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Stable Central Limit Theorems for Super Ornstein-Uhlenbeck Processes, II

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Abstract This paper is a continuation of our recent paper (Electron. J. Probab. **24** (2019), no. 141) and is devoted to the asymptotic behavior of a class of supercritical super Ornstein-Uhlenbeck processes $(X_t)_{t\geq 0}$ with branching mechanisms of infinite second moment. In the aforementioned paper, we proved stable central limit theorems for $X_t(f)$ for *some* functions f of polynomial growth in three different regimes. However, we were not able to prove central limit theorems for $X_t(f)$ for *all* functions f of polynomial growth. In this note, we show that the limiting stable random variables in the three different regimes are independent, and as a consequence, we get stable central limit theorems for $X_t(f)$ for *all* functions f of polynomial growth.

Keywords Superprocesses, Ornstein-Uhlenbeck processes, stable distribution, central limit theorem, law of large numbers, branching rate regime

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1 Introduction and main result

Let $d \in \mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{R}_+ := [0, \infty)$. Let $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in\mathbb{R}^d}\}$ be an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck process (OU process) with generator

$$Lf(x) = \frac{1}{2}\sigma^2\Delta f(x) - bx\cdot\nabla f(x), \quad x \in \mathbb{R}^d, f \in C^2(\mathbb{R}^d),$$

where $\sigma > 0$ and $b > 0$ are constants. Let ψ be a function on $\mathbb{R}_+$ of the form

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(\mathrm{d}y), \quad z \in \mathbb{R}_+,$$

where $\alpha > 0$, $\rho \geq 0$ and π is a measure on $(0, \infty)$ with $\int_{(0,\infty)}(y \wedge y^2)\pi(\mathrm{d}y) < \infty$. ψ is referred to as a branching mechanism and π is referred to as the Lévy measure of ψ. Denote by $\mathcal{M}(\mathbb{R}^d)$ ($\mathcal{M}_c(\mathbb{R}^d)$) the space of all finite Borel measures (of compact support) on $\mathbb{R}^d$. Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R})$ ($\mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$) the space of all $\mathbb{R}$-valued ($\mathbb{R}_+$-valued) Borel functions on $\mathbb{R}^d$. For $f, g \in \mathcal{B}(\mathbb{R}^d, \mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$, write $\mu(f) = \int f(x)\mu(\mathrm{d}x)$ and $\langle f, g\rangle = \int f(x)g(x)\mathrm{d}x$ whenever the integrals make sense. We say a real-valued Borel function f on $\mathbb{R}_+ \times \mathbb{R}^d$ is *locally bounded* if, for each $t \in \mathbb{R}_+$, we have $\sup_{s\in[0,t],x\in\mathbb{R}^d} |f(s, x)| < \infty$. For any $\mu \in \mathcal{M}(\mathbb{R}^d)$, we write $\|\mu\| = \mu(1)$. For any σ-finite signed measure μ, denote by $|\mu|$ the total variation measure of μ.

We say that an $\mathcal{M}(\mathbb{R}^d)$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ is a *super Ornstein-Uhlenbeck process (super-OU process)* with branching mechanism ψ, or a (ξ, ψ)-*superprocess*, if for each non-negative bounded Borel function f on $\mathbb{R}^d$, we have

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d),$$

where $(t, x) \mapsto V_t f(x)$ is the unique locally bounded non-negative solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t \psi(V_{t-s} f(\xi_s))\mathrm{d}s\Big] = \Pi_x[f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0.$$

The existence of such super-OU process X is well known, see [8, 16] for instance.

There have been many central limit theorem type results for branching processes, branching diffusions and superprocesses, under the second moment condition. See [1, 3–6, 9, 11–14, 18, 20–23]. For a detailed literature review, see [19, Section 1.1]. There are also central limit theorem type results for supercritical branching processes and branching Markov processes with branching mechanisms of infinite second moment. For earlier papers, see [2, 10]. Recently, Marks and Miłoś [17] established some spatial central limit theorems for supercritical branching OU processes with a special stable offspring distribution. In [19], we established stable central limit theorems for super-OU processes X with branching mechanisms ψ satisfying the following two assumptions.

Assumption 1 (Grey's condition) There exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and $\int_{z'}^\infty \psi(z)^{-1}\mathrm{d}z < \infty$.

Assumption 2 There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(\mathrm{d}y) - \frac{\eta\mathrm{d}y}{\Gamma(-1-\beta)y^{2+\beta}}\Big| < \infty$$

for some $\delta > 0$.

It is known (see [15, Theorems 12.5 & 12.7] for example) that, under Assumption 1, the *extinction event* $D := \{\exists t \geq 0 \text{ such that } \|X_t\| = 0\}$ is non-trivial with respect to $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}(\mathbb{R}^d) \setminus \{0\}$. It follows from [19, Lemma 2.2] that, if Assumption 2 holds, then η and β are uniquely determined by the Lévy measure π. Throughout this paper, β and η always stand for the constants in Assumption 2.

We now recall some notation and basic facts from [19]. We use $(P_t)_{t\geq 0}$ to denote the transition semigroup of ξ. Define $P_t^\alpha f(x) := e^{\alpha t} P_t f(x) = \Pi_x[e^{\alpha t} f(\xi_t)]$ for each $x \in \mathbb{R}^d$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$. It is known that $\mathbb{P}_\mu[X_t(f)] = \mu(P_t^\alpha f)$ for all $\mu \in \mathcal{M}(\mathbb{R}^d)$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$. The OU process ξ has an invariant probability on $\mathbb{R}^d$:

$$\varphi(x)\mathrm{d}x := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2} \exp\Big(-\frac{b}{\sigma^2}|x|^2\Big)\mathrm{d}x.$$

Let $L^2(\varphi)$ be the Hilbert space with inner product

$$\langle f_1, f_2\rangle_\varphi := \int_{\mathbb{R}^d} f_1(x) f_2(x)\varphi(x)\mathrm{d}x, \quad f_1, f_2 \in L^2(\varphi).$$

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. It is known that $(P_t)_{t\geq 0}$ is a strongly continuous semigroup in $L^2(\varphi)$ and its generator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. The eigenfunctions of L consists of a family of polynomials $\{\phi_p : p \in \mathbb{Z}_+^d\}$ which forms a complete orthonormal basis of $L^2(\varphi)$. For each $p \in \mathbb{Z}_+^d$, ϕ_p is an eigenfunction of L corresponding to the eigenvalue $b|p|$, where $|p| := \sum_{k=1}^d p_k$. For each function $f \in L^2(\varphi)$, define the order of f as $\kappa_f := \inf\big\{k \geq 0 : \exists\, p \in \mathbb{Z}_+^d,\ \text{s.t. } |p| = k \text{ and } \langle f, \phi_p\rangle_\varphi \neq 0\big\}$ with the convention that $\inf \emptyset = \infty$.

