Stable Central Limit Theorems for Super Ornstein-Uhlenbeck Processes

Zhenyao Sun[1]

Joint work with **Yan-Xia Ren**[1], **Renming Song**[2] and **Jianjie Zhao**[1]

[1]Peking University

[2]University of Illinois at Urbana-Champaign

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Model/Settings

- E: a locally compact separable metric space.
- $(\xi_t)_{t\geq 0}$: an E-valued Hunt process with semigroup $(P_t)_{t\geq 0}$.
- ψ: a function from $[0,\infty)$ to $\mathbb{R}$ given by

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(dy), \quad z \geq 0,$$

  where $\alpha > 0$, $\rho \geq 0$ and π is a measure on $(0,\infty)$ with

$$\int_{(0,\infty)} (y \wedge y^2)\pi(dy) < \infty.$$

- We call α the branching rate.

Model/Settings

- $\mathcal{M}(E)$: the space of all finite Borel measures on E equipped with the weak topology.
- $\langle f, \mu \rangle$: the integration of function f and measure μ whenever it make senses.
- Say a function f on $\mathbb{R}_+ \times E$ is *locally bounded* if

$$\sup_{s\in[0,t],x\in E} |f(s,x)| < \infty, \quad t \in \mathbb{R}_+.$$

Model/Superprocesses

Definition (Superprocess)

Say an $\mathcal{M}(E)$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is a (ξ,ψ)-superprocess if for each bounded non-negative measurable function f on E, we have

$$\mathbb{P}_\mu[e^{-\langle f, X_t\rangle}] = e^{-\langle V_t f, \mu\rangle}, \quad t\geq 0, \mu\in\mathcal{M}(E),$$

where $(t,x)\mapsto V_t f(x)$ is the unique locally bounded non-negative solution to the equation

$$V_t f(x) + \int_0^t P_{t-s}(\psi\circ V_s f)(x) ds = P_t f(x), \quad x\in E, t\geq 0.$$

Model/Superprocesses

Example

Let $\alpha > 0$ and $\beta \in (0,1)$. Consider a Branching OU-process with

- k initial particles;
- killing rate $2k^\beta$;
- offspring generating function

$$F_k(x) = x + \frac{1}{2}(1-x)^{1+\beta} + \frac{-\alpha(1-x)}{2k^\beta}.$$

Let $X_t^{(k)}(A)$ be number of particles in a Borel set A at time t. Then,

$$\Big(\frac{1}{k} X_t^{(k)}(\cdot)\Big)_{t\geq 0} \xrightarrow[k\to\infty]{w} (X_t)_{t\geq 0}$$

where $(X_t)_{t\geq 0}$ is a super OU process with branching mechanism

$$\psi(z) = -\alpha z + z^{1+\beta}.$$

Background/Motivation

Motivation

We are interested in the spatial laws of large numbers (LLNs) and central limit theorems (CLT) for superprocesses. More precisely, we want to find $(F_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ such that

$$\frac{\langle f, X_t\rangle - G_t}{F_t}$$

- converges in probability or in L^p to a non-degenerate random variable (weak LLNs);
- converges almost surely to a non-degenerate random variable (strong LLNs);
- converges weakly to some non-degenerate random variable (CLTs).

Background/LLNs of superprocesses

- Weak laws of large numbers under different settings: Englander 2009, Englander and Winter 2006, Englander and Turaev 2002.
- Strong law of large numbers under different settings: Chen, Ren and Wang 2008, Wang 2010, Liu, Ren and Song 2013, Kouritzin and Ren 2014, Chen, Ren, Song and Zhang 2015, Eckhoff, Kyprianou and Winkel 2015, Chen, Ren and Yang 2019.

Background/LLNs of superprocesses

Example (Chen, Ren and Yang 2019)

Suppose that X is a super OU process with branching mechanism $\psi(z) = -\alpha z + z^{1+\beta}$ with $\alpha > 0$ and $\beta \in (0,1)$. Then, for each $\mu \in \mathcal{M}(\mathbb{R}^d)$ and each continuous bounded non-negative function f on $\mathbb{R}^d$, we have

$$e^{-\alpha t}\langle f, X_t\rangle \xrightarrow[t\to\infty]{\mathbb{P}_\mu\text{-a.s.}, L^1(\mathbb{P}_\mu)} H_\infty \langle f, \varphi\rangle,$$

where H_∞ is the limit of the non-negative martingale $e^{-\alpha t}\langle 1, X_t\rangle$ and φ is the invariant density of the OU process ξ.

