Exact Expressions for Trigonometric Functions in Degrees

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Herman Robinson (7 Apr 1912 to 10 Oct 1986) was a scientist at the UC Berkeley Radiation Lab in Berkeley, CA. He worked there from 1945 until he retired in 1973. Herman was a chemical engineer by training, but his interests and skills ranged over a variety of science, math, and technology topics. He enjoyed "tinkering" with mathematical calculations after he retired with the Wang 720c calculator he purchased for himself, having used one at the office. In 1990, I discovered that he was my biological father.

In 2006, Herman's widow sent me a box of Herman's correspondence to sort through. Many of the things he had kept were numerical calculations that are relatively trivial to do with today's computer tools, but represent a significant amount of work with early calculators and computers. I saved a few pages from a notebook that contained algebraic expressions for trig functions in degrees. No doubt such things have been published elsewhere in the last few hundred years, but I haven't seen them (most of them are not in my CRC math handbook nor Abramowitz and Stegun) and I thought that someone somewhere might be able to use them, so I typed them up.

There were no notes explaining the calculations, but I would assume Herman was solving algebraic equations based on multiple-angle trig formulas. These calculations would have been done by hand, not with a computer algebra system. I used a python script to verify that the expressions were correct (see the appendix).

The formulas below were generated from the python script in the appendix by another script because they're more readable than code. I hope they're correct, but the definitive relations should be taken from the code.

$$\sin(3^\circ) = (\sqrt{6}+\sqrt{2})(\sqrt{5}-1)/16-(\sqrt{3}-1)\sqrt{5+\sqrt{5}}/8$$
$$\cos(3^\circ) = (\sqrt{6}-\sqrt{2})(\sqrt{5}-1)/16+(\sqrt{3}+1)\sqrt{5+\sqrt{5}}/8$$
$$\sin(6^\circ) = (\sqrt{30-6\sqrt{5}}-\sqrt{5}-1)/8$$
$$\cos(6^\circ) = (\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}})/8$$
$$\tan(6^\circ) = (\sqrt{10-2\sqrt{5}}+\sqrt{3}-\sqrt{15})/2$$
$$\cot(6^\circ) = (3\sqrt{3}+\sqrt{15}+2\sqrt{10-2\sqrt{5}}+\sqrt{50-10\sqrt{5}})/2$$
$$\sin(7.5^\circ) = \sqrt{8-2\sqrt{2}-2\sqrt{6}}/4$$
$$\cos(7.5^\circ) = \sqrt{8+2\sqrt{2}+2\sqrt{6}}/4$$
$$\tan(7.5^\circ) = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$$
$$\cot(7.5^\circ) = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$$
$$\sin(9^\circ) = (\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}})/8$$
$$\cos(9^\circ) = (\sqrt{10}+\sqrt{2}+2\sqrt{5-\sqrt{5}})/8$$
$$\tan(9^\circ) = 1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$$
$$\cot(9^\circ) = 1+\sqrt{5}+\sqrt{5+2\sqrt{5}}$$
$$\sin(11.25^\circ) = (\sqrt{2-\sqrt{2+\sqrt{2}}})/2$$
$$\cos(11.25^\circ) = (\sqrt{2+\sqrt{2+\sqrt{2}}})/2$$
$$\tan(11.25^\circ) = \sqrt{4+2\sqrt{2}}-\sqrt{2}-1$$
$$\sin(12^\circ) = (2\sqrt{3}-2\sqrt{15}+\sqrt{10-2\sqrt{5}}+\sqrt{50-10\sqrt{5}})/16$$
$$\cos(12^\circ) = (\sqrt{31+9\sqrt{5}+4\sqrt{30-6\sqrt{5}}-5\sqrt{6-2\sqrt{5}}})/8$$