For $p \in \mathbb{Z}_+^d$, define $H_t^p := e^{-(\alpha - |p|b)t} X_t(\phi_p), t \geq 0$. For each $u \neq -1$, we write $\tilde{u} = u/(1+u)$. We have shown in [19, Lemma 3.2] that for any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale. Furthermore, if $\alpha\tilde{\beta} > |p|b$, then for every $\gamma \in (0, \beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$; thus $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$. We will write H_t^0 and H_∞^0 as H_t and H_∞, respectively.

Denote by $\mathcal{P} \subset L^2(\varphi)$ the class of functions of polynomial growth on $\mathbb{R}^d$, i.e., $\mathcal{P} := \{f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : \exists C > 0, n \in \mathbb{Z}_+ \text{ s.t. } \forall x \in \mathbb{R}^d, |f(x)| \leq C(1+|x|)^n\}$. Define $\mathcal{C}_\mathrm{s} := \mathcal{P} \cap \overline{\mathrm{Span}}\{\phi_p : \alpha\tilde{\beta} < |p|b\}$, $\mathcal{C}_\mathrm{c} := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} = |p|b\}$, and $\mathcal{C}_\mathrm{l} := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} > |p|b\}$. Note that $\mathcal{C}_\mathrm{s}$ is an infinite dimensional space, $\mathcal{C}_\mathrm{l}$ and $\mathcal{C}_\mathrm{c}$ are finite dimensional spaces, and $\mathcal{C}_c$ might be empty. Define a semigroup

$$T_t f := \sum_{p\in\mathbb{Z}_+^d} e^{-\big||p|b - \alpha\tilde{\beta}\big|t} \langle f, \phi_p\rangle_\varphi \phi_p, \quad t \geq 0, f \in \mathcal{P}, \tag{1.1}$$

and a family of functionals

$$m_t[f] := \eta \int_0^t \mathrm{d}u \int_{\mathbb{R}^d} \big(-iT_u f(x)\big)^{1+\beta} \varphi(x)\mathrm{d}x, \quad 0 \leq t < \infty, f \in \mathcal{P}. \tag{1.2}$$

Let us recall the definition of the non-integer power of complex numbers here. For each $z \in \mathbb{C} \setminus (-\infty, 0]$, we define $\log z := \log|z| + i \arg z$ where $\arg z \in (-\pi, \pi)$ is uniquely determined so that $z = |z|e^{i\arg z}$. For each $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\gamma \in \mathbb{R}$, we define $z^\gamma := e^{\gamma \log z}$. For each $\gamma \geq 0$, we define $0^\gamma := \lim_{z\to 0, z\in\mathbb{C}\setminus(-\infty,0]} z^\gamma = \mathbf{1}_{\gamma=0}$.

We have shown in [19, Lemma 2.6 and Proposition 2.7] that,

for each $f \in \mathcal{P}$, there exists a $(1+\beta)$-stable random variable ζ^f with characteristic function $\theta \mapsto e^{m[\theta f]}, \theta \in \mathbb{R}$, where

$$m[f] := \begin{cases} \lim_{t\to\infty} m_t[f], & f \in \mathcal{C}_{\mathrm{s}} \oplus \mathcal{C}_{\mathrm{l}}, \\ \lim_{t\to\infty} \frac{1}{t} m_t[f], & f \in \mathcal{P} \setminus \mathcal{C}_{\mathrm{s}} \oplus \mathcal{C}_{\mathrm{l}}. \end{cases} \tag{1.3}$$

For each $\mu \in \mathcal{M}(\mathbb{R}^d) \setminus \{0\}$, write $\widetilde{\mathbb{P}}_\mu(\cdot) := \mathbb{P}_\mu(\cdot|D^c)$. We also proved in [19, Theorem 1.6] that

if $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$, then under $\widetilde{\mathbb{P}}_\mu$,

$$\begin{aligned} &e^{-\alpha t}\|X_t\| \xrightarrow[t\to\infty]{\text{a.s.}} \widetilde{H}_\infty; \quad \frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{s}}}; \\ &\frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{c}}}; \quad \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{-f_{\mathrm{l}}}, \end{aligned} \tag{1.4}$$

where $\widetilde{H}_\infty$ has the distribution of $\{H_\infty; \widetilde{\mathbb{P}}_\mu\}$; $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.3); and

$$\mathrm{x}_t(f) := \sum_{p\in\mathbb{Z}_+^d : \alpha\tilde{\beta} > |p|b} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_\infty^p, \quad t \geq 0, f \in \mathcal{P}.$$

Any $f \in \mathcal{P}$ can be decomposed as $f = f_s + f_c + f_l$ with $f_s \in \mathcal{C}_{\mathrm{s}}$, $f_c \in \mathcal{C}_{\mathrm{c}}$ and $f_l \in \mathcal{C}_{\mathrm{l}}$. In [19], we were not able to establish a central limit theorem for f if $\|f_{\mathrm{s}}\|_{L^2(\varphi)} > 0, \|f_{\mathrm{c}}\|_{L^2(\varphi)} = 0$ and $\|f_{\mathrm{l}}\|_{L^2(\varphi)} > 0$. We conjectured there that the limit random variables in (1.4) for $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$ are independent. Once this asymptotic independence is established, a central limit theorem for $X_t(f)$ for all $f \in \mathcal{P}$ would follow.

The main purpose of this note is to show that the limit random variables in (1.4) are independent.

