Sine Sticks

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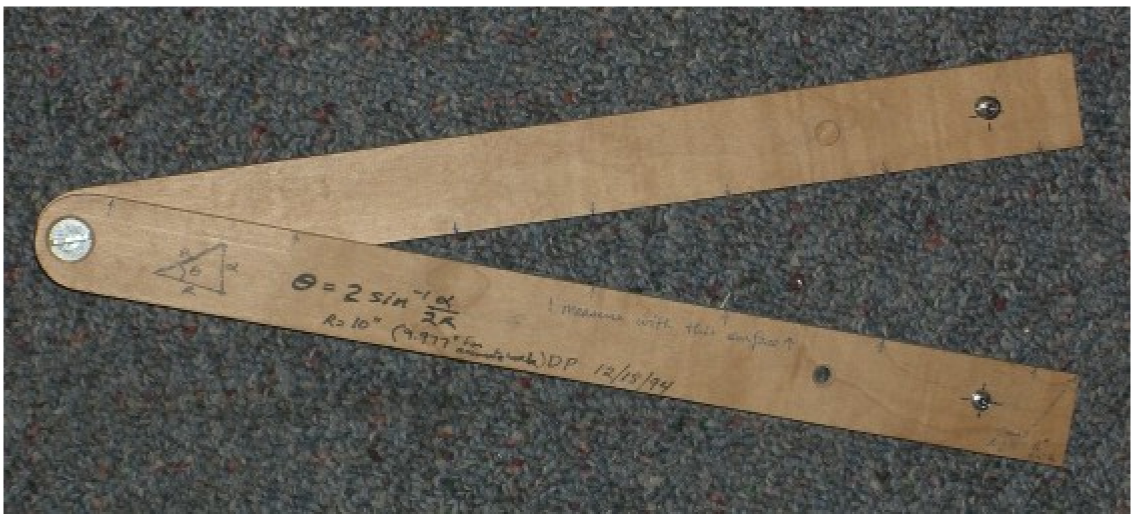
Here's a simple tool I call sine sticks you can make from scrap in your shop that will let you measure angles using a rule and a calculator or trig tables. These occurred to me one day in 1994 while I was trying to figure out a good way to measure tapers on a metal lathe. Of course, the idea isn't new (few shop things are); since then, I've seen pictures of hinged metal rules with tables on them that used this technique and were made in the early 1900's. Colvin & Stanley's *American Machinist's Handbook*, 8th ed., 1945 shows the same basic idea on page 814 and the 2nd edition of 1914 gives the method and table on page 403. See the *Line of chords rule* below for a commercial tool that does the same task.

Take two identical sticks and clamp them together. Drill a small hole through both sticks at one end and a hole through the opposite end to take a bolt and nut. Here's mine:



I used 1/4" thick maple and threaded the wood to receive two screws on the right, which I center punched at their center. If I measure the distance between the center punch marks, I can calculate the angle that the sticks are separated. Carefully made, it's a surprisingly accurate tool. When I want to make careful measurement of the separation of the screws, I use some machinist's trammels and can measure the distance to around ±0.01" or better. The characteristic distance is 10 inches; I've also made smaller sine sticks less than half this size that fit into confined spaces better.

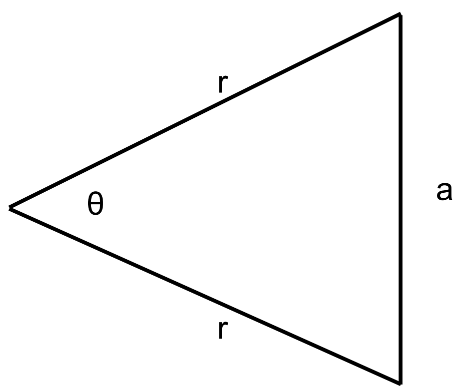
This was a prototype I made in 1994 to test the idea, but it has worked so well I'm still using it.

I made the joint as follows:

I adjust the bottom nut so the sticks have the desired tension, then tighten the top nut. A nylon lock nut could be used instead. Or, if you wanted to be able to adjust the tension better and be able to lock things down after a measurement, use a Belleville or wavy washer and a wing nut or thumbscrew.