$$\sin(15^\circ) = (\sqrt{6}-\sqrt{2})/4$$
$$\cos(15^\circ) = (\sqrt{6}+\sqrt{2})/4$$
$$\tan(15^\circ) = 2-\sqrt{3}$$
$$\cot(15^\circ) = 2+\sqrt{3}$$
$$\sin(18^\circ) = (\sqrt{5}-1)/4$$
$$\cos(18^\circ) = (\sqrt{10+2\sqrt{5}})/4$$
$$\tan(18^\circ) = \sqrt{1-2/\sqrt{5}}$$
$$\cot(18^\circ) = \sqrt{5+2\sqrt{5}}$$
$$\sin(21^\circ) = (\sqrt{3}+1)(\sqrt{5-\sqrt{5}})/8-(\sqrt{6}-\sqrt{2})(\sqrt{5}+1)/16$$
$$\cos(21^\circ) = (\sqrt{3}-1)(\sqrt{5-\sqrt{5}})/8+(\sqrt{6}+\sqrt{2})(\sqrt{5}+1)/16$$
$$\sin(22.5^\circ) = \sqrt{2-\sqrt{2}}/2$$
$$\cos(22.5^\circ) = \sqrt{2+\sqrt{2}}/2$$
$$\tan(22.5^\circ) = \sqrt{2}-1$$
$$\cot(22.5^\circ) = \sqrt{2}+1$$
$$\sin(24^\circ) = (\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}})/8$$
$$\cos(24^\circ) = (\sqrt{5}+1+\sqrt{30-6\sqrt{5}})/8$$
$$\sin(27^\circ) = (2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2})/8$$
$$\cos(27^\circ) = (2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2})/8$$
$$\sin(33^\circ) = (\sqrt{3}-1)(\sqrt{5+\sqrt{5}})/8+(\sqrt{6}+\sqrt{2})(\sqrt{5}-1)/16$$
$$\cos(33^\circ) = (\sqrt{3}+1)(\sqrt{5+\sqrt{5}})/8-(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)/16$$
$$\sin(36^\circ) = \sqrt{10-2\sqrt{5}}/4$$
$$\cos(36^\circ) = (\sqrt{5}+1)/4$$
$$\tan(36^\circ) = \sqrt{5-2\sqrt{5}}$$
$$\cot(36^\circ) = \sqrt{1+2/\sqrt{5}}$$
$$\sin(37.5^\circ) = (\sqrt{2-\sqrt{2-\sqrt{3}}})/2$$
$$\cos(37.5^\circ) = (\sqrt{2+\sqrt{2-\sqrt{3}}})/2$$
$$\tan(37.5^\circ) = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$
$$\cot(37.5^\circ) = \sqrt{6}-\sqrt{3}-\sqrt{2}+2$$
$$\sin(39^\circ) = (\sqrt{6}+\sqrt{2})(\sqrt{5}+1)/16-(\sqrt{3}-1)\sqrt{5-\sqrt{5}}/8$$
$$\cos(39^\circ) = (\sqrt{6}-\sqrt{2})(\sqrt{5}+1)/16+(\sqrt{3}+1)\sqrt{5-\sqrt{5}}/8$$

Here's an example of how these relationships were probably derived; let's calculate sin(22.5°). The half angle relationship for the sine is

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

Thus, we can write

$$\sin^2(22.5°) = \frac{1-\cos(45°)}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2}$$

Simplifying leads to $\sin(22.5°) = \sqrt{2-\sqrt{2}}/2$, the value shown in the table.

Appendix: python check script

If the python mpmath library is available, then the calculations will be done with extended precision.

