Analytic Geometry Formulas

someonesdad1@gmail.com

This file is AnalyticGeometry.pdf which was originally stored at http://code.google.com/p/hobbyutil/, but Google Code went defunct in 2014. It is now hosted on https://someonesdad1.github.io/hobbyutil/.

Revision history

21 Jun 2018	Changed license to Apache2. Produced PDF with Apache Open Office 4.1.3. Minor editing.
13 Dec 2017	Produced PDF with Apache Open Office 4.1.3.
6 Jan 2016	♦ Minor edits ♦ Added some material to helix section.
Oct 2014	♦ Minor edits. ♦ Updated to Libre Office 4.2.3.3. ♦ Worked on using an OO plugin to export to LaTeX, but I decided it isn't worth the effort. ♦ Added set algebra
16 Jan 2014	Link of chain in mensuration.
10 Sep 2012	Misc. constructions added. Additions to solid angle formulas.
5 Sep 2012	Added mensuration, trigonometric identities, series, and constants sections (I'm always looking these up, so I thought I'd put them here even though they're not really analytic geometry). Other minor changes.
6 Feb 2011	Original release.

This document is a collection of formulas from analytic geometry and a few other things I find useful to have on-hand. Since vectors are used for conciseness, there are also a few formulas from vector analysis.

If you find an error or obvious omission, please notify me at the above email address and it will get fixed in a subsequent revision.

Document maintainer: make the Check paragraph style visible to see document maintenance instructions and the details of the checks made on the formulas (but not all formulas have been checked). Originally written in Open Office 3.2.1.

Maintenance Plea

I originally wrote this document in an Open Office version in the late 2000's; I remember the OO version was something like 3.1.2. Unfortunately, OO split into two versions, Apache OO and Libre Office. I switched to a Linux box around then and used Libre Office. Then I went back to a Windows machine and used Open Office again (the current version is OO 4.1.3).

Unfortunately, the Open Office and Libre Office have broken functionality that used to work. The original document was a dream to maintain -- and you can imagine how much work it was to type in, check, and maintain all the original equations. The current version of OO has a nasty bug that screws up the white space around equations and figures, sometimes inserting numerous pages for some unknown reason. I can fix this manually, but it takes hours of time and I'm sick of doing it.

A number of years ago I emailed a guy (I think he was in Germany) who wrote an OO macro that would convert the OO document to a LaTeX form. This was a wonderful idea, but if I recall, it

wasn't ready for prime time, as it uncovered a number of bugs in his macro. But he fixed these, so maybe this tool (Writer2LaTeX) might be usable today. I chose not to proceed with LaTeX at the time due to other problems. However, it would be much nicer to maintain this document in a LaTeX form, I could use my programmer's editor, and the equation formatting would be much better.

Alas, this is too much work for me today; at least, I don't want to do this work anymore. Thus, if someone out there would like to take on this task, I'd love to turn the document over to you for such maintenance. Or, if you'd like to do the conversion to LaTeX, I'd be happy to continue maintaining the document, as it would be much easier than fighting the bugs in OO/Libre Office.

License

Copyright © 2010, 2018 Don Peterson

Licensed under the Apache License, Version 2.0 (the "License"); you may not use this file except in compliance with the License. You may obtain a copy of the License at

http://www.apache.org/licenses/LICENSE-2.0

Unless required by applicable law or agreed to in writing, software distributed under the License is distributed on an "AS IS" BASIS, WITHOUT WARRANTIES OR CONDITIONS OF ANY KIND, either express or implied. See the License for the specific language governing permissions and limitations under the License.

The author is identified by the string Don Peterson, a ten-digit number familiar to the author (Vernon's), and Gary E.'s password. These are concatenated and put into a UNIX-style file; the resulting SHA-256 hash is aea69710af3629341f95cafbb529a445275255b2270d92318e8e3dc59da0da08.

Notation

All angle measure is assumed to be in radians unless the degree symbol ° is present or a statement to that effect is given. mrad is milliradians and μrad is microradians.

log means the base 10 logarithm; *ln* means the natural logarithm.

A vector is given in bold: $\boldsymbol{A}$; the magnitude of $\boldsymbol{A}$ is A. All scalars are real unless otherwise stated.

$\boldsymbol{r}$ is the position vector of a general point in space = (x, y, z), $\boldsymbol{r}_1 = (x_1, y_1, z_1)$, etc.

A caret denotes a unit vector and it is gotten by dividing a vector by its magnitude:

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{v}$$

The usual Cartesian unit vectors are $(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}})$.

iff means "if and only if". Special formatting is used to make it stand out, as it can be easy to miss the second f.

$\boldsymbol{n}$ denotes a normal vector; what it is normal to will be clear from the context.

An overdot denotes differentiation (Newton's notation). It will either be respect to a single variable or a parameter. Multiple dots denote higher derivatives.

Partial derivatives: $f_x = \frac{\partial f}{\partial x} = \partial_x f$.

In parametric equations, s, t, and u are parameters and $s, t, u \in \mathbb{R}$.

$\boldsymbol{\Gamma} = (\alpha, \beta, \gamma)$ where α, β, and γ are the three direction cosines of a direction in space. Γ (not bolded) represents the gamma function.

The Levi-Civita symbol is

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if (i,j,k) is (1,2,3), (3,1,2), or (2,3,1) (even permutation)} \\ -1 & \text{if (i,j,k) is (1,3,2), (3,2,1), or (2,1,3) (odd permutation)} \\ 0 & \text{otherwise} \end{cases}$$

or

$$\begin{array}{ccc} i = 1 & i = 2 & i = 3 \\ \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \end{array}$$

When the Levi-Civita symbol is used, the usual Einstein summation convention is assumed over the three spacial indices 1, 2, and 3.

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

[*xyz*] is a reference to reference xyz. [*xyz:m*] is a reference to page m in reference xyz. [*xyz:m:p*] uses p to point to the page number in the PDF for documents that are accessible in PDF format.

Constants

The following pages contain some common constants. The first page gives the numbers in a conventional order and the second page gives them sorted by their significands. The beginning digit frequencies look a little like Benford's "Law". For a more extensive list, see e.g. H. Robinson and E. Potter, *Mathematical Constants*, Lawrence Radiation Laboratory, UCRL-20418, March 1971 (this was also published in *Mathematics of Computation*, Vol. 26, Jan 1972).

3.142	π	0.3010	$\log(2)$	0.2857	$2/7$	0.6429	$9/14$
6.283	2π	0.4771	$\log(3)$	0.4286	$3/7$	0.7857	$11/14$
9.425	3π	0.6021	$\log(4)$	0.5714	$4/7$	0.9286	$13/14$
12.57	4π	0.6990	$\log(5)$	0.7143	$5/7$	0.06667	$1/15$
15.71	5π	0.7782	$\log(6)$	0.8571	$6/7$	0.1333	$2/15$
18.85	6π	0.8451	$\log(7)$	0.1250	$1/8$	0.2667	$4/15$
21.99	7π	0.9031	$\log(8)$	0.3750	$3/8$	0.4667	$7/15$
25.13	8π	0.9542	$\log(9)$	0.6250	$5/8$	0.5333	$8/15$
28.27	9π	1.145	$\ln(\pi)$	0.8750	$7/8$	0.7333	$11/15$
1.571	$\pi/2$	0.6931	$\ln(2)$	0.1111	$1/9$	0.8667	$13/15$
1.047	$\pi/3$	1.099	$\ln(3)$	0.2222	$2/9$	0.9333	$14/15$
0.7854	$\pi/4$	1.386	$\ln(4)$	0.4444	$4/9$	0.06250	$1/16$
0.6283	$\pi/5$	1.609	$\ln(5)$	0.5556	$5/9$	0.1875	$3/16$
0.5236	$\pi/6$	1.792	$\ln(6)$	0.7778	$7/9$	0.3125	$5/16$
0.4488	$\pi/7$	1.946	$\ln(7)$	0.8889	$8/9$	0.4375	$7/16$
0.3927	$\pi/8$	2.079	$\ln(8)$	0.1000	$1/10$	0.5625	$9/16$
0.3491	$\pi/9$	2.197	$\ln(9)$	0.3000	$3/10$	0.6875	$11/16$
0.3183	$1/\pi$	2.303	$\ln(10)$	0.7000	$7/10$	0.8125	$13/16$
0.6366	$2/\pi$	0.4343	$\log(e)$	0.9000	$9/10$	0.9375	$15/16$
0.9549	$3/\pi$	2.718	e	0.09091	$1/11$	0.05882	$1/17$
1.273	$4/\pi$	0.8653	e/π	0.1818	$2/11$	0.1176	$2/17$
1.592	$5/\pi$	1.156	π/e	0.2727	$3/11$	0.1765	$3/17$
1.910	$6/\pi$	0.3679	$1/e$	0.3636	$4/11$	0.2353	$4/17$
2.228	$7/\pi$	1.649	$\sqrt{e}$	0.4545	$5/11$	0.2941	$5/17$
2.546	$8/\pi$	0.6065	$1/\sqrt{e}$	0.5455	$6/11$	0.3529	$6/17$
2.865	$9/\pi$	1.396	$\sqrt[3]{e}$	0.6364	$7/11$	0.4118	$7/17$
0.01745	$\pi/180$	0.7165	$1/\sqrt[3]{e}$	0.7273	$8/11$	0.4706	$8/17$
57.30	$180/\pi$	7.389	e^2	0.8182	$9/11$	0.5294	$9/17$
2.094	$2\pi/3$	0.1353	e^{-2}	0.9091	$10/11$	0.5882	$10/17$
4.189	$4\pi/3$	22.46	π^e	0.08333	$1/12$	0.6471	$11/17$
2.356	$3\pi/4$	23.14	e^π	0.4167	$5/12$	0.7059	$12/17$
0.1592	$1/(2\pi)$	0.04321	$e^{-\pi}$	0.5833	$7/12$	0.7647	$13/17$
0.1061	$1/(3\pi)$	535.5	$e^{2\pi}$	0.9167	$11/12$	0.8235	$14/17$
0.07958	$1/(4\pi)$	0.001867	$e^{-2\pi}$	0.07692	$1/13$	0.8824	$15/17$
9.870	π^2	4.810	$e^{\pi/2}$	0.1538	$2/13$	0.9412	$16/17$
31.01	π^3	0.2079	$e^{-\pi/2}$	0.2308	$3/13$	0.05556	$1/18$
97.41	π^4	0.5000	$1/2$	0.3077	$4/13$	0.2778	$5/18$
1.772	$\sqrt{\pi}$	0.3333	$1/3$	0.3846	$5/13$	0.3889	$7/18$
0.8862	$\sqrt{\pi}/2$	0.6667	$2/3$	0.4615	$6/13$	0.6111	$11/18$
0.5908	$\sqrt{\pi}/3$	0.2500	$1/4$	0.5385	$7/13$	0.7222	$13/18$
0.4431	$\sqrt{\pi}/4$	0.7500	$3/4$	0.6154	$8/13$	0.9444	$17/18$
1.465	$\sqrt[3]{\pi}$	0.2000	$1/5$	0.6923	$9/13$		
1.331	$\sqrt[4]{\pi}$	0.4000	$2/5$	0.7692	$10/13$		
0.5642	$1/\sqrt{\pi}$	0.6000	$3/5$	0.8462	$11/13$		
1.128	$2/\sqrt{\pi}$	0.8000	$4/5$	0.9231	$12/13$		
0.6828	$1/\sqrt[3]{\pi}$	0.1667	$1/6$	0.07143	$1/14$		
0.3989	$1/\sqrt{2\pi}$	0.8333	$5/6$	0.2143	$3/14$		
0.4971	$\log(\pi)$	0.1429	$1/7$	0.3571	$5/14$		

1.047	$\pi/3$	23.14	e^{π}	4.810	$e^{\pi/2}$	0.7500	$3/4$
0.1061	$1/(3\pi)$	0.2353	$4/17$	0.4971	$\log(\pi)$	0.7647	$13/17$
1.099	$\ln(3)$	2.356	$3\pi/4$	0.5000	$1/2$	0.07692	$1/13$
0.1111	$1/9$	0.2500	$1/4$	0.5236	$\pi/6$	0.7692	$10/13$
1.128	$2/\sqrt{\pi}$	25.13	8π	0.5294	$9/17$	0.7778	$7/9$
1.145	$\ln(\pi)$	2.546	$8/\pi$	0.5333	$8/15$	0.7782	$\log(6)$
1.156	π/e	0.2667	$4/15$	535.5	$e^{2\pi}$	0.7854	$\pi/4$
0.1176	$2/17$	2.718	e	0.5385	$7/13$	0.7857	$11/14$
0.1250	$1/8$	0.2727	$3/11$	0.5455	$6/11$	0.07958	$1/(4\pi)$
12.57	4π	0.2778	$5/18$	0.05556	$1/18$	0.8000	$4/5$
1.273	$4/\pi$	28.27	9π	0.5556	$5/9$	0.8125	$13/16$
1.331	$\sqrt[4]{\pi}$	0.2857	$2/7$	0.5625	$9/16$	0.8182	$9/11$
0.1333	$2/15$	2.865	$9/\pi$	0.5642	$1/\sqrt{\pi}$	0.8235	$14/17$
0.1353	e^{-2}	0.2941	$5/17$	0.5714	$4/7$	0.08333	$1/12$
1.386	$\ln(4)$	0.3000	$3/10$	57.30	$180/\pi$	0.8333	$5/6$
1.396	$\sqrt[3]{e}$	0.3010	$\log(2)$	0.5833	$7/12$	0.8451	$\log(7)$
0.1429	$1/7$	0.3077	$4/13$	0.05882	$1/17$	0.8462	$11/13$
1.465	$\sqrt[3]{\pi}$	31.01	π^3	0.5882	$10/17$	0.8571	$6/7$
0.1538	$2/13$	0.3125	$5/16$	0.5908	$\sqrt{\pi}/3$	0.8653	e/π
1.571	$\pi/2$	3.142	π	0.6000	$3/5$	0.8667	$13/15$
15.71	5π	0.3183	$1/\pi$	0.6021	$\log(4)$	0.8750	$7/8$
0.1592	$1/(2\pi)$	0.3333	$1/3$	0.6065	$1/\sqrt{e}$	0.8824	$15/17$
1.592	$5/\pi$	0.3491	$\pi/9$	0.6111	$11/18$	0.8862	$\sqrt{\pi}/2$
1.609	$\ln(5)$	0.3529	$6/17$	0.6154	$8/13$	0.8889	$8/9$
1.649	$\sqrt{e}$	0.3571	$5/14$	0.06250	$1/16$	0.9000	$9/10$
0.1667	$1/6$	0.3636	$4/11$	0.6250	$5/8$	0.9031	$\log(8)$
0.01745	$\pi/180$	0.3679	$1/e$	0.6283	$\pi/5$	0.9091	$10/11$
0.1765	$3/17$	0.3750	$3/8$	6.283	2π	0.09091	$1/11$
1.772	$\sqrt{\pi}$	0.3846	$5/13$	0.6364	$7/11$	0.9167	$11/12$
1.792	$\ln(6)$	0.3889	$7/18$	0.6366	$2/\pi$	0.9231	$12/13$
0.1818	$2/11$	0.3927	$\pi/8$	0.6429	$9/14$	0.9286	$13/14$
0.001867	$e^{-2\pi}$	0.3989	$1/\sqrt{2\pi}$	0.6471	$11/17$	0.9333	$14/15$
0.1875	$3/16$	0.4000	$2/5$	0.06667	$1/15$	0.9375	$15/16$
18.85	6π	0.4118	$7/17$	0.6667	$2/3$	0.9412	$16/17$
1.910	$6/\pi$	0.4167	$5/12$	0.6828	$1/\sqrt[3]{\pi}$	9.425	3π
1.946	$\ln(7)$	4.189	$4\pi/3$	0.6875	$11/16$	0.9444	$17/18$
0.2000	$1/5$	0.4286	$3/7$	0.6923	$9/13$	0.9542	$\log(9)$
0.2079	$e^{-\pi/2}$	0.04321	$e^{-\pi}$	0.6931	$\ln(2)$	0.9549	$3/\pi$
2.079	$\ln(8)$	0.4343	$\log(e)$	0.6990	$\log(5)$	97.41	π^4
2.094	$2\pi/3$	0.4375	$7/16$	0.7000	$7/10$	9.870	π^2
0.2143	$3/14$	0.4431	$\sqrt{\pi}/4$	0.7059	$12/17$	0.1000	$1/10$
2.197	$\ln(9)$	0.4444	$4/9$	0.07143	$1/14$		
21.99	7π	0.4488	$\pi/7$	0.7143	$5/7$		
0.2222	$2/9$	0.4545	$5/11$	0.7165	$1/\sqrt[3]{e}$		
2.228	$7/\pi$	0.4615	$6/13$	0.7222	$13/18$		
22.46	π^e	0.4667	$7/15$	0.7273	$8/11$		
2.303	$\ln(10)$	0.4706	$8/17$	0.7333	$11/15$		
0.2308	$3/13$	0.4771	$\log(3)$	7.389	e^2		

Common constants

I use the term "common" with a grain of salt because these are the numbers that I can usually remember when doing manual calculations.

Number	Value	Use
$\sqrt{2}$	1.414	Length of diagonal of square
$\sqrt{3}$	1.732	
cos (29°)	7/8 = 0.875	Cutting threads on lathe
$\frac{\sqrt{3}}{2}$	0.8660	sine of 60°
$\frac{1}{\sqrt{2}}$	0.7071	sine, cosine of 45°
1/6	$0.16\underline{6}$	
1/7	0.1428	
1/8	0.125	
1/9	$0.1\underline{1}$	
1/12	$0.083\underline{3}$	
1/16	0.0625	
1/20	0.05	
1/32	0.03125	
1/64	0.016	To nearest mil; can figure out exact value when needed.
22/7, 355/113	π	

A handy approximation for manually calculating $\sqrt{x}$ is to find the closest integer a such that $a^2 < x$. The approximation is then $b = a + (x - a^2)/(2a)$. You can often quickly get 2 significant figures in your head.

Example: $\sqrt{93}$ a is 9. The difference is 93 - 81 = 12. The approximation is $b \approx 9 + 12/18 = 9.67$, within about a quarter of a percent.

Refine the square root estimate by the Babylonian method (easily derived using Newton's root formula). After using the above approximation to get *b*, one iteration can give 5 or 6 figures:

$$b_{new} = \frac{1}{2}\left(b + \frac{x}{b}\right)$$

Sets

From https://en.wikipedia.org/wiki/Algebra_of_sets, accessed 6 Oct 2014.

All sets are subsets of a universe U. A^c denotes the complement of *A*.

DeMorgan's laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Complement properties

$$(A \cup A^c) = U$$
$$(A \cap A^c)^c = U^c = \emptyset$$
$$\emptyset^c = U$$
$$U^c = \emptyset$$
$$\text{if } A \subset B \text{ then } B^c \subset A^c$$
$$(A^c)^c = A$$

Relationships between relative and absolute complements

$$A - B = A \cap B^c$$
$$(A - B)^c = A^c \cup B$$

Algebra of relative complements

$$C - (A \cap B) = (C - A) \cup (C - B)$$
$$C - (A \cup B) = (C - A) \cap (C - B)$$
$$C - (B - A) = (A \cap C) \cup (C - B)$$
$$(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$$
$$(B - A) \cup C = (B \cup C) - (A - C)$$
$$A - A = \emptyset$$
$$\emptyset - A = \emptyset$$
$$A - \emptyset = \emptyset$$
$$B - A = A^c \cap B$$
$$(B - A)^c = A \cup B^c$$
$$U - A = A^c$$
$$A - U = \emptyset$$

Inequalities

[*bartsch:103*] $\mathbb{N} = \{ 1, 2, 3, \ldots \}$ and all symbols are real unless noted.

$A > B$ **iff** $B < A$

$|(A)| < B$ **iff** $-B < A < B$

$|(AB)| = |(A)|\,|(B)|$

$\left|\sum_i A_i\right| \le \sum_i |A_i|$ (triangle inequality; holds for complex numbers too)

$A^2 > 0$ if $A \neq 0$

If $A > B$ and $B > C$ then $A > C$ (transitivity)

$A > B$ **iff** $A + C > B + C$ (add arbitrary numbers to an inequality)

$A < B \Rightarrow A + C < B + C$

$A < B \Rightarrow AC < BC$ and $A > B \Rightarrow AC > BC$ if $C > 0$

$A < B \Rightarrow AC > BC$ and $A > B \Rightarrow AC < BC$ if $C < 0$

If A and B have the same sign and if $A < B$ then $\frac{1}{A} > \frac{1}{B}$

$0 < A < B \Rightarrow \frac{1}{A} > \frac{1}{B}$

$A < B$ and $C < D \Rightarrow A + C < B + D$

$A < B$ and $C < D \Rightarrow AC < BD$ if $B > 0$ and $C > 0$

$$A < B \text{ and } C < D \Rightarrow -A > -B$$

$$A^n + B^n \le (A+B)^n \text{ for } A > 0,\ B > 0,\ n \ne 0,\ n \in \mathbb{N}$$

$$(1+A)^n \ge 1 + nA,\ A \ge 0,\ n \in \mathbb{N}$$

$$2\sqrt{A+1} - 2\sqrt{A} < \frac{1}{\sqrt{A}} < 2\sqrt{A} - 2\sqrt{A-1}$$

$$2^k > k \text{ if } k \in \mathbb{N}$$

$$\sqrt[n]{1+A} < 1 + \frac{A}{n} \text{ if } A > 0,\ n \in \mathbb{N} \text{ and } n > 1$$

$$(1+x)^r \ge 1 + xr \text{ for all } r \ge 1 \text{ and } x \ge -1$$

Any monotonic increasing function *f* can be applied to both sides of an inequality and it will still be true. *f* leaves both ≤ and ≥ alone.

Applying a strictly monotonically decreasing function *g* to both sides of an inequality means the opposite inequality holds. *g* reverses ≤ to ≥ and vice versa.

Cauchy:

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)$$

Chebyshev: Let $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$ be two non-decreasing sequences of real numbers. Then

$$n \sum_i a_i b_i \ge \sum_i a_i \sum_i b_i \ge \sum_i a_i b_{n+1-i}$$

Cauchy-Schwartz: For any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$

$$\left(a_1^2 + \ldots + a_n^2\right)\left(b_1^2 + \ldots + b_n^2\right) \ge \left(a_1 b_1 + \ldots + a_n b_n\right)^2$$

with equality **iff** the two sequences are proportional.

Means: if $a_1, \ldots, a_n$ are positive real numbers, then

$$\frac{\sum_i a_i}{n} \ge \left(\prod_i a_i\right)^1 / n \ge \frac{1}{\left(\sum_i \frac{1}{a_i}\right) / n}$$

with equality **iff** $a_1 = a_2 = \cdots = a_n$

Jordan: for $0 \le x \le \pi/2$

$$\frac{2}{\pi} \le \sin x \le x$$

Napier: for $b > a > 0$

$$\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

Triangular: $|x - y| \le z \le x + y$. If they hold for one permutation of (x, y, z), they hold for all.

If you have a requirement that $a < x < b$ where you might also have $a > b$, a numerical test you can use to cover both cases is

$$(x - a)(b - x) > 0$$

Series, approximations, and formulas

Decimal expansions

Suppose we're given a number x that has a repeating decimal expansion. To convert it to a rational fraction, multiply by 10^n where n is the length of the repeating pattern in the decimal expansion. Then if d is the decimal expansion, we have

$$\begin{aligned} 10^n x &= 10^n d \\ x &= d \end{aligned}$$

If these two equations are subtracted, the repeating portion of the decimal expansion will be gone and manipulation yields a rational number.

For example, let $x = 14.2857\overline{142857}$. Here, $n = 6$, so we multiply by 10^6

$$10^6 x = 14285714.\overline{285714}$$

Subtracting x, we get $999999\,x = 14285700$ or

$$x = \frac{14285700}{999999} = \frac{2^2 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 37}{3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37} = \frac{100}{7}$$

If a number x does not have a repeating decimal expansion, then it is irrational.

Arithmetic progression

An **arithmetic progression** has terms that increase or decrease by a common difference. The nth term of an arithmetic progression is [*wp*]

$$a_n = a_1 + (n-1)\,d$$

where a_1 is the first term of the series, d is the common difference, and $n > 0$.

The sum of an arithmetic progression is

$$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\left[2a_1 + (n-1)\,d\right] = \frac{n}{2}(a_1 + L)$$

where L is the last term in the series $L = a_1 + (n-1)\,r$.

If a_1/d is not a negative integer or zero, then the product of the arithmetic progression's terms is

$$\prod_{k=1}^{n} a_k = d^n \frac{\Gamma\left(\frac{a_1}{d} + n\right)}{\Gamma\left(\frac{a_1}{d}\right)}$$

Geometric progression

A **geometric progression** has each term equal to the preceding term multiplied by a constant r, which can be complex:

$$a, ar, ar^2, ar^3, \ldots$$

The sum of terms is the geometric series [*wp*]

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \sum_{k=0}^{n} ar^k = \frac{a\left(1 - r^{n+1}\right)}{1 - r} \tag{1}$$

A recursion relation is

$$a_n = r\,a_{n-1} \text{ for } n > 0$$

Another relation is

$$a_n^2 = a_{n-1}\,a_{n+1}$$

i.e., the *n*th term is the geometric mean of the term before it and after it (and is the reason for the name of the geometric series).

The sum of an infinite number of terms is (the following series converge **iff** $|r|<1$)

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

For even powers, the finite sum is

$$\sum_{k=0}^{n} ar^{2k} = \frac{a\left(1-r^{2n+2}\right)}{1-r^2}$$

For odd powers, the finite sum is

$$\sum_{k=0}^{n} ar^{2k+1} = \frac{ar\left(1-r^{2n+2}\right)}{1-r^2}$$

If the sum does not start at k = 0

$$\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}$$

Differentiating equation (1) with a = 1 gives

$$\frac{d}{dr}\sum_{k=0}^{n} r^k = \sum_{k=1}^{n} kr^{k-1} = \frac{1-r^{n+1}}{(1-r)^2} - \frac{(n+1)\,r^n}{1-r}$$

Binomial theorem

The **binomial theorem** and its generalization lead to useful approximations. The theorem is [*gr:21*]

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n!}{(n-k)!\,k!}x^k + \ldots$$

If *n* is a positive integer, the series has a finite number of terms; otherwise there are an infinite number of terms. If *n* is not an integer, the series converges absolutely for $|x|<1$ and diverges for $|x|>1$. For $x=1$, the series converges for $n>-1$ and diverges for $n \le -1$ and the series converges absolutely for $n>0$. For $x=-1$, it converges absolutely for $n>0$ and diverges for $n<0$.

For integer n > 0 [*gr:21*]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where the binomial coefficient is

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Some special forms are [*gr:21*]

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{k=1}^{\infty} x^{k-1}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

$$\frac{1}{(1 \pm x)^2} = 1 \mp 2x + 3x^2 \mp 4x^3 + \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} k\, x^{k-1}$$

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k\, x^k \text{ where } x < 1$$

$$\sqrt{1 \pm x} = 1 \pm \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 \pm \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \pm \ldots$$

$$= 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \frac{7}{256}x^5 - \frac{21}{1024}x^6 \pm \ldots$$

$$\frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 \mp \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \mp \ldots$$

$$= 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \frac{35}{128}x^4 \mp \frac{63}{256}x^5 + \frac{231}{1024}x^6 \mp \ldots$$

Approximations

The above series expansions lead to commonly-used approximations when x is small, such as

$$\frac{1}{1 \pm x} \approx 1 \mp x$$

$$\frac{1}{(1 \pm x)^2} \approx 1 \mp 2x$$

$$\sqrt{1 \pm x} = 1 \pm \frac{1}{2}x$$

$$\frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{1}{2}x$$

These generalize to (all variables can be complex)

$$(1 \pm \delta)^a (1 \pm \epsilon)^b \approx 1 \pm a\delta \pm b\epsilon \text{ where } |\delta| \ll 1 \text{ and } |\epsilon| \ll 1$$

If x and y are nearly equal, another occasionally-useful approximation is $\sqrt{xy} \approx (x+y)/2$.

[*steinmetz:201:222*] gives a table of functions that are approximated by $1 \pm \delta$ when $\delta \ll 1$:

$\frac{1}{1 \mp \delta}$	$\left(1 \pm \frac{\delta}{n}\right)^n$	$\sqrt[n]{1 \pm n\delta}$	$\frac{1}{\sqrt[n]{1 \mp n\delta}}$	$\sqrt{\frac{1 \pm \delta}{1 \mp \delta}}$	
$e^{\pm\delta}$	$2 - e^{\mp\delta}$	$1 + \ln(1 \pm \delta)$	$1 - \ln(1 \mp \delta)$	$1 + n\ln\left(1 \pm \frac{\delta}{n}\right)$	$1 - n\ln\left(1 \mp \frac{\delta}{n}\right)$
$1 \pm n\sin\frac{\delta}{n}$	$1 \pm \frac{1}{n}\sin n\delta$	$\cos\sqrt{\mp 2\delta}$			

and numerous others could be generated.

Loss of significance

Subtraction or division of two nearly equal quantities can result in loss of significance. We'll look at two examples.

Suppose you want to calculate 297/283 with a slide rule; the typical 10 inch rule will give 1.05. Since these two numbers are nearly equal, the slide rule calculation (division by subtraction of logarithms) has lost significance. Rewrite the problem as 1 + 14/283; doing the latter term on the slide rule and adding 1 gives two more significant figures: 1.0495.

Another example is the quadratic equation for the roots of $x^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4c}}{2}=\frac{b}{2}\left[-1\pm\sqrt{1-\frac{4c}{b^2}}\right] \tag{2}$$

If $\frac{4c}{b^2}\ll 1$, then the radical will be nearly 1 and loss of significance will occur for the small root. To avoid losing significance in calculating the near-zero root, rewrite the formula using the approximation $\sqrt{1-x}\approx 1-\frac{x}{2}$:

$$x=\frac{b}{2}\left[-1+\sqrt{1-\frac{4c}{b^2}}\right]\approx\frac{b}{2}\left[-1+\left(1-\frac{4c}{2b^2}\right)\right]=-\frac{c}{b}$$

You can include higher terms in the approximation (from the binomial theorem) and get as many significant figures for the small root as you need.

You can try a numerical example: b = 10^n and c = 1 where *n* is larger than the number of significant digits that your calculator uses in its floating point representation. The standard formula (2) results in zero for the small root, but the approximate equation gives it as -10^{-n}.

Stirling's formula

$$\lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{n}}=\sqrt{2\pi}=2.5066$$

Convergence tests

Let the series be $\sum_{n=1}^{\infty} a_n$.

Comparison test

A series converges if each of its terms is equal to or less than the corresponding term of a known convergent series.

A series diverges if each of its terms is equal to or greater than the corresponding term of a known divergent series.

The geometric series can often be used for the comparison test:

$$\sum_{k=0}^{\infty} ar^k=\frac{a}{1-r}\text{ converges if }|r|<1\text{ and diverges if }|r|>1.$$

Remember *r* can be complex.

Ratio test

If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\to c$ where c is a constant, then the series converges if $c<1$, diverges if $c>1$, and is inconclusive if $c=1$.

Alternating series

If a series has terms that alternate in sign, it converges if $|a_n|<|a_{n-1}|$.

Sums of integers

[*gr:1*]

p	$\sum_{k=0}^{n} k^p$
1	$\frac{1}{2}n(n+1)$
2	$\frac{1}{6}n(n+1)(2n+1)$
3	$\left[\frac{1}{2}n(n+1)\right]^2$
4	$\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$
5	$\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

p	$\sum_{k=1}^{n} (2k-1)^p$
1	n^2
2	$\frac{1}{3}n(4n^2-1)$
3	$n^2(2n^2-1)$

Other series

$$\sum_{k=0}^{\infty}(a+kr)q^k = \frac{a}{1-q}+\frac{rq}{(1-q)^2} \text{ for } |q|<1$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$a^x = e^{x\ln a} = 1+x\ln a+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{3!}+\ldots$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots = x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}-\ldots$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+\ldots$$

$$\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\ldots \text{ for } |x|<1$$

$$= \frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\ldots \text{ for } |x|>1$$

$$\ln x = 2\left[\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^2+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^3+\ldots\right] \text{ where } x>0$$

$$\ln(1\pm x) = \pm x-\frac{x^2}{2}\pm\frac{x^3}{3}-\frac{x^4}{4}\pm\ldots$$

Summation by parts

For integer j

$$\sum_{0<j<n} u_j\, v_j = T_{n-1}\, v_n + \sum_{0<j<n} T_j \left(v_j - v_{j+1}\right) \qquad \text{where} \qquad T_j = \sum_{1 \le j \le n} u_j$$

Summing series with integrals

The trapezoidal formula for approximating an integral is

$$\frac{1}{2} f(a) + f(a+h) + f(a+2\,h) + \ldots + f(a+(n-1)\,h) + \frac{1}{2} f(a+nh) \;\approx\; \frac{1}{h} \int_a^b f(x)\, dx$$

Because h is small we also have e.g.

$$\frac{1}{2} f(a) \;\approx\; \frac{1}{h} \int_{a-\frac{h}{2}}^{a} f(x)\, dx$$

Thus, we can use these to estimate a sequence of function evaluations at abscissas that comprise an arithmetic series:

$$f(a) + f(a+h) + \ldots + f(b) \;\approx\; \frac{1}{h} \int_{a-\frac{h}{2}}^{b+\frac{h}{2}} f(x)\, dx$$

Example: compute $\sqrt{322} + \sqrt{323} + \ldots + \sqrt{579}$. The sum is approximated by the integral

$$\int_{321.5}^{579.5} \sqrt{x}\, dx \;=\; \frac{2}{3}\, x^{3/2} \Big|_{321.5}^{579.5} \;=\; 5457.04056$$

The correct value of the sum is 5457.04085419, so the approximation got better than 6 significant figures.

Euler-Maclaurin formula

The formula in the previous section is a special case of the Euler-Maclaurin formula which relates the value of sums and integrals with an error term involving Bernoulli numbers [*as:806*]. Let $f(x)$ have continuous first $2n$ derivatives on an interval (a, b). Divide the interval into m equal parts and let $h = (b-a)/m$. Then for some θ depending on $f^{(2n)}(x)$ on (a, b)

$$\begin{aligned}
\sum_{k=0}^{m} f(a+k\,h) \;&=\; \frac{1}{h} \int_a^b f(x)\, dx \;+\; \frac{1}{2}\left[f(a) + f(b)\right] \\
&\quad + \sum_{k=1}^{n-1} \frac{h^{2k-1}}{(2k)\,!}\, B_{2k} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\
&\quad + \frac{h^{2n}}{(2n)\,!}\, B_{2n} \sum_{k=0}^{m-1} f^{(2n)}(a + kh + \theta\, h)
\end{aligned}$$

Iteration

Suppose we wish to find the root of a univariate function $g(x) = 0$. Rewrite this as

$$x = f(x)$$

and use it to iterate for roots by the following scheme:

$$x_{i+1} \;=\; f(x_i)$$

This iteration will converge if $f'(x) < 1$ in the neighborhood of the root and it converges faster the smaller the derivative.

Iteration for roots

Newton's method applied to calculating roots: if $x^n = N$ and if x_k is an approximation of $N^{1/n}$, then the sequence [*as:18*]

$$x_{k+1} = \frac{1}{n}\left[\frac{N}{x_k^{n-1}} + (n-1)x_k\right]$$

will converge quadratically (i.e., about double the number of significant figures every iteration) to x. For square roots, this gives (sometimes called the Babylonian method)

$$x_{k+1} = \frac{1}{2}\left[\frac{N}{x_k} + x_k\right]$$

For cube roots

$$x_{k+1} = \frac{1}{3}\left[\frac{N}{x_k^2} + 2x_k\right]$$

Interpolation

Let a table contain equally-spaced tabular values x_i and a function $y_i = f(x_i)$ also tabulated. Let the first order-differences be Δy_i, the second-order differences be $\Delta_2 y_i$, etc. Then an interpolation formula for y_x such that $y_i < y_x < y_{i+1}$ is

$$y_x = y_i + \delta\Delta y_i + \frac{\delta(\delta-1)}{1\cdot 2}\Delta_2 y_i + \frac{\delta(\delta-1)(\delta-2)}{1\cdot 2\cdot 3}\Delta_3 y_i + \ldots$$

Here's an example of interpolating for the sine with six-figure values. Suppose we want the sine of 42.137° from interpolation in the following table (decimal points ignored):

$x \rightarrow$	42°		43°		44°		45°
$y \rightarrow$	669131		681998		694658		707107
Δ		12868		12660		12448	
Δ^2			-208		-212		
Δ^3				4			

where $\delta = \dfrac{y_x - y_i}{y_{i+1} - y_i}$. Here, $\delta = 0.137$. Then we have

$$\begin{aligned}\sin(42.137) &= 669131 + 0.137(12868) + \frac{0.137(0.137-1)}{2}(-208) + \frac{0.137(0.137-1)(0.137-2)}{6}(4)\\ &= 669131 + 1763 + 12 + 0.1\\ &= 670906\end{aligned}$$

which is 1/2 ppm from the true value. The third order difference didn't contribute anything useful.

Linear interpolation gives

$$\begin{aligned}\sin(41.237) &= 669131 + 0.137(12868)\\ &= 669131 + 1763\\ &= 670894\end{aligned}$$

which is 17 ppm low.

Proportion

[*wynne:2:17*] and [*hudm:2:15*]

If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$ and

$$\frac{a}{c}=\frac{b}{d}=\frac{a\pm b}{c\pm d} \qquad \frac{a}{b}=\frac{c}{d}=\frac{a\pm c}{b\pm d}$$
$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \qquad \frac{b+d}{b-d}=\frac{a+c}{a-c}$$
$$\frac{ma}{mb}=\frac{nc}{nd} \qquad \frac{ma}{nb}=\frac{mc}{nd}$$

If $\frac{a}{b}=\frac{c}{d}$ and $\frac{e}{f}=\frac{g}{h}$, then

$$\frac{ae}{bf}=\frac{cg}{dh} \quad \text{and} \quad \frac{ag}{bh}=\frac{ce}{df}.$$

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$

$$\frac{a+c+e}{b+d+f}=\frac{ma+nc+pe}{mb+nd+pf}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$$

Trigonometry

Many trigonometric identities can be derived using Euler's identity (see [*feynman*])

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Example: the addition formulas for the sine and cosine can be calculated from $e^{ix}e^{iy}$:

$$\begin{aligned}
&e^{ix}e^{iy}=(\cos x+i\sin x)(\cos y+i\sin y)\\
&e^{i(x+y)}=\cos x\cos y-\sin x\sin y+i(\sin x\cos y+\cos x\sin y)\\
&\cos(x+y)+i\sin(x+y)=\cos x\cos y-\sin x\sin y+i(\sin x\cos y+\cos x\sin y)
\end{aligned}$$

and the addition formulas can be read off by equating real and imaginary parts. Then, knowing the sine is odd and the cosine is even, substitute $-y$ for y and get the difference formulas.

The trig functions on the unit circle are (you can get the script that produces this drawing from http://code.google.com/p/hobbyutil/; download the `math/triguc.zip` file):

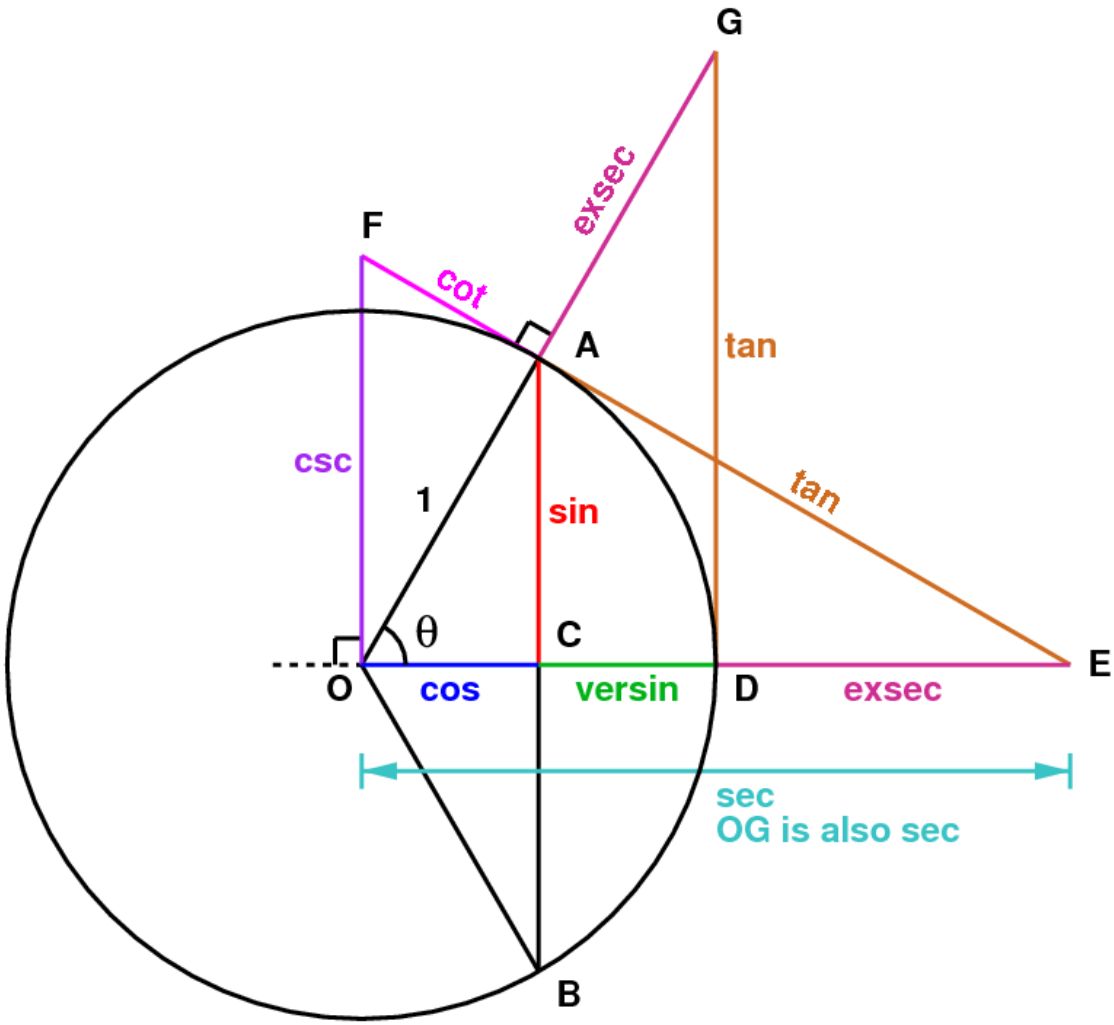
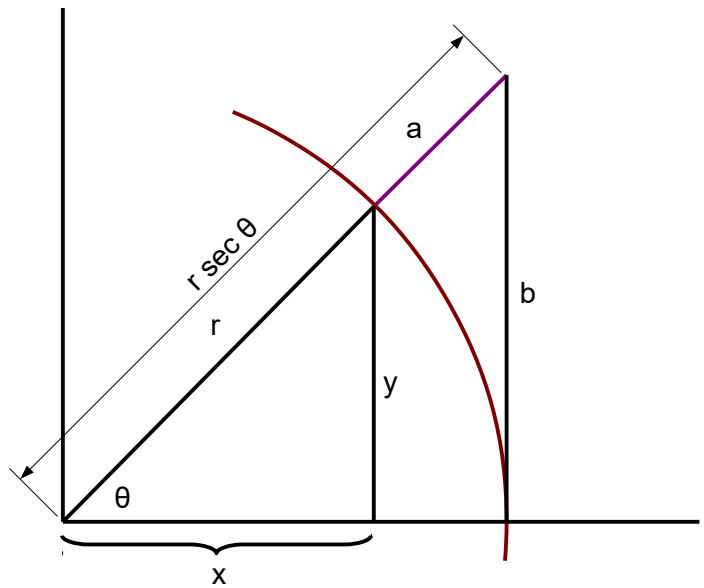


Figure 1

Some seldom-used trig functions are:

$$\begin{aligned}
&\operatorname{exsec}\theta = \sec\theta - 1 \\
&\operatorname{vers}\theta = 1 - \cos\theta \quad \text{(versed sine)} \\
&\operatorname{covers}\theta = 1 - \sin\theta \\
&\operatorname{hav}\theta = \frac{1}{2}\operatorname{vers}\theta \quad \text{(haversine)}
\end{aligned}$$

In the figure



$$\sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} = \cos\left(\frac{\pi}{2}-\theta\right) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -i\sinh i\theta$$
$$\cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} = \sin\left(\frac{\pi}{2}-\theta\right) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh i\theta$$
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta} = -i\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right) = -i\tanh i\theta$$
$$a = \sqrt{r^2+b^2}-r$$
$$b = r\tan\theta = r\frac{y}{x} = (r+a)\sin\theta$$

The distance r + a is immediately seen to be equal to $r\,sec\,\theta$ because of the Pythagorean theorem with the right triangle with the vertical $b = r\tan\theta$ leg and the other leg of r.

