STABLE CENTRAL LIMIT THEOREMS FOR SUPER ORNSTEIN-UHLENBECK PROCESSES

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Abstract. Let ξ be an Ornstein-Uhlenbeck process on $\mathbb{R}^d$ with generator $L = \frac{1}{2}\sigma^2\Delta - bx\cdot\nabla$, where $\sigma, b > 0$. Let ψ be a branching mechanism which is close to a function of the form $\widetilde{\psi}(z) = -\alpha z + \rho z^2 + \eta z^{1+\beta}$ with $\alpha > 0$, $\rho \geq 0$, $\eta > 0$ and $\beta \in (0,1)$, in some sense. In this paper, we study asymptotic behaviors of (ξ,ψ)-superprocesses $(X_t)_{t\geq 0}$. For any testing function f of polynomial growth, denote by κ_f the order of f in the spectral decomposition of f in terms of the spectrum of the mean semigroup of X. Conditioned on non-extinction, we establish some stable central limit theorems for $\langle f, X_t\rangle$ in three different regimes: the small branching rate regime $\alpha\beta < \kappa_f b(1+\beta)$; the critical branching rate regime $\alpha\beta = \kappa_f b(1+\beta)$; and the large branching rate regime $\alpha\beta > \kappa_f b(1+\beta)$.

1. Introduction

1.1. **Motivation.** Let $d \in \mathbb{N} := \{1,2,\dots\}$ and $\mathbb{R}_+ := [0,\infty)$. Let $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in\mathbb{R}^d}\}$ be an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck process (OU process) with generator

$$Lf(x) = \frac{1}{2}\sigma^2\Delta f(x) - bx\cdot\nabla f(x), \quad x\in\mathbb{R}^d, f\in C^2(\mathbb{R}^d),$$

where $\sigma > 0$ and $b > 0$ are constants. Let the *branching mechanism* ψ be a function on $\mathbb{R}_+$ which takes the form of

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(dy), \quad z\in\mathbb{R}_+, \tag{1.1}$$

where $\alpha > 0$, $\rho \geq 0$ and π is a measure on $(0,\infty)$ with $\int_{(0,\infty)}(y\wedge y^2)\pi(dy) < \infty$. We refer to π as the *Lévy measure* of the branching mechanism ψ. Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of all finite Borel measures on $\mathbb{R}^d$. For each $f,g\in\mathcal{B}(\mathbb{R}^d,\mathbb{R})$ and $\mu\in\mathcal{M}(\mathbb{R}^d)$, write $\langle f,\mu\rangle = \int f(x)\mu(dx)$ and $\langle f,g\rangle = \int f(x)g(x)dx$ whenever the integrals make sense. We say a real-valued Borel function $f : (t,x)\mapsto f(t,x)$ on $\mathbb{R}_+\times\mathbb{R}^d$ is *locally bounded* if for

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each $t \in \mathbb{R}_+$, we have

$$\sup_{s\in[0,t],x\in\mathbb{R}^d} |f(s,x)| < \infty.$$

We say that an $\mathcal{M}(\mathbb{R}^d)$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ on a measurable space $(\Omega, \mathcal{F})$ is a *super Ornstein-Uhlenbeck process (super-OU process)* with branching mechanism ψ, if for each non-negative bounded Borel function f on $\mathbb{R}^d$, we have

$$\mathbb{P}_\mu[e^{-\langle f,X_t\rangle}] = e^{-\langle V_t f,\mu\rangle}, \quad t \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d), \tag{1.2}$$

where $(t,x) \mapsto V_t f(x)$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t \psi\big(V_{t-s}f(\xi_s)\big)\, ds\Big] = \Pi_x[f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0.$$

The existence of such super-OU process X is well known, see [11] for instance.

Recently, there have been quite a few papers on laws of large numbers for superdiffusions. In [13, 14, 15], some weak laws of large numbers (convergence in law or in probability) were established. The strong law of large numbers for superprocesses was first studied in [7], followed by [8, 9, 12, 24, 28, 42] under different settings. For a good survey on recent developments in laws of large numbers for branching Markov processes and superprocesses, see [12].

The strong law of large numbers for the super-OU process X can be stated as follows: Under some conditions on the branching mechanism ψ (these conditions are satisfied under our Assumptions 1 and 2 below), there exists an Ω_0 of $\mathbb{P}_\mu$-full probability for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that on Ω_0, for every Lebesgue-almost everywhere continuous bounded non-negative function f on $\mathbb{R}^d$, we have $\lim_{t\to\infty} e^{-\alpha t}\langle f, X_t\rangle = H_\infty\langle f, \varphi\rangle$, where H_∞ is the limit of the non-negative martingale $e^{-\alpha t}\langle 1, X_t\rangle$ and φ is the invariant density of the OU process ξ defined in (1.5) below. See [9, Theorem 2.13 & Example 8.1] and [12, Theorem 1.2 & Example 4.1].

In this paper, we will establish some spatial central limit theorems (CLT) for the super-OU process X above. Our key assumption is that the branching mechanism $\psi(z)$ takes the form of $-\alpha z + \rho z^2 + \psi_1(z)$, where ψ_1 is close to $\eta z^{1+\beta}$ with $\eta > 0$ and $\beta \in (0,1)$, in some sense. We want to find $(F_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ such that

$$\frac{\langle f, X_t\rangle - G_t}{F_t}$$

converges weakly to some non-degenerate random variable as $t \to \infty$, for a large class of testing functions f. It turns out that the statements of the CLT are different in three regimes depending on the sign of $\alpha\beta - \kappa_f b(1+\beta)$: the small branching rate regime $\alpha\beta < \kappa_f b(1+\beta)$; the critical branching rate regime $\alpha\beta = \kappa_f b(1+\beta)$; and the large branching rate regime $\alpha\beta > \kappa_f b(1+\beta)$. Here, κ_f is the order of f in the spectral decomposition in terms of the spectrum of L. Note that, in the setting of this paper, $\langle f, X_t\rangle$ typically have infinite second moment.

There are many papers studying central limit theorems for branching processes, branching diffusions and superprocesses under the second moment condition. See [16, 18, 19] for supercritical Galton-Watson processes, [22, 23] for supercritical multi-type Galton-Watson processes, [4, 5, 6] for supercritical multi-type continuous time branching processes and [3] for general supercritical branching Markov processes under certain conditions. In [1], Adamczak and Miłoś proved some spatial central limit theorems for supercritical branching Ornstein-Uhlenbeck processes with binary branching mechanism and in [31] Miłoś proved some spatial central limit theorems for supercritical super-OU processes with branching mechanisms satisfying a fourth moment condition. These two papers made connections between the CLT and the branching rate regimes. In [34], Ren, Song and Zhang proved some spatial central limit theorems for supercritical super-OU processes with branching mechanisms satisfying only a second moment condition. Moreover, compared with the results of [1, 31], the limit distributions in [34] are non-degenerate. In [35, 36, 37], Ren, Song and Zhang also established a series of spatial central limit theorems for a large class of general supercritical branching Markov processes and superprocesses with spatially dependent branching mechanisms. The functional version of the central limit theorems was established in [21] for supercritical multitype branching processes, and in [38] for supercritical superprocesses.

There are also many limiting theorem type results for supercritical branching processes and branching Markov processes with branching mechanisms of infinite second moments. Heyde [17] established a central limit type theorem for supercritical Galton-Watson process when the offspring distribution belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, and proved that the limit law is a stable law. Similar results for supercritical multi-type Galton-Watson processes and supercritical continuous time branching processes, under some p-th ($p \in (1, 2]$) moment condition on the offspring distribution, were given in Asmussen [2]. Recently, Marks and Miloś [29] considered the limit behavior of supercritical branching Ornstein-Uhlenbeck processes with a special stable branching law. They established some spatial central limit theorems in the small and critical branching rate regimes, but they did not prove any central limit theorem type result in the large branching rate regime. We also mention here that very recently [20] considered stable fluctuations of Biggins' martingales in the context of branching random walks and [33] considered the asymptotic behavior of a class of critical superprocesses with spatially dependent stable branching.

As far as we know, this paper is the first to study spatial central limit theorems for supercritical superprocesses without the second moment condition.

1.2. Main results.

We will always assume that the following assumption holds.

Assumption 1. The branching mechanism satisfies Grey's condition, i.e., there exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and $\int_{z'}^{\infty} \psi(z)^{-1} dz < \infty$.

For each $\mu \in \mathcal{M}(\mathbb{R}^d)$, write $\|\mu\| = \langle 1, \mu \rangle$. It is known (see [25, Theorems 12.5 & 12.7] for example) that, under Assumption 1, the *extinction event* $D := \{\exists t \geq 0, \text{ s.t. } \|X_t\| = 0\}$ has positive probability, with respect to $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}(\mathbb{R}^d)$. In fact,

$$\mathbb{P}_\mu(D) = e^{-\bar{v}\|\mu\|}, \quad \mu \in \mathcal{M}(\mathbb{R}^d),$$

where

$$\bar{v} := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\} \in (0, \infty)$$

is the largest root of ψ.

Denote by Γ the gamma function. For any σ-finite signed measure μ, we use $|\mu|$ to denote the total variation measure of μ. In this paper, we will also assume the following:

Assumption 2. There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that

$$\int_{(1,\infty)} y^{1+\beta+\delta} \left|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\right| < \infty, \tag{1.3}$$

for some $\delta > 0$.

We will show in Subsection 2.1 that if Assumption 2 holds, then η and β are uniquely determined by the Lévy measure π. In the reminder of the paper, we will always use η and β to denote the constants in Assumption 2.

Remark 1.1. Note that δ is not uniquely determined by π. In fact, if $\delta > 0$ is a constant such that (1.3) holds, then replacing δ by any smaller positive number, (1.3) still holds. Therefore, Assumption 2 is equivalent to the following statement: There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that, for each small enough $\delta > 0$, (1.3) holds.

Assumption 2 says that there exist constants $\eta > 0$ and $\beta > 0$ such that the Lévy measure $\pi(dy)$ is not too far away from the measure $\eta\Gamma(-1-\beta)^{-1}y^{-2-\beta}dy$. In particular, if $\pi(dy)$ is equal to $\eta\Gamma(-1-\beta)^{-1}y^{-2-\beta}dy$, then the branching mechanism $\psi(z)$ takes the form of $-\alpha z + \rho z^2 + \eta z^{1+\beta}$. It will be proved in Lemma 2.3 that, under Assumption 2, the branching mechanism ψ satisfies the $L \log L$ condition, i.e.,

$$\int_{(1,\infty)} y \log y \; \pi(dy) < \infty.$$

This guarantees that H_∞, the limit of the non-negative martingale $(e^{-\alpha t}\|X_t\|)_{t\geq 0}$, is non-degenerate.

Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R})$ the space of all $\mathbb{R}$-valued Borel functions on $\mathbb{R}^d$. Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$ the space of all $\mathbb{R}_+$-valued Borel functions on $\mathbb{R}^d$. We use $(P_t)_{t\geq 0}$ to denote the transition semigroup of ξ. Define

$$P_t^\alpha f(x) := e^{\alpha t}P_t f(x) = \Pi_x[e^{\alpha t} f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0, f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+). \tag{1.4}$$

It is known that, see [26, Proposition 2.27] for example, $(P_t^\alpha)_{t\geq 0}$ is the *mean semigroup* of X, in the sense that

$$\mathbb{P}_\mu[\langle f, X_t\rangle] = \langle P_t^\alpha f, \mu\rangle, \quad t \geq 0, f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+), \mu \in \mathcal{M}(\mathbb{R}^d).$$

The limiting behavior of the super-OU process is closely related to the asymptotic property of this mean semigroup $(P_t^\alpha)_{t\geq 0}$, and therefore, to the property of the OU semigroup $(P_t)_{t\geq 0}$.

We now recall some basic spectral properties of $(P_t)_{t\geq 0}$ from [30]. It is known that the OU process ξ has an invariant density

$$\varphi(x)dx := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2} \exp\Big(-\frac{b}{\sigma^2}|x|^2\Big)dx, \quad x \in \mathbb{R}^d. \tag{1.5}$$

Let $L^2(\varphi) := \left\{h \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : \int_{\mathbb{R}^d} |h(x)|^2 \varphi(x)dx < \infty\right\}$. Then $L^2(\varphi)$ is a Hilbert space with inner product

$$\langle f_1, f_2 \rangle_\varphi := \int_{\mathbb{R}^d} f_1(x) f_2(x) \varphi(x) dx, \quad f_1, f_2 \in L^2(\varphi).$$

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For each $p = (p_k)_{k=1}^d \in \mathbb{Z}_+^d$, write $|p| := \sum_{k=1}^d p_k$, $p! := \prod_{k=1}^d p_k!$ and $\partial x^p := \prod_{k=1}^d \partial x_k^{p_k}$. The *Hermite polynomials* are defined by

$$H_p(x) := (-1)^{|p|} \exp(|x|^2) \frac{\partial^{|p|}}{\partial x^p} \exp(-|x|^2), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

It is known that $(P_t)_{t \geq 0}$ is a strongly continuous semigroup in $L^2(\varphi)$ and its generator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. For each $k \in \mathbb{Z}_+$, denote by $\mathcal{A}_k$ the eigenspace corresponding to the eigenvalue $-bk$, then $\mathcal{A}_k = \mathrm{Span}\{\phi_p : p \in \mathbb{Z}_+^d, |p| = k\}$, where

$$\phi_p(x) := \frac{1}{\sqrt{p! 2^{|p|}}} H_p\Big(\frac{\sqrt{b}}{\sigma} x\Big), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d. \tag{1.6}$$

In other words,

$$P_t \phi_p(x) = e^{-b|p|t} \phi_p(x), \quad t \geq 0, x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

Moreover, $\{\phi_p : p \in \mathbb{Z}_+^d\}$ forms a complete orthonormal basis of $L^2(\varphi)$. Thus for each $f \in L^2(\varphi)$, we have

$$f = \sum_{k=0}^{\infty} \sum_{p \in \mathbb{Z}_+^d : |p| = k} \langle f, \phi_p \rangle_\varphi \phi_p, \quad \text{in } L^2(\varphi). \tag{1.7}$$

For each function $f \in L^2(\varphi)$, denote by

$$\kappa_f := \inf\left\{k \geq 0 : \exists\, p \in \mathbb{Z}_+^d, \text{ s.t. } |p| = k \text{ and } \langle f, \phi_p \rangle_\varphi \neq 0\right\},$$

the order of the function f in the spectral decomposition (1.7). Note that $\kappa_f \geq 0$ and that, if $f \in L^2(\varphi)$ is non-trivial, then $\kappa_f < \infty$. In particular, the order of any constant non-zero function is zero. Denote by $\mathcal{P}$ the class of functions of polynomial growth on $\mathbb{R}^d$, i.e.,

$$\mathcal{P} := \left\{f \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}) : \exists\, C > 0, n \in \mathbb{Z}_+, \text{such that} \, \forall x \in \mathbb{R}^d,\ |f(x)| \leq C(1 + |x|)^n\right\}. \tag{1.8}$$

It is clear that $\mathcal{P} \subset L^2(\varphi)$. In this paper, we are mainly interested in the asymptotic behavior of $\langle f, X_t \rangle$ for non-trivial $f \in \mathcal{P}$. The asymptotic behavior varies with the following three regimes:

- The large branching rate regime: $\alpha\beta > \kappa_f b(1 + \beta)$;
- The critical branching rate regime: $\alpha\beta = \kappa_f b(1 + \beta)$;
- The small branching rate regime: $\alpha\beta < \kappa_f b(1 + \beta)$.

1.2.1. *Large branching rate regime.* Denote by $\mathcal{M}_c(\mathbb{R}^d)$ the space of all finite Borel measures of compact support on $\mathbb{R}^d$. For each $p \in \mathbb{Z}_+^d$, define

$$H_t^p := e^{-(\alpha-|p|b)t}\langle \phi_p, X_t\rangle, \quad t \geq 0.$$

If $\alpha\beta > |p|b(1+\beta)$, then for all $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, we will prove in Lemma 3.2 that $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$. Thus the limit $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$.

Theorem 1.2. *If $f \in \mathcal{P}$ satisfies $\alpha\beta > \kappa_f b(1+\beta)$, then for all $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$,*

$$e^{-(\alpha-\kappa_f b)t}\langle f, X_t\rangle \xrightarrow[t\to\infty]{} \sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f, \phi_p\rangle_\varphi H_\infty^p \quad \text{in } L^{1+\gamma}(\mathbb{P}_\mu).$$

Moreover, if f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$, then we also have almost sure convergence.

Remark 1.3. If $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$ is non-trivial and bounded, then $\kappa_f = 0$. Hence, Theorem 1.2 says that for any $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, as $t \to \infty$,

$$e^{-\alpha t}\langle f, X_t\rangle \to \langle f, \varphi\rangle H_\infty \quad \text{in } L^{1+\gamma}(\mathbb{P}_\mu).$$

Moreover, if f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$, then we also have almost sure convergence. However, to get almost sure convergence for bounded, non-negative, and Lebesgue-almost everywhere continuous testing functions f, we do not need f to be twice differentiable. See [9, Theorem 2.13 & Example 8.1] and [12, Theorem 1.2 & Example 4.1].

Remark 1.4. For general test functions $f \in \mathcal{P}$ with $\alpha\beta > \kappa_f b(1+\beta)$, Theorem 1.2 extends the strong laws of large numbers of [9] and [12] in which the first order asymptotic ($\kappa_f = 0$) was identified.

The fluctuation result associated to the law of large numbers in this case is given in Theorem 1.5 below. Define

$$\mathcal{N} := \{p \in \mathbb{Z}_+^d : \alpha\beta > |p|(1+\beta)b\}, \quad K := \sup\{|p| : p \in \mathcal{N}\} \tag{1.9}$$

and

$$\mathcal{C}_l := \Big\{f \in \mathcal{P} : f(x) = \sum_{p\in\mathcal{N}} \langle f, \phi_p\rangle_\varphi \phi_p(x)\Big\}. \tag{1.10}$$

For each $t \geq 0$, we define an operator I_t on $\mathcal{C}_l$ by

$$I_t f(x) := \sum_{p\in\mathcal{N}} \langle f, \phi_p\rangle_\varphi e^{-(\alpha-|p|b)t}\phi_p(x), \quad x \in \mathbb{R}^d, f \in \mathcal{C}_l. \tag{1.11}$$

It is easy to see that $I_s f(x) = \mathbb{P}_{\delta_x}[\langle I_t f, X_{t-s}\rangle] = P_{t-s}^\alpha I_t f(x)$. Define

$$\bar{m}[f] := \eta \int_0^\infty e^{\alpha s}\, ds \int_{\mathbb{R}^d} \big(iI_s f(x)\big)^{1+\beta} \varphi(x)\, dx, \quad f \in \mathcal{C}_l.$$

We will show in Lemma 2.6 that $\bar{m}[f]$ is well-defined for each $f \in \mathcal{C}_l$; and in Lemma 2.7 that, when $f \in \mathcal{C}_l$ is non-trivial, $\theta \mapsto \exp(\bar{m}[\theta f])$ is the characteristic function of a $(1+\beta)$-stable random variable.

Theorem 1.5. *For non-trivial $f \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that,*

$$\frac{\langle f, X_t\rangle - \sum_{p\in\mathcal{N}}\langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t}H_\infty^p}{\|X_t\|^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \bar{\zeta}, \tag{1.12}$$

where $\bar{\zeta}$ is a $(1+\beta)$-stable random variable with characteristic function

$$\mathbf{E}[e^{i\theta\bar{\zeta}}] = \exp(\bar{m}[\theta f]), \quad \theta \in \mathbb{R}.$$

1.2.2. *Critical branching rate regime.* For each $f \in \mathcal{P}$ satisfying $\alpha\beta = \kappa_f b(1+\beta)$, define

$$\widetilde{m}[f] := \eta \int_{\mathbb{R}^d} \Big(-i \sum_{p\in\mathbb{Z}_+^d : |p|=\kappa_f} \langle f, \phi_p\rangle \phi_p(x)\Big)^{1+\beta} \varphi(x)\, dx.$$

According to Lemma 2.6 below, $\widetilde{m}[f]$ is well-defined. We will prove in Lemma 2.7 that $\theta \mapsto \exp(\widetilde{m}[\theta f])$ is the characteristic function of a $(1+\beta)$-stable random variable.

Theorem 1.6. *If $f \in \mathcal{P}$ satisfies $\alpha\beta = \kappa_f b(1+\beta)$, then for any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that*

$$\frac{\langle f, X_t\rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \widetilde{\zeta},$$

where $\widetilde{\zeta}$ is a $(1+\beta)$-stable random variable with characteristic function

$$\mathbf{E}[e^{i\theta\widetilde{\zeta}}] = \exp(\widetilde{m}[\theta f]), \quad \theta \in \mathbb{R}.$$

1.2.3. *Small branching rate regime.* If $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta < \kappa_f b(1+\beta)$, we define

$$m[f] := \eta \int_0^\infty e^{-\alpha s}\, ds \int_{\mathbb{R}^d} \big(-iP_s^\alpha f(x)\big)^{1+\beta} \varphi(x)\, dx.$$

According to Lemma 2.6 below, $m[f]$ is well-defined. We will prove in Lemma 2.7 that $\theta \mapsto \exp(m[\theta f])$ is the characteristic function of a $(1+\beta)$-stable random variable.

Theorem 1.7. *If $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta < \kappa_f b(1+\beta)$, then for any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that*

$$\frac{\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \zeta,$$

where ζ is a $(1+\beta)$-stable random variable with

$$\mathbf{E}[e^{i\theta\zeta}] = \exp(m[\theta f]), \quad \theta \in \mathbb{R}.$$

1.3. **Some intuitive explanation.** Here we give some intuitive explanation of the central limit theorems above. The main ideas are similar to those given in [29] for branching OU processes. For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and any random variable Y with finite mean, we define

$$\mathcal{I}_s^t Y := \mathcal{I}_s^t[Y,\mu] := \mathbb{P}_\mu[Y|\mathscr{F}_t] - \mathbb{P}_\mu[Y|\mathscr{F}_s], \quad 0 \le s \le t < \infty. \tag{1.13}$$

We will use the shorter notation $\mathcal{I}_s^t Y$ when there is no danger of confusion. For each $f \in \mathcal{P}$, consider the following decomposition over the time interval $[0,t]$:

$$\langle f, X_t\rangle := \sum_{k=0}^{\lfloor t\rfloor -1} \mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle + \mathcal{I}_0^{t-\lfloor t\rfloor}\langle f, X_t\rangle + \langle P_t^\alpha f, X_0\rangle, \quad t \ge 0.$$

To find the fluctuation of $\langle f, X_t\rangle$, we will investigate the fluctuation of each term on the right hand side above.

Suppose that $(\zeta_k)_{k\in\mathbb{N}}$ are independent $(1+\beta)$-stable random variables with characteristic functions

$$\mathbf{E}[e^{i\theta\zeta_k}] = \exp(m_k[\theta f]), \quad k \in \mathbb{N},$$

where $m_k[f]$ is given in Lemma 2.6 below. Recall that $\|X_t\| \sim e^{\alpha t}$ as $t \to \infty$. Lemma 3.4 and Proposition 3.6 below imply that if $\alpha\beta \le \kappa_f b(1+\beta)$ then

$$\left(\frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \right)_{k=1}^n \xrightarrow[t\to\infty]{d} (\zeta_k)_{k=1}^n.$$

So if f is in the small branching rate regime, i.e., $\alpha\beta < \kappa_f b(1+\beta)$, we have roughly that

$$\frac{\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \zeta \overset{d}{=} \sum_{k=0}^\infty \zeta_k,$$

where ζ is a $(1+\beta)$-stable random variable with characteristic function

$$\mathbf{E}[e^{i\theta\zeta}] = \exp(m[\theta f]), \quad \theta \in \mathbb{R}.$$

with $m[\theta f]$ given in Lemma 2.6 below. In the explanation above, we have used the facts that $(\zeta_k)_{k\in\mathbb{N}}$, ζ are all well defined, and

$$m[f] = \sum_{k=0}^\infty m_k[f]. \tag{1.14}$$

These facts will be made clear in Subsection 2.4 below.

If f is in the critical branching rate regime, i.e., $\alpha\beta = \kappa_f b(1+\beta)$, then the right hand side of (1.14) may not converge. Instead, see Lemma 2.6 below, there is another quantity $\widetilde{m}[f]$ satisfying that

$$\widetilde{m}[f] = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{\lfloor t\rfloor} m_k[f].$$

In this case, we roughly have that

$$\frac{\langle f, X_t\rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \overset{d}{\approx} \frac{1}{t^{\frac{1}{1+\beta}}} \sum_{k=0}^{\lfloor t\rfloor} \zeta_k \xrightarrow[t\to\infty]{d} \widetilde{\zeta}$$

where $\widetilde{\zeta}$ is a $(1+\beta)$-stable random variable with characteristic function

$$\mathbf{E}[e^{i\theta\widetilde{\zeta}}] = \exp(\widetilde{m}[\theta f]), \quad \theta \in \mathbb{R}.$$

For testing functions $f \in \mathcal{C}_l$, the general idea is almost the same, except that we need to consider the decomposition over the time interval $[t,\infty)$. Taking $f = \phi_p$ with $\alpha\beta > |p|b(1+\beta)$ as an example, we do the following decomposition:

$$H_t^p - H_\infty^p = \sum_{n=1}^{\infty} (H_{t+n-1}^p - H_{t+n}^p).$$

The fluctuation behaviors of each of the terms $H_{t+n-1}^p - H_{t+n}^p$ and their asymptotic independence will be established in Lemma 3.7 and 3.8 below, respectively. These lemmas will eventually lead us to the fluctuation result in the large branching rate regime.

For a general f, we have a unique decomposition: $f = f_l + f_c + f_s$, where

$$f_l = \sum_{p\in\mathcal{N}} a_p\phi_p(x) \in \mathcal{C}_l, \quad x \in \mathbb{R}^d,$$

and $f_c = \langle f, \phi_p\rangle_\varphi \phi_p(x)$ with $p = \frac{\alpha\beta}{b(1+\beta)}$. Note that there may be no p such that $p = \frac{\alpha\beta}{b(1+\beta)}$. In this case $f_c = 0$. Our main results above give central limit type results for $\langle f_l, X_t\rangle$, $\langle f_c, X_t\rangle$ and $\langle f_s, X_t\rangle$, respectively. We conjecture that the limits of these three terms, normalized properly, are independent, because intuitively these limits come from small time intervals, intermediate time intervals and large time intervals, respectively. If this is valid, we can get a central limit type result for $\langle f, X_t\rangle$ for general $f \in \mathcal{P}$. This independence was proved under the second moment condition, see [36]. We leave the question of independence for stable branching mechanism to a future project.

This paper is our first attempt on the stable CLT for superprocesses. There are still many open questions. Ren, Song and Zhang have established some spatial central limit theorems in [36] for a class of superprocesses with general spatial motions under the assumption that the branching mechanisms satisfy a second moment condition. We hope to prove spatial CLT's for superprocesses with general motions without the second moment assumption on the branching mechanism in a future project.

Recall that our Assumption 2 says that the branching mechanism ψ is $-\alpha z+\rho z^2+\eta z^{1+\beta}$ plus a small perturbation

$$\psi_1(z) := \int_{(0,\infty)} (e^{-yz} - 1 + yz)\Big(\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big) \tag{1.15}$$

where ψ_1 satisfies (1.3) with some $\delta > 0$. It would be interesting to consider more general branching mechanisms.

