Limit theorems for a class of critical superprocesses with stable branching

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Outline

Background/Kolmogorov's result

Theorem (Kesten, Ney and Spitzer (1966))

Let $(Z_n)_{n\in\mathbb{N}}$ be a critical Galton-Watson process with offspring variance $\sigma^2 \in (0,\infty)$. Then

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2}.$$

- Kolmogorov (1938) obtained the above Theorem under a three moment condition.

Background/Yaglom's result

Theorem (Kesten, Ney and Spitzer (1966))

Let $(Z_n)_{n\in\mathbb{N}}$ be a critical Galton-Watson process with offspring variance $\sigma^2 \in (0,\infty)$. Then

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{d} \frac{\sigma^2}{2}\mathbf{e},$$

where $\mathbf{e}$ is an exponential random variable with mean 1.

- Yaglom (1947) obtained the above Theorem under a stronger condition.

Background/Slack's result

Theorem (Slack (1968))

Let $(Z_n)_{n\geq 0}$ be a critical Galton-Watson process with offspring generating function $f(s) = s + (1-s)^\alpha l(1-s)$ where $\alpha \in (1,2]$ and l is a slowly varing function at 0. Then $P(Z_n > 0) = n^{-1/(\alpha-1)}L(n)$ where L is slowly varying at ∞; and

$$\{P(Z_n > 0)Z_n; P(\cdot|Z_n > 0)\} \xrightarrow[n\to\infty]{d} \mathbf{z}^{(\alpha-1)},$$

where $\mathbf{z}^{(\alpha-1)}$ is a positive random variable with Laplace transform

$$E[e^{-u\mathbf{z}^{(\alpha-1)}}] = 1 - (1 + u^{-(\alpha-1)})^{-1/(\alpha-1)}, \quad u \geq 0.$$

- Zolotarev (1957) obtained the above Theorem under a stronger condition.

	$\alpha = 2$: Analytical	$\alpha = 2$: Probabilistic	$\alpha \in (1, 2)$
Galton-Watson (GW) processes	Kolmogorov (1938) Yaglom (1947) Kesten, Ney and Spitzer (1966)	Lyons, Pemantle and Peres (1995) Geiger (1999) Geiger (2000) Ren, Song and Sun (2018)	Zolotarev (1957) Slack (1968)
Multitype GW	Joffe and Spitzer (1967)	Vatutin and Dyakonova (2001)	Goldstein and Hoppe (1978)
Continuous time GW process	Athreya and Ney (1972)	-	Vatutin (1977)
Continuous time Multitype GW process	Athreya and Ney (1974)	-	Vatutin (1977)
Branching Markov processes	Asmussen and Hering (1983)	Powell (2015)	Asmussen and Hering (1983)
Continuous-state branching processes	Li (2000) Lambert (2007)	Ren, Song and Sun (2019)	Kyprianou and Pardo (2008) Ren, Yang and Zhao (2014)
Superprocesses	Evans and Perkins (1990) Ren, Song and Zhang (2015)	Ren, Song and Sun (2019)	Ren, Song and Sun (2019+)

Model/Settings

- E be a locally compact separable metric space;
- $\mathcal{M}$ be the collection of all the finite Borel measures on E;
- Spatial motion $\{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$ be an E-valued Hunt process with transition semigroup $(P_t)_{t\geq 0}$;
- Branching mechanism ψ be a function from $E \times [0,\infty)$ to $[0,\infty)$ s.t.

$$\psi(x,z) := -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy),$$

where β is a bounded measurable function on E, α is a bounded non-negative measurable function on E, and π is a kernel from E to $(0,\infty)$ s.t.

$$\sup_{x\in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x,dy) < \infty.$$

Model/Superprocesses

- For each measure μ and function f, write $\mu(f) := \int f d\mu$ whenever the integral make sense.
- We say a measurable function f on $\mathbb{R}_+ \times E$ is locally bounded if

$$\sup_{s\in[0,t],x\in E} |f(s,x)| < \infty, \quad t \in \mathbb{R}_+.$$

Definition (Superprocesses)

An $\mathcal{M}$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}}\}$ is called a (ξ,ψ)-superprocess if for each $\mu \in \mathcal{M}$, $f \in b\mathscr{B}_+$ and $t \geq 0$ we have