Background/CLTs for branching processes with 2rd moment conditions

- For CLT of supercritical Galton-Watson processes, see Heyde 1970, Heyde and Brown 1971, Heyde and Leslie 1971.
- For CLT of supercritical multi-type Galton-Watson processes, see Athreya 1969, Athreya 1969, Athreya 1971.
- For CLT of some general supercritical branching Markov processes, see Asmussen and Hering 1983.
- For CLT of supercritical branching OU processes with binary branching mechanism, see Adamczak and Milos 2015.

Background/CLTs for Superprocesses with 2rd moment conditions

- For CLT of some supercritical superprocesses with branching mechanisms satisfying a fourth moment condition, see Milos 2018.
- For CLT of supercritical super OU processes with branching mechanisms satisfying only a second moment condition, see Ren, Song and Zhang 2014.
- For a series of CLT for a large class of general supercritical branching Markov processes and superprocesses with spatially dependent branching mechanisms satisfying a second moment condition, see Ren, Song and Zhang 2014, Ren, Song and Zhang 2015, Ren, Song and Zhang 2017.

Background/Stable CLTs without the 2rd moment condition

- For stable CLT for supercritical Galton-Watson process whose offspring distribution belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, see Heyde 1971.
- For stable CLT for supercritical multi-type Galton-Watson process and supercritical continuous time branching processes, see Asmussen 1976.
- Recently, some stable CLTs for supercritical branching OU processes with offspring generating function

$$F(s) = ms - (m-1) + (m-1)(1-s)^{1+\beta}, \quad |s| < 1$$

  with $m > 1$ and $\beta \in (0, 1)$ are established in Marks and Milos 2018.
- Very recently, some stable fluctuation results of Biggins' martingales in the context of branching random walks are established in Iksanov, Kolesko and Meiners 2018.

Settings

- Let $E = \mathbb{R}^d$.
- Let $(\xi_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck (OU) process with generator

$$Lf(x) = \frac{1}{2}\sigma^2 \Delta f(x) - bx \cdot \nabla f(x), \quad x \in \mathbb{R}^d, f \in C^2(\mathbb{R}^d),$$

  where $\sigma, b > 0$.

Settings/Assumption 1

Assumption 1. (Grey's condition)

There exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and $\int_{z'}^{\infty} \psi(z)^{-1} dz < \infty$.

- It is known that, under Assumption 1, the *extinction event* $D := \{\exists t \geq 0, \text{ s.t. } \|X_t\| = 0\}$ has positive probability, with respect to $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}(\mathbb{R}^d)$.

Settings/Assumption 2

- Denote by Γ the gamma function. For any σ-finite signed measure μ, we use $|\mu|$ to denote the total variation measure of μ.

Assumption 2.

There exist constants $\eta > 0$ and $\beta \in (0,1)$ such that

$$\int_{(1,\infty)} y^{1+\beta+\delta} \left|\pi(dy) - \frac{\eta \, dy}{\Gamma(-1-\beta) y^{2+\beta}}\right| < \infty,$$

for some $\delta \in (0, 1-\beta)$.

Settings/Assumption 2

- Assumption 2 says that there exist constants $\eta > 0$ and $\beta > 0$ such that the Lévy measure $\pi(dy)$ is not too far away from the measure $\eta\Gamma(-1-\beta)^{-1}y^{-2-\beta}dy$.
- In particular, if

$$\pi(dy) = \eta\Gamma(-1-\beta)^{-1}y^{-2-\beta}dy,$$

then the branching mechanism takes the form:

$$\psi(z) = -\alpha z + \rho z^2 + \eta z^{1+\beta}, \quad z \geq 0.$$

Preliminary/Invariant measure of OU semigroup

- Recall that $(P_t)_{t\geq 0}$ is the transition semigroup of the OU process ξ.
- $(P_t)_{t\geq 0}$ has the invariant density

$$\varphi(x)dx := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2}\exp\Big(-\frac{b}{\sigma^2}|x|^2\Big)dx, \quad x\in\mathbb{R}^d.$$