Here are some examples of branching mechanisms satisfying Assumptions 1 and 2: If h is a complete Bernstein function which is regularly varying at 0 with index $\beta_1 \in (\beta, 1)$, then

$$\psi(z) := -\alpha z + \rho z^2 + \eta z^{1+\beta} + zh(z), \qquad z > 0$$

satisfies Assumptions 1 and 2. If $\beta_1 \in (\beta, 1)$, $c_1 \in (0, \eta/\Gamma(-1-\beta))$ and $c_2 \geq 1$, then

$$\psi(z) := -\alpha z + \rho z^2 + \eta z^{1+\beta} - \int_{c_2}^{\infty} (e^{-yz} - 1 + yz) \frac{c_1 dy}{y^{1+\beta_1}}, \qquad z \in \mathbb{R}_+$$

satisfies Assumptions 1 and 2.

The rest of the paper is organized as follows: In Subsection 2.1 we will give some preliminary results for our branching mechanism ψ. In Subsections 2.2 and 2.3 we will give some estimates for some operators related to the super-OU process X that will be used in our proofs. In Subsection 2.4 we will give the definitions of all the $(1+\beta)$-stable random variables involved in this paper. In Subsection 2.5 we will give some estimates for the small value probability of continuous state branching processes. In Subsection 2.6 we will give upper bounds for the $(1+\gamma)$-moments for our superprocesses. These estimates and upper bounds will be crucial in the proofs of our main results. In Subsection 3.1, we will give the proof of Theorem 1.2. In Subsection 3.2, we will give the proof of Theorem 1.6. In Subsection 3.3, we will give the proof of Theorem 1.7. In Subsection 3.4, we will give the proof of Theorem 1.5. In the Appendix, we consider a general superprocess $(X_t)_{t\geq 0}$, and we prove there that the characteristic exponent of $\langle f, X_t \rangle$ satisfies a complex-valued non-linear integral equation. This fact will be used at several places in this paper, and we think it is of independent interest.

We end this section with a list of frequently used notations:

- φ and ϕ_p are given in (1.5) and (1.6), respectively.
- P_t, P_t^α, V_t and U_t are given in (1.4), (1.4), (1.2), (2.3), respectively.
- $\mathcal{N}$, K, $\mathcal{C}_l$ and I_t are given in (1.9), (1.9), (1.10) and (1.11), respectively.
- ψ, ψ_1 and ψ_0 are given in (1.1), (1.15) and (2.1), respectively.
- Ψ, Ψ_1 and Ψ_0 are given in (2.2).
- Z_t, Z'_t, Z''_t and Z'''_t are given in (2.5).
- Q_κ and Q are given in (2.7) and (2.8), respectively.
- $m_t[f]$, $\bar{m}_t[f]$, $m[f]$, $\widetilde{m}[f]$ and $\bar{m}[f]$ are given in Lemma 2.6.

2. Preliminaries

2.1. **Branching mechanism.** Let ψ be the branching mechanism given in (1.1). Suppose that Assumptions 1 and 2 hold. In this subsection, we give some preliminary results about the branching mechanism ψ. Recall that η and β are the constants in Assumption 2. Let $\mathbb{C}_+ := \{x + iy : x \in \mathbb{R}_+, y \in \mathbb{R}\}$ and $\mathbb{C}_+^0 := \{x + iy : x \in (0, \infty), y \in \mathbb{R}\}$.

Lemma 2.1. *The function ψ_1 on $\mathbb{R}_+$ can be uniquely extended as a complex-valued continuous function on $\mathbb{C}_+$ which is holomorphic on $\mathbb{C}_+^0$. Moreover, for each $\delta > 0$ small enough, there exists a constant $C > 0$ such that for all $z \in \mathbb{C}_+$, we have $|\psi_1(z)| \leq C|z|^{1+\beta+\delta} + C|z|^2$.*

Proof. According to Lemma A.2 below and the uniqueness of holomorphic extensions, we know that ψ_1 can be uniquely extended as a complex-valued continuous function on $\mathbb{C}_+$ which is holomorphic on $\mathbb{C}_+^0$. The extended ψ_1 has the following form:

$$\psi_1(z) = \int_{(0,\infty)} (e^{-yz} - 1 + yz)\Big(\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big), \quad z \in \mathbb{C}_+.$$

Now, according to Assumption 2 and Remark 1.1, for each small enough $\delta > 0$, we have

$$\begin{aligned}|\psi_1(z)| &\le \int_{(0,\infty)} (|yz| \wedge |yz|^2)\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big| \\ &\le |z|^2 \int_{(0,1)} y^2\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big| \\ &\quad + |z|^{1+\beta+\delta}\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big|, \quad z \in \mathbb{C}_+,\end{aligned}$$

as desired. □

The following lemma says that the constants η, β in Assumption 2 are uniquely determined by the Lévy measure π.

Lemma 2.2. *Suppose that Assumption 2 holds. Suppose that there are constants $\eta' > 0, \beta' \in (0,1)$ and $\delta' > 0$ such that*

$$\int_{(1,\infty)} y^{1+\beta'+\delta'}\ \Big|\pi(dy) - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

Then $\eta' = \eta$ and $\beta' = \beta$.

Proof. Without loss of generality, we assume that $\beta + \delta \le \beta' + \delta'$. Using the fact that $y^{1+\beta+\delta} \le y^{1+\beta'+\delta'}$ with $y \ge 1$, we get

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(dy) - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

Comparing this with Assumption 2, we get

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}} - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

In other words, if we denote by $\widetilde{\pi}(dy)$ the measure $\eta'\Gamma(-1-\beta)^{-1}y^{-2-\beta'}dy$, then $\widetilde{\pi}$ is a Lévy measure which satisfies Assumption 2. Applying Lemma 2.1 to this measure $\widetilde{\pi}$, we have that there exists $c > 0$ such that

$$|\eta z^{1+\beta} - \eta' z^{1+\beta'}| \le c z^{1+\beta+\delta} + cz^2, \quad z \in \mathbb{R}_+.$$

Dividing both sides by $z^{1+\beta}$ we have

$$|\eta - \eta' z^{\beta'-\beta}| \le cz^{\delta} + cz^{1-\beta}, \quad z \in \mathbb{R}_+.$$

This implies that

$$\eta' z^{\beta'-\beta} \xrightarrow[\mathbb{R}^+ \ni z \to 0]{} \eta > 0.$$

So we must have $\beta' = \beta$ and $\eta' = \eta$. $\square$

We also have the following result:

Lemma 2.3. *If ψ satisfies Assumption 2, then ψ satisfies the $L\log L$ condition, i.e.,*

$$\int_{(1,\infty)} y \log y \ \pi(dy) < \infty.$$

Proof. Using Assumption 2 and the fact that $y \log y \le y^{1+\beta+\delta}$ for y large enough, we get

$$\int_{(1,\infty)} y \log y \ \left| \pi(dy) - \frac{\eta \ dy}{\Gamma(-1-\beta) y^{2+\beta}} \right| < \infty.$$

Therefore we have

$$\int_{(1,\infty)} y \log y \ \left(\pi(dy) - \frac{\eta \ dy}{\Gamma(-1-\beta) y^{2+\beta}} \right) < \infty.$$

Combining this with

$$\int_{(1,\infty)} \frac{\eta \log y \ dy}{\Gamma(-1-\beta) y^{1+\beta}} < \infty,$$

we immediately get the desired result. $\square$

2.2. **Definition of controller.** Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{C})$ the space of all $\mathbb{C}$-valued Borel functions on $\mathbb{R}^d$. Recall that $\mathcal{P}$ is defined in (1.8). Define $\mathcal{P}^+ := \mathcal{P} \cap \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$ and $\mathcal{P}^* := \{f \in \mathcal{B}(\mathbb{R}^d, \mathbb{C}) : |f| \in \mathcal{P}\}$. For any function $h : [0,\infty) \to [0,\infty)$, we say an operator $R : \mathcal{P}^+ \to \mathcal{P}^+$ is an *h-controller* if

- $f, g \in \mathcal{P}^+$ and $f \le g$ imply that $Rf \le Rg$; and
- $f \in \mathcal{P}^+$ and $\theta \in [0,\infty)$ imply that $R(\theta f) \le h(\theta) Rf$.

For a subset $\mathcal{D}$ of $\mathcal{P}^*$ and an h-controller R, we say an operator $A : \mathcal{D} \to \mathcal{P}^*$ is *h-controlled by R on $\mathcal{D}$* if $|Af| \le R|f|$ for each $f \in \mathcal{D}$. We say an operator $A : \mathcal{D} \to \mathcal{P}^*$ is *h-controllable* if there exists an h-controller R such that A is h-controlled by R. We say a family of operators $(A_s)_{s\in\Lambda}$ from $\mathcal{D}$ to $\mathcal{P}^*$ is *uniformly h-controllable on $\mathcal{D}$* if there exists an h-controller R such that, for each $s \in \Lambda$, A_s is h-controlled by R on $\mathcal{D}$.

For two operators $A : \mathcal{D}_A \subset \mathcal{P}^* \to \mathcal{P}^*$ and $B : \mathcal{D}_B \subset \mathcal{P}^* \to \mathcal{P}^*$, we define $(A \times B)f(x) := Af(x) \times Bf(x)$ for all $f \in \mathcal{D}_A \cap \mathcal{D}_B$ and $x \in \mathbb{R}^d$. For any $a \in \mathbb{R}$ and any operator $A : \mathcal{D}_A \to \mathcal{B}(\mathbb{R}^d, \mathbb{C} \setminus (-\infty, 0])$, define $A^{\times a} f(x) := (Af(x))^a$ for all $f \in \mathcal{D}_A$ and $x \in \mathbb{R}^d$.

Lemma 2.4. *Suppose that Λ is an index set and $(A_\lambda)_{\lambda\in\Lambda}$ is a family of operators from $\mathcal{D} \subset \mathcal{P}^*$ to $\mathcal{P}^*$. Assume that $(A_\lambda)_{\lambda\in\Lambda}$ is uniformly h-controllable on $\mathcal{D}$ for a given function $h : [0,\infty) \to [0,\infty)$.*

(1) *Suppose that $(\Lambda, \mathscr{F})$ is a measurable space and that $(\lambda, x) \mapsto A_\lambda f(x)$ is $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable for each $f \in \mathcal{D}$. For any probability measure ν on $(\Lambda, \mathscr{F})$, we write*

$$A_\nu f(x) := \int_\Lambda A_\lambda f(x) \ \nu(d\lambda), \quad f \in \mathcal{D}, x \in \mathbb{R}^d.$$

Then $\{A_\nu : \nu$ is a probability measure on $(\Lambda, \mathscr{F})\}$ is uniformly h-controllable on $\mathcal{D}$.

(2) *Suppose that Δ is another index set and $(B_\delta)_{\delta\in\Delta}$ is a family of operators from $\mathcal{D}_0 \subset \mathcal{P}^*$ to $\mathcal{D}$. Assume that $(B_\delta)_{\delta\in\Delta}$ is uniformly g-controllable on $\mathcal{D}_0$ for some function $g : [0,\infty) \to [0,\infty)$. Then $(A_\lambda B_\delta)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h\circ g)$-controllable on $\mathcal{D}_0$.*

(3) *Suppose that Δ is another index set and $(B_\delta)_{\delta\in\Delta}$ is a family of operators from $\mathcal{D}$ to $\mathcal{P}^*$. Assume that $(B_\delta)_{\delta\in\Delta}$ is uniformly g-controllable on $\mathcal{D}$ for some function $g : [0,\infty) \to [0,\infty)$. Then $(B_\delta \times A_\lambda)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h \times g)$-controllable on $\mathcal{D}$ and $(B_\delta + A_\lambda)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h \vee g)$-controllable on $\mathcal{D}$.*

(4) *Let $a > 0$. Suppose that, for each $\lambda \in \Lambda$, A_λ is an operator from $\mathcal{D}$ to $\mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$. Then $(A_\lambda^{\times a})_{\lambda\in\Lambda}$ is uniformly (h^a)-controllable.*

Proof. (1). Let $(A_\lambda)_{\lambda\in\Lambda}$ be uniformly controlled by an h-controller R on $\mathcal{D}$. For all $f \in \mathcal{D}$, $x \in \mathbb{R}^d$ and probability measure ν on $(\Lambda, \mathscr{F})$,

$$|A_\nu f(x)| \leq \int_\Lambda |A_\lambda f(x)|\nu(d\lambda) \leq \int_\Lambda R|f|(x)\nu(d\lambda) \leq R|f|(x).$$

(2). Let $(A_\lambda)_{\lambda\in\Lambda}$ be uniformly controlled by an h-controller R_A on $\mathcal{D}$. Let $(B_\delta)_{\delta\in\Delta}$ be uniformly controlled by a g-controller R_B on $\mathcal{D}_0$. Then we have

- For each $\lambda \in \Lambda$, $\delta \in \Delta$ and $f \in \mathcal{D}_0$, we have $|A_\lambda B_\delta f| \leq R_A|B_\delta f| \leq R_A R_B|f|$.
- If $f, g \in \mathcal{P}^+$ with $f \leq g$, then $R_A R_B f \leq R_A R_B g$.
- If $f \in \mathcal{P}^+$ and $\theta \in [0,\infty)$, then $R_A R_B(\theta f) \leq R_A(g(\theta) R_B f) \leq (h\circ g)(\theta) R_A R_B f$.

Therefore $(A_\lambda B_\delta)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h\circ g)$-controllable on $\mathcal{D}_0$.

(3). Let $(A_\lambda)_{\lambda\in\Lambda}$ be uniformly controlled by an h-controller R_A on $\mathcal{D}$. Let $(B_\delta)_{\delta\in\Delta}$ be uniformly controlled by a g-controller R_B on $\mathcal{D}$. Then we have

- For each $\lambda \in \Lambda$, $\delta \in \Delta$ and $f \in \mathcal{D}$, we have $|A_\lambda \times B_\delta f| \leq |A_\lambda f|\cdot|B_\delta f| \leq R_A \times R_B|f|$.
- If $f, g \in \mathcal{P}^+$ with $f \leq g$, then $R_A \times R_B f \leq R_A \times R_B g$.
- If $f \in \mathcal{P}^+$ and $\theta \in [0,\infty)$, then $R_A \times R_B(\theta f) \leq h(\theta)g(\theta)(R_A \times R_B)f$.

Therefore $(A_\lambda \times B_\delta)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h \times g)$-controllable on $\mathcal{D}$.

We also have the following:

- For each $\lambda \in \Lambda$, $\delta \in \Delta$ and $f \in \mathcal{D}$, we have $|(A_\lambda + B_\delta)f| \leq |A_\lambda f| + |B_\delta f| \leq (R_A + R_B)|f|$.
- If $f, g \in \mathcal{P}^+$ with $f \leq g$, then $(R_A + R_B)f \leq (R_A + R_B)g$.
- If $f \in \mathcal{P}^+$ and $\theta \in [0,\infty)$, then $(R_A + R_B)(\theta f) \leq h(\theta)R_A f + g(\theta)R_B f \leq (h(\theta) \vee g(\theta))(R_A + R_B)f$.

Therefore $(A_\lambda + B_\delta)_{\delta\in\Delta,\lambda\in\Lambda}$ is uniformly $(h \vee g)$-controllable on $\mathcal{D}$.

(4). Let $(A_\lambda)_{\lambda\in\Lambda}$ be uniformly controlled by an h-controller R on $\mathcal{D}$. Then we have the following:

- For each $\lambda \in \Lambda$, $f \in \mathcal{D}$, we have $|A_\lambda^{\times a} f| = |A_\lambda f|^a \leq R^{\times a}|f|$.
- If $f, g \in \mathcal{P}^+$ with $f \leq g$, then $R^{\times a} f \leq R^{\times a} g$.
- If $f \in \mathcal{P}^+$ and $\theta \in [0,\infty)$, then $R^{\times a}(\theta f) \leq (h(\theta)Rf)^a = h(\theta)^a R^{\times a} f$.

Therefore $(A_\lambda^{\times a})_{\lambda\in\Lambda}$ is uniformly (h^a)-controllable on $\mathcal{D}$. $\square$

2.3. **h-controller for super-OU processes.** Let X be the super-OU process introduced in Subsection 1.1 with branching mechanism ψ satisfying Assumptions 1 and 2. In this subsection, we will define several operators and study their properties that will be used in this paper.

Define

$$\psi_0(z) = \psi(z) + \alpha z, \quad z \in \mathbb{R}_+ \tag{2.1}$$

Using an argument similar to the proof of Lemma 2.1, both functions ψ, ψ_0 can be uniquely extended as complex-valued continuous functions on $\mathbb{C}_+$ which are holomorphic on $\mathbb{C}_+^0$. For all $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$ and $x \in \mathbb{R}^d$, we define

$$\begin{aligned} \Psi f(x) &:= -\alpha f(x) + \eta f(x)^{1+\beta} + \psi_1\big(f(x)\big), \\ \Psi_0 f(x) &:= \Psi f(x) + \alpha f(x), \\ \Psi_1 f(x) &:= \psi_1\big(f(x)\big). \end{aligned} \tag{2.2}$$

For all $t \in [0, \infty), x \in \mathbb{R}^d$ and $f \in \mathcal{P}$, let

$$U_t f(x) := \operatorname{Log} \mathbb{P}_{\delta_x}[e^{i\theta\langle f, X_t\rangle}]|_{\theta=1} \tag{2.3}$$

be the value of the characteristic exponent of the infinitely divisible random variable $\langle f, X_t\rangle$ at 1. It follows from (A.8) that $-U_t f(x)$ takes values in $\mathbb{C}_+$. Furthermore, we know from Proposition A.6 that

$$U_t f(x) - \int_0^t P^\alpha_{t-s}\Psi_0(-U_s f)(x)ds = iP^\alpha_t f(x), \quad t \in [0, \infty), x \in \mathbb{R}^d, f \in \mathcal{P}. \tag{2.4}$$

For all $t \geq 0$ and $f \in \mathcal{P}$, we define

$$\begin{aligned} Z_t f &:= \int_0^t P^\alpha_{t-s}\big(\eta(-iP^\alpha_s f)^{1+\beta}\big)ds, \\ Z'_t f &:= \int_0^t P^\alpha_{t-s}\big(\eta(-U_s f)^{1+\beta}\big)ds, \\ Z''_t f &:= \int_0^t P^\alpha_{t-s}\Psi_1(-U_s f)ds, \\ Z'''_t f &:= (Z'_t - Z_t + Z''_t)f. \end{aligned} \tag{2.5}$$

Then we have that

$$U_t - iP^\alpha_t = Z'_t + Z''_t = Z_t + Z'''_t, \quad t \geq 0. \tag{2.6}$$

For all $\kappa \in \mathbb{Z}_+$ and $f \in \mathcal{P}$, define

$$Q_\kappa f := \sup_{t\geq 0} e^{\kappa b t}|P_t f| \tag{2.7}$$

and

$$Qf := Q_{\kappa_f} f. \tag{2.8}$$

Then according to [29, Fact 1.2], Q is an operator from $\mathcal{P}$ to $\mathcal{P}$.

Lemma 2.5. *Under Assumptions 1 and 2, the following statements are true:*

(1) $(-U_t)_{0\le t\le 1}$ *is uniformly* θ*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^*\cap\mathcal{B}(\mathbb{R}^d,\mathbb{C}_+)$*.*
(2) $(P^\alpha_t)_{0\le t\le 1}$ *is uniformly* θ*-controllable on* $\mathcal{P}^*$*.*
(3) Ψ_0 *is* $(\theta^2\vee\theta^{1+\beta})$*-controllable on* $\mathcal{P}^*\cap\mathcal{B}(\mathbb{R}^d,\mathbb{C}_+)$*.*
(4) $(U_t-iP^\alpha_t)_{0\le t\le 1}$ *is uniformly* $(\theta^2\vee\theta^{1+\beta})$*-controllable on* $\mathcal{P}$*.*
(5) $(Z'_t-Z_t)_{0\le t\le 1}$ *is uniformly* $(\theta^{2+\beta}\vee\theta^{1+2\beta})$*-controllable on* $\mathcal{P}$*.*
(6) *For any* $\delta>0$ *small enough, we have that* $(Z''_t)_{0\le t\le 1}$ *is uniformly* $(\theta^2\vee\theta^{1+\beta+\delta})$*-controllable on* $\mathcal{P}$*.*
(7) *For any* $\delta>0$ *small enough, we have that* $(Z'''_t)_{0\le t\le 1}$ *is uniformly* $(\theta^{2+\beta}\vee\theta^{1+\beta+\delta})$*-controllable on* $\mathcal{P}$*.*

Proof. (1). According to (A.8), U_t is an operator from $\mathcal{P}$ to $\mathcal{B}(\mathbb{R}^d,\mathbb{C}_+)$. It follows from (A.9) that for all $g\in\mathcal{P}$, $0\le t\le 1$ and $x\in\mathbb{R}^d$,

$$|U_tg(x)|\le\sup_{0\le u\le 1}P^\alpha_u|g|(x).$$

We claim that $f\mapsto\sup_{0\le u\le 1}P^\alpha_uf$ is a map from $\mathcal{P}^+$ to $\mathcal{P}^+$. In fact, if $f\in\mathcal{P}^+$, there exists constant $c>0$ such that

$$0\le\sup_{0\le u\le 1}P^\alpha_uf\le\sup_{0\le u\le 1}P_u(e^{\alpha u}e^{-\kappa_f u}e^{\kappa_f u}f)\le c\sup_{0\le u\le 1}(e^{\kappa_f u}P_uf)\le cQf\in\mathcal{P}.$$

It is clear that $f\mapsto\sup_{0\le u\le 1}P^\alpha_uf$ is a θ-controller.

(2). Similar to the proof of (1).

(3). According to Lemma 2.1, there exist constants $C,\delta>0$ satisfying $\beta+\delta<1$ such that for all $f\in\mathcal{P}^*\cap\mathcal{B}(\mathbb{R}^d,\mathbb{C}_+)$,

$$|\Psi_0f|\le\eta|f|^{1+\beta}+|\Psi_1f|\le\eta|f|^{1+\beta}+C|f|^2+C|f|^{1+\beta+\delta}.$$

Note that

$$f\mapsto\eta f^{1+\beta}+Cf^2+Cf^{1+\beta+\delta},\quad f\in\mathcal{P}^+,$$

is a $(\theta^2\vee\theta^{1+\beta})$-controller.

(4). From parts (1)–(3) of this lemma and Lemma 2.4.(2), we know that the operators

$$f\mapsto P^\alpha_{t-s}\Psi_0(-U_sf),\quad 0\le s\le t\le 1,$$

are uniformly $(\theta^2\vee\theta^{1+\beta})$-controllable. Combining this with (2.4) and Lemma 2.4.(1), we get the desired result.

(5). Notice that from Lemma A.3,

$$|(-U_tf)^{1+\beta}-(-iP^\alpha_tf)^{1+\beta}|\le(1+\beta)|U_tf-iP^\alpha_tf|(|U_tf|^\beta+|iP^\alpha_tf|^\beta).$$

Now using parts (1), (2) and (4) of this lemma, and Lemma 2.4.(3)–(4), we get that operators

$$f\mapsto(-U_tf)^{1+\beta}-(-iP^\alpha_tf)^{1+\beta},\quad 0\le t\le 1,$$

are uniformly $(\theta^{2+\beta}\vee\theta^{1+2\beta})$-controllable. Combining this with Lemma 2.4.(1)-(2), and

$$(Z'_t-Z_t)f=\int_0^tP^\alpha_{t-s}\Big(\eta\big((-U_sf)^{1+\beta}-(-iT^\alpha_sf)^{1+\beta}\big)\Big)ds,\quad 0\le t\le 1,f\in\mathcal{P},$$

we get the desired result.

(6). According to Lemma 2.1, for any $\delta > 0$ small enough, there exists a constant $C > 0$ such that

$$|\Psi_1(f)| \le C(|f|^2 + |f|^{1+\beta+\delta}), \quad f \in \mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+).$$

Note that, for all $\delta, C > 0$,

$$f \mapsto C(f^2 + f^{1+\beta+\delta}), \quad f \in \mathcal{P}^+$$

is a $(\theta^2 \vee \theta^{1+\beta+\delta})$-controller. Therefore, for any $\delta > 0$ small enough, we have that Ψ_1 is a $(\theta^2 \vee \theta^{1+\beta+\delta})$-controllable operator from $\mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$ to $\mathcal{P}^*$. Combining this with parts (1) and (2) of this lemma, and Lemma 2.4.(1)–(2), we get that, for any $\delta > 0$ small enough, the operators

$$f \mapsto Z_t'' f = \int_0^t P_{t-s}^\alpha \Psi_1(-U_s f) ds, \quad 0 \le t \le 1,$$

are uniformly $(\theta^2 \vee \theta^{1+\beta+\delta})$-controllable from $\mathcal{P}$ to $\mathcal{P}^*$.

(7). Since $Z_t''' = (Z_t' - Z_t) + Z_t''$, the desired result follows from parts (5)– (6) of this lemma and Lemma 2.4.(3). □

2.4. **Stable distributions.** In this subsection, we will show that the $(1 + \beta)$-stable random variables in Theorem 1.5, 1.6 and 1.7 are all well defined.

Recall that $(P_t^\alpha)_{t\ge 0}$ is defined in (1.4), $\mathcal{P}$ is defined in (1.8), $\mathcal{N}$ is given in (1.9) and $\mathcal{C}_l$ is defined in (1.10).

Lemma 2.6. (1) *If $f \in \mathcal{P}$, then the following integrals are well defined:*

$$m_t[f] := \eta \int_t^{t+1} e^{-\alpha s}\, ds \int_{\mathbb{R}^d} (-iP_s^\alpha f(x))^{1+\beta} \varphi(x)\, dx, \quad t \ge 0.$$

Furthermore, there exists a constant $C > 0$ such that

$$|m_t[f]| \le Ce^{(\alpha\beta - \kappa_f b(1+\beta))t}, \quad t \ge 0.$$

(2) *If $f \in \mathcal{C}_l$, then the following integrals are well defined:*

$$\bar{m}_t[f] := \eta \int_{t-1}^t e^{\alpha s}\, ds \int_{\mathbb{R}^d} \big(iI_s f(x)\big)^{1+\beta} \varphi(x)\, dx, \quad t \ge 1.$$

Furthermore, there exists a constant $C > 0$ such that

$$|\bar{m}_t[f]| \le Ce^{-(\alpha\beta - K(1+\beta)b)t}, \quad t \ge 1,$$

where K is defined in (1.9).

(3) *If $f \in \mathcal{C}_l$, then the following integral is well defined:*

$$\bar{m}[f] := \eta \int_0^\infty e^{\alpha s}\, ds \int_{\mathbb{R}^d} \big(iI_s f(x)\big)^{1+\beta} \varphi(x)\, dx.$$

Furthermore, we have

$$\bar{m}[f] = \sum_{n=1}^\infty \bar{m}_n[f]. \tag{2.9}$$

(4) *If $f \in \mathcal{P}$ satisfies $\alpha\beta = \kappa_f b(1+\beta)$, then the following integral is well defined:*

$$\widetilde{m}[f] := \eta \int_{\mathbb{R}^d} \Big(-i \sum_{p \in \mathbb{Z}_+^d : |p| = \kappa_f} \langle f, \phi_p \rangle \phi_p(x) \Big)^{1+\beta} \varphi(x)\ dx.$$

Furthermore, we have

$$\widetilde{m}[f] = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{\lfloor t \rfloor} m_k[f]. \tag{2.10}$$

(5) *If $f \in \mathcal{P}$ satisfies $\alpha\beta < \kappa_f b(1+\beta)$, then the following integral is well defined:*

$$m[f] := \eta \int_0^\infty e^{-\alpha s}\ ds \int_{\mathbb{R}^d} \big(-iP_s^\alpha f(x) \big)^{1+\beta} \varphi(x)\ dx.$$

Furthermore, we have

$$m[f] = \sum_{k=0}^{\infty} m_k[f]. \tag{2.11}$$

Proof. (1). See [29, Lemma 2.7].