Theorem 1.1 *If $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$, then under $\widetilde{\mathbb{P}}_\mu$,*

$$\begin{aligned} S(t) &:= \left(e^{-\alpha t}\|X_t\|, \frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde{\beta}}} \right) \\ &\xrightarrow[t\to\infty]{d} (\widetilde{H}_\infty, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}}), \end{aligned} \tag{1.5}$$

where $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.4) with f replaced with f_{l}; $\widetilde{H}_\infty$ has the distribution of $\{H_\infty; \widetilde{\mathbb{P}}_\mu\}$; $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.3); $\widetilde{H}_\infty$, $\zeta^{f_{\mathrm{s}}}$, $\zeta^{f_{\mathrm{c}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent.

As a corollary of this theorem, we get central limit theorems for $X_t(f)$ for all $f \in \mathcal{P}$.

Corollary 1.2 *Let $\mu \in \mathcal{M}_c(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{P}$. Let $f = f_{\mathrm{s}} + f_{\mathrm{c}} + f_{\mathrm{l}}$ be the unique decomposition of f with $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$. Then under $\widetilde{\mathbb{P}}_\mu$, it holds that*

1. *if $f_{\mathrm{c}} \equiv 0$, then*

$$\frac{X_t(f) - \mathrm{x}_t(f)}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{s}}} + \zeta^{-f_{\mathrm{l}}},$$

where $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in (1.3), $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent;

2. *if $f_{\mathrm{c}} \not\equiv 0$, then*
$$\frac{X_t(f) - \mathrm{x}_t(f)}{\|tX_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{c}}},$$
where $\zeta^{f_{\mathrm{c}}}$ is the $(1+\beta)$-stable random variable described in (1.3).

Remark 1.3 We mention here that the branching mechanism ψ considered in this paper can be written in a form of $\psi(z) = -\alpha z + z^{1+\beta}(\eta + \epsilon(z))$ where $\epsilon(z) \to 0$ as $z \to 0$ (See [19, Remark 1.3]). It would be interesting to consider the more general cases $\psi(z) = -\alpha z + z^{1+\beta} l(z)$ where $l(z)$ is only assumed to be a slowly varying function. For this more general case, the arguments of [19] and this paper no longer work. It seems that new idea and treatment are needed.

2 Proof of main result

We first make some preparations before proving Theorem 1.1. For every $t \geq 0$ and $f \in \mathcal{P}$, define
$$Z_t f := \int_0^t P^\alpha_{t-s}\big(\eta(-iP^\alpha_s f)^{1+\beta}\big)\mathrm{d}s, \quad \Upsilon^f_t := \frac{X_{t+1}(f) - X_t(P^\alpha_1 f)}{\|X_t\|^{1-\tilde{\beta}}}.$$
From [19, Theorem 3.4] we know that, for each $f \in \mathcal{P}$, $\langle Z_1 f, \varphi\rangle$ is the characteristic exponent of the weak limit of Υ^f_t. For $g = g_{\mathrm{s}} + g_{\mathrm{c}} + g_{\mathrm{l}} \in \mathcal{P}$ with $g_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}, g_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $g_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$, we define $\overline{\mathcal{P}}_g := \{\theta_{\mathrm{s}} T_n g_{\mathrm{s}} + \theta_{\mathrm{c}} T_n g_{\mathrm{c}} + \theta_{\mathrm{l}} T_n g_{\mathrm{l}} : n \in \mathbb{Z}_+, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_{\mathrm{l}} \in [-1,1]\}$, where T_n is the operator defined in (1.1). The following Lemma 2.1 can be proved using an argument similar to that used in the proof of [19, Lemma 2.9]. We omit the details here.

Lemma 2.1 *For any $g \in \mathcal{P}$ there exists non-negative $h \in \mathcal{P}$ such that for all $f \in \overline{\mathcal{P}}_g$ and $t \geq 0$, we have $|P_t(Z_1 f - \langle Z_1 f, \varphi\rangle)| \leq e^{-bt} h$.*

The following result is a generalization of [19, Proposition 3.5], whose proof is similar to that of [19, Proposition 3.5], with Lemma 2.1 replacing the role of [19, Lemma 2.9]. Let $(\mathscr{F}_t)_{t\geq 0}$ be the natural filtration of X.

Proposition 2.2 *For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $g \in \mathcal{P}$, there exist $C, \delta > 0$ such that for all $t \geq 1$ and $f \in \overline{\mathcal{P}}_g$, we have*
$$\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon^f_t} - e^{\langle Z_1 f, \varphi\rangle}; D^c|\mathscr{F}_t]\big|\Big] \leq Ce^{-\delta t}.$$

The following generalization of [19, Proposition 3.5] will be used later in the proof of Theorem 2.5, a special case of Theorem 1.1. Note that the constants C and δ in the next result depend only on $f, g \in \mathcal{P}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, do not depend on n_1, n_2, f_j, g_j and t (as long as $t \geq n_1 + 1$).

Proposition 2.3 *For any $f, g \in \mathcal{P}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exist $C, \delta > 0$ such that for all $n_1, n_2 \in \mathbb{Z}_+$, $(f_j)_{j=0}^{n_1} \subset \overline{\mathcal{P}}_f$, $(g_j)_{j=0}^{n_2} \subset \overline{\mathcal{P}}_g$ and $t \geq n_1 + 1$, we have*
$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{k=0}^{n_1} e^{i\Upsilon^{f_k}_{t-k-1}}\Big)\Big(\prod_{k=0}^{n_2} e^{i\Upsilon^{g_k}_{t+k}}\Big)\Big] - \Big(\prod_{k=0}^{n_1} e^{\langle Z_1 f_k, \varphi\rangle}\Big)\Big(\prod_{k=0}^{n_2} e^{\langle Z_1 g_k, \varphi\rangle}\Big)\Big| \leq Ce^{-\delta(t-n_1)}. \tag{2.1}$$