- Let $L^2(\varphi)$ be the Hilbert space with inner product

$$\langle f_1, f_2\rangle_\varphi := \int_{\mathbb{R}^d} f_1(x)f_2(x)\varphi(x).$$

Preliminary/Hermite polynomials

- For each multi-index $p = (p_k)_{k=1}^d \in \mathbb{Z}_+^d$, write

$$|p| := \sum_{k=1}^d p_k; \quad p! := \prod_{k=1}^d p_k!; \quad \frac{\partial^p}{\partial x^p} := \prod_{k=1}^d \frac{\partial^{p_k}}{\partial x_k^{p_k}}.$$

- The *Hermite polynomials* are given by

$$H_p(x) := (-1)^{|p|} \exp(|x|^2) \frac{\partial^p}{\partial x^p} \exp(-|x|^2), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

Preliminary/Spectrum of OU semigroup

Lemma (Metafune, Pallara and Priola 2002)

$(P_t)_{t\geq 0}$ is a strongly continuous semigroup in $L^2(\varphi)$ and its generator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. For each $k \in \mathbb{Z}_+$, denote by $\mathcal{A}_k$ the eigenspace corresponding to the eigenvalue $-bk$, then

$$\mathcal{A}_k = \mathrm{Span}\{\phi_p : p \in \mathbb{Z}_+^d, |p| = k\},$$

where

$$\phi_p(x) := \frac{1}{\sqrt{p!2^{|p|}}} H_p\Big(\frac{\sqrt{b}}{\sigma}x\Big), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

Moreover, $\{\phi_p : p \in \mathbb{Z}_+^d\}$ forms a complete orthonormal basis for $L^2(\varphi)$.

Preliminary/Order of a testing function

- For each function $f \in L^2(\varphi)$, denote by

$$\kappa_f := \inf\left\{k \geq 0 : \exists\ p \in \mathbb{Z}_+^d,\ \text{s.t. } |p| = k \text{ and } \langle f, \phi_p\rangle_\varphi \neq 0\right\},$$

  the order of the function f.
- We say a function $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R})$ is of polynomial growth if there exists constants $C, n > 0$ such that

$$|f(x)| \leq C(1 + |x|)^n, \quad \forall x \in \mathbb{R}^d.$$

- $\mathcal{P}$: the collection of all functions of polynomial growth.

Results/Phase transition

- Recall that α is the branching rate.
- It turns out that the CLTs of $\langle f, X_t \rangle$ are different in three different regimes depending on the sign of $\alpha - \frac{\kappa_f b(1+\beta)}{\beta}$:
  - small branching rate regime: $\alpha < \frac{\kappa_f b(1+\beta)}{\beta}$;
  - critical branching rate regime: $\alpha = \frac{\kappa_f b(1+\beta)}{\beta}$;
  - large branching rate regime: $\alpha > \frac{\kappa_f b(1+\beta)}{\beta}$.

Results/Small branching rate regime/ CLT

Theorem (Ren, Song, S. and Zhao 2019+)

Let $f \in \mathcal{P} \setminus \{0\}$ satisfy $\alpha\beta < \kappa_f b(1+\beta)$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ have compact support. Then under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that

$$\|X_t\|^{-\frac{1}{1+\beta}} \langle f, X_t \rangle \xrightarrow[t\to\infty]{d} \zeta.$$

Here, ζ is a $(1+\beta)$-stable random variable with characteristic function $\theta \mapsto \exp(m[\theta f])$ with

$$m[\theta f] := \eta \int_0^\infty e^{-\alpha s}\, ds \int_{\mathbb{R}^d} \big(-i\theta P_s^\alpha f(x)\big)^{1+\beta} \varphi(x)\, dx.$$

Results/Critical branching rate regime/CLT

Theorem (Ren, Song, S. and Zhao 2019+)

Let $f \in \mathcal{P} \setminus \{0\}$ satisfy $\alpha\beta = \kappa_f b(1+\beta)$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ have compact support. Then under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that

$$(t\|X_t\|)^{-\frac{1}{1+\beta}} \langle f, X_t \rangle \xrightarrow[t\to\infty]{d} \widetilde{\zeta}.$$