(2). We first claim that there exists $h \in \mathcal{P}$ such that

$$|I_t f(x)| \le e^{-(\alpha - Kb)t} h(x), \quad t \ge 0, x \in \mathbb{R}^d.$$

In fact, if we put $h := \sum_{p\in\mathcal{N}} |\langle f, \phi_p\rangle_\varphi \phi_p|$, then

$$|I_t f(x)| = \Big| \sum_{p \in \mathcal{N}} \langle f, \phi_p \rangle_\varphi e^{-(\alpha - |p|b)t} \phi_p(x) \Big| \le e^{-(\alpha - Kb)t} h(x), \quad t \ge 0, x \in \mathbb{R}^d. \tag{2.12}$$

Therefore there exists a constant $C > 0$ such that

$$\begin{aligned} |\bar{m}_t[f]| &\le \eta \int_{t-1}^t e^{\alpha s}\ ds \int_{\mathbb{R}^d} |I_s f(x)|^{1+\beta} \varphi(x)\ dx \\ &\le \eta \int_{t-1}^t e^{\alpha s} e^{-(\alpha - Kb)(1+\beta)s}\ ds \int_{\mathbb{R}^d} h(x)^{1+\beta} \varphi(x)\ dx \\ &\le C e^{-(\alpha\beta - K(1+\beta)b)t}, \quad t \ge 1. \end{aligned}$$

(3). This is obvious from (2), noticing that $\alpha\beta > K(1+\beta)b$.

(4). See [29, Lemma 4.2].

(5). This is obvious from (1). □

The following proposition says that $m_t[f]$, $\bar{m}_t[f]$, $m[f]$, $\bar{m}[f]$ and $\widetilde{m}[f]$ are all related to $(1+\beta)$-stable distributions.

Proposition 2.7. *For all $t \ge 0$ and non-trivial $f \in \mathcal{P}$,*

(1) *$\theta \mapsto \exp(m_t[\theta f])$ is the characteristic function of an $\mathbb{R}$-valued $(1+\beta)$-stable random variable;*

(2) *$\theta \mapsto \exp(\bar{m}_t[\theta f])$ is the characteristic function of an $\mathbb{R}$-valued $(1+\beta)$-stable random variable, provided $f \in \mathcal{C}_l$.*

(3) $\theta \mapsto \exp(\bar{m}[\theta f])$ *is the characteristic function of an* $\mathbb{R}$*-valued* $(1+\beta)$*-stable random variable, provided* $f \in \mathcal{C}_l$*.*
(4) $\theta \mapsto \exp(\widetilde{m}[\theta f])$ *is the characteristic function of an* $\mathbb{R}$*-valued* $(1+\beta)$*-stable random variable, provided* $\alpha\beta = \kappa_f b(1+\beta)$*;*
(5) $\theta \mapsto \exp(m[\theta f])$ *is the characteristic function of an* $\mathbb{R}$*-valued* $(1+\beta)$*-stable random variable, provided* $\alpha\beta < \kappa_f b(1+\beta)$*.*

This proposition says that the $(1+\beta)$-stable random variables in Theorems 1.5, 1.6 and 1.7 are all well defined. The proof of this proposition relies on the following lemma:

Lemma 2.8. *Let* q *be a measure on* $\mathbb{R}^d \setminus \{0\}$ *with* $\int_{\mathbb{R}^d\setminus\{0\}} |x|^{1+\beta} q(dx) \in (0,\infty)$*. Then*

$$\theta \mapsto \exp\Big\{ \int_{\mathbb{R}^d\setminus\{0\}} (i\theta\cdot x)^{1+\beta} q(dx) \Big\}, \quad \theta \in \mathbb{R}^d$$

is the characteristic function of an $\mathbb{R}^d$*-valued* $(1+\beta)$*-stable random variable.*

Proof. It follows from disintegration that there exist a measure λ on $S := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ and a kernel $k(\xi, dt)$ from S to $\mathbb{R}_+$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} f(x) q(dx) = \int_S \lambda(d\xi) \int_{\mathbb{R}^+} f(\xi t) k(\xi, dt), \quad f \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}, \mathbb{R}_+).$$

We define another measure λ_0 on S by

$$\lambda_0(d\xi) := \frac{1}{\Gamma(-1-\beta)} \int_0^\infty t^{1+\beta} k(\xi, dt) \lambda(d\xi),$$

where Γ is the Gamma function. Then λ_0 is a non-zero finite measure, since

$$\begin{aligned}\lambda_0(S) &= \frac{1}{\Gamma(-1-\beta)} \int_S \lambda(d\xi) \int_0^\infty |t\xi|^{1+\beta} k(\xi, dt) \\ &= \frac{1}{\Gamma(-1-\beta)} \int_{\mathbb{R}^d\setminus\{0\}} |x|^{1+\beta} q(dx) \in (0,\infty).\end{aligned}$$

Define a measure ν on $\mathbb{R}^d \setminus \{0\}$ by

$$\int_{\mathbb{R}^d\setminus\{0\}} f(x) \nu(dx) = \int_S \lambda_0(d\xi) \int_0^\infty f(r\xi) \frac{dr}{r^{2+\beta}}.$$

Then, according to [39, Remark 14.4], ν is the Lévy measure of an $(1+\beta)$-stable distribution on $\mathbb{R}^d$, say μ, whose characteristic function is

$$\hat{\mu}(\theta) = \exp\Big\{ \int_{\mathbb{R}^d\setminus\{0\}} (e^{-i\theta\cdot y} - 1 + i\theta\cdot y) \nu(dy) \Big\}, \quad \theta \in \mathbb{R}.$$

Finally, according to (A.2), we have

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus\{0\}} (e^{-i\theta\cdot y} - 1 + i\theta\cdot y) \nu(dy) = \int_S \lambda_0(d\xi) \int_0^\infty (e^{-ir\theta\cdot\xi} - 1 + ir\theta\cdot\xi) \frac{dr}{r^{2+\beta}} \\ &= \int_S \lambda(d\xi) \int_0^\infty (e^{-ir\theta\cdot\xi} - 1 + ir\theta\cdot\xi) \frac{dr}{\Gamma(-1-\beta) r^{2+\beta}} \int_0^\infty t^{1+\beta} k(\xi, dt)\end{aligned}$$

$$= \int_S \lambda(d\xi) \int_0^\infty (i\theta \cdot \xi)^{1+\beta} t^{1+\beta} k(\xi, dt) = \int_S \lambda(d\xi) \int_0^\infty (i\theta \cdot t\xi)^{1+\beta} k(\xi, dt)$$
$$= \int_{\mathbb{R}^d} (i\theta \cdot x)^{1+\beta} q(dx). \qquad \square$$

Proof of Proposition 2.7. (1). Fix $t \geq 0$ and $f \in \mathcal{P}$. Note that $m_t[\theta f]$ can be rewritten as

$$m_t[\theta f] = \eta \int_t^{t+1} e^{-\alpha s}\, ds \int_{\mathbb{R}^d} \big(- i\theta (P_s^\alpha f)(x)\big)^{1+\beta} \varphi(x)\, dx, \quad \theta \in \mathbb{R}.$$

Therefore, according to Lemma 2.8, we only need to show that

$$\int_t^{t+1} e^{-\alpha s}\, ds \int_{\mathbb{R}^d} |P_s^\alpha f(x)|^{1+\beta} \varphi(x)\, dx \in (0, \infty). \tag{2.13}$$

According to Lemma 2.5.(2), Lemma 2.4.(1)–(2) and Lemma 2.4.(4), we know that

$$f \mapsto \int_t^{t+1} e^{-\alpha s} |P_s^\alpha f|^{1+\beta}\, ds$$

is a $\theta^{1+\beta}$-controllable operator on $\mathcal{P}$. This implies that

$$x \mapsto \int_t^{t+1} e^{-\alpha s} |P_s^\alpha f(x)|^{1+\beta}\, ds$$

is an element of $\mathcal{P}$. Therefore the integral on the left-hand side of (2.13) is finite. Since f is non-trivial, the integral on the left-hand side of (2.13) is positive.

The proofs of (2), (3), (4) and (5) are similar to that of (1). We omit the details here. $\square$

Before we end this subsection, we give two more lemmas which will be used in the proofs of our main results in Section 3.

Lemma 2.9. [29, Lemma 2.8] *For each $f \in \mathcal{P}$, let*

$$g_k := \frac{Z_1 P_k^\alpha f - \langle Z_1 P_k^\alpha f, \varphi\rangle}{e^{(\alpha - \kappa_f b)(1+\beta)t}}, \quad k \geq 0.$$

Then there exists $h \in \mathcal{P}$ such that

$$|P_s g_k| \leq e^{-bs} h, \quad s \geq 0, k \geq 0.$$

Lemma 2.10. *Suppose that $g \in \mathcal{C}_l$. Put*

$$\bar{g}_t := \frac{Z_1(-I_t g) - \langle Z_1(-I_t g), \varphi\rangle}{e^{-(\alpha - Kb)(1+\beta)t}}, \quad t \geq 0.$$

Then there exists $h \in \mathcal{P}$ such that

$$|P_s \bar{g}_t| \leq e^{-bs} h, \quad s \geq 0, t \geq 0.$$

Proof. First we claim that there exists $h_1 \in \mathcal{P}$ such that

$$\Big|\frac{\partial}{\partial x_i} I_t g(x)\Big| \le e^{-(\alpha-Kb)t} h_1(x), \quad t\ge 0, x\in\mathbb{R}^d. \tag{2.14}$$

In fact, if we put $h_1(x)=\sum_{p\in\mathcal{N}} |\langle g,\phi_p\rangle_\varphi \frac{\partial \phi_p}{\partial x_i}(x)| \in \mathcal{P}$, then

$$\Big|\frac{\partial}{\partial x_i} I_t g(x)\Big| = \Big|\sum_{p\in\mathcal{N}} \langle g,\phi_p\rangle_\varphi e^{-(\alpha-|p|b)t}\frac{\partial}{\partial x_i}\phi_p(x)\Big| \le e^{-(\alpha-Kb)t} h_1(x), \quad t\ge 0, x\in\mathbb{R}^d.$$

It is well known that

$$P_t f(x) = \int_{\mathbb{R}^d} f\big(xe^{-bt} + y\sqrt{1-e^{-2bt}}\big)\varphi(y)dy, \quad t\ge 0, x\in\mathbb{R}^d, f\in\mathcal{B}(\mathbb{R}^d,\mathbb{R}_+).$$

It is clear that the above equality also holds for each $f\in\mathcal{P}$. From this, one can easily check that for each differential $f\in\mathcal{P}$ with $|\nabla f|\in\mathcal{P}$, we have

$$\frac{\partial P_t^\alpha f(x)}{\partial x_i} = e^{-bt} P_t^\alpha\Big(\frac{\partial f}{\partial x_i}\Big)(x), \quad t\ge 0, x\in\mathbb{R}^d.$$

Note that if $g\in\mathcal{C}_l$, then $g^{1+\beta}\in\mathcal{P}$ and $|\nabla(g^{1+\beta})|\in\mathcal{P}$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x_i} P_{1-s}^\alpha (P_s^\alpha I_t g)^{1+\beta} &= \frac{\partial}{\partial x_i} P_{1-s}^\alpha (I_{t-s} g)^{1+\beta} = e^{-b(1-s)} P_{1-s}^\alpha \frac{\partial}{\partial x_i}(I_{t-s}g)^{1+\beta}\\ &= (1+\beta)e^{-b(1-s)} P_{1-s}^\alpha\Big((I_{t-s}g)^\beta \times \frac{\partial}{\partial x_i}(I_{t-s}g)\Big), \quad s\in[0,1], t\ge 0.\end{aligned}$$

Using (2.12), (2.14) and the above, we have that there exist a bounded function a on $[0,1]$ and an $h_2\in\mathcal{P}$ such that

$$\Big|\frac{\partial P_{1-s}^\alpha (P_s^\alpha I_t g)^{1+\beta}}{\partial x_i}(x)\Big| \le a(s) e^{-(\alpha-Kb)(1+\beta)t} h_2(x), \quad s\in[0,1], t\ge 0, x\in\mathbb{R}^d. \tag{2.15}$$

Therefore, for each compact subset $A\subset\mathbb{R}^d$, we have

$$\int_0^1 \sup_{x\in A}\Big|\frac{\partial P_{1-s}^\alpha (P_s^\alpha I_t g)^{1+\beta}}{\partial x_i}(x)\Big| ds < \infty, \quad t\ge 0.$$

Using this and [10, Theorem A.5.2], we can get

$$\frac{\partial}{\partial x_i} Z_1(-I_t g) = \eta i^{1+\beta}\frac{\partial}{\partial x_i}\int_0^1 P_{1-s}^\alpha (I_{t-s}g)^{1+\beta} ds = \eta i^{1+\beta}\int_0^1 \frac{\partial}{\partial x_i} P_{1-s}^\alpha (I_{t-s}g)^{1+\beta} ds.$$

Therefore, using (2.15) and the above, there exists a constant $C>0$ such that for all $t\ge 0$,

$$\Big|\frac{\partial}{\partial x_i}\bar g_t(x)\Big| = e^{(\alpha-Kb)(1+\beta)t}\Big|\frac{\partial}{\partial x_i} Z_1(-I_t g)\Big| \le C h_2(x), \quad x\in\mathbb{R}^d.$$

Therefore there exist constants $c,n>0$ such that

$$|\nabla \bar g_t(z)| \le c(1+|z|)^n, \quad t\ge 0, z\in\mathbb{R}^d.$$

Define

$$D(x,y) := \{ax+by : a,b\in[0,1]\}, \quad x,y\in\mathbb{R}^d.$$

Then for all $x, y \in \mathbb{R}^d$, $D(x,y)$ is a convex set on $\mathbb{R}^d$. In particular, it contains the interval connecting $xe^{-bs} + y\sqrt{1-e^{-2bs}}$ with y for each $s \geq 0$. Also, note that

$$|\sqrt{1-\theta} - 1| \leq \theta, \quad \theta \in [0,1]$$

and

$$1 + \theta + \lambda \leq (1+\theta)(1+\lambda), \quad \theta, \lambda \geq 0.$$

Using the fact that $\langle \bar{g}_t, \varphi \rangle = 0$, we get that for all $s, t \geq 0, x \in \mathbb{R}^d$,

$$\begin{aligned}
|P_s \bar{g}_t(x)| &= \Big| \int_{\mathbb{R}^d} \big(\bar{g}_t(xe^{-bs} + y\sqrt{1-e^{-2bs}}) - \bar{g}_t(y) \big) \varphi(y)\ dy \Big| \\
&\leq \int_{\mathbb{R}^d} \sup_{z \in D(x,y)} |\nabla \bar{g}_t(z)| \Big| xe^{-bs} + y\sqrt{1-e^{-2bs}} - y \Big| \varphi(y)\ dy \\
&\leq e^{-bs} \int_{\mathbb{R}^d} c(1+|x|+|y|)^n (|x|+|y|) \varphi(y)\ dy \\
&\leq ce^{-bs}(1+|x|)^n \Big(|x| \int_{\mathbb{R}^d} (1+|y|)^n \varphi(y) dy + \int_{\mathbb{R}^d} (1+|y|)^n |y| \varphi(y) dy \Big).
\end{aligned}$$

□

2.5. **Small value probability.** In this subsection, we digress briefly from our super-OU process and consider a (supercritical) *continuous-state branching process (CSBP)* $\{(Y_t)_{t\geq 0}; \mathbf{P}_x\}$ with branching mechanism ψ given by (1.1). Such a process $\{(Y_t)_{t\geq 0}; \mathbf{P}_x\}$ is defined as an $\mathbb{R}^+$-valued Hunt process satisfying

$$\mathbf{P}_x[e^{-\lambda Y_t}] = e^{-xv_t(\lambda)}, \quad x \in \mathbb{R}^+, t \geq 0, \lambda \in \mathbb{R}^+,$$

where for each $\lambda \geq 0$, $t \mapsto v_t(\lambda)$ is the unique positive solution to the equation

$$v_t(\lambda) - \int_0^t \psi(v_s(\lambda))\ ds = \lambda, \quad t \geq 0. \tag{2.16}$$

It can be verified that for each $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $x = \|\mu\|$, we have

$$\{(\|X_t\|)_{t\geq 0}; \mathbb{P}_\mu\} \overset{\text{law}}{=} \{(Y_t)_{t\geq 0}; \mathbf{P}_x\}.$$

Our goal in this subsection is to determine how fast the probability $\mathbf{P}_x(0 < e^{-\alpha t} Y_t \leq k_t)$ converges to 0 when $t \mapsto k_t$ is a strictly positive function on $[0, \infty)$ such that $k_t \to 0$ and $k_t e^{\alpha t} \to \infty$ as $t \to \infty$. Suppose that Grey's condition is satisfied, i.e., there is some constant $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$, and that $\int_{z'}^\infty \psi(z)^{-1} dz < \infty$. Also suppose that the $L \log L$ condition is satisfied, i.e.,

$$\int_1^\infty y \log y\ \pi(dr) < \infty.$$

We write $W_t = e^{-\alpha t} Y_t$ for each $t \geq 0$.

Proposition 2.11. *Suppose that $t \mapsto k_t$ is a strictly positive function on $[0, \infty)$ such that $k_t \to 0$ and $k_t e^{\alpha t} \to \infty$ as $t \to \infty$. Then, for each $x \geq 0$, there exist constants $C, \delta > 0$ such that*

$$\mathbf{P}_x(0 < W_t \leq k_t) \leq C(k_t^\delta + e^{-\delta t}), \quad t \geq 0.$$

Proof. Step 1. We recall some known facts about the CSBP (Y_t). For each $\lambda \geq 0$, we denote by $t \mapsto v_t(\lambda)$ the unique positive solution of (2.16). Letting $\lambda \to \infty$ in (2.16), we have by monotonicity that $\bar{v}_t := \lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0, \infty]$ for all $t \geq 0$, and that

$$\mathbf{P}_x(Y_t = 0) = e^{-x\bar{v}_t}, \quad t \geq 0, x \geq 0. \tag{2.17}$$

It is known, see [26, Theorems 3.5–3.8] for example, that under Grey's condition, the following statements are true:

- $0 \leq \bar{v}_t < \infty$ for all $t > 0$.
- $t \mapsto \bar{v}_t$ is decreasing and $\bar{v} := \lim_{t\to\infty} \bar{v}_t \in [0, \infty)$ is the largest root of $\psi(z) = 0$.

Letting $t \to \infty$ in (2.17), we have by monotonicity that

$$\mathbf{P}_x(\exists t \geq 0, Y_t = 0) = e^{-x\bar{v}}, \quad x \geq 0.$$

Note that ψ has derivative

$$\psi'(z) = -\alpha + 2\rho z + \int_{(0,\infty)} (1 - e^{-zy}) y\pi(dy), \quad z \geq 0,$$

which is increasing in z. This says that ψ is a convex function. Also notice that $\psi'(0+) = -\alpha < 0$ and that there exists $z > 0$ such that $\psi(z) > 0$. Therefore we have

- $\bar{v} > 0$,
- $\psi(z) < 0$ on $z \in (0, \bar{v})$,
- $\psi(z) > 0$ on $z \in (\bar{v}, \infty)$.

It is also known, see [26, Proposition 3.3] for example, that

- if $\lambda \in (0, \bar{v})$, then $0 < \lambda \leq v_t(\lambda) < \bar{v}$ and
$$\int_{\lambda}^{v_t(\lambda)} \frac{dz}{-\psi(z)} = t, \quad t \geq 0; \tag{2.18}$$
- if $\lambda \in (\bar{v}, \infty)$, then $\bar{v} < v_t(\lambda) \leq \lambda < \infty$ and
$$\int_{v_t(\lambda)}^{\lambda} \frac{dz}{\psi(z)} = t, \quad t \geq 0.$$

By monotonicity, we have that

$$\int_{\bar{v}_t}^{\infty} \frac{dz}{\psi(z)} = t, \quad t \geq 0. \tag{2.19}$$

Step 2. We will show that, for each $x \geq 0$ there exists a constant $c_1 > 0$ such that

$$\mathbf{P}_x(0 < W_t \leq k_t) \leq c_1\big(|\bar{v} - v_t(k_t^{-1}e^{-\alpha t})| + |\bar{v}_t - \bar{v}|\big), \quad t \geq 0.$$

In fact, for all $x \geq 0$ and $t \geq 0$, we have

$$\begin{aligned}
&\mathbf{P}_x(0 < W_t \leq k_t) = \mathbf{P}_x(e^{-k_t^{-1}W_t} \geq e^{-1}, W_t > 0) \\
&\leq e\mathbf{P}_x[e^{-k_t^{-1}W_t}; W_t > 0] = e\big(\mathbf{P}_x[e^{-k_t^{-1}W_t}] - \mathbf{P}_x(W_t = 0)\big) \\
&= e\big(e^{-xv_t(k_t^{-1}e^{-\alpha t})} - e^{-x\bar{v}_t}\big) \\
&\leq ex\big(|\bar{v} - v_t(k_t^{-1}e^{-\alpha t})| + |\bar{v}_t - \bar{v}|\big),
\end{aligned}$$

as desired in this step.

Step 3. We will show that there exist constants $c_2, \delta_1, t_0 > 0$ such that

$$|\bar{v}_t - \bar{v}| \leq c_2 e^{-\delta_1 t}, \quad t \geq t_0.$$

In fact, since ψ is a convex function, we must have $\tau := \psi'(\bar{v}) > 0$ and that $\psi(z) \geq (z-\bar{v})\tau$ for each $z \geq \bar{v}$. According to Grey's condition, we can find a constant $z_0 > \bar{v}$ such that

$$t_0 := \int_{z_0}^{\infty} \frac{dz}{\psi(z)} < \infty.$$

For each $t > t_0$, according to (2.19), we have

$$\begin{aligned} t - t_0 &= \int_{\bar{v}_t}^{\infty} \frac{dz}{\psi(z)} - \int_{z_0}^{\infty} \frac{dz}{\psi(z)} = \int_{\bar{v}_t}^{z_0} \frac{dz}{\psi(z)} \\ &\leq \int_{\bar{v}_t}^{z_0} \frac{dz}{(z-\bar{v})\tau} = \frac{1}{\tau}\big(\log(z_0 - \bar{v}) - \log(\bar{v}_t - \bar{v})\big). \end{aligned}$$

Rearranging, we get

$$\bar{v}_t - \bar{v} \leq (z_0 - \bar{v}) e^{-\tau(t-t_0)}, \quad t \geq t_0.$$

This implies the desired result in this step.

Step 4: We will show that there exist constants $c_3, \delta_2, t_1 > 0$ such that

$$|\bar{v} - v_t(k_t^{-1} e^{-\alpha t})| \leq c_3 k_t^{\delta_2}, \quad t \geq t_1.$$

Define

$$\rho_t := 1 + \frac{\log k_t}{t\alpha}, \quad t \geq 0.$$

According to the fact that

$$k_t^{-1} e^{-\alpha t} = e^{-\alpha \rho_t t}, \quad t \geq 0,$$

and the condition that $k_t e^{\alpha t} \xrightarrow[t\to\infty]{} \infty$, we have $\rho_t t \xrightarrow[t\to\infty]{} \infty$. Since the $L\log L$ condition is satisfied, we have (see [27] for example), $W_t \xrightarrow[t\to\infty]{a.s.} W_\infty$, where the martingale limit W_∞ is a non-degenerate positive random variable. This implies that

$$v_t(e^{-\alpha t}) = -\log \mathbf{P}_1[e^{-W_t}] \xrightarrow[t\to\infty]{} -\log \mathbf{P}_1[e^{-W_\infty}] =: z^* \in (0,\infty).$$

The $L\log L$ condition also guarantees that (see again [27] for example) $\{W_\infty = 0\} = \{\exists t \geq 0, X_t = 0\}$ a.s. in $\mathbf{P}_1$. This and the non-degeneracy of W_∞ imply that

$$z^* = -\log \mathbf{P}_1[e^{-W_\infty}] < -\log \mathbf{P}_1(W_\infty = 0) = \bar{v}.$$

Fix an arbitrary $\epsilon \in (0,\tau)$. According to the fact that $\tau = \psi'(\bar{v}) > 0$, there exists $z_0 \in (0,\bar{v})$ such that for all $z \in (z_0, \bar{v})$, we have $-\psi(z) \geq (\bar{v} - z)(\tau - \epsilon)$. Fix this z_0. For t large enough, we have $0 < k_t^{-1} e^{-\alpha t} < v_t(k_t^{-1} e^{-\alpha t}) < \bar{v}$. Then using (2.18) with $\lambda = k_t^{-1} e^{-\alpha t}$, we have for $t > 0$ large enough,

$$t = \int_{k_t^{-1} e^{-\alpha t}}^{v_t(k_t^{-1} e^{-\alpha t})} \frac{dz}{-\psi(z)}$$

$$
\begin{aligned}
&= \Big(\int_{e^{-\alpha\rho_t t}}^{v_{\rho_t t}(e^{-\alpha\rho_t t})} + \int_{v_{\rho_t t}(e^{-\alpha\rho_t t})}^{z_0} + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \Big) \frac{dz}{-\psi(z)} \\
&= \rho_t t + O(1) + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \frac{dz}{-\psi(z)},
\end{aligned}
$$

where we used the fact that

$$
\int_{v_{\rho_t t}(e^{-\alpha\rho_t t})}^{z_0} \frac{dz}{-\psi(z)} \xrightarrow[t\to\infty]{} \int_{z^*}^{z_0} \frac{dz}{-\psi(z)}.
$$

Now we have, for t large enough,

$$
\begin{aligned}
t &\le \rho_t t + O(1) + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \frac{dz}{(\bar v - z)(\tau-\epsilon)} \\
&= \rho_t t + O(1) - \frac{1}{\tau-\epsilon}\Big(\log\big(\bar v - v_t(e^{-\alpha\rho_t t})\big) - \log(\bar v - z_0)\Big).
\end{aligned}
$$

Rearranging, we get, for t large enough,

$$
e^{-t(\tau-\epsilon)} \ge e^{-\rho_t t(\tau-\epsilon)+O(1)}(\bar v - v_t(e^{-\alpha\rho_t t})).
$$

Therefore, there exist constants $c_3 > 0$ and $t_1 > 0$ such that for all $t \ge t_1$,

$$
\bar v - v_t(k_t^{-1}e^{-\alpha t}) \le e^{-t(\tau-\epsilon)+(1+\frac{\log k_t}{t\alpha})t(\tau-\epsilon)+O(1)} \le c_3 k_t^{\frac{\tau-\epsilon}{\alpha}}.
$$

This implies the desired result in this step.

Finally, according to the results in Steps 2-4, we have for each $x \ge 0$, there exist constant $c_4, \delta_3, t_2 > 0$ such that

$$
\mathbf{P}_x(0 < W_t \le k_t) \le c_4(k_t^{\delta_3} + e^{-\delta_3 t}), \quad t \ge t_2.
$$

Note that the left side is always bounded by 1, so we can take $t_2 = 0$ in the above statement. □

2.6. **Moments for super-OU processes.** In this subsection, we want to find some upper bound for the $(1+\gamma)$-th moment of $\langle g, X_t\rangle$, where $\gamma \in (0,\beta)$, $g \in \mathcal{P}$ and $\{(X_t)_{t\ge 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ is the super OU process considered in Subsection 1.2 satisfying Assumption 1 and 2. Recall that the operator $\mathcal{I}_s^t$ is defined in 1.13.

