14 top: Give an intuitive description of what the three bullet points formulate.

(15) p.17 l.-7: The RHS has a factor $e^{\lambda t}$ too. Also, you should explain that this follows from the ergodicity of the ϕ-transformed motion. (Two lines later you should refer to Proposition 3.1.)

(16) p.18 l. 8: ... with the same speed

(17) p.18 l.-11: Please remind us that $\nu = \phi^* m$.

(18) p. 19, above (3.10): On the other hand,...

(19) p. 19–20: It's easier to read the proof backwards. So perhaps you should tell at the beginning that your goal is to show (3.12) and (3.13), etc. For the same reason, I'd move the few lines "Proof of Thm 1.1(2)" from p. 22 to the beginning so that one can understand why the two propositions are needed.

(20) p. 21, l. 1: why is this "according to (3.2)"?

(21) p. 21, before (3.14) Make it one sentence: if ... then ...

(22) p. 21 (3.14): m is missing from the scalar product.

(23) p. 24, Lemma 3.5: Say that the assumption on γ_0 is $\gamma_0 \in (1,2)$. This looks like a standalone lemma, so perhaps the reader does not connect it with the previous definition of γ_0.

(24) p. 24 l.-2: When $0 < p < 1$, this is in fact not even a norm.

(25) p. 26, l. 1: explain a bit more: "Since this is true for all k, we get that $F(\alpha + \theta) = 0$ for all $\theta \leq 1/C, \alpha \leq \rho$, and so $F = 0$ on $[0, \rho + 1/C]$."

(26) pp. 24-26: Don't you want to move the analytical lemmas Lemma 3.5 and Lemma 3.6 move into an appendix?

(27) p. 26: Instead of (3.28), why don't you simply say that F satisfies the inequality in Lemma 3.6 (number that one) with $C = \gamma_0^{\frac{1}{\gamma_0 - 1}}$?

(28) p. 28, end of Step 1: $J_g(t, r, \xi)$ and the two other terms you upper estimate are random variables. So, do you mean the upper bounds almost surely?

(29) p. 29 l.-12: You should also recall that $\langle f, \phi^* \rangle_m = 1$.

(30) p. 32 l.-7: reverse instead of revers. Also "use this ... to have that..."

(31) p. 33 last sentence: I would say: "Finally, $M \equiv 0$ clearly implies that $\lim_t I_3(t, \theta, x) = 0$, and thus completes the verification of (3.29)."