A review of “Stable Central Limit Theorems for Super Ornstein-Uhlenbeck Processes”

This paper is concerned with the limit behavior of the super Ornstein-Uhlenbeck process $((X_t)_{t\geq 0}, P_\mu)$ with an infinite variance-branching mechanism. More precisely, the paper is interested in the limit behavior of $\langle f, X_t\rangle$ as $t \to \infty$, for functions f that have polynomial growth. There are three different behaviors identified, depending on the parameters α, β, κ_f, b. α and β are the parameters of the branching mechanism, b is a parameter of the Ornstein Uhlenbeck process, and κ_f is the order of the spectral decomposition of f with respect to the generator L of the Ornstein Uhlenbeck process. The authors prove a law of large numbers theorem for a case called “large branching rate regime”, which includes continuous and bounded test functions as a special case for which a law of large numbers was previously shown by other authors. The authors also prove a central limit theorem for all test functions f of polynomial growth in the cases “critical” and “small branching rate regimes” and a central limit theorem for all f with a specific finite spectral representation w.r.t L in the large branching rate regime.

I have not found any mathematical errors. I believe the results are important and worthwhile for publication; however, the presentation should be revised before the paper can be accepted. There are many duplications in the current presentation, see the comments below for more details. With a more general formulation, these duplications can be removed reducing the number of pages dramatically. For example, it is realistic that the same content can be presented in 45 pages rather than 60 pages.

Also, the authors should give stronger interpretations of the assumptions and the results. What do the the parameters α, β, κ_f, b represent, and why does the limit behavior change with these parameters? For example, what's the interpretation of the inequality $\alpha\beta > \kappa_f b(1+\beta)$? Why is this referred to this as the large branching rate regime?

- Page 2. line -11. Please provide an interpretation of the form of ψ. If you were to approximate this superprocess by a branching particle system, what are the implications on the distribution of the number of off-springs? Does the term ψ_1 imply that the offspring distribution is heavy tailed?

- Page 2. Line -8. Please provide an interpretation of the expression $\alpha\beta - \kappa_f b(1+\beta)$? Perhaps it might be easier to interpret if you wrote the inequality $\alpha\beta > \kappa_f b(1+\beta)$ as $\alpha\frac{\beta}{1+\beta} > \kappa_f b$. This way the parameters on the left are branching related, whereas the parameters on the right are related to the Ornstein-Uhlenbeck process. Is there an intuitive explanation why ρ doesn't play a role? Why does b come into play but σ does not?

- Page 4 equation 1.5. Please provide an interpretation of the form of the invariant density. For example, you can mention that it has a mode at the origin, it is symmetric around the origin and $\frac{b}{\sigma_2}$ determines the spread etc. Please also give some intuition about the role of the specific form of density in the results. If the invariant density had different properties, would the results still hold?

- Page 5 line -9. Please provide an interpretation of κ_f. What does it mean for κ_f to be large or small?

- Page 5 line -3. Please explain why the three cases are referred to as large, critical and small branching rate regimes?

- Page 6 Why is Theorem 1.5 restricted to test functions in $\mathcal{C}_l$, whereas Theorem 1.6 and 1.7 allow f to be in $\mathcal{P}$? Is this a limitation of the methodology used or there is a more substantial reason? Please explain.

- There is substantial repetition in the statements of Theorem 1.5, 1.6 and 1.7. These can be combined into a single Theorem, and differences can be managed as different cases in the same proof.

- Page 8 Definitions of $m(f)$, $\tilde{m}(f)$, $\bar{m}(f)$ can be combined into a single definition, and instead of using different notation $\tilde{m}$, $\bar{m}$, the definition of $m(f)$ can be extended to f in the critical and the large branching rate regimes by the formula given for $\tilde{m}$ and $\bar{m}$.

- Page 8. Instead of calling this section "some intuitive explanation", it would be more appropriate to call it "an outline of the methodology".

- Page 10 Line 1. Please provide an interpretation of the examples for ψ ? In which contexts do these examples arise?

- Page 12. Lemma 2.4 There is substantial repetition in statements 2 and 3, please combine these into one statement. In fact, it would be even better if all the hypothesis on the operators, index sets , etc can be given at the beginning of the lemma, and then all the results can be listed without saying "Suppose that" each time. Similarly assumptions don't need to be restated in the proof. There are duplications in the proof, these should be removed as well. For example you can say, "If $f, g \in \mathcal{P}^+$ with $f \leq g$, then $R_A R_B f \leq R_A R_B g$, and $R_A \times R_B f \leq R_A \times R_B g$ and $R_A + R_B f \leq R_A + R_B g$, and ...". These should reduce the length of Lemma 2.4 and and its proof by half.

- Page 16 17 Lemma 2.6, Proposition 2.7 If you use a single notation $m_t(f)$, instead of m_t, $\bar{m}$ and $\tilde{m}$, you can simply say that $\theta \mapsto exp(m_t \theta f]$ is the characteristic function of ..." without repeating it 5 times. No need to provide separate statements for m_t and m, since m can be considered as a special case of m_t for a suitable value of t, (e.g. $t = \infty$, or $t = \Delta$.)

- Page 19 Combine Lemma 2.9 and 2.10 into a single Lemma.

- Page 20 Line 3. Please give a reference after the statement "It is well known that".

- Page 22 Lines 8-12 The argument given here is not clear. Please provide more details.

- Page 22 Equations 2.18-2.19 These equations can be stated in a single line. For example, "if $0 < \lambda \leq v_t(\lambda) < \bar{v} < v_t(\lambda^*) < \lambda^* < \infty$, then ...".

- Page 24 Please give more details about the argument from [10, Theorem 3.3.6].
- Page 27 Line -2 Typo. *.. is an eigenfunction of L, ...*
- Page 31. Lemma 3.4 and Lemma 3.7 are nearly duplicates of each other. They should be written in a unifying way to cover both cases. Same applies to their proofs.
- Proof of Lemma 3.4. Please indicate clearly where $\alpha\beta \le \kappa_f b(1+\beta)$ is used. Same goes for the case $\alpha\beta > \kappa_f b(1+\beta)$
- Page 40. The proof of Theorem 1.7 is very similar to the proof of Theorem 1.6 with the exception of Step 2. Please combine these two proofs into one, and deal with differences as separate cases in the same proof (case 1: $\alpha\beta = \kappa_f b(1+\beta)$, case 2: $\alpha\beta < \kappa_f b(1+\beta)$.)
- Page 45. The proof of Lemma 3.8 is very similar to the proof of Proposition 3.6, please revise Proposition 3.6 to cover the case in Lemma 3.8.
- Page 46.Lemma 3.9 and Lemma 3.10 should be stated before the proof of Theorem 1.6 in more general way, that they can be used in the proof of both cases of CLT $\alpha\beta \le \kappa_f b(1+\beta)$ and $\alpha\beta > \kappa_f b(1+\beta)$.