I have an old folding Lufkin rule (model 85) that belonged to my grandfather. I drilled a couple of holes in it and it is also usable as sine sticks (the small holes I measure to are at 1/4"):

Here's the triangle we're interested in:



This is an isosceles triangle. The distance *r* is the distance between the center of rotation and the center marks on the screws. Given *a* and *r*, we want to know θ. Drop an altitude to the side of length *a* and you have the formula

$$\theta = 2\sin^{-1}\frac{a}{2r} \tag{1}$$

As shown in the picture, I wrote this relation on my device along with the value of *a*. I measure the distance *a* with a rule for normal work and a trammel or dividers for fussy work.

How sensitive is this expression to small measurement uncertainties? Using differentials, one can show the approximate relationship

$$\Delta\theta = \frac{\Delta a}{\sqrt{r^2 - \left(\frac{a}{2}\right)^2}} - \frac{a\Delta r}{r\sqrt{r^2 - \left(\frac{a}{2}\right)^2}}$$

If we assume no uncertainty in r ($\Delta r = 0$), the second term drops out. For an angle of 45° and $r = 10$ inches, we'd have a measurement of a of 7.654 inches. Suppose our measurement uncertainty was $\Delta a = 0.02$ inches (easily measured with a typical machinist's rule). This results in a $\Delta\theta$ of 0.12°. If Δa was a factor of 10 smaller, the angle uncertainty would be 0.01°. With care, you can measure angles as well or better than a Starrett machinist's vernier protractor, which resolves to 5 minutes of arc or 0.083°.

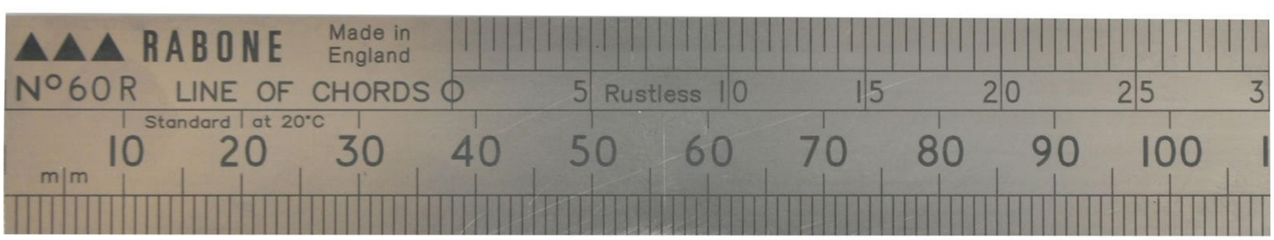
The uncertainty $\Delta\theta$ goes up drastically when the denominator approaches zero. This is when $a = 2r$ or when the angle θ is 180°. Practically, you can measure angles safely to 135° -- for angles larger than that, measure the supplement (180° - θ).

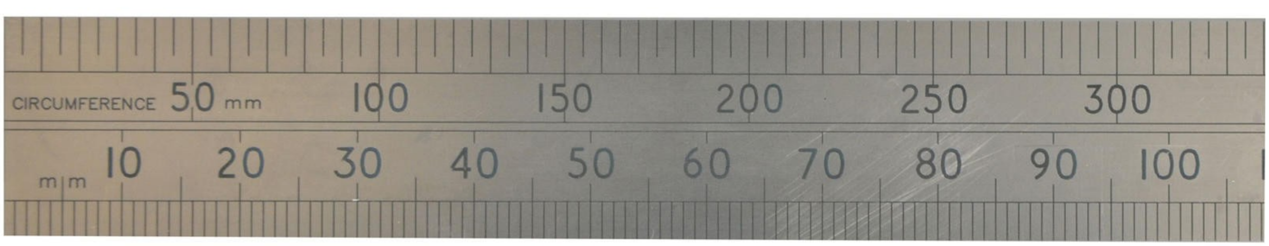
Since you'll probably have access to an accurate right angle, you can measure the right angle with the sine sticks and ensure that the distances *a* and *r* give the 90° per formula (1). If you have two well-known angles, you can measure them both and solve the resulting simultaneous equations for *r* and *a*, providing a check on your measurements of r. A combination square could be handy for this, as it has 45° and 90° angles.