There are also relationships with the hyperbolic functions (easy to derive from Euler's formula):

$$\sin ix = i\sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i\tanh x$$

$$\sinh ix = i\sin x$$
$$\cosh ix = \cos x$$
$$\tanh ix = i\tan x$$

Relations between the circular functions

Relations between the circular functions for $0 \le x \le \frac{\pi}{2}$. This table can also apply to the functions over the full circle, but you'll have to put a ± sign in front of the radicals and choose the proper sign depending on the quadrant.

	sin x = a	**cos x = a**	**tan x = a**	**cot x = a**	**sec x = a**	**csc x = a**
sin x	a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{a}$
cos x	$\sqrt{1-a^2}$	a	$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	$\frac{\sqrt{a^2-1}}{a}$
tan x	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$	a	$\frac{1}{a}$	$\sqrt{a^2-1}$	$\frac{1}{\sqrt{a^2-1}}$
cot x	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{1}{a}$	a	$\frac{1}{\sqrt{a^2-1}}$	$\sqrt{a^2-1}$
sec x	$\frac{1}{\sqrt{1-a^2}}$	$\frac{1}{a}$	$\sqrt{1+a^2}$	$\frac{\sqrt{1+a^2}}{a}$	a	$\frac{a}{\sqrt{a^2-1}}$
csc x	$\frac{1}{a}$	$\frac{1}{\sqrt{1-a^2}}$	$\frac{\sqrt{1+a^2}}{a}$	$\sqrt{1+a^2}$	$\frac{a}{\sqrt{a^2-1}}$	a

Inverse functions

	sin $^{-1}$	**cos $^{-1}$**	**tan $^{-1}$**	**cot $^{-1}$**	**sec $^{-1}$**	**csc $^{-1}$**
sin $^{-1}$ x		$\cos^{-1}\sqrt{1-x^2}$	$\tan^{-1}\frac{x}{\sqrt{1-x^2}}$	$\cot^{-1}\frac{\sqrt{1-x^2}}{x}$	$\sec^{-1}\frac{1}{\sqrt{1-x^2}}$	$\csc^{-1}\frac{1}{x}$
cos $^{-1}$ x	$\sin^{-1}\sqrt{1-x^2}$		$\tan^{-1}\frac{\sqrt{1-x^2}}{x}$	$\cot^{-1}\frac{x}{\sqrt{1-x^2}}$	$\sec^{-1}\frac{1}{x}$	$\csc^{-1}\frac{1}{\sqrt{1-x^2}}$
tan $^{-1}$ x	$\sin^{-1}\frac{x}{\sqrt{1+x^2}}$	$\cos^{-1}\frac{1}{\sqrt{1+x^2}}$		$\cot^{-1}\frac{1}{x}$	$\sec^{-1}\sqrt{1+x^2}$	$\csc^{-1}\frac{\sqrt{1+x^2}}{x}$
cot $^{-1}$ x	$\sin^{-1}\frac{1}{\sqrt{1+x^2}}$	$\cos^{-1}\frac{x}{\sqrt{1+x^2}}$	$\tan^{-1}\frac{1}{x}$		$\sec^{-1}\frac{\sqrt{1+x^2}}{x}$	$\csc^{-1}\sqrt{x^2-1}$
sec $^{-1}$ x	$\sin^{-1}\frac{\sqrt{x^2-1}}{x}$	$\cos^{-1}\frac{1}{x}$	$\tan^{-1}\sqrt{x^2-1}$	$\cot^{-1}\frac{1}{\sqrt{x^2-1}}$		$\csc^{-1}\frac{x}{\sqrt{x^2-1}}$
csc $^{-1}$ x	$\sin^{-1}\frac{1}{x}$	$\cos^{-1}\frac{\sqrt{x^2-1}}{x}$	$\tan^{-1}\frac{1}{\sqrt{x^2-1}}$	$\cot^{-1}\sqrt{x^2-1}$	$\sec^{-1}\frac{x}{\sqrt{x^2-1}}$	

$$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\frac{x \pm y}{1 \mp xy}$$

If $\sin^{-1}x$, $\tan^{-1}x$ and $\csc^{-1}$ are between -π/2 and π/2 and $\cos^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}$ are between 0 and π, then

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \qquad \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \qquad \sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

Magnitude and sign

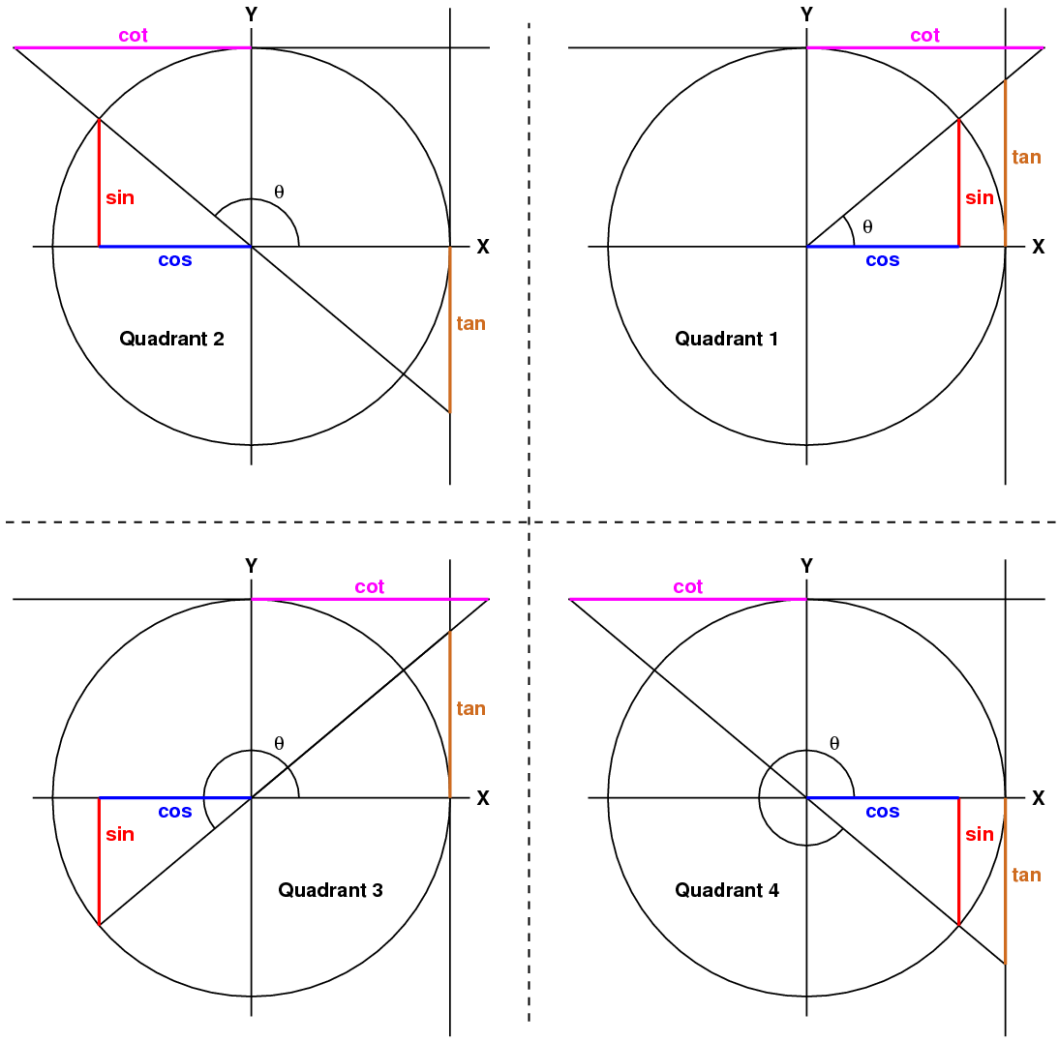
	If x is in quadrant			
	1	2	3	4
$\sin x$	$+0$ to $+1$	$+1$ to $+0$	-0 to -1	-1 to -0
$\cos x$	$+1$ to 0	-0 to -1	-1 to -0	$+0$ to $+1$
$\tan x$	$+0$ to $+\infty$	$-\infty$ to -0	$+0$ to $+\infty$	$-\infty$ to -0
$\cot x$	$+\infty$ to $+0$	-0 to $-\infty$	$+\infty$ to $+0$	-0 to $-\infty$
$\sec x$	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$	$+\infty$ to $+1$
$\csc x$	$+\infty$ to $+1$	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$

Angles in other quadrants

For an angle x in the first quadrant, the following table gives the functions in other quadrants:

	$-x$	$\pi/2 \pm x$	$\pi \pm x$	$3\pi/2 \pm x$	$n\pi \pm x$
$\sin x$	$-\sin x$	$+\cos x$	$\mp\sin x$	$-\cos x$	$\pm\sin x$
$\cos x$	$+\cos x$	$\mp\sin x$	$-\cos x$	$\pm\sin x$	$+\cos x$
$\tan x$	$-\tan x$	$\mp\cot x$	$\pm\tan x$	$\mp\cot x$	$\pm\tan x$
$\cot x$	$-\cot x$	$\mp\tan x$	$\pm\cot x$	$\mp\tan x$	$\pm\cot x$
$\sec x$	$+\sec x$	$\mp\csc x$	$-\sec x$	$\pm\csc x$	$+\sec x$
$\csc x$	$-\csc x$	$+\sec x$	$\mp\csc x$	$-\sec x$	$\pm\csc x$

The following pictures show the sine, cosine, tangent, and cotangent functions in different quadrants:



Small angle approximations

For small x in radians:

$$\sin x \approx x$$
$$\tan x \approx x$$
$$\cos x \approx 1 - \frac{x^2}{2}$$

Ordering relations

From [*perry:2-21*] for x in radians (note these are exact relationships)

$$\sin x < x < \tan x \qquad x\left(1-\frac{x^2}{2}\right) < \sin x < x$$

$$\cos x < \frac{\sin x}{x} < 1 \qquad x < \tan x < \frac{x}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} < \frac{\sin x}{x} < 1 \qquad \lim_{x\to\infty} \frac{\sin x}{x} = 1$$

$$x\sqrt{1-x^2} < \sin x < x \qquad \lim_{x\to\infty} \frac{\sin x}{\tan x} = 1$$

$$\cos x < \frac{x}{\tan x} < 1$$

Values for some special angles

The rows with the light gray background were derived by Herman Robinson (1912-1986).

Rad	Deg	sin x = a	cos x = a	tan x = a
0	0°	0	1	0
$\frac{\pi}{60}$	3°	See below	See below	--
$\frac{\pi}{30}$	6°	$\frac{\sqrt{30-6\sqrt{5}}-\sqrt{5}-1}{8}$	$\frac{\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{8}$	$\frac{\sqrt{10-2\sqrt{5}}+\sqrt{3}-\sqrt{15}}{2}$
$\frac{\pi}{24}$	7.5°	$\frac{\sqrt{8-2\sqrt{2}-2\sqrt{6}}}{4}$	$\frac{\sqrt{8+2\sqrt{2}+2\sqrt{6}}}{4}$	$\sqrt{6}-\sqrt{3}+\sqrt{2}-2$
$\frac{\pi}{20}$	9°	$\frac{\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}}{8}$	$\frac{\sqrt{10}+\sqrt{2}+2\sqrt{5-\sqrt{5}}}{8}$	$1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$
$\frac{\pi}{16}$	11.25°	$\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$	$\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$	$\sqrt{4+2\sqrt{2}}-\sqrt{2}-1$
$\frac{\pi}{15}$	12°	See below	See below	--
$\frac{\pi}{12}$	15°	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$2-\sqrt{3}$
$\frac{\pi}{10}$	18°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{1-\frac{2}{\sqrt{5}}}$
$\frac{\pi}{8}$	22.5°	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\sqrt{2}-1$
$\frac{3\pi}{20}$	27°	$\frac{2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2}}{8}$	$\frac{2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2}}{8}$	--
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{5}$	36°	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\sqrt{5-2\sqrt{5}}$
$\frac{5\pi}{24}$	37.5°	$\frac{\sqrt{2-\sqrt{2-\sqrt{3}}}}{2}$	$\frac{\sqrt{2+\sqrt{2-\sqrt{3}}}}{2}$	$\sqrt{6}+\sqrt{3}-\sqrt{2}-2$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{5\pi}{12}$	75°	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	$2+\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	∞

379 $$\sin\left(\frac{\pi}{60}\right)=\sin(3°)=\frac{(\sqrt{6}+\sqrt{2})(\sqrt{5}-1)}{16}-\frac{(\sqrt{3}-1)\sqrt{5+\sqrt{5}}}{8}$$

380 $$\cos\left(\frac{\pi}{60}\right)=\cos(3°)=\frac{(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)}{16}+\frac{(\sqrt{3}+1)\sqrt{5+\sqrt{5}}}{8}$$

381 $$\sin\left(\frac{\pi}{15}\right)=\sin(12°)=\frac{2\sqrt{3}-2\sqrt{15}+\sqrt{10-2\sqrt{5}}+\sqrt{50-10\sqrt{5}}}{16}$$

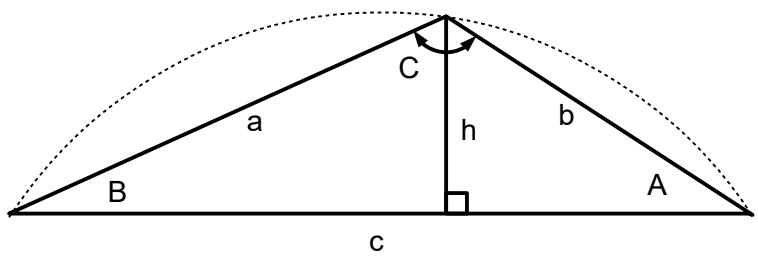
$$\cos\left(\frac{\pi}{15}\right)=\cos\left(12°\right)=\frac{\sqrt{31+9\sqrt{5}+4\sqrt{30-6\sqrt{5}}-5\sqrt{6-2\sqrt{5}}}}{8}$$

Pythagorean relationships

$$\sin^2 x+\cos^2 x=1 \qquad 1+\tan^2 x=\sec^2 x \qquad 1+\cot^2 x=\csc^2 x$$

Sine law, cosine law, etc.

In the following triangle [*maor:216:224*]



project the length of sides a and b on the altitude h; since they're equal, we have

$$b\sin A=a\sin B$$

which immediately leads to

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad \text{Law of sines}$$

The sine law is really a theorem about circles and their chords [*maor:90:99*]: three noncollinear points uniquely determine a circumscribed circle about the triangle; define this circle to have a unit <u>diameter</u>. Then the law of sines tells us that $a=\sin A$, $b=\sin B$, and $c=\sin C$.

Project the length of sides a and b onto the side of length c

$$c=a\cos B+b\cos A$$

The triangle inequality $c\le a+b$ follows immediately from this with equality **iff** $A=B=0$. Square this equation and substitute from the sine law and you can derive

$$c^2=a^2+b^2-2ab\cos C \qquad \text{Law of cosines}$$

From the sine law and the expression for $\sin A\pm\sin B$ [*maor:152:161*]

$$\frac{a+b}{a-b}=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} \qquad \text{Law of tangents}$$

In any acute triangle (all three interior angles are $\le\pi/2$) [*maor:153:162*]

$$\tan A\ \tan B\ \tan C=\tan A+\tan B+\tan C\ge 3\sqrt{3}$$

The minimum value is attained **iff** the triangle is equilateral.

Sums and products

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \cos^2 y = \cos^2 y - \sin^2 x$$
$$\sin x \pm \sin y = 2\sin\frac{1}{2}(x \pm y)\cos\frac{1}{2}(x \mp y)$$
$$\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$
$$\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$$
$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$$

$$2\sin x \sin y = \cos(x-y) - \cos(x+y)$$
$$2\cos x \cos y = \cos(x-y) + \cos(x+y)$$
$$2\sin x \cos y = \sin(x+y) + \sin(x-y)$$

Hand calculation formulas

To extend tables or make slide rule calculations simpler, the following identities can be used:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\sin\left(\frac{\pi}{4} \pm x\right) = \cos\left(\frac{\pi}{4} \mp x\right)$$

$$\tan\left(x \pm \frac{\pi}{4}\right) = -\cot\left(x \mp \frac{\pi}{4}\right)$$

The following relations help calculate tangents for angles between 45° and 90°:

$$\tan x = \frac{1+b}{1-b} \quad \text{where} \quad b = \tan\left(x - \frac{\pi}{4}\right)$$

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{1+\tan x}{1-\tan x} \qquad \tan\left(x - \frac{\pi}{4}\right) = \frac{1-\tan x}{1+\tan x}$$

$$\tan x = \frac{1}{\tan\left(\frac{\pi}{2} - x\right)} \approx \frac{1}{\frac{\pi}{2} - x} \text{ for x near } \frac{\pi}{2} \text{ and } x \text{ in radians}$$

Example: tan(80°) can be calculated from tan(80° - 45 °) = 0.70 as 1.7/0.3 = 17/3 or about 5.7.

Sum of sinusoids

One can derive from the angle sum relation above that

$$a \sin \omega t + b \cos \omega t = V_0 \cos(\omega t - \delta)$$

where

$$V_0 = \sqrt{a^2 + b^2}$$
$$\delta = \tan^{-1}\frac{a}{b}$$

Multiple angles

$$\sin 2x = 2\sin x\cos x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\sin 4x = 4\sin x\cos x - 8\sin^2 x\cos x$$

$$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$$

$$\sin nx = 2\sin(n-1)x\cos x - \sin(n-2)x$$

$$\cos nx = 2\cos(n-1)x\cos x - \cos(n-2)x$$

$$\tan nx = \frac{\tan(n-1)x + \tan x}{1-\tan(n-1)x\tan x}$$

Half angles

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

where the sign is to be chosen according to the sign of the left-hand side.

Powers

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$$

Inverse functions

Principal values

	Domain	Range
$\sin^{-1} x$	$-1 \le x \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \le x \le 1$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\mathbb{R}$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$sec^{-1} x$	$x \ge 1$ $x \le -1$	$\left[0, \frac{\pi}{2}\right)$ $\left[-\pi, \frac{\pi}{2}\right)$
$csc^{-1} x$	$x \ge 1$ $x \le -1$	$\left(0, \frac{\pi}{2}\right]$ $\left(-\pi, \frac{\pi}{2}\right]$

Relations

These are for the principal values.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \csc^{-1}\frac{1}{x}$$

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x} = \csc^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \csc^{-1}\frac{\sqrt{1+x^2}}{x}$$

$$\cot^{-1} x = \tan^{-1}\frac{1}{x}$$

$$\sec^{-1} x = \cos^{-1}\frac{1}{x}$$

$$\csc^{-1} x = \sin^{-1}\frac{1}{x}$$

$$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\frac{x \pm y}{1 \mp xy}$$

$$\cos(\sin^{-1} x) = \sin(\cos^{-1} x) = \sqrt{1-x^2}$$

$$\tan(\sec^{-1} x) = \cot(\csc^{-1} x) = \sqrt{x^2-1}$$

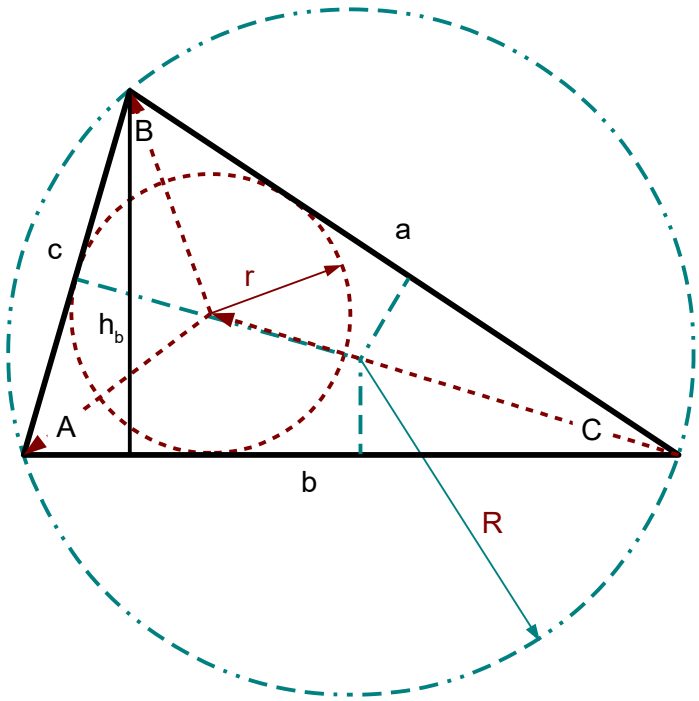
$$\sec(\tan^{-1} x) = \csc(\cot^{-1} x) = \sqrt{x^2+1}$$

$$\sin^{-1} x = \cos^{-1}(-x) - \frac{\pi}{2} = \frac{\pi}{2} - \cos^{-1} x = -\sin^{-1}(-x) = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} - \cot^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$\cos^{-1} x = \frac{\pi}{2} + \sin^{-1}(-x) = \frac{\pi}{2} - \sin^{-1} x = \pi - \cos^{-1}(-x) = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} - \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$\tan^{-1} x = \cot^{-1}(-x) - \frac{\pi}{2} = \frac{\pi}{2} - \cot^{-1} x = -\tan^{-1}(-x) = \sin^{-1}\frac{x}{\sqrt{x^2+1}} = = \frac{\pi}{2} - \cos^{-1}\frac{x}{\sqrt{x^2+1}}$$

Triangles



We'll use the notation

a, *b*, *c* = triangle's sides
A, *B*, *C* = triangle's corresponding angles
K = area
r = radius of inscribed circle (incircle)
R = radius of circumscribed circle (circumcircle)
h_b = altitude that intersections side b (and analogous for the other sides)

A formula in bold denotes a formula that has other forms which are gotten by cyclically permuting the variables (*a*, *b*, *c*) and (*A*, *B*, *C*).

450 $$s = \frac{1}{2}(a + b + c)$$

451 $$Q = (s - a)(s - b)(s - c)$$

452 $$A + B + C = \pi$$

453 $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

454 $$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

455 $$a^2 = b^2 + c^2 - 2bc\cos A$$

456 $$a = b\cos C + c\cos B$$

457 $$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

458 $$\sin A = \frac{2}{bc}\sqrt{sQ}$$

459 $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

460 $$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

461 $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

462 $$h_b = c\sin A = a\sin C = \frac{2}{b}\sqrt{sQ}$$

463 $$r = \sqrt{\frac{Q}{s}} = (s-a)\tan\frac{A}{2} = \frac{K}{s}$$

464 $$R = \frac{ac}{2h_b} = \frac{a}{2\sin A} = \frac{abc}{4K}$$

$$K = \frac{1}{2} b\, h_b = \frac{1}{2} a\, b \sin C = \frac{a^2 \sin B \sin C}{2 \sin A} = \sqrt{sQ} = r\, s$$
$$= 2R^2 \sin A \sin B \sin C = s^2 \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$$

$a+b > c$, $b+c > a$, $a+c > b$, $|a-b| < c$, $|b-c| < a$, $|a-c| < b$

The center of the **circumscribed circle** is at the intersection of the perpendicular bisectors of the sides. The center of the **inscribed circle** is at the intersection of the bisectors of the internal angles. Suppose the triangle passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Then the center of the circumscribed circle and radius R are

$$\left(-\frac{d}{2a}, -\frac{e}{2a}\right) \qquad R = \sqrt{\frac{d^2+e^2}{4a^2} - \frac{f}{a}}$$

where

$$a = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad d = -\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix} \qquad d = \begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix} \qquad d = -\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix}$$

Distance of center of circumscribed circle to center of inscribed circle $= \sqrt{R^2 - Rr}$

The lines from the angles' vertex to the midpoints of the sides intersect at the centroid. The distance of the centroid from one side is equal to one-third of the side's associated altitude. The length of the line from A to the midpoint of a is $\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$ with analogous formulas for the other two sides/angles.

The length of the angle bisector for angle A is $\frac{1}{b+c}\sqrt{bc[(b+c)^2-a^2]}$ with analogous formulas for the other two sides.

The altitudes are in the ratios $h_a : h_b : h_c = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

For the angles of a triangle A, B, and C (i.e., A, B, and C sum to π) [*huntington:132:139*]

$$\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

$$\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A\cos B\cos C)$$

$$\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$$

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

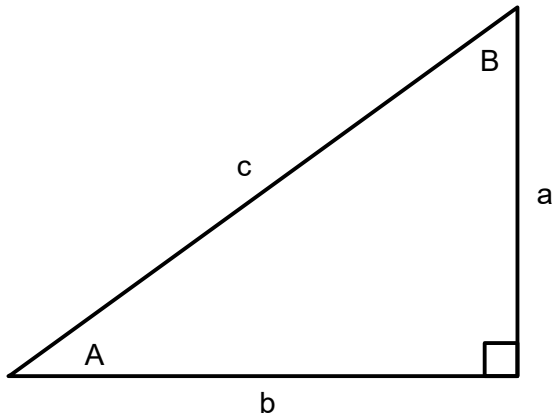
$$\cot A\cot B + \cot A\cot C + \cot B\cot C = 1$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$$

$$\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$$

Solution of right triangles

Given any two sides or one side and an acute angle A, find the remaining parts of the triangle.



$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b}, \qquad B = \frac{\pi}{2} - A$$

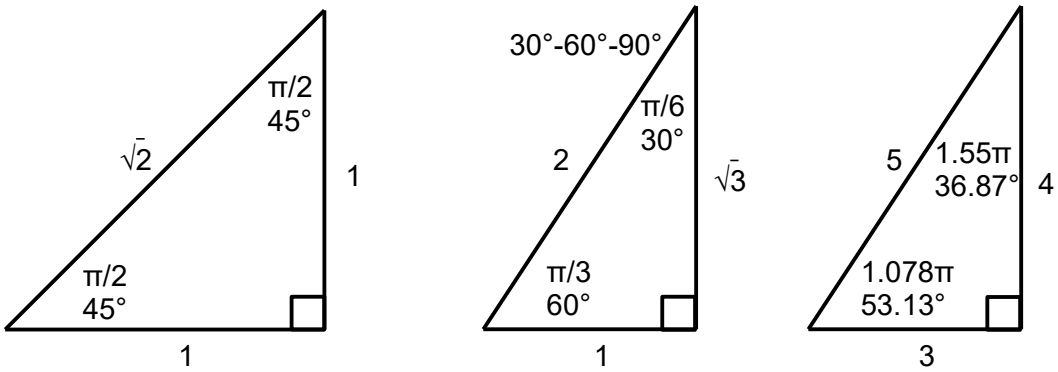
$$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)} = c \sin A = b \tan A$$

$$b = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)} = c \cos A = \frac{a}{\tan A}$$

$$c = \frac{a}{\sin A} = \frac{b}{\cos A} = \sqrt{a^2 + b^2}$$

$$\text{Area} = \frac{1}{2} ab = \frac{a^2}{2 \tan A} = \frac{b^2 \tan A}{2} = \frac{c^2 \sin 2A}{4}$$

Some commonly-used right triangles are:

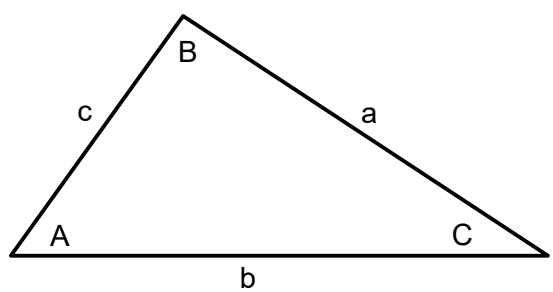


The Pythagorean triples that are less than 300 are:

a	b	c	a	b	c
3	4	5	48	55	73
5	12	13	51	140	149
7	24	25	52	165	173
8	15	17	60	91	109
9	40	41	60	221	229
11	60	61	65	72	97
12	35	37	68	285	293
13	84	85	69	260	269
15	112	113	84	187	205
16	63	65	85	132	157
17	144	145	88	105	137
19	180	181	95	168	193
20	21	29	96	247	265
20	99	101	104	153	185
21	220	221	105	208	233
23	264	265	115	252	277
24	143	145	119	120	169
28	45	53	120	209	241
28	195	197	133	156	205
32	255	257	140	171	221
36	77	85	160	231	281
39	80	89	161	240	289
44	117	125			

A Pythagorean triple of numbers (a, b, c) is such that $a^2+b^2=c^2$. This list only contains triples that are not integer multiples of other triples (i.e., a and b are relatively prime).

Solution of oblique triangles



ASA

Given angles A and B and side c:

$$C=\pi-(A+B))$$

$$a = \frac{c \sin A}{\sin C}$$

$$b = \frac{c \sin B}{\sin C}$$

SSA

Given sides a and c and angle A:

$$\sin C = \frac{c \sin A}{a}$$

$$B = \pi - (A + C)$$

$$b = \frac{a \sin B}{\sin A}$$

C may have two values: $C_1 < \frac{\pi}{2}$ and $C_2 = \pi - C_1 > \frac{\pi}{2}$. If $A + C_2 > \pi$, use only C_1.

SAS

Given any two sides b and c and their included angle A. Use any one of the following sets of formulas:

(1)

$$\frac{1}{2}(B + C) = \pi - \frac{A}{2}$$

$$\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \tan \frac{1}{2}(B + C)$$

$$B = \frac{1}{2}(B + C) + \frac{1}{2}(B - C)$$

$$C = \frac{1}{2}(B + C) - \frac{1}{2}(B - C)$$

$$a = \frac{b \sin A}{\sin B}$$

(2)

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

$$\sin B = \frac{b \sin A}{a}$$

$$C = \pi - (A + B)$$

(3)

$$\tan C = \frac{c \sin A}{b - c \cos A}$$

$$B = \pi - (A + C)$$

$$a = \frac{c \sin A}{\sin C}$$

SSS

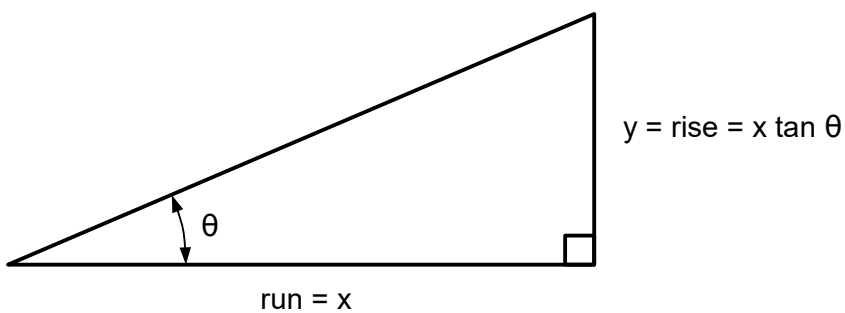
Given the three sides a, b, and c. Use either of the following sets of formulas.

(1)
$$s = \frac{1}{2}(a+b+c)$$
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$
$$\tan\frac{A}{2} = \frac{r}{s-a}$$
$$\tan\frac{B}{2} = \frac{r}{s-b}$$
$$\tan\frac{C}{2} = \frac{r}{s-c}$$

(2)
$$\cos A = \frac{b^2+c^2-a^2}{2\text{bc}}$$
$$\cos B = \frac{c^2+a^2-b^2}{2\text{ca}}$$
$$C = \pi-(A+B)$$

Slope as angle measure

Let m = tan θ where θ is the polar angle of a line. Slopes are useful for measuring angles for practical work because you can lay out angles on work directly by using the rise and run:



This method can give better resolution than protractors. Suppose dx represents our ability to discriminate a linear measurement; since $\theta = \tan^{-1} m$

$$d\theta = \frac{dm}{1+m^2} \quad \text{and} \quad m = \frac{y}{x} \Rightarrow dm = -\frac{y}{x^2}dx$$
$$d\theta = -\frac{y\,dx}{x^2(1+m^2)}$$

For example, in the shop, if you use a run of around x = 250 mm and can measure to dx = 0.5 mm, for a 45° angle, we have

$$d\theta = -\frac{250(0.5)}{250^2(2)} = -\frac{1}{250}\frac{1}{4} = 1 \text{ milliradian} = 0.057°$$

which is better than you can do with a machinist's vernier protractor.

Given the slope m, you can calculate the sine and cosine from

$$\cos\theta = A \quad \text{and} \quad \sin\theta = mA \quad \text{where } A = \frac{1}{\sqrt{1+m^2}}$$

Derivatives and integrals

```
Let D = d/dx, u = u(x), v = v(x).
Note 1/a*b = (1/a)*b

Function            Derivative
--------            ----------

log_a u             log_a(e)*u*Du
ln u                (1/u)*Du
a^u                 a^u*(ln a)*Du
u^v                 v*u^(v-1)*Du + ln(u)*u^v*Dv

sin(u)              cos(u)*Du
cos(u)              -sin(u)*Du
tan(u)              sec^2(u)*Du
cot(u)              -csc^2(u)*Du
sec(u)              sec(u)*tan(u)*Du
csc(u)              -csc(u)*cot(u)*Du
vers(u)             sin(u)*Du

asin(u)             Du/sqrt(1 - u^2), abs(asin(u)) <= pi/2
acos(u)             -Du/sqrt(1 - u^2), 0 <= acos(u) <= pi
atan(u)             Du/(1 + u^2), abs(atan(u)) <= pi/2
acot(u)             -Du/(1 + u^2), 0 <= acot(u) <= pi

sinh(u)             cosh(u)*Du
cosh(u)             sinh(u)*Du
tanh(u)             sech^2(u)*Du
coth(u)             csch^2(u)*Du
sech(u)             sech(u)*tanh(u)*Du
csch(u)             csch(u)*coth(u)*Du

asinh(u)            Du/sqrt(u^2 + 1)
acosh(u)            Du/sqrt(u^2 - 1), u > 1, acosh(u) > 0
atanh(u)            Du/(1 - u^2), u^2 < 1
acoth(u)            Du/(1 - u^2), u^2 > 1
asech(u)            -Dx/(u*sqrt(1 - u*u)), 0 < u < 1, asech(u) > 0
acsch(u)            -Dx/(abs(u)*sqrt(1 + u*u))

Integrals
b^(a*x)             b^(a*x)/(a*ln(b)), b > 0
ln(x)               x*ln(x) - x
a^x*log(a)          a^x, a > 0
1/(a^2 + x^2)       1/a*atan(x/a)
1/(a^2 - x^2)       1/a*atanh(x/a) = 1/(2*a)*ln((a + x)/(a - x)), a^2 > x^2
1/(x^2 - a^2)       -1/a*acoth(x/a) = 1/(2*a)*ln((x - a)/(x + a)), x^2 > a^2
1/sqrt(a^2 - x^2)   asin(x/abs(a)) = -acos(x/abs(a)), a^2 > x^2
1/sqrt(x^2 + a^2)   ln(x + sqrt(x^2 + a^2))
1/sqrt(x^2 - a^2)   ln(x + sqrt(x^2 - a^2))
sqrt(a^2 - x^2)     1/2*(x*sqrt(a^2 - x^2) + a^2*asin(x/abs(a)))
sqrt(a^2 + x^2)     1/2*(x*A + a^2*ln(x + A)), A = sqrt(a^2 + x^2)

sin(a*x)            -1/a*cos(a*x)
cos(a*x)            1/a*sin(a*x)
tan(a*x)            -1/a*ln(cos(a*x)) = 1/a*ln(sec(a*x))
cot(a*x)            1/a*ln(sin(a*x)) = -1/a*ln(csc(a*x))
sec(a*x)            1/a*ln(sec(a*x) + tan(a*x)) = 1/a*ln(tan(pi/4 + a*x/2))
csc(a*x)            1/a*ln(csc(a*x) - cot(a*x)) = 1/a*ln(tan(a*x/2))

sin^2(a*x)          -1/(2*a)*sin(a*x)*cos(a*x) + x/2
cos^2(a*x)          1/(2*a)*sin(a*x)*cos(a*x) + x/2
1/sin^2(a*x)        -1/a*cot(a*x)
1/cos^2(a*x)        1/a*tan(a*x)

sin(m*x)*sin(n*x)   sin((m-n)*x)/(2*(m-n)) - sin((m+n)*x)/(2*(m+n)), m^2 != n^2
cos(m*x)*cos(n*x)   sin((m-n)*x)/(2*(m-n)) + sin((m+n)*x)/(2*(m+n)), m^2 != n^2
```

```
sin(m*x)*cos(n*x)   cos((m-n)*x)/(2*(m-n)) - cos((m+n)*x)/(2*(m+n)), m^2 != n^2

sin(a*x)*cos(a*x)   1/(2*a)*sin^2(a*x)
1/sin(a*x)*cos(a*x) 1/a*ln(tan(a*x))

sin(a + b*x)        -1/b*cos(a + b*x)
cos(a + b*x)        1/b*sin(a + b*x)
```

Hyperbolic functions

[*huntington:135:142*]

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x \qquad 1 - \coth^2 x = -\operatorname{csch}^2 x$$

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x \qquad \tanh(-x) = -\tanh x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh 2x = 2 \sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x \qquad \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\sinh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x - 1)} \qquad \cosh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)} \qquad \tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh^{-1} \frac{x}{a} = \ln\left(x + \sqrt{x^2 + a^2}\right) - \ln a$$

$$\cosh^{-1} \frac{x}{a} = \ln\left(x + \sqrt{x^2 - a^2}\right) - \ln a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln \frac{a + x}{a - x}$$

The Gudermannian function gd(x) can relate the circular and hyperbolic functions without using complex arguments:

$$\operatorname{gd}(x) = 2 \tan^{-1} e^x - \frac{\pi}{2}$$

$$\operatorname{gd}^{-1}(x) = \ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$$

$$\sinh x = \tan(\operatorname{gd}(x))$$

$$\cosh x = \sec(\operatorname{gd}(x))$$

$$\tanh x = \sin(\operatorname{gd}(x))$$

$$\tanh \frac{x}{2} = \tan\left(\frac{\operatorname{gd}(x)}{2}\right)$$

Spherical trigonometry

[*casey:19:39*]

Let the triangle have sides a, b, c and corresponding angles A, B, C; the sphere's radius is *r*. Unless otherwise specified, two additional formulas can be gotten by cyclic permutations of the sides and angles.

The fundamental formula for spherical triangles is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

If $a+b+c = 2s$, then [*casey:21:41*]

$$\sin\frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \qquad \cos\frac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$$

$$\tan\frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}} \qquad \tan\frac{1}{2}A \tan\frac{1}{2}B = \frac{\sin(s-c)}{\sin s}$$

$$n = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)} \text{ where } 2s = a+b+c$$

$$\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c} = \frac{4n^2}{\sin^2 b \sin^2 c}$$

Let m_a be the median, the arc which joins the vertex at angle A to the opposite side. Let h_a be the altitude of the triangle from the vertex at angle A to side a, intersecting the arc a perpendicularly.

$$\cos b + \cos c = 2\cos\frac{a}{2}\cos m_a$$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad \text{Sine law}$$

$$\sin a \sin h_a = \sin b \sin h_b = \sin c \sin h_c = 2n$$

$$\frac{\sin b}{\tan a} = \frac{\sin C}{\tan A} + \cos C \cos b$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$2E = A+B+C-\pi = \text{spherical excess}$$

$$\text{Area of spherical triangle} = E r^2$$

$$\sin\frac{1}{2}a = \sqrt{\frac{\sin E \sin(A-E)}{\sin B \sin C}} \qquad \cos\frac{1}{2}a = \sqrt{\frac{\sin(B-E)\sin(C-E)}{\sin B \sin C}}$$

$$\tan\frac{1}{2}a = \sqrt{\frac{\sin E \sin(A-E)}{\sin(B-E)\sin(C-E)}}$$

Right spherical triangles

Let the sides be a, b, c with corresponding angles A, B, C and where angle C is the right angle.

From the fundamental formula above, we have

$$\cos A = \cos a \sin B$$

and, thus, the three sides are all acute (less than $\pi/2$) or one is acute and the other two obtuse (greater than $\pi/2$).

$$\sin a = \frac{\tan b}{\tan B} = \sin A \sin c \qquad \sin b = \frac{\tan a}{\tan A} = \sin B \sin c$$

$$\cos A = \frac{\tan b}{\tan c} = \cos a \sin B \qquad \cos B = \frac{\tan a}{\tan c} = \cos b \sin A$$

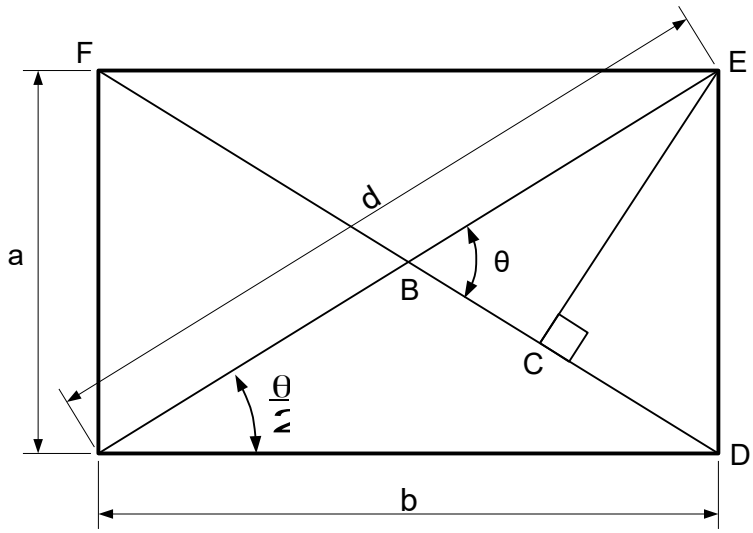
Mensuration

In the following

n = number of sides
r = radius of inscribed circle
R = radius of circumscribed circle
A = area
K = area (used on e.g. triangles where A is one of the angles)
p = perimeter
V = volume
h = height

Two-dimensional figures

Rectangle



$$A = a\,b = a\sqrt{d^2 - a^2} = b\sqrt{d^2 - b^2} = \frac{1}{2} d^2 \sin\theta$$

$$d = \text{length of diagonal} = \sqrt{a^2 + b^2}$$

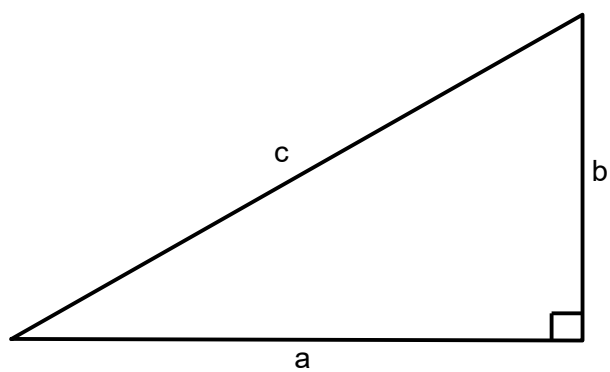
$$\theta = 2\,\tan^{-1}\left(\frac{a}{b}\right)$$

$$p = 2(a + b)$$

$$a = \sqrt{d^2 - b^2}$$

$$b = \sqrt{d^2 - a^2}$$

Right triangle

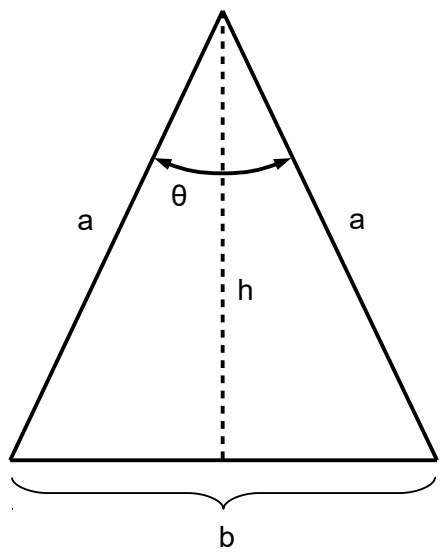


$$A = \frac{1}{2} a\,b$$

$$p = a + b + c$$

$$c^2 = a^2 + b^2$$

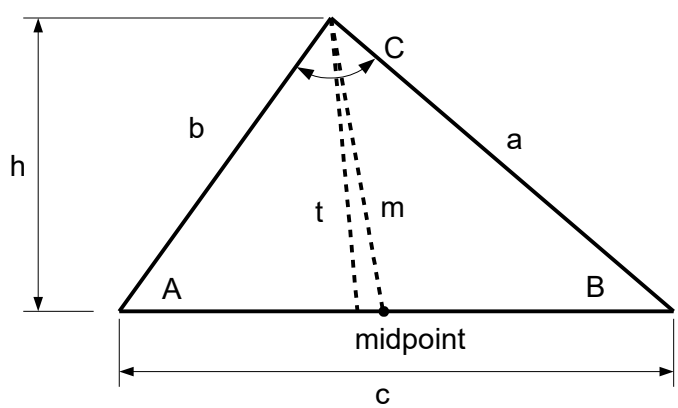
Isosceles triangle



$$A = \frac{1}{2} b h$$
$$p = 2a + b$$
$$b = a\sqrt{2(1-\cos\theta)} = a\sin\frac{\theta}{2}$$
$$h = a\cos\frac{\theta}{2}$$

General triangle

We use K for the area because A is used for one of the angles. Note that a number of the formulas can have their arguments permuted cyclically to produce more formulas.



$$K = \text{area} = \frac{1}{2}c\,h = \frac{1}{2}ab\sin C = \frac{c^2\sin A\sin B}{2\sin C} = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = 2R^2\sin A\sin B\sin C$$

$$= \frac{1}{2}\left[(x_1y_2-x_2y_2)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\right]$$

$$= \text{where } (x_1,y_1),(x_2,y_2), \text{ and } (x_3,y_3) \text{ are coordinates of the vertices}$$

$$= \frac{1}{2}\sqrt{a^2c^2-\left(\frac{a^2+c^2-b^2}{2}\right)^2} \text{ where } a \geq b \geq c$$

$$s = \frac{1}{2}(a+b+c) = \frac{p}{2}$$

$$r = c\sin\frac{A}{2}\sin\frac{B}{2}\sec\frac{C}{2} = \frac{ab\sin C}{2s} = (s-c)\tan\frac{C}{2} = \frac{abc}{4Rs}$$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{K}{s} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4K}$$

$$h = a\sin B = b\sin A = \frac{2K}{c} = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$

$$t = \text{length of bisector of angle C} = \frac{2ab}{a+b}\cos\frac{C}{2} = \sqrt{ab\left[1-\frac{c^2}{(a+b)^2}\right]}$$

$$m = \text{length of line connecting to c's midpoint} = \sqrt{\frac{1}{2}(a^2+b^2)-\frac{1}{4}c^2}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{(Sine law)}$$

$$c^2 = a^2+b^2-2ab\cos C \quad \text{(Cosine law)}$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \quad \text{(Tangent law)}$$

$$\tan\frac{1}{2}A = \frac{r}{s-a}$$

$$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\frac{b-c}{a} = \frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A} \quad \text{(Mollweide)}$$

$$\frac{b+c}{a} = \frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A} \quad \text{(Newton)}$$

The median lines *m* that join each vertex with the midpoint of the side opposite meet in the centroid of the triangle; the centroid trisects each median.

The altitudes meet at a point called the **orthocenter**. It lies within the triangle **iff** the triangle is acute (i.e., doesn't have an angle larger than a right angle). If the triangle is a right triangle, the orthocenter is the vertex of the right angle. If the vertexes of the triangle are the points A, B, C and the orthocenter is point H, then we have the relation

$$a^2+b^2+c^2+AH^2+BH^2+CH^2=12R^2$$

The perpendicular bisectors of the sides meet at the center of the **circumscribed circle** with radius R.

The angle bisectors meet at the center of the **inscribed circle** with radius r. If the three vertices are located at (x_a, y_a), (x_b, y_b), and (x_c, y_c) and the sides opposite these vertices have corresponding lengths *a*, *b*, and *c*, then the center of the inscribed circle is at

$$\left(\frac{a\,x_a+b\,x_b+c\,x_c}{a+b+c},\frac{a\,y_a+b\,y_b+c\,y_c}{a+b+c}\right)$$

If *c* is the largest side of the triangle, then C is the largest angle and $|a-b|<c<a+b$.

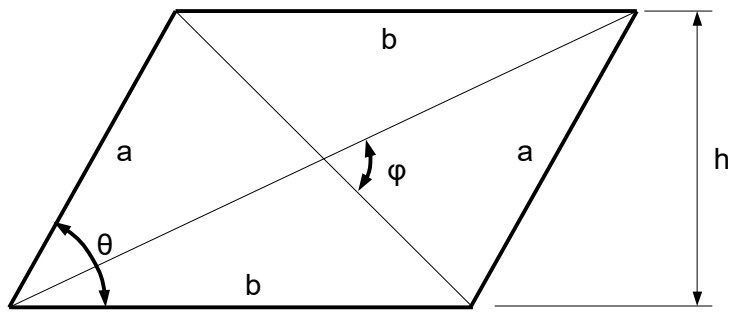
The formula for the area $K=\sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula) is numerically unstable for e.g. skinny triangles. A numerically more stable formula is [*corral:57:65*]

$$K=\frac{1}{4}\sqrt{[a+(b+c)][c-(a-b)][c+(a-b)][a+(b-c)]}$$

where the order of operations should be done as indicated by the parentheses and braces. Note this is Heron's formula rewritten (replace *s* by its definition).

For a right triangle, the hypotenuse is a diameter of the circumscribed circle and the midpoint of the hypotenuse is the center of the circle.

