STABLE CENTRAL LIMIT THEOREMS FOR SUPER ORNSTEIN-UHLENBECK PROCESSES, II

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Abstract. This paper is a continuation of our recent paper (Elect. J. Probab. **24** (2019), no. 141) and is devoted to the asymptotic behavior of a class of supercritical super Ornstein-Uhlenbeck processes $(X_t)_{t\geq 0}$ with branching mechanisms of infinite second moment. In the aforementioned paper, we proved stable central limit theorems for $X_t(f)$ for *some* functions f of polynomial growth in three different regimes. However, we were not able to prove central limit theorems for $X_t(f)$ for *all* functions f of polynomial growth. In this note, we show that the limit stable random variables in the three different regimes are independent, and as a consequence, we get stable central limit theorems for $X_t(f)$ for *all* functions f of polynomial growth.

1. Introduction and main result

Let $d \in \mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{R}_+ := [0, \infty)$. Let $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in\mathbb{R}^d}\}$ be an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck process (OU process) with generator

$$Lf(x) = \frac{1}{2}\sigma^2 \Delta f(x) - bx \cdot \nabla f(x), \quad x \in \mathbb{R}^d, f \in C^2(\mathbb{R}^d),$$

where $\sigma > 0$ and $b > 0$ are constants. Let ψ be a function on $\mathbb{R}_+$ of the form

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(\mathrm{d}y), \quad z \in \mathbb{R}_+,$$

where $\alpha > 0$, $\rho \geq 0$ and π is a measure on $(0, \infty)$ with $\int_{(0,\infty)}(y \wedge y^2)\pi(\mathrm{d}y) < \infty$. ψ is referred to as a branching mechanism and π is referred to as the Lévy measure of ψ. Denote by $\mathcal{M}(\mathbb{R}^d)$ ($\mathcal{M}_c(\mathbb{R}^d)$) the space of all finite Borel measures (of compact support) on $\mathbb{R}^d$. Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R})$ ($\mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$) the space of all $\mathbb{R}$-valued ($\mathbb{R}_+$-valued) Borel functions on $\mathbb{R}^d$. For $f, g \in \mathcal{B}(\mathbb{R}^d, \mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$, write $\mu(f) = \int f(x)\mu(\mathrm{d}x)$ and $\langle f, g\rangle = \int f(x)g(x)\mathrm{d}x$ whenever the integrals make sense. We say a real-valued Borel function f on $\mathbb{R}_+ \times \mathbb{R}^d$ is *locally bounded* if, for each $t \in \mathbb{R}_+$, we have $\sup_{s\in[0,t],x\in\mathbb{R}^d} |f(s, x)| < \infty$.

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For any $\mu \in \mathcal{M}(\mathbb{R}^d)$, we write $\|\mu\| = \mu(1)$. For any σ-finite signed measure μ, denote by $|\mu|$ the total variation measure of μ.

We say that an $\mathcal{M}(\mathbb{R}^d)$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ is a *super Ornstein-Uhlenbeck process (super-OU process)* with branching mechanism ψ, or a (ξ, ψ)-*superprocess*, if for each non-negative bounded Borel function f on $\mathbb{R}^d$, we have

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d),$$

where $(t, x) \mapsto V_t f(x)$ is the unique locally bounded non-negative solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t \psi(V_{t-s} f(\xi_s))\mathrm{d}s\Big] = \Pi_x[f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0.$$

The existence of such super-OU process X is well known, see [8] for instance.

There have been many central limit theorem type results for branching processes, branching diffusions and superprocesses, under the second moment condition. See [9, 11, 12] for supercritical Galton-Watson processes (GW processes), [13, 14] for supercritical multi-type GW processes, [4, 5, 6] for supercritical multi-type continuous time branching processes and [3] for general supercritical branching Markov processes under certain conditions. Some spatial central limit theorems for supercritical branching OU processes with binary branching mechanism were proved in [1], and some spatial central limit theorems for supercritical super-OU processes with branching mechanisms satisfying a fourth moment condition were proved in [19]. These two papers made connections between central limit theorems and branching rate regimes. The results of [19] were extended and refined in [21]. Since then, a series of spatial central limit theorems for a large class of general supercritical branching Markov processes and superprocesses with spatially dependent branching mechanisms were proved in [22, 23, 24].

There are also central limit theorem type results for supercritical branching processes and branching Markov processes with branching mechanisms of infinite second moment. For earlier papers, see [2, 10]. Recently, Marks and Miloś [17] established some spatial central limit theorems in the small and critical branching rate regimes, for some supercritical branching OU processes with a special stable offspring distribution. In [20], we established stable central limit theorems for super-OU processes X with branching mechanisms ψ satisfying the following two assumptions.

Assumption 1 (Grey's condition)**.** There exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and $\int_{z'}^\infty \psi(z)^{-1}\mathrm{d}z < \infty$.

Assumption 2. There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(\mathrm{d}y) - \frac{\eta\mathrm{d}y}{\Gamma(-1-\beta)y^{2+\beta}}\Big| < \infty$$

for some $\delta > 0$.

It is known (see [15, Theorems 12.5 & 12.7] for example) that, under Assumption 1, the *extinction event*

$$D := \{\exists t \geq 0, \text{ such that } \|X_t\| = 0\}$$

is non-trivial with respect to $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}(\mathbb{R}^d)\setminus\{0\}$. In fact, $\mathbb{P}_\mu(D) = e^{-\bar{v}\|\mu\|}$, where $\bar{v} := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\} \in (0,\infty)$ is the largest root of ψ. Assumption 2 says that ψ is "not too far away" from $\widetilde{\psi}(z) := -\alpha z + \eta z^{1+\beta}$ near 0, see [20, Remark 1.3]. It follows from [20, Lemma 2.2] that, if Assumption 2 holds, then η and β are uniquely determined by the Lévy measure π. In [20, Lemma 2.3], we have shown that, under Assumption 2, ψ satisfies the $L\log L$ condition, i.e., $\int_{(1,\infty)} y\log y\pi(\mathrm{d}y) < \infty$. In the reminder of the paper, we will always use η and β to denote the constants in Assumption 2. Note that δ is not uniquely determined by π.