There are a number of things that could be done to improve this tool. If the tool is carefully constructed, the measuring pins could stick out a bit and let you measure the pin heights with a height gauge, the tool then operating like a sine bar (line the pins up vertically with a square). With a sheet metal clamp to hold the arms securely after a measurement, the pin distance could be measured with a dial caliper or micrometer for improved accuracy. Guy Lautard suggested putting clear plastic inserts in with cross hairs to make it easier to read the measurement on a rule.

Line of chords rule

There have been rules available that provide both a folding rule and a scale to convert the distance between the two points to the equivalent angle; these rules are called line of chords rules. I've seen pictures of ones made in the early 1900's. Examples are sold branded by Stanley and Rabone in the UK:





These are 300 mm folding rules that let you measure to 600 mm. Note the circumference scale which reads out π times the linear distance in mm.

The idea isn't new; see *Popular Mechanics*, Jan. 1931, pg. 162 and you can read about the sector, an instrument invented a bit before 1600 (some sectors have a line of chords scale).

Update Jan 2018: I purchased one of these line of chord rules and have been disappointed with it, for a number of reasons.

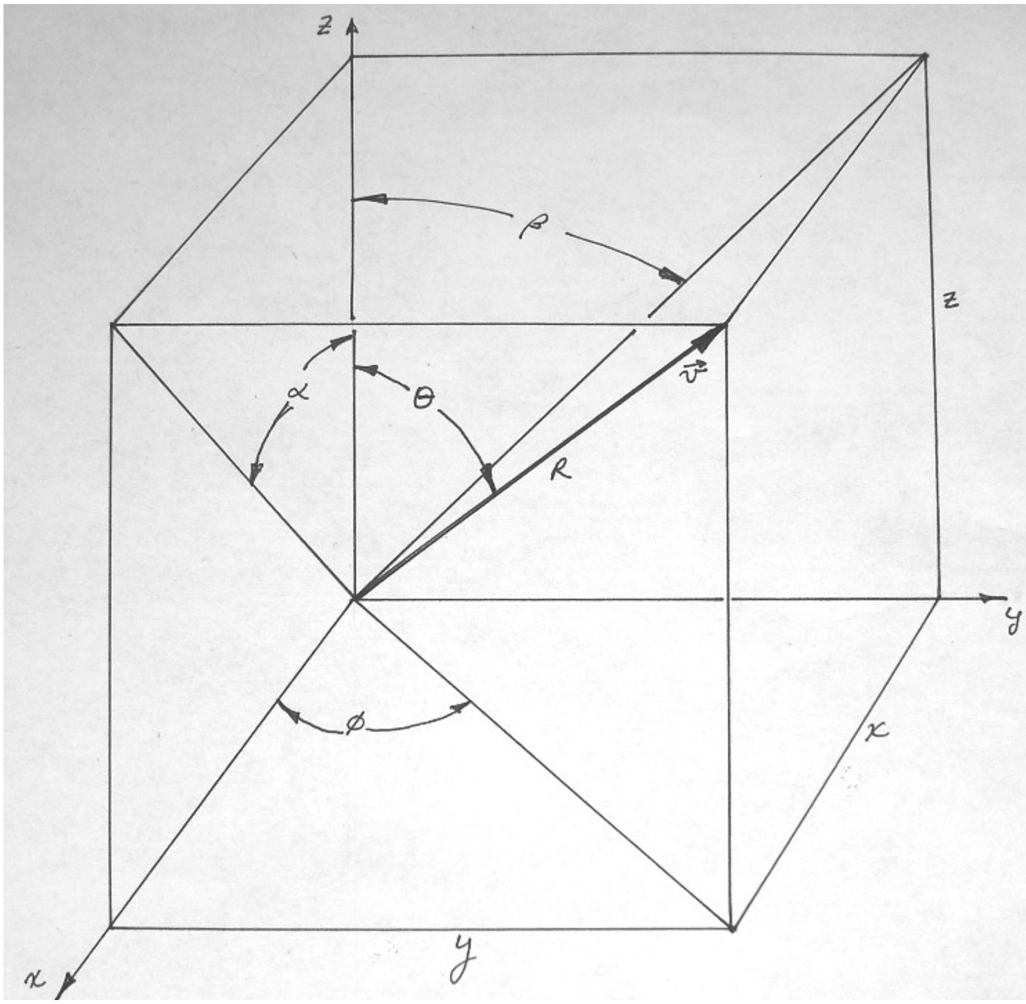
- ♦ It is worth (in my opinion) about half of what I paid for it.
- ♦ It appears to be photoengraved on shiny ground flat stock, then coated with something to prevent rust. This makes for a hard-to-read-rule, as it would be better to have a matte chrome finish with black marks like most machinist rules are made today. I'd like to see the graduations be roughly 50% wider. This rule is difficult to read in dim light.
- ♦ The line of chords feature is not calibrated accurately, as I checked it with a precision machinist's square at 90° and it was off by around half a degree.
- ♦ The line of chord marks should be small diameter holes or at least prick punch marks to allow divider tips to index into the holes. They appear to be coated with plastic or paint, which is a very poor design (clearly, the designer never had to use this rule for practical tasks).