Parallelogram

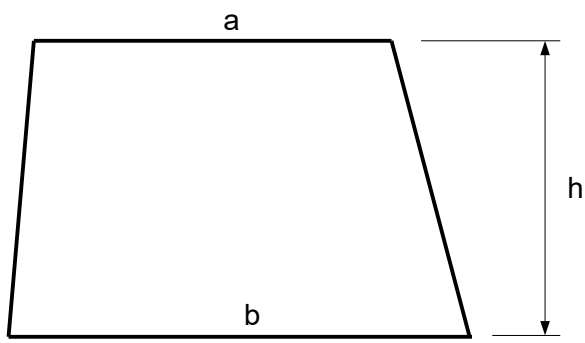


$$A = bh = ab\sin\theta = \frac{1}{2}d_1 d_2 \sin\varphi$$

where d_1 and d_2 are the diagonals and φ is the angle between them

$$p = 2(a+b)$$

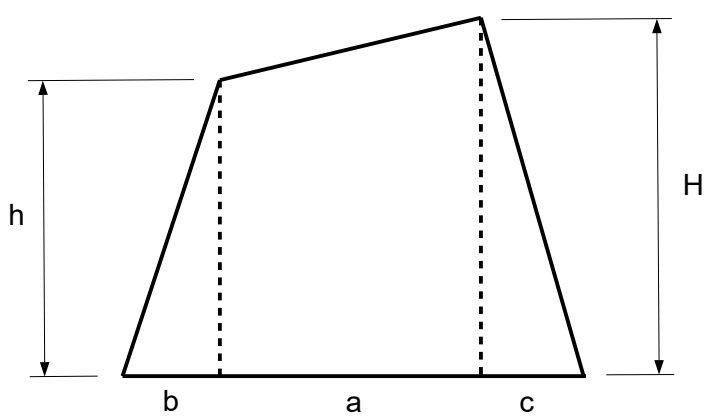
Trapezoid



$$A = \frac{1}{2}h(a+b)$$

Note the two vertical sides do not have to intersect the base at the same angle.

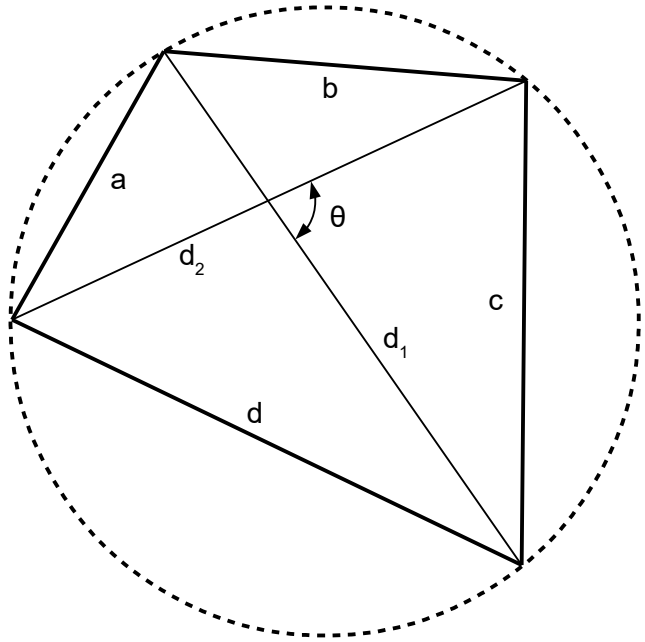
Trapezium



$$A = \frac{(H+h)a + bh + cH}{2}$$

Quadrilateral

For a quadrilateral inscribed in a circle



$$A = \frac{1}{2} d_1 d_2 \sin\theta = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \frac{1}{2}(ac+bd)\sin\theta$$

where d_1 and d_2 are the diagonals and θ is the angle between them

$$p = a+b+c+d$$

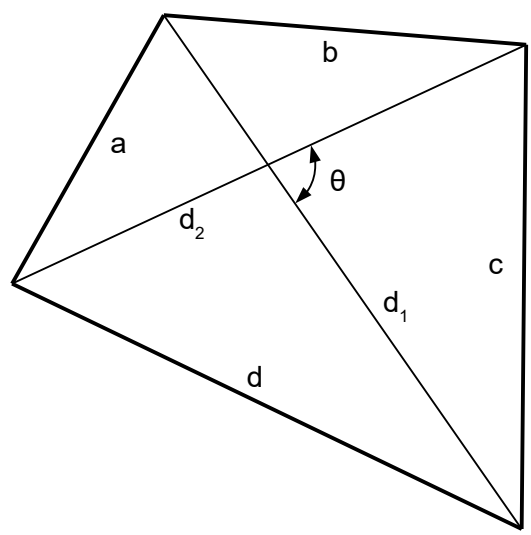
$$s = \frac{p}{2}$$

If the quadrilateral is convex (a line between all pairs of points in the quadrilateral lies within or on the quadrilateral), then Ptolemy's theorem is

$$ac+bd = d_1 d_2$$

(This is implied in the formulas for the area.)

For a general quadrilateral



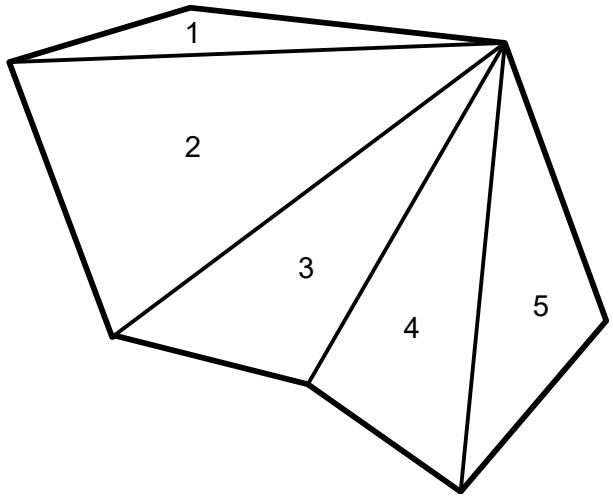
$$A = \frac{1}{2} d_1 d_2 \sin\theta$$

where d_1 and d_2 are the diagonals and θ is the angle between them. If m is the distance between

the midpoints of d_1 and d_2, then

$$a^2+b^2+c^2+d^2 = d_1^2+d_2^2+4\,m^2$$

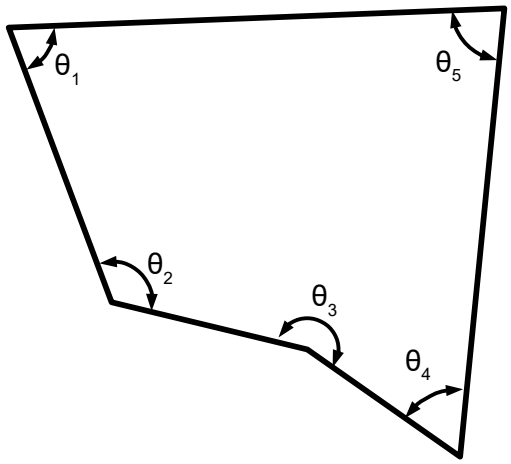
General polygon



Divide the polygon into triangles and sum their areas; for example:

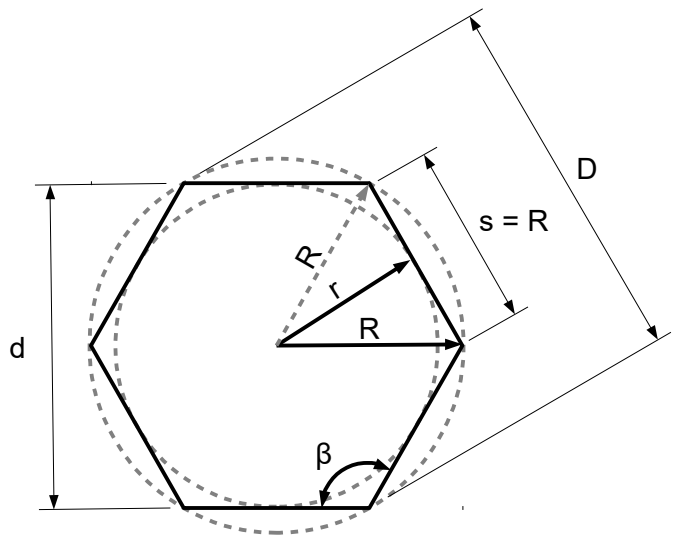
$$A = A_1+A_2+A_3+A_4+A_5$$

If the interior angles of a planar polygon of n sides are $\theta_1, \theta_2, \ldots, \theta_n$, as in the following picture (here, n is 5)



then the sum of these interior angles is $\pi(n-2)$.

Hexagon (regular)



$$r = \text{radius of inscribed circle} = \frac{\sqrt{3}}{2}R = 0.8660R$$

$$R = \text{radius of circumscribed circle} = \frac{2}{\sqrt{3}}r = 1.155r$$

$$d = \text{distance across flats} = 2r = \frac{\sqrt{3}}{2}D = 0.8660D$$

$$D = \text{distance across points} = 2R = \frac{2}{\sqrt{3}}d = 1.155d$$

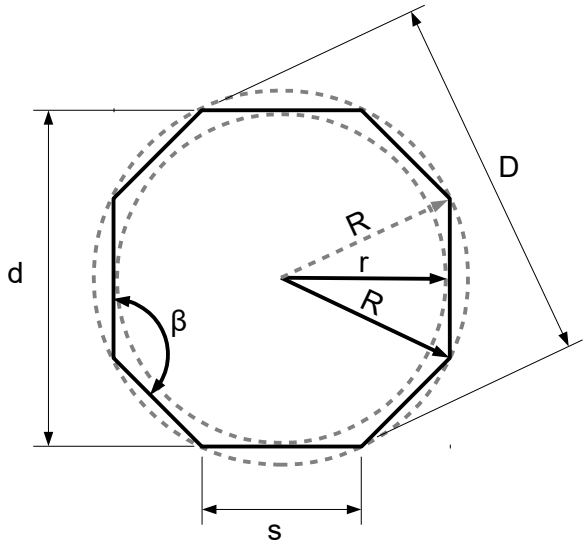
$$\beta = \frac{2}{3}\pi$$

$$A = 6\left(\frac{1}{2}Rr\right) = \frac{\sqrt{3}}{2}d^2 = 0.8660d^2 = \frac{3\sqrt{3}}{8}D^2 = 0.6459D^2$$

$$= 2\sqrt{3}r^2 = 3.454r^2 = \frac{3\sqrt{3}}{2}R^2 = 2.598R^2$$

$$p = 6s = 6R = 3D = \frac{6}{\sqrt{3}}d = 3.464d$$

Octagon (regular)



r = radius of inscribed circle $= R\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}R = 0.9239R$

R = radius of circumscribed circle $= \frac{2r}{\sqrt{2+\sqrt{2}}} = 1.0324r$

d = distance across flats $= 2r = D\sqrt{2+\sqrt{2}} = 1.8478D$

D = distance across points $= 2R = \frac{4d}{\sqrt{2+\sqrt{2}}} = 2.1648d$ ¿

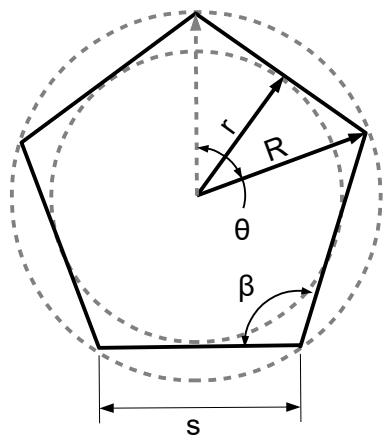
$\beta = \frac{3}{4}\pi$

$A = 8\left(\frac{1}{2}Rr\right) = 2d^2(\sqrt{2}-1) = 0.8284d^2 = \frac{D^2}{\sqrt{2}} = 0.7071D^2 = 2s^2(\sqrt{2}+1)$

$s = d(\sqrt{2}-1) = 0.4142d$

$p = 8s$

Polygon (regular)



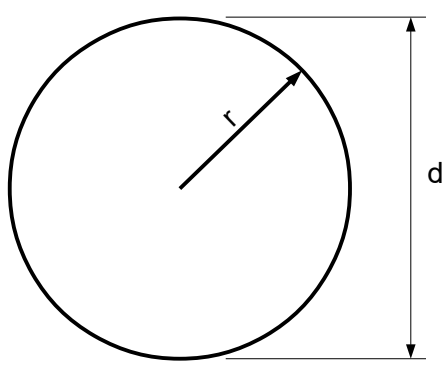
$$n = \text{number of sides}$$
$$\theta = \frac{2\pi}{n}$$
$$\beta = \frac{n-2}{n}\pi$$
$$s = 2r\tan\frac{\pi}{n} = 2R\sin\frac{\pi}{n} = d\tan\frac{\pi}{n} = D\sin\frac{\pi}{n}$$
$$p = ns$$
$$A = \frac{1}{4}ns^2\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{2}nR^2\sin\frac{2\pi}{n} = \frac{nd^2}{4}\tan\frac{\pi}{n} = \frac{1}{8}nD^2\sin\frac{2\pi}{n}$$
$$r = \frac{s}{2}\cot\frac{\pi}{n} = R\cos\frac{\pi}{n}$$
$$d = s\cot\frac{\pi}{n} = D\cos\frac{\pi}{n}$$
$$R = \frac{s}{2}\csc\frac{\pi}{n} = r\sec\frac{\pi}{n}$$
$$D = s\csc\frac{\pi}{n} = d\sec\frac{\pi}{n}$$

The following table gives various polygon characteristics; multiply by the column's denominator to get the value for a polygon of arbitrary size. The symbols are defined above. [*marks:1-39*]

n	θ	θ, °	A/s²	A/D²	A/d²	D/s	D/d	s/D	s/d	d/D	d/s
3	2.094	120	.4330	.3248	1.299	.8660	2.000	.8660	1.732	.5000	.5774
4	1.571	90	1.000	.5000	1.000	.7071	1.414	.7071	1.000	.7071	1.000
5	1.257	72	1.720	.5944	.9082	.5878	1.236	.5878	.7265	.8090	1.376
6	1.047	60	2.598	.6495	.8660	.5000	1.155	.5000	.5774	.8660	1.732
7	.8976	51.43	3.634	.6841	.8428	.4339	1.110	.4339	.4816	.9010	2.077
8	.7854	45	4.828	.7071	.8284	.3827	1.082	.3827	.4142	.9239	2.414
9	.6981	40	6.182	.7231	.8189	.3420	1.064	.3420	.3640	.9397	2.747
10	.6283	36	7.694	.7347	.8123	.3090	1.051	.3090	.3249	.9511	3.078
12	.5236	30	11.20	.7500	.8038	.2588	1.035	.2588	.2679	.9659	3.732
15	.4189	24	17.64	.7626	.7971	.2079	1.022	.2079	.2126	.9781	4.705
16	.3927	22.5	20.11	.7654	.7956	.1951	1.020	.1951	.1989	.9808	5.027
20	.3142	18	31.57	.7725	.7919	.1564	1.012	.1564	.1584	.9877	6.314
24	.2618	15	45.57	.7765	.7899	.1305	1.009	.1305	.1317	.9914	7.596
32	.1963	11.25	81.23	.7804	.7879	.09802	1.005	.09802	.09849	.9952	10.15
48	.1309	7.50	183.1	.7832	.7865	.06540	1.002	.06540	.06554	.9979	15.26
64	.09817	5.625	325.7	.7841	.7860	.04907	1.001	.04907	.04913	.9988	20.36

Example: I wish to make a concrete stepping stone 3 inches thick that is a hexagon with an inscribed circle diameter of 16 inches. What will this stepping stone weigh? The density of concrete is about 0.075 lb/in^3. The inscribed radius r is 8 inches. Under the column A/r^2, we get the number 3.464. Calculate (3.464)8^2 to get 221.696 in^2 for the area. Thus the volume is 3 times this or 665.088 in^3 and the weight is 0.075 times this or 49.88 lb.

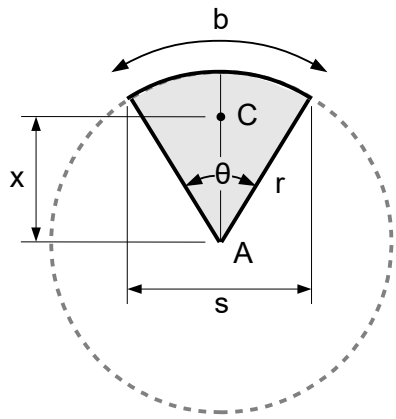
Circle



$$C = \text{circumference} = 2\pi r = \pi d = 2\sqrt{\pi A} = \frac{2A}{r}$$

$$A = \pi r^2 = \frac{1}{4}\pi d^2 = \frac{C^2}{4\pi} = \frac{1}{2} C r$$

Circular sector



$b = r\,\theta$

$d = 2\mathrm{r}$

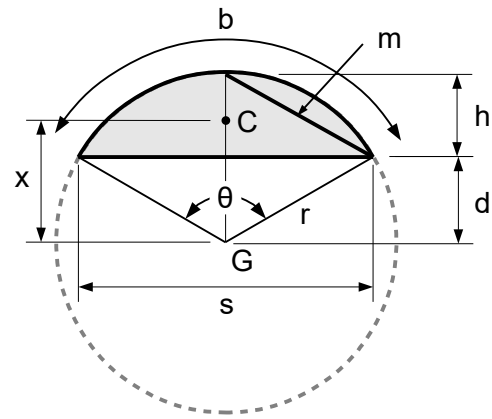
$$A = \pi r^2 \left(\frac{\theta}{2\pi}\right) = \frac{r^2\theta}{2} = \frac{br}{2} = \frac{1}{8} d^2 \theta$$

$$s = 2r \sin\frac{\theta}{2}$$

Centroid is distance x up from center A on symmetry axis

$$x = \frac{2}{3}\frac{rs}{b} = \frac{4r}{3\theta}\sin\frac{\theta}{2}$$

Circular segment



$$b = r\theta$$

$$r = \frac{s^2 + 4h^2}{8h}$$

$$d = r - h = r\cos\frac{\theta}{2} = \frac{1}{2}s\cot\frac{\theta}{2} = \frac{1}{2}\sqrt{4r^2 - s^2}$$

$$h = r - d = r - \frac{1}{2}\sqrt{4r^2 - s^2} \quad \text{for } h < r$$

$$= r\left(1 - \cos\frac{\theta}{2}\right)$$

$$s = 2r\sin\frac{\theta}{2} = 2(r - h)\tan\frac{\theta}{2} = 2\sqrt{2hr - h^2} = r\sqrt{2(1 - \cos\theta)}$$

$$\theta = 2\sin^{-1}\frac{s}{2r}$$

$$m = \sqrt{2rh}$$

$$A = \frac{1}{2}r\left(b - r\sin\frac{b}{r}\right) = r^2\sin^{-1}\frac{s}{2r} - \frac{s}{4}\sqrt{4r^2 - s^2}$$

$$A = \frac{1}{2}[br - s(r - h)] = \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= r^2\cos^{-1}\frac{d}{r} - d\sqrt{r^2 - d^2} = r^2\cos^{-1}\frac{r - h}{r} - (r - h)\sqrt{2rh - h^2}$$

The following approximation is accurate within 1 part in 1000 for angles $< \frac{\pi}{11} = 16°$

$A \approx \frac{2}{3}hs$ (i.e., the area of the segment is about 2/3 of the enclosing rectangle's area)

The centroid of the segment is a distance x up from center G on symmetry axis:

$$x = \frac{s^3}{12A}$$

For measurements on a segment where the center of the arc may not be accessible, see *Arcs*.

For arcs where *s* is small, Huygen's approximation is [*marks:2-17*]

$$b \approx \frac{1}{3}(8m - s)$$

Given two of the variables r, s, b, θ, and h, the following table [*mh04:70*] gives formulas for the other variables[1]:

1 There were a number of errors in the Machinery's Handbook table; hopefully, I've fixed them.

Given	Formulas		
θ, r	$s = 2r\sin\frac{\theta}{2}$	$h = r\left(1-\cos\frac{\theta}{2}\right)$	$b = r\theta$
θ, s	$r = \frac{s}{2\sin\frac{\theta}{2}}$	$h = \frac{s}{2}\tan\frac{\theta}{4}$	$b = \frac{s\theta}{2\sin\frac{\theta}{2}}$
θ, h	$r = \frac{h}{1-\cos\frac{\theta}{2}}$	$s = \frac{2h}{\tan\frac{\theta}{4}}$	$b = \frac{h\theta}{1-\cos\frac{\theta}{2}}$
θ, b	$r = \frac{b}{\theta}$	$s = \frac{2b\sin\frac{\theta}{2}}{\theta}$	$h = \frac{b\left(1-\cos\frac{\theta}{2}\right)}{\theta}$
r, s	$\theta = 2\sin^{-1}\frac{s}{2r}$	$h = r - \frac{\sqrt{4r^2-s^2}}{2}$	$b = 2r\sin^{-1}\left(\frac{s}{2r}\right)$
r, h	$\theta = 2\cos^{-1}\left(1-\frac{h}{r}\right)$	$s = 2\sqrt{h(2r-h)}$	$b = 2r\cos^{-1}\left(1-\frac{h}{r}\right)$
r, b	$\theta = \frac{b}{r}$	$s = 2r\sin\frac{b}{2r}$	$h = r\left(1-\cos\frac{b}{2r}\right)$
s, h	$\theta = 4\tan^{-1}\frac{2h}{s}$	$r = \frac{s^2+4h^2}{8h}$	$b = \frac{s^2+4h^2}{2h}\tan^{-1}\frac{2h}{s}$
s, b	Iterate the following equation for θ: $\frac{2b}{s} = \frac{\theta}{\sin\frac{\theta}{2}}$ Then use the other formulas for the remaining terms.		
h, b	Iterate the following equation for θ: $\frac{b}{h} = \frac{\theta}{1-\cos\frac{\theta}{2}}$ Then use the other formulas for the remaining terms.		

The maximum % deviation from the correct value for the segment's area depends on the maximum allowed θ:

a	Maximum % deviation for $0 < \theta < \pi/a$	π/a in °
1	15	180
2	3.2	90
4	0.76	45
8	0.18	22
16	0.043	11
32	0.012	5.5

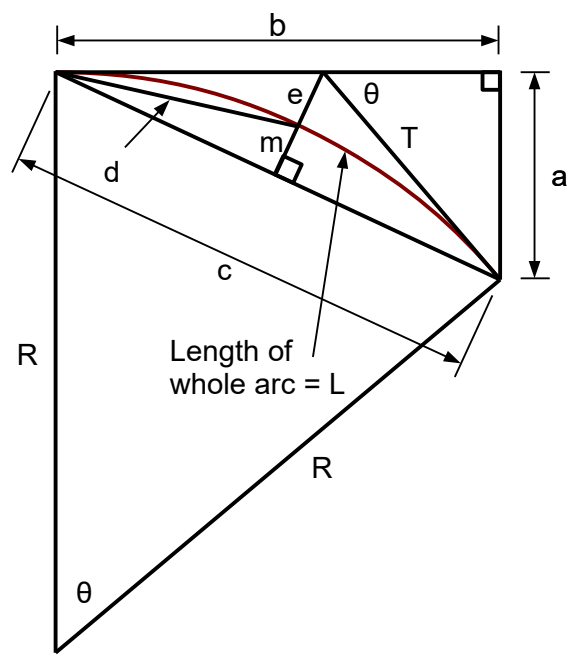
In other words, for angles less than $\pi/8$, you can expect the approximation to be within 0.18% of the true value with a good chance of it being substantially better, as the median is somewhat less than half of the given maximum value.

Thus, this is a good approximation for small angles. For hand calculations, a rule of thumb is that the approximation's error is less than 1 part in 1000 for angles less than $\pi/11 = 16\ °$.

The length of the chord *s* is often needed given the diameter of the circle and the subtended angle θ. In the following table, multiply the value under the Chord column by the diameter to get the length of the chord:

θ, °	Chord	θ, °	Chord	θ, °	Chord	θ, °	Chord	θ, °	Chord
1	.00873	21	.18224	41	.35021	61	.50754	81	.64945
2	.01745	22	.19081	42	.35837	62	.51504	82	.65606
3	.02618	23	.19937	43	.36650	63	.52250	83	.66262
4	.03490	24	.20791	44	.37461	64	.52992	84	.66913
5	.04362	25	.21644	45	.38268	65	.53730	85	.67559
6	.05234	26	.22495	46	.39073	66	.54464	86	.68200
7	.06105	27	.23345	47	.39875	67	.55194	87	.68835
8	.06976	28	.24192	48	.40674	68	.55919	88	.69466
9	.07846	29	.25038	49	.41469	69	.56641	89	.70091
10	.08716	30	.25882	50	.42262	70	.57358	90	.70711
11	.09585	31	.26724	51	.43051	71	.58070	91	.71325
12	.10453	32	.27564	52	.43837	72	.58779	92	.71934
13	.11320	33	.28402	53	.44620	73	.59482	93	.72537
14	.12187	34	.29237	54	.45399	74	.60182	94	.73135
15	.13053	35	.30071	55	.46175	75	.60876	95	.73728
16	.13917	36	.30902	56	.46947	76	.61566	96	.74314
17	.14781	37	.31730	57	.47716	77	.62251	97	.74896
18	.15643	38	.32557	58	.48481	78	.62932	98	.75471
19	.16505	39	.33381	59	.49242	79	.63608	99	.76041
20	.17365	40	.34202	60	.50000	80	.64279	100	.76604

In [*kent:59:112*], the following diagram is given, along with the following formulas (the only change I made was to use θ for I. Kent credits J. C. Locke, *Eng. News*, March 16, 1908.



$$c = \sqrt{2\mathrm{Ra}} = \sqrt{a^2+b^2} = \sqrt{2\mathrm{R}\left(R-\sqrt{R^2-b^2}\right)} = 2\sqrt{m(2\mathrm{R}-m)} = 2\mathrm{R}\sin\frac{\theta}{2} = 2\mathrm{T}\cos\frac{\theta}{2}$$

$$e = R\tan\frac{\theta}{2}\tan\frac{\theta}{4} = T\tan\frac{\theta}{4}$$

$$b = \sqrt{a(2\mathrm{R}-a)} = c\sqrt{1-\left(\frac{c}{2\mathrm{R}}\right)^2} = R\sin\theta = a\cot\frac{\theta}{2}$$

$$R = \frac{a^2+b^2}{2\mathrm{a}} = \frac{c^2}{2\mathrm{a}} = \frac{d^2}{2\mathrm{m}} = \frac{c^2+4\mathrm{m}^2}{8\mathrm{m}} = \frac{1}{2}\left(\frac{c^2}{4\mathrm{m}}+m\right)$$

$$d = \sqrt{2\mathrm{Rm}} = \sqrt{R(2\mathrm{R}-\sqrt{4\mathrm{R}^2-c^2})} = 2\mathrm{R}\sin\frac{\theta}{4}$$

785

$$m = \frac{d^2}{2\mathrm{R}} = R\mp\sqrt{R(2\mathrm{R}-\sqrt{4\mathrm{R}^2-c^2})} = R\sin\frac{\theta}{2}\tan\frac{\theta}{4} = \frac{c}{2}\tan\frac{\theta}{4}$$

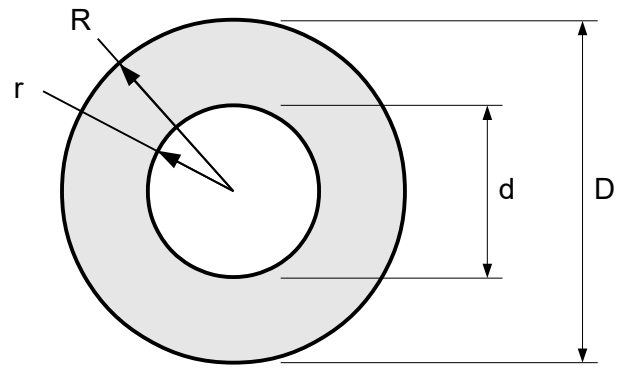
$$a = \frac{c^2}{2\mathrm{R}} = R-\sqrt{R^2-b^2} = 2\mathrm{R}\sin^2\frac{\theta}{2} = R\sin\theta\tan\frac{\theta}{2} = b\tan\frac{\theta}{2} = T\sin\theta$$

$$T = R\tan\frac{\theta}{2}$$

$$L = \theta R = \frac{8\mathrm{d}-c}{3}$$

$$\text{Area of segment} = \frac{LR}{2} - \frac{R^2\sin\theta}{2} = \frac{LR}{2} - \frac{Rb}{2}$$

Annulus



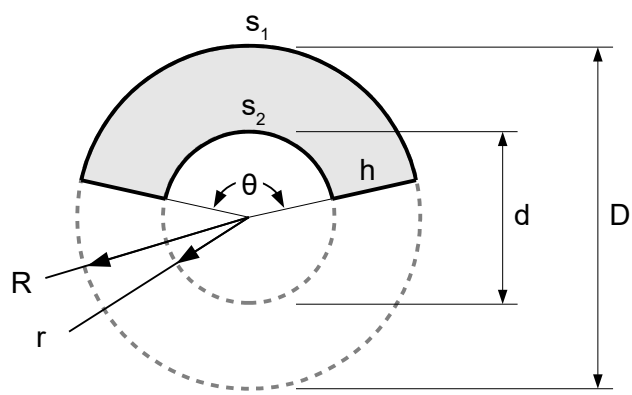
$$A = \pi(R^2 - r^2) = \pi(R+r)(R-r) = \frac{\pi}{4}(D^2 - d^2)$$

$$R = \sqrt{\frac{A}{\pi} + r^2} = \sqrt{\frac{4A}{\pi} + d^2}$$

$$r = \sqrt{R^2 - \frac{A}{\pi}} = \sqrt{D^2 - \frac{4A}{\pi}}$$

A ring is where one circle completely contains another; they don't need to be concentric. It has the same formula for its area as an annulus.

Annular sector



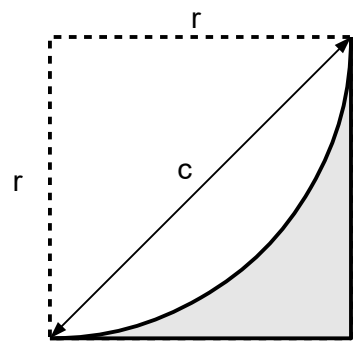
$$h = R - r$$

$$A = \frac{1}{2}\theta(R^2 - r^2) = \frac{1}{8}\theta(D^2 - d^2) = \theta h \frac{R+r}{2} = \frac{1}{2}h(s_1 + s_2)$$

$$s_1 = R\theta$$

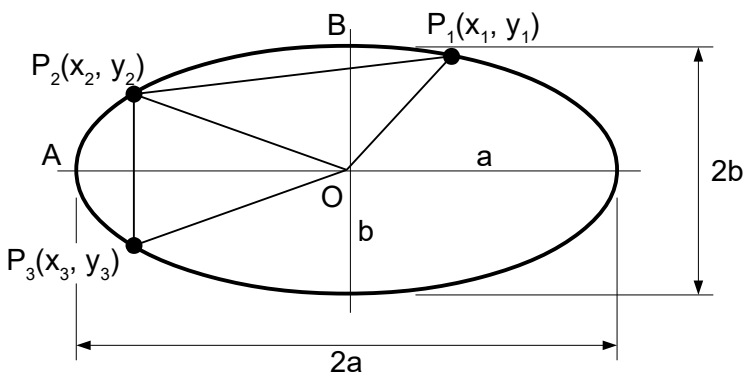
$$s_2 = r\theta$$

Spandrel



$$A = r^2\left(1-\frac{\pi}{4}\right) = 0.2146\,r^2 = \frac{1}{2}c^2\left(1-\frac{\pi}{4}\right) = 0.1073\,c^2$$
$$c = r\sqrt{2}$$

Ellipse



Note: $b \le a$

$A = \pi\, a\, b$

$e = \text{eccentricity} = \sqrt{1-\left(\frac{b}{a}\right)^2}$

Area of elliptical segment $P_1 B P_2 = \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{ab}{2}\left(\sin^{-1}\frac{x_2}{a} - \sin^{-1}\frac{x_1}{a}\right)$

Area of elliptical segment $P_2 P_3 A = ab\cos^{-1}\frac{x_2}{a} - x_2 y_2$

Area of elliptical sector $O P_2 A P_3 = ab\cos^{-1}\frac{x_2}{a}$

Area of elliptical sector $O P_2 B P_1 = \frac{ab}{2}\left(\sin\frac{x_2}{a} - \sin^{-1}\frac{x_1}{a}\right)$

The perimeter of the ellipse is expressed in terms of the complete elliptic function of the second kind; see the *Ellipse* section. Ramanujan's 1914 approximation is

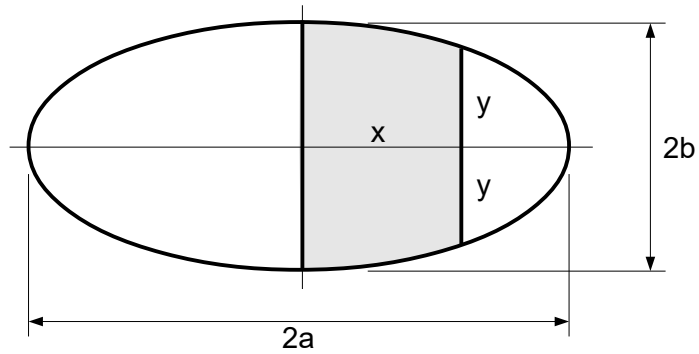
$$p \approx \pi(a+b)\left[1+\frac{\xi}{10+\sqrt{4-\xi}}\right], \text{ where } \xi=3\left(\frac{a-b}{a+b}\right)^2$$

[*hudson:17:26*] gives the Gauss-Kummer series

$$\begin{aligned}
p &= \pi(a+b)\left[1+\frac{x^2}{4}+\frac{x^4}{64}+\frac{x^6}{256}+\frac{x^8}{16384}+\frac{x^{10}}{65536}+\ldots\right] \\
&= \pi(a+b)\left[1+\frac{x^2}{2^2}+\frac{x^4}{8^2}+\frac{x^6}{16^2}+\frac{x^8}{128^2}+\frac{x^{10}}{256^2}+\ldots\right] \\
&= {}_2F_1\left(-\frac{1}{2},-\frac{1}{2};1;x^2\right) \\
&= \sum_{n=0}^{\infty}\binom{1/2}{n}^2 x^{2n} \\
&\text{where } x = \frac{a-b}{a+b}
\end{aligned}$$

${}_2F_1$ is the hypergeometric function.

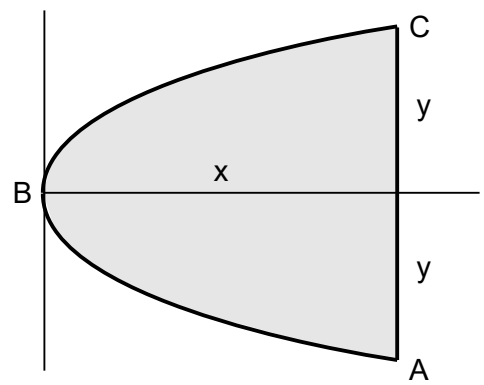
For a typical non-flat ellipse where, say, b = 0.7a, x is 0.176 and the 6th power term is about 1e-7, so the series will converge quickly for practical problems with ellipses that are only slightly flattened. For a quite flattened ellipse where b = 0.1a, the 6th power term is around 1e-3, so it will take a few more terms to converge.



Area of shaded segment is

$$A = xy + ab\sin^{-1}\frac{x}{a}$$

Parabola



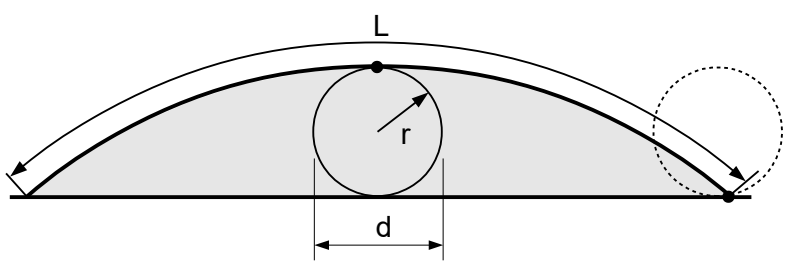
The function is $y = a\sqrt{x}$, so the area is the integral from 0 to x or

$$A = \frac{4}{3}xy$$

The length of the arc ABC is

$$\sqrt{4x^2+y^2}+\frac{y^2}{2x}\ln\left(\frac{2x+\sqrt{4x^2+y^2}}{y^2}\right)$$

Cycloid



$$A = 3\pi r^2 = \frac{3}{4}\pi d^2$$
$$L = 8r = 4d$$

Approximate area

Let $y_0, y_1, \ldots, y_n$ be equidistant ordinates separated by distance h. Then the following approximate rules for the area enclosed by the boundary defined by these points are:

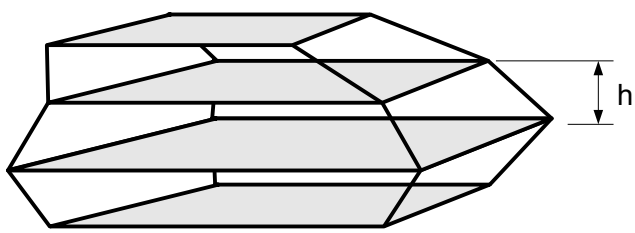
Trapezoidal rule:

$$A = h\left[\frac{1}{2}(y_0+y_n)+y_1+y_2+\ldots+y_n\right]$$

Simpson's rule (n must be even)

$$A = h\left[\frac{1}{3}(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})\right]$$

[*marks:2-23*] Simpson's rule can also be used to compute volumes if the ordinates y_i are interpreted as the areas of plane sections of the volume at a constant distance *h* apart.



[*huntington:106:113*] The error of Simpson's rule is

$$-\frac{1}{180}\frac{(b-a)^5}{n^4}f^{(4)}(\xi)$$

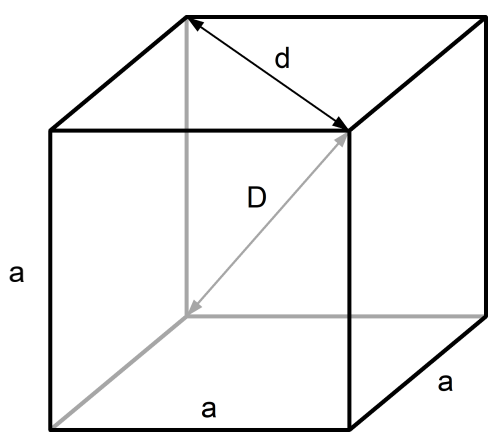
where a is the abscissa of y_0, b is the abscissa of y_{n-1}, and $y = f(x)$.

Three-dimensional figures

Symbols used are

A = surface area
V = volume

Cube



$$A = 6a^2 = 3d^2 = 2D^2$$

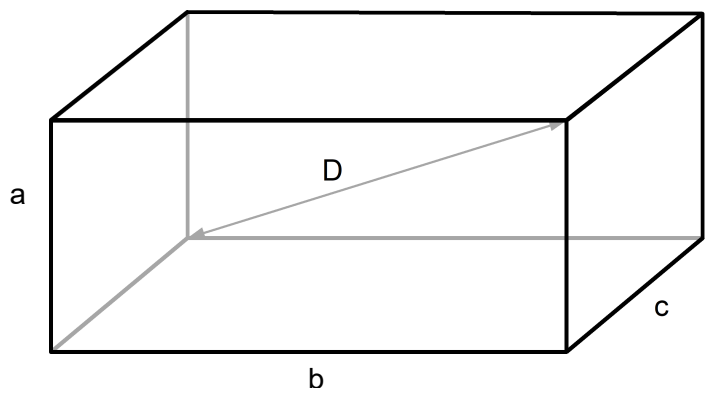
$$V = a^3 = \frac{d^3}{2\sqrt{2}} = \frac{D^3}{3\sqrt{3}}$$

$$a = \sqrt[3]{V} = \frac{A}{6} = \frac{d}{\sqrt{2}} = \frac{D}{\sqrt{3}}$$

$$d = \text{diagonal of cube face} = a\sqrt{2}$$

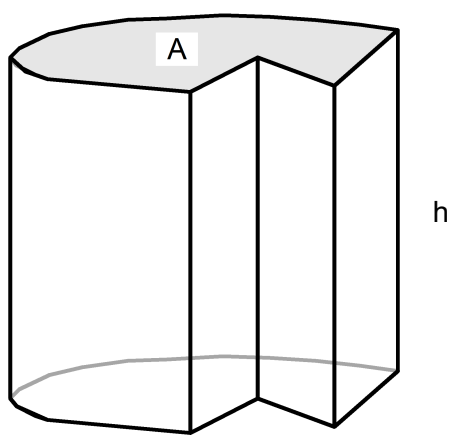
$$D = \text{diagonal of cube } = \sqrt{\frac{3d^2}{2}} = a\sqrt{3}$$

Rectangular prism



$$A = 2(ab + ac + bc)$$
$$V = abc$$
$$D = \text{diagonal} = \sqrt{a^2 + b^2 + c^2}$$

Cylinder



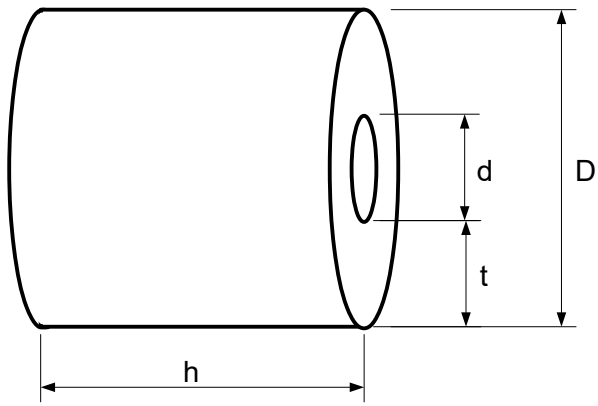
A generalized cylinder can have ends of any planar surface with area A. The height h must be measured perpendicular to the planar surfaces.

$$V = Ah$$

If the perimeter of area A is p, then the surface area A_s of the cylinder is

$$A_s = ph$$

Hollow circular cylinder



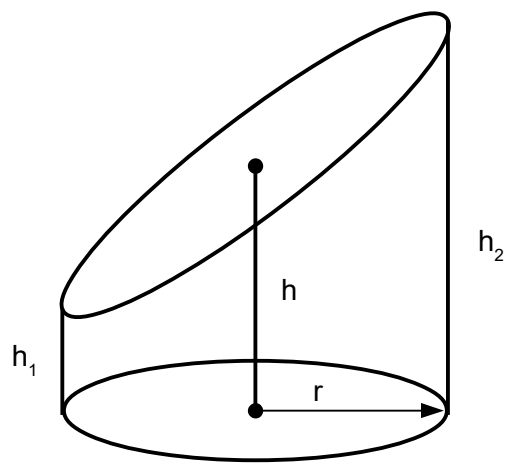
In terms of diameters:

$$t = \frac{1}{2}(D-d)$$
$$d = D-2\mathrm{t}$$
$$D = d+2\mathrm{t}$$
$$V = \frac{\pi}{4}\left(D^2-d^2\right)$$
$$= \pi t(d+t)$$
$$= \pi t(D-t)$$

In terms of radii $r = d/2$ and $R = D/2$

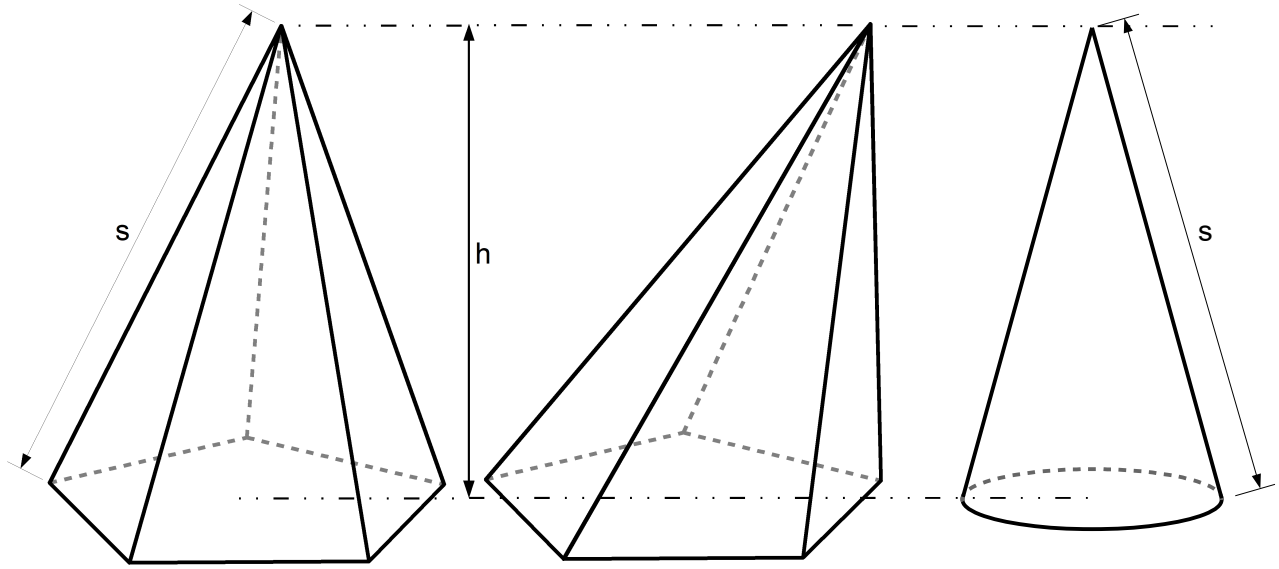
$$t = R-r$$
$$r = R-t$$
$$R = r+t$$
$$V = \pi(R^2-r^2)$$
$$= \pi t(2\mathrm{r}+t)$$
$$= \pi t(2\mathrm{R}-t)$$

Truncated right circular cylinder



$$V = \pi r^2 h = Bh$$
$$h = \text{mean height} = \frac{1}{2}(h_1+h_2)$$
$$A = \text{lateral area} = ph$$
$$B = \textit{area of base} = \pi r^2$$
$$p = \textit{perimeter of base} = 2\pi r$$

Pyramid or cone

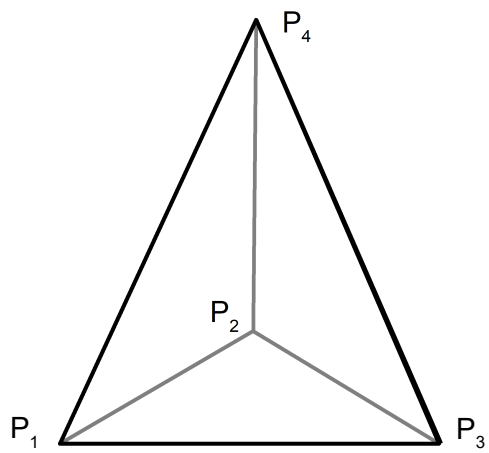


$$V = \frac{1}{3} A h$$

A = area of base
h = altitude
s = slant height
p = perimeter of base

The lateral area of the regular figure is $\frac{1}{2} ps$.