The limit behavior of X is closely related to the spectral property of the OU semigroup $(P_t)_{t\geq 0}$ which we now recall (see [18] for more details). We use $(P_t)_{t\geq 0}$ to denote the transition semigroup of ξ. Define $P_t^\alpha f(x) := e^{\alpha t}P_t f(x) = \Pi_x[e^{\alpha t} f(\xi_t)]$ for each $x \in \mathbb{R}^d$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$. It is known that, see [16, Proposition 2.27] for example, $(P_t^\alpha)_{t\geq 0}$ is the *mean semigroup* of X in the sense that $\mathbb{P}_\mu[X_t(f)] = \mu(P_t^\alpha f)$ for all $\mu \in \mathcal{M}(\mathbb{R}^d)$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$. It is known that the OU process ξ has an invariant probability on $\mathbb{R}^d$

$$\varphi(x)\mathrm{d}x := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2} \exp\Big(-\frac{b}{\sigma^2}|x|^2\Big)\mathrm{d}x$$

which is a symmetric multivariate Gaussian distribution. Let $L^2(\varphi)$ be the Hilbert space with inner product

$$\langle f_1, f_2\rangle_\varphi := \int_{\mathbb{R}^d} f_1(x) f_2(x)\varphi(x)\mathrm{d}x, \quad f_1, f_2 \in L^2(\varphi).$$

Let $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. For each $p = (p_k)_{k=1}^d \in \mathbb{Z}_+^d$, write $|p| := \sum_{k=1}^d p_k$, $p! := \prod_{k=1}^d p_k!$ and $\partial_p := \prod_{k=1}^d (\partial^{p_k}/\partial x_k^{p_k})$. The *Hermite polynomials* are defined by

$$\mathcal{H}_p(x) := (-1)^{|p|} e^{|x|^2}\partial_p e^{-|x|^2}, \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

It is known that $(P_t)_{t\geq 0}$ is a strongly continuous semigroup in $L^2(\varphi)$ and its generator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. For $k \in \mathbb{Z}_+$, denote by $\mathcal{A}_k$ the eigenspace corresponding to the eigenvalue $-bk$, then $\mathcal{A}_k = \mathrm{Span}\{\phi_p : p \in \mathbb{Z}_+^d, |p| = k\}$ where

$$\phi_p(x) := \frac{1}{\sqrt{p!2^{|p|}}}\mathcal{H}_p\Big(\frac{\sqrt{b}}{\sigma}x\Big), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

In other words, $P_t\phi_p(x) = e^{-b|p|t}\phi_p(x)$ for all $t \geq 0$, $x \in \mathbb{R}^d$ and $p \in \mathbb{Z}_+^d$. Moreover, $\{\phi_p : p \in \mathbb{Z}_+^d\}$ forms a complete orthonormal basis of $L^2(\varphi)$. Thus for each $f \in L^2(\varphi)$, we have

$$f = \sum_{k=0}^\infty \sum_{p\in\mathbb{Z}_+^d : |p|=k} \langle f, \phi_p\rangle_\varphi \phi_p, \quad \text{in } L^2(\varphi). \tag{1.1}$$

For each function $f \in L^2(\varphi)$, define the order of f as

$$\kappa_f := \inf\big\{k \geq 0 : \exists\, p \in \mathbb{Z}_+^d, \text{ s.t. } |p| = k \text{ and } \langle f, \phi_p\rangle_\varphi \neq 0\big\}$$

which is the lowest non-trivial frequency in the eigen-expansion (1.1). Note that $\kappa_f \geq 0$ and that, if $f \in L^2(\varphi)$ is non-trivial, then $\kappa_f < \infty$. In particular, the order of any constant non-zero function is zero. For $p \in \mathbb{Z}_+^d$, define

$$H_t^p := e^{-(\alpha - |p|b)t} X_t(\phi_p), \qquad t \geq 0.$$

We will write H_t^0 as H_t. For each $u \neq -1$, we write $\tilde{u} = u/(1+u)$. We have shown in [20, Lemma 3.2] the following:

(1.2) For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale. Futhermore, if $\alpha\tilde{\beta} > |p|b$, then for every $\gamma \in (0, \beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$; thus $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$.

We will write H_∞^0 as H_∞.

Let us also recall some results from [20] before we formulate our main theorem. Denote by $\mathcal{P}$ the class of functions of polynomial growth on $\mathbb{R}^d$, i.e.,

$$\mathcal{P} := \{f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : \exists C > 0, n \in \mathbb{Z}_+ \text{ s.t. } \forall x \in \mathbb{R}^d, |f(x)| \leq C(1+|x|)^n\}.$$

It is clear that $\mathcal{P} \subset L^2(\varphi)$. Define

$$\mathcal{C}_{\mathrm{s}} := \mathcal{P} \cap \overline{\mathrm{Span}}\{\phi_p : \alpha\tilde{\beta} < |p|b\}, \quad \mathcal{C}_{\mathrm{c}} := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} = |p|b\}, \text{ and}$$
$$\mathcal{C}_{\mathrm{l}} := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} > |p|b\}.$$

Note that $\mathcal{C}_{\mathrm{s}}$ is an infinite dimensional space, $\mathcal{C}_{\mathrm{l}}$ and $\mathcal{C}_{\mathrm{c}}$ are finite dimensional spaces, and $\mathcal{C}_c$ might be empty. Define a semigroup

$$T_t f := \sum_{p \in \mathbb{Z}_+^d} e^{-\left||p|b - \alpha\tilde{\beta}\right| t} \langle f, \phi_p \rangle_\varphi \phi_p, \quad t \geq 0, f \in \mathcal{P},$$

and a family of functionals

$$m_t[f] := \eta \int_0^t \mathrm{d}u \int_{\mathbb{R}^d} \big(- iT_u f(x)\big)^{1+\beta} \varphi(x) \mathrm{d}x, \quad 0 \leq t < \infty, f \in \mathcal{P}. \tag{1.3}$$

For each $\mu \in \mathcal{M}(\mathbb{R}^d) \setminus \{0\}$, write $\widetilde{\mathbb{P}}_\mu(\cdot) := \mathbb{P}_\mu(\cdot | D^c)$. We have shown in [20, Lemma 2.6 and Propsoition 2.7] that,

(1.4) for each $f \in \mathcal{P}$, there exists a $(1+\beta)$-stable random variable ζ^f with characteristic function $\theta \mapsto e^{m[\theta f]}, \theta \in \mathbb{R}$, where

$$m[f] := \begin{cases} \lim_{t\to\infty} m_t[f], & f \in \mathcal{C}_{\mathrm{s}} \oplus \mathcal{C}_{\mathrm{l}}, \\ \lim_{t\to\infty} \frac{1}{t} m_t[f], & f \in \mathcal{P} \setminus \mathcal{C}_{\mathrm{s}} \oplus \mathcal{C}_{\mathrm{l}}. \end{cases}$$