Here, $\widetilde{\zeta}$ is a $(1+\beta)$-stable random variable with characteristic function $\theta \mapsto \exp(\widetilde{m}[\theta f])$ where

$$\widetilde{m}[\theta f] := \eta \int_{\mathbb{R}^d} \left(-i\theta \sum_{p \in \mathbb{Z}_+^d : |p| = \kappa_f} \langle f, \phi_p \rangle_\varphi \phi_p(x) \right)^{1+\beta} \varphi(x)\, dx.$$

Results/Large branching rate regime/ Martingale limit

Lemma (Ren, Song, S. and Zhao 2019)

For each multi-index $p \in \mathbb{Z}_+^d$ *with* $\alpha\beta > |p|b(1+\beta)$, *each* $\gamma \in (0, \beta)$ *and each* $\mu \in \mathcal{M}(\mathbb{R}^d)$ *with compact support, the following martingale*

$$H_t^p := e^{-(\alpha - |p|b)t} \langle \phi_p, X_t \rangle, \quad t \geq 0,$$

is bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$. *Thus the limit* $H_\infty^p := \lim_{t\to\infty} H_t^p$ *exists* $\mathbb{P}_\mu$*-almost surely and in* $L^{1+\gamma}(\mathbb{P}_\mu)$.

Results/Large branching rate regime/LLNs

Theorem (Ren, Song, S. and Zhao 2019+)

Let $f \in \mathcal{P} \setminus \{0\}$ satisfy $\alpha\beta > \kappa_f b(1+\beta)$. Then for each $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ with compact support, it holds that

$$e^{-(\alpha-\kappa_f b)t}\langle f, X_t\rangle \xrightarrow[t\to\infty]{} \sum_{p\in\mathbb{Z}_+^d : |p|=\kappa_f} \langle f, \phi_p\rangle_\varphi H_\infty^p \quad \text{in } L^{1+\gamma}(\mathbb{P}_\mu).$$

Moreover, if f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$, then we also have almost sure convergence.

Results/Spectrum decomposition

- Define

$$\mathcal{C}_l := \overline{Span}\{\phi_p : \alpha\beta > |p|(1+\beta)b\}.$$
$$\mathcal{C}_c := \overline{Span}\{\phi_p : \alpha\beta = |p|(1+\beta)b\}.$$
$$\mathcal{C}_s := \overline{Span}\{\phi_p : \alpha\beta > |p|(1+\beta)b\}.$$

- For any $f \in \mathcal{P} \setminus \{0\}$, there is a unique decomposition:

$$f := f_l + f_c + f_s$$

  where $f_l \in \mathcal{C}_l, f_c \in \mathcal{C}_c$ and $f_s \in \mathcal{C}_s$.
- CLTs for f_s and f_c are already established.

Results/Large branching rate regime/CLTs

Theorem (Ren, Song, S. and Zhao 2019+)

Let $f \in \mathcal{C}_l \setminus \{0\}$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ have compact support. Then under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that

$$\frac{\langle f, X_t\rangle - \sum_{p\in\mathcal{N}}\langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H^p_\infty}{\|X_t\|^{\frac{1}{1+\beta}}} \xrightarrow[t\to\infty]{d} \bar{\zeta}.$$

Here $\bar{\zeta}$ is a $(1+\beta)$-stable random variable with characteristic function $\theta \mapsto \exp(\bar{m}[\theta f])$ where

$$\bar{m}[\theta f] := \eta \int_0^\infty e^{\alpha s}\, ds \int_{\mathbb{R}^d} \left(i\theta I_s f(x)\right)^{1+\beta} \varphi(x)\, dx, \quad f \in \mathcal{C}_l$$

and $(I_t)_{t\ge 0}$ are the inverse operator of $(P^\alpha_t)_{t\ge0}$ on $\mathcal{C}_l$.

Remark/Phase transition

- Similar phase transition phenomenon in the context of branching OU processes is first observed by Adamczak and Milos 2015.
- Coarsening effect (increasing of the spatial inequalities): A consequence of branching. Simply an area with more particles will produce more offspring.
- Smoothing effect (decreasing of the spatial inequalities): A consequence of the mixing property of the OU process. Simply each OU particles will "forget" its initial position exponentially fast.
- The phase transition phenomenon is due to an interplay of the coarsening effect and the smoothing effect.

Thanks!