Proof In this proof, we fix $f, g \in \mathcal{P}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $n_1, n_2 \in \mathbb{Z}_+$, $(f_j)_{j=0}^{n_1} \subset \overline{\mathcal{P}}_f$, $(g_j)_{j=0}^{n_2} \subset \overline{\mathcal{P}}_g$ and $t \geq n_1 + 1$. For any $k_1 \in \{-1, 0, \ldots, n_1\}$ and $k_2 \in \{-1, 0, \ldots, n_2\}$, define
$$a_{k_1,k_2} := \widetilde{\mathbb{P}}_\mu\Big[\Big(\prod_{j=k_1+1}^{n_1} e^{i\Upsilon^{f_j}_{t-j-1}}\Big)\Big(\prod_{j=0}^{k_2} e^{i\Upsilon^{g_j}_{t+j}}\Big)\Big]\Big(\prod_{j=0}^{k_1} e^{\langle Z_1 f_j, \varphi\rangle}\Big)\Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j, \varphi\rangle}\Big),$$

where we used the convention that $\prod_{j=0}^{-1} = 1$. Then for all $k_2 \in \{0, \dots, n_2\}$, we have

$$\begin{aligned} a_{-1,k_2} - a_{-1,k_2-1} = \frac{1}{\mathbb{P}_\mu(D^c)} \Big(\prod_{j=k_2+1}^{n_2} e^{\langle Z_1 g_j, \varphi \rangle} \Big) \times \\ \mathbb{P}_\mu \Big[\Big(\prod_{j=0}^{n_1} e^{i \Upsilon_{t-j-1}^{f_j}} \Big) \Big(\prod_{j=0}^{k_2-1} e^{i \Upsilon_{t+j}^{g_j}} \Big) \mathbb{P}_\mu [e^{i \Upsilon_{t+k_2}^{g_{k_2}}} - e^{\langle Z_1 g_{k_2}, \varphi \rangle}; D^c | \mathscr{F}_{t+k_2}] \Big]. \end{aligned} \tag{2.2}$$

By Proposition 2.2, there exist $C_0, \delta_0 > 0$, depending only on μ and g, such that for each $k_2 \in \{0, \dots, n_2\}$,

$$\begin{aligned} & |a_{-1,k_2} - a_{-1,k_2-1}| \overset{(2.2)}{\le} \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu \Big[\big| \mathbb{P}_\mu [e^{i \Upsilon_{t+k_2}^{g_{k_2}}} - e^{\langle Z_1 g_{k_2}, \varphi \rangle}; D^c | \mathscr{F}_{t+k_2}] \big| \Big] \\ & \le C_0 e^{-\delta_0 (t+k_2)}. \end{aligned} \tag{2.3}$$

Similarly, for any $k_1 \in \{0, \dots, n_1\}$,

$$\begin{aligned} a_{k_1-1,-1} - a_{k_1,-1} = \frac{1}{\mathbb{P}_\mu(D^c)} \Big(\prod_{j=0}^{k_1-1} e^{\langle Z_1 f_j, \varphi \rangle} \Big) \Big(\prod_{j=0}^{n_2} e^{\langle Z_1 g_j, \varphi \rangle} \Big) \times \\ \mathbb{P}_\mu \Big[\mathbb{P}_\mu \big[e^{i \Upsilon_{t-k_1-1}^{f_{k_1}}} - e^{\langle Z_1 f_{k_1}, \varphi \rangle}; D^c \big| \mathscr{F}_{t-k_1-1} \big] \prod_{j=k_1+1}^{n_1} e^{i \Upsilon_{t-j-1}^{f_j}} \Big]. \end{aligned} \tag{2.4}$$

By Proposition 2.2, there exist $C_1, \delta_1 > 0$, depending only on μ and f, such that for any $k_1 \in \{0, \dots, n_1\}$,

$$\begin{aligned} & |a_{k_1-1,-1} - a_{k_1,-1}| \overset{(2.4)}{\le} \frac{1}{\mathbb{P}_\mu(D^c)} \mathbb{P}_\mu \Big[\big| \mathbb{P}_\mu [e^{i \Upsilon_{t-k_1-1}^{f_{k_1}}} - e^{\langle Z_1 f_{k_1}, \varphi \rangle}; D^c | \mathscr{F}_{t-k_1-1}] \big| \Big] \\ & \le C_1 e^{-\delta_1 (t-k_1)}. \end{aligned} \tag{2.5}$$

Therefore, there exist $C, \delta > 0$, depending only on f, g and μ, such that

$$\begin{aligned} & \text{LHS of (2.1)} = |a_{-1,n_2} - a_{n_1,-1}| \le \sum_{k=0}^{n_1} |a_{k-1,-1} - a_{k,-1}| + \sum_{k=0}^{n_2} |a_{-1,k} - a_{-1,k-1}| \\ & \overset{(2.3),(2.5)}{\le} \sum_{k=0}^{n_1} C_1 e^{-\delta_1 (t-k)} + \sum_{k=0}^{n_2} C_0 e^{-\delta_0 (t+k)} \le C e^{-\delta (t-n_1)}. \end{aligned}$$

□

The following analytic result is elementary, and will also be used in the proof of Theorem 2.5.

Lemma 2.4 *There exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}$,*

$$|(ix+iy)^{1+\beta} - (ix)^{1+\beta} - (iy)^{1+\beta}| \le C(|x||y|^\beta + |x|^\beta |y|).$$

Proof Note the desired result holds if $x = 0$ or $y = 0$. So, we can assume $|x| > 0$ and $|y| > 0$. Noticing also the symmetry between x and y, we can assume without loss of generality that $|x| \ge |y|$. Also note that $\overline{z^{1+\beta}} = \bar{z}^{1+\beta}$ for each $z \in \mathbb{C} \setminus (-\infty, 0)$ where $\bar{z}$ is the complex conjugate of z. It is easy to see that

$$|(ix+iy)^{1+\beta} - (ix)^{1+\beta} - (iy)^{1+\beta}| = |(-ix-iy)^{1+\beta} - (-ix)^{1+\beta} - (-iy)^{1+\beta}|.$$

Thus we can further assume without loss of generality that $y > 0$. Note that for any $z \in \mathbb{C}$,

$$\lim_{r\to\infty, r>0} \frac{(r\pm 1)^{1+\beta} - r^{1+\beta} - z}{r^\beta} = \lim_{r\to\infty, r>0} \frac{(r\pm 1)^{1+\beta} - r^{1+\beta}}{r^\beta}$$
$$= \lim_{r\to\infty, r>0} \big((1\pm r^{-1})^{1+\beta} - 1\big) r = \pm(1+\beta).$$