Furthermore, we proved in [20, Theorem 1.6] that

if $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}} \setminus \{0\}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}} \setminus \{0\}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}} \setminus \{0\}$, then under $\widetilde{\mathbb{P}}_\mu$,

$$
\begin{aligned}
&e^{-\alpha t}\|X_t\| \xrightarrow[t\to\infty]{\text{a.s.}} \widetilde{H}_\infty; \quad \frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{s}}};\\
&\frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{c}}}; \quad \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{-f_{\mathrm{l}}},
\end{aligned} \tag{1.5}
$$

where $\widetilde{H}_\infty$ has the distribution of $\{H_\infty; \widetilde{\mathbb{P}}_\mu\}$; $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.4); and

$$
\mathrm{x}_t(f) := \sum_{p\in\mathbb{Z}_+^d : \alpha\tilde{\beta} > |p|b} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_\infty^p, \quad t \geq 0, f \in \mathcal{P}.
$$

The above result gives the central limit theorem for $X_t(f)$ if $f \in \mathcal{P}\setminus\{0\}$ satisfies $\alpha\tilde{\beta} \leq \kappa_f b$. A general $f \in \mathcal{P}$ can be decomposed as $f_s + f_c + f_l$ with $f_s \in \mathcal{C}_{\mathrm{s}}$, $f_c \in \mathcal{C}_{\mathrm{c}}$ and $f_l \in \mathcal{C}_{\mathrm{l}}$; and if $f \in \mathcal{P}$ satisfies $\alpha\tilde{\beta} > \kappa_f b$, then f_{c} and f_{l} maybe non-zero. In [20], we were not able to establish a central limit theorem in this case. We conjectured there that the limit random variables in (1.5) for $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$ are independent. Once this asymptotic independence is established, a central limit theorem for $X_t(f)$ for all $f \in \mathcal{P}$ would follow.

The main purpose of this note is to show that the limit random variables in (1.5) are independent.

Theorem 1.1. *If $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}} \setminus \{0\}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}} \setminus \{0\}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}} \setminus \{0\}$, then under $\widetilde{\mathbb{P}}_\mu$,*

$$
\begin{aligned}
S(t) &:= \left(e^{-\alpha t}\|X_t\|, \frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde{\beta}}}\right)\\
&\xrightarrow[t\to\infty]{d} (\widetilde{H}_\infty, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}}),
\end{aligned} \tag{1.6}
$$

where $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.5) with f replaced with f_{l}; $\widetilde{H}_\infty$ has the distribution of $\{H_\infty; \widetilde{\mathbb{P}}_\mu\}$; $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.4); $\widetilde{H}_\infty$, $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent.

As a corollary of this theorem, we get central limit theorems for $X_t(f)$ for all $f \in \mathcal{P}$.

Corollary 1.2. *Let $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{P} \setminus \{0\}$. Let $f = f_{\mathrm{s}} + f_{\mathrm{c}} + f_{\mathrm{l}}$ be the unique decomposition of f with $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$. Then under $\widetilde{\mathbb{P}}_\mu$, it holds that*

(1) *if $f_{\mathrm{c}} = 0$, then*

$$
\frac{X_t(f) - \mathrm{x}_t(f)}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{s}}} + \zeta^{-f_{\mathrm{l}}},
$$

where $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.4), $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent;

(2) *if* $f_{\mathrm{c}} \neq 0$, *then*

$$\frac{X_t(f) - \mathrm{x}_t(f)}{\|tX_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{c}}}.$$

where $\zeta^{f_{\mathrm{c}}}$ *is the* $(1+\beta)$*-stable random variables described in* (1.4).
Here $\mathrm{x}_t(f)$ *is defined in* (1.5).

2. Proof of main result

We first make some preparations before proving Theorem 1.1. For every $t \geq 0$ and $f \in \mathcal{P}$, define

$$Z_t f := \int_0^t P^\alpha_{t-s}\big(\eta(-iP^\alpha_s f)^{1+\beta}\big)\mathrm{d}s, \quad \Upsilon^f_t := \frac{X_{t+1}(f) - X_t(P^\alpha_1 f)}{\|X_t\|^{1-\tilde{\beta}}}.$$

Form [20, Theorem 3.4] we know that, for each $f \in \mathcal{P}$, $\langle Z_1 f, \varphi\rangle$ is the characteristic exponent of the limit of Υ^f_t. For $g \in \mathcal{P}$, define $\mathcal{P}_g := \{\theta T_n g : n \in \mathbb{Z}_+, \theta \in [-1,1]\}$. The following generalization of [20, Proposition 3.5] will be used later in the proof of Theorem 2.3, a special case of Theorem 1.1.

Proposition 2.1. *For each* $f, g \in \mathcal{P}$ *and* $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, *there exist* $C, \delta > 0$ *such that for all* $n_1, n_2 \in \mathbb{Z}_+$, $(f_j)_{j=0}^{n_1} \subset \mathcal{P}_f$, $(g_j)_{j=0}^{n_2} \subset \mathcal{P}_g$ *and* $t \geq n_1 + 1$, *we have*

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{k=0}^{n_1} e^{i\Upsilon^{f_k}_{t-k-1}}\Big)\Big(\prod_{k=0}^{n_2} e^{i\Upsilon^{g_k}_{t+k}}\Big)\Big] - \Big(\prod_{k=0}^{n_1} e^{\langle Z_1 f_k, \varphi\rangle}\Big)\Big(\prod_{k=0}^{n_2} e^{\langle Z_1 g_k, \varphi\rangle}\Big)\Big| \leq Ce^{-\delta(t-n_1)}. \tag{2.1}$$