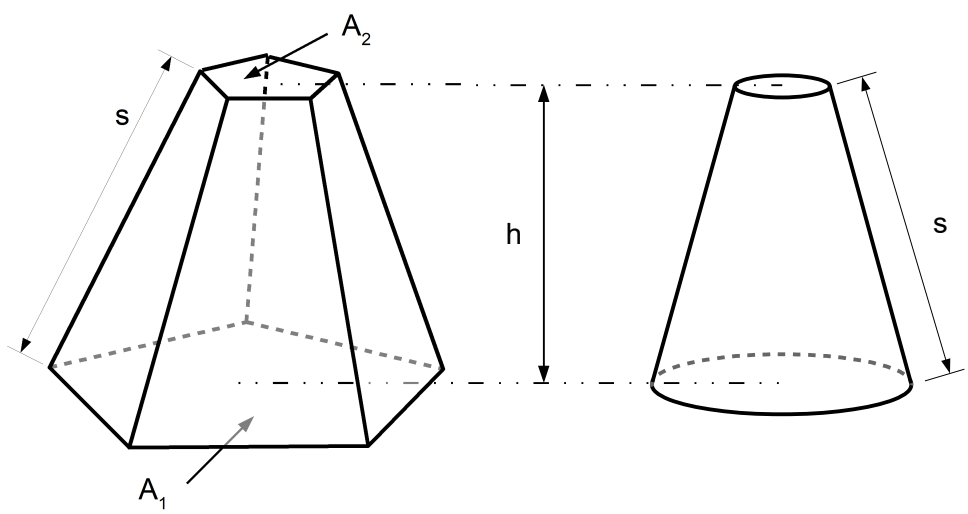
Tetrahedron



If a tetrahedron's four corners are $P_i = (x_i, y_i, z_i)$, i = 1, 2, 3, 4, then the volume is [*salmon:22:47*] [*wilson:114:141*]

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Frustum of pyramid or cone



$$V = \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$

A_1 = area of lower base
A_2 = area of upper base
h = altitude
s = slant height
p = perimeter of base

The lateral surface area of the regular figure is $\frac{1}{2}(p_1 + p_2)s$. Note the bases can be moved around

in the planes containing them and the figure's volume doesn't change.

A right frustum[2] made from two similar regular polygons of n sides has a lateral area of (http://en.wikipedia.org/wiki/Frustum)

$$A = \frac{n}{4}\left[\left(a_1^2+a_2^2\right)\cot\frac{\pi}{n}+\sqrt{(a_1^2-a_2^2)^2\sec^2\frac{\pi}{2}+4h^2(a_1+a_2)^2}\right]$$

where a_1 and a_2 are the sides of the bases.

Regular polyhedra

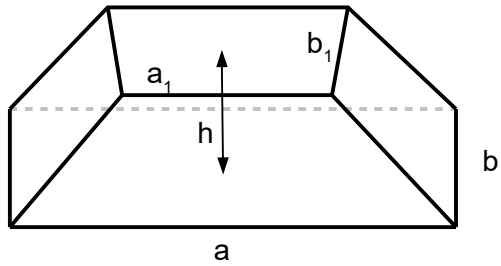
[*perry:2-7*] *and* [*wikipedia*] Volume and surface area of regular polyhedra with edge s:

		Volume $= a s^3$	**Surface area** $= a s^2$
Surface	**Name**	**a**	**a**
4 equilateral triangles	Tetrahedron	$\sqrt{2}/12 = 0.1179$	$\sqrt{3} = 1.7321$
6 squares	Hexahedron (cube)	1	6
8 equilateral triangles	Octahedron	$\sqrt{2}/3 = 0.4714$	$2\sqrt{3} = 3.4641$
12 pentagons	Dodecahedron	$(15+7\sqrt{5})/4 = 7.6631$	$3\sqrt{25+10\sqrt{5}} = 20.6458$
20 equilateral triangles	Icosahedron	$5(3+\sqrt{5})/12 = 2.1817$	$5\sqrt{3} = 8.6603$

2 The word is Latin for "piece" or "crumb". Note it's often misspelled "frustrum".

Obelisk

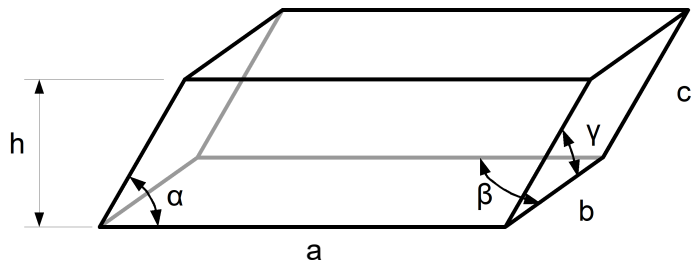
The obelisk is the frustum of a rectangular pyramid:



$$V = \frac{h}{6}\left[(2a+a_1)b+(2a_1+a)b_1\right] = \frac{h}{6}\left[ab+(a+a_1)(b+b_1)+a_1b_1\right]$$

parallelopiped

Ref. http://en.wikipedia.org/wiki/Parallelepiped



$$V = abc\sqrt{1+2\cos(\alpha)\cos(\beta)\cos(\gamma)-\cos^2(\alpha)-\cos^2(\beta)-\cos^2(\gamma)}$$

If ***a***, ***b***, and ***c*** are vectors lying along the length of the corresponding sides, then the volume can be gotten from the scalar triple product

$$
\begin{aligned}
V &= |\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})| = h|\boldsymbol{a}\times\boldsymbol{b}| \\
&= \left|\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right|
\end{aligned}
$$

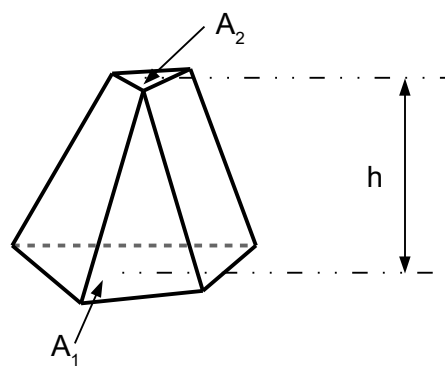
where the indexed quantities are the Cartesian components of the vectors. The area of the base is

$$A_{\text{base}} = |\boldsymbol{a} \times \boldsymbol{b}| = a\, b \sin\beta$$

and similar formulas hold for the other sides. Thus, the surface area is

$$A = 2\left[\, |\boldsymbol{a} \times \boldsymbol{b}| + |\boldsymbol{a} \times \boldsymbol{c}| + |\boldsymbol{b} \times \boldsymbol{c}| \,\right]$$

Prismoidal formula



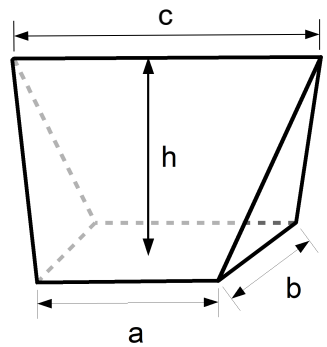
Bases are parallel planes, lateral faces are triangles or trapezoids. [*hudson:19:28*]

$$V = \frac{h}{6}\left(A_1 + A_2 + 4\,A_m\right)$$

$A_1 =$ area of lower base
$A_2 =$ area of upper base
$A_m =$ area of midsection
$h =$ altitude

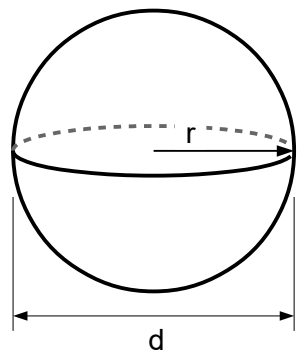
[*marks:2.22*] This formula is exact for any solid lying between two parallel planes and such that the area of a section at a distance x from one of these planes is expressible as a polynomial not higher than the third degree in x. It is approximately true for many other solids.

Wedge



$$V = \frac{bh}{6}(2a + c)$$

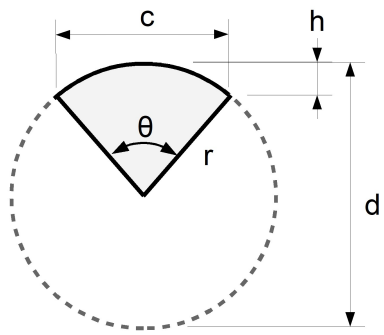
Sphere



$$V = \frac{4}{3}\pi r^3 = 4.1888 r^3 = \frac{\pi}{6} d^3 = 0.5236 d^3$$
$$A = 4\pi r^2 = 12.57 r^2 = \pi d^2 = 3.1416 d^2$$
$$d = \sqrt[3]{\frac{6V}{\pi}} = 1.2407 \sqrt[3]{V} = \sqrt{\frac{A}{\pi}} = 0.5642 \sqrt{A}$$

The volume of the sphere is $\frac{2}{3}$ of a circumscribed cylinder whose height is the sphere's diameter. The surface area is four times the area of a great circle.

Spherical sector



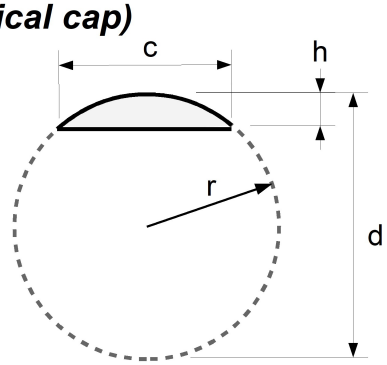
$$V = \frac{2}{3}\pi r^2 h = 2.0944\, r^2 h = \frac{1}{6}\pi d^2 h = 0.5236\, d^2 h$$

$$\text{Volume of wedge} = \frac{\theta}{2\pi}\frac{4}{3}\pi r^3 = \frac{2}{3}\theta r^3 = \frac{1}{12}\theta d^3$$

$$A = \pi r\left(2\mathrm{h} + \frac{c}{2}\right) = \text{area of conical surface and spherical cap}$$

$$c = 2\sqrt{h(2\mathrm{r} - h)}$$

Spherical segment (spherical cap)



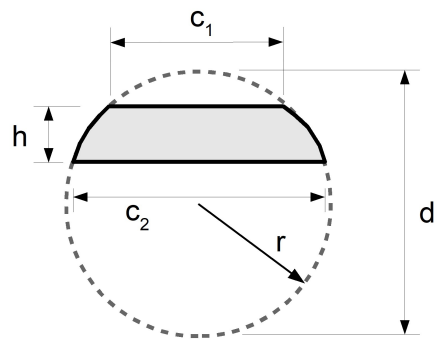
$$V = \pi h^2\left(r - \frac{h}{3}\right) = \pi h\left(\frac{c^2}{8} + \frac{h^2}{6}\right)$$

$$A = 2\pi r h = \pi\left(\frac{c^2}{4} + h^2\right)$$

$$c = 2\sqrt{h(2\mathrm{r} - h)}$$

$$r = \frac{c^2 + 4\mathrm{h}^2}{8\mathrm{h}}$$

Spherical zone

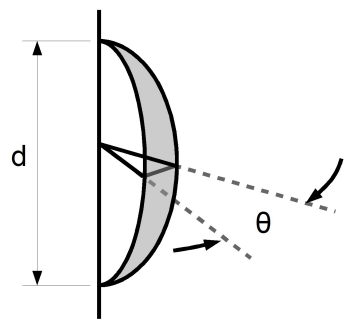


$$V = \frac{\pi}{6}\left(\frac{3}{4}(c_1^2+c_2^2)+h^2\right)$$

$$A = 2\pi r h = \text{area of spherical surface}$$

$$r = \sqrt{\frac{c_2^2}{4}+\left(\frac{c_2^2 - c_1^2 - 4h^2}{8h}\right)^2}$$

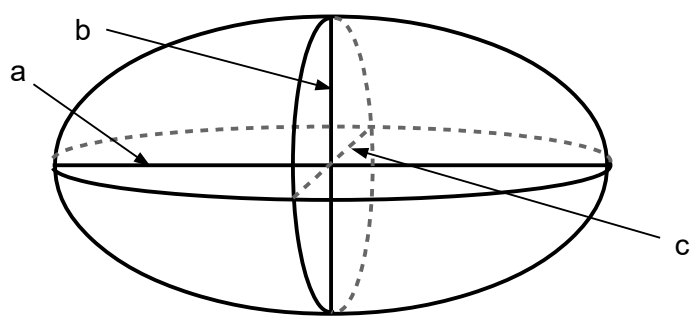
Lune



The lune is bounded by two semicircles that are on a common diameter of the sphere; they are separated by the dihedral angle θ. If $r = d/2$ is the diameter of the sphere, then

$$V = \frac{\theta}{2\pi}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\theta r^3 = \frac{1}{12}\theta d^3$$
$$A = A = \frac{\theta}{2\pi}\left(4\pi r^2\right) = 2\theta r^2 = \frac{1}{2}\theta d^2$$

Ellipsoid



$$V = \frac{4}{3}\pi a b c$$

Note a, b, and c are the semidiameters, not the diameters.

The ellipsoid's equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Spheroid

[*perry:2-7*]

Prolate spheroid: formed by rotating an ellipse about its major axis 2*a*; minor axis is 2*b*:

$$A = \text{surface area} = 2\pi\left[b^2 + \frac{ab}{e}\sin^{-1}e\right]$$

$$V = \frac{4}{3}\pi ab^2$$

$$e = \text{eccentricity} = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

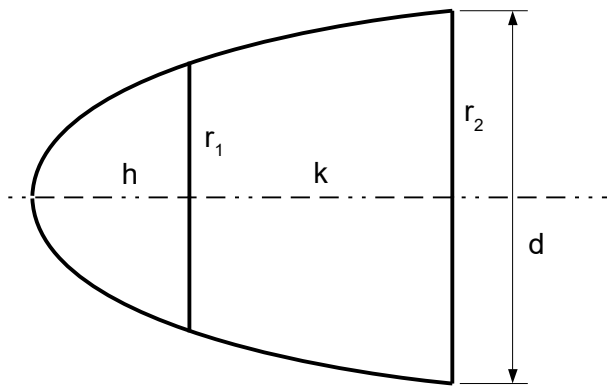
Oblate spheroid: formed by rotating an ellipse about its minor axis 2*b*; major axis is 2*a*:

$$A = \text{surface area} = \pi\left[2a^2 + \frac{b^2}{e}\ln\frac{1+e}{1-e}\right]$$

$$V = \frac{4}{3}\pi a^2 b$$

Paraboloidal segment

The volume is shown in a side view:



The paraboloid can have 1 or 2 bases. If 1 base, the length is h, the radius is r_1 and $d = 2r_1$.

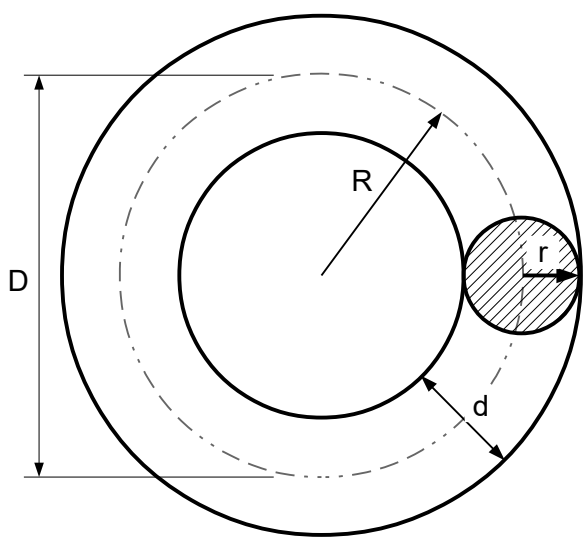
$$V \text{ for one base} = \frac{1}{2}\pi r_1^2 h$$

$$V \text{ for two bases} = \frac{1}{2}\pi k\,(r_1^2 + r_2^2)$$

$$A = \text{surface area for one base} = \frac{2\pi}{3\mathrm{p}}\left[\left(\frac{d^2}{4} + p^2\right)^{3/2} - p^3\right]$$

$$p = \frac{d^2}{8\mathrm{h}}$$

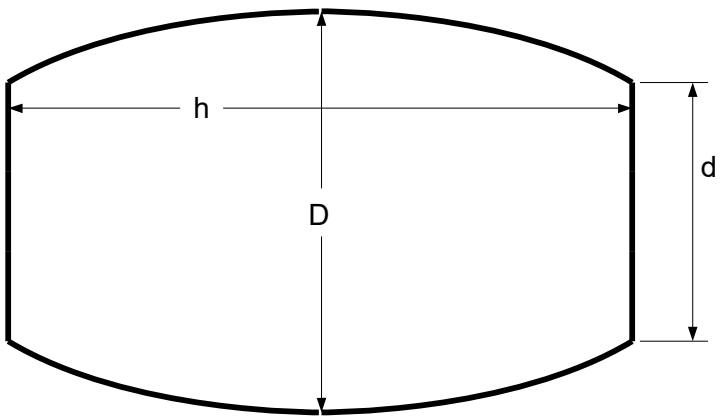
Torus



$$V = 2\pi^2 R r^2 = 19.739\,R\,r^2 = \frac{\pi^2}{4} D d^2 = 2.4674\,D d^2$$

$$A = 4\pi^2 R r = 39.478\,R\,r = \pi^2 D d = 9.8696\,D d$$

Barrel



The following are from [*marks:2-22*]. If the sides are bent to the arc of a circle, the approximate volume is

$$V = \frac{1}{12}\pi h(2\mathrm{D}^2+d^2) = 0.262h(2\mathrm{D}^2+d^2)$$

If the sides are bent to the arc of a parabola, the exact volume is

$$V = \frac{1}{15}\pi h\left[2\mathrm{D}^2+Dd+\frac{3}{4}d^2\right]$$

The number of gallons in a partially-filled cask standing on end is approximately $0.0034\,n^2 h$ where n is number of inches in the mean diameter (i.e., $\frac{1}{2}(D+d)$) and h is the height of the liquid level in inches. If n and h are measured in mm and the volume in liters, the formula is $7.85\times10^{-7}n^2h$.

Regular polyhedra

Ref. http://www.math.rutgers.edu/~erowland/polyhedra.html

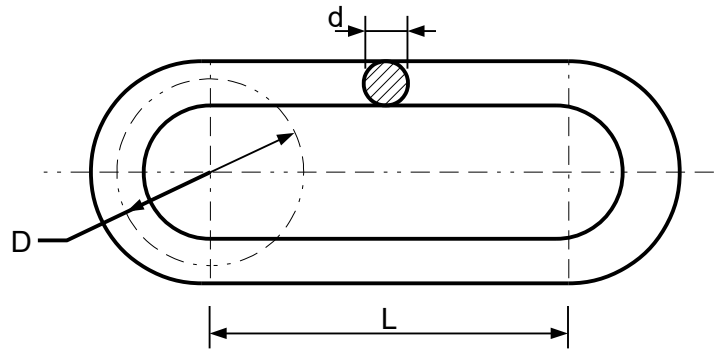
Length of side = s, V = number of vertices, E = number of edges, F = number of faces. These values satisfy Euler's formula $V-E+F=2$.

Name	F	V	E	Surface area	Volume	Internal diagonals
Tetrahedron	4	4	6	$\sqrt{3}s^2$	$\frac{s^3}{6\sqrt{2}}$	0
Octahedron	8	6	12	$2\sqrt{3}s^2$	$\frac{\sqrt{2}}{3}s^3$	3
Dodecahedron	12	20	30	$3\sqrt{5(5+2\sqrt{5})}s^2$	$\frac{1}{4}(15+7\sqrt{5})s^3$	100
Icosahedron	20	12	30	$5\sqrt{3}s^2$	$\frac{5}{12}(3+\sqrt{5})s^3$	36

A regular tetrahedron has no internal diagonals.

Link

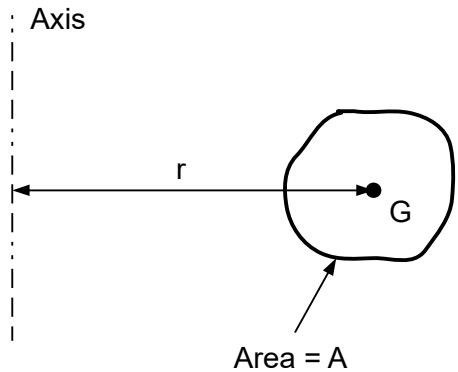
[*haswell:353:358*]



$$C = \text{circumference} = \pi D + 2L$$

$$\text{volume} = \frac{1}{4}\pi D^2 C = \frac{1}{4}\pi D^2 (\pi D + 2L)$$

Solid or surface of revolution



Theorem of Pappus: suppose a plane area A is rotated about an axis in its plane that doesn't cross the area. The distance r is to the center of mass G of the area. Then the volume of the figure generated for a rotation of the area by an angle θ is

$$V = As$$

where $s = r\theta$ is the distance along the arc that the center of gravity traveled. For a full revolution, the volume is thus

$$V = 2\pi r A$$

If the area is replaced by an arc of length L in the same plane and the arc is rotated by an angle θ around the axis, the resulting area of the generated surface is

$$A = Ls$$

where again $s = r\theta$ is the distance along the arc that the center of gravity traveled. For a full revolution, the surface area is thus

$$A = 2\pi r L$$

As an easy check, suppose we have a line of length h parallel to the axis of rotation and a distance r from it. Then the surface area of the resulting cylinder is $2\pi r h$. Note you can rotate the line about its center of mass G about an axis that passes through G perpendicular to the plane and the surface area remains constant.

The **generalized Theorem of Pappus** is: suppose you have any curved path of length s. If a plane figure of area A or a plane curve of length L move so that its center of gravity slides along the curved path while the area or the curve remain perpendicular to the path, then the volume generated is As and the area generated is Ls. The path is assumed to curve slowly enough that

successive positions of A or L will not intersect.

Suppose a curve is rotated around the x-axis; the curve is defined by a regular function, parametric equations, and a polar equation:

$$y = f(x) \qquad \begin{matrix} x = u(t) \\ y = v(t) \end{matrix} \quad r = r(\theta)$$

When this curve is rotated one revolution around the x-axis, the surface area of the resulting surface is [*bartsch:371*]

$$\begin{aligned} A &= 2\pi \int_{x_1}^{x_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{t_1}^{t_2} v(t) \sqrt{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2}\, dt \\ &= 2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

When this curve is rotated one revolution around the x-axis, the volume of the resulting volume of revolution is

$$\begin{aligned} V &= \pi \int_{x_1}^{x_2} y^2 dx \\ &= \pi \int_{t_1}^{t_2} v^2 \frac{dx}{dt} dt \\ &= \pi \int_{\theta_1}^{\theta_2} r^2 \sin^2\theta \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) d\theta \end{aligned}$$

When $y = f(x)$ is rotated around the y axis, the volume is

$$\begin{aligned} V &= \pi \int_{y_1}^{y_2} g(y)^2\, dy \\ &= \pi \int_{x_1}^{x_2} x^2 \dot{y}\, dx \end{aligned}$$

where $y = f(x)$ and $x = g(y)$

When $r = r(\theta)$ is rotated around the y axis, the volume is

$$V = \pi \int_{\theta_1}^{\theta_2} r^2 \cos^2\theta \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) d\theta$$

The method of shells sometimes makes it easier to calculate a volume gotten from rotating a function $y = f(x)$ between $x = a$ and $x = b$ about the y axis

$$V = 2\pi \int_a^b x\, f(x) dx$$

Cavalieri's theorem

[*marks:2-23*] Assume two solids have their bases in the same plane. If the plane section of one solid at every distance x above the plane is equal in area to the plane section of the other solid at height x above the base, then the two solids have equal volumes.

Plane geometry

Classifying a triangle

Let the length of the sides of the triangle be a, b, c, where c is the length of the longest side. Then calculate $\alpha = a^2 + b^2$. Classify the triangle by

$$\begin{array}{ll} \alpha = c^2 & \text{Right triangle} \\ \alpha < c^2 & \text{Acute triangle} \\ \alpha > c^2 & \text{Obtuse triangle} \end{array}$$

The classification using the first equality is the converse of the Pythagorean theorem.

Area of a triangle

If $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are the vectors of the sides of a triangle, then [*wilson:81:108*]

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$$

and the area K of the triangle is [*wilson:81:108*]

$$2K = |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{c}| = \frac{a^2 \sin B \sin C}{\sin A} = \frac{b^2 \sin A \sin C}{\sin B} = \frac{c^2 \sin A \sin B}{\sin C}$$

where A, B, and C are the angles opposite the corresponding sides. These relationships hold in three dimensions also as long as the vectors form a closed polygon.

Suppose we have a parallelogram in the plane with corners at the origin and vectors $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{p} + \boldsymbol{q}$. Then the area of the parallelogram is the absolute value of the determinant

$$\begin{vmatrix} p_x & p_y \\ q_x & q_y \end{vmatrix}$$

The area of the triangle with corners at the origin, $\boldsymbol{p}$, and $\boldsymbol{p}+\boldsymbol{q}$ is obviously half of this.

For a triangle with corners (x_1, y_1), (x_2, y_2), and (x_3, y_3), the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Here's how to see this (from Micah Fogel's *Application of the Determinant* document): the vector from (x_3, y_3) to (x_2, y_2) is (x_3-x_2, y_3-y_2) and similarly for the vector from (x_3, y_3) to (x_1, y_1). By subtracting the third row from the first and second in the above determinant, we effectively slide the triangle along so that one of its corners is at the origin. Then the resulting determinant

$$\begin{vmatrix} x_1-x_3 & y_1-y_3 & 0 \\ x_2-x_3 & y_2-y_3 & 0 \\ x_3 & y_3 & 1 \end{vmatrix}$$

can be expanded along the third column to get the area.

Area of a polygon

Let a polygon be defined by its vertices at (x_1, y_1), (x_2, y_2), $\ldots (x_n, y_n)$. The area of this polygon can be found from the shoelace algorithm (the picture makes the name obvious).

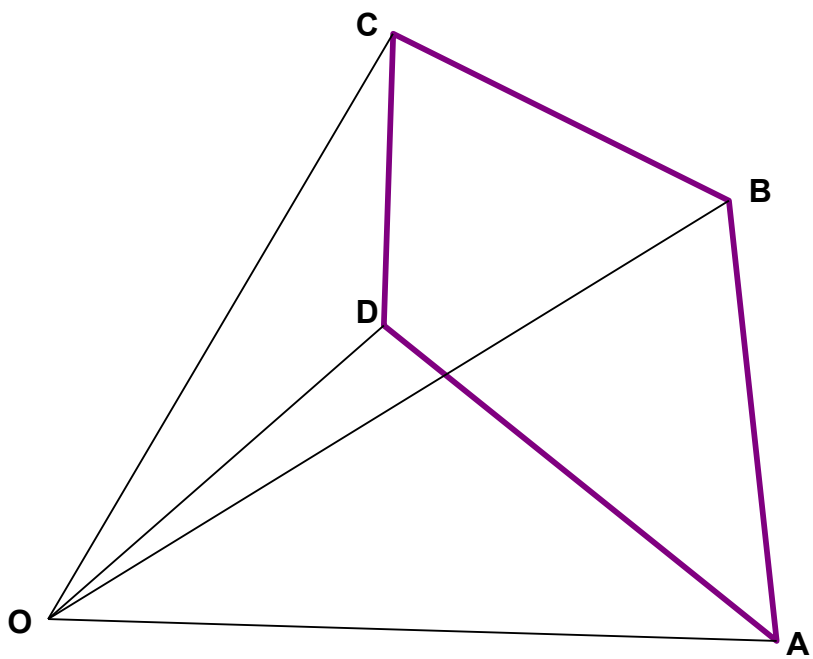
$$\frac{1}{2}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{bmatrix}$$

Take all the "down-and-to-the-right" products $x_1 y_2$, $x_2 y_3$, … and add them. Then subtract the "down-and-to-the-left" products $y_1 x_2$, $y_2 x_3$, … and halve the result.

Here's some python code that shows how simple the algorithm is to program (n is the length of the x and y arrays):

```
area = 0
for i in range(n):
    area += (x[i]*y[(i + 1) % n] - x[(i + 1) % n]*y[i])/2
```

For a correct area, the sides of the polygon must not intersect each other. Here's a picture from Micah Fogel's *Application of the Determinant* document



The area of the quadrilateral ABCD is the sum of triangles OBC and OAB minus the sum of the areas of the triangles ODC and OAD.

Inscribed circle of a triangle

Given the vertex points $P_1(x_1, y_1), P_2(x_2, y_2),$ and $P_3(x_3, y_3)$ of a triangle, the location of the center of the inscribed circle is

$$(x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

where a is the length of P_2P_3, b is the length of P_1P_3, and c is the length of P_1P_2. See the General triangle section for the radius of the inscribed circle.

Circumscribed circle of a triangle

Given the vertex points $P_1(x_1, y_1), P_2(x_2, y_2),$ and $P_3(x_3, y_3)$ of a triangle, the location of the center of the inscribed circle is

$$(x, y) = \left(\frac{\begin{vmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{vmatrix}}{D}, \frac{\begin{vmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{vmatrix}}{D} \right)$$

$$D = 2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

See the General triangle section for the radius of the inscribed circle.

Equation of a line

Given (x_1, y_1) and (x_2, y_2) on a line, the equation of the line is [*schmall:49:64*]

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

which can be written

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

The symmetrical form in terms of the intercepts of the axes is [*schmall:50:65*]

$$\frac{x}{a} + \frac{y}{b} = 1$$

With a slope *m* and the y intercept of *b*, the equation is

$$y = mx + b$$

The slope *m* is the tangent of the angle between the line and the x axis. The slope of the line perpendicular to this line is $-\frac{1}{m}$.

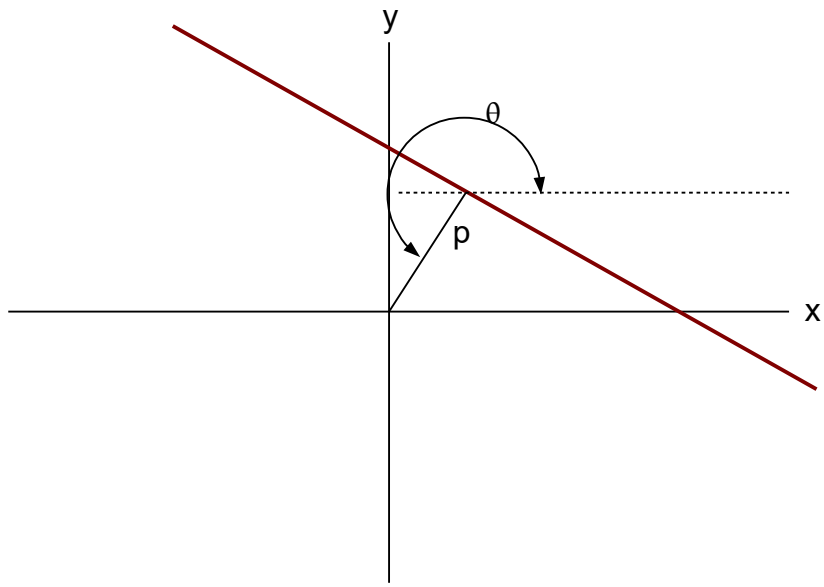
The point-slope form is

$$y - y_1 = m(x - x_1)$$

The normal form of the equation is [*schmall:53:68*]

$$x\cos\theta + y\sin\theta = p$$

where *p* is the distance of the line from the origin and θ is the angle of the perpendicular to the line from the x axis. The normal's angle with the x axis will be from $\pi/2$ to $3\pi/2$.



This can be rewritten as

$$y = -\frac{x}{\tan\theta} + \frac{p}{\sin\theta}$$

which shows that the y intercept is $b = \frac{p}{\sin\theta}$.

Any first degree equation in two variables represents a straight line

$$Ax + By + C = 0$$

These are obvious simplifications of the corresponding equations for three dimensions.

If a line passes through the two points (x_1, y_1) and (x_2, y_2), parametric equations where the two points are given for the parameter t = 0 and t = 1 respectively are

$$x = x_1 + (x_2 - x_1)t$$
$$y = y_1 + (y_2 - y_1)t$$

Polar coordinates

By converting the slope-intercept form of the equation of a line, we get the polar form

$$\rho(\sin\theta - m\cos\theta) = b$$

In polar coordinates, the equation of a straight line between two points (ρ_1, θ_1) and (ρ_2, θ_2) is [*schmall:66:81*]

$$\frac{\sin(\theta_1 - \theta_2)}{\rho} + \frac{\sin(\theta_2 - \theta)}{\rho_1} + \frac{\sin(\theta - \theta_1)}{\rho_2} = 0$$

or

$$\begin{vmatrix} \frac{1}{\rho} & \cos\theta & \sin\theta \\ \frac{1}{\rho_1} & \cos\theta_1 & \sin\theta_1 \\ \frac{1}{\rho_2} & \cos\theta_2 & \sin\theta_2 \end{vmatrix} = 0$$

Angle between two lines

If $y = m_1x + b_1$ and $y = m_2x + b_2$, we have [*schmall:60:75*]

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Distance from a line to a point

If the equation of the line is $x\cos\alpha + y\sin\alpha = p$, the distance from the point $\boldsymbol{r_1}$ is [*schmall:63:78*]

$$|x_1\cos\alpha + y_1\sin\alpha - p|$$

If the equation of the line is Ax + By + C = 0 and the point is (x, y), the distance is [*schmall:63:78*]

$$d = \frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

Bisectors of angles between two lines

Given the lines Ax + By + C = 0 and $A_1x + B_1y + C_1 = 0$. The point (x, y) on the bisector is [*schmall:64:79*]

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}$$

The + sign is used if the bisector and the origin lies within the same one of the four possible angles.

If the equations of the lines are

$$x\cos\alpha + y\sin\alpha = p$$
$$x\cos\alpha_1 + y\sin\alpha_1 = p_1$$

then the equation of the bisector is

$$x(\cos\alpha_1 \pm \cos\alpha) + y(\sin\alpha_1 \pm \sin\alpha) = p_1 \pm p$$

Equation of circle

A circle with center at (x_0, y_0) and radius r has the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Every equation of the form $x^2 + y^2 + 2ax + 2by + c = 0$ represents a circle with center at (-a, -b) and

radius $\sqrt{a^2+b^2-c}$ (which can be real, zero, or imaginary). [*schmall:87:102*]

The equation of the tangent line to a circle at point (x_1, y_1) is [*schmall:321:355*]

$$x\,x_1+y\,y_1=r^2$$

Polar equation of circle

If the circle's center is (ρ_1, θ_1) and the radius is r, then the equation is [*schmall:117:131*]

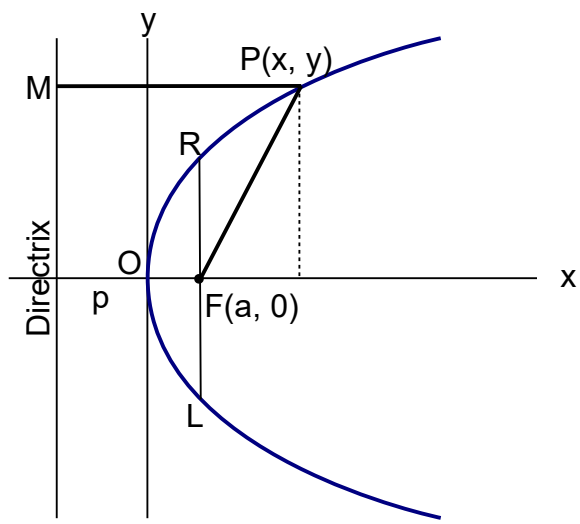
$$\rho^2-2\rho\,\rho_1\cos(\theta-\theta_1)+\rho_1^2-r^2=0$$

For the circle $x^2 + y^2 + 2Gx + 2Fy + C = 0$, the polar form is [*schmall:119:134*]

$$\rho^2+2(G\cos\theta+F\sin\theta)=0$$

where the pole is on the circle (i.e., C = 0).

Equation of parabola



The equation is $y^2=4ax$. Because $FP=\epsilon MP$ is always true and $\epsilon=1$ for the parabola, the focus is at a distance a from the origin on the x axis, the distance p = a. For any two points (x_1, y_1) and (x_2, y_2) on the parabola, we have $\frac{y_1^2}{y_2^2}=\frac{x_1}{x_2}$. The tangent at P bisects the angle FPM. LR is the latus rectum and its length is 4a.

The equation of the tangent line to the parabola at (x_1, y_1) is [*schmall:137:152*]

$$y\,y_1=2a(x+x_1)$$

Given a point (x_1, y_1) we have that there is two distinct, one, or no lines tangent to the parabola corresponding to whether the point is outside, on, or inside of the parabola. The slope(s) of the line(s) are [*schmall:143:158*]

$$m=\frac{y_1\pm\sqrt{y_1^2-4ax_1}}{2x_1}$$

Two tangents to the parabola that are perpendicular to each other intersect on the directrix *[schmall:144:159]*.

The line $y=mx+\frac{a}{m}$ is tangent to the parabola for all nonzero values of m [*schmall:138:153*].

The polar equation for a right-opening parabola with the origin at the focus is [*schmall:153:168*]

$$r = \frac{2a}{1-\cos\theta}$$

With the origin at the vertex, the polar equation is [*schmall:153:168*]

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

Any section of a right circular cone made by a plane parallel to a tangent plane of the cone will be a parabola.

Equation of ellipse

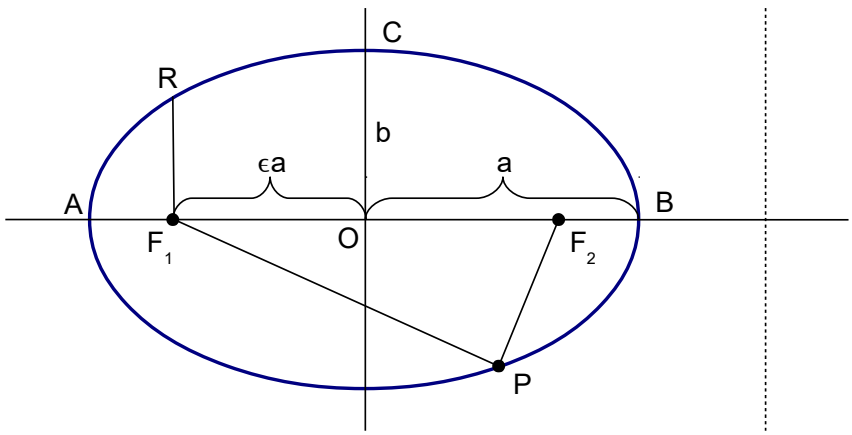


Figure 2

Also see the section *Ellipse* on page 191. [*http://en.wikipedia.org/wiki/Ellipse*]

F_1 and F_2 are called the foci and have the property that if a light ray is emitted at one foci, it will be reflected to the other foci if the inside of the ellipse is perfectly reflecting. It is also true that the distance F_1P+F_2P is a constant for any point P on the ellipse.

The general equation of an ellipse is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$$

provided $F \neq 0$ and $F(B^2-4\mathrm{AC}) > 0$; this ellipse is at an angle to the axes. If B = 0 and AC > 0, then the ellipse's axes are parallel to the coordinate axes.

Another form is

$$Ax^2+Bxy+Cy^2+Dx+Ey = 1$$

with $B^2-4\mathrm{AC} < 0$.

By a translation and rotation, the ellipse's equation can be put in canonical form (ellipse's center at origin and the major diameter lies along the x axis):

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 = 1$$

The major diameter is 2a and the minor diameter is 2b. This equation can also be written

$$y = \pm b\sqrt{1-\left(\frac{x}{a}\right)^2} = \pm\sqrt{(a^2-x^2)(1-\epsilon^2)}$$

The foci are (-ϵa, 0) and (ϵa, 0) where ϵ is the eccentricity

$$\epsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

A point is outside, on, or inside the ellipse if $(x/a)^2+(y/b)^2$ is > 1, = 1, or < 1, respectively.

The equation of a tangent line with nonzero slope m to the ellipse is $y = mx \pm \sqrt{(am)^2+b^2}$.

The distance to the directrixes from the origin is a/ϵ. F_1R is called the latus rectum and is $\frac{2b^2}{a}$.

The distance from the center to any focus is $\sqrt{a^2-b^2}$. The dashed line is the directrix (there's one at the other end of the ellipse also) and is a distance $\frac{a}{\epsilon}$ from the center.

An ellipse centered at (h, k) and with the major axis parallel to the x axis has the equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1$$

which can also be given in parametric form ($-\pi \le t \le \pi$)

$$\begin{aligned} x &= h+a\cos t \\ y &= k+b\sin t \end{aligned}$$

A general parametric form is

$$\begin{aligned} x &= x_0+a\cos t\cos\phi-b\sin t\sin\phi \\ y &= y_0+a\cos t\sin\phi-b\sin t\cos\phi \\ &\text{where } 0 \le t \le 2\pi \end{aligned}$$

(x_0, y_0) is the center of the ellipse and ϕ is the angle between the x axis and the major diameter axis of the ellipse. For an ellipse in canonical position (center at origin, major diameter along the x axis), the equation is

$$\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}$$

In polar coordinates with the origin at the center of the ellipse and the polar angle θ measured from the major diameter axis, the equation is

$$r = \frac{ab}{\sqrt{(a\sin\theta)^2+(b\cos\theta)^2}} = \frac{b}{\sqrt{1-\epsilon^2\cos^2\theta}}$$

If instead the origin is translated to a focus, then the other focus is rotated to an angle ϕ, the polar equation is

$$r = \frac{a(1-\epsilon)}{1\pm\epsilon\cos(\theta-\phi)}$$

The general polar form is

$$r = \frac{P+Q}{R}$$

where

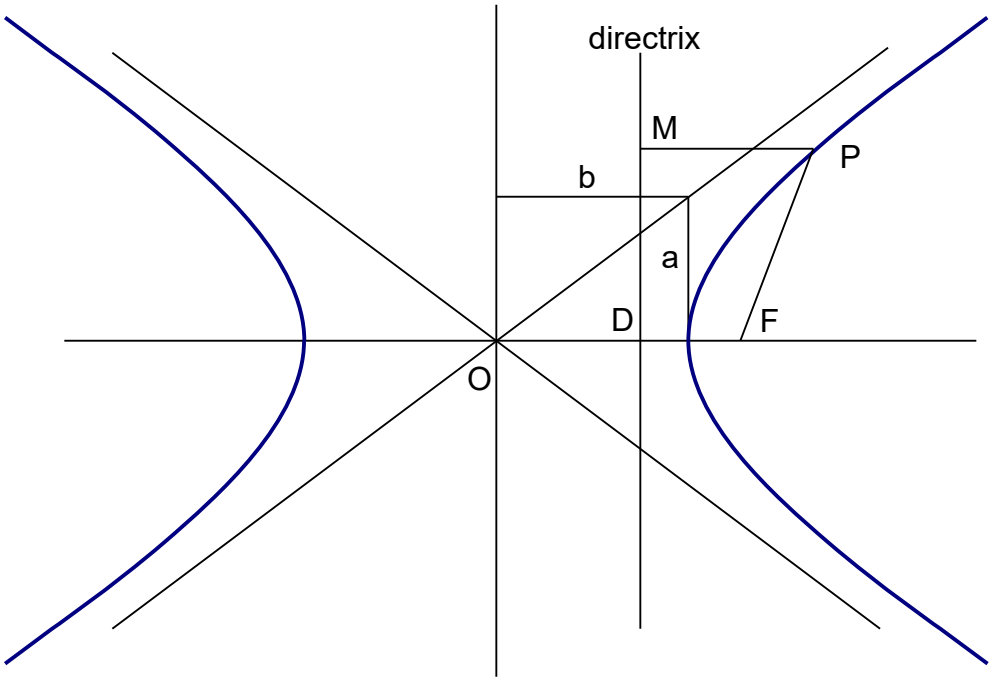
$$\begin{aligned} P &= r_0\left[(b^2-a^2)\cos(\theta+\theta_0-2\phi)+(a^2+b^2)\cos(\theta-\theta_0)\right] \\ Q &= \sqrt{2}ab\sqrt{R-2r_0^2\sin^2(\theta-\theta_0)} \\ R &= (b^2-a^2)\cos(2\theta-2\phi)+a^2+b^2 \end{aligned}$$

and the center of the ellipse is at (r_0, θ_0), the major diameter is 2a, the minor diameter is 2b, and the major diameter axis is rotated by ϕ relative to the polar axis.

The circumference of the ellipse is $4aE(\epsilon)$ where E is the complete elliptic integral of the second kind. A power series is

$$
\begin{aligned}
C &= 2\pi\, a \sum_{n=0}^{\infty} \left\{ -\left[\prod_{m=1}^{n} \left(\frac{2m-1}{2m} \right) \right]^2 \frac{\epsilon^{2n}}{2n-1} \right\} \\
&= 2\pi\, a \left[1 - \left(\frac{1}{2} \right)^2 \epsilon^2 - \left(\frac{1\cdot 3}{2\cdot 4} \right)^2 \frac{\epsilon^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} \right)^2 \frac{\epsilon^6}{5} - \ldots \right]
\end{aligned}
$$

Equation of hyperbola



The basic property of the hyperbola is that $MP = \epsilon FP$. The equation of the asymptotes are $y = \pm \frac{b}{a} x$.

The hyperbola is represented by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where $B^2 > 4AC$. For the hyperbolas opening in the x direction shown in the figure, the equation is $x^2 - y^2 = 1$. For hyperbolas opening in the y direction, the equation is $y^2 - x^2 = 1$. From here on, the formulas are for a hyperbola opening in the x direction. For a hyperbola centered at (h, k), the equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The eccentricity is $\epsilon = \sqrt{1 + \left(\frac{b}{a}\right)^2}$. The foci are at (h + c, k) where $c^2 = a^2 + b^2$. The directrix is located at x = a/ϵ.

The polar equation is $r^2 = \frac{a}{\cos 2\theta}$ where a is a constant. If the origin is at the left focus, the equation is $r = \frac{a(\epsilon^2 - 1)}{\epsilon \cos\theta - 1}$ [*schmall:216:231*].

Parametric equations are

$$x = \frac{a}{\cos t} + h \qquad \text{or} \qquad x = \pm a \cosh t + h$$
$$y = b \tan t + k \qquad\qquad y = b \sinh t + k$$

The equation of the tangent to the hyperbola at (x_1, y_x) is [*schmall:202:217*]

$$\frac{x\,x_1}{a^2}-\frac{y\,y_1}{b^2}=1$$

The line $y=m\,x\pm\sqrt{(ma)^2-b^2}$ with nonzero slope m is tangent to the hyperbola for all values of m [*schmall:203:218*]. The tangent lines to the conjugate hyperbola are $y=m\,x\pm\sqrt{b^2-(ma)^2}$.

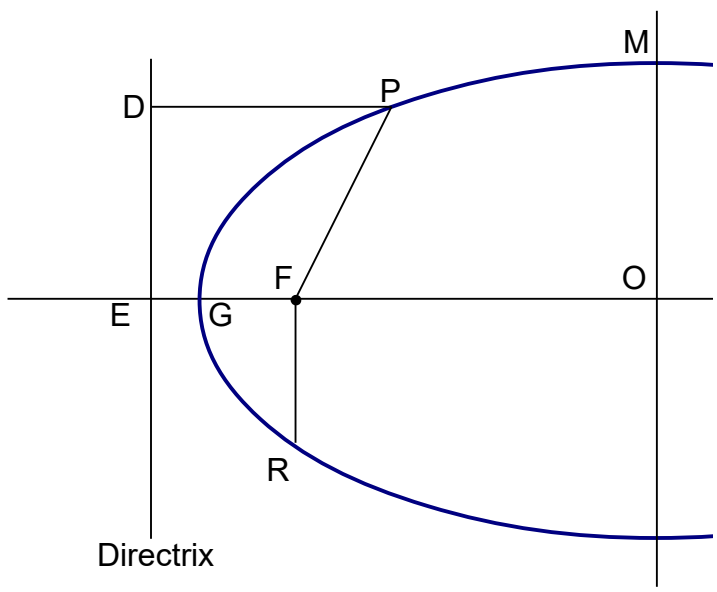
Given the asymptotes $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$, the equation of the hyperbola is $(A_1x+B_1y+C_1)(A_2x+B_2y+C_2)=k$ where k is some constant [*schmall:211:226*]. The area of the triangle formed by any tangent and the asymptotes is constant [*schmall:216:231*].