In the rule's favor, the mm scale was right on with a machinist's scale I trust. However, I don't think this rule is a good value for the money and I don't recommend buying one.

Here's a picture with the camera about 30° off the normal showing that the rule is difficult to read compared to a Starrett rule with a satin chrome finish.

Projection Angles in Space

The sine sticks can be used to measure angles in space. Consider the following diagram:



We have a vector ***v*** in space with a length *r* (R in the figure). If we're given α and β in the figure, we can calculate θ and ϕ (and vice versa). Note θ and ϕ are the angles used in spherical coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2}$$

Since $x = z\tan\alpha$ and $y = z\tan\beta$, we get

$$\tan\phi = \frac{\tan\beta}{\tan\alpha} \tag{3}$$

Let $\rho^2 = x^2 + y^2$. Then $\frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{\rho^2}{z^2}$ and since $\frac{x}{z} = \tan\alpha$, etc., we have

$$\tan^2\theta = \tan^2\alpha + \tan^2\beta \tag{4}$$

In $\tan\alpha = \frac{x}{z}$, substitute for *x* and *z* from equation (2) to get

$$\tan\alpha = \tan\theta\cos\phi \tag{5}$$

Similarly,

$$\tan\beta = \tan\theta\sin\phi \tag{6}$$

Now, you're wondering why bother with this? Here's an example -- I wanted to duplicate a children's chair and table set. Here's a picture of one of the chairs:

Here's a picture of the legs:

I wanted to construct some templates and tooling that made it easy to drill the holes for the legs and the dowels making up the back. Using the angle tool of the previous section, I measured the angles α and β (actually, their complements) and was then able to calculate the angle ϕ, which I used to lay out and drill the fixture holes on the drill press.

By the way, that chair design is very sturdy -- those chairs have been in use for nearly 50 years and have been used by lots of kids and grandkids. The leg dowels are 1-1/4" in diameter and the seat is made from 3/4" stock. The back uses 5/8", 3/4", and 1-1/4" dowels. The chair will easily hold a 200 pound adult, but they're nicely sized for a young child. I've had to epoxy a few of the legs back in after heavy use or abuse.

This technique of measuring angles in space from the projection angles is general and useful. Another way to utilize it is to take a picture of the object of interest with a telephoto lens -- pull back quite a ways from the object and take pictures from two directions 90° apart. Then print these pictures on a piece of paper and use a protractor to measure the projection angles to get the coordinates of the angle in space of interest. This telephoto technique is also useful with a scale in the picture to allow you to scale the drawing. Using a telephoto lens and high magnification minimizes perspective and distortion effects (except, of course, those inherent in the lens).