Combining the above with continuity, we get that for each $z \in \mathbb{C}$ there exists $C(z) > 0$ such that for each $r \geq 1$,

$$|(r\pm 1)^{1+\beta} - r^{1+\beta} - z| \leq C(z)|r|^\beta.$$

One can easily check by definition that $(iur)^\gamma = (iu)^\gamma r^\gamma$ for any $u \in \mathbb{R}\setminus\{0\}, r > 0$ and $\gamma \geq 0$. Therefore for each $r \geq 1$ we have

$$\big|\big(i(1+r)\big)^{1+\beta} - (ir)^{1+\beta} - i^{1+\beta}\big| = \big|i^{1+\beta}\big((r+1)^{1+\beta} - r^{1+\beta} - 1\big)\big|$$
$$= |(r+1)^{1+\beta} - r^{1+\beta} - 1| \leq C(1)|r|^\beta;$$

and for each $r \leq -1$ we have

$$\big|\big(i(1+r)\big)^{1+\beta} - (ir)^{1+\beta} - i^{1+\beta}\big| = \big|(-i)^{1+\beta}\big((-r-1)^{1+\beta} - (-r)^{1+\beta} - i^{2+2\beta}\big)\big|$$
$$= |(-r-1)^{1+\beta} - (-r)^{1+\beta} - i^{2+2\beta}| \leq C(i^{2+2\beta})|r|^\beta.$$

Summarizing, we have that there exists a $C_1 > 0$ such that for each $r \in \mathbb{R}$ with $|r| \geq 1$,

$$\big|\big(i(1+r)\big)^{1+\beta} - (ir)^{1+\beta} - i^{1+\beta}\big| \leq C_1|r|^\beta.$$

Now it follows immediately that

$$|(ix+iy)^{1+\beta} - (ix)^{1+\beta} - (iy)^{1+\beta}| = \big|y^{1+\beta}\big((i(1+x/y))^{1+\beta} - (ix/y)^{1+\beta} - i^{1+\beta}\big)\big|$$
$$= |y|^{1+\beta}\big|\big(i(1+x/y)\big)^{1+\beta} - (ix/y)^{1+\beta} - i^{1+\beta}\big| \leq C_1|y||x|^\beta.$$

□

In the remainder of this section, we fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)\setminus\{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$. For every $t \geq 1$, define

$$R(t) := \Big(\frac{X_t(f_{\mathrm{s}})}{\|X_t\|^{1-\tilde\beta}}, \frac{X_t(f_{\mathrm{c}})}{\|tX_t\|^{1-\tilde\beta}}, \frac{X_t(f_{\mathrm{l}}) - \mathrm{x}_t(f_{\mathrm{l}})}{\|X_t\|^{1-\tilde\beta}}\Big),$$
$$R'(t) := \big(R'_{\mathrm{s}}(t), R'_{\mathrm{c}}(t), R'_{\mathrm{l}}(t)\big)$$
$$:= \Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k\tilde f_{\mathrm{s}}}, t^{\tilde\beta-1}\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k\tilde f_{\mathrm{c}}}, \sum_{k=0}^{\lfloor t^2\rfloor} \Upsilon_{t+k}^{-T_k\tilde f_{\mathrm{l}}}\Big),$$

where T_k is the operator defined in (1.1), $\mathrm{x}_t(f_{\mathrm{l}})$ is defined in (1.4) with f replaced with f_{l}, $\tilde f_{\mathrm{s}} := e^{\alpha(\tilde\beta-1)}f_{\mathrm{s}}$, $\tilde f_{\mathrm{c}} := e^{\alpha(\tilde\beta-1)}f_{\mathrm{c}}$ and $\tilde f_{\mathrm{l}} := \sum_{p\in\mathbb{Z}_+^d:\alpha\tilde\beta>|p|b} e^{-(\alpha-|p|b)}\langle f_{\mathrm{l}}, \phi_p\rangle_\varphi \phi_p$. The following result is a special case of Theorem 1.1.

Theorem 2.5 *Under* $\widetilde{\mathbb{P}}_\mu$, $R(t) \xrightarrow[t\to\infty]{d} (\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_{\mathrm{l}}})$, *where* $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ *and* $\zeta^{-f_{\mathrm{l}}}$ *are the* $(1+\beta)$*-stable random variables described in* (1.3)*, and* $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ *and* $\zeta^{-f_{\mathrm{l}}}$ *are independent.*

Proof In this proof, we always work under $\widetilde{\mathbb{P}}_\mu$. According to the proof of [19, Theorem 1.6] and the fact that the convergence in probability of random vectors to the zero vector is equivalent to the convergence of each components of the random vectors to zero, we have

$$R(t) - R'(t) \xrightarrow[t\to\infty]{\text{in probability}} 0.$$

With the help of Slutsky's theorem, what is left to show is that,

$$R'(t) \xrightarrow[t\to\infty]{d} (\zeta^{f_{\rm s}}, \zeta^{f_{\rm c}}, \zeta^{-f_{\rm l}}). \tag{2.6}$$

Now we prove (2.6). Since Υ_t^f is linear in f, for each $t \geq 1$,

$$\widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{j={\rm s,c,l}} R'_j(t)\Big)\Big] = \widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k(\tilde{f}_{\rm s}+t^{\tilde{\beta}-1}\tilde{f}_{\rm c})}\Big)\exp\Big(i\sum_{k=0}^{\lfloor t^2\rfloor}\Upsilon_{t+k}^{-T_k\tilde{f}_{\rm l}}\Big)\Big].$$

Note that $\{T_k(\tilde{f}_{\rm s}+t^{\tilde{\beta}-1}\tilde{f}_{\rm c}) : k\in\mathbb{Z}_+, t\geq 1\}\subset\overline{\mathcal{P}}_{\tilde{f}_{\rm s}+\tilde{f}_{\rm c}}$ and $\{-T_k\tilde{f}_{\rm l} : k\in\mathbb{Z}_+\}\subset\overline{\mathcal{P}}_{\tilde{f}_{\rm l}}$. Therefore, we can use Proposition 2.3 with f taken as $\tilde{f}_{\rm s}+\tilde{f}_{\rm c}$ and g taken as $\tilde{f}_{\rm l}$ to get that there exist $C_1, \delta_1 > 0$ such that for every $t > e$ (which implies $t \geq \lfloor t-\ln t\rfloor + 1$),

$$\begin{aligned}&\Big|\widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{j={\rm s,c,l}} R'_j(t)\Big)\Big] - \\ &\qquad \exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\big\langle Z_1\big(T_k(\tilde{f}_{\rm s}+t^{\tilde{\beta}-1}\tilde{f}_{\rm c})\big),\varphi\big\rangle\Big)\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\rm l}),\varphi\rangle\Big)\Big| \\ &\leq C_1 e^{-\delta_1(t-\lfloor t-\ln t\rfloor)}.\end{aligned}$$