Conic sections

http://en.wikipedia.org/wiki/Conic_section

A conic section is the intersection of a plane with a right circular cone. The conic section is represented by the general algebraic equation of degree 2. Conic sections have the geometrical property that the horizontal distance DP is equal to the distance FP for any point P on the conic section. The point F is the focus.

Conics have polar equation $r=L/(1\pm\epsilon\cos\theta)$ where L is the semilatus rectum and ϵ is the eccentricity.



Name	Equation‡	ϵ	c = FO	L = FR	p = EF
circle	$x^2+y^2=a^2$	0	0	a	∞
ellipse	$(x/a)^2+(y/b)^2=1$	$\sqrt{1-(b/a)^2}$	$\sqrt{a^2-b^2}$	b^2/a	$b^2/\sqrt{a^2-b^2}$
parabola	$y^2=4ax$	1	a	2a	2a
hyperbola	$(x/a)^2-(y/b)^2=1$	$\sqrt{1+(b/a)^2}$	$\sqrt{a^2+b^2}$	b^2/a	$b^2/\sqrt{a^2+b^2}$

‡ The origin is at the vertex.

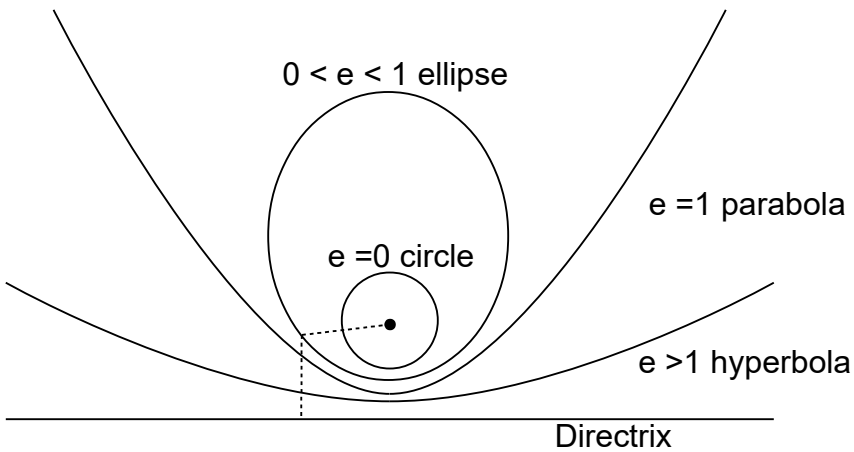
Note the relations $p\epsilon=L$ and $ae=c$.

For the case where the origin is at a focus, we have ($\alpha = a^2+b^2$, $\beta = a^2-b^2$)

Name	x‡	y‡	Polar	Cartesian
circle	$a\cos t$	$a\sin t$	$r = a$	$x^2+y^2=a^2$
ellipse	$a\cos t$	$b\sin t$	$b^2/(a-\sqrt{\beta}\cos\theta)$	$((x-\sqrt{\beta})/a)^2+(y/b)^2 = 1$
parabola	at^2	2at	$2a/(1-\cos\theta)$	$y^2 = 4a(x+a)$
hyperbola	$a/\cos t$	$b\tan t$	$b^2/(a-\sqrt{\alpha}\cos\theta)$	$((x+\sqrt{\alpha})/a)^2-(y/b)^2 = 1$

‡ Parametric equations.

Five points determine a conic if no three are collinear; i.e., there is one unique conic passing through them.



The equation $Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$ can be rearranged to give

$$Ax^2+Bxy+Cy^2 = -(Dx+Ey+F)$$

which shows that the quadratic form $z = Ax^2+Bxy+Cy^2$ and the plane $z = -(Dx+Ey+F)$ intersect to yield a conic section. Parabolas and hyperbolas are gotten by a horizontal plane ($D = E = 0$) while ellipses require the plane to intersect the x-y plane at an angle. Degenerate conics correspond to degenerate intersections.

Conic sections with matrix notation

http://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections

Conic sections in the Cartesian plane are represented by the equation

$$ax^2+bxy+cy^2+dx+ey+f = 0$$

or in homogeneous coordinates as $\boldsymbol{r}^{\mathrm{T}}Q\boldsymbol{r} = 0$ where Q is the symmetric matrix

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

Let $M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, D = det(M), $N = \begin{pmatrix} c & e/2 \\ e/2 & f \end{pmatrix}$, and E = det(N).

If D = 0, the conic is degenerate:

$$\begin{aligned} D < 0 &: \text{two intersecting lines} \\ D = 0 &: \text{two (possibly coincident) parallel straight lines} \\ D > 0 &: \text{empty} \end{aligned}$$

If D is not zero, the conics can be classified by the determinant of the minor Q_{11}:

$$\begin{aligned} E < 0 &: \text{hyperbola} \\ E = 0 &: \text{parabola} \\ E > 0 &: \text{ellipse (a circle if } Q_{11} = Q_{22} \end{aligned}$$

The center of the conic is

$$x_c = \frac{be - 2\mathrm{cd}}{4\mathrm{D}}$$
$$y_c = \frac{db - 2\mathrm{ae}}{4\mathrm{D}}$$

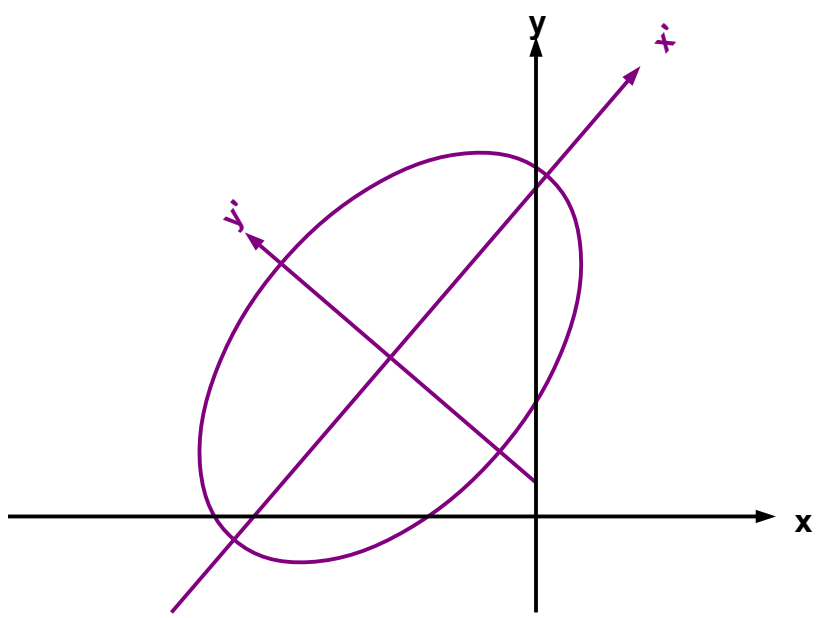
The major and minor axes pass through the center of the conic and their directions are the eigenvectors of M.

Through a point p, there are generally two tangents. The points of tangency are the intersection of the conic with the line $\boldsymbol{p}^{\mathrm{T}}\mathrm{Q}\boldsymbol{r} = 0$. When $\boldsymbol{p}$ is on the conic, the line is tangent there. When $\boldsymbol{p}$ is inside an ellipse, the line is the set of all points whose own associated line passes through $\boldsymbol{p}$.

To write the canonical form of the equation of the conic, the conic is translated and rotated so that the conic's center is at the origin and its axes are parallel to the coordinate axes. Let λ_1 and λ_2 be the eigenvalues of M. We have

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \frac{D}{det(M)} = 0$$

where the coordinates are



Divide through by D/det(M) to get the canonical form.

Circle through three points

For each point, write the equation $x^2 + y^2 + 2Gx + 2Fy + C = 0$. This represents three equations in three unknowns G, F, and C. Solve for them. Equivalently, use the equation $(x - h)^2 + (y - k)^2 = r^2$ and solve for the constants.

For the three points (x_1, y_1), (x_2, y_2), (x_3, y_3), the equation is [*schmall:97:112*]

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

From http://mathworld.wolfram.com/Circle.html, we get the solution:

$$\text{Center } = (x_{0,}\ y_0) = \left(-\frac{d}{2\,a}, -\frac{e}{2\,a}\right)$$

and the radius is

$$r = \sqrt{\frac{d^2+e^2}{4\,a^2} - \frac{f}{a}}$$

where

$$a = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$d = -\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix}$$

$$e = \begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix}$$

$$f = -\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix}$$

Tangent to a circle

Given a circle $x^2 + y^2 = r^2$, the equation of the tangent at (x_1, y_1) is [*schmall:89:104*]

$$xx_1+yy_1 = r^2$$

For the circle $x^2 + y^2 + 2Gx + 2Fy + C = 0$, the equation of the tangent at (x_1, y_1) is [*schmall:94:109*]

$$xx_1+yy_1+G(x+x_1)+F(y+y_1)+C = 0$$

For the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, the equation of the tangent at (x_1, y_1) is [*schmall:96:111*]

$$(x_1-x_0)(x-x_0)+(y_1-y_0)(y-y_0) = r^2$$

For the line $y = mx + b$ intersecting the circle $x^2 + y^2 = r^2$, the line

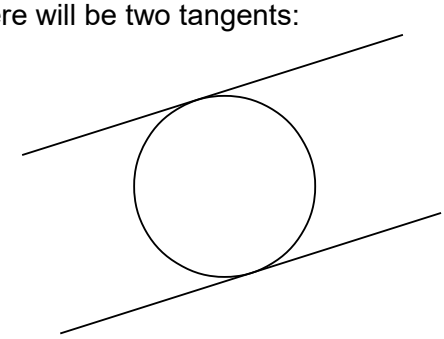
$$y=mx\pm r\sqrt{1+m^2}$$

is tangent to the circle for all values of m [*schmall:98:113*].

For the line $y = mx + b$ intersecting the circle $x^2 + y^2 + 2Gx + 2Fy + C = 0$, the equation of the tangent is [*schmall:96:113*]

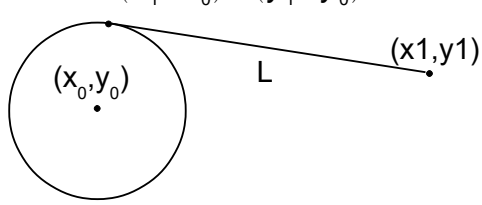
$$y+F = m(x+G)\pm\sqrt{G^2+F^2-C}\cdot\sqrt{1+m^2}$$

The ± is used because there will be two tangents:



If we have a point $P = (x_1, y_1)$ and a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, the length L of the tangent from the point P to the point of tangency is [*schmall:98*]

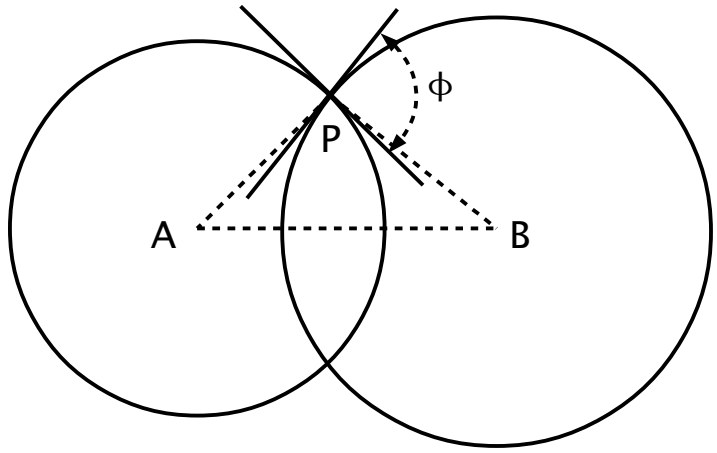
$$L^2 = (x_1-x_0)^2+(y_1-y_0)^2-r^2$$



If the equation was $x^2 + y^2 + 2Gx + 2Fy + C = 0$, the tangent's length L is [*schmall:99:114*]

$$L^2 = x_1^2+y_1^2+2Gx_1+2Fy_1+C$$

Angle of intersection of two circles



Let [*schmall:110:125*]

$$\begin{aligned} AB &= \delta \\ AP &= r_1 \\ BP &= r_2 \end{aligned}$$

Suppose the equations of the two circles are

$$\begin{aligned} x^2+y^2+2G_1 x+2F_1 y+C_1 &= 0 \\ x^2+y^2+2G_2 x+2F_2 y+C_2 &= 0 \end{aligned}$$

Then

$$\cos\phi = \frac{2G_1 G_2+2F_1 F_2-C_1-C_2}{2r_1 r_2}$$

The condition that the two circles cut orthogonally is

$$2G_1 G_2+2F_1 F_2-C_1-C_2 = 0$$

Center of triangle's inscribed circle

For a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), the center of the inscribed circle is [*schmall:122:137*]

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the triangle's sides.

Euclidean Transformations

The relevant group is E(2), comprised of rotations and translations. As usual, translations and rotations do not commute, as shown by inspection of the infinitesimal generators ∂_x, ∂_y, and $x\partial_y - y\partial_x$.

Translations

To transform from (x, y) to (X, Y) with a translation to (x_0, y_0), the transformation equations are

$$\begin{aligned} X &= x+x_0 \\ Y &= y+y_0 \end{aligned}$$

The inverse transformation is

$$x = X - x_0$$
$$y = Y - y_0$$

In polar coordinates, suppose we have the new (translated) origin at $(r_0,\ \theta_0)$, where r and θ are in the original coordinate system. The new coordinate system's coordinates will be denoted by carets. The old coordinates are related to the coordinates in the new system by [*lawrence:17*]:

$$r^2 = r_0^2 + \hat{r}^2 + 2\, r\, \hat{r} \cos(\theta_0 - \theta)$$
$$\tan\theta = \frac{\hat{r}\sin\hat{\theta} + r_0 \sin\theta_0}{\hat{r}\cos\hat{\theta} + r_0 \cos\theta_0}$$

Rotations

If the axes are rotated by an angle θ, the transformation equations can be used to find the old and new coordinates of a point P (which doesn't rotate). If the old coordinates are (x, y), use the following transformation to get things in terms of the new coordinates

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

The inverse transformation is

$$X = \ \ x\cos\theta + y\sin\theta$$
$$Y = -x\sin\theta + y\cos\theta$$

Note the transformation is orthogonal: in the matrix representation of the rotation group, the inverse transformation matrix is the transpose of the forward transformation matrix.

Oblique transformations

Suppose the positive direction of the new x axis (call it X) makes an angle α with the old x axis; analogously, the new y axis (call it Y) makes an angle β with the old x axis. Then we have the transformation [*schmall:126:141*]

$$x = X\cos\alpha + Y\cos\beta$$
$$y = X\sin\alpha + Y\sin\beta$$

See the section *Three noncoplanar vectors as a base (reciprocal system)* on page 104 for the extension to three dimensions.

Oblique coordinates

If the x and y axes are separated by an angle ω and $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two points, then [*crc:252*]

Distance between P_1 and P_2:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega}$$

Point dividing P_1P_2 in ratio r/s:

$$\left(\frac{rx_2+sx_1}{r+s}, \frac{ry_2+sy_1}{r+s}\right)$$

Midpoint of P_1P_2:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Area of triangle $P_1P_2P_3$:

$$\frac{1}{2}\sin\omega\,(x_1y_2+x_2y_3+x_3y_1-y_1x_2-y_2x_3-y_3x_1)$$

Circle: center at (h, k), radius r:

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega = r^2$$

Archimedean spiral

Polar equation $r = a\theta$. Arc length $s(\theta) = \int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \frac{a}{2}\left[\theta\alpha+\ln(\theta+\alpha)\right]$ where $\alpha = \sqrt{1+\theta^2}$.

Note $\ln(\theta+\alpha) = \sinh^{-1}\theta$.

Curvature

For a function $y = f(x)$, the curvature is [*lawrence:22*]

$$K = \frac{u_{xx}(x)}{\left(1+[u_x(x)]^2\right)^{3/2}}$$

The radius of curvature is $\rho = 1/|K|$.

If the curve is given in parametric form $x = f(t)$ and $y = g(t)$, the curvature is

$$K = \frac{\dot{f}\ddot{g}-\ddot{f}\dot{g}}{\left(\dot{f}^2+\dot{g}^2\right)^{3/2}}$$

If the curve is $f(x, y) = 0$, the curvature is

$$K = \frac{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{\left(f_x^2+f_y^2\right)^{3/2}}$$

In polar coordinates, the curvature is

$$K = \frac{r^2+2\dot{r}^2-r\ddot{r}}{\left(r^2+(\dot{r})^2\right)^{3/2}}$$

where $\dot{r} = \frac{dr}{d\theta}$.

The center of curvature is the point (α, β) that is the center of the circle tangent to the curve at (x, y) with the indicated radius of curvature. In Cartesian coordinates:

$$\alpha = x-\frac{\dot{y}\left(1+\dot{y}^2\right)}{\ddot{y}}$$
$$\beta = y+\frac{1+\dot{y}^2}{\ddot{y}}$$

In parametric form:

$$\begin{aligned}\alpha &= f-\gamma\dot{g}\\ \beta &= g+\gamma\dot{f}\\ \gamma &= \frac{\dot{f}^2+\dot{g}^2}{\dot{f}\ddot{g}-\ddot{f}\dot{g}}\end{aligned}$$

In polar coordinates:

$$\begin{aligned}\alpha &= r\cos\theta-\gamma\left(r^2+\dot{r}^2 d(r\sin\theta)\right)\\ \beta &= r\sin\theta+\gamma\left(r^2+\dot{r}^2 d(r\cos\theta)\right)\\ \gamma &= \left(r^2+2\dot{r}^2-r\ddot{r}\right)\end{aligned}$$

Products

Pseudovectors and pseudoscalars

Pseudovectors and pseudoscalars change signs under parity transformations (a reflection where one coordinate changes sign).

The cross product is a pseudovector. The dot product of two vectors is a scalar; the dot product of a vector and a pseudovector is a pseudoscalar. Thus, the scalar triple product is a pseudoscalar.

An example of a pseudovector is angular momentum. If you are watching an auto drive away from you along the +x axis of a right-handed coordinate system, the angular momentum of each tire is a pseudovector pointing in the -y direction. If you switch to a left-handed coordinate system, the pseudovector will be pointing in the +y direction. Because of the parity transformation from the right to left coordinate system, the angular momentum must transform as a pseudovector because the angular momentum of a rotating object not subject to a torque points in a fixed direction, regardless of what coordinate system you put on the physical space.

Scalar (dot) product

$$\boldsymbol{a}\cdot\boldsymbol{b} \stackrel{\text{def}}{=} ab\cos\theta$$

where θ is the smallest angle between the vectors. You can see $0 \le |\boldsymbol{a}\cdot\boldsymbol{b}| \le ab$ and $\boldsymbol{a}\cdot\boldsymbol{b} \in \mathbb{R}$. This form emphasizes the invariant nature of the scalar product under Euclidean group coordinate transformations‡ because it depends only on the magnitudes of the vectors and the angle between them. Properties:

$$\begin{aligned}&\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}\\ &\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{b}+\boldsymbol{c})\cdot\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}\\ &(\mu\,\boldsymbol{a})\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\mu\,\boldsymbol{b})\\ &\boldsymbol{a}\cdot\boldsymbol{a} = a^2\end{aligned}$$

Two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular **iff** $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. If $\boldsymbol{a}\cdot\boldsymbol{a} = 0$, then $\boldsymbol{a}$ must be the zero vector.

In terms of components:

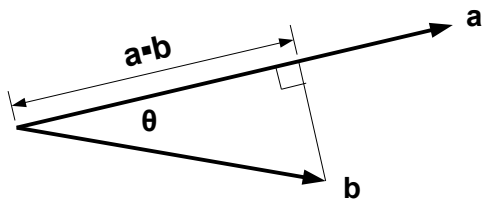
$$\boldsymbol{a}\cdot\boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

Suppose $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are vectors that make a closed triangle. Then we can get the cosine law:

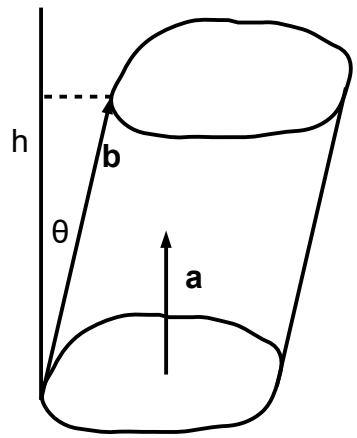
$$\begin{aligned}&\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = 0 \text{ and}\\ &(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) \stackrel{\text{def}}{=} (\boldsymbol{a}-\boldsymbol{b})^2 = a^2+b^2-2ab\cos\theta\end{aligned}$$

The geometric picture of the dot product is that it yields the projection of one vector along another:

‡ See the comments on invariance in the **Invariance** section.



If ***a*** represents a plane area and if ***b*** is a vector inclined to that plane, then ***a***·***b*** is the volume of the slanted cylinder as shown in the following figure [*wilson:57:84*]

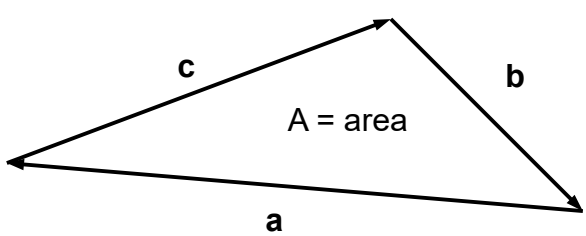


This is because ***a*** is parallel to the vertical axis and the altitude *h* is $b\cos\theta$.

Vector (cross) product

$$\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b}$$
$$c = ab\sin\theta$$
$$c_i = \epsilon_{ijk} a_j b_k$$

where the direction of ***c*** is found by the usual right hand rule and θ is the angle between ***a*** and ***b***. Again, this is an invariant with respect to Euclidean group coordinate transformations‡, but ***c*** is a pseudovector because of its behavior with respect to parity transformations. If three vectors form a triangle (closed polygon), then the cross product between any two of the sides is twice the area of the triangle. The summation convention in the last form is used over the three space indices.



We have $2A = |\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{b}\times\boldsymbol{c}| = |\boldsymbol{a}\times\boldsymbol{c}|$. Note $|\boldsymbol{a}\times\boldsymbol{b}|$ is the area of the parallelogram formed by ***a*** and ***b***.

Properties:

‡ See the comments on invariance under the **Invariance** section.

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$
$$(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$$
$$\boldsymbol{a}\times\boldsymbol{a} = 0$$
$$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \quad \hat{\boldsymbol{j}}\times\hat{\boldsymbol{k}} = \hat{\boldsymbol{i}} \quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$$
$$d(\boldsymbol{a}\times\boldsymbol{b}) = (d\,\boldsymbol{a})\times b+\boldsymbol{a}\times(d\,\boldsymbol{b})$$
$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = 0$$
$$\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{c}\times\boldsymbol{d} = (\boldsymbol{a}-\boldsymbol{c})\times(\boldsymbol{b}-\boldsymbol{d})+\boldsymbol{a}\times\boldsymbol{d}+\boldsymbol{c}\times\boldsymbol{b}$$
$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot d)(\boldsymbol{b}\cdot\boldsymbol{c})$$
$$\frac{d}{dt}(\boldsymbol{a}\times\boldsymbol{b}) = \dot{\boldsymbol{a}}\times\boldsymbol{b}+\boldsymbol{a}\times\dot{\boldsymbol{b}}$$

Note $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{a}\times\boldsymbol{c}$ does not imply $\boldsymbol{b} = \boldsymbol{c}$, but rather $\boldsymbol{a}\times(\boldsymbol{b}-\boldsymbol{c}) = 0$.

If M is a 3x3 matrix, the cross product obeys the following identity:

$$(M\,\boldsymbol{a})\times(M\,\boldsymbol{b}) = (det\,M)\left(M^{-1}\right)^T(\boldsymbol{a}\times\boldsymbol{b})$$

Two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel **iff** their cross product vanishes.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are the sides of a parallelogram, $\boldsymbol{a}\times\boldsymbol{b}$ is the area of the parallelogram.

In terms of components:

$$(\boldsymbol{a}\times\boldsymbol{b})_x = a_y b_z - a_z b_y$$
$$(\boldsymbol{a}\times\boldsymbol{b})_y = a_z b_x - a_x b_z$$
$$(\boldsymbol{a}\times\boldsymbol{b})_z = a_x b_y - a_y b_x$$

which can be remembered by symbolically expanding the determinant along the first row:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

The component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is

$$\boldsymbol{b}_\perp = \frac{\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{a})}{\boldsymbol{a}\cdot\boldsymbol{a}}$$

and the component parallel is $\boldsymbol{b}_\parallel = \boldsymbol{a}\cdot\boldsymbol{b}$.

The Pythagorean theorem relates the cross and dot products:

$$|\boldsymbol{a}\times\boldsymbol{b}|^2 = a^2b^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2 = a^2b^2\left(1-\cos^2\theta\right) = (a\,b\sin\theta)^2$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Scalar triple product

$$[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] \stackrel{\text{def}}{=} \boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$
$$= abc\cos\phi\sin\theta = \epsilon_{ijk}\,a^i b^j c^k$$

where θ is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ and ϕ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}\times\boldsymbol{c}$. This quantity is invariant under coordinate rotation (it's a pseudoscalar because the sign can change under a coordinate system parity change). The absolute value is the volume of the parallelopiped whose edges are $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

The three nonzero vectors are coplanar **iff** $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = 0$.

Other properties:

$$(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{a} = (\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})$$
$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a} = \boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}$$
$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = -\boldsymbol{a}\cdot\boldsymbol{c}\times\boldsymbol{b}$$
$$\frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}) = \dot{\boldsymbol{a}}\cdot\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{a}\cdot\dot{\boldsymbol{b}}\times\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{b}\times\dot{\boldsymbol{c}}$$

In the scalar triple product, the dot and cross can be interchanged and the components permuted cyclically; a change of cyclic order changes the sign. This is most easily seen by virtue of the interpretation as the volume of a parallelopiped.

Four vectors have a linear relationship [*wilson:76:103*]*:*

$$[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\,\boldsymbol{d}+[\boldsymbol{c}\ \boldsymbol{d}\ \boldsymbol{a}]\,\boldsymbol{b} = [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]\,\boldsymbol{a}+[\boldsymbol{d}\ \boldsymbol{a}\ \boldsymbol{b}]\,\boldsymbol{c}$$

Another way of saying and writing this is that any vector $\boldsymbol{r}$ may be represented in terms of three others $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ by [*coffin:232:261*]

$$\boldsymbol{r}[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = [\boldsymbol{r}\ \boldsymbol{b}\ \boldsymbol{c}]\,\boldsymbol{a}+[\boldsymbol{r}\ \boldsymbol{c}\ \boldsymbol{a}]\,\boldsymbol{b}+[\boldsymbol{r}\ \boldsymbol{a}\ \boldsymbol{b}]\,\boldsymbol{c}$$

A reduction formula [*wilson:86:113*]

$$[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}][\boldsymbol{d}\ \boldsymbol{e}\ \boldsymbol{f}] = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{d} & \boldsymbol{a}\cdot\boldsymbol{e} & \boldsymbol{a}\cdot\boldsymbol{f} \\ \boldsymbol{b}\cdot\boldsymbol{d} & \boldsymbol{b}\cdot\boldsymbol{e} & \boldsymbol{b}\cdot\boldsymbol{f} \\ \boldsymbol{c}\cdot\boldsymbol{d} & \boldsymbol{c}\cdot\boldsymbol{e} & \boldsymbol{c}\cdot\boldsymbol{f} \end{vmatrix}$$

Other reduction formulas [*wilson:113:139*]

$$\boldsymbol{a}\times(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})) = [\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d}]\,\boldsymbol{b}-\boldsymbol{a}\cdot\boldsymbol{b}\ \boldsymbol{c}\times\boldsymbol{d} = \boldsymbol{b}\cdot\boldsymbol{d}\ \boldsymbol{a}\times\boldsymbol{c}-\boldsymbol{b}\cdot\boldsymbol{c}\ \boldsymbol{a}\times\boldsymbol{d}$$
$$[\boldsymbol{a}\times\boldsymbol{b}\ \boldsymbol{b}\times\boldsymbol{c}\ \boldsymbol{c}\times\boldsymbol{a}] = [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\cdot[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$$

Three noncoplanar vectors as a base (reciprocal system)

The scalar triple product is of use in expressing a given vector in terms of three noncoplanar vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ [*wilson:81:108*]. Let

$$\boldsymbol{r} = a\,\boldsymbol{a}+b\,\boldsymbol{b}+c\,\boldsymbol{c}$$

By multiplying by $\cdot\boldsymbol{b}\times\boldsymbol{c}$ and the other corresponding terms, we find

$$\boldsymbol{r} = \frac{[\boldsymbol{r}\ \boldsymbol{b}\ \boldsymbol{c}]}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]}\boldsymbol{a} + \frac{[\boldsymbol{r}\ \boldsymbol{c}\ \boldsymbol{a}]}{[\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{a}]}\boldsymbol{b} + \frac{[\boldsymbol{r}\ \boldsymbol{a}\ \boldsymbol{b}]}{[\boldsymbol{c}\ \boldsymbol{a}\ \boldsymbol{b}]}\boldsymbol{c}$$

We can define

$$\boldsymbol{A} = \frac{\boldsymbol{b}\times\boldsymbol{c}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]},\quad \boldsymbol{B} = \frac{\boldsymbol{c}\times\boldsymbol{a}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]},\quad \boldsymbol{C} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]}$$

These vectors constitute the reciprocal base. Note $\boldsymbol{a}\cdot\boldsymbol{A} = \boldsymbol{b}\cdot\boldsymbol{B} = \boldsymbol{c}\cdot\boldsymbol{C} = 1$ and all other dot products between the two sets are zero (this is easy to recognize because of the cross products in the definitions). ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) is reciprocal to ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$) **iff** these dot product relations hold [*wilson:85:112*].

Then the expression for the arbitrary vector $\boldsymbol{r}$ is

$$\boldsymbol{r} = \boldsymbol{r}\cdot\boldsymbol{A}\,\boldsymbol{a}+\boldsymbol{r}\cdot\boldsymbol{B}\,\boldsymbol{b}+\boldsymbol{r}\cdot\boldsymbol{C}\,\boldsymbol{c}$$

These represent a concise formulation to convert between Cartesian and oblique coordinate systems in three dimensional space. We can also express $\boldsymbol{r}$ in the reciprocal base as

$$\boldsymbol{r} = \boldsymbol{r}\cdot\boldsymbol{a}\,\boldsymbol{A}+\boldsymbol{r}\cdot\boldsymbol{b}\,\boldsymbol{B}+\boldsymbol{r}\cdot\boldsymbol{c}\,\boldsymbol{C}$$

The previous two equations can be used to write down the transformation and its inverse between the two coordinate systems. Note $(\hat{\boldsymbol{i}},\hat{\boldsymbol{j}},\hat{\boldsymbol{k}})$ is its own reciprocal system; this is the only basis for which this is true (along with the equivalent left-handed basis) [*wilson:87:114*].

These reciprocal vectors are used in crystallography to discuss reciprocal lattices (see http://en.wikipedia.org/wiki/Reciprocal_lattice). A reciprocal lattice vector $\boldsymbol{k}$ has the property that $e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{r}} = 1$ for all lattice point positions $\boldsymbol{r}$; i.e., $\boldsymbol{k}\cdot\boldsymbol{r}$ is an integer.

If ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) is reciprocal to ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$), then

$$[\boldsymbol{A}\ \boldsymbol{B}\ \boldsymbol{C}][\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 1$$

Vector triple product

The definition is $\boldsymbol{d} = \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$. The parentheses are necessary because the cross product is nonassociative: $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) \neq (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$. We have [*wilson:76:103*] [*wilson:111:137*] [*wilson:113:139*]

$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= \boldsymbol{b}\ \boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{c}\ \boldsymbol{a}\cdot\boldsymbol{b} \\
(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} &= \boldsymbol{a}\cdot\boldsymbol{c}\ \boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{c}\ \boldsymbol{a} \\
(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} &= -\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) \\
\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{a} - a^2\,\boldsymbol{b} \\
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) &= 0 \\
\boldsymbol{a}\times(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})) &= [\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d}]\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}\times\boldsymbol{d} \\
&= \boldsymbol{b}\cdot\boldsymbol{d}\,\boldsymbol{a}\times\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{c}\,\boldsymbol{a}\times\boldsymbol{d}
\end{aligned}$$

The first equation yields the "bac cab" mnemonic to help remember the formula.

The vector triple product is a vector perpendicular to $\boldsymbol{a}$ lying in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$.

In terms of Levi-Civita (summation convention)

$$(\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}))_i = \epsilon_{ijk}\,a^j\,\epsilon_{klm}\,b^l\,c^m = \epsilon_{ijk}\,\epsilon_{klm}\,a^j\,b^l\,c^m$$

The vector $\boldsymbol{d}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}\times\boldsymbol{c}$; since $\boldsymbol{b}\times\boldsymbol{c}$ is perpendicular to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{d}$ must lie in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$ and thus takes the form

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = s\boldsymbol{b} + t\,\boldsymbol{c} \qquad s, t \in \mathbb{R}$$

and, from the rule above, $s = \boldsymbol{a}\cdot\boldsymbol{c}$ and $t = -\boldsymbol{a}\cdot\boldsymbol{b}$.

Also [*wilson:76:103*]

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = \boldsymbol{a}\cdot\boldsymbol{c}\ \boldsymbol{b}\cdot\boldsymbol{d} - \boldsymbol{a}\cdot\boldsymbol{d}\ \boldsymbol{b}\cdot\boldsymbol{c} = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{c} & \boldsymbol{a}\cdot\boldsymbol{d} \\ \boldsymbol{b}\cdot\boldsymbol{c} & \boldsymbol{b}\cdot\boldsymbol{d} \end{vmatrix}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]\ \boldsymbol{a} - [\boldsymbol{c}\ \boldsymbol{d}\ \boldsymbol{a}]\ \boldsymbol{b} + [\boldsymbol{d}\ \boldsymbol{a}\ \boldsymbol{b}]\ \boldsymbol{c} - [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\ \boldsymbol{d}$$

An arbitrary combination of dot and cross products can be broken into a sum of terms each involving only one cross product [*salmon:77:104*].

Direction cosines

A direction in space can be specified using direction cosines. Suppose a vector $\boldsymbol{r}$ points in the desired direction. Then the direction cosines are

$$\alpha = \frac{\boldsymbol{r}\cdot\hat{\boldsymbol{i}}}{r} = \frac{r_x}{r} \qquad \beta = \frac{\boldsymbol{r}\cdot\hat{\boldsymbol{j}}}{r} = \frac{r_y}{r} \qquad \gamma = \frac{\boldsymbol{r}\cdot\hat{\boldsymbol{k}}}{r} = \frac{r_z}{r}$$

We will use $\Gamma = (\alpha, \beta, \gamma)$ to indicate the vector made up of three direction cosines. Note $|\Gamma| = 1$. Direction numbers are $\mu\,\Gamma$ where μ is nonzero.

Suppose $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1)$ and $\Gamma_2 = (\alpha_2, \beta_2, \gamma_2)$. These two directions are parallel **iff** $\Gamma_1 = \Gamma_2$. These directions are perpendicular **iff** $\Gamma_1\cdot\Gamma_2 = 0$. [*drmath*]

Three directions are parallel to a common plane **iff** [*drmath*]

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0$$

If we are given two lines in space with direction cosines $\mathbf{\Gamma_1}$ and $\mathbf{\Gamma_2}$ we can calculate the direction cosines of the line that is perpendicular to both lines from [*snyder:24:44*]

$$\frac{\alpha}{\beta_1\gamma_2-\beta_2\gamma_1} = \frac{\beta}{\gamma_1\alpha_2-\gamma_2\alpha_1} = \frac{\gamma}{\alpha_1\beta_2-\alpha_2\beta_1} = \frac{\pm 1}{\sin\theta}$$

where θ is the angle between the two lines [*snyder:8:28*]

$$\cos\theta = \mathbf{\Gamma_1}\cdot\mathbf{\Gamma_2}$$

and

$$\sin^2\theta = 1-\mathbf{\Gamma_1}\cdot\mathbf{\Gamma_2} = (\mathbf{\Gamma_1}\times\mathbf{\Gamma_2})^2$$

where the square of a vector $\boldsymbol{a}$ is $\boldsymbol{a}\cdot\boldsymbol{a}$ (this relation can be proved by writing $1-\mathbf{\Gamma_1}\cdot\mathbf{\Gamma_2} = \Gamma_1^2\Gamma_2^2-\mathbf{\Gamma_1}\cdot\mathbf{\Gamma_2}$ and expanding the right hand side in components).

Equation of a line

Forms

Intersection of two planes

Two planes that intersect determine a line:

$$\begin{aligned} A_1x+B_1y+C_1z+D &= 0 \\ A_2x+B_2y+C_2z+D &= 0 \end{aligned}$$

Projecting planes

There are two planes which contain the line and intersect the xz and yz planes (these are projecting planes) and lead two the two simultaneous equations for the line [*schmall:291:306*]

$$\begin{aligned} x &= mz+b \\ y &= nz+c \end{aligned}$$

Note by setting z = 0 in these equations you can find the point where the line pierces the xy plane; similarly for the other planes:

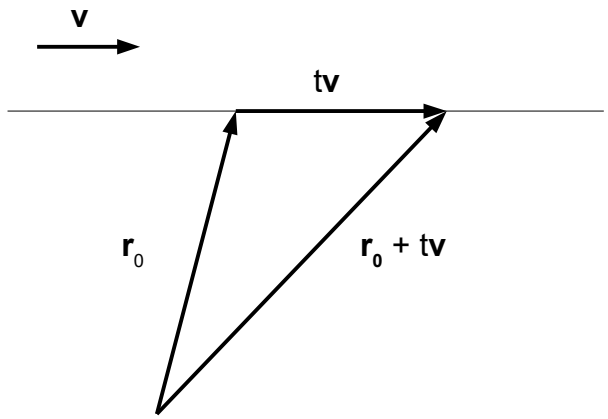
$$\begin{aligned} \text{xy piercing point} &= (b,c,0) \\ \text{xz piercing point} &= (-mc/n+b,0,-c/n) \\ \text{yz piercing point} &= (0,-nb/m+c,-b/m) \end{aligned}$$

Vector

If $\boldsymbol{r_0}$ is a point on the line and a vector $\boldsymbol{v}$ is parallel to the line, then all points $\boldsymbol{r}$ on the line can be found from [*corral*]

$$\boldsymbol{r} = \boldsymbol{r_0} + t\,\boldsymbol{v} \qquad t \in \mathbb{R}$$

A picture makes this obvious



Endpoints of two vectors

Given two vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, the equation for the line between the ends of the two vectors is

$$\boldsymbol{r} = t\,\boldsymbol{r_1} + (1-t)\,\boldsymbol{r_2} \tag{3}$$

Parametric

This form is composed of the components of the vector form:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

where $\boldsymbol{v}$ = (a, b, c).

Symmetric

By solving the parametric equations for the parameter and equating in pairs, we get [*snyder:20:41*]

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

If, e.g., a = 0, we can't divide by zero, but the parametric form gives $x = x_0$. This means the line lies in the plane $x = x_0$. a, b, and c are direction numbers of the line and can be converted to direction cosines by dividing by $\sqrt{a^2+b^2+c^2}$.

Two-point form

Using a second point $\boldsymbol{r_1}$ with the symmetric form, we have [*snyder:20:41*][*drmath*]

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

Equation from two points on line

If $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are points on the line, then all points on the line can be found from [*corral*]

$$\boldsymbol{r_1} + t(\boldsymbol{r_2} - \boldsymbol{r_1}) \qquad t \in \mathbb{R}$$

(the previous equation is the same as that given in section *Endpoints of two vectors* above. The

parametric form is

$$x = x_1+(x_2-x_1)t, \quad y = y_1+(y_2-y_1)t, \quad z = z_1+(z_2-z_1)t$$

and the symmetric representation is [*schmall:292:307*]

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

From point and direction angles

If the line passes through the point (x_1, y_1, z_1) and has direction angles (a, b, c), the equation is [*schmall:291:305*]

$$\frac{x-x_1}{\cos a} = \frac{y-y_1}{\cos b} = \frac{z-z_1}{\cos c}$$

From two planes

Two planes that intersect result in a line. This implies the line is the locus of points that satisfy the two simultaneous equations [*snyder*]

$$\begin{aligned} A_1 x+B_1 y+C_1 z+D_1 &= 0 \\ A_2 x+B_2 y+C_2 z+D_2 &= 0 \end{aligned}$$

See more details in the section **Intersection of two planes** on page 116.

Projecting planes

[*schmall:292:307*] The projecting planes from a line are those planes that contain the line and intersect the xy, xz, and yz planes perpendicularly. If the equations of the line are given by the two equations

$$\begin{aligned} S_1 &= A_1 x+B_1 y+C_1 z+D = 0 \\ S_2 &= A_2 x+B_2 y+C_2 z+D = 0 \end{aligned}$$

then the equation

$$a S_1+b S_2 = 0$$

represents any plane passing through the intersection of the planes S_1 and S_2. If a and b be chosen so that one of the variables will vanish, then $a S_1+b S_2 = 0$ will represent one of the projecting planes of the given line.

Equation of a plane

General form

Every equation of the first degree in x, y, and z represents a plane

$$Ax+By+Cz+D = 0 \quad \text{or} \quad \boldsymbol{r}\cdot\boldsymbol{Q} = -D$$

if $\boldsymbol{Q}$ is (A, B, C). [*wilson:88:115*] Any scalar equation of the first degree in an unknown vector $\boldsymbol{r}$ may be reduced to this form and, hence, any scalar equation of the first degree in $\boldsymbol{r}$ represents a plane. Let

$$\Gamma = \sqrt{A^2+B^2+C^2}$$

If we divide the general equation by Γ, we get

$$\alpha x+\beta y+\gamma z+p = 0$$

where α, β, and γ are the direction cosines of the normal to the plane and

$$p = \frac{D}{\Gamma}$$

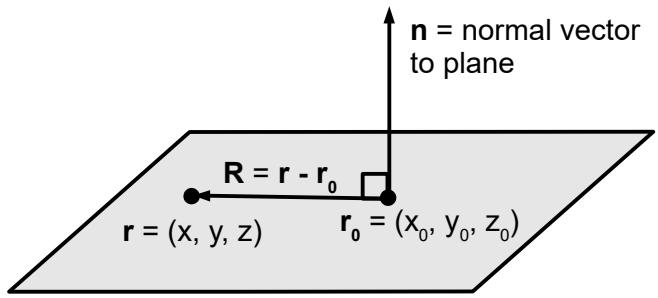
Hessian normal form

The preceding allows us to put the equation of the plane in the Hessian normal form:

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{r} = p$$

where p is the perpendicular distance from the origin to the plane (note p can be positive, zero, or negative) and $\hat{\boldsymbol{n}} = (\alpha, \beta, \gamma)$ is a unit normal to the plane. p can be positive or negative, depending on where the plane is with respect to the origin. Thus, in words: **the position vector of any point on the plane dotted with a unit normal to the plane is a constant.**

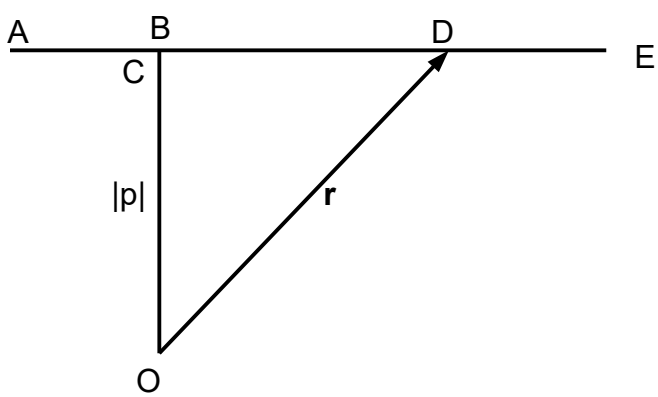
Here's a picture that illustrates the notion that any point in the plane dotted with a normal to the plane is a constant [*corral:35:43*]



Let $\boldsymbol{r}$ and $\boldsymbol{r_0}$ be any two points in the plane. Then $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_0}$. Since $\boldsymbol{R}$ lies in the plane, we have $\boldsymbol{n} \cdot \boldsymbol{R} = \boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r0}) = 0$. Thus, $\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{n} \cdot \boldsymbol{r_0}$. Since $\boldsymbol{r}$ and $\boldsymbol{r_0}$ are arbitrary points in the plane, we have that $\boldsymbol{n} \cdot \boldsymbol{r}$ is the same constant for any point in the plane.

Let $\hat{\boldsymbol{n}} \cdot \boldsymbol{r} = p$. Note we're now using the unit normal $\hat{\boldsymbol{n}}$. Since we now know this is a constant, let's look at the case where $\boldsymbol{r}$ is parallel to $\hat{\boldsymbol{n}}$ -- this is the case when $\boldsymbol{r}$ is the perpendicular to the plane from the origin. Thus we have $r = p$, so we see p is the distance from the origin to the plane. Since it's a dot product, we could also have $\boldsymbol{r}$ and $\hat{\boldsymbol{n}}$ be antiparallel, so p could be negative. You could take the convention that the unit normal should always point away from the origin, so then p would always be positive.

Here's a geometrical picture that illustrates this further. Draw the line OB perpendicular to the plane AE from the origin O. A cross section perpendicular to the plane will be:



The line OD represents the position vector $\boldsymbol{r}$ of any point in the plane AE. This vector dotted with the unit normal in the direction OB (i.e., its projection onto the normal direction) is the length OB. Since the point D is arbitrary, you can see that $\boldsymbol{n} \cdot \boldsymbol{r}$ is constant for all points in the plane because you can rotate the line AE about OB to get to all possible points in the plane with position vector $\boldsymbol{r}$.

Note that $OB = |p|$.

Dual form

The plane can be represented by a single vector [*wilson:108:135*] ***p***. The direction of ***p*** will be from the origin perpendicularly to the plane and the magnitude will be the reciprocal of the length of that perpendicular The equation of the plane in Hessian form is then

$$\boldsymbol{r} \cdot \boldsymbol{p} = 1$$

The components of ***p*** are the reciprocals of the intercepts of the plane on the axes. Planes through the origin will of course be problematic.

From axial intercepts

If a, b, and c are the intercepts on the x, y, and z axes, then the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

If the plane's equation is $Ax + By + Cz = D$, then

$$a = \frac{D}{A}, \quad b = \frac{D}{B}, \quad c = \frac{D}{C}$$

or, in terms of direction cosines and the Hessian normal p parameter

$$a = \frac{p}{\alpha}, \quad b = \frac{p}{\beta}, \quad c = \frac{p}{\gamma}$$

Through origin and parallel to two vectors

$$\boldsymbol{r} = s\,\boldsymbol{r}_1 + t\,\boldsymbol{r}_2$$

where s and t are parameters.

From three noncollinear points

Let the points be $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$. Define

$$\boldsymbol{n} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3)$$

If the three points are collinear, this vector is identically zero. Otherwise, it's a normal to the plane containing the three points.