Proof. In this proof, we fix $f, g \in \mathcal{P}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $n_1, n_2 \in \mathbb{Z}_+$, $(f_j)_{j=0}^{n_1} \subset \mathcal{P}_f$, $(g_j)_{j=0}^{n_2} \subset \mathcal{P}_g$ and $t \geq n_1 + 1$. For any $k_1 \in \{-1, 0, \dots, n_1\}$ and $k_2 \in \{-1, 0, \dots, n_2\}$, define

$$a_{k_1,k_2} := \widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{j=k_1+1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2} e^{i\Upsilon^{g_j}_{t+j}}\Big)\Big]\Big(\prod_{j=0}^{k_1} e^{\langle Z_1 f_j, \varphi\rangle}\Big)\Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j, \varphi\rangle}\Big),$$

where we used the convention that $\prod_{j=0}^{-1} = 1$. Then for all $k_2 \in \{0, \dots, n_2\}$, we have

$$\begin{aligned}
&a_{-1,k_2} - a_{-1,k_2-1} \\
&= \widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{j=0}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2} e^{i\Upsilon^{g_j}_{t+j}}\Big)\Big]\Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j, \varphi\rangle}\Big) \\
&\quad - \widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{j=0}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2-1} e^{i\Upsilon^{g_j}_{t+j}}\Big)\Big]\Big(\prod_{j=k_2}^{n_2} e^{\langle Z_1 g_j, \varphi\rangle}\Big) \\
&= \frac{1}{\mathbb{P}_\mu(D^c)}\Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j, \varphi\rangle}\Big) \times \\
&\quad \mathbb{P}_\mu\Big[\Big(\prod_{j=0}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2-1} e^{i\Upsilon^{g_j}_{t+j}}\Big)\big(e^{i\Upsilon^{g_{k_2}}_{t+k_2}} - e^{\langle Z_1 g_{k_2}, \varphi\rangle}\big); D^c\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{P}_\mu(D^c)}\Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j,\varphi\rangle}\Big)\times \\
&\mathbb{P}_\mu\Big[\Big(\prod_{j=0}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2-1} e^{i\Upsilon^{g_j}_{t+j}}\Big)\mathbb{P}_\mu[e^{i\Upsilon^{g_{k_2}}_{t+k_2}} - e^{\langle Z_1 g_{k_2},\varphi\rangle}; D^c|\mathscr{F}_{t+k_2}]\Big].
\end{aligned} \tag{2.2}
$$

Therefore, there exist $C_0, \delta_0 > 0$, depending only on μ and g, such that for each $k_2 \in \{0,\dots,n_2\}$,

$$
\begin{aligned}
|a_{-1,k_2} - a_{-1,k_2-1}| &\overset{(2.2)}{\leq} \mathbb{P}_\mu(D^c)^{-1}\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon^{g_{k_2}}_{t+k_2}} - e^{\langle Z_1 g_{k_2},\varphi\rangle}; D^c|\mathscr{F}_{t+k_2}]\big|\Big] \\
&\overset{[20,\text{ Proposition 3.5}]}{\leq} C_0 e^{-\delta_0(t+k_2)}.
\end{aligned} \tag{2.3}
$$

Notice that, for any $k_1 \in \{0,\dots,n_1\}$,

$$
\begin{aligned}
&a_{k_1-1,-1} - a_{k_1,-1} \\
&= \widetilde{\mathbb{P}}_\mu\Big[\prod_{j=k_1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big]\Big(\prod_{j=0}^{k_1-1} e^{\langle Z_1 f_j,\varphi\rangle}\Big)\Big(\prod_{j=0}^{n_2} e^{\langle Z_1 g_j,\varphi\rangle}\Big) - \\
&\quad \widetilde{\mathbb{P}}_\mu\Big[\prod_{j=k_1+1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big]\Big(\prod_{j=0}^{k_1} e^{\langle Z_1 f_j,\varphi\rangle}\Big)\Big(\prod_{j=0}^{n_2} e^{\langle Z_1 g_j,\varphi\rangle}\Big) \\
&= \widetilde{\mathbb{P}}_\mu\Big[\big(e^{i\Upsilon^{f_{k_1}}_{t-k_1-1}} - e^{\langle Z_1 f_{k_1},\varphi\rangle}\big)\prod_{j=k_1+1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big]\Big(\prod_{j=0}^{k_1-1} e^{\langle Z_1 f_j,\varphi\rangle}\Big)\Big(\prod_{j=0}^{n_2} e^{\langle Z_1 g_j,\varphi\rangle}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{P}_\mu(D^c)}\Big(\prod_{j=0}^{k_1-1} e^{\langle Z_1 f_j,\varphi\rangle}\Big)\Big(\prod_{j=0}^{n_2} e^{\langle Z_1 g_j,\varphi\rangle}\Big)\times \\
&\quad \mathbb{P}_\mu\Big[\mathbb{P}_\mu\big[e^{i\Upsilon^{f_{k_1}}_{t-k_1-1}} - e^{\langle Z_1 f_{k_1},\varphi\rangle}; D^c\big|\mathscr{F}_{t-k_1-1}\big]\prod_{j=k_1+1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big].
\end{aligned} \tag{2.4}
$$

Therefore, there exist $C_1, \delta_1 > 0$, depending only on μ and f, such that for any $k_1 \in \{0,\dots,n_1\}$,

$$
\begin{aligned}
|a_{k_1-1,-1} - a_{k_1,-1}| &\overset{(2.4)}{\leq} \frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon^{f_{k_1}}_{t-k_1-1}} - e^{\langle Z_1 f_{k_1},\varphi\rangle}; D^c|\mathscr{F}_{t-k_1-1}]\big|\Big] \\
&\overset{[20,\text{ Proposition 3.5}]}{\leq} C_1 e^{-\delta_1(t-k_1)}.
\end{aligned} \tag{2.5}
$$

Therefore, there exist $C,\delta > 0$, depending only on f, g and μ, such that

$$
\begin{aligned}
&\text{LHS of (2.1)} = |a_{-1,n_2} - a_{n_1,-1}| \leq \sum_{k=0}^{n_1} |a_{k-1,-1} - a_{k,-1}| + \sum_{k=0}^{n_2} |a_{-1,k} - a_{-1,k-1}| \\
&\overset{(2.3),(2.5)}{\leq} \sum_{k=0}^{n_1} C_1 e^{-\delta_1(t-k)} + \sum_{k=0}^{n_2} C_0 e^{-\delta_0(t+k)} \leq C e^{-\delta(t-n_1)}.
\end{aligned}
$$

□

The following elementary result will also be used in the proof of Theorem 2.3.