Lemma 2.12. *There is a $(\theta^2 \vee \theta^{1+\beta})$-controller R such that for all $0 \le t \le 1$, $g \in \mathcal{P}$, $\lambda > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, we have*

$$
\mathbb{P}_\mu(|\mathcal{I}_0^t\langle g, X_t\rangle| > \lambda) \le \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} \langle R|\theta g|, \mu\rangle d\theta.
$$

Proof. Denote by R the $(\theta^2 \vee \theta^{1+\beta})$-controller in Lemma 2.5.(4). Using Lemma A.1 and the argument in the proof of [10, Theorem 3.3.6], we get

$$
\big|\mathbb{P}_\mu(|\mathcal{I}_0^t\langle g, X_t\rangle| > \lambda)\big| \le \Big|\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}(1 - \mathbb{P}_\mu[e^{i\theta\mathcal{I}_0^t\langle g, X_t\rangle}])d\theta\Big|
$$

$$\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} |1-e^{\langle U_t(\theta g)-iP_t^\alpha(\theta g),\mu\rangle}|d\theta \leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\langle |U_t(\theta g)-iP_t^\alpha(\theta g)|,\mu\rangle d\theta$$
$$\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\langle R|\theta g|,\mu\rangle d\theta. \qquad \square$$

Lemma 2.13. *For all $h\in\mathcal{P}^+$ and $\mu\in\mathcal{M}_c(\mathbb{R}^d)$, there exists a constant $C>0$ such that for all $\kappa\in\mathbb{Z}_+$, $\lambda>0$ and $0\leq r\leq s\leq t<\infty$ with $s-r\leq 1$, we have*
$$\sup_{g\in\mathcal{P}:Q_\kappa g\leq h}\mathbb{P}_\mu(|\mathcal{I}_r^s\langle g,X_t\rangle|>\lambda)\leq Ce^{\alpha r}\bigg(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^2\bigg).$$

Proof. Denote by R the $(\theta^2\vee\theta^{1+\beta})$-controller in Lemma 2.12. Fix $h\in\mathcal{P}^+$, $\mu\in\mathcal{M}_c(\mathbb{R}^d)$ $\kappa\in\mathbb{Z}_+$ and $0\leq r\leq s\leq t<\infty$ with $s-r\leq 1$. Suppose that $g\in\mathcal{P}$ satisfies $Q_\kappa g\leq h$. Using the Markov property of X, we get
$$\begin{aligned}
&\mathbb{P}_\mu(|\mathcal{I}_r^s\langle g,X_t\rangle|>\lambda)=\mathbb{P}_\mu\Big[\mathbb{P}_\mu\big[|\langle P_{t-s}^\alpha g,X_s\rangle-\langle P_{t-r}^\alpha g,X_r\rangle|>\lambda\big|\mathscr{F}_r\big]\Big]\\
&=\mathbb{P}_\mu\big[\mathbb{P}_{X_r}(|\langle P_{t-s}^\alpha g,X_{s-r}\rangle-\langle P_{t-r}^\alpha g,X_0\rangle|>\lambda)\big]\\
&=\mathbb{P}_\mu\big[\mathbb{P}_{X_r}(|\mathcal{I}_0^{s-r}\langle P_{t-s}^\alpha g,X_{s-r}\rangle|>\lambda)\big]\\
&\leq\mathbb{P}_\mu\Big[\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\langle R|\theta P_{t-s}^\alpha g|,X_r\rangle d\theta\Big]\\
&\leq\mathbb{P}_\mu\Big[\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\langle R|\theta e^{(t-s)(\alpha-\kappa b)}h|,X_r\rangle d\theta\Big]\\
&\leq\mathbb{P}_\mu[\langle Rh,X_r\rangle]\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}(|\theta e^{(t-s)(\alpha-\kappa b)}|^{1+\beta}+|\theta e^{(t-s)(\alpha-\kappa b)}|^2)d\theta\\
&=\langle P_r^\alpha Rh,\mu\rangle\bigg(\frac{2^{2+\beta}}{2+\beta}\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\frac{2^3}{3}\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^2\bigg)\\
&\leq Ce^{\alpha r}\bigg(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^2\bigg),
\end{aligned}$$
where the constant $C>0$ above is chosen as
$$C:=\Big(\frac{2^{2+\beta}}{2+\beta}+\frac{2^3}{3}\Big)\langle Q_0Rh,\mu\rangle. \qquad \square$$

Lemma 2.14. *For all $h\in\mathcal{P}$, $\mu\in\mathcal{M}_c(\mathbb{R}^d)$ and $\gamma\in(0,\beta)$, there exists a constant $C>0$ such that for all $\kappa\in\mathbb{Z}_+$ and $0\leq r\leq s\leq t<\infty$ with $s-r\leq 1$, we have*
$$\sup_{g\in\mathcal{P}:Q_\kappa g\leq h}\mathbb{P}_\mu\big[|\mathcal{I}_r^s\langle g,X_t\rangle|^{1+\gamma}\big]\leq Ce^{t\alpha+(t-s)(\gamma\alpha-(1+\gamma)\kappa b)}.$$

Proof. Fix $h\in\mathcal{P}$ and $\mu\in\mathcal{M}_c(\mathbb{R}^d)$. Let C_0 be the constant in the Lemma 2.13. For all $\kappa\in\mathbb{Z}_+$, $0\leq r\leq s\leq t$ with $s-r\leq 1$, $g\in\mathcal{P}$ with $Q_\kappa g\leq h$, and $c>0$, we have
$$\mathbb{P}_\mu\big[|\mathcal{I}_r^s\langle g,X_t\rangle|^{1+\gamma}\big]=(1+\gamma)\int_0^\infty\lambda^\gamma\mathbb{P}_\mu(|\mathcal{I}_r^s\langle g,X_t\rangle|>\lambda)d\lambda$$

$$
\begin{aligned}
&\le (1+\gamma)\int_0^c \lambda^\gamma d\lambda + (1+\gamma)\int_c^\infty \lambda^\gamma \mathbb{P}_\mu(|\mathcal{I}_r^s\langle g, X_t\rangle| > \lambda) d\lambda \\
&\le c^{1+\gamma} + C_0 e^{\alpha r}(1+\gamma)\int_c^\infty \Big(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta} + \Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^2\Big)\lambda^\gamma d\lambda \\
&\le c^{1+\gamma}e^{\alpha r} + C_0 e^{\alpha r}(1+\gamma)\Big(\frac{e^{(1+\beta)(t-s)(\alpha-\kappa b)}}{(\beta-\gamma)c^{\beta-\gamma}} + \frac{e^{2(t-s)(\alpha-\kappa b)}}{(1-\gamma)c^{1-\gamma}}\Big).
\end{aligned}
$$

Taking $c = e^{(t-s)(\alpha-\kappa b)}$, we get

$$
\mathbb{P}_\mu\big[|\mathcal{I}_r^s\langle g, X_t\rangle|^{1+\gamma}\big] \le e^{(1+\gamma)(t-s)(\alpha-\kappa b)} e^{\alpha r}\Big(1 + C_0\frac{1+\gamma}{\beta-\gamma} + C_0\frac{1+\gamma}{1-\gamma}\Big).
$$

Note that

$$
\begin{aligned}
&(1+\beta)(t-s)(\alpha-\kappa b) + \alpha r = (t-s)\alpha + (t-s)(\beta\alpha - (1+\beta)\kappa b) \\
&\le t\alpha + (t-s)(\beta\alpha - (1+\beta)\kappa b).
\end{aligned}
$$

So the desired result is true with

$$
C := 1 + C_0\frac{1+\gamma}{\beta-\gamma} + C_0\frac{1+\gamma}{1-\gamma}. \qquad \square
$$

For each random variable $\{Y;\mathbb{P}\}$ and $p \in [1,\infty)$, we write

$$
\|Y\|_{\mathbb{P};p} := \mathbb{P}[|Y|^p]^{1/p}.
$$

Lemma 2.15. *For all $h \in \mathcal{P}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $\gamma \in (0,\beta)$ and $\kappa \in \mathbb{Z}_+$, there exists a constant $C > 0$ such that for each $t \ge 0$, we have*

(1) $\sup_{g\in\mathcal{P}:Q_\kappa g\le h} \|\langle g, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} \le Ce^{(\alpha-\kappa b)t}$ *provided $\alpha\gamma > \kappa(1+\gamma)b$;*
(2) $\sup_{g\in\mathcal{P}:Q_\kappa g\le h} \|\langle g, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} \le Cte^{\frac{\alpha}{1+\gamma}t}$ *provided $\alpha\gamma = \kappa(1+\gamma)b$;*
(3) $\sup_{g\in\mathcal{P}:Q_\kappa g\le h} \|\langle g, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} \le Ce^{\frac{\alpha}{1+\gamma}t}$ *provided $\alpha\gamma < \kappa(1+\gamma)b$.*

Proof. Fix $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Let C be the constant in Lemma 2.14. Using the triangle inequality, for all $\kappa \in \mathbb{Z}_+$, $g \in \mathcal{P}$ with $Q_\kappa g \le h$ and $t \ge 0$, we have

$$
\begin{aligned}
&\|\langle g, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} \\
&\le \sum_{l=0}^{\lfloor t\rfloor-1} \big\|\mathcal{I}_{t-l-1}^{t-l}\langle g, X_t\rangle\big\|_{\mathbb{P}_\mu;1+\gamma} + \big\|\mathcal{I}_0^{t-\lfloor t\rfloor}\langle g, X_t\rangle\big\|_{\mathbb{P}_\mu;1+\gamma} + |\langle P_t^\alpha g, \mu\rangle| \\
&\le C^{\frac{1}{1+\gamma}} e^{\frac{\alpha}{1+\gamma}t}\sum_{l=0}^{\lfloor t\rfloor} e^{\frac{\gamma\alpha-\kappa(1+\gamma)b}{1+\gamma}l} + e^{(\alpha-\kappa b)t}\langle h, \mu\rangle.
\end{aligned}
$$

By calculating the sum on the right, we get the desired result. $\square$

3. Proofs of main results

In this section, we will prove the main results of this paper. Recall that $\mathcal{M}_c(\mathbb{R}^d)$ is the space of all finite Borel measures of compact support on $\mathbb{R}^d$. For simplicity, we will write $\widetilde{\mathbb{P}}_\mu = \mathbb{P}_\mu(\cdot|D^c)$ in this section.

3.1. **The large branching rate regime: law of large numbers.** In this subsection, we prove Theorem 1.2. We will return to the large branching rate regime in Subsection 3.4 to prove Theorem 1.5.

To prove Theorem 1.2, we first prove the almost sure and $L^{1+\gamma}(\mathbb{P}_\mu)$ convergence of a family of martingales for $\gamma \in (0, \beta)$. Recall that L is the infinitesimal generator of the OU-process. For $f \in \mathcal{P} \cap C^2(\mathbb{R}^d)$ and $a \in \mathbb{R}$, we define

$$M_t^{f,a} := e^{-(\alpha-ab)t}\langle f, X_t\rangle - \int_0^t e^{-(\alpha-ab)s}\langle (L+ab)f, X_s\rangle \, ds. \tag{3.1}$$

Let $(\mathscr{F}_t)_{t\geq 0}$ be the natural filtration of X. The following lemma says that $\{M_t^{f,a} : t \geq 0\}$ is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.

Lemma 3.1. *For all $f \in \mathcal{P} \cap C^2(\mathbb{R}^d)$, $a \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, the process $(M_t^{f,a})_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

Proof. Put$\bar{f} := (L + ab)f$. It follows easily from Ito's formula that

$$P_t^{ab} f(x) = f(x) + \int_0^t P_s^{ab}\bar{f}(x)\, ds, \quad t \geq 0, x \in \mathbb{R}^d, \tag{3.2}$$

where $P_t^{ab} := e^{abt}P_t$. For $0 \leq s \leq t$, we have

$$\begin{aligned} &\mathbb{P}_\mu[M_t^{f,a}|\mathscr{F}_s] \\ &= e^{-(\alpha-ab)t}\mathbb{P}_\mu\left[\langle f, X_t\rangle|\mathscr{F}_s\right] - \mathbb{P}_\mu\Big[\int_0^t e^{-(\alpha-ab)u}\langle \bar{f}, X_u\rangle\, du\Big|\mathscr{F}_s\Big] \\ &= e^{-(\alpha-ab)t}\langle P_{t-s}^\alpha f, X_s\rangle - \int_0^s e^{-(\alpha-ab)u}\langle \bar{f}, X_u\rangle\, du \\ &\quad - \int_s^t e^{-(\alpha-ab)u}\langle P_{u-s}^\alpha \bar{f}, X_s\rangle\, du. \end{aligned} \tag{3.3}$$

Using (3.2) and Fubini's theorem, we have

$$\begin{aligned} &\int_s^t e^{-(\alpha-ab)u}\langle P_{u-s}^\alpha \bar{f}, X_s\rangle\, du = e^{-(\alpha-ab)s}\int_s^t \langle P_{u-s}^{ab}\bar{f}, X_s\rangle\, du \\ &= e^{-(\alpha-ab)s}\Big\langle \int_0^{t-s} P_u^{ab}\bar{f}\, du, X_s\Big\rangle = e^{-(\alpha-ab)s}\left(\langle P_{t-s}^{ab} f, X_s\rangle - \langle f, X_s\rangle\right) \\ &= e^{-(\alpha-ab)t}\langle P_{t-s}^\alpha f, X_s\rangle - e^{-(\alpha-ab)s}\langle f, X_s\rangle. \end{aligned}$$

Using this and (3.3), we get the desired result. □

Recall that, for $p = (p_1, ..., p_d) \in \mathbb{Z}_+^d$, ϕ_p is an eigenfunctions of L corresponding to the eigenvalue $-|p|b$. Define

$$H_t^p := e^{-(\alpha-|p|b)t}\langle \phi_p, X_t\rangle, \quad t \geq 0.$$

Lemma 3.2. *For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t\ge 0}$ is a $\mathbb{P}_\mu$-martingale with respect to $(\mathscr{F}_t)$. Moreover, if $\alpha\beta > |p|b(1+\beta)$, then, for all $\gamma \in (0,\beta)$, we have $\sup_{t\ge 0}\|H_t^p\|_{\mathbb{P}_\mu;1+\gamma} < \infty$ and*

$$H_\infty^p := \lim_{t\to\infty} H_t^p$$

exists $\mathbb{P}_\mu$-a.s and in $L^{1+\gamma}(\mathbb{P}_\mu)$.

Proof. It follows from Lemma 3.1 that $(H_t^p)_{t\ge 0}$ is a $\mathbb{P}_\mu$-martingale for any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. There exists $\gamma_0 \in (0,\beta)$ close enough to β such that for $\gamma \in [\gamma_0,\beta)$, we have $\alpha\gamma > |p|(1+\gamma)b$. Using Lemma 2.15 and the fact $\kappa_{\phi_p} = |p|$, we get that, for $\gamma \in [\gamma_0,\beta)$, there exists a constant $C_{\gamma,\mu,p} > 0$ (depending on γ, μ and p) such that

$$\|H_t^p\|_{\mathbb{P}_\mu;1+\gamma} \le C_{\gamma,\mu,p} e^{-(\alpha-|p|b)t} e^{(\alpha-|p|b)t} = C_{\gamma,\mu,p}, \quad t \ge 0.$$

For all $\gamma \in (0,\gamma_0)$ and any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, we have

$$\|H_t^p\|_{\mathbb{P}_\mu;1+\gamma} \le \|H_t^p\|_{\mathbb{P}_\mu;1+\gamma_0} < C_{\gamma_0,\mu,p}, \quad t \ge 0.$$

Hence, for all $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, the martingale is bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$ and hence converges in $L^{1+\gamma}(\mathbb{P}_\mu)$ and almost surely, by [10, Theorem 5.4.5]. □

In particular, when $p = 0$, H_t^0 reduces to $H_t := e^{-\alpha t}\|X_t\|$, thus, as $t \to \infty$, H_t converges to H_∞, $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$ for any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$.

Lemma 3.3. *Let $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. For all $\gamma \in (0,\beta)$ and $p \in \mathbb{Z}_+^d$ with $\alpha\gamma > |p|b(1+\gamma)$, there exists $C > 0$ such that for all $0 \le s < t$,*

$$\|H_t^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma} \le C e^{-\frac{1}{1+\gamma}(\alpha\gamma - |p|b(1+\gamma))s}.$$

Moreover, we have

$$\|H_\infty^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma} \le C e^{-\frac{1}{1+\gamma}(\alpha\gamma - |p|b(1+\gamma))s}, \quad s \ge 0.$$

Proof. The second assertion follows immediately from the first, so we only prove the first assertion. Fix $\gamma \in (0,\beta)$, $p \in \mathbb{Z}_+^d$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. It follows from Lemma 2.14 with $g = \phi_p$ and $k = |p|$ that there exists a constant $C_1 > 0$ such that for all $0 \le r \le s$ with $s - r \le 1$, we have

$$\mathbb{P}_\mu\Big[\big|e^{(\alpha-|p|b)(t-s)}\langle \phi_p, X_s\rangle - e^{(\alpha-|p|b)(t-r)}\langle \phi_p, X_r\rangle\big|^{1+\gamma}\Big] \le C_1 e^{\alpha t + (t-s)(\alpha\gamma - (1+\gamma)|p|b)}.$$

Dividing both sides by $e^{(\alpha-|p|b)t(1+\gamma)}$, we get

$$\mathbb{P}_\mu\big[|H_s^p - H_r^p|^{1+\gamma}\big] \le C_1 e^{-(\alpha\gamma-(1+\gamma)|p|b)s}.$$

Thus there exists $C_2 > 0$ such that for any $0 \le s < t$,

$$\begin{aligned}
&\|H_t^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma} \\
&\le \|H_{\lfloor s\rfloor+1}^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma} + \sum_{k=\lfloor s\rfloor+1}^{\lfloor t\rfloor} \|H_{k+1}^p - H_k^p\|_{\mathbb{P}_\mu;1+\gamma} + \|H_t^p - H_{\lfloor t\rfloor+1}^p\|_{\mathbb{P}_\mu;1+\gamma}
\end{aligned}$$

$$\le C_1^{\frac{1}{1+\gamma}}\Big(e^{-\frac{(\alpha\gamma-(1+\gamma)|p|b)s}{1+\gamma}}+\sum_{k=\lfloor s\rfloor+1}^{\lfloor t\rfloor}e^{-\frac{(\alpha\gamma-(1+\gamma)|p|b)k}{1+\gamma}}+e^{-\frac{(\alpha\gamma-(1+\gamma)|p|bt}{1+\gamma}}\Big)$$
$$\le C_2e^{-\frac{(\alpha\gamma-(1+\gamma)|p|b)}{1+\gamma}s}. \qquad \square$$

Proof of Theorem 1.2. Fix $f\in\mathcal{P}$ such that $\alpha\beta>\kappa_f b(1+\beta)$ and $\mu\in\mathcal{M}_c(\mathbb{R}^d)$. Write

$$f=\sum_{p\in\mathbb{Z}_+^d:|p|\ge\kappa_f}\langle f,\phi_p\rangle_\varphi\phi_p=:\sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f}\langle f,\phi_p\rangle_\varphi\phi_p+\widetilde{f}.$$

Then

$$e^{-(\alpha-\kappa_f b)t}\langle f,X_t\rangle=\sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f}\langle f,\phi_p\rangle_\varphi H_t^p+e^{-(\alpha-\kappa_f b)t}\langle\widetilde{f},X_t\rangle,\quad t\ge 0.$$

According to Lemma 3.2, we have

$$\sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f}\langle f,\phi_p\rangle_\varphi H_t^p\xrightarrow[t\to\infty]{}\sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f}\langle f,\phi_p\rangle_\varphi H_\infty^p,$$

$\mathbb{P}_\mu$-a.s. and in $L^{1+\gamma}(\mathbb{P}_\mu)$ for any $\gamma\in(0,\beta)$. Therefore, it suffices to show that

$$J_t:=e^{-(\alpha-\kappa_f b)t}\langle\widetilde{f},X_t\rangle,\quad t\ge 0,$$

converges in $L^{1+\gamma}(\mathbb{P}_\mu)$ for all $\gamma\in(0,\beta)$, and converges a.s. provided $f\in C^2(\mathbb{R}^d)$ satisfies $D^2f\in\mathcal{P}$.

Step 1. Let $g\in\mathcal{P}$. Let $\kappa>0$ be such that $\kappa<\kappa_g$ and $\kappa<\frac{\alpha\beta}{b(1+\beta)}$. We will show that for each $\gamma\in(0,\beta)$ there exist constants $C_1,\delta_1>0$ such that

$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le C_1e^{-\delta_1 t},\quad t\ge 0.$$

In order to do this, we choose a $\gamma_0\in(0,\beta)$ close enough to β such that, for each $\gamma\in[\gamma_0,\beta)$, we have $\kappa<\frac{\alpha\gamma}{b(1+\gamma)}$. Then, according to Lemma 2.15, we have, for each $\gamma\in(0,\beta)$,

(1) if $\gamma\in[\gamma_0,\beta)$ and $\alpha\gamma>\kappa_g(1+\gamma)b$, then there exists $C_2>0$ such that
$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le C_2e^{-(\alpha-\kappa b)t}e^{(\alpha-\kappa_g b)t}\le C_2e^{-(\kappa_g-\kappa)bt},\quad t\ge 0;$$

(2) if $\gamma\in[\gamma_0,\beta)$ and $\alpha\gamma=\kappa_g(1+\gamma)b$, then there exists $C_3>0$ such that
$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le C_3te^{-(\alpha-\kappa b)t}e^{\frac{\alpha}{1+\gamma}t}=C_3te^{-(\frac{\alpha\gamma}{1+\gamma}-\kappa b)t},\quad t\ge 0;$$

(3) if $\gamma\in[\gamma_0,\beta)$ and $\alpha\gamma<\kappa_g(1+\gamma)b$, then there exists $C_4>0$ such that
$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le C_4e^{-(\alpha-\kappa b)t}e^{\frac{\alpha}{1+\gamma}t}=C_4e^{-(\frac{\alpha\gamma}{1+\gamma}-\kappa b)t},\quad t\ge 0;$$

(4) if $\gamma\in(0,\gamma_0)$ then, thanks to (1)–(3) above and the fact that
$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma_0},$$
there exist $C_5,\delta_2>0$ such that
$$\|e^{-(\alpha-\kappa b)t}\langle g,X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\le C_5e^{-\delta_2 t},\quad t\ge 0.$$

Thus, the desired conclusion in this step is valid. In particular, by taking $g = \widetilde{f}$ and $\kappa = \kappa_f$, we get that J_t converges to 0 in $L^{1+\gamma}(\mathbb{P}_\mu)$ for any $\gamma \in (0, \beta)$.

Step 2. We further assume that $f \in C^2(\mathbb{R}^d)$ and $D^2 f \in \mathcal{P}$. We will show that J_t converges to 0 almost surely. For $a \geq 0$, $t \geq 0$, and $g \in \mathcal{P} \cap C^2(\mathbb{R}^d)$ satisfying $D^2 g \in \mathcal{P}$, we define

$$L_t^{g,a} := \int_0^t e^{-(\alpha - ab)s} \langle (L + ab)g, X_s \rangle ds$$

and

$$Y_t^{g,a} := \int_0^t e^{-(\alpha - ab)s} |\langle (L + ab)g, X_s \rangle| ds.$$

Now choose $a_0 \in (\kappa_f, \kappa_f + 1)$ close enough to κ_f so that $a_0 < \frac{\alpha\beta}{b(1+\beta)}$. According to (3.1), we have

$$J_t = e^{-(a_0 - \kappa_f)bt} (M_t^{\widetilde{f}, a_0} + L_t^{\widetilde{f}, a_0}), \quad t \geq 0.$$

So we only need to show that

$$e^{-(a_0 - \kappa_f)bt} M_t^{\widetilde{f}, a_0} \xrightarrow[t \to \infty]{} 0, \quad e^{-(a_0 - \kappa_f)bt} L_t^{\widetilde{f}, a_0} \xrightarrow[t \to \infty]{} 0 \quad \mathbb{P}_\mu\text{-a.s.}$$

Notice that $\kappa_{(L + a_0 b)\widetilde{f}} \geq \kappa_{\widetilde{f}} \geq \kappa_f + 1 > a_0$. By Step 1, for any fixed $\gamma \in (0, \beta)$, there exist $C_6, \delta_3 > 0$ such that for each $t \geq 0$,

$$\|e^{-(\alpha - a_0 b)t} \langle \widetilde{f}, X_t \rangle)\|_{\mathbb{P}_\mu; 1+\gamma} \leq C_6 e^{-\delta_3 t}, \quad \|e^{-(\alpha - a_0 b)t} \langle L\widetilde{f} + a_0 b \widetilde{f}, X_t \rangle\|_{\mathbb{P}_\mu; 1+\gamma} \leq C_6 e^{-\delta_3 t}.$$

Now, by the triangle inequality, for each $t \geq 0$,

$$\begin{aligned}
&\|L_t^{\widetilde{f}, a_0}\|_{\mathbb{P}_\mu; 1+\gamma} \leq \|Y_t^{\widetilde{f}, a_0}\|_{\mathbb{P}_\mu; 1+\gamma} \\
&\leq \int_0^t \|e^{-(\alpha - a_0 b)s} \langle L\widetilde{f} + a_0 b \widetilde{f}, X_s \rangle\|_{\mathbb{P}_\mu; 1+\gamma} ds \leq C_6 \int_0^t e^{-\delta_3 s} ds \leq \frac{C_6}{\delta_3}.
\end{aligned}$$

Since $Y_t^{\widetilde{f}, a_0}$ is increasing in t, it converges to some finite random variable $Y_\infty^{\widetilde{f}, a_0}$ almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$. Consequently, we have

$$\lim_{t \to \infty} e^{-(a_0 - \kappa_f)bt} |L_t^{\widetilde{f}, a_0}| \leq \lim_{t \to \infty} e^{-(a_0 - \kappa_f)bt} |Y_t^{\widetilde{f}, a_0}| = 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

On the other hand, the martingale $M_t^{\widetilde{f}, a_0}$ satisfies

$$\|M_t^{\widetilde{f}, a_0}\|_{\mathbb{P}_\mu; 1+\gamma} \leq \|e^{-(\alpha - a_0 b)t} \langle \widetilde{f}, X_t \rangle)\|_{\mathbb{P}_\mu; 1+\gamma} + \|L_t^{\widetilde{f}, a_0}\|_{\mathbb{P}_\mu; 1+\gamma} \leq C_6 (e^{-\delta_3 t} + \frac{1}{\delta_3}), \quad t \geq 0.$$

This implies that the martingale converges almost surely. Consequently,

$$\lim_{t \to \infty} e^{-(a_0 - \kappa_f)bt} M_t^{\widetilde{f}, a_0} = 0, \quad \mathbb{P}_\mu\text{-a.s.}. \qquad \square$$

3.2. **The critical branching rate regime.** Recall that $\mathcal{I}_s^t$ is defined in (1.13).