[*coffin:232:261, 17:46*]:

$$(\boldsymbol{r} - \boldsymbol{r}_1) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) = 0$$

or

$$\boldsymbol{\phi} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

where

$$\boldsymbol{\phi} = \boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1$$

The condition that the points lie in the plane $Ax + By + Cz + D = 0$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Two parametric equation forms are

$$\boldsymbol{r} = \boldsymbol{r}_1 + s(\boldsymbol{r}_2 - \boldsymbol{r}_1) + t(\boldsymbol{r}_2 - \boldsymbol{r}_3)$$
$$\boldsymbol{r} = s\boldsymbol{r}_1 + t\boldsymbol{r}_2 + (1 - s - t)\boldsymbol{r}_3$$

Alternative equations are [*crc:356*]

$$r = s\boldsymbol{r}_1 + t\boldsymbol{r}_2 + u\boldsymbol{r}_3$$
or
$$[\boldsymbol{r} - \boldsymbol{r}_1 \;\; \boldsymbol{r}_1 - \boldsymbol{r}_2 \;\; \boldsymbol{r}_2 - \boldsymbol{r}_3] = 0$$
or
$$[\boldsymbol{r} \;\; \boldsymbol{r}_1 \;\; \boldsymbol{r}_2] + [\boldsymbol{r} \;\; \boldsymbol{r}_2 \;\; \boldsymbol{r}_3] + [\boldsymbol{r} \;\; \boldsymbol{r}_3 \;\; \boldsymbol{r}_1] - [\boldsymbol{r}_1 \;\; \boldsymbol{r}_2 \;\; \boldsymbol{r}_3] = 0$$

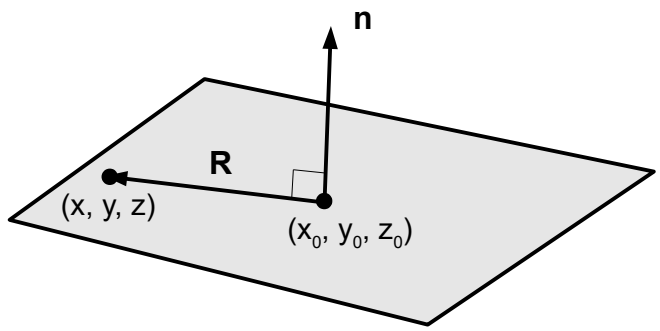
where $s, t, u \in \mathbb{R}$.

Through a point and perpendicular to a vector

The equation of a plane containing the point $\boldsymbol{r_0}$ = (x_0, y_0, z_0) and perpendicular to the nonzero vector $\boldsymbol{n}$ (i.e., $\boldsymbol{n}$ is a normal to the plane) is [*corral:35:43*]

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

Here, $\boldsymbol{r}$ = (x, y, z) is any point in the plane. This is seen in the following picture:



For any point in the plane, $\boldsymbol{n}\cdot\boldsymbol{R} = 0$ and $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_0}$. This can also be written as [*crc:356*]

$$\boldsymbol{r}\cdot\boldsymbol{n}=c$$

where c is a constant and is $\boldsymbol{n}\cdot\boldsymbol{r_0}$. The perpendicular distance from any point $\boldsymbol{r}$ to this plane is

$$\frac{c-\boldsymbol{n}\cdot\boldsymbol{r}}{r}$$

From two simultaneous equations: projecting planes

Suppose we have the simultaneous equations

$$\begin{aligned} x &= mz+b \\ y &= nz+c \end{aligned}$$

are two planes which intersect in a straight line. They are perpendicular to the xz and yz planes, respectively. These two planes are called **projecting planes**.

To find the projecting plane of a given line

Let the line be given by the Hessian forms

$$\begin{aligned} S_1 &= \hat{\boldsymbol{n}}_1\cdot\boldsymbol{r}-\boldsymbol{d}_1 = 0 \\ S_2 &= \hat{\boldsymbol{n}}_2\cdot\boldsymbol{r}-\boldsymbol{d}_2 = 0 \end{aligned}$$

Then the equation

$$\mu S_1 + \nu S_2 = 0$$

represents any plane passing through the intersection of the above two planes. Choose μ and ν so that one of the variables will vanish, then we'll have an equation for one of the projecting planes of the given line. In practice, the desired variable is simply eliminated from the two equations.

Points

Collinear points

r₁, ***r₂***, ***r₃*** are collinear **iff** [*drmath*]

$$\frac{x_2 - x_1}{x_3 - x_1} = \frac{y_2 - y_1}{y_3 - y_1} = \frac{z_2 - z_1}{z_3 - z_2}$$

From equation (3) above for the parametric equation for a line between the ends of two vectors, one can write [*coffin:13:42*]

$$t\,\boldsymbol{r}_1 + (1-t)\,\boldsymbol{r}_2 - \boldsymbol{r}_3 = 0$$

if the ends of these vectors fall on a line (i.e., they must satisfy equation (3). Note the coefficients of the three vectors sum to zero. Thus, if we have

$$\begin{aligned} x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} &= 0 \\ x + y + z &= 0 \end{aligned}$$

then the ends of the three vectors ***a***, ***b***, and ***c*** must end on the same line.

Coplanar points

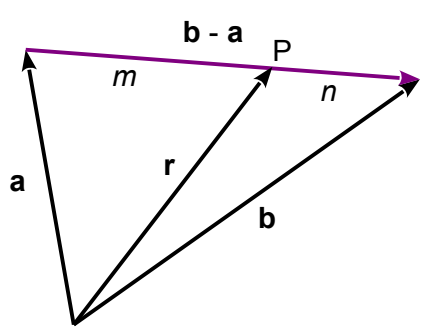
r₁, ***r₂***, ***r₃***, ***r₄*** are coplanar **iff** [*drmath*]

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Lines and Planes

Point dividing a line segment into a given ratio

Suppose we have two points ***a*** and ***b*** and we want to find the point P on the line between these two points such that their length ratios are m/n:



We have [*coffin:19:48*]

$$\boldsymbol{r} = \boldsymbol{a} + \frac{m}{m+n}(\boldsymbol{b} - \boldsymbol{a}) = \frac{n\boldsymbol{a} + m\boldsymbol{b}}{m+n}$$

Lines in space

Suppose line L_1 is given by $\boldsymbol{r_1} + s\boldsymbol{v_1}$ and L_2 is given by $\boldsymbol{r_2} + t\boldsymbol{v_2}$. Then these lines are either

1. Identical
2. Parallel: $\boldsymbol{v_1}$ is parallel to $\boldsymbol{v_2}$.
3. Intersect
4. Skew: they don't intersect, but lie in parallel planes.

For the lines to be perpendicular, we must have $\boldsymbol{v_1}$ perpendicular to $\boldsymbol{v_2}$. Note that skew lines can be perpendicular.

Line in a plane

Given a line

$$\begin{aligned} x &= mz + a \\ y &= nz + b \end{aligned}$$

and a plane $Ax + By + Cz + D = 0$. For the line to be in the plane, we must have [*salmon:29:55*]

$$\begin{aligned} Am + Bn + C &= 0 \quad \text{(line is parallel to plane)} \\ &\quad \text{and} \\ Aa + Bb + D &= 0 \quad \text{(one of the line's points is in the plane)} \end{aligned}$$

Line through a point parallel to a vector

We desire the equation of a line through the point $\boldsymbol{r_0}$ and parallel to a vector $\boldsymbol{A}$. If $\boldsymbol{r}$ is the vector to any point on the line, then $\boldsymbol{r} - \boldsymbol{r_0}$ is parallel to $\boldsymbol{A}$; thus, the vector product vanishes and we have [*wilson:107:134*] [*coffin:232:261*]

$$\boldsymbol{A} \times (\boldsymbol{r} - \boldsymbol{r_0}) = 0$$

Intersection of two lines

To determine intersection, use the parametric representation of the lines, then set the two (x, y, z) triples equal [*corral:34:42*]. This will result in a system of three equations in two unknowns (the two parameters).

For two non-parallel lines (i.e., $\boldsymbol{r_2}$ and $\boldsymbol{r_4}$ are not parallel)

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r_1} + s\boldsymbol{r_2} \\ \boldsymbol{r} &= \boldsymbol{r_3} + t\boldsymbol{r_4} \end{aligned}$$

a necessary and sufficient condition that the lines intersect is [*crc:355*]

$$[(\boldsymbol{r_1} - \boldsymbol{r_3})\ \boldsymbol{r_2}\ \boldsymbol{r_4}] = (\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \boldsymbol{r_2} \times \boldsymbol{r_4} = 0$$

The equation of the plane holding the two intersection lines is found from the normal, which is $\boldsymbol{r_2} \times \boldsymbol{r_4}$ and the intersecting point.

If the two lines are given by the equations

$$\begin{aligned} x &= mz + b \qquad & y &= nz + c \\ x &= m_1 z + b_1 \qquad & y &= n_1 z + c_1 \end{aligned}$$

then the lines will intersect if [*schmall:293:308*]

$$\frac{b_1-b}{m-m_1}=\frac{c_1-c}{n-n_1}$$

Intersection of two planes

Let the two planes be given by the equations [*drmath*] [*osgood:477:502*]

$$\begin{aligned} P_1 &= A_1 x+B_1 y+C_1 z+D_1=0 \\ P_2 &= A_2 x+B_2 y+C_2 z+D_2=0 \end{aligned}$$

The line of intersection is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

where

$$a=\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \qquad b=\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \qquad c=\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

$$\mu = a^2+b^2+c^2$$

$$\mu x_1 = b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}, \quad \mu y_1 = c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix},$$
$$\mu z_1 = a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}$$

If $a=b=c=0$, then the planes are parallel.

Intersection of three planes

The equations of the three planes are

$$\begin{aligned} A_1 x+B_1 y+C_1 z+D_1 &= 0 \\ A_2 x+B_2 y+C_2 z+D_2 &= 0 \\ A_3 x+B_3 y+C_3 z+D_3 &= 0 \end{aligned}$$

and one must solve these simultaneous equations for the point (x, y, z). [*salmon:21:46*]

Intersection of four planes

For the four planes

$$\begin{aligned} A_1 x+B_1 y+C_1 z+D_1 &= 0 \\ A_2 x+B_2 y+C_2 z+D_2 &= 0 \\ A_3 x+B_3 y+C_3 z+D_3 &= 0 \\ A_4 x+B_4 y+C_4 z+D_4 &= 0 \end{aligned}$$

to meet in a point, we must have [*salmon:21:46*]

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0$$

Intersection of a line and plane

Let $\boldsymbol{r} = \boldsymbol{r}_l + t\boldsymbol{n}_l$ be the equation of a line where $\boldsymbol{r}_l$ is a point on the line and $\boldsymbol{n}_l$ is a vector of the direction cosines of the line. Let $\boldsymbol{n}_p \cdot (\boldsymbol{r} - \boldsymbol{r}_p) = 0$ be the equation of a plane where $\boldsymbol{n}_p$ is the vector of the direction cosines of the normal to the plane and $\boldsymbol{r}_p$ is a point in the plane. Then the intersection point is $\boldsymbol{r}_i$ given by

$$\boldsymbol{r}_i = \boldsymbol{r}_l + G\boldsymbol{n}_l$$

where

$$G = \frac{\boldsymbol{n}_p \cdot (\boldsymbol{r}_p - \boldsymbol{r}_l)}{\boldsymbol{n}_l \cdot \boldsymbol{n}_p}$$

G won't exist if the plane is parallel to the line (i.e., the plane's normal is perpendicular to the line).

Another formulation [*wilson:107:134*]: let the equation of the plane be

$$\boldsymbol{r} \cdot \boldsymbol{n} = \mu$$

where μ is a constant and let us have a line that passes through the point $\boldsymbol{r}_0$ and is parallel to the vector $\boldsymbol{A}$; the equation is

$$\boldsymbol{A} \times (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

Then the point of intersection is given by

$$\frac{(\boldsymbol{A} \times \boldsymbol{r}_0) \times \boldsymbol{n} + \mu \boldsymbol{A}}{\boldsymbol{A} \cdot \boldsymbol{n}}$$

This is simpler if $\boldsymbol{n}$ and $\boldsymbol{A}$ are unit vectors

$$(\hat{\boldsymbol{A}} \times \boldsymbol{r}_0) \times \hat{\boldsymbol{n}} + \mu \hat{\boldsymbol{A}}$$

Angle between two lines in space

Suppose line L_1 is given by $\boldsymbol{r}_1 + s\boldsymbol{v}_1$ and L_2 is given by $\boldsymbol{r}_2 + t\boldsymbol{v}_2$. Then the angle between them (if necessary, they must be translated so as to intersect) is

$$\cos\theta = \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{v_1 v_2}$$

If the two lines are characterized by vectors of their direction cosines $\boldsymbol{d_1} = (\alpha_1, \beta_1, \gamma_1)$ and $\boldsymbol{d_2} = (\alpha_2, \beta_2, \gamma_2)$, then we have

$$\cos\theta = \boldsymbol{d_1}\cdot\boldsymbol{d_2}$$

Angle between a line and a plane

Given a plane $Ax+By+Cz+D=0$ and the line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

The angle θ between the line and the plane is the complement of the angle between the line and the normal to the plane:

$$\sin\theta = \frac{aA+bB+cC}{\sqrt{A_2+B^2+C^2}\sqrt{a^2+b^2+c^2}}$$

Angle between two planes

Given two planes with normals $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$. The angle θ between these two planes is given by

$$\cos\theta = \frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{n_1 n_2}$$

Parallel planes

If we have two planes

$$\begin{aligned} A_1 x+B_1 y+C_1 z+D_1 &= 0 \\ A_2 x+B_2 y+C_2 z+D_2 &= 0 \end{aligned}$$

the condition that the planes are parallel is

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

If you convert to Hessian normal form, you can see that parallel planes will have the same direction cosines in their equations. Equivalently, the dot product of their unit normal vectors is unity

$$\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2 = 1$$

Perpendicular planes

Two planes are perpendicular if the dot product of their unit normal vectors is zero:

$$\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2 = 0$$

Plane through a point parallel to two vectors

The plane parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$ and through $\boldsymbol{r_0}$ is [*coffin:232:261*]

$$[(\boldsymbol{r}-\boldsymbol{r_0})\ \boldsymbol{a}\ \boldsymbol{b}] = 0$$

If we're given the direction cosines $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ of the two vectors then the equation is

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0$$

Plane through a point parallel to another plane

The plane through $\boldsymbol{r}_1$ parallel to the plane of $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ is [*crc:356*]

$$\boldsymbol{r} = \boldsymbol{r}_1 + s\boldsymbol{r}_2 + t\boldsymbol{r}_3$$

or

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot \boldsymbol{r}_2 \times \boldsymbol{r}_3 = 0$$

or

$$[(\boldsymbol{r} - \boldsymbol{r}_1)\ \boldsymbol{r}_2\ \boldsymbol{r}_3] = [\boldsymbol{r}\ \boldsymbol{r}_2\ \boldsymbol{r}_3] - [\boldsymbol{r}_1\ \boldsymbol{r}_2\ \boldsymbol{r}_3] = 0$$

Plane through a line parallel to another line

Let the lines have direction cosines $\boldsymbol{\Gamma}_i = (\alpha_i, \beta_i, \gamma_i)$ and pass through $\boldsymbol{r}_i = (x_i, y_i, z_i)$ for i = 1, 2. The equations of the planes containing one line and parallel to the other line are [*salmon:31:56*]

$$(x-x_i)(\beta_1\gamma_2-\beta_2\gamma_1)+(y-y_i)(\gamma_1\alpha_2-\gamma_2\alpha_1)$$
$$+(z-z_i)(\alpha_1\beta_2-\alpha_2\beta_1)=0 \quad \text{for i = 1, 2}$$

or, in vector form,

$$(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{\Gamma}_1\times\boldsymbol{\Gamma}_2=0 \quad \text{for i = 1, 2}$$

The perpendicular distance between these two planes is

$$\frac{(\boldsymbol{r}_1-\boldsymbol{r}_2)\cdot\boldsymbol{\Gamma}_1\times\boldsymbol{\Gamma}_2}{\sin\theta}$$

where θ is the angle between the lines

$$\sin\theta=\sqrt{1-\boldsymbol{\Gamma}_1\cdot\boldsymbol{\Gamma}_2}$$

Plane through a given point and line

From the parametric equations for the line, obtain two points P1 and P2. Then with the given point, find the plane from the formulas for a plane from noncollinear points above.

From [*schmall:296:311*]. Let the point be (x_2, y_2, z_2) and the line be given by

$$\frac{x-x_1}{A_1}=\frac{y-y_1}{B_1}=\frac{z-z_1}{C_1}$$

The required plane is $Ax+By+Cz+D=0$. Then, since the required plane must pass through the given point and through the given line, we have the four equations

$$\begin{aligned}Ax+By+Cz+D&=0\\Ax_2+By_2+Cz_2&=D\\AA_1+BB_1+CC_1&=0\\Ax_1+By_1+Cz_1&=D\end{aligned}$$

Eliminate *A*, *B*, *C*, and *D* from these equations to get the equation of the required plane.

Plane through a point and normal to a vector

The plane normal to $\boldsymbol{n}$ and passing through $\boldsymbol{r_0}$ is

$$\boldsymbol{n}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)=0$$

Line perpendicular to a plane containing three points

Suppose we have three noncollinear points $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r_3}$. The equation of the line perpendicular to the plane containing these points is [*crc:355*]

$$\boldsymbol{r}=\boldsymbol{r}_1\times\boldsymbol{r}_2+\boldsymbol{r}_2\times\boldsymbol{r}_3+\boldsymbol{r}_3\times\boldsymbol{r}_1$$

Line perpendicular to two lines

If we have two lines with direction cosines $\boldsymbol{\Gamma_1}=(\alpha_1,\beta_1,\gamma_1)$ and $\boldsymbol{\Gamma_2}=(\alpha_2,\beta_2,\gamma_2)$ and we want the direction cosines $\boldsymbol{\Gamma}=(\alpha,\beta,\gamma)$ of the line perpendicular to both, we have [*salmon:8:33*]

$$\Gamma=\frac{\Gamma_1\times\Gamma_2}{\sin\theta}=\frac{\Gamma_1\times\Gamma_2}{\sqrt{1-\Gamma_1\cdot\Gamma_2}}$$

where θ is the angle between the two lines. Written out in components, this is

$$\begin{aligned}\alpha\sin\theta&=\beta_1\gamma_2-\beta_2\gamma_1\\ \beta\sin\theta&=\gamma_1\alpha_2-\gamma_2\alpha_1\\ \gamma\sin\theta&=\alpha_1\beta_2-\alpha_2\beta_1\end{aligned}$$

Distance of a plane to the origin

Suppose we have three noncollinear points $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r_3}$. The distance of the plane containing these three points from the origin is [*crc:355*]

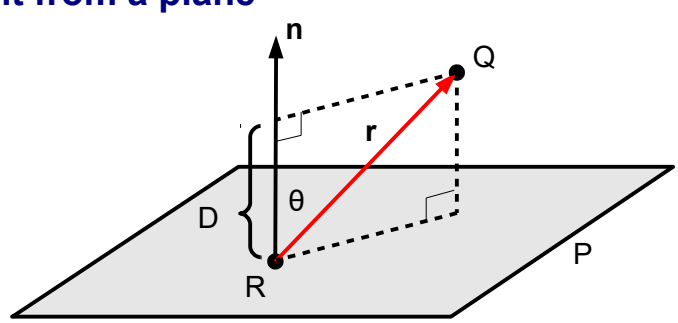
$$d=\frac{\boldsymbol{r}_1\cdot(\boldsymbol{r}_2\times\boldsymbol{r}_3)}{(\boldsymbol{r}_2-\boldsymbol{r}_1)\times(\boldsymbol{r}_3-\boldsymbol{r}_1)}$$

Distance of a point to a line

Given the direction cosines $\Gamma=(\alpha,\beta,\gamma)$ and a line through $\boldsymbol{r_1}$ in this direction. Then the distance *d* from a point $\boldsymbol{r_2}$ to the line is [*drmath*]

$$d=\sqrt{(\Gamma\times(\boldsymbol{r}_2-\boldsymbol{r}_1))\cdot(\Gamma\times(\boldsymbol{r}_2-\boldsymbol{r}_1))}$$

Distance of a point from a plane



Let $Q=(x_0,y_0,z_0)$ and $\boldsymbol{n}$ be a normal to the plane. The plane has the equation $ax+by+cz+d=0$ and it does not contain the point *Q*. Now $\boldsymbol{r}=(x-x_0,y-y_0,z-z_0)$. Then

$$D = r\,|\cos\theta| = \frac{|\boldsymbol{n}\cdot\boldsymbol{r}|}{|\boldsymbol{n}|} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a_2+b^2+c^2}}$$

If D is positive, the point and the origin are on opposite sides of the plane.

Distance between two non-intersecting lines

Let the direction cosines of the two lines be $\boldsymbol{\Gamma_1} = (\alpha_1, \beta_1, \gamma_1)$ and $\boldsymbol{\Gamma} = (\alpha_2, \beta_2, \gamma_2)$ and let the lines pass through the points (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. The shortest distance between the lines is

$$d = \alpha(x_1-x_2)+\beta(y_1-y_2)+\gamma(z_1-z_2)$$

where

$$\frac{\alpha}{\beta_1\gamma_2-\gamma_1\beta_2} = \frac{\beta}{\gamma_1\alpha_2-\gamma_2\alpha_1} = \frac{\gamma}{\alpha_1\beta_2-\beta_1\alpha_2} = \frac{\pm 1}{\sin\theta}$$

where $\boldsymbol{\Gamma} = (\alpha, \beta, \gamma)$ are the direction cosines of the shortest line connecting the two lines and θ is the angle between the two lines. We have

$$\boldsymbol{\Gamma} = \frac{\boldsymbol{\Gamma_1}\times\boldsymbol{\Gamma_2}}{\sin\theta} = \frac{\boldsymbol{\Gamma_1}\times\boldsymbol{\Gamma_2}}{\sqrt{1-\boldsymbol{\Gamma_1}\cdot\boldsymbol{\Gamma_2}}}$$

where θ is the angle between the lines. In components, this is

$$\begin{aligned}\alpha \sin\theta &= \beta_1\gamma_2 - \beta_2\gamma_1\\ \beta \sin\theta &= \gamma_1\alpha_2 - \gamma_2\alpha_1\\ \gamma \sin\theta &= \alpha_1\beta_2 - \alpha_2\beta_1\end{aligned}$$

The distance is [*drmath*]

$$\pm\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ \alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{vmatrix}}{\sqrt{\begin{vmatrix}\beta_1 & \gamma_1\\ \beta_2 & \gamma_2\end{vmatrix}^2+\begin{vmatrix}\gamma_1 & \alpha_1\\ \gamma_2 & \alpha_2\end{vmatrix}^2+\begin{vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{vmatrix}^2}}$$

The lines intersect **iff** the numerator is zero.

Circle in space

[http://mathforum.org/library/drmath/view/63755.html]

A parametric equation for a circle in space can be found from:

$$\begin{aligned}\hat{\boldsymbol{n}} &= \text{unit normal vector for the plane containing the circle}\\ \boldsymbol{C} &= \text{circle center}\\ r &= \text{circle radius}\\ \hat{\boldsymbol{u}} &= \text{unit vector from C toward a point on the circle}\\ \hat{\boldsymbol{v}} &= \hat{\boldsymbol{n}}\times\hat{\boldsymbol{u}}\\ t &= \text{parameter}\end{aligned}$$

Then a point **P** is on the circle if

$$\boldsymbol{P} = \boldsymbol{C} + r\left(\hat{\boldsymbol{u}}\cos t + \hat{\boldsymbol{v}}\sin t\right)$$

Vectors

First degree vector equation

[*wilson:90:117*] An equation of the first degree in an unknown vector ***r*** has each term as a vector quantity containing the unknown vector not more than once. An example is

$$\boldsymbol{a}\times\boldsymbol{r}+\boldsymbol{b}\cdot\boldsymbol{c}\,\boldsymbol{r}+\alpha\,\boldsymbol{r}-\boldsymbol{F}=0$$

The equation may be solved for $\boldsymbol{r}$ by dotting it successively with three noncoplanar vectors to get three scalar equations. If the equation is of the form

$$\boldsymbol{A}\,\boldsymbol{a}\cdot\boldsymbol{r}+\boldsymbol{B}\,\boldsymbol{b}\cdot\boldsymbol{r}+\boldsymbol{C}\,\boldsymbol{c}\cdot\boldsymbol{r}=\boldsymbol{D}$$

then one can dot multiply by $\boldsymbol{A'}$, $\boldsymbol{B'}$, and $\boldsymbol{C'}$ where the prime denotes the reciprocal basis vectors to ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) (of course, they must be noncoplanar). Then one will get

$$\boldsymbol{r}=\boldsymbol{D}\cdot\boldsymbol{A'}\,\boldsymbol{a'}+\boldsymbol{D}\cdot\boldsymbol{B'}\,\boldsymbol{b'}+\boldsymbol{D}\cdot\boldsymbol{C'}\,\boldsymbol{c'}$$

which is the solution.

Other terms in the equation may be of the form $\alpha\,\boldsymbol{r}$ or $\boldsymbol{E}\times\boldsymbol{r}$. These can be expressed as [*wilson:91:118*]

$$\alpha\,\boldsymbol{r}=\alpha\,\boldsymbol{a'}\,\boldsymbol{a}\cdot\boldsymbol{r}+\boldsymbol{b'}\,\boldsymbol{b}\cdot\boldsymbol{r}+\boldsymbol{c'}\,\boldsymbol{c}\cdot\boldsymbol{r}$$
$$\boldsymbol{E}\times\boldsymbol{r}=\boldsymbol{E}\times\boldsymbol{a'}\,\boldsymbol{a}\cdot\boldsymbol{r}+\boldsymbol{E}\times\boldsymbol{b'}\,\boldsymbol{b}\cdot\boldsymbol{r}+\boldsymbol{E}\times\boldsymbol{c'}\,\boldsymbol{c}\cdot\boldsymbol{r}$$

By such means, one will be able to reduce the equation to the form

$$\boldsymbol{L}\,\boldsymbol{a}\cdot\boldsymbol{r}+\boldsymbol{M}\,\boldsymbol{b}\cdot\boldsymbol{r}+\boldsymbol{N}\,\boldsymbol{c}\cdot\boldsymbol{r}=\boldsymbol{K}$$

and this can solved as above using the reciprocal basis (see page 104) as

$$\boldsymbol{r} = \boldsymbol{K}\cdot\boldsymbol{L}'\boldsymbol{a}'+\boldsymbol{K}\cdot\boldsymbol{M}'\boldsymbol{b}'+\boldsymbol{K}\cdot\boldsymbol{N}'\boldsymbol{c}'$$

Invariance

Vectors that represent physical quantities need to be invariant under common coordinate transformations. You can mentally picture the vector as an arrow (either as a free vector or attached to a certain point, such as in a torque). Thus, for example, imagine you're applying a torque to a lug nut with a socket wrench. A second person must measure the same force and lever arm and calculate the same torque you're applying, regardless from where he views you.

A way to view this is that the torque vector has an "existence" independent of the coordinate system that has been put on the space. Thus, we would only want to use mathematical constructs and models that have the same behavior. The fact that the torque is independent of the observer can only be found by experimentation, not deduced from first principles.

Another example is the angular momentum of a car's tire driving away from you. If you're at the origin of a right-handed coordinate system and the car is driving down the +x axis, the angular momentum vector of a tire points down the +y direction. If you switch to a left-handed coordinate system, the angular momentum vector still has to point in the same direction. Since the angular momentum is defined as a cross product and a cross product is a pseudovector, it has this requisite behavior. In other words, the physical arrow in space representing the angular momentum doesn't care what coordinate system it's in.

This invariance applies to the usual Euclidean transformations of everyday experience. Such invariances can lead to discovering quantities that don't change under transformations (i.e., conserved quantities). Examples from mechanics are that time translation invariance can imply energy conservation, invariance under spatial translations implies linear momentum conservation, and invariance under rotation implies angular momentum conservation. This is detailed in one of the most beautiful theorems in applied math/physics: Noether's theorem (see [*gf:81*]).

There are other cases where you can envision the vector being attached to the space. An example would be making measurements of an object by photographing it. The position vectors you measure will be affected by distortions from the imaging system. If you used a photograph to measure the lever arm of the torque above, the perspective transformation (and lens distortion effects) would be inherent in the physical measurement. Clearly, the calculated torque is not an invariant measured value in this case -- we assume it is a physical invariant, but our measurement methods include some distortion and you'll possibly get different values when you make the measurement from different points in space.

[*coffin:22:51*] If we have a linear relation amongst some vectors:

$$\sum_i \mu_i \boldsymbol{a}_i = 0$$

then this relationship will be invariant to translations **iff**

$$\sum_i \mu_i = 0$$

This is easily seen by adding an arbitrary vector $\boldsymbol{r}$ to each $\boldsymbol{a}_i$. Thus, for example [*coffin:24:53*], if

$$\begin{aligned} x\,\boldsymbol{a}+y\,\boldsymbol{b}+z\,\boldsymbol{c}+w\,\boldsymbol{d} &= 0 \\ x+y+z+w &= 0 \end{aligned}$$

then the four vectors lie in the same plane.

Spheres

The general equation of a sphere is [*drmath*]

$$x^2+y^2+z^2+2d\,x+2e\,y+2f\,z+m = 0$$

The center is $(-d, -e, -f)$ and the radius is $r = \sqrt{(d^2+e^2+f^2-m)}$.

The equation of a sphere at the origin with radius R is [*coffin:232:261*]

$$\boldsymbol{r}\cdot\boldsymbol{r} = R^2$$

The equation of a sphere with center at $\boldsymbol{r_1}$ and radius R is [*crc:357*]

$$(\boldsymbol{r}-\boldsymbol{r_1})\cdot(\boldsymbol{r}-\boldsymbol{r_1}) = R^2$$

The equation of a sphere having points $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ as the endpoints of a diameter is [*crc:357*]

$$(\boldsymbol{r}-\boldsymbol{r_1})\cdot(\boldsymbol{r}-\boldsymbol{r_2}) = 0$$

Four point form [*drmath*]

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

For the plane $\boldsymbol{r}\cdot\boldsymbol{r_1}$ = s to be tangent to the sphere $(\boldsymbol{r} - \boldsymbol{r_2})\cdot(\boldsymbol{r} - \boldsymbol{r_2}) = R^2$, we must have [*crc:357*]

$$(s-\boldsymbol{r_1}\cdot\boldsymbol{r_2})\cdot(s-\boldsymbol{r_1}\cdot\boldsymbol{r_2}) = r_1^2 R^2$$

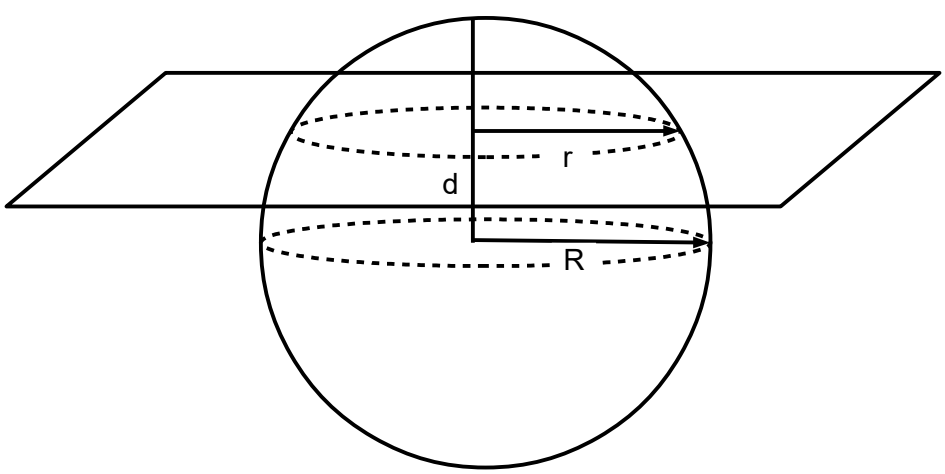
The equation of the tangent plane at $\boldsymbol{r_2}$ on the surface of sphere $(\boldsymbol{r} - \boldsymbol{r_1})\cdot(\boldsymbol{r} - \boldsymbol{r_1}) = R^2$ is *[crc:357]*

$$(\boldsymbol{r}-\boldsymbol{r}_2)\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)=0$$

The area of the surface of a sphere of radius *r* illuminated by a point source a distance *a* away from the surface of the sphere is [*hawkes:491:210*]

$$\frac{2\pi\, a r^2}{a+r}$$

If a hole of length *L* is drilled through a sphere, the remaining volume of material is independent of the radius of the sphere [*old puzzle*].



If a sphere of radius R is intersected by a plane, the result is a circle of radius r or a point. The circle is a great circle if the plane intersects the sphere's center. If the normal distance to the plane

from the center of the sphere is d, then $R^2 = r^2 + d^2$.

Miscellaneous

Center of mass

The vector $\boldsymbol{r}$ to the center of masses m_i at the points $\boldsymbol{r}_i$ is

$$\boldsymbol{r} = \frac{\Sigma m_i \boldsymbol{r}_i}{\Sigma m_i}$$

Projections

Isometric

Ref. http://en.wikipedia.org/wiki/Isometric_projection

The following is the isometric transformation for transforming a point $\boldsymbol{r}$ in the first octant to a two dimensional point (b_x, b_y) in the xy plane:

$$\begin{bmatrix} b_x \\ b_y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 & -\sqrt{3} \\ 1 & 2 & 1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\alpha = \tan^{-1}\frac{1}{\sqrt{2}} = \sin^{-1}\left(\tan\frac{\pi}{6}\right) \approx 35.26\,^\circ$ and $\beta = \frac{\pi}{4}$. The leading matrix of 1's and 0's is an orthographic transformation to the xy plane.

The transformation matrix is thus

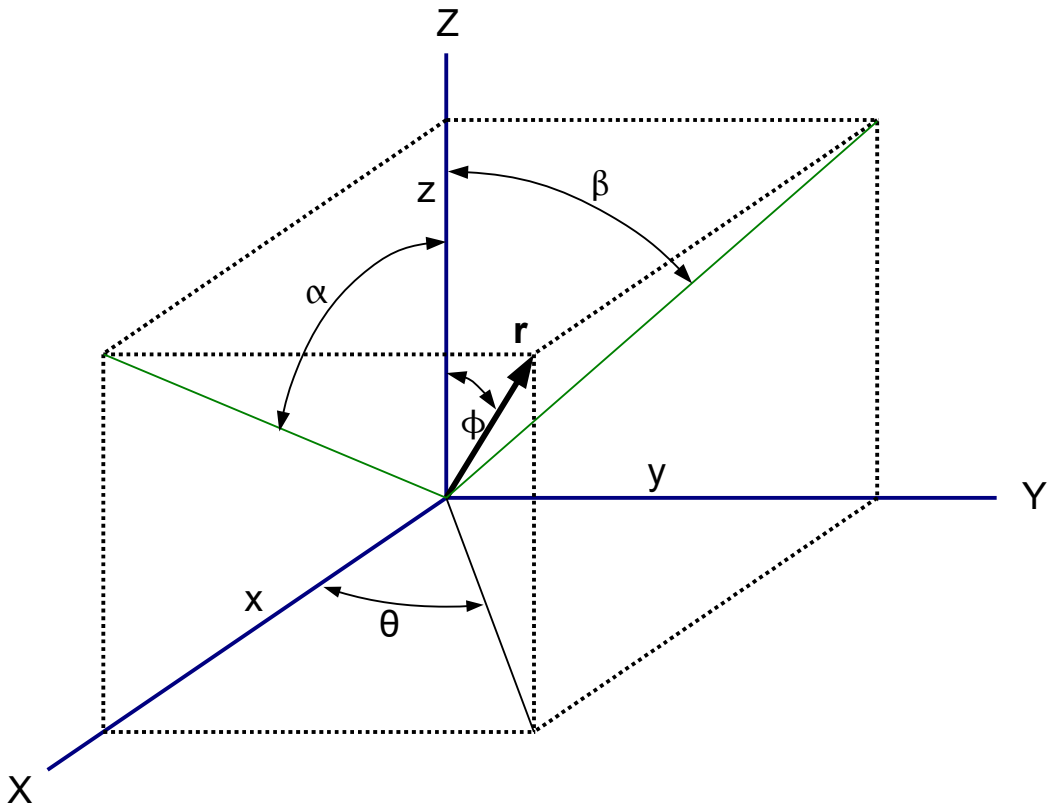
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{6}} \\ 0 & 0 & 0 \end{bmatrix}$$

The determinant is zero, which makes sense because it's a many-to-one transformation and, thus, not invertible. The transformation equations are

$$b_x = \frac{x - z}{\sqrt{2}}$$

$$b_y = \frac{x + 2y + z}{\sqrt{6}}$$

Projection angles



In the figure, the vector **r** has the spherical coordinates (r, θ, ϕ). It also has the angles α and β that are the projection angles onto the xz and yz planes, respectively.

We have

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
x &= r\sin\phi\cos\theta \\
y &= r\sin\phi\sin\theta \\
z &= r\cos\phi \\
x &= z\tan\alpha \\
y &= z\tan\beta
\end{aligned}
$$

From

$$
\begin{aligned}
x &= r\sin\phi\cos\theta = z\tan\alpha = r\cos\phi\tan\alpha \\
y &= r\sin\phi\sin\theta = z\tan\beta = r\cos\phi\tan\beta
\end{aligned}
$$

we get

$$
\begin{aligned}
\tan\alpha &= \tan\phi\cos\theta \\
\tan\beta &= \tan\phi\sin\theta
\end{aligned}
\tag{4}
$$

with inverses

$$
\begin{aligned}
\tan\theta &= \frac{\tan\beta}{\tan\alpha} \\
\tan^2\phi &= \tan^2\alpha+\tan^2\beta
\end{aligned}
\tag{5}
$$

These relationships are useful in the shop to document how something is constructed. You can

use sine sticks (see below) to measure the angles of a feature from two orthogonal directions and reconstruct the direction in space. This also can be done by taking a photograph from the two orthogonal directions and measuring the angles from the pictures. It's best to back away from the object and use a telephoto lens to minimize added distortions.

[*schmall:295:310*] The green lines in the figure above are called *traces* of the plane; i.e., the lines of intersection of the plane with the coordinate planes. If the plane's equation is

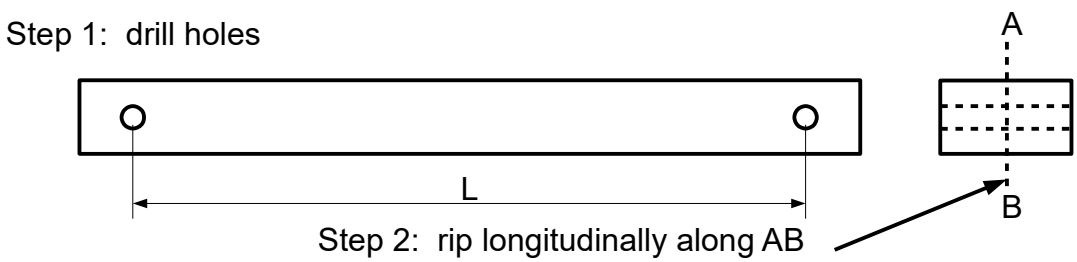
$$Ax + By + Cz = D$$

then the equations of the traces are

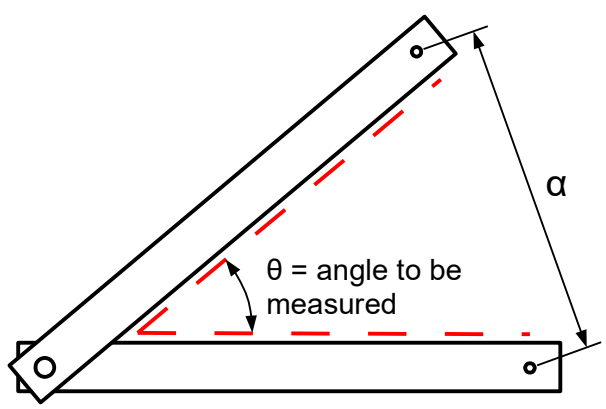
$$\begin{aligned} Ax + By &= D \\ By + Cz &= D \\ Ax + Cz &= D \end{aligned}$$

Sine sticks

A simple-to-make device can be used to measure projections angles and, thus, measure angles in space.



Take a board and drill two holes through it a distance L apart. Then rip the board in half longitudinally to get two identical pieces. Join the two boards with a snug-fitting bolt or rivet through the holes so that the two boards can pivot about the hole center. Tighten the bolt enough to let the sticks' angle be adjusted but held in place with normal handling.



To use this device, adjust the edges on the two boards to coincide with an angle.

Measure the distance α and calculate the angle θ by

$$\theta = 2\sin^{-1}\frac{\alpha}{2L}$$

On a practical note, a 12 inch or larger metal folding rule is convenient to make such an angle measuring device (one such rule was the Lufkin blacksmith's rule). Just superimpose the two legs, clamp, and drill a small hole through both legs -- these holes will automatically be at the same distance from the pivot. You can then carefully calibrate the device to a right angle to determine the distance L. In use, dividers can be used to set or measure the distance α. The author has seen such a rule from the early 1900's that had such marks and a table to help measure angles.

Some reflection will show you this angle measuring device can be surprisingly accurate. For an angle of 60°, if you made a measurement error of 0.02 inches for an L of 10 inches, you'll see that this affects the angle by 0.13°. Thus, ordinary shop work can be done to a tenth of a degree without undue effort.

For closer work, these sine sticks can be made from metal such as some 0.25" by 0.5" aluminum rectangular bar stock. Drill and ream the holes, then press in (or use Loctite) some pins with center marks located on a lathe. Using trammels, you should be able to measure the separation's α to 0.005" or less. The above 60° angle will be in error by 0.033°, which is nearly 3 times better than the usual machinist's vernier protractor can read (which is typically 5 minutes of arc). Even more accurate measurements could be made by using a sine bar and using the sine sticks as an angle transfer device.

Rotations

See "Rotation matrix" at http://en.wikipedia.org/wiki/Rotation_representation_(mathematics) and http://en.wikipedia.org/wiki/Rotation_matrix. Matrix representations of SO(n) have a determinant of +1 and are orthogonal matrices, meaning their transpose is equal to the inverse. Orthogonal matrices have the defining properties that their column vectors (or row vectors) form an orthonormal set.

For n > 2, the group is non-Abelian (i.e., rotation matrices do not commute). This is easy to see from the infinitesimal generators; e.g., for n = 3, calculate the commutator for $y\partial_x - x\partial_y$ and $z\partial_x - x\partial_z$.

> For less mathematically-inclined folks, you can demonstrate this behavior by holding a book in your hands at waist level. Establish a 3D Cartesian coordinate system for the book, then perform two rotations about two perpendicular axes. Note the book's orientation. Then repeat the rotations but in the opposite order. In general, the two rotations result in different orientations of the book.
>
> This is one of the easiest examples I can think of to introduce people to the notion of abstraction in mathematics, how mathematicians have studied the behavior of these more abstract objects (rotations), and let them contact an idea of "higher mathematics" (I guess it's called "higher" because if you fall out of the tree, you're more likely to break something). Grokking that the notion of the abstract special orthogonal group in three dimensions is different from the *representations* of that group is another plateau of knowledge and points to the mathematics as extracting the essence of something and removing all the fluff. This can lead to beautiful theorems that can excite our cognition -- but it's a shame that these great ideas are unknown to most people.

The following matrices are SO(3) representations and represent rotations about the x, y, and z axes in 3-space:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that if $\theta = \pi/2$, the R_x takes $\hat{\boldsymbol{j}}$ into $\hat{\boldsymbol{k}}$, R_y takes $\hat{\boldsymbol{i}}$ into $\hat{\boldsymbol{k}}$, and R_z takes $\hat{\boldsymbol{i}}$ into $\hat{\boldsymbol{k}}$.

Other rotation matrices can be gotten from these three using matrix multiplication. Thus, $R_x(\gamma)R_y(\beta)R_z(\alpha)$ represents a rotation whose yaw, pitch, and roll are α, β, and γ, respectively. Similarly, $R_z(\gamma)R_x(\beta)R_z(\alpha)$ represents a rotation whose Euler angles are α, β, and γ.

For using Euclidean transformations, it is handy to use homogeneous coordinates: the general position vector is (x, y, z, 1). The transformation matrices are then

$$T(a,b,c) = \text{translation} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2068

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

One can also write the rotation matrix as the direction cosine matrix. Suppose we have three unit vectors $\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}$ as the basis of the rotated system

$$R = \begin{bmatrix} \hat{\boldsymbol{u}}_x & \hat{\boldsymbol{v}}_x & \hat{\boldsymbol{w}}_x \\ \hat{\boldsymbol{u}}_y & \hat{\boldsymbol{v}}_y & \hat{\boldsymbol{w}}_y \\ \hat{\boldsymbol{u}}_z & \hat{\boldsymbol{v}}_z & \hat{\boldsymbol{w}}_z \end{bmatrix}$$

where each element is the cosine of the angle between a rotated unit basis vector and one of the reference axes. R is a real, orthogonal matrix with unity determinant and eigenvalues

$$(1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta)$$

The eigenvalue of 1 corresponds to the rotation axis, as it is the only vector unchanged by the rotation. R has three degrees of freedom; they are constrained by the relations

$$\begin{aligned} |\hat{\boldsymbol{u}}| &= |\hat{\boldsymbol{v}}| = 1 \\ \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} &= 0 \\ \hat{\boldsymbol{w}} &= \hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}} \end{aligned}$$

A rotation matrix preserves distances between points. Let $\boldsymbol{r}$ be an arbitrary vector; then $\boldsymbol{r}^T\boldsymbol{r}$ is the square of its magnitude. If R$\boldsymbol{r}$ is a rotated vector corresponding to the rotation matrix R, we have $(R\boldsymbol{r})^T(R\boldsymbol{r}) = (\boldsymbol{r}^T R^T)(R\boldsymbol{r}) = \boldsymbol{r}^T(R^T R)\boldsymbol{r} = \boldsymbol{r}^T\boldsymbol{r}$, which again is the square of $\boldsymbol{r}$'s magnitude. $R^T R$ is the identity matrix because R is orthogonal.

Rotating about a given axis

Given a unit vector $\hat{\boldsymbol{u}} = (x, y, z)$, the matrix for a rotation by an angle θ about an axis in the direction of **u** is (http://en.wikipedia.org/wiki/Rotation_matrix#Finding_the_rotation_matrix)

$$R = \begin{bmatrix} x^2 + (1 - x^2)\cos\theta & x\,y\,(1 - \cos\theta) - z\sin\theta & x\,z\,(1 - \cos\theta) + y\sin\theta \\ x\,y\,(1 - \cos\theta) + z\sin\theta & y^2 + (1 - y^2)\cos\theta & y\,z\,(1 - \cos\theta) - x\sin\theta \\ x\,z\,(1 - \cos\theta) - y\sin\theta & y\,z\,(1 - \cos\theta) + x\sin\theta & z^2(1 - z^2)\cos\theta \end{bmatrix}$$

In a normal right handed Cartesian coordinate system, this rotation will be counterclockwise for an observer placed so that the axis **u** goes in the observer's direction (i.e., the tip of the arrow that represents **u** pokes the observer in the eye).

Rodrigues' formula can be used instead:

$$R = P + (I - P)\cos\theta + Q\sin\theta$$

where

$$P = \begin{bmatrix} x^2 & x\,y & x\,z \\ x\,y & y^2 & y\,z \\ x\,z & y\,z & z^2 \end{bmatrix} = \boldsymbol{u}\,\boldsymbol{u}^T, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Q is the skew-symmetric representation of the cross product, P is the projection onto the axis of rotation, and *I* - *P* is the projection onto the plane orthogonal to the axis.