We claim that

$$\begin{aligned}&\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\big\langle Z_1\big(T_k(\tilde{f}_{\rm s}+t^{\tilde{\beta}-1}\tilde{f}_{\rm c})\big),\varphi\big\rangle\Big)\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\rm l}),\varphi\rangle\Big) \\ &= \exp(m[f_{\rm s}]+m[f_{\rm c}]+m[-f_{\rm l}]).\end{aligned} \tag{2.7}$$

Given this claim, we have

$$\widetilde{\mathbb{P}}_\mu\Big[\exp\Big(i\sum_{j={\rm s,c,l}} R'_j(t)\Big)\Big] \xrightarrow[t\to\infty]{} \exp(m[f_{\rm s}]+m[f_{\rm c}]+m[-f_{\rm l}]).$$

Since $R'_j(t)$ are linear in $f_j\in\mathcal{C}_j$ $(j={\rm s,c,l})$, replacing f_j with $\theta_j f_j$, we immediately get (2.6).

Now we prove the claim (2.7). For every $f\in\mathcal{C}_{\rm s}\oplus\mathcal{C}_{\rm c}$ and $n\in\mathbb{Z}_+$,

$$\begin{aligned}\sum_{k=0}^n\langle Z_1 T_k\tilde{f},\varphi\rangle &= \sum_{k=0}^n\int_0^1\big\langle P_u^\alpha\big(\eta(-iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta}\big),\varphi\big\rangle{\rm d}u \\ &= \sum_{k=0}^n\int_0^1 e^{\alpha u}\langle\eta(-iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta},\varphi\rangle{\rm d}u = \sum_{k=0}^n\int_0^1\langle\eta(-iT_{k+1-u}f)^{1+\beta},\varphi\rangle{\rm d}u \\ &= \int_0^{n+1}\langle\eta(-iT_u f)^{1+\beta},\varphi\rangle{\rm d}u = m_{n+1}[f],\end{aligned}$$

where $\tilde{f} = e^{\alpha(\tilde{\beta}-1)}f$. Therefore, for any $t\geq 1$,

$$\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\rm s}+t^{\tilde{\beta}-1}\tilde{f}_{\rm c}),\varphi\rangle = \eta\int_0^{\lfloor t-\ln t\rfloor+1}\big\langle\big(-iT_u(f_{\rm s}+t^{\tilde{\beta}-1}f_{\rm c})\big)^{1+\beta},\varphi\big\rangle{\rm d}u. \tag{2.8}$$

Note that for each $u\geq 0$, $T_u f_{\rm c} = f_{\rm c}$. Also note that according to Step 1 in the proof of [19, Lemma 2.6], there exist $\delta > 0$ and $h\in\mathcal{P}$ (depending only on $f_{\rm s}$) such that for each $u\geq 0$, $|T_u f_{\rm s}|\leq e^{-\delta u}h$. It follows from Lemma 2.4 that there exists $C > 0$ such that for all $u\geq 0$ and $t\geq 0$,

$$|(-i(T_u f_{\rm s}+t^{\tilde{\beta}-1}T_u f_{\rm c}))^{1+\beta} - (-iT_u f_{\rm s})^{1+\beta} - (-it^{\tilde{\beta}-1}T_u f_{\rm c})^{1+\beta}|$$

$$
\begin{aligned}
&\overset{\text{Lemma 2.4}}{\leq} C(t^{-\frac{\beta}{1+\beta}}|T_u f_{\mathrm{s}}||T_u f_{\mathrm{c}}|^{\beta} + t^{-\frac{1}{1+\beta}}|T_u f_{\mathrm{s}}|^{\beta}|T_u f_{\mathrm{c}}|) \\
&\leq C(t^{-\frac{\beta}{1+\beta}}e^{-\delta u}h|f_{\mathrm{c}}|^{\beta} + t^{-\frac{1}{1+\beta}}e^{-\delta\beta u}h^{\beta}|f_{\mathrm{c}}|). \qquad (2.9)
\end{aligned}
$$

This means that there exists $C_1 > 0$ such that for all $t \geq 1$,

$$
\begin{aligned}
&\Big|\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_c), \varphi\rangle\Big) - m_{\lfloor t-\ln t\rfloor+1}[f_{\mathrm{s}}] - \frac{1}{t}m_{\lfloor t-\ln t\rfloor+1}[f_{\mathrm{c}}]\Big| \\
&\overset{(2.8),(1.2)}{\leq} \Big|\eta\int_0^{\lfloor t-\ln t\rfloor+1}\big\langle\big(-iT_u(f_{\mathrm{s}} + t^{\tilde{\beta}-1}f_{\mathrm{c}})\big)^{1+\beta}, \varphi\big\rangle \mathrm{d}u - \\
&\eta\int_0^{\lfloor t-\ln t\rfloor+1}\langle(-iT_u f_{\mathrm{s}})^{1+\beta}, \varphi\rangle \mathrm{d}u - \frac{1}{t}\eta\int_0^{\lfloor t-\ln t\rfloor+1}\langle(-iT_u f_{\mathrm{c}})^{1+\beta}, \varphi\rangle \mathrm{d}u\Big| \\
&\overset{(2.9)}{\leq} C_1\int_0^{\lfloor t-\ln t\rfloor+1}\langle t^{-\frac{\beta}{1+\beta}}e^{-\delta u}h|f_{\mathrm{c}}|^{\beta} + t^{-\frac{1}{1+\beta}}e^{-\delta\beta u}h^{\beta}|f_{\mathrm{c}}|, \varphi\rangle \mathrm{d}u \\
&\leq C_1 t^{-\frac{\beta}{1+\beta}}\langle h|f_{\mathrm{c}}|^{\beta}, \varphi\rangle\int_0^{\infty}e^{-\delta u}\mathrm{d}u + C_1 t^{-\frac{1}{1+\beta}}\langle h^{\beta}|f_{\mathrm{c}}|, \varphi\rangle\int_0^{\infty}e^{-\delta\beta u}\mathrm{d}u \\
&\xrightarrow[t\to\infty]{} 0.
\end{aligned}
$$