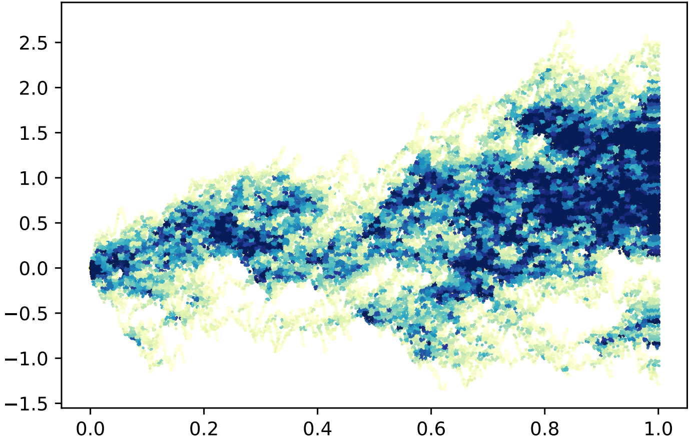
$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}.$$

Here, $(t,x) \mapsto V_t f(x)$ on $[0,\infty) \times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \int_0^t P_{t-s}\psi(\cdot, V_s f(\cdot))(x) ds = P_t f(x).$$

Model/Superprocesses

- Superprocess arose as high-density limits of branching particle systems. (Watanabe 1968, Dawson 1975, Dynkin 1991).



Model/Assumptions/Spatial motion

- The mean semigroup of the superprocess:

$$\mathbf{P}_{\delta_x}[X_t(f)] = P_t^{\beta} f(x) := \Pi_x[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)].$$

Assumption 1.

There exist a σ-finite measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot) : t > 0\}$ on $E \times E$ such that

- $P_t f(x) = \int_E p_t(x,y) f(y) m(dy)$;
- $\int_E p_t(y,x) m(dy) \leq 1$;
- $\int_E \int_E p_t(x,y)^2 m(dx) m(dy) < \infty$;
- $x \mapsto \int_E p_t(x,y)^2 m(dy)$ and $x \mapsto \int_E p_t(y,x)^2 m(dy)$ are both continuous.

Model/Assumptions/Spatial motion

Under Assumption 1, we can say the following:

- $(P_t^{\beta})_{t\geq 0}$ and its disjoint $(P_t^{\beta *})_{t\geq 0}$ are strongly continuous semigroups of compact operators in $L^2(E, m)$.
- Let L and L^* be the generators of $(P_t^{\beta})_{t\geq 0}$ and $(P_t^{\beta *})_{t\geq 0}$, respectively. Then $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$ is a common eigenvalue of multiplicity 1 for both L and L^*.
- The corresponding eigenfunctions ϕ of L and ϕ^* of L^* can be chosen to be strictly positive and continuous everywhere on E.
- Normalize ϕ and ϕ^* by $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$ so that they are unique.
- Operator P_t^{β} has transition density $p_t^{\beta}(x, y)$ with respect to measure m.

Model/Assumptions/Mean semigroup and branching mechanism

Assumption 2. (Critical and Intrinsic Ultracontractive)

- $\lambda = 0.$
- $\forall t > 0, \exists c_t > 0, \forall x, y \in E, \quad p_t^\beta(x, y) \le c_t \phi(x)\phi^*(y).$

Assumption 3 (Stable branching)

The branching mechanism ψ is of the form:

$$\psi(x, z) = -\beta(x)z + \kappa(x)z^{\gamma(x)},$$

where $\beta \in \mathscr{B}_b(E), \gamma \in \mathscr{B}_b^+(E)$, $\kappa \in \mathscr{B}_b^+(E)$ with $1 < \gamma(\cdot) < 2$. We also assume that

$$\gamma_0 := \operatorname{ess\,inf}_{m(dx)} \gamma(x) > 1$$

and $\kappa_0 := \operatorname{ess\,inf}_{m(dx)} \kappa(x) > 0.$

Theorem 1

Under Assumptions 1,2 and 4, we have

(1) $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$, for each $t > 0$ and $x \in E$.

(2) For each $\mu \in \mathcal{M}$, $\mathbf{P}_\mu(\|X_t\| \neq 0) = t^{-\frac{1}{\gamma_0 - 1}} L(t)$ where $L(t)$ is a slowly varing function at ∞.