Lemma 3.4. *Let $f \in \mathcal{P}$ be non-trivial and satisfy $\alpha\beta \le \kappa_f b(1+\beta)$. Then for all $k \ge 0$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, under $\mathbb{P}_\mu(\cdot|D^c)$, we have*

$$\gamma_{t,k} := \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}} \xrightarrow{d} \zeta_k, \quad t \to \infty,$$

where ζ_k is a $(1+\beta)$-stable random variable with characteristic function

$$\mathbf{E}(e^{i\theta\zeta_k}) = \exp(m_k[\theta f]), \quad \theta \in \mathbb{R}.$$

Proof. We only need to show that for all $\mu \in \mathcal{M}_c(\mathbb{R}^d), f \in \mathcal{P}, k \ge 0$,

$$\mathbb{P}_\mu[\exp(i\gamma_{t,k}); D^c] \xrightarrow[t\to\infty]{} \mathbb{P}_\mu(D^c)\exp(m_k[f]).$$

In fact, once we prove this, we can replace f by θf, with $\theta \in \mathbb{R}$ being arbitrary, to get the desired result. In the rest of the proof we fix a $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and an $f \in \mathcal{P}$. Define

$$A_t(\epsilon) := \{\|X_t\| \ge e^{(\alpha-\epsilon)t}\}, \quad t \ge 0, \epsilon > 0.$$

Step 1. We will show that for all $\epsilon > 0, k \ge 0$ and $t > k+1$, we have

$$\big|\mathbb{P}_\mu\big[e^{i\gamma_{t,k}} - e^{m_k[f]}; D^c\big]\big| \le J_1(t,k,\epsilon) + J_2(t,k,\epsilon) + J_3(t,k,\epsilon),$$

where

$$\begin{aligned} J_1(t,k,\epsilon) &:= \mathbb{P}_\mu\big[|\langle Z_1'''(\theta_{t,k}P_k^\alpha f), X_{t-k-1}\rangle|; A_{t-k-1}(\epsilon)\big], \\ J_2(t,k,\epsilon) &:= \mathbb{P}_\mu\big[|\langle Z_1(\theta_{t,k}P_k^\alpha f), X_{t-k-1}\rangle - m_k[f]|; A_{t-k-1}(\epsilon)\big], \\ J_3(t,k,\epsilon) &:= 2\mathbb{P}_\mu(A_{t-k-1}(\epsilon)\Delta D^c), \\ \theta_{t,k} &:= (e^{\alpha(k+1)}\|X_{t-k-1}\|)^{-\frac{1}{1+\beta}}. \end{aligned} \tag{3.4}$$

In fact, it follows from (2.6), the definitions of U_1, Z_1''' and Z_1, that for all $k \ge 0, t \ge k+1$,

$$\begin{aligned} \mathbb{P}_\mu[e^{i\gamma_{t,k}}|\mathscr{F}_{t-k-1}] &= \mathbb{P}_\mu[e^{i\theta_{t,k}\langle P_k^\alpha f, X_{t-k}\rangle - i\theta_{t,k}\langle P_{k+1}^\alpha f, X_{t-k-1}\rangle}|\mathscr{F}_{t-k-1}] \\ &= e^{\langle (U_1 - iP_1^\alpha)(\theta_{t,k}P_k^\alpha f), X_{t-k-1}\rangle} = e^{\langle (Z_1+Z_1''')(\theta_{t,k}P_k^\alpha f), X_{t-k-1}\rangle}. \end{aligned} \tag{3.5}$$

From Proposition 2.7.(1), we have

$$\operatorname{Re} m_t[f] < 0. \tag{3.6}$$

Using (3.5), (A.8), (3.6) and the fact that

$$|e^{-x} - e^{-y}| \le |x-y|, \quad x, y \in \mathbb{C}_+,$$

we get that for all $k \ge 0$, $t \ge k+1$ and $\epsilon > 0$,

$$\begin{aligned} &\big|\mathbb{P}_\mu\big[e^{i\gamma_{t,k}} - e^{m_k[f]}; D^c\big]\big| \\ &\le \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\gamma_{t,k}} - e^{m_k[f]}; D^c|\mathscr{F}_{t-k-1}]\big|\Big] \\ &\le \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\gamma_{t,k}} - e^{m_k[f]}; A_{t-k-1}(\epsilon)|\mathscr{F}_{t-k-1}]\big| + 2\mathbb{P}_\mu(A_{t-k-1}(\epsilon)\Delta D^c|\mathscr{F}_{t-k-1})\Big] \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&= \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\gamma_{t,k}}|\mathscr{F}_{t-k-1}] - e^{m_k[f]}\big|; A_{t-k-1}(\epsilon)\Big] + J_3(t,k,\epsilon)\\
&\leq \mathbb{P}_\mu\big[|e^{\langle (Z_1+Z_1''')(\theta_{t,k}P^\alpha_k f), X_{t-k-1}\rangle} - e^{m_k[f]}|; A_{t-k-1}(\epsilon)\big] + J_3(t,k,\epsilon)\\
&\leq \mathbb{P}_\mu\big[|\langle (Z_1+Z_1''')(\theta_{t,k}P^\alpha_k f), X_{t-k-1}\rangle - m_k[f]|; A_{t-k-1}(\epsilon)\big] + J_3(t,k,\epsilon)\\
&\leq J_1(t,k,\epsilon)+J_2(t,k,\epsilon)+J_3(t,k,\epsilon).
\end{aligned}
$$

Step 2. We will show that for each $\epsilon>0$ small enough, there exist constants $C_1,\delta_1>0$ such that

$$J_1(t,k,\epsilon)\leq C_1 e^{-\delta_1(t-k)},\quad k\geq 0, t\geq k+1.$$

In fact, let $\delta_0>0$ be the constant in Lemma 2.5.(7) and let R be the corresponding $(\theta^{2+\beta}\vee\theta^{1+\beta+\delta_0})$-controller. Then, we have for all $k\geq 0$, $t\geq k+1$ and $\epsilon>0$,

$$
\begin{aligned}
&|Z_1'''(\theta_{t,k}P^\alpha_k f)|\mathbf{1}_{A_{t-k-1}(\epsilon)} \leq R(|\theta_{t,k}P^\alpha_k f|)\mathbf{1}_{A_{t-k-1}(\epsilon)}\\
&\leq R\Big(\frac{e^{(\alpha-\kappa_f b)k}Qf}{\big(e^{\alpha(k+1)}e^{(\alpha-\epsilon)(t-k-1)}\big)^{\frac{1}{1+\beta}}}\Big)\\
&\leq \sum_{\rho\in\{\delta_0,1\}}\Big(\frac{e^{(\alpha-\kappa_f b)k}}{\big(e^{\alpha(k+1)}e^{(\alpha-\epsilon)(t-k-1)}\big)^{\frac{1}{1+\beta}}}\Big)^{1+\beta+\rho}RQf\\
&\leq \sum_{\rho\in\{\delta_0,1\}} e^{\frac{1+\beta+\rho}{1+\beta}(\alpha\beta-\kappa_f b(1+\beta))k}e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t-k-1)}RQf\\
&\leq \sum_{\rho\in\{\delta_0,1\}} e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t-k-1)}RQf,
\end{aligned}
$$

where Q is defined by (2.8). Thus for all $k\geq 0$, $t\geq k+1$ and $\epsilon>0$,

$$
\begin{aligned}
J_1(t,k,\epsilon) &\leq \sum_{\rho\in\{\delta_0,1\}} e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t-k-1)}\mathbb{P}_\mu[\langle RQf, X_{t-k-1}\rangle]\\
&\leq \sum_{\rho\in\{\delta_0,1\}}\langle Q_0 RQf,\mu\rangle e^{-(\alpha\frac{\rho}{1+\beta}-\epsilon\frac{1+\beta+\rho}{1+\beta})(t-k-1)},
\end{aligned}
\tag{3.8}
$$

where Q is defined by (2.8). By taking $\epsilon>0$ small enough, we get the desired result in this step.

Step 3. We will show that for $\epsilon>0$ small enough, there exist constants $C_2,\delta_2>0$ such that

$$J_2(t,k,\epsilon)\leq C_2 e^{-\delta_2(t-k)},\quad k\geq 0, t\geq k+1.$$

We first claim that for each $k\geq 0$ and $g\in\mathcal{P}$, $m_k[g]=e^{-\alpha(k+1)}\langle Z_1 P^\alpha_k g,\varphi\rangle$. In fact, by Fubini's theorem and the definitions of Z_1 and m_k, we have for all $k\geq 0$ and $g\in\mathcal{P}$.

$$m_k[g]=\eta\int_k^{k+1} e^{-\alpha s}\,ds\int_{\mathbb{R}^d}(-iP^\alpha_s g(x))^{1+\beta}\varphi(x)\,dx$$

$$
\begin{aligned}
&= \eta \int_0^1 e^{-\alpha(k+s)}\ ds \int_{\mathbb{R}^d} (-iP^\alpha_{s+k} g)^{1+\beta} \varphi(x)\ dx \\
&= e^{-\alpha(k+1)} \cdot \eta \int_0^1 e^{\alpha(1-s)}\ ds \int_{\mathbb{R}^d} (-iP^\alpha_{s+k} g(x))^{1+\beta} \varphi(x)\ dx \\
&= e^{-\alpha(k+1)} \langle \eta \int_0^1 P^\alpha_{1-s} (-iP^\alpha_{s+k} g(x))^{1+\beta}\ ds, \varphi\rangle \\
&= e^{-\alpha(k+1)} \langle Z_1(P^\alpha_k g), \varphi\rangle.
\end{aligned}
$$

Therefore, for all $k \geq 0$, $t \geq k+1$ and $\epsilon > 0$,

$$
\begin{aligned}
&\langle Z_1(\theta_{t,k} P^\alpha_k f), X_{t-k-1}\rangle - m_k[f] \\
&= \theta_{t,k}^{1+\beta} \langle Z_1 P^\alpha_k f, X_{t-k-1}\rangle - e^{-\alpha(k+1)} \langle Z_1 P^\alpha_k f, \varphi\rangle \\
&= e^{-\alpha(k+1)} \Big(\frac{\langle Z_1 P^\alpha_k f, X_{t-k-1}\rangle}{\|X_{t-k-1}\|} - \langle Z_1 P^\alpha_k f, \varphi\rangle \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
J_2(t,k,\epsilon) &= \mathbb{P}_\mu\big[|\langle Z_1(\theta_{t,k} P^\alpha_k f), X_{t-k-1}\rangle - m_k[f]|; A_{t-k-1}(\epsilon)\big] \\
&= e^{-\alpha(k+1)} \mathbb{P}_\mu\Big[\Big| \frac{\langle Z_1 P^\alpha_k f, X_{t-k-1}\rangle}{\|X_{t-k-1}\|} - \langle Z_1 P^\alpha_k f, \varphi\rangle \Big|; A_{t-k-1}(\epsilon)\Big] \\
&\leq e^{-\alpha(k+1)} e^{-(\alpha-\epsilon)(t-k-1)} e^{(\alpha-\kappa_f b)(1+\beta)k} \mathbb{P}_\mu\left[|\langle g_k, X_{t-k-1}\rangle| \right],
\end{aligned}
$$

where

$$
g_k = \frac{Z_1 P^\alpha_k f - \langle Z_1 P^\alpha_k f, \varphi\rangle}{e^{(\alpha-\kappa_f b)(1+\beta)k}}, \quad k \geq 0.
$$

It follows from Lemma 2.9 that there exists $h \in \mathcal{P}$ such that

$$
Q_1(\operatorname{Re} g_k) \leq h \text{ and } Q_1(\operatorname{Im} g_k) \leq h, \quad k \geq 0,
$$

where Q_1 is defined by (2.7) with $\kappa = 1$.

Chose a $\gamma \in (0,\beta)$ small enough such that $\alpha\gamma < b < (1+\gamma)b$. According to Lemma 2.15.(3) (with $\kappa = 1$), there exists $C_3 > 0$ such that for all $t \geq 0$ and $k \geq 0$,

$$
\begin{aligned}
&\mathbb{P}_\mu\left[|\langle g_k, X_t\rangle| \right] \leq \|\langle \operatorname{Re} g_k, X_t\rangle\|_{\mathbb{P}_\mu,1+\gamma} + \|\langle \operatorname{Im} g_k, X_t\rangle\|_{\mathbb{P}_\mu,1+\gamma} \\
&\leq 2 \sup_{g\in\mathcal{P}: Q_1 g \leq h} \|\langle g, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} \leq C_3 e^{\frac{\alpha t}{1+\gamma}}.
\end{aligned}
$$

Therefore, for all $k \geq 0$, $t \geq k+1$ and $\epsilon > 0$, we have

$$
\begin{aligned}
J_2(t,k,\epsilon) &\leq e^{-\alpha(k+1)} e^{-(\alpha-\epsilon)(t-k-1)} e^{(\alpha-\kappa_f b)(1+\beta)k} \mathbb{P}_\mu\left[|\langle g_k, X_{t-k-1}\rangle| \right] \\
&\leq C_3 e^{-\alpha k} e^{-(\alpha-\epsilon)(t-k-1)} e^{(\alpha-\kappa_f b)(1+\beta)k} e^{\frac{\alpha}{1+\gamma}(t-k-1)} \\
&= C_3 e^{(\alpha\beta-\kappa_f b(1+\beta))k} e^{-(\frac{\alpha\gamma}{1+\gamma}-\epsilon)(t-k-1)} \\
&\leq C_3 e^{-(\frac{\alpha\gamma}{1+\gamma}-\epsilon)(t-k-1)}.
\end{aligned}
\tag{3.9}
$$

By taking $\epsilon > 0$ small enough, we get the required result in this step.

Step 4. We will show that, for each $\epsilon \in (0,\alpha)$, there exist $C_4, \delta_3 > 0$ such that for all $k \geq 0, t \geq k+1$,

$$J_3(t,k,\epsilon) \leq C_4 e^{-\delta_3(t-k)}.$$

In fact, we have for all $t \geq 0, \epsilon > 0$,

$$\begin{aligned}\mathbb{P}_\mu(A_t(\epsilon), D) &= \mathbb{P}_\mu[\mathbb{P}_\mu(D|\mathscr{F}_t); A_t(\epsilon)] \\ &= \mathbb{P}_\mu[e^{-\bar{v}\|X_t\|}; A_t(\epsilon)] \leq \exp(-\bar{v}\|\mu\| e^{(\alpha-\epsilon)t}).\end{aligned}$$

According to Proposition 2.11, for each $\epsilon \in (0,\alpha)$, there exists $C_5, \delta_4 > 0$ such that for all $t \geq 0$,

$$\mathbb{P}_\mu(A_t(\epsilon)^c, D^c) \leq \mathbb{P}_\mu(0 < e^{-\alpha t}\|X_t\| \leq e^{-\epsilon t}) \leq C_5(e^{-\epsilon\delta_4 t} + e^{-\delta_4 t}).$$

Combining these results, we get the desired result in step 4.

Finally, combining the results in Steps 1–4, noticing that, if $\epsilon > 0$ is chosen small enough then $J_i(t,k,\epsilon), i = 1,2,3$ converge to 0 exponentially fast as $t \to \infty$, we immediately get the desired result. $\square$

Corollary 3.5. *Suppose that $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta \leq \kappa_f b(1+\beta)$. Then for all $\Theta > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist $C > 0$ and $\delta > 0$ such that for all $k \geq 0, t \geq k+1$ and $|\theta| \leq \Theta$,*

$$\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\theta\gamma_{t,k}} - e^{m_k[\theta f]}; D^c|\mathscr{F}_{t-k-1}]\big|\Big] \leq Ce^{-\delta(t-k)}.$$

Proof. For $f \in \mathcal{P}$, define $J_1^f(t,k,\epsilon), J_2^f(t,k,\epsilon)$ and $J_3(t,k,\epsilon)$ as the J_1, J_2 and J_3 in (3.4). Now, fix a $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and an $f \in \mathcal{P}$. According to (3.7), we have for all $\theta \in \mathbb{R}$, $k \geq 0$, $t \geq k+1$ and $\epsilon > 0$,

$$\begin{aligned}&\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\theta\gamma_{t,k}} - e^{m_k[\theta f]}; D^c|\mathscr{F}_{t-k-1}]\big|\Big] \\ &\leq J_1^{\theta f}(t,k,\epsilon) + J_2^{\theta f}(t,k,\epsilon) + J_3(t,k,\epsilon).\end{aligned}$$

Let $\delta_0 > 0$ be the constant in Lemma 2.5.(7) and let R be the corresponding $(\theta^{2+\beta} \vee \theta^{1+\beta+\delta_0})$-controller. According to (3.8), we have for all $\theta \in \mathbb{R}$, $k \geq 0$, $t \geq k+1$ and $\epsilon > 0$,

$$\begin{aligned}J_1^{\theta f}(k,t,\epsilon) &\leq \sum_{\rho\in\{\delta_0,1\}} \langle Q_0 R Q(\theta f), \mu\rangle e^{-(\alpha\frac{\rho}{1+\beta} - \epsilon\frac{1+\beta+\rho}{1+\beta})(t-k-1)} \\ &\leq (|\theta|^{2+\beta} \vee |\theta|^{1+\beta+\delta_0}) \sum_{\rho\in\{\delta_0,1\}} \langle Q_0 R Q f, \mu\rangle e^{-(\alpha\frac{\rho}{1+\beta} - \epsilon\frac{1+\beta+\rho}{1+\beta})(t-k-1)}.\end{aligned}$$

From the definitions of Z_1 and m_t we get that for all $g \in \mathcal{P}, \theta \geq 0, t \geq 0$,

$$Z_1(\pm\theta g) = \theta^{1+\beta} Z_1(\pm g), \quad m_t[\pm\theta g] = \theta^{1+\beta} m_t[\pm g].$$

Therefore, we have for all $\theta > 0, k \geq 0, t \geq k+1, \epsilon > 0$,

$$J_2^{\pm\theta f}(t,k,\epsilon) := \theta^{1+\beta} J_2^{\pm f}(t,k,\epsilon).$$

According to this and (3.9), we have that there exists a constant $C > 0$ such that for all $\theta \in \mathbb{R}$, $k \geq 0$, $t \geq k+1$ and $\epsilon > 0$,

$$J_2^{\theta f}(t,k,\epsilon) \leq C|\theta|^{1+\beta} \exp\Big(-(\frac{\alpha\gamma}{1+\gamma} - \epsilon)(t-k-1)\Big).$$

Finally, noticing that $|\theta| < \Theta$, using the estimates of $J_i^{\theta f}, i = 1, 2$ and the estimate of J_3 in Step 4 of the proof of the previous lemma, we get the desired result by choosing ϵ small enough. $\square$

Proposition 3.6. *Suppose that $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta \leq \kappa_f b(1+\beta)$. Then for all $\Theta > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist constants $C, \delta > 0$ such that for all $t \geq 0$, $n \in \{0, \cdots, \lfloor t \rfloor\}$ and $(\theta_0, \cdots, \theta_n) \in \mathbb{R}^{n+1}$satisfying $|\theta_i| \leq \Theta$, we have*

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\prod_{k=0}^{n} \exp\Big(i\theta_k \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}}\Big) - \prod_{k=0}^{n} \exp(m_k[\theta_k f])\Big]\Big| \leq Ce^{-\delta(t-n)}. \tag{3.10}$$

Proof. Recall that

$$\gamma_{t,k} := \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}}, \quad k \geq 0, t \geq k+1.$$

Fix $t \geq 0$, $n \in \{0, \cdots, \lfloor t \rfloor\}$ and $(\theta_0, \cdots, \theta_n) \in \mathbb{R}^{n+1}$satisfying $|\theta_i| \leq \Theta$. For each $k \in \{-1, ..., n\}$, we define

$$a_k := \Big(\prod_{l=0}^{k} \exp(m_l[\theta_l f])\Big) \times \widetilde{\mathbb{P}}_\mu\Big(\prod_{l=k+1}^{n} \exp\left(i\theta_l\gamma_{t,l}\right)\Big),$$

where by convention the first product is 1 for $k = -1$. Then we get for each $k > -1$,

$$\begin{aligned}
&a_{k-1} - a_k \\
&= \mathbb{P}_\mu(D^c)^{-1}\Big(\prod_{l=0}^{k-1} e^{m_l[\theta_l f]}\Big) \times \mathbb{P}_\mu\Big[(e^{i\theta_k\gamma_{t,k}} - e^{m_k[\theta_k f]}) \prod_{l=k+1}^{n} e^{i\theta_l\gamma_{t,l}}; D^c\Big] \\
&= \mathbb{P}_\mu(D^c)^{-1}\Big(\prod_{l=0}^{k-1} e^{m_l[\theta_l f]}\Big) \times \mathbb{P}_\mu\Big[\mathbb{P}_\mu[e^{i\theta_k\gamma_{t,k}} - e^{m_k[\theta_k f]}; D^c|\mathscr{F}_{t-k-1}] \prod_{l=k+1}^{n} e^{i\theta_l\gamma_{t,l}}\Big].
\end{aligned}$$

According to Corollary 3.5, there exist $C, \delta > 0$ such that for any $k \in \{0, 1, \cdots, n\}$, we have

$$\begin{aligned}
|a_{k-1} - a_k| &\leq \frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\theta_k\gamma_{t,k}} - e^{m_k[\theta_k f]}; D^c\big|\mathscr{F}_{t-k-1}]\big|\Big] \\
&\leq Ce^{-\delta(t-k)}.
\end{aligned}$$

Therefore,

$$\text{LHS of (3.10)} = |a_{-1} - a_n| \leq \sum_{k=0}^{n} |a_{k-1} - a_k| \leq \sum_{k=0}^{n} Ce^{-\delta(t-k)}.$$

Recall that $C, \delta > 0$ are independent of the choice of $t \geq 0$, $n \in \{0, ..., \lfloor t \rfloor\}$ and $(\theta_0, ..., \theta_n) \in \mathbb{R}^{n+1}$ with $|\theta_i| \leq \Theta$. This implies the desired result. $\square$

We now present the proof of Theorem 1.6.

Proof of Theorem 1.6. By assumption, $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta = \kappa_f b(1+\beta)$. Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Choose $t_0 > 0$ large enough so that $\lceil t_0 - \ln t_0 \rceil \leq \lfloor t_0 \rfloor - 1$. We write

$$\begin{aligned} &(t\|X_t\|)^{-\frac{1}{1+\beta}} \langle f, X_t \rangle \\ &= \sum_{k=0}^{\lfloor t-\ln t \rfloor} \frac{\mathcal{I}_{t-k-1}^{t-k} \langle f, X_t \rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} + \Big(\sum_{k=\lceil t-\ln t \rceil}^{\lfloor t \rfloor - 1} \frac{\mathcal{I}_{t-k-1}^{t-k} \langle f, X_t \rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} + \frac{\mathcal{I}_0^{t-\lfloor t \rfloor} \langle f, X_t \rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \Big) + \frac{\langle P_t^\alpha f, X_0 \rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \\ &=: I_1(t) + I_2(t) + I_3(t), \quad t \geq t_0. \end{aligned}$$

Define

$$\widetilde{I}_1(t) := \sum_{k=0}^{\lfloor t-\ln t \rfloor} \frac{\mathcal{I}_{t-k-1}^{t-k} \langle f, X_t \rangle}{(te^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}}, \quad t \geq t_0.$$

Fix $\theta \in \mathbb{R}$. Taking $\theta_k = t^{-\frac{1}{1+\beta}}\theta$ and $n = \lfloor t - \ln t \rfloor$ in Proposition 3.6, we get that there exist $C_1, \delta_1 > 0$ such that,

$$\Big| \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1(t)}] - \exp\Big(\frac{1}{t} \sum_{k=0}^{\lfloor t-\ln t \rfloor} m_k[\theta f] \Big) \Big| \leq \frac{C_1}{t^{\delta_1}}, \quad t \geq t_0.$$

According to (2.10), $\widetilde{I}_1(t) \xrightarrow[t\to\infty]{d} \widetilde{\zeta}$ under $\widetilde{\mathbb{P}}_\mu$. Therefore, we only need to prove

$$|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1(t)}]| \xrightarrow[t\to\infty]{} 0 \tag{3.11}$$

and

$$I_i(t) \xrightarrow[t\to\infty]{d} 0, \quad i = 2, 3, \text{ under } \widetilde{\mathbb{P}}_\mu.$$

By [10, Lemma 3.4.3],

$$|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1(t)}]| \leq \sum_{k=0}^{\lfloor t-\ln t \rfloor} \widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|\big], \quad t \geq t_0,$$

where for all $k \geq 0$ and $t \geq k+1$,

$$Y_{t,k} := \exp\Big(i\theta \frac{\mathcal{I}_{t-k-1}^{t-k} \langle f, X_t \rangle}{(te^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}} \Big) - \exp\Big(i\theta \frac{\mathcal{I}_{t-k-1}^{t-k} \langle f, X_t \rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \Big).$$

Let $\gamma \in (0, \beta)$ be close enough to β such that

$$\frac{\alpha\gamma}{1+\gamma} > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0.$$

Fix this γ, then choose $\eta_0, \eta_1 > 0$ such that

$$\frac{\alpha\gamma}{1+\gamma} > \eta_0 > \eta_0 - 3\eta_1 > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0.$$

Recall that $H_t := e^{-\alpha t}\|X_t\|$. Define for all $k \geq 0$ and $t \geq k+1$,

$$\mathcal{D}_{t,k} := \left\{|H_t - H_{t-k-1}| \leq e^{-\eta_0(t-k-1)}, H_{t-k-1} > 2e^{-\eta_1(t-k-1)}\right\}. \tag{3.12}$$

Step 1. We will show that there exist $C_2, \delta_2 > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\widetilde{\mathbb{P}}_\mu\left[|Y_{t,k}|; \mathcal{D}^c_{t,k}\right] \leq C_2 e^{-\delta_2(t-k)}. \tag{3.13}$$

It follows from Proposition 2.11, Lemma 3.3 with $|p| = 0$ and Chebyshev's inequality that there exist $C_3, \delta_3 > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_\mu(\mathcal{D}^c_{t,k}) \\
&\leq \widetilde{\mathbb{P}}_\mu(|H_t - H_{t-k-1}| > e^{-\eta_0(t-k-1)}) + \widetilde{\mathbb{P}}_\mu(H_{t-k-1} \leq 2e^{-\eta_1(t-k-1)}), \\
&\leq \mathbb{P}_\mu(D^c)^{-1} e^{\eta_0(t-k-1)} \mathbb{P}_\mu[|H_t - H_{t-k-1}|] \\
&\quad + \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu(H_{t-k-1} \leq 2e^{-\eta_1(t-k-1)}; D^c) \\
&\leq \mathbb{P}_\mu(D^c)^{-1} e^{\eta_0(t-k-1)} \|H_t - H_{t-k-1}\|_{\mathbb{P}_\mu;1+\gamma} \\
&\quad + \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu(0 < H_{t-k-1} \leq 2e^{-\eta_1(t-k-1)}) \\
&\leq C_3 e^{-(\frac{\alpha\gamma}{1+\gamma} - \eta_0)(t-k-1)} + C_3 e^{-\delta_3(t-k-1)}.
\end{aligned} \tag{3.14}$$

This implies the desired result in this step, since $|Y_{t,k}| \leq 2$ a.s..