Otherwise, the built-in floating point math library is used.

```
'''
Algebraic expressions for trigonometric functions in degrees
'''

from __future__ import division

try:
    from mpmath import mp, pi, mpf, sin as s, cos as c, tan as t, sqrt
    ndigits = 100
    mp.dps = ndigits
    eps, D = mpf(10)**(-ndigits + 1), pi/180
except ImportError:
    from math import pi, sin as s, cos as c, tan as t, sqrt
    mpf = float
    eps, D = 10**(-14), pi/180

s2, s3, s5, s6, s10, s15 = map(sqrt, (2, 3, 5, 6, 10, 15))

def eq(a, b):
    if abs(a - b) > eps:
        raise ValueError()

# 3 deg
eq(s(3*D), ((s6+s2)*(s5-1)/16 - (s3-1)*sqrt(5 + s5)/8))
eq(c(3*D), ((s6-s2)*(s5-1)/16 + (s3+1)*sqrt(5 + s5)/8))
# 6 deg
eq(s(6*D), (sqrt(30 - 6*s5) - s5 - 1)/8)
eq(c(6*D), (s15 + s3 + sqrt(10 -2*s5))/8)
eq(t(6*D), (sqrt(10 - 2*s5) + s3 - s15)/2)
eq(1/t(6*D), (3*s3 + s15 + 2*sqrt(10 - 2*s5) + sqrt(50 - 10*s5))/2)
# 7.5 deg
eq(s(mpf("7.5")*D), (sqrt(8 - 2*s2 - 2*s6)/4))
eq(c(mpf("7.5")*D), (sqrt(8 + 2*s2 + 2*s6)/4))
eq(t(mpf("7.5")*D), (s6 - s3 + s2 - 2))
eq(1/t(mpf("7.5")*D), (s6 + s3 + s2 + 2))
# 9 deg
eq(s(9*D), (s10 + s2 - 2*sqrt(5 - s5))/8)
eq(c(9*D), (s10 + s2 + 2*sqrt(5 - s5))/8)
eq(t(9*D), (1 + s5 - sqrt(5 + 2*s5)))
eq(1/t(9*D), (1 + s5 + sqrt(5 + 2*s5)))
# 11.25 deg
eq(s(mpf("11.25")*D), (sqrt(2 - sqrt(2 + s2)))/2)
eq(c(mpf("11.25")*D), (sqrt(2 + sqrt(2 + s2)))/2)
eq(t(mpf("11.25")*D), sqrt(4 + 2*s2) - s2 - 1)
# 12 deg
eq(s(12*D), (2*s3 - 2*s15 + sqrt(10 - 2*s5) + sqrt(50 - 10*s5))/16)
eq(c(12*D), (sqrt(31 + 9*s5 + 4*sqrt(30 - 6*s5) - 5*sqrt(6 - 2*s5)))/8)
# 15 deg
eq(s(15*D), (s6 - s2)/4)
eq(c(15*D), (s6 + s2)/4)
eq(t(15*D), (2 - s3))
eq(1/t(15*D), (2 + s3))
# 18 deg
eq(s(18*D), (s5 - 1)/4)
eq(c(18*D), (sqrt(10 + 2*s5))/4)
eq(t(18*D), sqrt(1 - 2/s5))
eq(1/t(18*D), sqrt(5 + 2*s5))
# 21 deg
eq(s(21*D), (s3+1)*(sqrt(5-s5))/8 - (s6-s2)*(s5+1)/16)
eq(c(21*D), (s3-1)*(sqrt(5-s5))/8 + (s6+s2)*(s5+1)/16)
# 22.5 deg
eq(s(mpf("22.5")*D), sqrt(2 - s2)/2)
eq(c(mpf("22.5")*D), sqrt(2 + s2)/2)
eq(t(mpf("22.5")*D), s2 - 1)
eq(1/t(mpf("22.5")*D), s2 + 1)
# 24 deg
```

```
eq(s(24*D), (s15 + s3 - sqrt(10 - 2*s5))/8)
eq(c(24*D), (s5 + 1 + sqrt(30 - 6*s5))/8)
# 27 deg
eq(s(27*D), (2*sqrt(5 + s5) - s10 + s2)/8)
eq(c(27*D), (2*sqrt(5 + s5) + s10 - s2)/8)
# 33 deg
eq(s(33*D), (s3-1)*(sqrt(5+s5))/8 + (s6+s2)*(s5-1)/16)
eq(c(33*D), (s3+1)*(sqrt(5+s5))/8 - (s6-s2)*(s5-1)/16)
# 36 deg
eq(s(36*D), sqrt(10 - 2*s5)/4)
eq(c(36*D), (s5 + 1)/4)
eq(t(36*D), sqrt(5 - 2*s5))
eq(1/t(36*D), sqrt(1 + 2/s5))
# 37.5 deg
eq(s(mpf("37.5")*D), sqrt(2 - sqrt(2 - s3))/2)
eq(c(mpf("37.5")*D), sqrt(2 + sqrt(2 - s3))/2)
eq(t(mpf("37.5")*D), s6 + s3 - s2 - 2)
eq(1/t(mpf("37.5")*D), s6 - s3 - s2 + 2)
# 39 deg
eq(s(39*D), (s6+s2)*(s5+1)/16 - (s3-1)*sqrt(5 - s5)/8)
eq(c(39*D), (s6-s2)*(s5+1)/16 + (s3+1)*sqrt(5 - s5)/8)

print "Tests passed"
```