Lemma 2.2. *There exists a constant $C > 0$, such that for any $x, y \in \mathbb{R}$,*

$$|(x+y)^{1+\beta} - x^{1+\beta} - y^{1+\beta}| \le C(|x||y|^\beta + |x|^\beta|y|).$$

Proof. Note that

$$\lim_{|y|\to\infty} \frac{(y+1)^{1+\beta} - y^{1+\beta} - 1}{y^\beta} = \lim_{|y|\to\infty} \frac{(y+1)^{1+\beta} - y^{1+\beta}}{y^\beta} = \lim_{|y|\to\infty} \big((1+\frac{1}{y})^{1+\beta} - 1\big)y = 1+\beta.$$

Using this and continuity, we get that there exists $C_1 > 0$ such that for all $|y| \ge 1$,

$$|(1+y)^{1+\beta} - y^{1+\beta} - 1| \le C_1|y|^\beta.$$

Note that if $x = 0$ or $y = 0$, then the desired result is trivial. So we only need to consider the case that $x \ne 0$ and $y \ne 0$. In this case, if $|x| \ge |y|$, we have

$$|(x+y)^{1+\beta} - x^{1+\beta} - y^{1+\beta}| \le |y|^{1+\beta}\left(\left|\left(1+\frac{x}{y}\right)^{1+\beta} - \left(\frac{x}{y}\right)^{1+\beta} - 1\right|\right) \le C_1|y||x|^\beta;$$

and if $|x| \le |y|$, we have

$$|(x+y)^{1+\beta} - x^{1+\beta} - y^{1+\beta}| \le |x|^{1+\beta}\left(\left|\left(1+\frac{y}{x}\right)^{1+\beta} - \left(\frac{y}{x}\right)^{1+\beta} - 1\right|\right) \le C_1|x||y|^\beta.$$

Combining the above, we immediately get the desired result. $\square$

In the remainder of this section, we always fix $\mu \in \mathcal{M}_c(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}} \setminus \{0\}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}} \setminus \{0\}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}} \setminus \{0\}$. For any random variable Y with finite mean under $\mathbb{P}_\mu$, we define

$$\mathcal{I}_r^t Y := \mathbb{P}_\mu[Y|\mathscr{F}_{t\vee 0}] - \mathbb{P}_\mu[Y|\mathscr{F}_{r\vee 0}], \quad -\infty < r, t < \infty.$$

For each $t \ge 1$, we have the following decomposition.

$$\begin{aligned}
I^{f_{\mathrm{s}}}(t) &:= \frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}} = I_1^{f_{\mathrm{s}}}(t) + I_2^{f_{\mathrm{s}}}(t) + I_3^{f_{\mathrm{s}}}(t) \\
&:= \Big(\sum_{k\in\mathbb{N}\cap[0,t-\ln t]} \frac{\mathcal{I}_{t-k-1}^{t-k} X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}}\Big) + \Big(\sum_{k\in\mathbb{N}\cap(t-\ln t,t]} \frac{\mathcal{I}_{t-k-1}^{t-k} X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}}\Big) + \Big(\frac{X_0(P_t^\alpha f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}}\Big), \\
I^{f_{\mathrm{c}}}(t) &:= \frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}} = I_1^{f_{\mathrm{c}}}(t) + I_2^{f_{\mathrm{c}}}(t) + I_3^{f_{\mathrm{c}}}(t) \\
&:= \Big(\sum_{k\in\mathbb{N}\cap[0,t-\ln t]} \frac{\mathcal{I}_{t-k-1}^{t-k} X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}}\Big) + \Big(\sum_{k\in\mathbb{N}\cap(t-\ln t,t]} \frac{\mathcal{I}_{t-k-1}^{t-k} X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}}\Big) + \Big(\frac{X_0(P_t^\alpha f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}}\Big), \\
I^{f_{\mathrm{l}}}(t) &:= \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde\beta}} = I_1^{f_{\mathrm{l}}}(t) + I_2^{f_{\mathrm{l}}}(t) + I_3^{f_{\mathrm{l}}}(t) \\
&:= \Big(\sum_{k\in\mathbb{N}\cap[0,t^2]} \frac{\mathcal{I}_{t+k+1}^{t+k} \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde\beta}}\Big) + \Big(\sum_{k\in\mathbb{N}\cap(t^2,\infty)} \frac{\mathcal{I}_{t+k+1}^{t+k} \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde\beta}}\Big) + 0,
\end{aligned}$$

where $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.5) with f replaced with f_{l}. For every $t \geq 1$, define

$$
\begin{aligned}
R_j(t) &:= \big(I_j^{f_{\mathrm{s}}}(t), I_j^{f_{\mathrm{c}}}(t), I_j^{f_{\mathrm{l}}}(t)\big), \quad j = 1,2,3, \\
R(t) &:= \Big(\frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}}, \frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}}, \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde\beta}}\Big), \\
R_0(t) &= \big(I_0^{f_{\mathrm{s}}}(t), I_0^{f_{\mathrm{c}}}(t), I_0^{f_{\mathrm{l}}}(t)\big) \\
&:= \Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k\tilde f_{\mathrm{s}}}, t^{\tilde\beta-1}\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k\tilde f_{\mathrm{c}}}, \sum_{k=0}^{\lfloor t^2\rfloor} \Upsilon_{t+k}^{-T_k\tilde f_{\mathrm{l}}}\Big),
\end{aligned}
$$

where $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.5) with f replaced with f_{l}. $\tilde f_{\mathrm{s}} := e^{\alpha(\tilde\beta-1)} f_{\mathrm{s}}$, $\tilde f_{\mathrm{c}} := e^{\alpha(\tilde\beta-1)} f_{\mathrm{c}}$ and $\tilde f_{\mathrm{l}} := \sum_{p\in\mathcal N} e^{-(\alpha-|p|b)}\langle f_{\mathrm{l}}, \phi_p\rangle_\varphi \phi_p$. The following result is a special case of Theorem 1.1.

Theorem 2.3. *Under $\widetilde{\mathbb P}_\mu$, $R(t) \xrightarrow[t\to\infty]{d} (\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}})$, where $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.4), and $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent.*

Proof. In this proof, we always work under $\widetilde{\mathbb P}_\mu$. Note that for each $t \geq 1$,

$$R(t) = R_0(t) + \big(R_1(t) - R_0(t)\big) + R_2(t) + R_3(t).$$

Note that

$$(R_1(t) - R_0(t)) = (I_1^{f_{\mathrm{s}}}(t) - I_0^{f_{\mathrm{s}}}(t), I_1^{f_{\mathrm{c}}}(t) - I_0^{f_{\mathrm{c}}}(t), I_1^{f_{\mathrm{l}}}(t) - I_0^{f_{\mathrm{l}}}(t)).$$