Step 2. We will show that there exist constant $C_4, \delta_4 > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\widetilde{\mathbb{P}}_\mu\left[|Y_{t,k}|\mathbf{1}_{\mathcal{D}_{t,k}}\right] \leq C_4 e^{-\delta_4(t-k)}. \tag{3.15}$$

In fact, since $|e^{ix} - e^{iy}| \leq |x-y|$ for all $x, y \in \mathbb{R}$, we have for all $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_\mu\left[|Y_{t,k}|\mathbf{1}_{\mathcal{D}_{t,k}}\right] \\
&\leq |\theta| t^{-\frac{1}{1+\beta}} \widetilde{\mathbb{P}}_\mu\left[|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle| \cdot \left|\frac{1}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}} - \frac{1}{\|X_t\|^{\frac{1}{1+\beta}}}\right| \mathbf{1}_{\mathcal{D}_{t,k}}\right] \\
&\leq |\theta| e^{-\frac{\alpha}{1+\beta}t} \widetilde{\mathbb{P}}_\mu\left[|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle| \cdot K_{t,k}\right],
\end{aligned} \tag{3.16}$$

where

$$K_{t,k} := \left|\frac{H_t^{\frac{1}{1+\beta}} - H_{t-k-1}^{\frac{1}{1+\beta}}}{H_t^{\frac{1}{1+\beta}} H_{t-k-1}^{\frac{1}{1+\beta}}}\right| \mathbf{1}_{\mathcal{D}_{t,k}}. \tag{3.17}$$

Note that, since $\eta_1 < \eta_0$, we have on $\mathcal{D}_{t,k}$,

$$H_t \geq H_{t-k-1} - e^{-\eta_0(t-k-1)} \geq 2e^{-\eta_1(t-k-1)} - e^{-\eta_0(t-k-1)} \geq e^{-\eta_1(t-k-1)}.$$

Therefore, for each $k \geq 0$ and $t \geq k+1$, on $\mathcal{D}_{t,k}$,

$$\begin{aligned}\left|H_t^{\frac{1}{1+\beta}} - H_{t-k-1}^{\frac{1}{1+\beta}}\right| &\leq \frac{1}{1+\beta}\max\left\{H_t^{-\frac{\beta}{1+\beta}}, H_{t-k-1}^{-\frac{\beta}{1+\beta}}\right\}|H_t - H_{t-k-1}| \\ &\leq \frac{1}{1+\beta}\max\{e^{\eta_1(t-k-1)}, \frac{1}{2}e^{\eta_1(t-k-1)}\}^{\frac{\beta}{1+\beta}} e^{-\eta_0(t-k-1)} \\ &\leq \frac{1}{1+\beta}e^{\eta_1(t-k-1)}e^{-\eta_0(t-k-1)} = \frac{1}{1+\beta}e^{-(\eta_0-\eta_1)(t-k-1)}\end{aligned}$$

and

$$|H_t^{\frac{1}{1+\beta}} H_{t-k-1}^{\frac{1}{1+\beta}}| \geq 2^{\frac{1}{1+\beta}} e^{-2\eta_1(t-k-1)}.$$

Thus, there exists $C_5 > 0$ such that for all $k \geq 0, t \geq k+1$,

$$K_{t,k} \leq C_5 e^{-(\eta_0-3\eta_1)(t-k-1)}. \tag{3.18}$$

According to Lemma 2.14, (3.16) and (3.18), there exist constant $C_6 > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}\widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|\mathbf{1}_{\mathcal{D}_{t,k}}\big] &\leq C_5|\theta|e^{-\frac{\alpha}{1+\beta}t}\widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle|\big]e^{-(\eta_0-3\eta_1)(t-k-1)} \\ &\leq \frac{C_5}{\mathbb{P}_\mu(D^c)}|\theta|e^{-\frac{\alpha}{1+\beta}t}\mathbb{P}_\mu\big[|\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle|\big]e^{-(\eta_0-3\eta_1)(t-k-1)} \\ &\leq \frac{C_5}{\mathbb{P}_\mu(D^c)}|\theta|e^{-\frac{\alpha}{1+\beta}t}\|\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}e^{-(\eta_0-3\eta_1)(t-k-1)} \\ &\leq C_6 e^{-\frac{\alpha}{1+\beta}t}e^{\frac{\alpha}{1+\gamma}t}e^{\frac{\gamma\alpha-\kappa_f(1+\gamma)b}{1+\gamma}k}e^{-(\eta_0-3\eta_1)(t-k)} \\ &= C_6 e^{(\frac{\alpha}{1+\gamma}-\frac{\alpha}{1+\beta})(t-k)}e^{-(\eta_0-3\eta_1)(t-k)},\end{aligned}$$

as desired in this step. In the last equality, we used the fact that

$$-(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta}) = \alpha(1-\frac{1}{1+\gamma}) - \alpha(1-\frac{1}{1+\beta}) = \frac{\gamma\alpha}{1+\gamma} - k_f b = \frac{\alpha\gamma - \kappa_f(1+\gamma)b}{1+\gamma}.$$

Step 3. We will show that there exist $C_7, \delta_5 > 0$ such that for t large enough,

$$|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1(t)}]| \leq C_7 t^{-\delta_5}. \tag{3.19}$$

In fact, according to (3.13) and (3.15), there exist $C_8, \delta_6 > 0$ such that for all $k \geq 0$ and $t \geq k+1$ we have,

$$\widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|\big] \leq C_8 e^{-\delta_6(t-k)}.$$

Therefore, there exists $C_9 > 0$ such that for all $t \geq 1$,

$$\begin{aligned}\sum_{k=0}^{\lfloor t-\ln t\rfloor}\widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|\big] &\leq C_8\sum_{k=0}^{\lfloor t-\ln t\rfloor}e^{-\delta_6(t-k)} = C_8 e^{-\delta_6 t}\sum_{k=0}^{\lfloor t-\ln t\rfloor}e^{\delta_6 k} \\ &\leq C_9 e^{-\delta_6 t}e^{\delta_6(t-\ln t)} = \frac{C_9}{t^{\delta_6}}.\end{aligned}$$

Letting $t \to \infty$ in (3.19), we get (3.11).

Step 4. We will show that $I_2(t) \xrightarrow[t\to\infty]{d} 0$ with respect to $\widetilde{\mathbb{P}}_\mu$. In fact, let $\mathcal{E}_t := \{\|X_t\| > t^{-1/2}e^{\alpha t}\}$. According to Proposition 2.11, there exist $C_{10}, \delta_7 > 0$ such that

$$\widetilde{\mathbb{P}}_\mu(\mathcal{E}_t^c) \le \frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu(0 < e^{-\alpha t}\|X_t\| \le t^{-1/2}) \le C_{10}(t^{-\delta_7} + e^{-\delta_7 t}), \quad t \ge 0.$$

Therefore,

$$|\widetilde{\mathbb{P}}_\mu[(e^{i\theta I_2(t)} - 1)\mathbf{1}_{\mathcal{E}_t^c}]| \le 2\widetilde{\mathbb{P}}_\mu(\mathcal{E}_t^c) \le C_{10}(t^{-\delta_7} + e^{-\delta_7 t}), \quad t \ge t_0. \tag{3.20}$$

Choose a $\gamma \in (0, \beta)$ close enough to β such that $\alpha(\frac{1}{1+\gamma} - \frac{1}{1+\beta}) \le \frac{1}{2(1+\beta)}$. According to Lemma 2.14, there exist $C_{11}, C_{12}, C_{13} > 0$ such that

$$\begin{aligned}
&|\widetilde{\mathbb{P}}_\mu[(e^{i\theta I_2(t)} - 1)\mathbf{1}_{\mathcal{E}_t}]| \le |\theta|\widetilde{\mathbb{P}}_\mu\big[|I_2(t)|\mathbf{1}_{\mathcal{E}_t}\big] \\
&\le |\theta| t^{-\frac{1}{2(1+\beta)}} e^{-\frac{\alpha}{1+\beta}t}\Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle|\big] + \widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}_0^{t-\lfloor t\rfloor}\langle f, X_t\rangle|\big]\Big) \\
&\le C_{11}|\theta| t^{-\frac{1}{2(1+\beta)}} e^{-\frac{\alpha}{1+\beta}t}\Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \|\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} + \|\mathcal{I}_0^{t-\lfloor t\rfloor}\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\Big) \\
&\le C_{12}|\theta| t^{-\frac{1}{2(1+\beta)}} e^{-\frac{\alpha}{1+\beta}t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{\frac{\alpha}{1+\gamma}t} e^{\frac{\alpha\gamma - \kappa_f(1+\gamma)b}{1+\gamma}k} \\
&= C_{12}|\theta| t^{-\frac{1}{2(1+\beta)}} e^{(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{-(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})k} \\
&\le C_{12}|\theta| t^{-\frac{1}{2(1+\beta)}} e^{(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})(t-\lceil t-\ln t\rceil)} \sum_{j=0}^{\infty} e^{-(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})j} \\
&\le C_{13}|\theta| t^{-\frac{1}{2(1+\beta)}} t^{(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})}, \quad t \ge t_0.
\end{aligned}$$

From this and (3.20), we get the desired result in this step.

Step 5. We will show that $I_3(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0$. In fact, we have

$$\begin{aligned}
|I_3(t)| &\le \frac{\langle |P_t^\alpha f|, X_0\rangle}{(t\|X_t\|)^{\frac{1}{1+\beta}}} \le \frac{\langle e^{\alpha t - \kappa_f bt} Qf, X_0\rangle}{(te^{\alpha t}H_t)^{\frac{1}{1+\beta}}} \\
&= t^{-\frac{1}{1+\beta}} e^{\frac{\beta\alpha t}{1+\beta} - k_f bt} H_t^{-\frac{1}{1+\beta}} \langle Qf, X_0\rangle \\
&= t^{-\frac{1}{1+\beta}} H_t^{-\frac{1}{1+\beta}} \langle Qf, X_0\rangle \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0.
\end{aligned}$$

Finally, combining Steps 3–5, we complete the proof of Theorem 1.6. $\square$

3.3. **The small branching rate regime.** In this subsection, we prove the central limit theorem for the small branching rate regime.

Proof of Theorem 1.7. By assumption, $f \in \mathcal{P}$ is non-trivial and satisfies $\alpha\beta < \kappa_f b(1+\beta)$. Let $t_0 > 1$ be large enough so that

$$\lceil t - \ln t \rceil \le \lfloor t \rfloor - 1, \quad t \ge t_0.$$

Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and write

$$\begin{aligned}
&\frac{\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \\
&= \sum_{k=0}^{\lfloor t-\ln t\rfloor} \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} + \Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} + \frac{\mathcal{I}_{0}^{t-\lfloor t\rfloor}\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \Big) + \frac{\langle P_t^\alpha f, X_0\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \\
&=: I_1'(t) + I_2'(t) + I_3'(t), \quad t \ge t_0.
\end{aligned}$$

Define

$$\widetilde{I}_1'(t) := \sum_{k=0}^{\lfloor t-\ln t\rfloor} \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}}, \quad t > t_0.$$

Fix a $\theta \in \mathbb{R}$. Taking $\theta_k = \theta$ and $n = \lfloor t - \ln t\rfloor$ in Proposition 3.6, we get that there exist $C_1, \delta_1 > 0$ such that

$$\Big| \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1'(t)}] - \exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor} m_k[\theta f]\Big)\Big| \le C_1 e^{-\delta_1(t-\lfloor t-\ln t\rfloor)} \le \frac{C_1}{t^{\delta_1}}, \quad t \ge 0.$$

Hence, according to (2.11), we have $\widetilde{I}_1'(t) \xrightarrow[t\to\infty]{d} \zeta$ under $\widetilde{\mathbb{P}}_\mu$. So we only need to prove that $|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1'(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1'(t)}]| \xrightarrow[t\to\infty]{} 0$ and $I_i'(t) \xrightarrow[t\to\infty]{d} 0$, $i = 2, 3$, under $\widetilde{\mathbb{P}}_\mu$.

Step 1. We will show that $|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1'(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1'(t)}]| \xrightarrow[t\to\infty]{} 0$. Define for $k \ge 0$ and $t \ge k+1$,

$$Y_{t,k}' := \exp\Big(i\theta \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{(e^{\alpha(k+1)}\|X_{t-k-1}\|)^{\frac{1}{1+\beta}}}\Big) - \exp\Big(i\theta \frac{\mathcal{I}_{t-k-1}^{t-k}\langle f, X_t\rangle}{\|X_t\|^{\frac{1}{1+\beta}}}\Big).$$

Then we have

$$|\widetilde{\mathbb{P}}_\mu[e^{i\theta I_1'(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}_1'(t)}]| \le \sum_{k=0}^{\lfloor t-\ln t\rfloor} \widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}'|\big], \quad t \ge t_0.$$

Let $\gamma \in (0, \beta)$ be close enough to β such that

$$\frac{\alpha\gamma}{1+\gamma} > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0.$$

Fix this γ, then chose $\eta_0, \eta_1 > 0$ such that

$$\frac{\alpha\gamma}{1+\gamma} > \eta_0 > \eta_0 - 3\eta_1 > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0.$$

Define $\mathcal{D}_{t,k}$ and $K_{t,k}$ as in (3.12) and (3.17) respectively. Using Lemma 2.14, (3.14), (3.18) and an argument similar to that used in proving (3.16), we get that there exist $C_2, C_3, \delta_2 > 0$ such that for $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
\widetilde{\mathbb{P}}_\mu\big[|Y'_{t,k}|\big] &= \widetilde{\mathbb{P}}_\mu\big[|Y'_{t,k}|;\mathcal{D}_{t,k}\big] + \widetilde{\mathbb{P}}_\mu\big[|Y'_{t,k}|;\mathcal{D}^c_{t,k}\big] \\
&\leq |\theta| e^{-\frac{\alpha}{1+\beta}t}\widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle| \cdot K_{t,k}\big] + 2\widetilde{\mathbb{P}}_\mu(\mathcal{D}^c_{t,k}) \\
&\leq C_2 e^{-\frac{\alpha}{1+\beta}t}\|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} e^{-(\eta_0-3\eta_1)(t-k)} + Ce^{-\delta_2(t-k)} \\
&\leq C_3\big(e^{-\frac{\alpha}{1+\beta}t} e^{\frac{\alpha}{1+\gamma}t} e^{\frac{\gamma\alpha-\kappa_f(1+\gamma)b}{1+\gamma}k} e^{-(\eta_0-3\eta_1)(t-k)} + e^{-\delta_2(t-k)}\big).
\end{aligned}$$

Since $\alpha\beta < \kappa_f(1+\beta)b$, we have

$$\begin{aligned}
-\Big(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta}\Big) &= \alpha\Big(1 - \frac{1}{1+\gamma}\Big) - \alpha\Big(1 - \frac{1}{1+\beta}\Big) \\
&> \frac{\gamma\alpha}{1+\gamma} - k_f b = \frac{\alpha\gamma - \kappa_f(1+\gamma)b}{1+\gamma}.
\end{aligned} \tag{3.21}$$

Using this, we have that there exist $C_4, \delta_3 > 0$ such that for $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
\widetilde{\mathbb{P}}_\mu\big[|Y'_{t,k}|\big] &\leq C_3\big(e^{(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta})(t-k)} e^{-(\eta_0-3\eta_1)(t-k)} + e^{-\delta_2(t-k)}\big) \\
&\leq C_4 e^{-\delta_3(t-k)}.
\end{aligned}$$

Now we can use the same argument used in the proof of Theorem 1.6 to prove $|\widetilde{\mathbb{P}}_\mu[e^{i\theta I'_1(t)}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta \widetilde{I}'_1(t)}]| \xrightarrow[t\to\infty]{} 0$.

Step 2. We will show that $I'_2(t) \xrightarrow[t\to\infty]{d} 0$. Let $\gamma \in (0,\beta)$. According to (3.21), we can choose $\epsilon > 0$ small enough so that

$$q := -\frac{\alpha\gamma - \kappa_f(1+\gamma)b}{1+\gamma} > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} + \frac{2\epsilon}{1+\beta} > 0.$$

Recall that

$$A_t(\epsilon) := \{\|X_t\| > e^{(\alpha-\epsilon)t}\}, \quad t \geq 0.$$

According to (2.11), there exist $C_5, \delta_4 > 0$ such that

$$\begin{aligned}
|\widetilde{\mathbb{P}}_\mu[(e^{i\theta I'_2(t)} - 1)\mathbf{1}_{A_t(\epsilon)^c}]| &\leq 2\widetilde{\mathbb{P}}_\mu(A_t(\epsilon)^c) \leq \frac{2}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu(0 < e^{-\alpha t}\|X_t\| \leq e^{-\epsilon t}) \\
&\leq C_5(e^{-\epsilon\delta_4 t} + e^{-\delta_4 t}), \quad t \geq t_0.
\end{aligned}$$

According to Lemma 2.14, there exist $C_6, C_7, C_8 > 0$ such that for all $t \geq t_0$,

$$\begin{aligned}
&\big|\widetilde{\mathbb{P}}_\mu[(e^{i\theta I'_2(t)} - 1)\mathbf{1}_{A_t(\epsilon)}]\big| \leq |\theta|\widetilde{\mathbb{P}}_\mu\big[|I'_2(t)|\mathbf{1}_{A_t(\epsilon)}\big] \\
&\leq |\theta| e^{-\frac{(\alpha-\epsilon)t}{1+\beta}}\Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle|\big] + \widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}^{t-\lfloor t\rfloor}_0\langle f, X_t\rangle|\big]\Big) \\
&\leq C_6 e^{-\frac{(\alpha-\epsilon)t}{1+\beta}}\Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \|\mathcal{I}^{t-k}_{t-k-1}\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma} + \|\mathcal{I}^{t-\lfloor t\rfloor}_0\langle f, X_t\rangle\|_{\mathbb{P}_\mu;1+\gamma}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq C_7 e^{-\frac{(\alpha-\epsilon)t}{1+\beta}} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{\frac{\alpha}{1+\gamma}t} e^{\frac{\alpha\gamma-\kappa_f(1+\gamma)b}{1+\gamma}k} \\
&\leq C_7 e^{-\frac{\epsilon}{1+\beta}t} e^{(\frac{\alpha}{1+\gamma}-\frac{\alpha}{1+\beta}+\frac{2\epsilon}{1+\beta})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{\frac{\alpha\gamma-\kappa_f(1+\gamma)b}{1+\gamma}k} \\
&\leq C_7 e^{-\frac{\epsilon}{1+\beta}t} e^{qt} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{-qk} \\
&\leq C_7 e^{-\frac{\epsilon}{1+\beta}t} e^{q(t-\lceil t-\ln t\rceil)} \sum_{j=0}^{\infty} e^{-qj} \leq C_8 e^{-\frac{\epsilon}{1+\beta}t} t^q.
\end{aligned}
$$

Therefore, we get the desired result in this step.

Step 3. We will show that $I_3'(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0$. In fact, we have

$$
\begin{aligned}
|I_3'(t)| &\leq \frac{\langle |P_t^\alpha f|, X_0\rangle}{\|X_t\|^{\frac{1}{1+\beta}}} \leq \frac{\langle e^{\alpha t-\kappa_f bt} Qf, X_0\rangle}{(e^{\alpha t}H_t)^{\frac{1}{1+\beta}}} \\
&= e^{(\frac{\beta\alpha}{1+\beta}-k_f b)t} H_t^{-\frac{1}{1+\beta}} \langle Qf, X_0\rangle \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0. \qquad \square
\end{aligned}
$$

3.4. **The large branching rate regime: CLT.** In this subsection, we revisit the large branching rate regime and prove Theorem 1.5. For $g \in \mathcal{C}_l$, recall the definition of $I_t g$ in (1.11). By Fubini's theorem, we have

$$
\begin{aligned}
\bar{m}_n[g] &= e^{\alpha(n-1)} \int_0^1 e^{\alpha s} \langle \eta(iI_{s+n-1}g)^{1+\beta}, \varphi\rangle ds \\
&= e^{\alpha(n-1)} \langle \int_0^1 P_s^\alpha \langle \eta(iI_{s+n-1}g)^{1+\beta} ds, \varphi\rangle \\
&= e^{\alpha(n-1)} \langle \int_0^1 P_{1-s}^\alpha \langle \eta(iI_{n-s}g)^{1+\beta} ds, \varphi\rangle = e^{\alpha(n-1)} \langle Z_1(-I_n g), \varphi\rangle, \quad n \geq 1.
\end{aligned} \tag{3.22}
$$

Recall that H_t^p are defined before Lemma 3.2. We write $a_p := \langle \phi_p, g\rangle_\varphi$ for each $p \in \mathbb{Z}_+^d$.

Lemma 3.7. *For all $n \geq 1$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, under $\mathbb{P}(\cdot|D^c)$,*

$$
\bar{\gamma}_{t,n} := \sum_{p\in\mathcal{N}} a_p \frac{H_{t+n-1}^p - H_{t+n}^p}{e^{-(\alpha-|p|b)t}\big(e^{-\alpha(n-1)}\|X_{t+n-1}\|\big)^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \bar{\zeta}_n,
$$

where $\bar{\zeta}_n$ is a $(1+\beta)$-stable random variable with characteristic function

$$
\mathbf{E}[e^{i\theta\bar{\zeta}_n}] = \exp(\bar{m}_n[\theta g]), \quad \theta \in \mathbb{R}.
$$

Proof. Fix $n \geq 1$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. We only need to show that

$$
\mathbb{P}_\mu[\exp(i\bar{\gamma}_{t,n}); D^c] \xrightarrow[t\to\infty]{} \mathbb{P}_\mu(D^c)\exp(\bar{m}_n[g]).
$$

Put $\theta_{t,p,n} := a_p \frac{e^{-(\alpha-|p|b)n}}{(e^{-\alpha(n-1)}\|X_{t+n-1}\|)^{\frac{1}{1+\beta}}}$ and $A_t(\epsilon) := \{\|X_t\| > e^{(\alpha-\epsilon)t}\}$. We have

$$\bar{\gamma}_{t,n} = \sum_{p\in\mathcal{N}} \theta_{t,p,n}(e^{\alpha-|p|b}\langle\phi_p, X_{t+n-1}\rangle - \langle\phi_p, X_{t+n}\rangle).$$

Step 1. We will show that for all $\epsilon > 0, n \geq 1$, and $t \geq 0$, we have

$$\left|\mathbb{P}_\mu\left[e^{i\bar{\gamma}_{t,n}} - e^{\bar{m}_n[g]}; D^c\right]\right| \leq J_1'(t,n,\epsilon) + J_2'(t,n,\epsilon) + J_3'(t,n,\epsilon),$$

where

$$\begin{aligned}
J_1'(t,n,\epsilon) &:= \mathbb{P}_\mu\big[|\langle Z_1'''(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p), X_{t+n-1}\rangle|; A_{t+n-1}(\epsilon)\big],\\
J_2'(t,n,\epsilon) &:= \mathbb{P}_\mu\big[|\langle Z_1(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p), X_{t+n-1}\rangle - \bar{m}_n[g]|; A_{t+n-1}(\epsilon)\big],\\
J_3'(t,n,\epsilon) &:= 2\mathbb{P}_\mu(A_{t+n-1}(\epsilon)\Delta D^c).
\end{aligned}$$

In fact, it follows from (2.6) that

$$\begin{aligned}
\mathbb{P}_\mu[e^{i\bar{\gamma}_{t,n}}|\mathscr{F}_{t+n-1}] &= \mathbb{P}_\mu[e^{i\sum_{p\in\mathcal{N}}\theta_{t,p,n}(e^{\alpha-|p|b}\langle\phi_p,X_{t+n-1}\rangle-\langle\phi_p,X_{t+n}\rangle)}|\mathscr{F}_{t+n-1}]\\
&= e^{\langle (Z_1'''+Z_1)(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p),X_{t+n-1}\rangle}.
\end{aligned}\tag{3.23}$$

According to (3.23) and the fact that

$$|e^{-x} - e^{-y}| \leq |x-y|, \quad x, y \in \mathbb{C}_+,$$

we have for all $n \geq 1, t \geq 0$ and $\epsilon > 0$,

$$\begin{aligned}
&\left|\mathbb{P}_\mu\left[e^{i\bar{\gamma}_{t,n}} - e^{\bar{m}_n[g]}; D^c\right]\right|\\
&\leq \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\bar{\gamma}_{t,n}} - e^{\bar{m}_n[g]}; D^c|\mathscr{F}_{t+n-1}]\big|\Big]\\
&\leq \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\bar{\gamma}_{t,n}} - e^{\bar{m}_n[g]}; A_{t+n-1}(\epsilon)|\mathscr{F}_{t+n-1}]\big| + 2\mathbb{P}_\mu(A_{t+n-1}(\epsilon)\Delta D^c|\mathscr{F}_{t+n-1})\Big]\\
&= \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\bar{\gamma}_{t,n}}|\mathscr{F}_{t+n-1}] - e^{\bar{m}_n[g]}\big|; A_{t+n-1}(\epsilon)\Big] + J_3'(t,n,\epsilon)\\
&= \mathbb{P}_\mu\Big[\big|e^{\langle (Z_1'''+Z_1)(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p),X_{t+n-1}\rangle} - e^{\bar{m}_n[g]}\big|; A_{t+n-1}(\epsilon)\Big] + J_3'(t,n,\epsilon)\\
&\leq \mathbb{P}_\mu\Big[\big|\langle (Z_1'''+Z_1)(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p), X_{t+n-1}\rangle - \bar{m}_n[g]\big|; A_{t+n-1}(\epsilon)\Big] + J_3'(t,n,\epsilon)\\
&\leq J_1'(t,n,\epsilon) + J_2'(t,n,\epsilon) + J_3'(t,n,\epsilon).
\end{aligned}$$

Step 2. We will show that for $\epsilon > 0$ small enough, there exist constants $C_1, \delta_1 > 0$ such that for all $n \geq 1$ and $t \geq 0$,

$$J_1'(t,n,\epsilon) \leq C_1 e^{-\delta_1(t+n)}.$$

Let δ_0 be the constant in Lemma 2.5.(7) and R be the corresponding $(\theta^{2+\beta}\vee\theta^{1+\beta+\delta_0})$-controller. Then we have for all $n\geq 1, t\geq 0, \epsilon>0$,

$$
\begin{aligned}
&|Z_1'''(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p)|\mathbf{1}_{A_{t+n-1}(\epsilon)}\leq R(|\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p|)\mathbf{1}_{A_{t+n-1}(\epsilon)}\\
&\leq R\Big(\frac{\sum_{p\in\mathcal{N}}|a_p|e^{-(\alpha-|p|b)n}|\phi_p|}{\big(e^{-\alpha(n-1)}e^{(\alpha-\epsilon)(t+n-1)}\big)^{\frac{1}{1+\beta}}}\Big)\\
&\leq \sum_{\rho\in\{\delta_0,1\}}\Big(\frac{e^{-(\alpha-Kb)n}}{\big(e^{-\alpha(n-1)}e^{(\alpha-\epsilon)(t+n-1)}\big)^{\frac{1}{1+\beta}}}\Big)^{1+\beta+\rho}Rh\\
&=\sum_{\rho\in\{\delta_0,1\}}e^{-\alpha\frac{1+\beta+\rho}{1+\beta}}e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha\beta-K(1+\beta)b)n}e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t+n-1)}Rh\\
&\leq \sum_{\rho\in\{\delta_0,1\}}e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t+n-1)}Rh,
\end{aligned}
$$

where $h=\sum_{p\in\mathcal{N}}|a_p\phi_p|$. Thus, there exists constant $C_3>0$ such that for all $n\geq 1, t\geq 0$ and $\epsilon>0$,

$$
\begin{aligned}
J_1'(t,n,\epsilon)&\leq \sum_{\rho\in\{\delta_0,1\}}e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)(t+n-1)}\mathbb{P}_\mu[\langle Rh, X_{t+n-1}\rangle]\\
&\leq C_1\sum_{\rho\in\{\delta_0,1\}}\exp\Big\{-\Big(\alpha\frac{\rho}{1+\beta}-\epsilon\frac{1+\beta+\rho}{1+\beta}\Big)(t+n-1)\Big\}.
\end{aligned}
$$

By choosing $\epsilon>0$ small enough, we get the desired result in this step.