Combining this with (1.3), we get that

$$
\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1 T_k(\tilde{f}_{\mathrm{s}} + t^{\tilde{\beta}-1}\tilde{f}_c), \varphi\rangle\Big) = \exp(m[f_{\mathrm{s}}] + m[f_{\mathrm{c}}]). \qquad (2.10)
$$

Also note that according to the Step 1 in the Proof of Theorem 1.6.(3) in [19], we have

$$
\lim_{t\to\infty}\exp\Big(\sum_{k=0}^{\lfloor t^2\rfloor}\langle Z_1(-T_k\tilde{f}_{\mathrm{l}}), \varphi\rangle\Big) = \exp(m[-f_{\mathrm{l}}]). \qquad (2.11)
$$

Thus the desired claim follows from (2.10) and (2.11). □

Proof of Theorem 1.1 We first recall some facts about weak convergence which will be used later. For any bounded Lipschitz function $f : \mathbb{R}^d \mapsto \mathbb{R}$, let

$$
\|f\|_L := \sup_{x\neq y}\frac{|f(x) - f(y)|}{|x - y|}
$$

and $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. For any probability distributions μ_1 and μ_2 on $\mathbb{R}^d$, define

$$
d(\mu_1, \mu_2) := \sup\Big\{\Big|\int f\mathrm{d}\mu_1 - \int f\mathrm{d}\mu_2\Big| : \|f\|_{BL} \leq 1\Big\}.
$$

Then d is a metric. It follows from [7, Theorem 11.3.3] that the topology generated by d is equivalent to the weak convergence topology. Using the definition, we can easily see that, if μ_1 and μ_2 are the distributions of two $\mathbb{R}^d$-valued random variables X and Y respectively, defined on same probability space, then

$$
d(\mu_1, \mu_2) \leq \mathbb{E}|X - Y|. \qquad (2.12)
$$

In this proof, let us fix $\mu \in \mathcal{M}_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$, $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$. Recall that $S(t)$ $(t \geq 0)$ is given by (1.5). For every $r, t > 0$, let

$$
S(t, r) := \Big(e^{-\alpha t}\|X_t\|, \frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde{\beta}}}, \frac{X_{t+r}(f_{\mathrm{c}})}{\|(t+r)X_{t+r}\|^{1-\tilde{\beta}}}, \frac{X_{t+r}(f_{\mathrm{l}}) - \mathrm{x}_{t+r}(f_{\mathrm{l}})}{\|X_{t+r}\|^{1-\tilde{\beta}}}\Big),
$$

and

$$\widetilde{S}(t,r) := (e^{-\alpha(t+r)}\|X_{t+r}\| - e^{-\alpha t}\|X_t\|, 0, 0, 0),$$

where, for any $t > 0$, $\mathrm{x}_t(f_1)$ is defined in (1.4) with f replaced with f_1. Then $S(t+r) = S(t,r) + \widetilde{S}(t,r)$. We claim that

for each $t > 0$, under $\widetilde{\mathbb{P}}_\mu$, we have

$$S(t,r) \xrightarrow[r\to\infty]{d} (\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_1}), \tag{2.13}$$

where $\widetilde{H}_t$ has the distribution of $\{e^{-\alpha t}\|X_t\|; \widetilde{\mathbb{P}}_\mu\}$, $\zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and ζ^{-f_1} are the $(1+\beta)$-stable random variables described in (1.3), and $\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}$ and ζ^{-f_1} are independent.

For every $r, t \geq 0$, let $\mathcal{D}(r)$ and $\mathcal{D}(r,t)$ be the distributions of $S(r)$ and $S(t,r)$ under $\widetilde{\mathbb{P}}_\mu$ respectively; let $\widetilde{\mathcal{D}}(t)$ and $\mathcal{D}$ be the distributions of $(\widetilde{H}_t, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_1})$ and $(\widetilde{H}_\infty, \zeta^{f_{\mathrm{s}}}, \zeta^{f_{\mathrm{c}}}, \zeta^{-f_1})$, respectively. Then for each $\gamma \in (0,\beta)$, there exist constant $C > 0$ such that for every $t > 0$,

$$\begin{aligned}
&\overline{\lim_{r\to\infty}}\, d(\mathcal{D}(t+r), \mathcal{D}) \\
&\overset{\text{triangle inequality}}{\leq} \overline{\lim_{r\to\infty}} \Big(d\big(\mathcal{D}(t+r), \mathcal{D}(t,r)\big) + d\big(\mathcal{D}(t,r), \widetilde{\mathcal{D}}(t)\big) + d\big(\widetilde{\mathcal{D}}(t), \mathcal{D}\big) \Big) \\
&\overset{(2.12)}{\leq} \overline{\lim_{r\to\infty}}\, \widetilde{\mathbb{P}}_\mu[|S(t+r) - S(t,r)|] + \overline{\lim_{r\to\infty}}\, d\big(\mathcal{D}(t,r), \widetilde{\mathcal{D}}(t)\big) + \widetilde{\mathbb{P}}_\mu[|H_t - H_\infty|] \\
&\overset{(2.13)}{\leq} \overline{\lim_{r\to\infty}}\, \widetilde{\mathbb{P}}_\mu[|H_t - H_{t+r}|] + \widetilde{\mathbb{P}}_\mu[|H_t - H_\infty|] \\
&\overset{\text{Hölder inequality}}{\leq} \overline{\lim_{r\to\infty}}\, \mathbb{P}_\mu(D^c)^{-1}(\|H_t - H_{t+r}\|_{L_{1+\gamma}(\mathbb{P}_\mu)} + \|H_t - H_\infty\|_{L_{1+\gamma}(\mathbb{P}_\mu)}) \\
&\overset{[19,\text{ Lemma 3.3}]}{\leq} Ce^{-\alpha\tilde{\gamma}t}.
\end{aligned} \tag{2.14}$$

Therefore,

$$\overline{\lim_{r\to\infty}}\, d\big(\mathcal{D}(r), \mathcal{D}\big) = \overline{\lim_{t\to\infty}}\, \overline{\lim_{r\to\infty}}\, d\big(\mathcal{D}(t+r), \mathcal{D}\big) \overset{(2.14)}{\leq} \overline{\lim_{t\to\infty}}\, Ce^{-\alpha\tilde{\gamma}t} = 0.$$

The desired result now follows immediately.