In the proof of Theorem 1.6(1) in [20], we proved that $I_1^{f_{\mathrm{s}}}(t) - I_0^{f_{\mathrm{s}}}(t) \xrightarrow[t\to\infty]{d} 0$, $I_2^{f_{\mathrm{s}}}(t) \xrightarrow[t\to\infty]{d} 0$ and $I_3^{f_{\mathrm{s}}}(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb P}_\mu-a.s.} 0$. In the proof of Theorem 1.6(2) in [20], we proved that $I_1^{f_{\mathrm{c}}}(t) - I_0^{f_{\mathrm{c}}}(t) \xrightarrow[t\to\infty]{d} 0$, $I_2^{f_{\mathrm{c}}}(t) \xrightarrow[t\to\infty]{d} 0$ and $I_3^{f_{\mathrm{c}}}(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb P}_\mu-a.s.} 0$. In the proof of Theorem 1.6(3) in [20], we proved that $I_1^{f_{\mathrm{l}}}(t) - I_0^{f_{\mathrm{l}}}(t) \xrightarrow[t\to\infty]{d} 0$ and $I_2^{f_{\mathrm{l}}}(t) \xrightarrow[t\to\infty]{d} 0$. Thus we have $R_1(t) - R_0(t) \xrightarrow[t\to\infty]{d} (0,0,0)$, $R_2(t) \xrightarrow[t\to\infty]{d} (0,0,0)$ and $R_3(t) \xrightarrow[t\to\infty]{d} (0,0,0)$. Combining the above results and using Slutsky's theorem, we only need to show that, under $\widetilde{\mathbb P}_\mu$,

$$R_0(t) \xrightarrow[t\to\infty]{d} (\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}}). \tag{2.6}$$

Now we prove (2.6). Since Υ_t^f is linear in f, for each $t \geq 1$,

$$\widetilde{\mathbb P}_\mu\Big[\exp\Big(i\sum_{j=\mathrm{s,c,l}} I_0^{f_j}(t)\Big)\Big] = \widetilde{\mathbb P}_\mu\Big[\exp\Big(i\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k(\tilde f_{\mathrm{s}} + t^{\tilde\beta-1}\tilde f_{\mathrm{c}})}\Big)\exp\Big(i\sum_{k=0}^{\lfloor t^2\rfloor} \Upsilon_{t+k}^{-T_k\tilde f_{\mathrm{l}}}\Big)\Big].$$

Using Proposition 2.1 with $f = \tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_{\mathrm{c}}$ and $g = -\tilde{f}_{\mathrm{l}}$, we get that there exist $C_1, \delta_1 > 0$ such that for every $t \geq 1$,

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{j=\mathrm{s,c,l}} I_0^{f_j}(t)\Big)\Big] - \exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1(T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_{\mathrm{c}})),\varphi\rangle\Big)\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\mathrm{l}}),\varphi\rangle\Big)\Big|$$

$$\leq C_1 e^{-\delta_1(t-\lfloor t-\ln t\rfloor)}.$$

We claim that

$$\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1(T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_{\mathrm{c}})),\varphi\rangle\Big)\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\mathrm{l}}),\varphi\rangle\Big) = \exp(m[f_{\mathrm{s}}] + m[f_{\mathrm{c}}] + m[-f_{\mathrm{l}}]). \tag{2.7}$$

Given this claim, we have

$$\widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{j=\mathrm{s,c,l}} I_0^{f_j}(t)\Big)\Big] \xrightarrow[t\to\infty]{} \exp(m[f_{\mathrm{s}}] + m[f_{\mathrm{c}}] + m[-f_{\mathrm{l}}]).$$

Since $I_0^{f_j}(t)$ are linear in $f_j \in \mathcal{C}_j (j = \mathrm{s,c,l})$, replacing f_j with $\theta_j f_j$, we immediately get (2.6).

Now we prove the claim (2.7). For every $f \in \mathcal{C}_{\mathrm{s}} \oplus \mathcal{C}_{\mathrm{c}}$ and $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\sum_{k=0}^{n}\langle Z_1 T_k\tilde{f},\varphi\rangle &= \sum_{k=0}^{n}\int_0^1 \langle P_u^\alpha(\eta(-iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta}),\varphi\rangle \mathrm{d}u\\
&= \sum_{k=0}^{n}\int_0^1 e^{\alpha u}\langle \eta(-iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta},\varphi\rangle \mathrm{d}u\\
&= \sum_{k=0}^{n}\int_0^1 \langle \eta(-iT_{k+1-u}f)^{1+\beta},\varphi\rangle \mathrm{d}u = \int_0^{n+1}\langle \eta(-iT_u f)^{1+\beta},\varphi\rangle \mathrm{d}u = m_{n+1}[f],
\end{aligned}$$

where $\tilde{f} = e^{\alpha(\tilde{\beta}-1)}f$. Therefore, for any $t \geq 1$,

$$\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_{\mathrm{c}}),\varphi\rangle = \eta\int_0^{\lfloor t-\ln t\rfloor+1}\big\langle\big(-iT_u(f_{\mathrm{s}} + t^{\tilde{\beta}-1}f_{\mathrm{c}})\big)^{1+\beta},\varphi\big\rangle \mathrm{d}u. \tag{2.8}$$

Note that for each $u \geq 0$, $T_u f_{\mathrm{c}} = f_{\mathrm{c}}$. Also note that according to Step 1 in the proof of [20, Lemma 2.6], there exist $\delta > 0$ and $h \in \mathcal{P}$ (depending only on f_{s}) such that for each $u \geq 0$, $|T_u f_{\mathrm{s}}| \leq e^{-\delta u}h$. It follows from Lemma 2.2 that there exists $C > 0$ such that for all $u \geq 0$ and $t \geq 0$,

$$\begin{aligned}
&|(-i(T_u f_{\mathrm{s}} + t^{\tilde{\beta}-1}T_u f_{\mathrm{c}}))^{1+\beta} - (-iT_u f_{\mathrm{s}})^{1+\beta} - (-it^{\tilde{\beta}-1}T_u f_{\mathrm{c}})^{1+\beta}|\\
&= |-i|^{1+\beta}|(T_u f + t^{\tilde{\beta}-1}T_u f_{\mathrm{c}})^{1+\beta} - (T_u f_{\mathrm{s}})^{1+\beta} - (t^{\tilde{\beta}-1}T_u f_{\mathrm{c}})^{1+\beta}|
\end{aligned}$$

$$\overset{\text{Lemma 2.2}}{\leq} C(t^{-\frac{\beta}{1+\beta}}|T_u f_{\mathrm{s}}||T_u f_{\mathrm{c}}|^\beta + t^{-\frac{1}{1+\beta}}|T_u f_{\mathrm{s}}|^\beta |T_u f_{\mathrm{c}}|)$$

$$\leq C(t^{-\frac{\beta}{1+\beta}}e^{-\delta u}h|f_{\mathrm{c}}|^\beta + t^{-\frac{1}{1+\beta}}e^{-\delta\beta u}h^\beta |f_{\mathrm{c}}|). \tag{2.9}$$