Step 3. We will show that for each $\epsilon>0$ small enough, there exist $C_2,\delta_2>0$ such that for all $n\geq 1, t\geq 0$,

$$
J_2'(t,n,\epsilon)\leq C_2e^{-\delta_2(t+n)}.
$$

In fact, according to the definitions of Z_1 and $\bar{m}_n$, and (3.22), we have for all $t\geq 0, n\geq 1$,

$$
\begin{aligned}
&\langle Z_1(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p), X_{t+n-1}\rangle-\bar{m}_n[g]\\
&=e^{\alpha(n-1)}\Big(\frac{\langle Z_1(-I_ng), X_{t+n-1}\rangle}{\|X_{t+n-1}\|}-\langle Z_1(-I_ng),\varphi\rangle\Big).
\end{aligned}
$$

Therefore, for all $t\geq 0, n\geq 1$ and $\epsilon>0$,

$$
\begin{aligned}
J_2'(t,n,\epsilon)&=\mathbb{P}_\mu\big[|\langle Z_1(-\sum_{p\in\mathcal{N}}\theta_{t,p,n}\phi_p), X_{t+n-1}\rangle-\bar{m}_n[g]|; A_{t+n-1}(\epsilon)\big]\\
&\leq e^{\alpha(n-1)}\mathbb{P}_\mu\Big[\Big|\frac{\langle Z_1(-I_ng), X_{t+n-1}\rangle}{\|X_{t+n-1}\|}-\langle Z_1(-I_ng),\varphi\rangle\Big|; A_{t+n-1}(\epsilon)\Big]\\
&\leq e^{\alpha(n-1)}e^{-(\alpha-Kb)(1+\beta)n}e^{-(\alpha-\epsilon)(t+n-1)}\mathbb{P}_\mu[\langle \bar{g}_n, X_{t+n-1}\rangle],
\end{aligned}
$$

where

$$\bar{g}_n := \frac{Z_1(-I_n g) - \langle Z_1(-I_n g), \varphi\rangle}{e^{-(\alpha-Kb)(1+\beta)n}}.$$

Fix a $\gamma > 0$ small enough such that $\alpha\gamma < (1+\gamma)b$. Using Lemma 2.10 and Lemma 2.15.(3) with $\kappa = 1$, there exists $C_3 > 0$ such that

$$\mathbb{P}_\mu[\langle \bar{g}_n, X_{t+n-1}\rangle] \le C_3 e^{\frac{\alpha}{1+\gamma}(t+n-1)}, \quad t \ge 0, n \ge 1.$$

Therefore, for all $t \ge 0, n \ge 1$ and $\epsilon > 0$,

$$\begin{aligned}
J_2'(t,n,\epsilon) &\le e^{\alpha(n-1)} e^{-(\alpha-Kb)(1+\beta)n} e^{-(\alpha-\epsilon)(t+n-1)} \mathbb{P}_\mu[\langle \bar{g}_n, X_{t+n-1}\rangle] \\
&\le C_3 e^{\alpha(n-1)} e^{-(\alpha-Kb)(1+\beta)n} e^{-(\alpha-\epsilon)(t+n-1)} e^{\frac{\alpha}{1+\gamma}(t+n-1)} \\
&= C_3 e^{-\alpha} e^{-(\alpha\beta - K(1+\beta)b)n} e^{-(\frac{\alpha\gamma}{1+\gamma}-\epsilon)(t+n-1)} \\
&\le C_3 e^{-(\frac{\alpha\gamma}{1+\gamma}-\epsilon)(t+n-1)}.
\end{aligned}$$

Now, by choosing $\epsilon > 0$ small enough, we get the desired result in this step.

Step 4. According to the Step 4 of the proof of Lemma 3.4, we have for any $\epsilon \in (0,\alpha)$ there exist $C_4, \delta_3 > 0$ such that

$$J_3'(t,n,\epsilon) \le C_4 e^{-\delta_3(t+n)} \quad t \ge 0, n \ge 1.$$

Finally, combining the results in Steps 1–4. and noticing that, if $\epsilon > 0$ is chosen small enough then $J_i, i = 1,2,3$, converge to 0 exponentially fast when $t \to \infty$, we get the desired result. ☐

Lemma 3.8. *For all $\theta \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist constants $C, \delta > 0$ such that for all $t \ge 0$, $m \in \mathbb{N}$, we have*

$$\left| \widetilde{\mathbb{P}}_\mu \Big[\prod_{l=1}^m \exp(i\theta \bar{\gamma}_{t,l}) - \prod_{l=1}^m \exp(\bar{m}_l[\theta g]) \Big] \right| \le C e^{-\delta t}. \tag{3.24}$$

Proof. Step 1. Fix $\theta \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Similar to the proof of Corollary 3.5, we can show that for all $\theta \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist $C_1, \delta_1 > 0$ such that, for any $t \ge 0$, $n \ge 1$, we have

$$\mathbb{P}_\mu\Big[\Big|\mathbb{P}_\mu[e^{i\theta\bar{\gamma}_{t,n}} - e^{\bar{m}_n[\theta g]}; D^c | \mathscr{F}_{t+n-1}]\Big|\Big] \le C_1 e^{-\delta_1(t+n-1)}.$$

Step 2. Fix $t \ge 0$ and $m \in \mathbb{N}$. For each $n = 0, \cdots, m$, we define

$$\widetilde{a}_n := \widetilde{\mathbb{P}}_\mu \Big[\prod_{l=0}^n \exp(i\theta\bar{\gamma}_{t,l}) \Big] \times \prod_{l=n+1}^m \exp(\bar{m}_l[\theta g]),$$

where by convention the first product is 1 for $n = 0$. Then we get for each $n \ge 1$,

$$\begin{aligned}
&\widetilde{a}_{n-1} - \widetilde{a}_n \\
&= \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu \Big[\Big(\prod_{l=0}^{n-1} e^{i\theta\bar{\gamma}_{t,l}} \Big) \times (e^{\bar{m}_n[\theta g]} - e^{i\theta\bar{\gamma}_{t,n}}); D^c \Big] \times \prod_{l=n+1}^m e^{\bar{m}_l[\theta g]}
\end{aligned}$$

$$= \mathbb{P}_\mu(D^c)^{-1}\mathbb{P}_\mu\Big[\Big(\prod_{l=0}^{n-1} e^{i\theta\bar{\gamma}_{t,l}}\Big) \times \mathbb{P}_\mu[e^{\bar{m}_n[\theta g]} - e^{i\theta\bar{\gamma}_{t,n}}; D^c|\mathscr{F}_{t+n-1}]\Big] \times \prod_{l=n+1}^{m} e^{\bar{m}_l[\theta g]}.$$

According to Step 1 and Proposition 2.7, there exists $C_2 > 0$ such that for any $n = 1, \cdots, m$ and $t \geq 0$, we have

$$\begin{aligned}|\widetilde{a}_{n-1} - \widetilde{a}_n| &\leq \frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\theta\bar{\gamma}_{t,n}} - e^{\bar{m}_n[\theta g]}; D^c\big|\mathscr{F}_{t+n-1}]\big|\Big]\\ &\leq C_2 e^{-\delta_1(t+n-1)}.\end{aligned}$$

Therefore,

$$\text{LHS of (3.24)} = |\widetilde{a}_0 - \widetilde{a}_m| \leq \sum_{n=1}^{m} |\widetilde{a}_{n-1} - \widetilde{a}_n| \leq \sum_{n=1}^{m} C_2 e^{-\delta_1(t+n-1)}.$$

Notice that $C_2, \delta_1 > 0$ are independent of the choice of $t \geq 0$, $m \in \mathbb{N}$. $\square$

Now we define

$$\widetilde{\gamma}_{t,n} := \sum_{p\in\mathcal{N}} a_p \frac{H^p_{t+n-1} - H^p_{t+n}}{e^{-(\alpha-|p|b)t}\|X_t\|^{\frac{1}{1+\beta}}}, \quad t \geq 0, n \geq 1.$$

Lemma 3.9. *For all $\theta \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist constants $C, \delta > 0$ such that for all $m \in \mathbb{N}$ and $t \geq 0$, we have*

$$\Big|\widetilde{\mathbb{P}}_\mu[\prod_{n=1}^{m} e^{i\theta\widetilde{\gamma}_{t,n}}] - \widetilde{\mathbb{P}}_\mu[\prod_{n=1}^{m} e^{i\theta\bar{\gamma}_{t,n}}]\Big| \leq Cme^{-\delta t}.$$

Proof. Fix a $\theta \in \mathbb{R}$ and a $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. According to [10, Lemma 3.4.3],

$$\Big|\widetilde{\mathbb{P}}_\mu[\prod_{n=1}^{m} e^{i\theta\widetilde{\gamma}_{t,n}}] - \widetilde{\mathbb{P}}_\mu[\prod_{n=1}^{m} e^{i\theta\bar{\gamma}_{t,n}}]\Big| \leq \sum_{n=1}^{m} \widetilde{\mathbb{P}}_\mu[|Y''_{t,n}|], \tag{3.25}$$

where

$$Y''_{t,n} := e^{i\theta\widetilde{\gamma}_{t,n}} - e^{i\theta\bar{\gamma}_{t,n}}, \quad t \geq 0, n \geq 1.$$

Let $\gamma \in (0, \beta)$ be close enough to β such that

$$\frac{\alpha\gamma}{1+\gamma} > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0$$

and

$$\alpha\gamma > K(1+\gamma)b,$$

where K is defined in (1.9). Fix this γ, and then choose $\eta_0, \eta_1 > 0$ such that

$$\frac{\alpha\gamma}{1+\gamma} > \eta_0 > \eta_0 - 3\eta_1 > \frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} > 0.$$

Define for all $n \geq 1$ and $t \geq 0$,

$$\mathcal{D}_{t,n} := \left\{|H_t - H_{t+n-1}| \leq e^{-\eta_0 t}, H_t > 2e^{-\eta_1 t}\right\}.$$

Step 1. Similar to Step 1 in the proof of Theorem 1.6, we can show that there exist $C_1, \delta_1 > 0$ such that for all $n \geq 1$ and $t \geq 0$,

$$\widetilde{\mathbb{P}}_\mu\big[|Y''_{t,n}|\mathbf{1}_{\mathcal{D}^c_{t,n}}\big] \leq C_1 e^{-\delta_1 t}.$$

Step 2. We will show that there exist $C_2, \delta_2 > 0$ such that for all $n \geq 1$ and $t \geq 0$,

$$\widetilde{\mathbb{P}}_\mu\big[|Y''_{t,n}|\mathbf{1}_{\mathcal{D}_{t,n}}\big] \leq C_1 e^{-\delta_2 t}. \tag{3.26}$$

In fact, since $|e^{ix} - e^{iy}| \leq |x - y|$ for all $x, y \in \mathbb{R}$, we have, for all $t \geq 0$ and $n \geq 1$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_\mu\big[|Y''_{t,n}|\mathbf{1}_{\mathcal{D}_{t,n}}\big] \\
&\leq |\theta| \sum_{p\in\mathcal{N}} |a_p| e^{(\alpha-|p|b)t} \widetilde{\mathbb{P}}_\mu\Big[|H^p_{t+n-1} - H^p_{t+n}| \cdot \Big|\frac{1}{(e^{-\alpha(n-1)}\|X_{t+n-1}\|)^{\frac{1}{1+\beta}}} - \frac{1}{\|X_t\|^{\frac{1}{1+\beta}}}\Big|\mathbf{1}_{\mathcal{D}_{t,n}}\Big] \\
&\leq |\theta| \sum_{p\in\mathcal{N}} |a_p| e^{(\alpha-|p|b)t} e^{-\frac{\alpha}{1+\beta}t} \widetilde{\mathbb{P}}_\mu\Big[|H^p_{t+n-1} - H^p_{t+n}| \cdot K'_{t,n}\Big],
\end{aligned}$$

where

$$K'_{t,n} := \Big|\frac{H_t^{\frac{1}{1+\beta}} - H_{t+n-1}^{\frac{1}{1+\beta}}}{H_t^{\frac{1}{1+\beta}} H_{t+n-1}^{\frac{1}{1+\beta}}}\Big|\mathbf{1}_{\mathcal{D}_{t,n}}.$$

Since $\eta_1 < \eta_0$, using an argument similar to that used in Step 2 of the proof of Theorem 1.6, we can show that, there is a constant $C_3 > 0$ such that,

$$K'_{t,n} \leq C_3 e^{-(\eta_0 - 3\eta_1)t}, \quad t \geq 0, n \geq 1.$$

According to Lemma 3.3, there exists a constant $C_4 > 0$ such that for all $t \geq 0$ and $n \geq 1$,

$$\begin{aligned}
\widetilde{\mathbb{P}}_\mu\big[|Y''_{t,n}|\mathbf{1}_{\mathcal{D}_{t,n}}\big] &\leq \frac{C_3}{\mathbb{P}_\mu(D^c)} |\theta| \sum_{p\in\mathcal{N}} |a_p| e^{(\alpha-|p|b)t} e^{-\frac{\alpha}{1+\beta}t} e^{-(\eta_0-3\eta_1)t} \mathbb{P}_\mu\Big[|H^p_{t+n-1} - H^p_{t+n}|\Big] \\
&\leq \frac{C_3}{\mathbb{P}_\mu(D^c)} |\theta| \sum_{p\in\mathcal{N}} |a_p| e^{(\alpha-|p|b)t} e^{-\frac{\alpha}{1+\beta}t} e^{-(\eta_0-3\eta_1)t} \|H^p_{t+n-1} - H^p_{t+n}\|_{\mathbb{P}_\mu;1+\gamma} \\
&\leq C_4 |\theta| \sum_{p\in\mathcal{N}} |a_p| e^{(\alpha-|p|b)t} e^{-\frac{\alpha}{1+\beta}t} e^{-(\eta_0-3\eta_1)t} e^{-\frac{1}{1+\gamma}(\alpha\gamma-|p|(1+\gamma)b)(t+n-1)} \\
&= C_4 |\theta| \sum_{p\in\mathcal{N}} |a_p| \exp\Big\{\Big(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} - (\eta_0 - 3\eta_1)\Big)t\Big\} \cdot e^{-\frac{1}{1+\gamma}(\alpha\gamma-|p|(1+\gamma)b)(n-1)}.
\end{aligned}$$

(3.26) now follows since $\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} < \eta_0 - 3\eta_1$ and $\alpha\gamma > Kb(1+\gamma) \geq |p|b(1+\gamma)$.

Combining (3.25) with Steps 1-2, we get the result immediately. Notice that δ is independent of m and t. $\square$

Lemma 3.10. *For each $\theta \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist constants $C, \delta_1, \delta_2 > 0$ such that*

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\exp(i\theta \sum_{n=m}^\infty \widetilde{\gamma}_{t,n}) - 1\Big]\Big| \leq C(e^{-\delta_1 t} + e^{\delta_1 t} e^{-\delta_2 m}), \quad m \in \mathbb{N}, t \geq 0.$$

Proof. Fix $\theta \in \mathbb{R}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $\epsilon \in (0,\alpha)$ and $\gamma \in (0,\beta)$ with $\alpha\gamma > Kb(1+\gamma)$. Recall that $A_t(\epsilon) := \{\|X_t\| > e^{(\alpha-\epsilon)t}\}$. According to Proposition 2.11, there exist $C_1, \delta > 0$ such that for all $t \geq 0$ and $m \in \mathbb{N}$,

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\big(\exp(i\theta\sum_{n=m}^{\infty}\widetilde{\gamma}_{t,n}) - 1\big)\mathbf{1}_{A_t(\epsilon)^c}\Big] \leq 2\widetilde{P}_\mu(A_t(\epsilon)^c) \leq C_1 e^{-\delta t}. \tag{3.27}$$

According to Lemma 3.3, there exist $C_2, C_3 > 0$ such that for all $m \in \mathbb{N}, t \geq$,

$$\begin{aligned}
&\Big|\widetilde{\mathbb{P}}_\mu\Big[\big(\exp(i\theta\sum_{n=m}^{\infty}\widetilde{\gamma}_{t,n}) - 1\big)\mathbf{1}_{A_t(\epsilon)}\Big]\Big| \\
&\leq \frac{1}{\mathbb{P}_\mu(D^c)}|\theta|\sum_{n=m}^{\infty}\sum_{p\in\mathcal{N}}|a_p|\mathbb{P}_\mu\Big[\frac{|H^p_{t+n-1} - H^p_{t+n}|}{e^{-(\alpha-|p|b)t}\|X_t\|^{\frac{1}{1+\beta}}}\mathbf{1}_{A_t(\epsilon)}\Big] \\
&\leq \frac{1}{\mathbb{P}_\mu(D^c)}|\theta|\sum_{p\in\mathcal{N}}|a_p|e^{(\alpha-|p|b)t}e^{-\frac{\alpha-\epsilon}{1+\beta}t}\sum_{n=m}^{\infty}\|H^p_{t+n-1} - H^p_{t+n}\|_{\mathbb{P}_\mu;1+\gamma} \\
&\leq C_2|\theta|\sum_{p\in\mathcal{N}}|a_p|e^{(\alpha-|p|b)t}e^{-\frac{\alpha-\epsilon}{1+\beta}t}\sum_{n=m}^{\infty}e^{-\frac{1}{1+\gamma}(\alpha\gamma-|p|(1+\gamma)b)(t+n-1)} \\
&\leq C_3|\theta|\exp\Big\{\Big(\frac{\alpha}{1+\gamma} - \frac{\alpha}{1+\beta} + \frac{\epsilon}{1+\beta}\Big)t\Big\}e^{-\frac{1}{1+\gamma}(\alpha\gamma-K(1+\gamma)b)m}.
\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28), we complete the proof. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $\theta \in \mathbb{R}$. Let δ_0 be the constant δ in Lemma 3.9. Take $m = \lfloor e^{\frac{\delta_0}{2}t}\rfloor$. First note that for $t \geq 0$,

$$\text{LHS of (1.12)} = \sum_{n=1}^{\infty}\sum_{p\in\mathcal{N}}a_p\frac{H^p_{t+n-1} - H^p_{t+n}}{e^{-(\alpha-|p|b)t}\|X_t\|^{\frac{1}{1+\beta}}} = \sum_{n=1}^{m}\widetilde{\gamma}_{t,n} + \sum_{n=m+1}^{\infty}\widetilde{\gamma}_{t,n}. \tag{3.29}$$

According to Lemma 3.8, there exist $C_1, \delta_1 > 0$ such that for all $t \geq 0$,

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\exp(i\theta\sum_{n=1}^{m}\gamma_{t,n}) - \exp(\sum_{n=1}^{m}\bar{m}_n[\theta g])\Big]\Big| \leq C_1 e^{-\delta_1 t}. \tag{3.30}$$

According to Lemma 3.9, there exists $C_2 > 0$ such that for all $t \geq 0$,

$$\Big|\widetilde{\mathbb{P}}_\mu[e^{i\theta\sum_{n=1}^{m}\widetilde{\gamma}_{t,n}}] - \widetilde{\mathbb{P}}_\mu[e^{i\theta\sum_{n=1}^{m}\gamma_{t,n}}]\Big| \leq C_2 e^{-\frac{\delta_0}{2}t}. \tag{3.31}$$

According to Lemma 3.10, there exist $C_3, \delta_2, \delta_3 > 0$ such that for all $t \geq 0$,

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\exp(i\theta\sum_{n=m}^{\infty}\widetilde{\gamma}_{t,n}) - 1\Big]\Big| \leq C_3(e^{-\delta_2 t} + e^{\delta_2 t}e^{-\delta_3 e^{\frac{\delta_0}{2}t}}). \tag{3.32}$$

Also note that $\sum_{n=1}^{\infty}\bar{m}_n[\theta g] = \bar{m}[\theta g]$, see (2.9).

Combining (3.29),(3.30), (3.31) and (3.32), and letting $t \to \infty$, we get the desired result of Theorem 1.5. $\square$

Appendix A.

A.1. **Analytic facts.** In this subsection, we collect some useful analytic facts.

Lemma A.1. *For $z \in \mathbb{C}_+$, we have*

$$\Big| e^{-z} - \sum_{k=0}^{n} \frac{(-z)^k}{k!} \Big| \le \frac{|z|^{n+1}}{(n+1)!} \wedge \frac{2|z|^n}{n!}, \quad n \in \mathbb{Z}_+. \tag{A.1}$$

Proof. Notice that $|e^{-z}| = e^{-\operatorname{Re} z} \le 1$. Therefore,

$$|e^{-z} - 1| = \Big| \int_0^1 e^{-\theta z} z d\theta \Big| \le |z|.$$

Also, notice that $|e^{-z} - 1| \le |e^{-z}| + 1 \le 2$. Thus (A.1) is true when $n = 0$. Now, suppose that (A.1) is true when $n = m$ for some $m \in \mathbb{Z}_+$. Then

$$\begin{aligned} \Big| e^{-z} - \sum_{k=0}^{m+1} \frac{(-z)^k}{k!} \Big| &= \Big| \int_0^1 \Big(e^{-\theta z} - \sum_{k=0}^{m} \frac{(-\theta z)^k}{k!} \Big) z d\theta \Big| \\ &\le \Big(\int_0^1 \frac{|\theta z|^{m+1}}{(m+1)!} |z| d\theta \Big) \wedge \Big(\int_0^1 \frac{2|\theta z|^m}{m!} |z| d\theta \Big) = \frac{|z|^{m+2}}{(m+2)!} \wedge \frac{2|z|^{m+1}}{(m+1)!}, \end{aligned}$$

which says that (A.1) is true for $n = m + 1$. $\square$

Lemma A.2. *Suppose that π is a measure on $(0,\infty)$ with $\int_{(0,\infty)} (y \wedge y^2) \pi(dy) < \infty$. Then the functions*

$$h(z) = \int_{(0,\infty)} (e^{-zy} - 1 + zy) \pi(dy), \quad z \in \mathbb{C}_+$$

and

$$h'(z) = \int_{(0,\infty)} (1 - e^{-zy}) y \pi(dy), \quad z \in \mathbb{C}_+$$

are well defined, continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. Moreover,

$$\frac{h(z) - h(z_0)}{z - z_0} \xrightarrow[\mathbb{C}_+ \ni z \to z_0]{} h'(z_0), \quad z_0 \in \mathbb{C}_+.$$

Proof. It follows from Lemma A.1 that h and h' are well defined on $\mathbb{C}_+$. According to [40, Theorems 3.2. & Proposition 3.6], h' is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$.

It follows from Lemma A.1 that, for each $z_0 \in \mathbb{C}_+$, there exists $C > 0$ such that for $z \in \mathbb{C}_+$ close enough to z_0 and any $y > 0$,

$$\Big| \frac{e^{-zy} - e^{-z_0 y} + (z - z_0) y}{z - z_0} \Big| = \frac{1}{|z - z_0|} \Big| \int_0^1 \big(-y e^{-(\theta z + (1-\theta) z_0) y} + y \big) (z - z_0) d\theta \Big|$$

$$\leq y \int_0^1 |1 - e^{-(\theta z + (1-\theta) z_0) y}| d\theta \leq (2y) \wedge \Big(y^2 \int_0^1 |\theta z + (1-\theta) z_0| d\theta \Big) \leq C(y \wedge y^2).$$

Using this and the dominated convergence theorem, we have

$$\frac{h(z) - h(z_0)}{z - z_0} = \int_{(0,\infty)} \frac{e^{-zy} + zy - (e^{-z_0 y} + z_0 y)}{z - z_0} \pi(dy)$$

$$\xrightarrow[\mathbb{C}_+ \ni z \to z_0]{} \int_{(0,\infty)} (1 - e^{-z_0 y}) y \pi(dy) = h'(z_0),$$

which says that h is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. □

For each $z \in \mathbb{C} \setminus (-\infty, 0]$, we define $\log z := \log |z| + i \arg z$ where $\arg z \in (-\pi, \pi)$ is uniquely determined by $z = |z| e^{i \arg z}$. For all $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\gamma \in \mathbb{C}$, we define $z^\gamma := e^{\gamma \log z}$. Then it is known, see [41, Theorem 6.1] for example, that $z \mapsto \log z$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Therefore, for each $\gamma \in \mathbb{C}$, $z \mapsto z^\gamma$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. (We use the convention that $0^\gamma := \mathbf{1}_{\gamma = 0}$.) Using the definition above we can easily show that $(z_1 z_0)^\gamma = z_1^\gamma z_0^\gamma$ provided $\arg(z_1 z_0) = \arg(z_1) + \arg(z_0)$.

Recall that the Gamma function Γ is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

It is known, see, for instance, [41, Theorem 6.1.3] and the remark following it, that the function Γ has an unique analytic extension in $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ and that

$$\Gamma(z+1) = z \Gamma(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Using this recursively, one gets that

$$\Gamma(x) := \int_0^\infty t^{x-1} \Big(e^{-t} - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \Big) dt, \quad -n < x < -n+1, n \in \mathbb{N}.$$

Fix a $\beta \in (0,1)$. Using the uniqueness of holomorphic extension and Lemma A.2, we get that

$$z^\beta = \int_0^\infty (e^{-zy} - 1) \frac{dy}{\Gamma(-\beta) y^{1+\beta}}, \quad z \in \mathbb{C}_+,$$

by showing that the both sides

- are extension of the real function $x \mapsto x^\beta$ defined on $[0, \infty)$;
- are holomorphic on $\mathbb{C}_+^0$;
- are continuous on $\mathbb{C}_+$.

Similarly, we get that

$$z^{1+\beta} = \int_0^\infty (e^{-zy} - 1 + zy) \frac{dy}{\Gamma(-1-\beta) y^{2+\beta}}, \quad z \in \mathbb{C}_+. \tag{A.2}$$

Lemma A.2 also says that the derivative of $z^{1+\beta}$ is $(1+\beta) z^\beta$ on $\mathbb{C}_+^0$.

Lemma A.3. *For all $z_0, z_1 \in \mathbb{C}_+$, we have*

$$|z_0^{1+\beta} - z_1^{1+\beta}| \leq (1+\beta)(|z_0|^\beta + |z_1|^\beta)|z_0 - z_1|.$$

Proof. Since $z^{1+\beta}$ is continuous on $\mathbb{C}_+$, we only need to prove the lemma assuming $z_0, z_1 \in \mathbb{C}_+^0$. Notice that

$$|z^\beta| = |e^{\beta \log |z| + i\beta \arg z}| = e^{\beta \log |z|} = |z|^\beta, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Define a path $\gamma : [0,1] \to \mathbb{C}_+^0$ such that

$$\gamma(\theta) = z_0(1-\theta) + \theta z_1, \quad \theta \in [0,1].$$

Then, we have

$$\begin{aligned} |z_0^{1+\beta} - z_1^{1+\beta}| &\leq (1+\beta) \int_0^1 |\gamma(\theta)^\beta| \cdot |\gamma'(\theta)| d\theta \leq (1+\beta) \sup_{\theta \in [0,1]} |\gamma(\theta)|^\beta \cdot |z_1 - z_0| \\ &\leq (1+\beta)(|z_1|^\beta + |z_0|^\beta)|z_1 - z_0|. \end{aligned}$$ □

Suppose that $\varphi(\theta)$ is a continuous function from $\mathbb{R}$ into $\mathbb{C}$ such that $\varphi(0) = 1$ and $\varphi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. Then according to [39, Lemma 7.6], there is a unique continuous function $f(\theta)$ from $\mathbb{R}$ into $\mathbb{C}$ such that $f(0) = 0$ and $e^{f(\theta)} = \varphi(\theta)$. Such a function f is called the distinguished logarithm of the function φ and is denoted as $\operatorname{Log} \varphi(\theta)$. In particular, when φ is the characteristic function of an infinitely divisible random variable Y, $\operatorname{Log} \varphi(\theta)$ is called the Lévy exponent of Y. This distinguished logarithm should not be confused with the log function defined on $\mathbb{C} \setminus (-\infty, 0]$. See the paragraph immediately after [39, Lemma 7.6].