Direction cosine matrix to Euler angles

Given a 3x3 rotation matrix *R*, one can calculate the zxz Euler angles ϕ, θ, ψ (rotation around *z*, *x*, then *z*) as:

$$
\begin{aligned}
\phi &= \operatorname{atan2}(R_{31}, R_{32}) \\
\theta &= \cos^{-1} R_{33} \quad (\theta \text{ must be in } [0, \pi)) \\
\psi &= -\operatorname{atan2}(R_{13}, R_{23})
\end{aligned}
$$

If $R_{33} = 0$, ϕ and ψ should be calculated from R_{11}, R_{12} instead. atan2 is the usual atan2(y, x) found in programming languages (note the y component is the first argument).

Direction cosine matrix to Euler axis and angle

If the Euler angle θ is not a multiple of π, then the Euler axis **u** = (x, y, z) and angle θ are

$$
\begin{aligned}
\cos\theta &= \frac{trace(R) - 1}{2} \\
x &= \frac{R_{32} - R_{23}}{2\sin\theta} \\
y &= \frac{R_{13} - R_{31}}{2\sin\theta} \\
z &= \frac{R_{21} - R_{12}}{2\sin\theta}
\end{aligned}
$$

This is the eigenvector associated with the eigenvalue of 1.

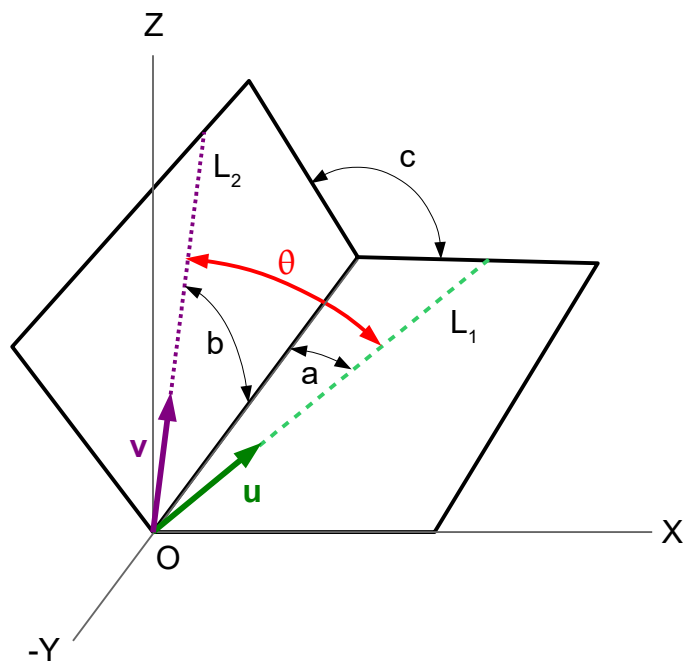
This is of use in practical problems. If you have an arbitrary SO(3) rotation matrix R that is the product of a number of rotations, you can use the above formulas to determine the axis (x, y, z) and the angle of rotation θ around that axis that is equivalent to the product of the rotations.

Rotating lines in space

(Motivation: http://www.mathforum.org/library/drmath/view/56462.html.)

Suppose you have a notebook (see figure below). Open it up and draw a line L_1 from a point O on the spine to anywhere on the right page; call the angle between the spine and the dashed green line a. On the left page, draw a line L_2 from O to anywhere; call the angle between the spine and the left dotted line b. Now, open the notebook so that the angle between the two pages is c. Then the angle θ in space between the two drawn lines L_1 and L_2 is

$$\cos\theta = \cos a \cos b + \sin a \sin b \cos c$$



Useful sub-cases are:

c = 0°: $\theta = a - b$
c = 90°: $\cos\theta = \cos a \cos b$
c = 180°: $\theta = a + b$

Derivation: suppose that the dashed and dotted lines both pass through the origin O as shown and a customary right-hand Cartesian coordinate system is put where the lines intersect, as shown. Let ***u*** be a unit vector pointing in the dashed line's direction and ***v*** be a unit vector pointing in the dotted line's direction. Then the cosine of the angle θ is

$$\cos\theta = \boldsymbol{u} \cdot \boldsymbol{v}$$

Since ***u*** lies in the xy plane, its direction cosines are easy to write down: $(\sin a, \cos a, 0)$.

To get ***v***'s direction cosines, let's start with *c* = 0. Then ***v***'s direction cosines are $(\sin b, \cos b, 0)$. Rotate the coordinate system about the y axis counterclockwise by an angle *c*. Clearly, only the x and z components will change. The y-axis rotation matrix is

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Here are the direction cosines of the rotated direction cosine vector of the dotted line L_2 up out of the xy plane by the angle *c*:

$$\begin{bmatrix} \cos c & 0 & -\sin c \\ 0 & 1 & 0 \\ \sin c & 0 & \cos c \end{bmatrix} \begin{pmatrix} \sin b \\ \cos b \\ 0 \end{pmatrix} = \begin{pmatrix} \sin b \cos c \\ \cos b \\ \sin b \sin c \end{pmatrix}$$

Now we calculate the angle θ:

$$\cos\theta = (\sin a, \cos a, 0) \cdot (\sin b \cos c, \cos b, \sin b \sin c)$$

or

$$\cos\theta = \sin a \sin b \cos c + \cos a \cos b$$

Helix

Ref. Weisstein, Eric W. "Helix." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Helix.html

A helix is a curve in space on a right circular cylinder that advances at a uniform rate as the angle around the cylinder's axis changes. A screw thread is the prototypical example. The parametric equations of a helix are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= \rho\theta \end{aligned}$$

r is the radius of the cylinder and $p = 2\pi\rho$ is the pitch of the helix. The tangent line to the helix makes a fixed angle with an arbitrary line. The helix is the shortest path between two points on a cylinder (cut the cylinder, roll it out flat, and draw a line between the two points).

Formulas:

$$\alpha = r^2 + \rho^2$$

$$\text{Arc length} = \theta\sqrt{\alpha}$$

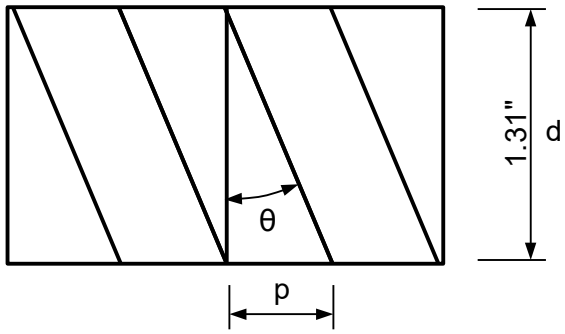
$$\text{Curvature} = \kappa = \frac{r}{\alpha}$$

$$\text{Torsion} = \tau = \frac{\rho}{\alpha}$$

$$\frac{\kappa}{\tau} = \frac{r}{\rho} = \text{constant}$$

A curve is a helix **iff** the ratio of curvature to torsion is constant.

If the following is a side view of a helical thread of diameter *d* and pitch *p*

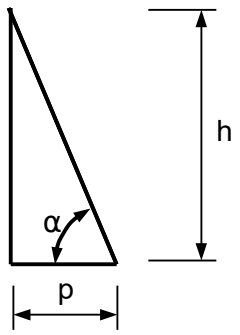


then the angle θ is given by

$$\tan\theta = \frac{p}{d}$$

Note that there are thread measures called *thread angle*, *lead angle*, and *helix angle*, but this angle θ is not the same as them. The angle θ is relevant if you use thread calipers (an old measuring tool mostly unknown to modern machinists) to measure the root diameter. Because of the lean of the thread, the calipers measure a slightly longer distance than the actual root diameter; the measured value is multiplied by the cosine of θ to correct the reading. Example: for a 3/4-10 UNC thread, the correction is on the order of 1%. Since 0.01(0.75) = 0.0075 inches is below the typical resolution gotten with a rule, virtually no one (including me) ever makes the correction. It would possibly be relevant for a high lead or multiple start thread.

The helix angle is gotten by unrolling the helix onto a plane to get the triangle

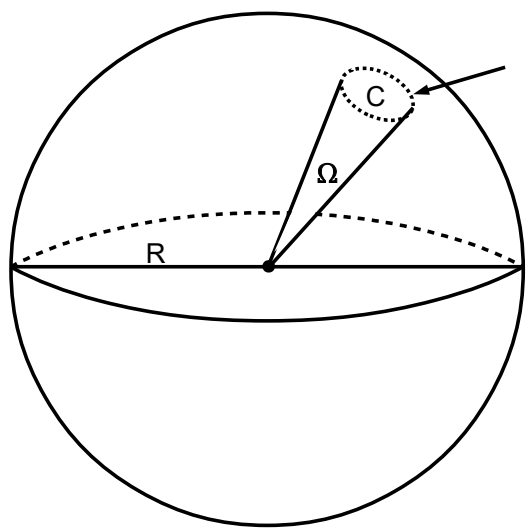


where $h = \pi d$ and p is the pitch (how much it advances along its axis in one revolution); then the helix angle α is

$$\alpha = \tan^{-1}\frac{h}{p}$$

Solid angles

For a solid angle Ω in three dimensions, we have the following picture:



A small circle C is on the surface of the sphere of radius R. The solid angle is then

$$\Omega = \frac{A}{R^2}$$

where A is the area of the spherical cap with the circle C as its border. A solid angle is a dimensionless number; it is also given a name, the *steradian*. If you make the circle C a great circle of the sphere, you can see that the solid angle is 2π and it's 4π for the whole sphere. If you make the area of the cap equal to R^2, then you have a unit solid angle. You can convert steradians to square degrees by multiplying by $(180/\pi)^2$, analogous to converting from radians to degrees by multiplying radians by $180/\pi$,

Suppose I have an area A that subtends a solid angle Ω from point S as shown in the figure below. If I rotate the area A about the axis PQ by an angle θ, the solid angle subtended by the circle from point S is reduced to $\Omega \cos\theta$.

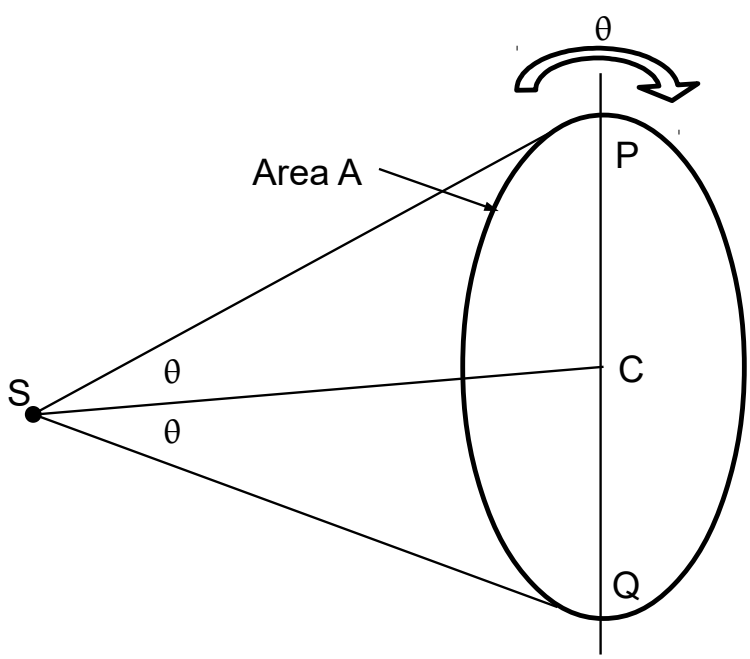


Figure 3

If θ is the angle subtended by the radius of the circle and the plane of the circle is perpendicular to SC in Figure 3, the solid angle subtended by the cone with vertex at S and intersecting the circle's

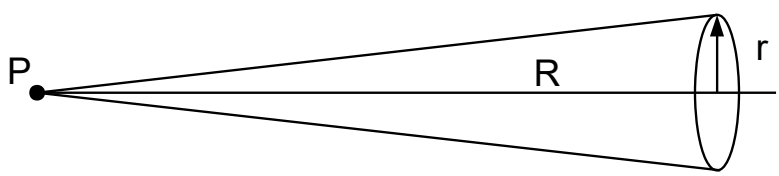
circumference is

$$\Omega = 2\pi(1-\cos\theta) = 4\pi\sin^2\frac{\theta}{2}$$

If $\theta \ll 1$, then

$$\Omega \approx \pi\theta^2$$

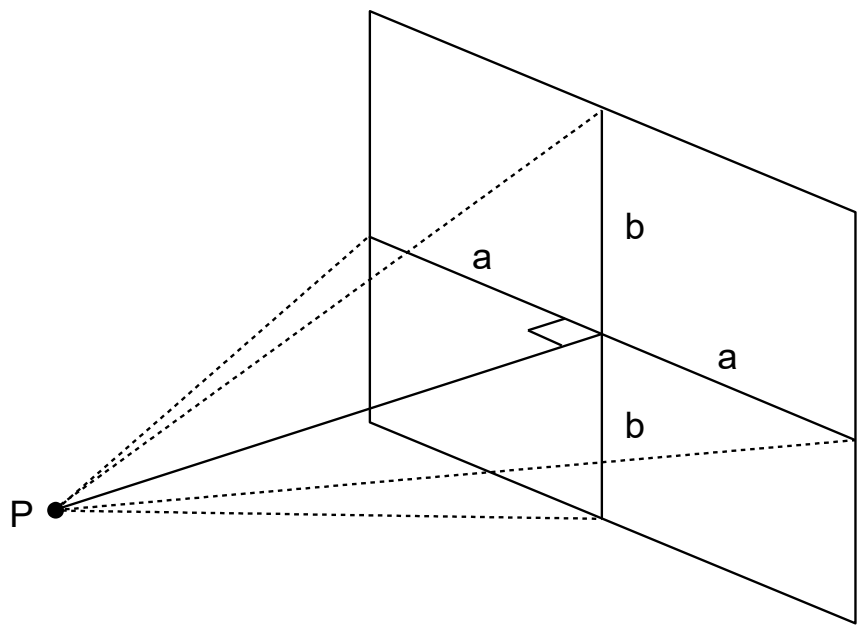
For the case where $R \gg r$ in the following figure



then the solid angle of the disk from P is

$$\Omega \approx \pi\left(\frac{r}{R}\right)^2$$

For a rectangular area with angular widths *a* and *b* as in



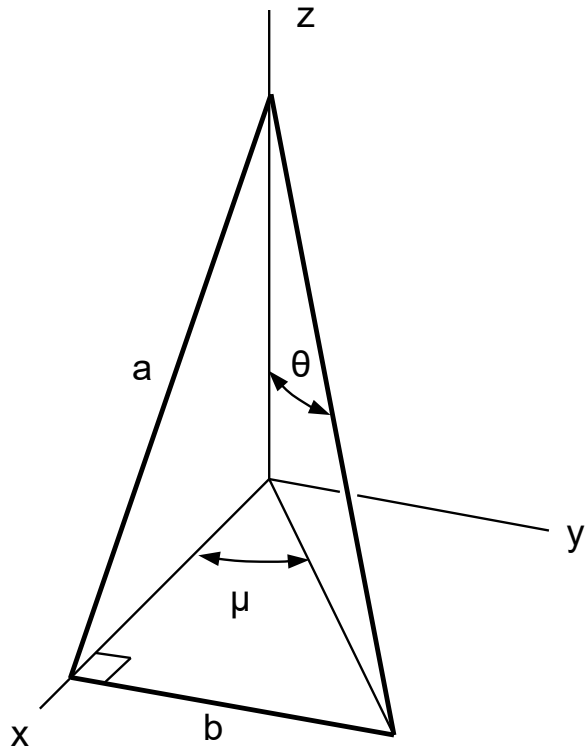
the solid angle of the rectangle from point P is

$$\Omega = 4\sin^{-1}(\sin a \sin b)$$

For a regular polygon of n sides with a circumscribed circle that subtends an angle θ at a point P normal to the polygon's plane and passing through its center, the solid angle at P is

$$\Omega = 2\pi - n\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$$

For a triangle in space specified by two angles θ and μ with one side *a* in the *xz* plane and another side *b* in the *xy* plane perpendicular to the *xz* plane



the solid angle subtended by the triangle viewed from the origin is

$$\Omega = \mu - \tan^{-1}(\cos\theta \tan\mu)$$

For a triangle in space whose three points are $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$, the solid angle subtended by the triangle as viewed from the origin is (equation 8 in [*vo*])

$$\tan\frac{\Omega}{2} = \frac{\boldsymbol{r_1}\cdot\boldsymbol{r_2}\times\boldsymbol{r_3}}{r_1 r_2 r_3 + r_1(\boldsymbol{r_2}\cdot\boldsymbol{r_3}) + r_2(\boldsymbol{r_3}\cdot\boldsymbol{r_1}) + r_3(\boldsymbol{r_1}\cdot\boldsymbol{r_2})}$$

For a spherical triangle with angles A, B, and C on a sphere of radius r, the area is [*casey:16:36*] Er^2 where $E =$ spherical excess $= \pi - (A+B+C)$. Hence, the solid angle subtended at the center of the sphere of this spherical triangle is E.

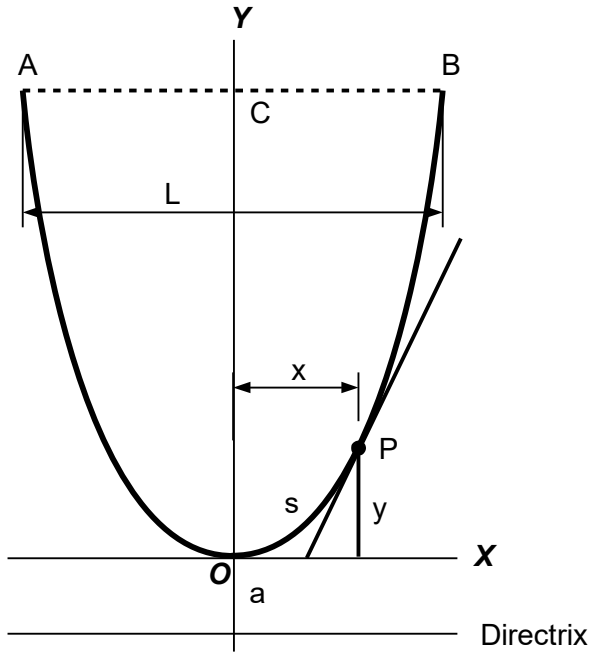
For a spherical polygon of n sides A_i, the area is [*casey:17:37*]

$$Br^2 = \left(\sum_i A_i + (2-n)\pi\right) r^2$$

and thus the solid angle is B.

Catenary

[*hudm:67:80*] If a perfectly flexible non-stretchable chain, cord, or rope of uniform linear mass density is suspended freely between two points at the same level, it will assume the shape of a catenary



Let the origin of a Cartesian coordinate system be at O at the vertex of the catenary and

(x, y) = coordinates of any point P on the catenary
s = arc length of OP
a = the parameter of the catenary = T_x/w
d = sag of catenary = CO
w = weight per unit length of the catenary = distance from vertex to directrix
L = width of catenary = AB
T = tension at any point P
T_x = the horizontal component of T = constant for all points
T_y = the vertical component of T

The catenary's equation is

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) = a\cosh\frac{x}{a}$$

When *a* is large, the curve is nearly flat. If the width of the span is *L* and the sag is *d*, then the length *s* of the arc of the catenary is gotten as follows. Calculate z numerically from

$$\cosh z = \frac{2\mathrm{d}}{L}z + 1$$

Then the arc length *s* is

$$s = \frac{1}{z}\sinh z$$

[http://mathworld.wolfram.com/Catenary.html] The parametric equations of a catenary are

$$\begin{aligned} x &= t \\ y &= \frac{1}{2}\left(e^{t/a} + e^{-t/a}\right) = a\cosh(t/a) \end{aligned}$$

where $t = 0$ corresponds to the vertex. The arc length as a function of t is $s(t) = a\sinh(t/a)$. The slope of the curve at a point is proportional to the arc length of the point from the vertex. [http://en.wikipedia.org/wiki/Catenary] Over any horizontal interval, the ratio of the area under the catenary to its arc length on that interval is a. The differential equation of the catenary curve is

$$\frac{dy}{dx} = \frac{s}{a} \quad \text{and} \quad s = a\sinh\frac{x}{a} \quad \Rightarrow \quad \frac{dy}{dx} = \sinh\frac{x}{a}$$

which is not surprising, given that the Cartesian equation of the catenary is $y = \frac{1}{2}\cosh\left(\frac{x}{a}\right)$.

For a suspension bridge with a horizontal roadway with a weight of cable supporting it (and the weight of the cable is negligible), the cable follows a parabola, not a catenary.

[*marks1916:147:176*] gives some practical tools to help solve catenary problems. Since this reference is in the public domain, I copy heavily from it. The following sections solve problems associated with the catenary where you're given one set of parameters and need to solve for the others.

Given x and y

Compute *y*/*x* and find the value of z from the following table:

Roots of $(\cosh(z)-1)/z = y/x$

Table entries are *z* values

y/x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0200	0.0400	0.0600	0.0800	0.0999	0.1199	0.1398	0.1597	0.1795
0.1	0.1993	0.2191	0.2389	0.2586	0.2782	0.2978	0.3173	0.3368	0.3562	0.3756
0.2	0.3948	0.4141	0.4332	0.4522	0.4712	0.4901	0.5089	0.5276	0.5463	0.5648
0.3	0.5833	0.6016	0.6199	0.6381	0.6561	0.6741	0.6919	0.7097	0.7274	0.7449
0.4	0.7624	0.7797	0.7969	0.8140	0.8311	0.8480	0.8648	0.8814	0.8980	0.9145
0.5	0.9308	0.9471	0.9632	0.9792	0.9951	1.0109	1.0266	1.0422	1.0577	1.0730
0.6	1.0883	1.1034	1.1184	1.1334	1.1482	1.1629	1.1775	1.1920	1.2064	1.2207

Then

$$\begin{aligned} a &= \frac{x}{z} \\ s &= a\sinh z \\ T &= w\,a\cosh z \end{aligned}$$

Example: a rope is stretched between two points 65 m apart. The cable has a linear mass density of 0.2 kg/m and the droop of the rope at the center is 1.35 m. What is the tension in the rope?

We calculate w = (0.2 kg/m)(9.8 m/s^2) = 1.96 N/m and y/x = (1.35 m)/(65/2 m) = 0.0415. From the above table, we linearly interpolate to get z = 0.0830. Then a = (65/2 m)/0.0830 = 391.6 m and the tension is

$$(1.96\ \text{N/m})(391.6\ \text{m})\cosh 0.0830 = \underline{770\ \text{N}}$$

Given x and T/w

Compute *wx*/*T* and find the value of *z* from the following table:

Roots of $z/\cosh(z) = wx/T$

Table entries are *z* values

wx/T	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0100	0.0200	0.0300	0.0400	0.0501	0.0601	0.0702	0.0803	0.0904
0.1	0.1005	0.1107	0.1209	0.1311	0.1414	0.1517	0.1621	0.1725	0.1830	0.1936
0.2	0.2042	0.2149	0.2256	0.2365	0.2474	0.2584	0.2695	0.2807	0.2920	0.3035
0.3	0.3150	0.3267	0.3385	0.3505	0.3626	0.3749	0.3873	0.4000	0.4128	0.4259
0.4	0.4392	0.4527	0.4665	0.4806	0.4950	0.5097	0.5248	0.5403	0.5562	0.5725
0.5	0.5894	0.6068	0.6249	0.6436	0.6632	0.6836	0.7050	0.7277	0.7518	0.7775
0.6	0.8053	0.8357	0.8695	0.9081	0.9540	1.0131	1.1109			

Then

$$a = \frac{x}{z}$$
$$y = a(\cosh(z) - 1)$$
$$s = a\sinh(z)$$

Example: from the example in the previous section, suppose we have a rope stretched between points 65 m apart; the rope is 0.2 kg/m and the tension in the rope at x = 65/2 m is measured at 770 N. Find the sag in the rope.

Calculate wx/T as (0.2 kg/m)(9.8 m/s^2)(65/2 m)/(770 N) = 0.0827. From the table, we linearly interpolate to get z = 0.0830 (the same value as in the previous example, as we would expect). Then a = (65/2 m)/0.0830 = 391.6 m and the sag in the rope is y = (391.6 m)(cosh(0.0830) - 1) = 1.35 m. The arc length s is 32.54 m.

Given x and s

Compute s/x and find the value of z from the following tables:

Roots of $\sinh(z)/z = s/x$

Table entries are z values

s/x	0	1	2	3	4	5	6	7	8	9
1.000	.....	0.0245	0.0346	0.0424	0.0490	0.0548	0.0600	0.0648	0.0693	0.0735
1.001	0.0774	0.0812	0.0848	0.0883	0.0916	0.0948	0.0980	0.1010	0.1039	0.1067
1.002	0.1095	0.1122	0.1149	0.1174	0.1200	0.1224	0.1249	0.1272	0.1296	0.1319
1.003	0.1341	0.1363	0.1385	0.1406	0.1428	0.1448	0.1469	0.1489	0.1509	0.1529
1.004	0.1548	0.1567	0.1586	0.1605	0.1624	0.1642	0.1660	0.1678	0.1696	0.1713
1.005	0.1731	0.1748	0.1765	0.1782	0.1799	0.1815	0.1831	0.1848	0.1864	0.1880
1.006	0.1896	0.1911	0.1927	0.1942	0.1958	0.1973	0.1988	0.2003	0.2018	0.2033
1.007	0.2047	0.2062	0.2076	0.2091	0.2105	0.2119	0.2133	0.2147	0.2161	0.2175
1.008	0.2188	0.2202	0.2215	0.2229	0.2242	0.2255	0.2269	0.2282	0.2295	0.2308
1.009	0.2321	0.2333	0.2346	0.2359	0.2372	0.2384	0.2397	0.2409	0.2421	0.2434

Roots of $\sinh(z)/z = s/x$

Table entries are z values

s/x	0	1	2	3	4	5	6	7	8	9
1.01	0.2446	0.2565	0.2678	0.2787	0.2892	0.2993	0.3091	0.3186	0.3278	0.3367
1.02	0.3454	0.3539	0.3621	0.3702	0.3781	0.3859	0.3934	0.4009	0.4082	0.4153
1.03	0.4224	0.4293	0.4361	0.4428	0.4494	0.4559	0.4623	0.4686	0.4748	0.4809
1.04	0.4870	0.4930	0.4989	0.5047	0.5105	0.5162	0.5218	0.5274	0.5329	0.5383
1.05	0.5437	0.5490	0.5543	0.5595	0.5647	0.5698	0.5749	0.5799	0.5849	0.5898
1.06	0.5947	0.5996	0.6044	0.6091	0.6139	0.6186	0.6232	0.6278	0.6324	0.6369
1.07	0.6414	0.6459	0.6504	0.6548	0.6591	0.6635	0.6678	0.6721	0.6763	0.6806
1.08	0.6848	0.6889	0.6931	0.6972	0.7013	0.7053	0.7094	0.7134	0.7174	0.7213
1.09	0.7253	0.7292	0.7331	0.7369	0.7408	0.7446	0.7484	0.7522	0.7559	0.7597
1.10	0.7634									

Then

$$a = \frac{x}{z}$$
$$y = a(\cosh(z) - 1)$$
$$T = wa\cosh(z)$$

Example: from the example in the previous section, suppose we have a rope of length 32.54 m, which is the arc length from the vertex to the point where the rope is attached at x = 65/2 m. The rope is 0.2 kg/m. Find the sag and the tension in the rope.

Enter the table with 32.54/(65/2) = 1.00123 and get z = 0.0858 using linear interpolation. Then

$$a = \frac{65/2\ \text{m}}{0.0858} = 378.8\ \text{m}$$
$$y = (378.8\ \text{m})(\cosh(0.0858) - 1) = 1.395\ \text{m}$$
$$T = (0.2\ \text{kg/m})(9.8\ \text{m/s}^2)(378.8\ \text{m})\cosh(0.0858) = 745.2\ \text{N}$$

The calculated tension is about 3% low from the 770 N and the sag is about 3% high from the numbers given in the previous example.

In a practical sense, this problem would be difficult to do with rope, as the rope would be likely to stretch, making it harder to measure the actual arc length. It would probably be best used for dealing with steel cables and chain which will have less stretch than rope, especially synthetic fiber ropes.

Given y and s

$$\frac{T}{w} = \frac{s^2}{2y} + \frac{y}{2}$$

$$x = \left(\frac{s^2}{y} - y\right)\tanh^{-1}\frac{y}{s}$$

$$\approx s\left[1 - \frac{2}{3}\left(\frac{y}{s}\right)^2 - \frac{2}{15}\left(\frac{y}{s}\right)^4 - \ldots\right] \text{ if } y/s \text{ is small}$$

$$a = \frac{s^2}{2y} - \frac{y}{2}$$

Given y and T/w

$$a = \frac{T}{w} - y$$

$$x = \left(\frac{T}{w} - y\right)\cosh^{-1}\frac{T/w}{(T/w) - y}$$

$$s = \sqrt{\frac{2yT}{w} - y^2}$$

If $\mu = \dfrac{y}{T/w}$ is small (a rope/chain with small sag and large tension or a point near the vertex),

$$x = \sqrt{\frac{2yT}{w}}\left(1 - \frac{7}{12}\mu - \ldots\right)$$

$$\frac{s-x}{s} \approx \frac{1}{3}\mu$$

$$s = \sqrt{\frac{2yT}{w}}\left(1 - \frac{1}{4}\mu - \frac{1}{32}\mu^2 - \frac{1}{128}\mu^3 - \ldots\right)$$

Given s and T/w

Let $\mu = \frac{ws}{T}$; then

$$a = \frac{T}{w}\sqrt{1-\mu^2}$$

$$x = a \tanh^{-1}\mu$$

$$y = \frac{T}{w} - a$$

If $\mu \ll 1$,

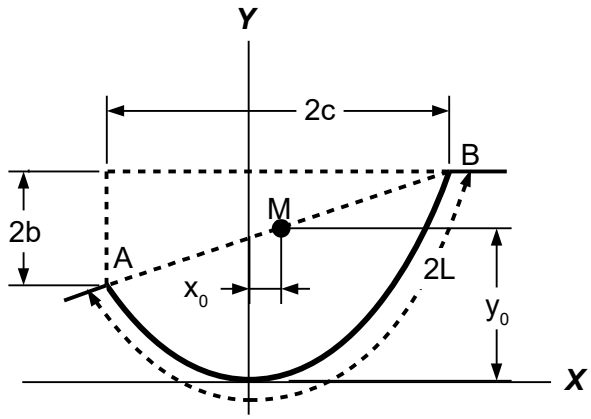
2297

$$x = s\left[1 - \frac{1}{6}\mu^2 - \frac{11}{120}\mu^4 - \ldots\right]$$

$$y = s\left[\frac{1}{2}\mu + \frac{1}{8}\mu^3 + \ldots\right]$$

$$a = \frac{T}{w}\left[1 - \frac{1}{2}\mu^2 - \frac{1}{8}\mu^4 - \ldots\right]$$

Chain supported at two points A and B not at the same level



Suppose *b* and *c* are known; find *a*. Let

$$\frac{\sqrt{L^2-b^2}}{c}=\frac{s}{x}$$

and enter the tables in the *Given x and s* section with this value of *s*/*x* and find the corresponding value of *z*. Then

$$a=\frac{c}{z}$$

The coordinates of the midpoint *M* of the line segment *AB* are

$$x_0 = a \tanh^{-1}\left(\frac{b}{L}\right)$$
$$y_0 = \frac{L}{\tanh(z)} - a$$

This midpoint is the lowest point on the catenary.

Correction for sag in chaining uphill

Also see http://en.wikipedia.org/wiki/Tape_correction_%28surveying%29#Correction_due_to_sag for a derivation of the proper formula.

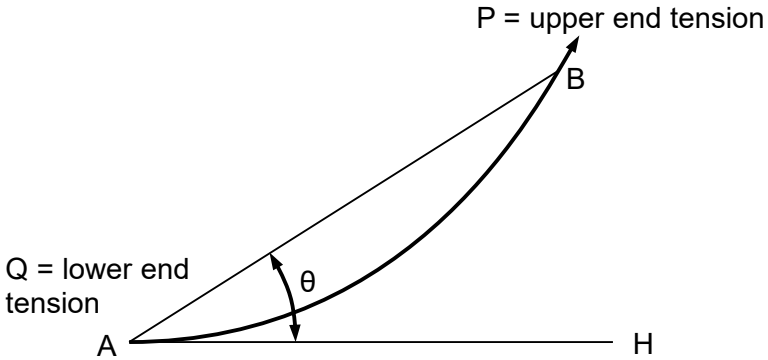


Figure 4

Let

L = the length of the tape corrected for stretch and temperature
w = weight per unit length of the tape
θ = angle between the line segment AB and the horizontal AH

If the tension P at the upper end B is known, compute wL/P and find $k = k_1$ from the following table:

Sag constant k_1

	Angle θ in degrees								
wl/P	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
0.01	0	0	0	0	0	0	0	0	0
0.02	2	2	1	1	1	1	0	0	0
0.03	4	4	3	3	2	2	1	0	0
0.04	7	6	6	5	4	3	2	1	0
0.05	11	10	9	8	6	4	3	1	0
0.06	15	15	13	12	9	6	4	2	0
0.07	20	20	18	16	12	9	5	3	1
0.08	27	26	24	21	16	12	7	3	1
0.09	34	33	31	26	21	15	9	4	1
0.10	42	41	38	33	26	19	11	5	1
0.11	51	50	46	40	32	23	14	7	2
0.12	60	60	55	48	38	27	17	8	2
0.13	70	70	65	57	45	32	20	9	2
0.14	82	81	76	66	53	38	23	11	3
0.15	94	94	87	76	61	44	27	13	3
0.16	107	107	100	87	70	50	31	15	4
0.17	121	121	113	99	79	57	35	17	4
0.18	136	136	128	112	90	65	40	19	5
0.19	151	152	143	125	101	73	45	21	6
0.20	168	168	159	140	113	82	50	24	6

If the tension Q at the lower end A is known, compute wL/Q and find $k = k_2$ from the following table:

Sag constant k_2

	Angle θ in degrees								
wl/Q	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
0.01	0	0	0	0	0	0	0	0	0
0.02	2	2	1	1	1	1	0	0	0
0.03	4	4	3	3	2	1	1	0	0
0.04	7	6	6	5	4	3	2	1	0
0.05	11	10	9	8	6	4	2	1	0
0.06	15	14	13	11	8	6	4	2	0
0.07	20	20	18	15	11	8	5	2	1
0.08	27	26	23	19	15	11	6	3	1
0.09	34	32	29	24	19	13	8	4	1
0.10	42	40	36	30	23	16	10	4	1
0.11	51	48	43	36	28	19	11	5	1
0.12	60	57	51	43	33	23	14	6	2
0.13	70	67	60	50	38	26	16	7	2
0.14	82	78	69	57	44	30	18	8	2
0.15	94	89	79	66	50	35	21	10	2
0.16	107	101	90	74	57	39	22	11	3
0.17	121	114	101	84	64	44	26	12	3
0.18	136	128	113	92	71	49	29	13	3
0.19	151	142	125	103	79	54	32	15	4
0.20	168	157	138	114	87	60	35	16	4

Then for either case, the corrected length of the tape is chord $AB = L\left(1 - 10^{-5}k\right)$.

Correction for stretch in chaining uphill

Using Figure 4 above, let

L = unstretched length of tape at working temperature
w = weight of tape per unit length
θ = angle between chord AB and the horizontal AH
F = cross-sectional area of tape
E = Young's modulus of tape material (for steel, use 29 Mpsi = 200 GPa)
λ = stretched length of tape

If the tension P at the upper end is known, compute wL/P and find m from the following table:

Uphill tape stretch constant m

Angle θ in degrees

wl/P	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
0.00	0	0	0	0	0	0	0	0	0	0
0.01	1	10	18	26	33	39	44	47	49	50
0.02	3	21	38	53	67	78	88	94	99	100

Then

$$\lambda = L\left[1+\frac{P}{FE}\left(1-\frac{m}{1000}\right)\right]$$

If the tension P at the upper end is known, compute *wL*/*P* and find *m* from the following table:

Uphill tape stretch constant *n*

Angle θ in degrees

wl/Q	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
0.00	0	0	0	0	0	0	0	0	0
0.01	8	16	24	32	38	43	47	49	50
0.02	14	31	47	62	75	86	94	99	100

Then

$$\lambda = L\left[1+\frac{Q}{FE}\left(1+\frac{n}{1000}\right)\right]$$

Practical square root calculation

Sometimes when doing shop work I don't have a calculator handy. Since I do my trigonometric work using the slope *m* (see *Slope as angle measure*), I can calculate the sine and cosine if I can evaluate $\sqrt{1+m^2}$. This usually isn't too hard for practical angles, which often have slopes of 1 or less, meaning I need to evaluate square roots of numbers between 1 and 2. To make the arithmetic a little easier, I use

$$\frac{1}{10}\sqrt{100(1+m^2)}$$

which means I have to find the square roots of numbers between 100 and 200. You first make up a table:

x	x^2
10	100
11	121
12	144
13	169
14	196

Then you inverse linearly interpolate for the desired square root. Let's use $m^2 = 0.37$ as an example. We need $\sqrt{113.7}$. Inverse linear interpolation gives the root as

$$10 + \frac{13.7}{21}(11-10) = 10.65$$

Your calculator will show this is off by about 1 part out of 1000. This root can be refined by a few more significant figures with one pass of the Babylonian method (derive it using Newton's method):

$$x_{new} = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

where x is the estimated root of $\sqrt{a}$. This requires doing the long division a/x, but you already know 3 or 4 digits of the answer, so it goes relatively quickly.

The new root estimate from this is 10.6630; your calculator will show this is correct to 6 significant figures. The cost was two long divisions.

Hand multiplication and division

[*huntington:89:96*] gives methods for multiplying and dividing numbers such that you need only keep the significant figures you need.

Suppose we want to multiply 4956 by 8372 and get four significant figures in the answer. We should then carry the calculation to 5 significant figures and round at the end. Here's a non-traditional way to do this multiplication. Remove the decimal points and fix the answer's decimal point with the usual order of magnitude estimation. Here, 5000 times 8000 will give 40 million, so we know the size of the answer. The calculation is (note we **multiply the single digits from left to right**):

		4	9	5	6				
		8	3	7	2				
	3	9	6	4	8				A
		1	4	8	6	***8***			B
			3	4	6	***9***	***2***		C
					9	***9***	***1***	***2***	D
	4	1	4	9	2	***x***	***x***	***x***	
Column	8	7	6	5	4	3	2	1	

The rows are labeled by letters and the columns by numbers. Draw a vertical line to indicate which digits will be significant -- the significant ones will go to the left of this line and the insignificant ones to the right.

The first row A is gotten by multiplying the 8 in the multiplier by 4956. The answer is written down from right to left as is usual, starting in column 4. Row B is gotten by multiplying by the 3 in the multiplier and its result is **shifted right** one column so it starts in column 3. Rows C and D are gotten analogously. The x's indicate digits we don't really care about.

Next, we inspect column 3 and 2 to see about what column 3 will add up to. We see it will result in a 6 and a carry of 2. Then we add the 2 to column 4's numbers and get 31. We write down 2 because the 6 will cause us to round up. Then add up columns 5 through 8 in the usual way. The answer, rounded to 4 significant figures, is 4149. Thus, the multiplication's answer is 41,490,000.

If the original numbers had decimals points in them, you could have written the power of 10 to the right of the original numbers in, say, column 1 that would place the decimal point in the right location. Suppose the two numbers being multiplied were 4.956 and 0.08372 (so the answer should be about 0.5). Then we would have written -3 for the exponent of the multiplicand and -5 for the multiplier's exponent. In other words, $8372 \times 10^{-5} = 0.08372$. These exponents add together to get -8, which tells us we must shift the decimal point to the left 8 times in the answer. The result would be 0.4149. This method is somewhat like the slide rule, as you need to fix the decimal point correctly. The order of magnitude calculation is critical for this.

You can see that this method would be a bit faster than the traditional method, especially if you only wanted a few significant figures. You'd probably only write down numbers one column to the right of the vertical line. If you're willing to be off a digit or two, you'll probably learn to shorten things a bit more and not write any of the numbers in bold italics down. By the way, this method of multiplication is due to Oughtred (see [*claudel:60:77*]), a contemporary of Newton. Oughtred also gave us sin, cos, and tan and is generally considered the inventor of the slide rule.

There's a method for long division where you only want a certain number of significant figures. Here's the example [*hunt*] gives for π/ln(10) = 3.1416/2.3026. Suppose we want 5 significant figures in the answer, which is implied by the problem 31416/23026 (again, decimal points are dropped and calculations done with integers, so the first step, as usual, is to approximate the answer). Here's the calculation:

```
23026 ) 3 1 4 1 6 (1    A
      - 2 3 0 2 6       B
 2303 )   8 3 9 0 (3    C
        - 6 9 0 9       D
  230 )   1 4 8 1 (6    E
        - 1 3 8 0       F
   23 )     1 0 1 (4    G
          -   9 2       H
    2 )         9 (4    J
```

The quotient is thus 13644 (without a decimal point). Let's see how it's done.

The divisor 23026 is written on the left in the first column on row A. Then the dividend 31416 is written down; so far this looks like a traditional division problem. The 1 at the right in row A is how many times 23026 goes into 31416. Then the two numbers are subtracted to give 8390, which is written down with its least significant digit in the least significant digit of the dividend. Here's the key step: **the next divisor brought down has the first right-hand digit of the original divisor chopped off**. Then the number of times it can go into 8390 is written down as the 3 after the left parenthesis. Subtract and bring down the divisor again, chopping off the right hand digit.

We inspect the original problem to get the decimal point and thus write the answer as 1.3644.

I put in the parentheses and the - signs as clues to what's going on. However, when you write the problem on paper, you'll probably do it like so:

```
23026|3 1 4 1 6|1          1 3 6 4 3
     |2 3 0 2 6|       23026|3 1 4 1 6 0 0 0 0
 2303|  8 3 9 0|3            2 3 0 2 6
     |  6 9 0 9|              8 3 9 0 0
  230|  1 4 8 1|6              6 9 0 7 8
     |  1 3 8 0|              1 4 8 2 2 0
   23|    1 0 1|4              1 3 8 1 5 6
     |      9 2|                1 0 0 6 4 0
    2|        9|4                 9 2 1 0 4
                                     8 5 3 6 0
                                     6 9 0 7 8
                                     1 6 2 8 2
```

The right-hand calculation is the traditional calculation, showing that the calculation needs to be done with 5 and 6 digit numbers. The traditional long division method has the advantage of giving the answer to any number of figures desired -- but in practical work, it's rare to need more than 3 or 4 significant digits.

Here's the same division when only 3 figures are wanted:

```
230|3 1 4|1
   |2 3 0|
 23|  8 4|3
   |  6 9|
  2|  1 4|6
   |  1 2|
```

I like how the arithmetic gets simpler in the approximate method as you get successive digits in the answer: that means less work -- and that can translate into fewer errors.

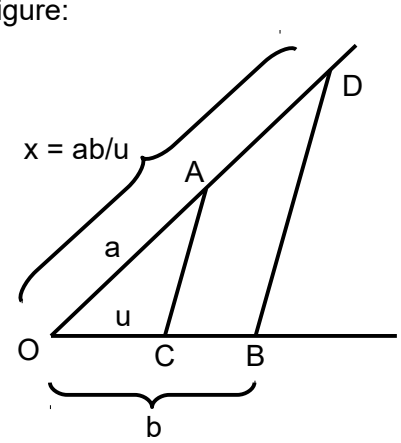
Layout

This section contains various methods of geometrical layout using simple tools.

Graphical arithmetic

Multiplication and division

[*morris:250:259*] Suppose we want the product *ab* in terms of a unit of length *u*. To obtain this graphically, use the following figure:



Construct lines OB and OD at any angle to each other. Draw length *u* as OC. Draw length *a* as OA and OB as distance *b*. Draw line BD parallel to AC. The distance *x* (line OD) has length *ab/u*.

The truth of this construction is seen from OCA and OBD being similar triangles:

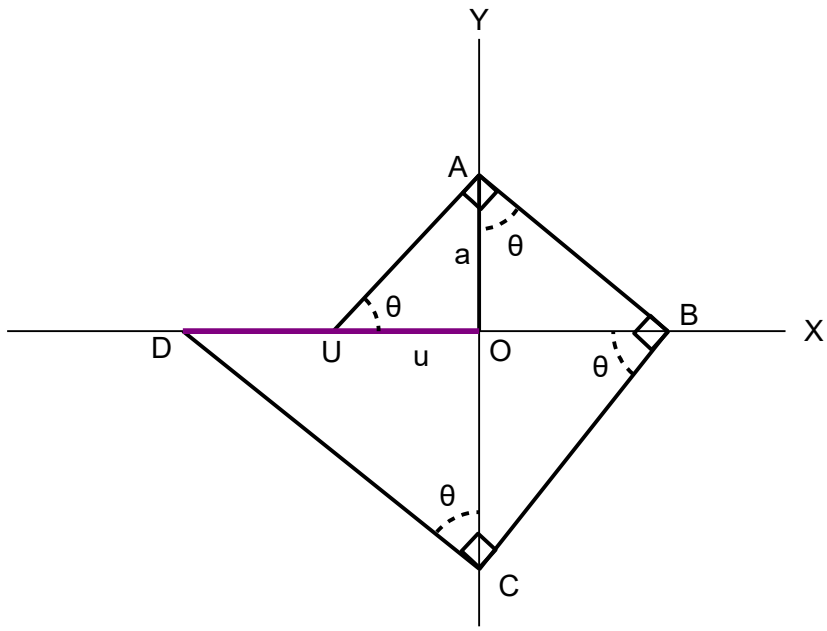
$$\frac{OD}{OB} = \frac{OA}{OC} \quad \text{or} \quad \frac{x}{b} = \frac{a}{u}. \quad \text{Thus } x = \frac{ab}{u}$$

No generality is lost in making angle OCA and OBD right angles; this makes for easier construction with tee square and triangle.

Division can be done with the same figure.

Integer powers

[*morris:250:259*] Suppose we want a^4 in terms of a unit *u*. In the following figure,



construct a set of Cartesian axes with origin O. Draw the unit length u as line OU on the x axis. Draw the length a up the Y axis as line OA. Draw the perpendicular AB to line AU passing through the point A. Then the distance OB is $(a/u)^2$. Continue with successive perpendiculars to get the desired power. Triangles OUA and OAB are similar and $\sphericalangle OUA = \sphericalangle OAB$. Therefore

$$\frac{a}{u} = \tan\theta = \frac{\mathrm{OB}}{a} \quad \Rightarrow \quad \mathrm{OB} = \frac{a^2}{u} = a\left(\frac{a}{u}\right)$$

In triangle OBC,

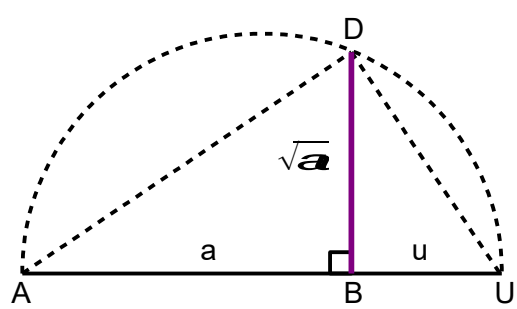
$$\tan\theta = \frac{\mathrm{OC}}{a^2/u} = \frac{a}{u} \quad \Rightarrow \quad \mathrm{OC} = \frac{a^3}{u^2} = a^2\left(\frac{a}{u}\right)$$

and in triangle OCD,

$$\tan\theta = \frac{\mathrm{OD}}{a^3/u^2} = \frac{a}{u} \quad \Rightarrow \quad \mathrm{OD} = \frac{a^4}{u^3} = a^3\left(\frac{a}{u}\right)$$

Square root

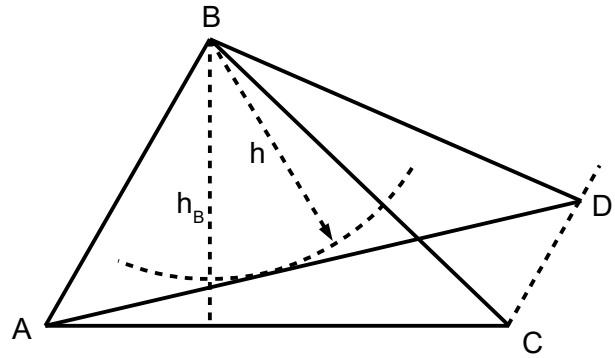
[*morris:254:263*] Given a, construct $\sqrt{a}$ with the distance u being unity. In the figure, construct line AU such that $\mathrm{AB} = a$ and $\mathrm{BU} = u$. Construct semicircle ADU with diameter AD and draw line BD.