This means that there exists $C_1 > 0$ such that for all $t \geq 1$,

$$\begin{aligned}
&\Big|\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_c), \varphi\rangle\Big) - m_{\lfloor t-\ln t\rfloor+1}[f_{\mathrm{s}}] - \frac{1}{t}m_{\lfloor t-\ln t\rfloor+1}[f_{\mathrm{c}}]\Big| \\
&\overset{(2.8),(1.3)}{\leq} \Big|\eta\int_0^{\lfloor t-\ln t\rfloor+1}\big\langle\big(-iT_u(f_{\mathrm{s}} + t^{\tilde{\beta}-1}f_{\mathrm{c}})\big)^{1+\beta}, \varphi\big\rangle \mathrm{d}u - \\
&\eta\int_0^{\lfloor t-\ln t\rfloor+1}\langle(-iT_u f_{\mathrm{s}})^{1+\beta}, \varphi\rangle \mathrm{d}u - \eta\int_0^{\lfloor t-\ln t\rfloor+1}\langle(-iT_u f_{\mathrm{c}})^{1+\beta}, \varphi\rangle \mathrm{d}u\Big| \\
&\overset{(2.9)}{\leq} C_1\int_0^{\lfloor t-\ln t\rfloor+1}\langle t^{-\frac{\beta}{1+\beta}}e^{-\delta u}h|f_{\mathrm{c}}|^\beta + t^{-\frac{1}{1+\beta}}e^{-\delta\beta u}h^\beta|f_{\mathrm{c}}|, \varphi\rangle \mathrm{d}u \\
&\leq C_1 t^{-\frac{\beta}{1+\beta}}\langle h|f_{\mathrm{c}}|^\beta, \varphi\rangle\int_0^\infty e^{-\delta u}\mathrm{d}u + C_1 t^{-\frac{1}{1+\beta}}\langle h^\beta|f_{\mathrm{c}}|, \varphi\rangle\int_0^\infty e^{-\delta\beta u}\mathrm{d}u \\
&\xrightarrow[t\to\infty]{} 0.
\end{aligned}$$

Combining this with (1.4), we get that

$$\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_c), \varphi\rangle\Big) = \exp(m[f_{\mathrm{s}}] + m[f_{\mathrm{c}}]). \tag{2.10}$$

Also note that according to the Step 1 in the Proof of Theorem 1.6.(3) in [20], we have

$$\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\mathrm{l}}), \varphi\rangle\Big) = \exp(m[-f_{\mathrm{l}}]). \tag{2.11}$$

Thus the desired claim follows from (2.10) and (2.11). □

Proof of Theorem 1.1. We first recall some facts about weak convergence which will be used later. For $f : \mathbb{R}^d \mapsto \mathbb{R}$, let

$$\|f\|_L := \sup_{x\neq y}\frac{|f(x) - f(y)|}{|x-y|}$$

and $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. For any probability distributions μ_1 and μ_2 on $\mathbb{R}^d$, define

$$d(\mu_1, \mu_2) := \sup\Big\{\Big|\int f\mathrm{d}\mu_1 - \int f\mathrm{d}\mu_2\Big| : \|f\|_{BL} \leq 1\Big\}.$$

Then d is a metric. It follows from [7, Theorem 11.3.3] that the topology generated by d is equivalent to the weak convergence topology. Using the definition, we can easily see that, if μ_1 and μ_2 are the distributions of two $\mathbb{R}^d$-valued random variables X and Y respectively, defined on same probability space then

$$d(\mu_1, \mu_2) \leq \mathbb{E}|X - Y|. \tag{2.12}$$

In this proof, let us fix $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}} \setminus \{0\}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}} \setminus \{0\}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}} \setminus \{0\}$. Recall that $S(t)$ $(t \geq 0)$ is given by (1.6). For every $r, t > 0$, let

$$S(t,r) := \Big(e^{-\alpha t}\|X_t\|, \frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde{\beta}}}, \frac{X_{t+r}(f_{\mathrm{c}})}{\|(t+r)X_{t+r}\|^{1-\tilde{\beta}}}, \frac{X_{t+r}(f_{\mathrm{l}}) - \mathrm{x}_{t+r}(f_{\mathrm{l}})}{\|X_{t+r}\|^{1-\tilde{\beta}}}\Big),$$

and

$$\widetilde{S}(t,r) = \Big(e^{-\alpha(t+r)}\|X_{t+r}\| - e^{-\alpha t}\|X_t\|, 0, 0, 0\Big),$$

where, for any $t > 0$, $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.5) with f replaced with f_{l}. Then $S(t+r) = S(t,r) + \widetilde{S}(t,r)$. We claim that

(2.13) for each $t > 0$, under $\widetilde{\mathbb{P}}_\mu$, we have
$$S(t,r) \xrightarrow[r\to\infty]{d} (\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}}),$$
where $\widetilde{H}_t$ has the distribution of $\{e^{-\alpha t}\|X_t\|; \widetilde{\mathbb{P}}_\mu\}$, $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.4), and $\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent.

For every $r, t \geq 0$, let $\mathcal{D}(r)$ and $\mathcal{D}(r,t)$ be the distributions of $S(r)$ and $S(t,r)$ under $\widetilde{\mathbb{P}}_\mu$ respectively; let $\widetilde{\mathcal{D}}(t)$ and $\mathcal{D}$ be the distributions of $(\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}})$ and $(\widetilde{H}_\infty, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}})$, respectively. Then for each $\gamma \in (0,\beta)$, there exist constant $C > 0$ such that for every $t > 0$,

$$\begin{aligned}
&\varlimsup_{r\to\infty} d(\mathcal{D}(t+r), \mathcal{D}) \\
&\overset{\text{triangle inequality}}{\leq} \varlimsup_{r\to\infty} \Big(d\big(\mathcal{D}(t+r), \mathcal{D}(t,r)\big) + d\big(\mathcal{D}(t,r), \widetilde{\mathcal{D}}(t)\big) + d\big(\widetilde{\mathcal{D}}(t), \mathcal{D}\big)\Big) \\
&\overset{(2.12)}{\leq} \varlimsup_{r\to\infty} \widetilde{\mathbb{P}}_\mu[|S(t+r) - S(t,r)|] + \varlimsup_{r\to\infty} d\big(\mathcal{D}(t,r), \widetilde{\mathcal{D}}(t)\big) + \widetilde{\mathbb{P}}_\mu[|H_t - H_\infty|] \\
&\overset{(2.13)}{\leq} \varlimsup_{r\to\infty} \widetilde{\mathbb{P}}_\mu[|H_t - H_{t+r}|] + \widetilde{\mathbb{P}}_\mu[|H_t - H_\infty|] \\
&\overset{\text{Hölder inequality}}{\leq} \varlimsup_{r\to\infty} \mathbb{P}_\mu(D^c)^{-1}(\|H_t - H_{t+r}\|_{L_{1+\gamma}(\mathbb{P}_\mu)} + \|H_t - H_\infty\|_{L_{1+\gamma}(\mathbb{P}_\mu)}) \\
&\overset{[20,\text{ Lemma 3.3}]}{\leq} Ce^{-\alpha\tilde{\gamma}t}. \qquad (2.14)
\end{aligned}$$