A.2. **Feynman-Kac formula with complex valued functions.** In this subsection we give a version of the Feynman-Kac formula with complex valued functions. Suppose that $\{(\xi_t)_{t \in [r,\infty)}; (\Pi_{r,x})_{r \in [0,\infty), x \in E}\}$ is a (possibly non-homogeneous) Hunt process in a locally compact separable metric space E. We write

$$H_{(s,t)}^{(h)} := \exp\Big\{ \int_s^t h(u, \xi_u) du \Big\}, \quad 0 \leq s \leq t, h \in \mathcal{B}_b([0,t] \times E, \mathbb{C}).$$

Lemma A.4. *Let $t \geq 0$. Suppose that $\rho_1, \rho_2 \in \mathcal{B}_b([0,t] \times E, \mathbb{C})$ and $f \in \mathcal{B}_b(E, \mathbb{C})$. Then*

$$g(r,x) := \Pi_{r,x}[H_{(r,t)}^{(\rho_1+\rho_2)} f(\xi_t)], \quad r \in [0,t], x \in E, \tag{A.3}$$

is the unique locally bounded solution to the equation

$$g(r,x) = \Pi_{r,x}[H_{(r,t)}^{(\rho_1)} f(\xi_t)] + \Pi_{r,x}\Big[\int_r^t H_{(r,s)}^{(\rho_1)} \rho_2(s, \xi_s) g(s, \xi_s) \ ds \Big], \quad r \in [0,t], x \in E.$$

Proof. The proof is similar to that of [11, Lemma A.1.5]. We include it here for the sake of completeness. We first verify that (A.3) is a solution. Notice that

$$\Pi_{r,x}\Big[\int_r^t |H_{(r,t)}^{(\rho_1)} \rho_2(s, \xi_s) H_{(s,t)}^{(\rho_2)} f(\xi_t)| \ ds \Big] \leq \int_r^t e^{(t-r)\|\rho_1\|_\infty} e^{(t-s)\|\rho_2\|_\infty} \|\rho_2\|_\infty \|f\|_\infty \ ds < \infty.$$

Also notice that

$$\frac{\partial}{\partial s} H^{(\rho_2)}_{(s,t)} = -H^{(\rho_2)}_{(s,t)} \rho_2(s,\xi_s), \quad s \in (0,t).$$

Therefore, from the Markov property of ξ and Fubini's theorem we get that

$$\begin{aligned}
&\Pi_{r,x}\Big[\int_r^t H^{(\rho_1)}_{(r,s)}\ (\rho_2 g)(s,\xi_s)\ ds\Big] = \Pi_{r,x}\Big[\int_r^t H^{(\rho_1)}_{(r,s)} \rho_2(s,\xi_s)\Pi_{s,\xi_s}[H^{(\rho_1+\rho_2)}_{(s,t)} f(\xi_t)]\ ds\Big] \\
&= \Pi_{r,x}\Big[\int_r^t H^{(\rho_1)}_{(r,t)} \rho_2(s,\xi_s) H^{(\rho_2)}_{(s,t)} f(\xi_t)\ ds\Big] = \Pi_{r,x}[H^{(\rho_1)}_{(r,t)} f(\xi_t)(H^{(\rho_2)}_{(r,t)} - 1)] \\
&= g(r,x) - \Pi_{r,x}[H^{(\rho_2)}_{(r,t)} f(\xi_t)].
\end{aligned}$$

For uniqueness, suppose $\widetilde{g}$ is another solution. Put $h(r) = \sup_{x\in E} |g(r,x) - \widetilde{g}(r,x)|$. Then

$$h(r) \le e^{t\|\rho_1\|_\infty} \|\rho_2\|_\infty \int_r^t h(s) ds, \quad r \le t.$$

Applying Gronwall's inequality, we get that $h(r) = 0$ for $r \in [0,t]$. $\square$

A.3. **Superprocesses.** In this subsection, we will give the definition of a general superprocess. Let E be locally compact separable metric space. Denote by $\mathcal{M}(E)$ the collection of all the finite measures on E equipped with the topology of weak convergence. For each function $F(x,z)$ on $E \times \mathbb{R}_+$ and each $\mathbb{R}_+$-valued function f on E, we use the following convention in this subsection:

$$F(x,f) := F(x, f(x)), \quad x \in E.$$

A process $X = \{(X_t)_{t\ge 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is said to be a (ξ,ψ)-superprocess if

- the spatial motion $\xi = \{(\xi_t)_{t\ge0}; (\Pi_x)_{x\in E}\}$ is an E-valued Hunt process with its lifetime denoted by ζ;
- the branching mechanism $\psi : E \times [0,\infty) \to \mathbb{R}$ is given by

$$\psi(x,z) = -\rho_1(x) z + \rho_2(x) z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy). \tag{A.4}$$

  where $\rho_1 \in \mathcal{B}_b(E)$, $\rho_2 \in \mathcal{B}_b(E,\mathbb{R}_+)$ and $\pi(x,dy)$ is a kernel from E to $(0,\infty)$ such that $\sup_{x\in E} \int_{(0,\infty)} (y\wedge y^2)\pi(x,dy) < \infty$;
- $X = \{(X_t)_{t\ge0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is an $\mathcal{M}(E)$-valued Hunt process with transition probability determined by

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t\ge 0, \mu \in \mathcal{M}(E), f \in \mathcal{B}_b^+(E),$$

  where for each $f \in \mathcal{B}_b(E)$, the function $(t,x) \mapsto V_t f(x)$ on $[0,\infty)\times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \psi(\xi_s, V_{t-s} f) ds\Big] = \Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}], \quad t \ge 0, x \in E.$$

We refer our readers to [26] for more discussions about the definition and the existence of superprocesses. To avoid triviality, we assume that $\psi(x,z)$ is not identically equal to $-\rho_1(x)z$.

Notice that the branching mechanism ψ can be extended into a map from $E\times\mathbb{C}_+$ to $\mathbb{C}$ using (A.4). Define

$$\psi'(x,z) := -\rho_1(x) + 2\rho_2(x)z + \int_{(0,\infty)} (1-e^{-zy})y\pi(x,dy), \quad x\in E, z\in\mathbb{C}_+.$$

Then according to Lemma A.2, for each $x\in E$, $z\mapsto\psi(x,z)$ is a holomorphic function on $\mathbb{C}_+^0$ with derivative $z\mapsto\psi'(x,z)$. Define $\psi_0(x,z):=\psi(x,z)+\rho_1(x)z$ and $\psi_0'(x,z):=\psi'(x,z)+\rho_1(x)$.

Denote by $\mathbb{W}$ the space of $\mathcal{M}(E)$-valued càdlàg paths with its canonical path denoted by $(W_t)_{t\geq 0}$. We say X is *non-persistent* if $\mathbf{P}_{\delta_x}(\|X_t\|=0)>0$ for all $x\in E$ and $t>0$. Suppose that $(X_t)_{t\geq 0}$ is non-persistent, then according to [26, Section 8.4], there is a unique family of measures $(\mathbb{N}_x)_{x\in E}$ on $\mathbb{W}$ such that

- $\mathbb{N}_x(\forall t>0, \|W_t\|=0)=0$;
- $\mathbb{N}_x(\|W_0\|\neq 0)=0$;
- for any $\mu\in\mathcal{M}(E)$, if $\mathcal{N}$ is a Poisson random measure defined on some probability space with intensity $\mathbb{N}_\mu(\cdot):=\int_E \mathbb{N}_x(\cdot)\mu(dx)$, then the superprocess $\{X;\mathbf{P}_\mu\}$ can be realized by $\widetilde{X}_0:=\mu$ and $\widetilde{X}_t(\cdot):=\mathcal{N}[W_t(\cdot)]$ for each $t>0$.

We refer to $(\mathbb{N}_x)_{x\in E}$ as the *Kuznetsov measures* of X.

A.4. **Semigroups for superprocesses.** Let X be a non-persistent superprocess with its Kuznetsov measure denoted by $(\mathbb{N}_x)_{x\in E}$. We define the mean semigroup

$$P_t^{\rho_1}f(x) := \Pi_x[e^{\int_0^t \rho_1(\xi_s)ds} f(\xi_t)\mathbf{1}_{t<\zeta}], \quad t\geq 0, x\in E, f\in\mathcal{B}_b(E,\mathbb{R}_+).$$

It is known from [26, Proposition 2.27] and [25, Theorem 2.7] that for all $t>0$, $\mu\in\mathcal{M}(E)$ and $f\in\mathcal{B}_b(E,\mathbb{R}_+)$,

$$\mathbb{N}_\mu[\langle W_t,f\rangle] = \mathbf{P}_\mu[\langle X_t,f\rangle] = \mu(P_t^{\rho_1}f). \tag{A.5}$$

Define

$$L_1(\xi) := \{f\in\mathcal{B}(E): \forall x\in E, t\geq 0, \quad \Pi_x[|f(\xi_t)|]<\infty\},$$
$$L_2(\xi) := \{f\in\mathcal{B}(E): |f|^2\in L_1(\xi)\}.$$

Using monotonicity and linearity, we get from (A.5) that

$$\mathbb{N}_x[\langle W_t,f\rangle] = \mathbf{P}_{\delta_x}[\langle f,X_t\rangle] = P_t^{\rho_1}f(x)\in\mathbb{R}, \quad f\in L_1(\xi), t>0, x\in E.$$

This says that the random variable $\langle X_t,f\rangle$ is well defined under probability $\mathbf{P}_{\delta_x}$ provided $f\in L_1(\xi)$. By the branching property of the superprocess, $\langle X_t,f\rangle$ is an infinitely divisible random variable. Therefore, we can write

$$U_t(\theta f)(x) := \operatorname{Log}\mathbf{P}_{\delta_x}[e^{i\theta\langle X_t,f\rangle}], \quad t\geq 0, f\in L_1(\xi), \theta\in\mathbb{R}, x\in E,$$

as its characteristic exponent. According to Campbell's formula, see [25, Theorem 2.7] for example, we have

$$\mathbf{P}_{\delta_x}[e^{i\theta\langle X_t,f\rangle}] = \exp(\mathbb{N}_x[e^{i\theta\langle W_t,f\rangle} - 1]), \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E.$$

Noticing that $\theta \mapsto \mathbb{N}_x[e^{i\theta W_t(f)} - 1]$ is a continuous function on $\mathbb{R}$ and that $\mathbb{N}_x[e^{i\theta\langle W_t,f\rangle} - 1] = 0$ if $\theta = 0$, according to [39, Lemma 7.6], we have

$$U_t(\theta f)(x) = \mathbb{N}_x[e^{i\langle W_t,\theta f\rangle} - 1], \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E. \tag{A.6}$$

Lemma A.5. *There exists a constant $C \geq 0$ such that for all $f \in L_1(\xi), x \in E$ and $t \geq 0$, we have*

$$\big|\psi\big(x, -U_t f\big)\big| \leq C P_t^{\rho_1}|f|(x) + C(P_t^{\rho_1}|f|(x))^2. \tag{A.7}$$

Proof. Noticing that

$$e^{\operatorname{Re} U_t f(x)} = |e^{U_t f(x)}| = |\mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]| \leq 1,$$

we have

$$\operatorname{Re} U_t f(x) \leq 0. \tag{A.8}$$

Therefore, we can speak of $\psi(x, -U_t f)$ since $z \mapsto \psi(x, z)$ is well defined on $\mathbb{C}_+$. According to Lemma A.1, we have that

$$|U_t f(x)| \leq \mathbb{N}_x[|e^{i\langle W_t,f\rangle} - 1|] \leq \mathbb{N}_x[|i\langle W_t, f\rangle|] \leq (P_t^{\rho_1}|f|)(x). \tag{A.9}$$

Notice that, for any compact $K \subset \mathbb{R}$,

$$\mathbb{N}_x\Big[\sup_{\theta\in K}\Big|\frac{\partial}{\partial\theta}(e^{i\theta\langle W_t,f\rangle} - 1)\Big|\Big] \leq \mathbb{N}_x[|\langle W_t, f\rangle|]\sup_{\theta\in K}|\theta| \leq (P_t^{\rho_1}|f|)(x)\sup_{\theta\in K}|\theta| < \infty.$$

Therefore, according to [10, Theorem A.5.2] and (A.6), $U_t(\theta f)(x)$ is differentiable in $\theta \in \mathbb{R}$ with

$$\frac{\partial}{\partial\theta}U_t(\theta f)(x) = i\mathbb{N}_x[\langle W_t, f\rangle e^{i\theta\langle W_t,f\rangle}], \quad \theta \in \mathbb{R}.$$

Moreover, from the above, it is clear that

$$\sup_{\theta\in\mathbb{R}}\Big|\frac{\partial}{\partial\theta}U_t(\theta f)(x)\Big| \leq (P_t^{\rho_1}|f|)(x). \tag{A.10}$$

It follows from the dominated convergence theorem that $(\partial/\partial\theta)U_t(\theta f)(x)$ is continuous in θ. In other words, $\theta \mapsto -U_t(\theta f)(x)$ is a C^1 map from $\mathbb{R}$ to $\mathbb{C}_+$. Thus,

$$\psi(x, -U_t f) = -\int_0^1 \psi'\big(x, -U_t(\theta f)\big)\frac{\partial}{\partial\theta}U_t(\theta f)(x)\, d\theta. \tag{A.11}$$

Notice that

(A.12)

$$|\psi'(x, -U_t f)|$$

$$= \Big| -\rho_1(x) - 2\rho_2(x)U_t f(x) + \int_{(0,\infty)} y(1 - e^{yU_t f(x)})\pi(x, dy)\Big|$$

$$= \Big| -\rho_1(x) - 2\rho_2(x)\mathbb{N}_x[e^{i\langle W_t,f\rangle} - 1] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[1 - e^{i\langle X_t,f\rangle}]\pi(x,dy)\Big|$$

$$\leq \|\rho_1\|_\infty + 2\rho_2(x)\mathbb{N}_x[\langle W_t,|f|\rangle] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[2 \wedge \langle X_t,|f|\rangle]\pi(x,dy)$$

$$\leq \|\rho_1\|_\infty + 2\|\rho_2\|_\infty P_t^{\rho_1}|f|(x) + \Big(\sup_{x\in E}\int_{(0,1]} y^2\pi(x,dy)\Big)\ P_t^{\rho_1}|f|(x) + 2\sup_{x\in E}\int_{(1,\infty)} y\pi(x,dy)$$

$$=: C_1 + C_2(P_t^{\rho_1}|f|)(x),$$

where C_1, C_2 are constants independent on f, x and t. Now, combining (A.11), (A.10) and (A.12), we get the desired result. □

This lemma also says that if $f \in L^2(\xi)$ then

$$\Pi_x\Big[\int_0^t \psi(\xi_s, -U_{t-s}f)ds\Big] \in \mathbb{C}, \quad x \in E, t \geq 0.$$

is well defined. In fact, using Jensen's inequality and the Markov property, we have

$$\begin{aligned}
\text{(A.13)}\qquad & \Pi_x\Big[\int_0^t \big|\psi\big(\xi_s, -U_{t-s}f\big)\big|ds\Big] \\
&\leq \Pi_x\Big[\int_0^t \big(C_1 P_{t-s}^{\rho_1}|f|(\xi_s) + C_2 P_{t-s}^{\rho_1}|f|(\xi_s)^2\big)ds\Big] \\
&\leq \int_0^t \big(C_1 e^{t\|\rho_1\|}\Pi_x\big[\Pi_{\xi_s}[|f(\xi_{t-s})|]\big] + C_2 e^{2t\|\rho_1\|}\Pi_x\big[\Pi_{\xi_s}[|f(\xi_{t-s})|]^2\big]\big)\ ds \\
&\leq \int_0^t (C_1 e^{t\|\rho_1\|}\Pi_x[|f(\xi_t)|] + C_2 e^{2t\|\rho_1\|}\Pi_x[|f(\xi_t)|^2])\ ds < \infty.
\end{aligned}$$

A.5. **A complex-valued non-linear integral equation.** Let X be a non-persistent superprocess. In this subsection, we will prove the following:

Proposition A.6. *If $f \in L_2(\xi)$, then for all $t \geq 0$ and $x \in E$,*

$$U_t f(x) - \Pi_x\Big[\int_0^t \psi\big(\xi_s, -U_{t-s}f\big)ds\Big] = i\Pi_x[f(\xi_t)] \tag{A.14}$$

and

$$U_t f(x) - \int_0^t P_{t-s}^{\rho_1}\psi_0\big(\cdot, -U_s f\big)(x)\ ds = iP_t^{\rho_1} f(x). \tag{A.15}$$

To prove this, we will need the generalized spine decomposition theorem from [32]. Let $f \in \mathcal{B}_b(E, \mathbb{R}_+)$, $T > 0$ and $x \in E$. Suppose that $\mathbf{P}_{\delta_x}[\langle X_T, f\rangle] = \mathbb{N}_x[\langle W_T, f\rangle] = P_T^{\rho_1} f(x) \in$

$(0,\infty)$, then we can define the following probability transforms:

$$d\mathbf{P}_{\delta_x}^{\langle X_T,f\rangle} := \frac{\langle X_T,f\rangle}{P_T^{\rho_1}f(x)}d\mathbf{P}_{\delta_x}; \quad d\mathbb{N}_x^{\langle W_T,f\rangle} := \frac{\langle W_T,f\rangle}{P_T^{\rho_1}f(x)}d\mathbb{N}_x.$$

Following the definition in [32], we say that $\{\xi,\mathbf{n};\mathbf{Q}_x^{(f,T)}\}$ is a spine representation of $\mathbb{N}_x^{\langle W_T,f\rangle}$ if

- the spine process $\{(\xi_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$ is a copy of $\{(\xi_t)_{0\le t\le T};\Pi_x^{(f,T)}\}$, where
$$d\Pi_x^{(f,T)} := \frac{f(\xi_T)e^{\int_0^T\rho_1(\xi_s)ds}}{P_T^{\rho_1}f(x)}d\Pi_x;$$
- given $\{(\xi_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$, the immigration measure
$$\{\mathbf{n}(\xi,ds,dw);\mathbf{Q}_x^{(f,T)}[\cdot|(\xi_t)_{0\le t\le T}]\}$$
is a Poisson random measure on $[0,T]\times\mathbb{W}$ with intensity
$$\mathbf{m}(\xi,ds,dw) := 2\rho_2(\xi_s)ds\cdot\mathbb{N}_{\xi_s}(dw) + ds\cdot\int_{y\in(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}(X\in dw)\pi(\xi_s,dy); \tag{A.16}$$
- $\{(Y_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$ is an $\mathcal{M}(E)$-valued process defined by
$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s}\mathbf{n}(\xi,ds,dw), \quad 0\le t\le T.$$

According to the spine decomposition theorem in [32], we have that

$$\{(X_s)_{s\ge0};\mathbf{P}_{\delta_x}^{\langle X_T,f\rangle}\} \overset{f.d.d.}{=} \{(X_s+W_s)_{s\ge0};\mathbf{P}_{\delta_x}\otimes\mathbb{N}_x^{\langle W_T,f\rangle}\} \tag{A.17}$$

and

$$\{(W_s)_{0\le s\le T};\mathbb{N}_x^{\langle W_T,f\rangle}\} \overset{f.d.d.}{=} \{(Y_s)_{s\ge0};\mathbf{Q}_x^{(f,T)}\}. \tag{A.18}$$

Proof of Proposition A.6. Assume that $f\in\mathcal{B}_b(E)$. Fix $t>0, r\in[0,t), x\in E$ and a strictly positive $g\in\mathcal{B}_b(E)$. Denote by $\{\xi,\mathbf{n};\mathbf{Q}_x^{(g,t)}\}$ the spine representation of $\mathbb{N}_x^{\langle W_t,g\rangle}$. Conditioned on $\{\xi;\mathbf{Q}_x^{(g,t)}\}$, denote by $\mathbf{m}(\xi,ds,dw)$ the conditional intensity of $\mathbf{n}$ in (A.16). Denote by $\Pi_{r,x}$ the probability of Hunt process $\{\xi;\Pi\}$ initiated at time r and position x. From Lemma A.1, we have $\mathbf{Q}_x^{(g,t)}$-almost surely

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{W}}|e^{i\langle w_{t-s},f\rangle}-1|\mathbf{m}(\xi,ds,dw) \le \int_{[0,t]\times\mathbb{W}}\big(|\langle w_{t-s},f\rangle|\wedge 2\big)\mathbf{m}(\xi,ds,dw)\\
&\le \int_0^t\Big(2\rho_2(\xi_s)\mathbb{N}_{\xi_s}\big(\langle W_{t-s},|f|\rangle\big) + \int_{(0,1]}y\mathbf{P}_{y\delta_{\xi_s}}[\langle X_{t-s},|f|\rangle]\pi(\xi_s,dy)\\
&\qquad + 2\int_{(1,\infty)}y\pi(\xi_s,dy)\Big)ds\\
&\le \int_0^t(P_{t-s}^{\rho_1}|f|)(\xi_s)\Big(2\rho_2(\xi_s)+\int_{(0,1]}y^2\pi(\xi_s,dy)\Big)ds + 2t\sup_{x\in E}\int_{(1,\infty)}y\pi(x,dy)
\end{aligned}$$

$$\leq \Big(2\|\rho_2\|_\infty + \sup_{x\in E}\int_{(0,1]} y^2\pi(x,dy)\Big) te^{t\|\rho_1\|_\infty}\|f\|_\infty + 2t\sup_{x\in E}\int_{(1,\infty)} y\pi(x,dy) < \infty.$$

Using this, Fubini's theorem, (A.6) and (A.8) we have $\mathbf{Q}_x^{(g,t)}$-almost surely,

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{N}} (e^{i\langle w_{t-s},f\rangle}-1)\mathbf{m}(\xi,ds,dw)\\
&= \int_0^t \Big(2\rho_2(\xi_s)\mathbb{N}_{\xi_s}(e^{i\langle W_{t-s},f\rangle}-1) + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}[e^{i\langle X_{t-s},f\rangle}-1]\pi(\xi_s,dy)\Big)ds\\
&= \int_0^t \Big(2\rho_2(\xi_s)U_{t-s}f(\xi_s) + \int_{(0,\infty)} y(e^{yU_{t-s}f(\xi_s)}-1)\pi(\xi_s,dy)\Big)ds\\
&= -\int_0^t \psi_0'\big(\xi_s,-U_{t-s}f\big)ds.
\end{aligned}$$

Therefore, according to (A.18), Campbell's formula and above, we have that

$$\begin{aligned}
\mathbb{N}_x^{\langle W_t,g\rangle}[e^{i\langle W_t,f\rangle}] &= \mathbf{Q}_x^{(g,t)}\Big[\exp\Big\{\int_{[0,t]\times\mathbb{N}}(e^{i\langle w_{t-s},f\rangle}-1)\mathbf{m}(\xi,ds,dw)\Big\}\Big]\\
&= \Pi_x^{(g,t)}[e^{-\int_0^t \psi_0'(\xi_s,-U_{t-s}f)ds}]\\
&= \frac{1}{P_t^{\rho_1}g(x)}\Pi_x[g(\xi_t)e^{-\int_0^t\psi'(\xi_s,-U_{t-s}f)ds}].
\end{aligned}\tag{A.19}$$

Let $\epsilon>0$. Define $f^+ = (f\vee 0)+\epsilon$ and $f^- = (-f)\vee 0+\epsilon$, then $f^\pm$ are strictly positive and $f = f^+ - f^-$. According to (A.17), we have that

$$\frac{\mathbf{P}_{\delta_x}[\langle X_t,f^\pm\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[\langle X_t,f^\pm\rangle]} = \mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]\mathbb{N}_x^{\langle W_t,f^\pm\rangle}[e^{i\langle X_t,f\rangle}].$$

Using (A.19) and the above, we have

$$\begin{aligned}
\frac{\mathbf{P}_{\delta_x}[\langle X_t,f\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]} &= \mathbf{P}_{\delta_x}[\langle X_t,f^+\rangle]\mathbb{N}_x^{\langle W_t,f^+\rangle}[e^{i\langle X_t,f\rangle}] - \mathbf{P}_{\delta_x}[\langle X_t,f^-\rangle]\mathbb{N}_x^{\langle W_t,f^-\rangle}[e^{i\langle X_t,f\rangle}]\\
&= \Pi_x[f(\xi_t)e^{-\int_0^t\psi'(\xi_s,-U_{t-s}f)ds}].
\end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial\theta}U_t(\theta f)(x) = \frac{\mathbf{P}_{\delta_x}[i\langle X_t,f\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]} = \Pi_x[if(\xi_t)e^{-\int_0^t\psi'(\xi_s,-U_{t-s}(\theta f))ds}].$$

Since $\{(\xi_{r+t})_{t\geq 0};\Pi_{r,x}\} \overset{d}{=} \{(\xi_t)_{t\geq0};\Pi_x\}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) = \Pi_x[if(\xi_{t-r})e^{-\int_0^{t-r}\psi'(\xi_s,-U_{t-r-s}(\theta f))ds}]\\
&= \Pi_{r,x}[if(\xi_t)e^{-\int_0^{t-r}\psi'(\xi_{r+s},-U_{t-r-s}(\theta f))ds}] = \Pi_{r,x}[if(\xi_t)e^{-\int_r^t\psi'(\xi_s,-U_{t-s}(\theta f))ds}].
\end{aligned}$$

From (A.12), we know that for each $\theta \in \mathbb{R}$, $(t,x) \mapsto |\psi'(x, -U_t f(x))|$ is locally bounded (i.e. bounded on $[0,T] \times E$ for each $T \geq 0$). Therefore, we can apply Lemma A.4 and get that

$$\frac{\partial}{\partial\theta} U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t \psi'\big(\xi_s, -U_{t-s}(\theta f)\big) \frac{\partial}{\partial\theta} U_{t-s}(\theta f)(\xi_s)\ ds\Big] = \Pi_{r,x}[if(\xi_t)]$$

and

$$\frac{\partial}{\partial\theta} U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \rho_1(\xi_u)du} \psi_0'\big(\xi_s, -U_{t-s}(\theta f)\big) \frac{\partial}{\partial\theta} U_{t-s}(\theta f)(\xi_s)\ ds\Big]$$
$$= \Pi_{r,x}[ie^{\int_r^t \rho_1(\xi_s)ds} f(\xi_t)].$$

Integrating the two displays above with respect to θ on [0,1], using (A.11), (A.12), (A.10) and Fubini's theorem, we get

$$U_{t-r} f(x) - \Pi_{r,x}\Big[\int_r^t \psi\big(\xi_s, -U_{t-s} f\big)\ ds\Big] = i\theta \Pi_{r,x}[f(\xi_t)]$$

and

$$U_{t-r} f(x) - \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \rho_1(\xi_u)du} \psi_0\big(\xi_s, -U_{t-s} f\big)\ ds\Big] = i\Pi_{r,x}[e^{\int_r^t \rho_1(\xi_u)du} f(\xi_t)].$$

Taking $r = 0$, we get that (A.14) and (A.15) are true if $f \in \mathcal{B}_b(E)$.

The rest of the proof is to evaluate (A.14) and (A.15) for all $f \in L_2(\xi)$. We only do this for (A.14) since the argument for (A.15) is similar. Let $n \in \mathbb{N}$. Writing $f_n := (f^+ \wedge n) - (f^- \wedge n)$, then $f_n \xrightarrow[n\to\infty]{} f$ pointwise. From what we have proved, we have

$$U_t f_n(x) - \Pi_x\Big[\int_0^t \psi\big(\xi_s, -U_{t-s} f_n\big)\ ds\Big] = i\Pi_x[f_n(\xi_t)]. \tag{A.20}$$

Notice the following:

- It is clear that $\Pi_x[f_n(\xi_t)] \xrightarrow[n\to\infty]{} \Pi_x[f(\xi_t)]$.
- $U_t f_n(x) \xrightarrow[n\to\infty]{} U_t f(x)$ due to (A.6), the dominated convergence theorem and the fact that
$$|e^{iW_t(f_n)} - 1| \leq \langle W_t, |f|\rangle; \quad \mathbb{N}_x[\langle W_t, |f|\rangle] = (P_t^{\rho_1}|f|)(x) < \infty.$$
- $\Pi_x[\int_0^t \psi(\xi_s, -U_{t-s} f_n) ds] \xrightarrow[n\to\infty]{} \Pi_x[\int_0^t \psi(\xi_s, -U_{t-s} f) ds]$ due to the dominated convergence theorem, (A.13) and the fact (see (A.7)) that
$$\big|\psi(\xi_s, -U_{t-s} f_n)\big| \leq C_1 P_{t-s}^{\rho_1}|f|(\xi_s) + C_2 P_{t-s}^{\rho_1}|f|(\xi_s)^2.$$

Using the above arguments, letting $n \to \infty$ in (A.20), we get the desired result. □

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