Now we prove the claim (2.13). For every $r, t > 0$, let

$$\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1 \in \mathbb{R} \mapsto k(\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1, r, t)$$

be the characteristic function of $S(t,r)$ under $\widetilde{\mathbb{P}}_\mu$. Then for each $\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1 \in \mathbb{R}$ and $r, t > 0$,

$$\begin{aligned}
k(\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1, r, t) &= \widetilde{\mathbb{P}}_\mu\big[\exp\big(i\theta e^{-\alpha t}\|X_t\| + A(\theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1, r, t, \infty)\big)\big] \\
&\overset{\text{bounded convergence}}{=} \lim_{u\to\infty} \frac{1}{\mathbb{P}_\mu(D^c)} \mathbb{P}_\mu\big[\exp\big(i\theta e^{-\alpha t}\|X_t\| + A(\theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1, r, t, u)\big); D^c\big],
\end{aligned} \tag{2.15}$$

where for each $u \in [0,\infty]$,

$$\begin{aligned}
&A(\theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_1, r, t, u) \\
&:= i\theta_{\mathrm{s}} \frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde{\beta}}} + i\theta_{\mathrm{c}} \frac{X_{t+r}(f_{\mathrm{c}})}{\|(t+r)X_{t+r}\|^{1-\tilde{\beta}}} + i\theta_1 \frac{X_{t+r}(f_1) - \mathbb{P}_\mu[\mathrm{x}_{t+r}(f_1)|\mathscr{F}_u]}{\|X_{t+r}\|^{1-\tilde{\beta}}}
\end{aligned}$$

$$
\begin{aligned}
&= i\theta_{\mathrm{s}} \frac{X_{t+r}(f_{\mathrm{s}})}{\|X_{t+r}\|^{1-\tilde{\beta}}} + \frac{i\theta_{\mathrm{c}}}{(t+r)^{1-\tilde{\beta}}} \frac{X_{t+r}(f_{\mathrm{c}})}{\|X_{t+r}\|^{1-\tilde{\beta}}} + \\
&\quad i\theta_{\mathrm{l}} \frac{X_{t+r}(f_{\mathrm{l}}) - \sum_{p \in \mathbb{Z}_+^d : \alpha\tilde{\beta} > |p|b} \langle f_{\mathrm{l}}, \phi_p \rangle_\varphi e^{(\alpha - |p|b)(t+r)} H_u^p}{\|X_{t+r}\|^{1-\tilde{\beta}}}.
\end{aligned} \tag{2.16}
$$

Now for each $t > 0$, we get

$$
\begin{aligned}
&\lim_{r\to\infty} k(\theta, \theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_{\mathrm{l}}, r, t) \\
&\overset{(2.15)}{=} \lim_{r\to\infty} \lim_{u\to\infty} \frac{1}{\mathbb{P}_\mu(D^c)} \mathbb{P}_\mu \big[\exp\{i\theta e^{-\alpha t} \|X_t\|\} \mathbf{1}_{\|X_t\|>0} \mathbb{P}_\mu [\exp\{A(\theta_{\mathrm{s}}, \theta_{\mathrm{c}}, \theta_{\mathrm{l}}, r, t, u)\} \mathbf{1}_{D^c} | \mathscr{F}_t] \big] \\
&\overset{(2.16),\ \text{Markov property}}{=} \lim_{r\to\infty} \lim_{u\to\infty} \frac{1}{\mathbb{P}_\mu(D^c)} \mathbb{P}_\mu \Bigg[\exp\{i\theta e^{-\alpha t} \|X_t\|\} \mathbf{1}_{\|X_t\|>0} \times \\
&\qquad \mathbb{P}_{X_t} \left[\exp\left\{ A\left(\theta_{\mathrm{s}}, \theta_{\mathrm{c}} \Big(\frac{r}{t+r} \Big)^{1-\tilde{\beta}}, \theta_{\mathrm{l}}, r, 0, u-t \right) \right\} \mathbf{1}_{D^c} \right] \Bigg] \\
&\overset{\text{bounded convergence}}{=} \lim_{r\to\infty} \mathbb{P}_\mu \Bigg[\exp\{i\theta e^{-\alpha t} \|X_t\|\} \mathbf{1}_{\|X_t\|>0} \frac{\mathbb{P}_{X_t}(D^c)}{\mathbb{P}_\mu(D^c)} \times \\
&\qquad \widetilde{\mathbb{P}}_{X_t} \left[\exp\left\{ A\left(\theta_{\mathrm{s}}, \theta_{\mathrm{c}} \Big(\frac{r}{t+r} \Big)^{1-\tilde{\beta}}, \theta_{\mathrm{l}}, r, 0, \infty \right) \right\} \right] \Bigg]. \\
&\overset{\text{Theorem 2.5}}{=} \mathbb{P}_\mu \Big[\exp\{i\theta e^{-\alpha t} \|X_t\|\} \mathbf{1}_{\|X_t\|>0} \frac{\mathbb{P}_{X_t}(D^c)}{\mathbb{P}_\mu(D^c)} \Big] \Big(\prod_{j=\mathrm{s},\mathrm{c}} \exp\{m[\theta_j f_j]\} \Big) \exp\{m[-\theta_{\mathrm{l}} f_{\mathrm{l}}]\} \\
&= \widetilde{\mathbb{P}}_\mu [\exp\{i\theta e^{-\alpha t} \|X_t\|\}] \Big(\prod_{j=\mathrm{s},\mathrm{c}} \exp\{m[\theta_j f_j]\} \Big) \exp\{m[-\theta_{\mathrm{l}} f_{\mathrm{l}}]\}.
\end{aligned}
$$

□

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