Therefore,

$$\varlimsup_{r\to\infty} d\big(\mathcal{D}(r), \mathcal{D}\big) = \varlimsup_{t\to\infty} \varlimsup_{r\to\infty} d\big(\mathcal{D}(t+r), \mathcal{D}\big) \overset{(2.14)}{\leq} \varlimsup_{t\to\infty} Ce^{-\alpha\tilde{\gamma}t} = 0.$$

The desired result now follows immediately.

Now we prove the claim (2.13). For every $r, t > 0$, let

$$\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_{\mathrm{l}} \in \mathbb{R} \mapsto k(\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_{\mathrm{l}}, r, t)$$

be the characteristic function of $S(t,r)$ under $\widetilde{\mathbb{P}}_\mu$. Then for each $\theta,\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}}\in\mathbb{R}$ and $r,t>0$,

$$
\begin{aligned}
k(\theta,\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t) &= \widetilde{\mathbb{P}}_\mu\big[\exp\big(i\theta e^{-\alpha t}\|X_t\| + A(\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t,\infty)\big)\big]\\
&\overset{\text{bounded convergence}}{=} \lim_{u\to\infty}\frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\big[\exp\big(i\theta e^{-\alpha t}\|X_t\| + A(\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t,u)\big);D^c\big],
\end{aligned}
\tag{2.15}
$$

where for each $u\in[0,\infty]$,

$$
\begin{aligned}
&A(\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t,u)\\
&:= i\theta_{\mathrm{s}}\frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde\beta}} + i\theta_{\mathrm{c}}\frac{X_{t+r}(f_{\mathrm{c}})}{\|(t+r)X_{t+r}\|^{1-\tilde\beta}} + i\theta_{\mathrm{l}}\frac{X_{t+r}(f_{\mathrm{l}})-\mathbb{P}_\mu[\mathrm{x}_{t+r}(f_{\mathrm{l}})|\mathscr{F}_u]}{\|X_{t+r}\|^{1-\tilde\beta}}\\
&\overset{(1.2)}{=} i\theta_{\mathrm{s}}\frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde\beta}} + \frac{i\theta_{\mathrm{c}}}{(t+r)^{1-\tilde\beta}}\frac{X_{t+r}(f_{\mathrm{c}})}{\|X_{t+r}\|^{1-\tilde\beta}} +\\
&\qquad i\theta_{\mathrm{l}}\frac{X_{t+r}(f_{\mathrm{l}})-\sum_{p\in\mathbb{Z}_+^d:\alpha\tilde\beta>|p|b}e^{(\alpha-|p|b)(t+r)}e^{-(\alpha-|p|b)u}X_u(\phi_p)}{\|X_{t+r}\|^{1-\tilde\beta}}.
\end{aligned}
\tag{2.16}
$$

Now for each $t>0$, we get

$$
\begin{aligned}
&\lim_{r\to\infty}k(\theta,\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t)\\
&\overset{(2.15)}{=} \lim_{r\to\infty}\lim_{u\to\infty}\frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\big[\exp\{i\theta e^{-\alpha t}\|X_t\|\}\mathbf{1}_{\|X_t\|>0}\mathbb{P}_\mu[\exp\{A(\theta_{\mathrm{s}},\theta_{\mathrm{c}},\theta_{\mathrm{l}},r,t,u)\}\mathbf{1}_{D^c}|\mathscr{F}_t]\big]\\
&\overset{\text{(2.16), Markov property}}{=} \lim_{r\to\infty}\lim_{u\to\infty}\frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu\Bigg[\exp\{i\theta e^{-\alpha t}\|X_t\|\}\mathbf{1}_{\|X_t\|>0}\times\\
&\qquad\qquad \mathbb{P}_{X_t}\Bigg[\exp\Bigg\{A\Bigg(\theta_{\mathrm{s}},\theta_{\mathrm{c}}\Big(\frac{r}{t+r}\Big)^{1-\tilde\beta},\theta_{\mathrm{l}},r,0,u-t\Bigg)\Bigg\}\mathbf{1}_{D^c}\Bigg]\Bigg]\\
&\overset{\text{bounded convergence}}{=} \lim_{r\to\infty}\mathbb{P}_\mu\Bigg[\exp\{i\theta e^{-\alpha t}\|X_t\|\}\mathbf{1}_{\|X_t\|>0}\frac{\mathbb{P}_{X_t}(D^c)}{\mathbb{P}_\mu(D^c)}\times\\
&\qquad\qquad \widetilde{\mathbb{P}}_{X_t}\Bigg[\exp\Bigg\{A\Bigg(\theta_{\mathrm{s}},\theta_{\mathrm{c}}\Big(\frac{r}{t+r}\Big)^{1-\tilde\beta},\theta_{\mathrm{l}},r,0,\infty\Bigg)\Bigg\}\Bigg]\Bigg].\\
&\overset{\text{Theorem 2.3}}{=} \mathbb{P}_\mu\Big[\exp\{i\theta e^{-\alpha t}\|X_t\|\}\mathbf{1}_{\|X_t\|>0}\frac{\mathbb{P}_{X_t}(D^c)}{\mathbb{P}_\mu(D^c)}\Big]\Big(\prod_{j=\mathrm{s},\mathrm{c}}\exp\{m[\theta_j f_j]\}\Big)\exp\{m[-\theta_{\mathrm{l}}f_{\mathrm{l}}]\}\\
&= \widetilde{\mathbb{P}}_\mu[\exp\{i\theta e^{-\alpha t}\|X_t\|\}]\Big(\prod_{j=\mathrm{s},\mathrm{c}}\exp\{m[\theta_j f_j]\}\Big)\exp\{m[-\theta_{\mathrm{l}}f_{\mathrm{l}}]\}.
\end{aligned}
$$

$\square$

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