Write $C_X := \langle \mathbf{1}_{\gamma(\cdot)=\gamma_0} \kappa \cdot \phi^{\gamma_0}, \phi^* \rangle_m$ and $\eta_t := \big(C_X(\gamma_0 - 1)t\big)^{-1/(\gamma_0 - 1)}$. Further assume that $m(x : \gamma(x) = \gamma_0) > 0$, then

(3) $\lim_{t\to\infty} \eta_t^{-1} \mathbf{P}_\mu(\|X_t\| \neq 0) = \mu(\phi)$;

(4) for each $f \in \mathscr{B}^+(E)$ with $\langle f, \phi^* \rangle_m > 0$ and $\|\phi^{-1} f\|_\infty < \infty$,

$$\{\eta_t X_t(f); \mathbf{P}_\mu(\cdot | \|X_t\| \neq 0)\} \xrightarrow[t\to\infty]{d} \langle f, \phi^* \rangle_m \mathbf{z}^{(\gamma_0 - 1)}.$$

Remarks/Slack type result

- The asymptotic behavior of the critical superprocesses with spatially dependent stable branching is dominated by the heaviest tail γ_0.
- The weak limit is universal: the distribution of $\mathbf{z}^{(\gamma_0 - 1)}$ is only related to γ_0.
- We proof the above results by characterizing some measure transform of the superprocesses.

Definition (Size-biased transform)

Let G be a non-negative measurable function which is integrable with respect to a σ-finite measure Q. A probability measure Q^G is called the G-transform of Q if

$$dQ^G = \frac{G}{Q[G]} dQ.$$

Methods: Size-biased add-on

- Let X be an non-negative r.v. with finite mean. Define

$$\color{red}{F(\theta)} := \frac{P^X[e^{-\theta X}]}{P[e^{-\theta X}]}, \quad \theta \geq 0.$$

  Then it holds that

$$-\log \mathbf{P}[e^{-\theta X}] = \mathbf{P}[X] \int_0^\theta \color{red}{F(r)} dr, \quad \theta \geq 0.$$

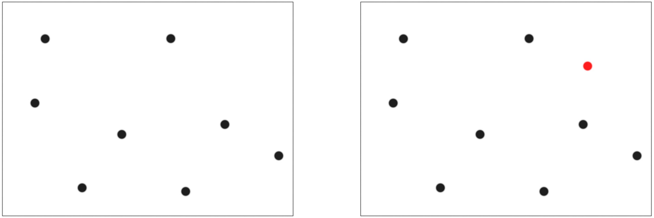
- We call $\color{red}{F(\theta)}$ the size-biased add-on function of the random variable X.
- The distribution of X is characterized by its mean and its size-biased add-on function.

Methods/Poisson random measures

Lemma 1

Let $\mathcal{N}$ be a Poisson random measure with intensity measure N. Let F be a non-negative testing function with $0 < N[F] < \infty$. Then

$$\{\mathcal{N}; P^{\mathcal{N}(F)}\} \overset{d}{=} \{\mathcal{N} + \delta_s; P \otimes N^F(ds)\}.$$



Methods/Kuznetsov measures

- The Kuznetsov measures $(\mathbb{N}_x)_{x\in E}$ of superprocess $(X_t)_{t\geq 0}$ is given by the following Theorem:

Theorem (Li (2011) Theorem 8.24)

There is a family of σ-finite measures $(\mathbb{N}_x)_{x\in E}$ on space

$$\mathbb{D} := \{\mathcal{M}\text{-valued càdlàg functions on } [0,\infty) \text{ with the null measure as a trap}\}$$

such that for each $x \in E$,

$$\{(X_t)_{t>0}; \mathbf{P}_{\delta_x}\} \overset{d}{=} \left(\int_{\mathbb{D}} w_t \mathcal{N}(dw)\right)_{t>0},$$

where $\mathcal{N}$ is a Poisson random measure on $\mathbb{D}$ with intensity measure $\mathbb{N}_x$.

Methods/Measure transform of superprocesses

Theorem 3.

Let $\mu \in \mathcal{M}$ and write $\mathbb{N}_\mu(\cdot) := \int_E \mathbb{N}_x(\cdot)\mu(dx)$. For each non-negative measurable function F on $\mathbb{D}$ with $\mathbb{N}_\mu[F] \in (0, \infty)$, we have

$$\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathcal{N}(F)}\} \overset{d}{=} \{(X_t)_{t\geq 0}; \mathbf{P}_\mu\} \otimes \{(w_t)_{t\geq 0}; \mathbb{N}_\mu^F(dw)\}.$$

We can characterize $\{(w_t)_{t\geq 0}; \mathbb{N}_\mu^F(dw)\}$ while

- $F(w) = w_t(\phi)$ using the classical Spine Decomposition Theorem.
- $F(w) = w_t(f)$ using a generalized Spine Decomposition Theorem.
- $F(w) = w_t(\phi)^2$ using a 2-Spine Decomposition Theorem.

Few References

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感谢!