**Referee report on**
**STABLE CENTRAL LIMIT THEOREMS**
**FOR SUPER ORNSTEIN-UHLENBECK PROCESSES, II**
**by Yan-Xia Ren, Renming Song, Zhenyao Sun and Jianjie Zhao**

The aim of the paper under review is to answer a question left open in a paper by the same authors recently published in EJP. The setting is that of a super Ornstein-Uhlenbeck process $(X_t)_{t\geq 0}$ with spatially independent near-$(1+\beta)$-stable branching mechanism. Specifically, the authors had shown that $\int f(x)X_t(dx)$, suitably scaled and/or centered, converges to a $(1+\beta)$-stable distribution, for $f \in (\mathcal{C}_\mathrm{s} \oplus \mathcal{C}_\mathrm{c}) \cup \mathcal{C}_\mathrm{l}$, where $\mathcal{C}_\mathrm{s} \oplus \mathcal{C}_\mathrm{c} \oplus \mathcal{C}_\mathrm{l}$ is a decomposition of the vector space $\mathcal{P}$ of measurable functions on $\mathbb{R}^d$ with polynomial growth. This result did not cover functions of the form $f = f_\mathrm{s} + f_\mathrm{c} + f_\mathrm{l}$ with $f_\mathrm{l} \in \mathcal{C}_\mathrm{l} \setminus \{0\}$ and $f_\mathrm{s} + f_\mathrm{c} \in (\mathcal{C}_\mathrm{s} \oplus \mathcal{C}_\mathrm{c}) \setminus \{0\}$. To extend their stable CLT to this case, they conjectured that the convergence already established separately for $f = f_\mathrm{s}$, $f = f_\mathrm{c}$ and $f = f_\mathrm{l}$, holds jointly and exhibits asymptotic independence.

In the paper under review, the authors claim to provide a complete answer to their conjecture. The approach is a variation of the techniques of their recent paper. The interface does not work well. Reading in detail up to the top of page 10, I encountered four pages of Introduction that were copied almost verbatim from their EJP paper, about a page of new statements, with the remainder providing slight extensions of results from their EJP paper, including pages of notation needed to pull intermediate results out of proofs. While some of this may be unavoidable, much of this can be improved to make this paper friendlier to readers.

In any case, **I cannot recommend the present version for publication**, because it's unclear how to apply Proposition 2.1 to $f = \tilde{f}_\mathrm{s} + t^{\tilde{\beta}-1}\tilde{f}_\mathrm{c}$ at the top of page 10. Specifically, that proposition yields $C$ and $\delta$ depending on $f$ and $\mu$ to give bounds for all $t$ and $n_1$. But with $f$ depending on $t$, this looks **circular**. It's quite possible that this can be fixed, and since I do consider both solving the conjecture and establishing the general CLT worthwhile, **I would be in favour of allowing the authors to resubmit a revised version for a further full round of refereeing**. Indeed, I think this detail is only relevant for the full conjecture, but not for the general CLT since $X_t(f_\mathrm{c})$ dominates $X_t(f_\mathrm{s} + f_\mathrm{l})$ anyway so Slutsky's theorem already yields Corollary 1.2(2), and the difficult part is Corollary 1.2(1) when $f_\mathrm{c} = 0$, since $X_t(f_\mathrm{s})$ and $X_t(f_\mathrm{l})$ have the same order. It seems that their arguments achieve this. Below is a list of more detailed suggestions and typos.

**Detailed suggestions and typos**

1. I wonder if this paper is currently within the ECP page limit.

2. In any case, the Introduction is too similar to the introduction of the EJP paper, too long and too technical before anything happens. It'd be good if the introduction finished before starting page 5.

   - I suggest moving Assumptions 1 and 2, as well as the definition of $\mathcal{P}$ up to p.2, l.10 and then formulating a version of Corollary 2.1 as a first theorem on p.2, leaving $\mathrm{x}_t(f)$ as "suitable centering".
   - There clearly needs to be a discussion of what has already been done in [20], but I'd suggest taking the most direct route to (1.1), the definitions of $\mathcal{C}_{\mathrm{s}}$, $\mathcal{C}_{\mathrm{c}}$ and $\mathcal{C}_{\mathrm{l}}$, as well as $\mathrm{x}_t(f)$.
   - This then allows formulating Theorem 1.1 as a second theorem, which doesn't have to state the constants in the limit distributions. The main novelty is the asymptotic independence.
   - I suggest referring the reader to [20] for a literature review.

3. Notation of linear and nonlinear functionals seems rather arbitrary. $f \mapsto Z_t f$ and $f \mapsto m_t[f]$ are not linear, while $f \mapsto T_t f$, $f \mapsto X_t(f)$, $f \mapsto \Upsilon_t^f$ and $f \mapsto I^f(t)$ are. I have not found the filtration introduced, which appears in (2.2)-(2.5), nor $\mathcal{N}$, which appears just above the statement of Theorem 2.3. Notation $\mathrm{x}_t(f_{\mathrm{l}})$ is recalled repeatedly in short succession, while the (substochastic!) semigroup $(T_t)$ was used in passing in (1.3) and then resurfaces as $T_n$ on p.6, l.10 and as $T_k$ on p.9, l.5, without reference back to its definition.

4. To improve the interface and reduce the overlap with the EJP paper, I suggest the following.

   - Shorten the proof of Proposition 2.1. The seven- and six-line displays are quite elementary and virtually identical. We don't need the second and third lines of the seven-line display nor lines 2-4 of the six-line display. The logical structure can be improved by placing the reference "By [20, Proposition 3.5]" above display (2.3), replacing "Therefore". The same change would improve the passage from (2.4) to (2.5). Above the six-line display, "Similarly" could replace "Notice that".

- The notation introduced on the second half of p.8 and top of p.9 is only used to pull intermediate steps out of proofs in the EJP paper. This can be achieved by just defining $R_0$ and $R$, then referring to the proof of [20, Theorem 1.6] as showing component by component that $R(t) - R_0(t) \to 0$ in probability.

5. There are various places, where some $f_{\mathrm{s}}$, $f_{\mathrm{c}}$, $f_{\mathrm{l}}$ should be allowed to be the 0 function. This includes p.5, l.2 and l.18, and also p.8, l.14f and p.12, l.1.

6. On p.11, l.-9ff, the regularity of $f$ is unclear. Should $f$ be just bounded measurable? or continuous? or Lipschitz? or differentiable?

7. Typos

   - p.1, Abstract, l.7: “limit stable random variables” should be “limiting stable random variables”.
   - p.2, l.9: “process” should be “processes”.
   - p.3, l.6: “reminder” should be “remainder”.
   - p.4, (1.2): “Futhermore” should be “Furthermore”.
   - p.4, l. -5: “Propsoition” should be “Proposition”
   - p.6, l.3: “variables” should be “variable”.
   - p.10, l.10: there should be space between “$\mathcal{C}_j$” and “($j$ = s, c, l)”.