From the section *Geometric mean* below, we know the length of BD is $\sqrt{au}$. Since the numerical value of u is 1, this is $\sqrt{a}$.

Triangular area

[*morris:254:263*] Given a triangle ABC, construct another triangle with the same area that has an altitude of h.



Draw the indicated arc with center B and radius *h* and draw line CD parallel to AB. Draw line AD tangent to the arc; the triangle ADB has altitude *h* and the same area as triangle ABC.

The triangles ABC and ABD have identical areas because they have a common base AB and share a common height, the perpendicular distance between lines AB and CD.

Cutting pipe ends for welding

Suppose cylindrical pipe 1 of radius R runs along the y axis and another cylindrical pipe 2 of radius r where $r \le R$ runs along the x axis. Rotate pipe 2 by an angle α about the z axis (the positive angle is counterclockwise when looking at the xy plane from the +z direction) and then translate the pipe in the +z direction by a distance *a*. Since the generator for the rotation is $x\,\partial_y - y\,\partial_x$ and the translation is ∂_z, these transformations commute.

You can lay out a pattern on a sheet of paper and use it as a template to cut the end of the smaller pipe. The formula for the y distance to trim off given the angle θ around the circumference of the smaller pipe is

$$y(\theta) = sec\,\alpha\left(\sqrt{R^2-(r\sin\theta+a)^2}+r\cos\theta\sin\alpha\right)$$

The angle $\theta = 0$ is in the plane containing the pipes' axes.

To derive this, start with the parametric equations for one cylinder that lies on the y axis:

$$f(u,\theta) = (r\cos\theta, u, r\sin\theta) \tag{6}$$

Start with the other cylinder lying along the x axis:

$$g(v,\psi) = (v, R\cos\psi, R\sin\psi)$$

Use a matrix to rotate this vector an angle α around the z axis:

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This leaves the cylinder in the xy plane. Then translate it in the +z direction.

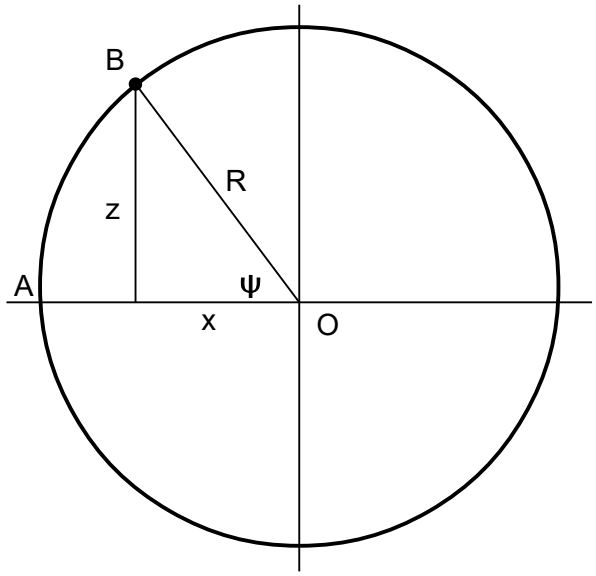
Equate the components *f* and *g*, then solve them for θ, *u*, and *v* in terms of ψ and substitute into (6) to get a one-parameter vector for the curve of the intersection. Then use the inverse of *M* to rotate the curve back to it around the x axis and translate back down to the xy plane. This will yield the above formula.

The site http://www.harderwoods.com/pipetemplate.php provides an on-line solution via this method.

To make a template to trace the hole on the larger pipe, start with the parametric vector describing the curve:

$$\begin{aligned} x &= \frac{\sqrt{R^2-(r\sin\theta+a)^2}+r\cos\theta\sin\alpha}{\cos\alpha} \\ y &= r\cos\theta \\ z &= r\sin\theta - a \end{aligned}$$

Looking down the y axis (the axis of the larger pipe), you'll see

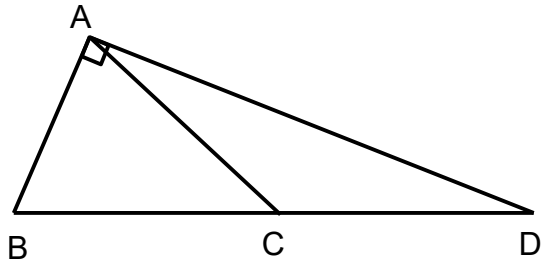


The point B is on the curve of intersection. The angle $\psi = \tan^{-1}\frac{z}{x}$. The arc length of the arc AB is $s = R\psi$. Then when we plot the points $(R\tan^{-1}\frac{z(\theta)}{x(\theta)}, y(\theta))$ as θ goes from 0 to 2π, the cutout will be traced in the plane.

Laying out a right angle

The usual practical method is to use a 3-4-5 triangle. However, here's a technique that only requires a straight stick (Colvin & Stanley, *American Machinist's Handbook*, McGraw-Hill, 1914):

Draw a line AC and at one end, draw line BD at any angle:

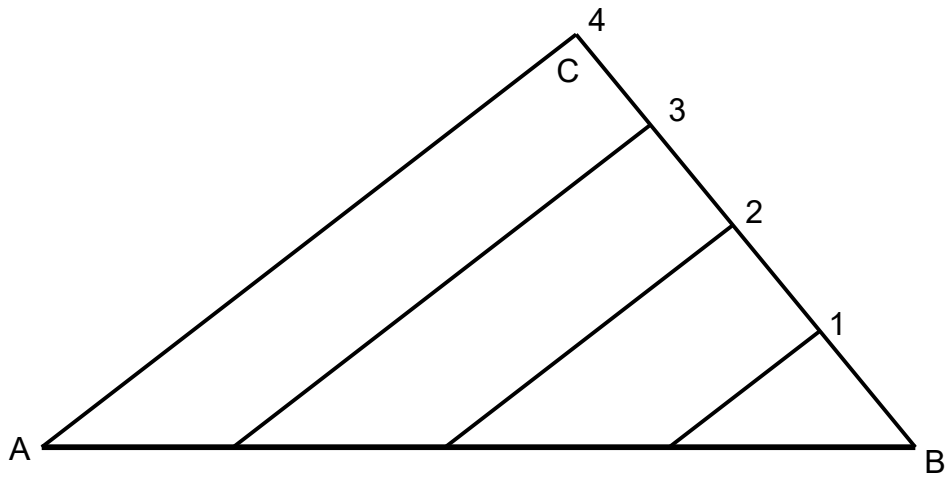


Make sure that BC = AC and CD = AC. Then $\sphericalangle BAD$ is a right angle. OK, you have to mark on the stick too, so this will take an additional tool like a pencil.

Dividing a line

The quickest method is to lay a rule at an angle to the line, and mark off points along a line at n major divisions. Then draw parallel lines from the marked points to intersect the original line (this is most easily done with two drafting triangles, sliding one on the other). In the figure below, AB is the line to be divided equally into 4 parts. Suppose we set a rule along line BC that makes BC exactly 4 units. Then we mark the points 1, 2, 3, and 4. Draw a line from point 4 (i.e., C) to A. Then draw lines through 1, 2, and 3 parallel to AC. The intersections divide line AB up as desired. This is

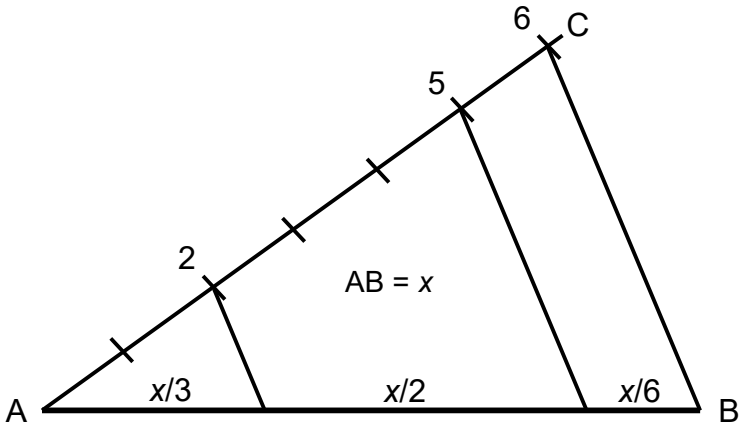
known as the draftsman's method.



You can also calculate AB/n and use your rule to mark successive points on the line AB. But this method can be tedious and subject to reading errors because the required points are e.g. not integers. The above method, used for thousands of years, avoids the tedium and reading mistakes (you can also use a compass or dividers to mark off equal lengths on line AC).

Because calculators are common, you can calculate AB/n and set the dividers to this value, then step off the intermediate points. The first method can be more accurate, as there's no cumulative error due to setting the dividers slightly wrong. However, you can practically eliminate the cumulative error if you first step off the divider along AB and ensure it comes out the proper length; if not, make small adjustments and repeat. This is most easily done with screw-adjusted bow spring dividers (machinist or drafting dividers). If the error at the end of stepping off *n* points is δ, then the divider setting error is δ/n. With a little care, you can divide AB up into n equal divisions with no sensible error.

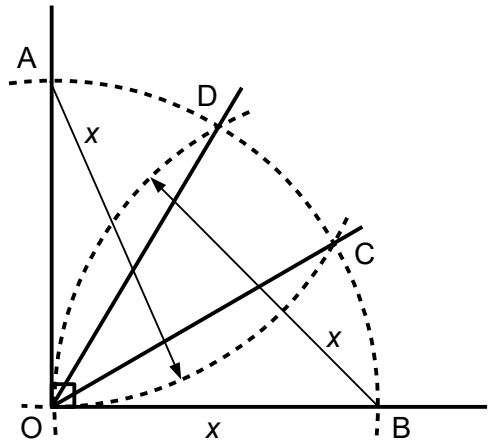
This naturally extends to a method to divide a line into any desired proportions.



Suppose we want to divide the line AB = *x* into the proportions 2, 5, and 6. Construct the line AC at any convenient angle and mark off 2, 5, and 6 units of length on AC. Connect point 6 to B and the other lines as needed parallel to 6B.

Trisect a right angle

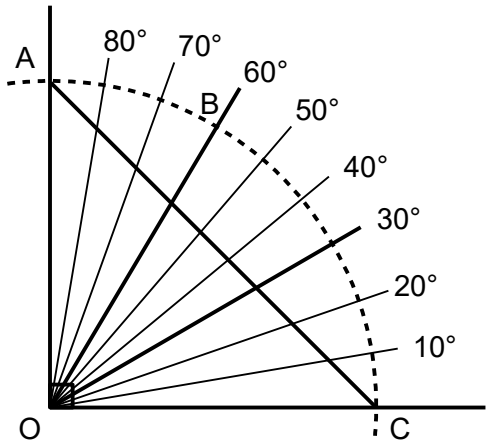
Draw right angle arc AB with arbitrary radius *x:*



Without changing compass setting, draw arc OD with center at B and arc OC with center at A. The lines OD and OB are the trisectors of the angle.

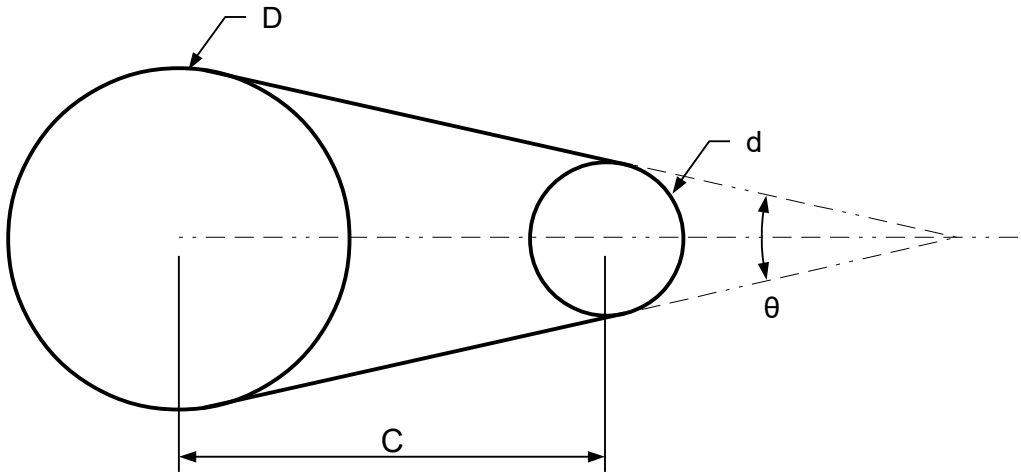
Scale of chords

This is a graphical tool that can be constructed to help measure angles. Construct a right angle AOC and draw the arc ABC. Trisect the right angle as given in the section *Trisect a right angle* and use dividers to trisect each of the three resulting angles. Subdivide the 10° angles as needed.



Belt length on pulleys

Given two pulleys of diameters D and d whose centers are separated by a distance C:



The belt length L is

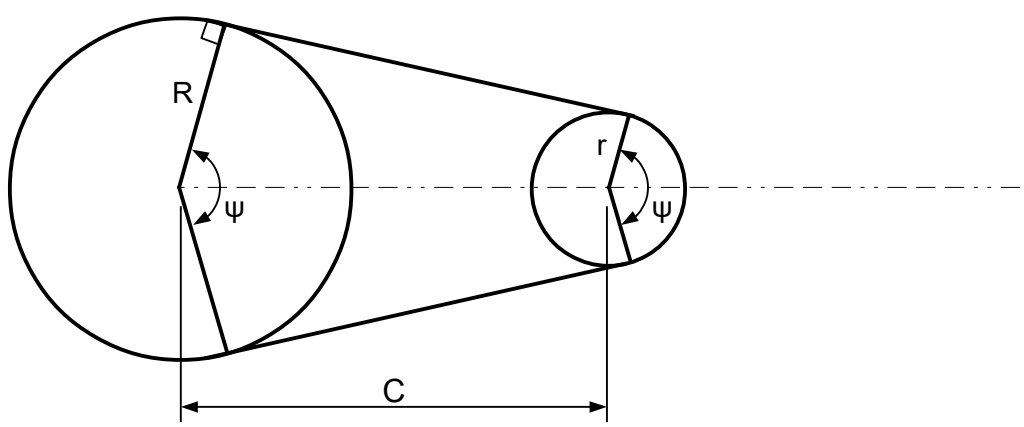
$$L = \sqrt{4C^2 - (D-d)^2} + D\left(\frac{\pi}{2} + \theta\right) + d\left(\frac{\pi}{2} - \theta\right)$$

where

$$\theta = 2\sin^{-1}\frac{D-d}{2C}$$

When D and d are equal, this reduces to $2C + \pi D$, as expected.

From the following diagram



we get an equivalent formulation:

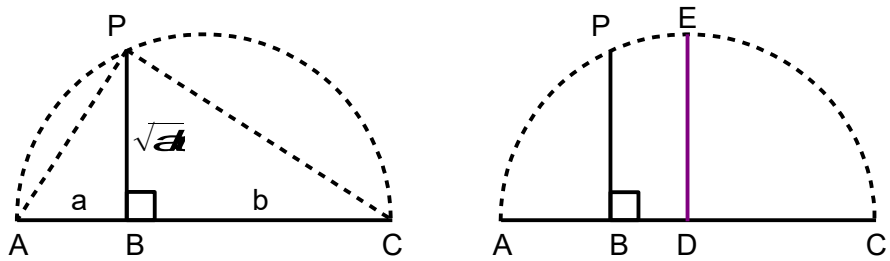
$$L = 2C\sin\left(\frac{\psi}{2}\right) + R(2\pi - \psi) + r\psi$$

where

$$\psi = 2\cos^{-1}\frac{R-r}{C}$$

Geometric mean

To construct the geometric mean of two numbers *a* and *b*, construct the following figure



The length of BP is the geometric mean of *a* and *b* or $\sqrt{ab}$.

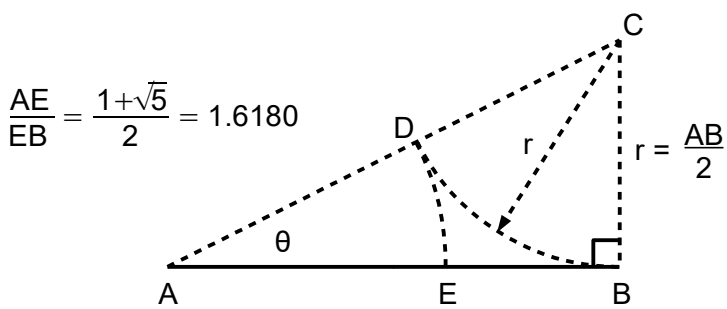
Proof: angle APC is a right angle; thus, angles APB and PCB are the same. The tangents of APB and PBC give *a*/PB = PB/*b* and the relation follows.

Note that the colored line EB is equal to $\frac{a+b}{2}$. This is an immediate picture proof that the arithmetic mean (i.e., the radius of the circle) is always greater than or equal to the geometric mean:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Golden ratio

[*marks:2-14*]



To divide a line AB into two segments AE and EB that have the golden ratio

$$\frac{AE}{EB} = \frac{1+\sqrt{5}}{2} = 1.6180$$

construct BC perpendicular to the line AB at B such that the perpendicular's length r is half of the line AB's length. Draw AC and scribe arc DB with center at C and radius r. Then scribe arc DE with center at A. E is the required point that divides AB into the desired ratio.

Proof: Let $AB = a$; then $AC = r\sqrt{5}$ and $AD = AE = AC - r = r(\sqrt{5}-1)$. Then

$$EB = a - AE = 2r - r(\sqrt{5}-1) = r(3-\sqrt{5})$$

Hence

$$\frac{AE}{EB} = \frac{r(\sqrt{5}-1)}{r(3-\sqrt{5})} = \frac{1+\sqrt{5}}{2}$$

Inscribed/circumscribed circle

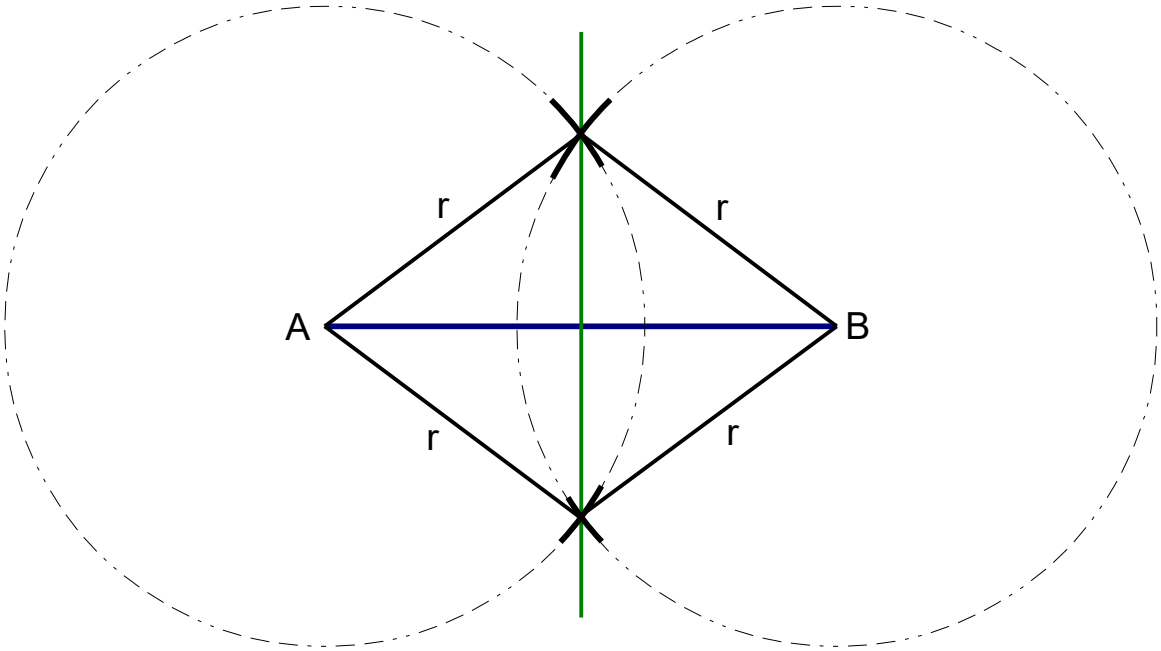
To construct the circumscribed circle of a given triangle, the center of the circle is the intersection of the perpendicular bisectors of the sides. For the inscribed circle, the center is intersection of the bisectors of the interior angles.

To inscribe a hexagon in a circle, draw a diameter. Then scribe the radius from the intersections of

the diameter and the circle; the intersection points are on the hexagon.

Line segment perpendicular bisector

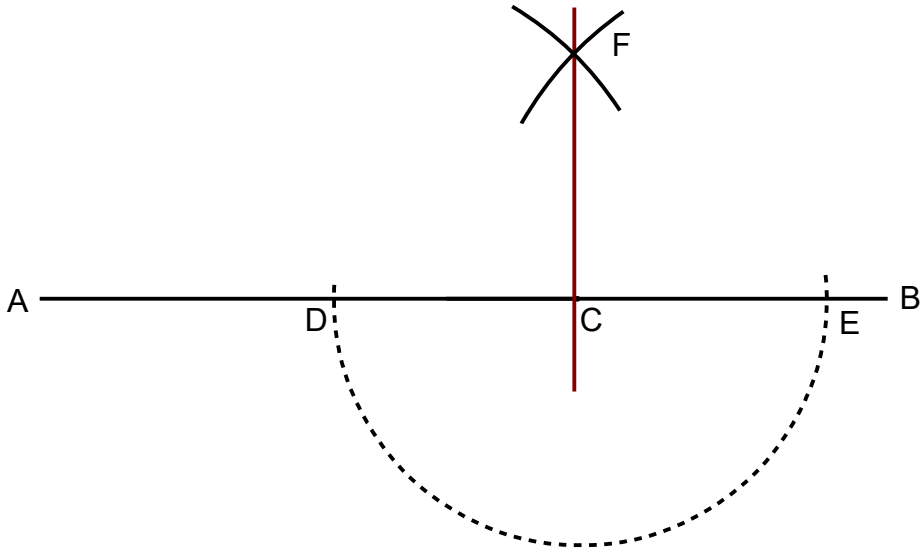
Given line segment AB, construct the perpendicular bisector by scribing equal arcs from the endpoints of the segment. Set dividers to 3/4 or more of the line segment length. The intersections of the two equal circular arcs lie on the perpendicular bisector.



The method works because the diagonals of a rhombus are perpendicular bisectors of each other.

Perpendicular to a point interior to a line segment

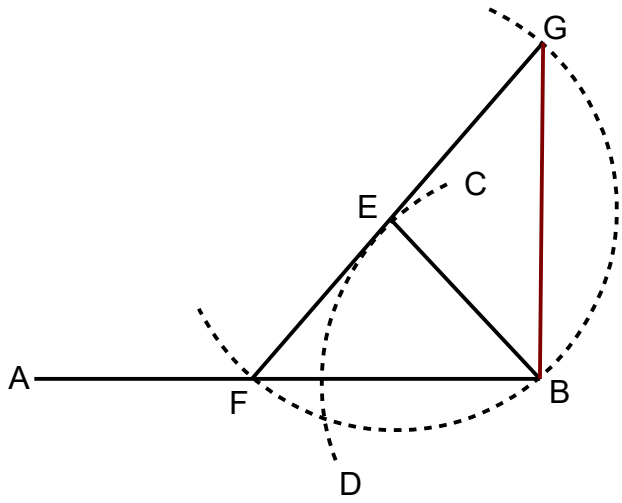
Given line AB and point C to which a perpendicular to AB must pass. Using dividers, construct arc DE with center at C:



Then construct equal radius arcs from centers D and E that pass through point F. Line CF is perpendicular to AB.

Perpendicular at end of line segment

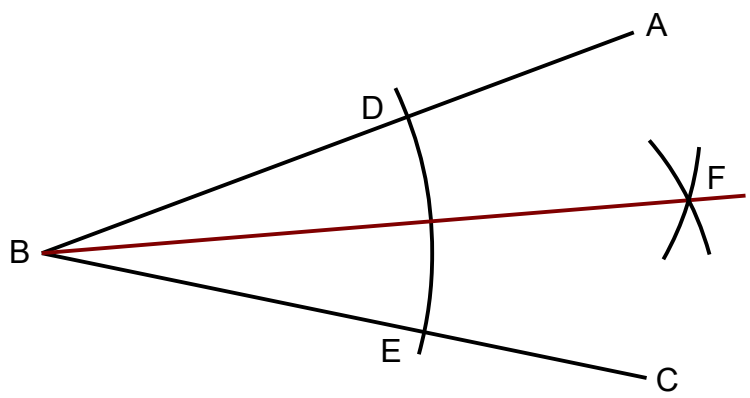
Given line segment AB. To construct a perpendicular at B, set dividers to a reasonable size and scribe arc CD with center at B. Locate point E on arc CD approximately 45° above the line AB. Scribe the dashed circle FBG with radius EB. Construct line FG and extend it to intersect the dashed circle at G. The line BG is perpendicular to AB.



The principle is based on the fact that if any two chords FB and GB in a circle meet along a diameter EF, then angle FBG is a right angle.

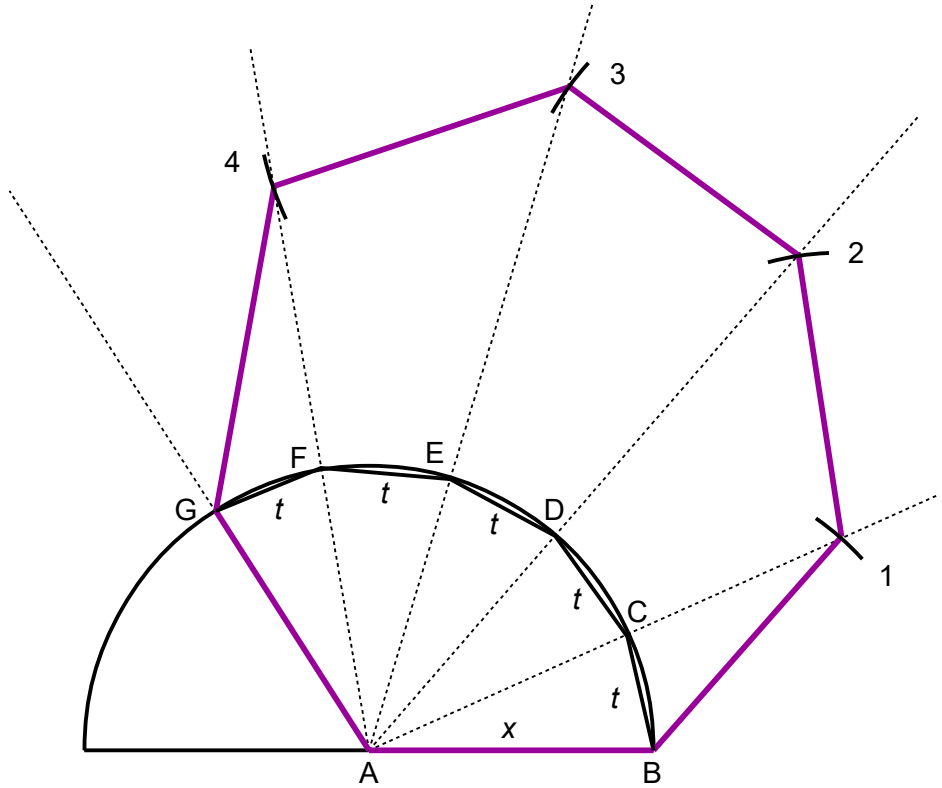
Bisecting an angle

To bisect angle ABC, construct arc DE with center B. Construct arcs DF and EF that have the same radius. BF is the angle's bisector.



Construct *n*-sided regular polygon

[*huntington:103:110*] Construct an *n*-sided regular polygon with a side length of *x*.

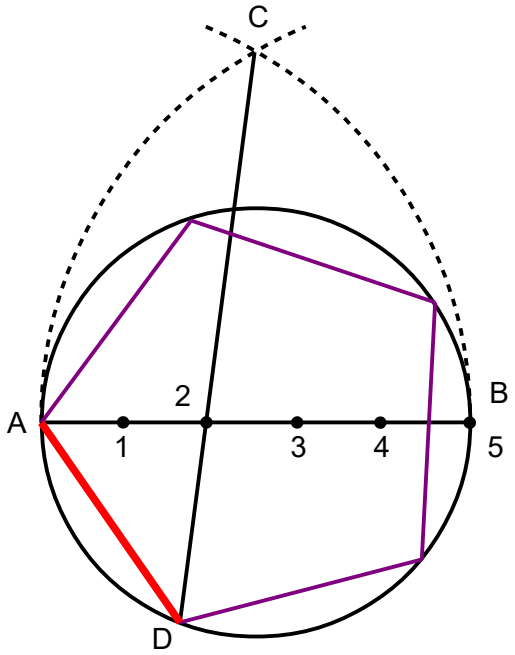


With A as center and AB = *x* as radius, draw semicircle and divide it into *n* parts, of which *n* - 2 parts counting from B will be used. If r = AB, set your dividers to

$$t = 2r\sin\frac{\pi}{2n}$$

Step off the distance *t* starting at B to lay out the points C, D, E, F, and G on the semicircle (the n = 7 case is shown). Draw the dashed lines and set the dividers to *x*. Starting at B, step off the arcs 1, 2, 3, and 4. Draw the connecting lines.

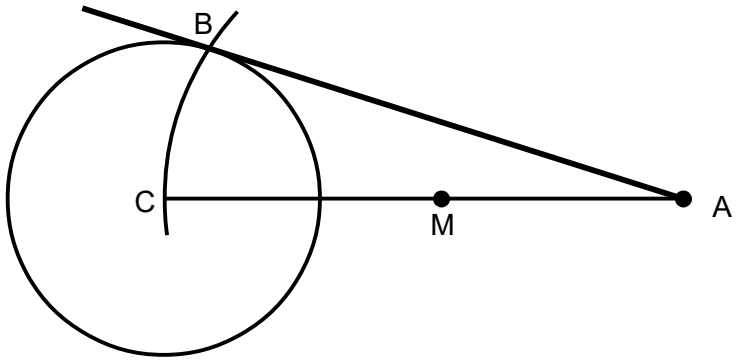
[*carroll:28:29*] An approximate method of drawing an *n*-sided polygon in a circle is as follows (example is for a pentagon *n* = 5):



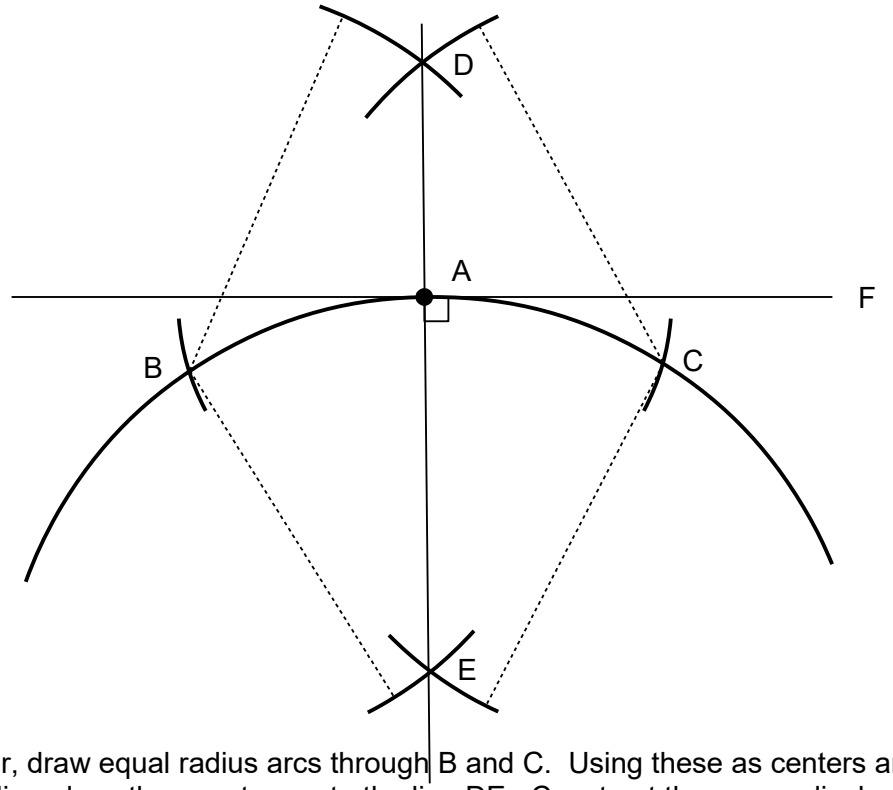
Divide the diameter into *n* equal segments (5 in this example). Draw arcs AC and BC with radius AB from points A and B. Draw line CD that passes through point 2 on the diameter. This creates one side of the polygon AD; step the other sides off with the dividers.

Tangent to a circle

[*huntington:103:110*] To draw a tangent to a circle from an external point A, bisect AC to get M. With M as center, draw arc BC with radius CM. Intersection point of arc and circle at B is the required tangent point.



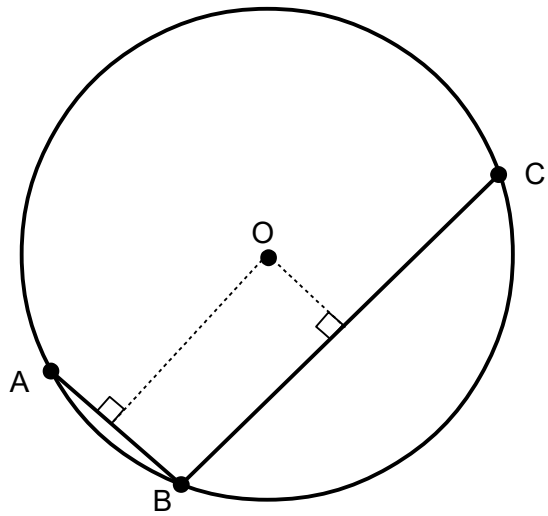
When the center of the circle isn't accessible, construct a tangent to the circle through A [*morris:34:43*]:



With A as center, draw equal radius arcs through B and C. Using these as centers and a constant but arbitrary radius, draw the arcs to create the line DE. Construct the perpendicular bisector of DE to get the tangent AF.

Circle through 3 points

[*huntington:103:110*]



Draw chords AB and BC and their perpendicular bisectors, which will intersect in the circle's center at O.

Find the center of a circle

There are a variety of methods, each has its advantages and disadvantages.

In the shop, the fastest method for bar stock is the center head attachment for a combination square.

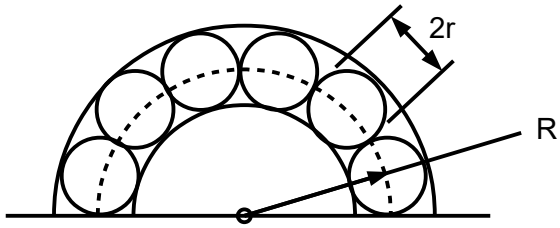
You can clamp a chunk of ground flat stock to a carpenter's square so that it bisects the right angle and use it for circles in the plane.

The geometrical construction is to draw two or more chords of the circle. Their perpendicular bisectors (construct with dividers or compass) will intersect in the center of the circle.

A fast and practical method is to set some dividers to approximately the radius of the circle. Put the dividers' point on the circle at approximately 90° spacing on the circumference and scribe an arc near the center. Draw a line through the arcs' intersections; the intersection point of those two lines is the center of the circle.

A practical shop method uses a drafting scale. Lay the scale across the circle and, without moving the scale, draw a pencil line using each side. Measure the length of the lines to where they intersect the circle; using the half-size scale, mark the center points of the line. Use a straightedge to connect these center points; extend this line to the circle on each side. Measure the length of this line and mark it's center; this is the center of the circle.

Annulus containing n circles

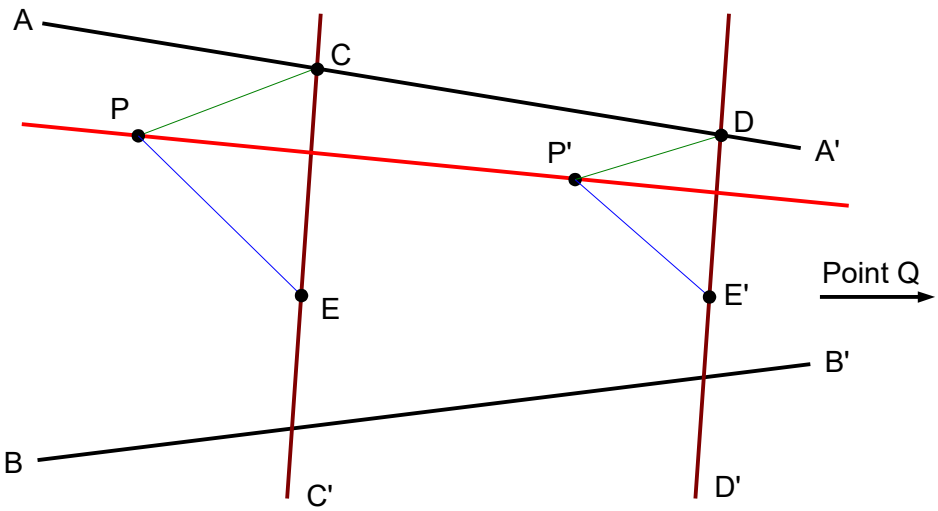


[*huntington:105:112*] Suppose you want *n* circles of radius *r* tangent to each other with their centers at radius *R*. Then the relationship between *r*, *R*, and *n* is

$$r = R \sin \frac{\pi}{n}$$

Inaccessible point

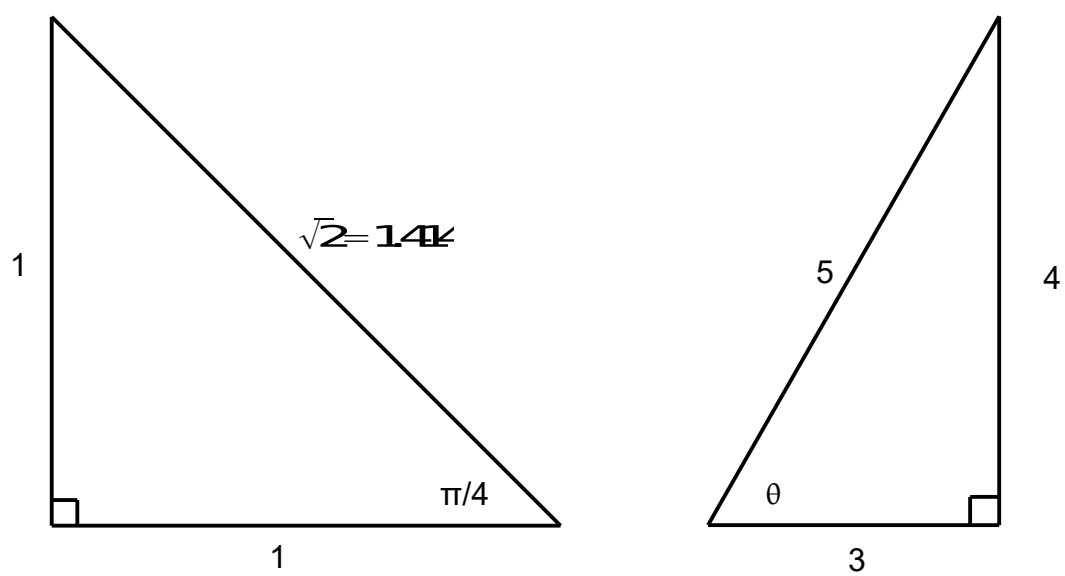
A point P is given along with two lines AA' and BB' that intersect at an inaccessible point Q. Construct the line PQ. The method is to use similar triangles:



Draw any line CC' and construct DD' parallel to it. Draw CP and construct DP' parallel to it. Draw PE and construct P'E' parallel to it. Now you have the point P', so construct the desired line through PP'.

Laying out right angles

Two triangles are commonly used for right angle layout: the isosceles and the 3:4:5 triangle.

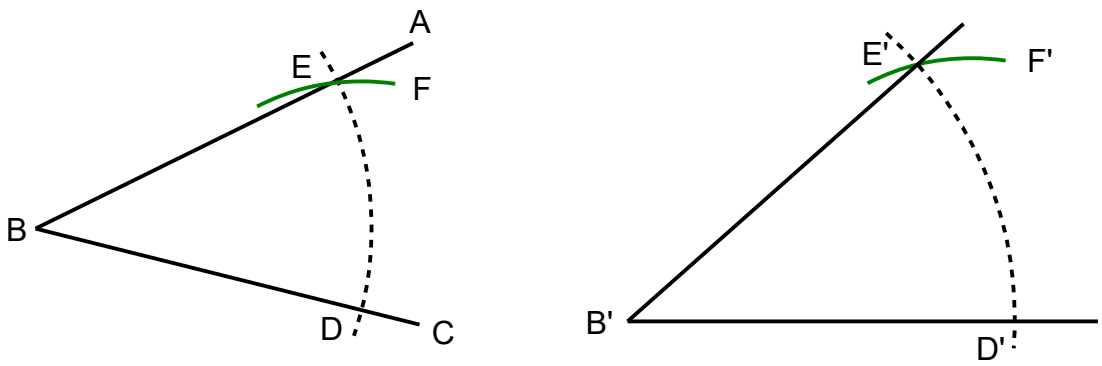


These are often used in carpentry, yard work, and surveying. The advantage is that all that's needed is a rule or tape measure. The isosceles triangle method can be extended to measuring the diagonals of a rectangle -- when they are equal, then all four corners are right angles. This method doesn't even need a tape measure -- mark one diagonal on a stick and compare it to the other diagonal.

The 3-4-5 triangle is also useful with dividers and a rule. Be mindful that angle θ isn't any "common" angle -- it's $\tan^{-1}\frac{4}{3} = 0.9273$ radians $= 53.13°$.

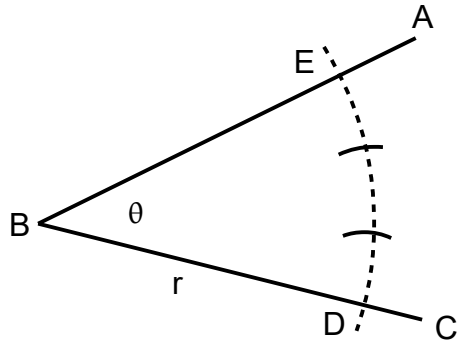
Copying an existing angle

Suppose angle ABC is given and it is desired to copy this angle. The new vertex is B' and one side of the angle is to fall on line B'D'. On the ABC angle, draw an arc ED and, without changing the divider settings, draw the arc E'D' from the new vertex B'. On the ABC angle, set the dividers to the distance DE and scribe D'E' on the new drawing. Set the dividers to center D and scribe the arc EF; scribe the corresponding arc E'F'. Then draw the line B'E'.



Dividing an angle

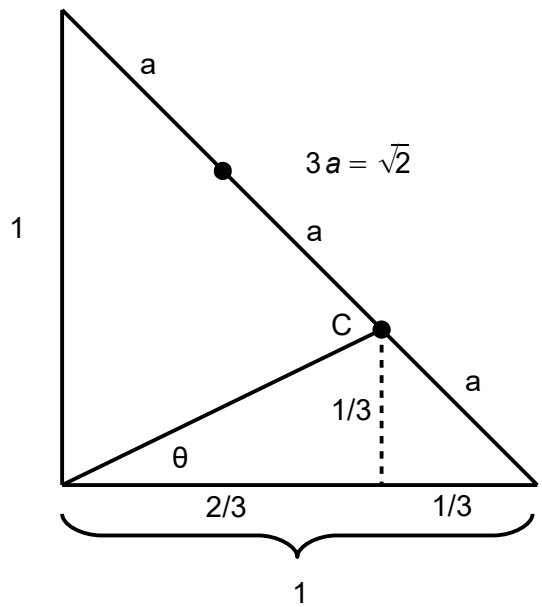
To divide an angle ABC into *n* equal parts, draw arc DE with center at B, then set the dividers to an estimate of the arc length DE/*n*. Step off n points on the arc DE. Iterate until the last point falls exactly on point D or E. The diagram shows the case when *n* = 3.



You cannot draw a line from DE and divide that into equal segments to equally divide the angle. However, if you have a rule and calculator handy, you can measure the line segment DE and *r*, then calculate the angle θ by the cosine law. Then the divider setting is

$$2r\sin\frac{\theta}{2n}$$

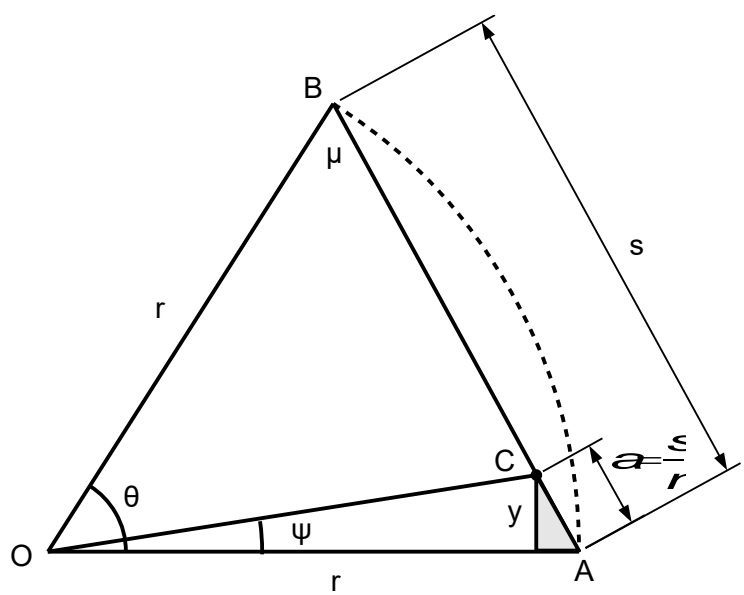
That using a straight line doesn't work is shown by trying to divide a right angle into three equal angles. The following sketch shows the problem:



Since $a = \sqrt{2}/3 = 0.4714$, we see that $\theta = \tan^{-1}\frac{1/3}{2/3} = 26.56\,°$, which doesn't lead to the desired angle of 30°. However, the formula above gives a divider setting of $2\sin\frac{90\,°}{6} = 2\sin 15\,° = 0.5176$. The height of the endpoint C of the chord with this length is $\beta = \frac{2}{\sqrt{2}}\sin 15\,°$. Then we have

$$\tan\theta = \frac{\beta}{1-\beta} \quad \text{or} \quad \theta = 30\,°$$

Here's the general case assuming θ is less than a right angle. Suppose we want to divide $\sphericalangle AOB = \theta$ into n equal parts, so we divide the chord into n equal pieces:



Then

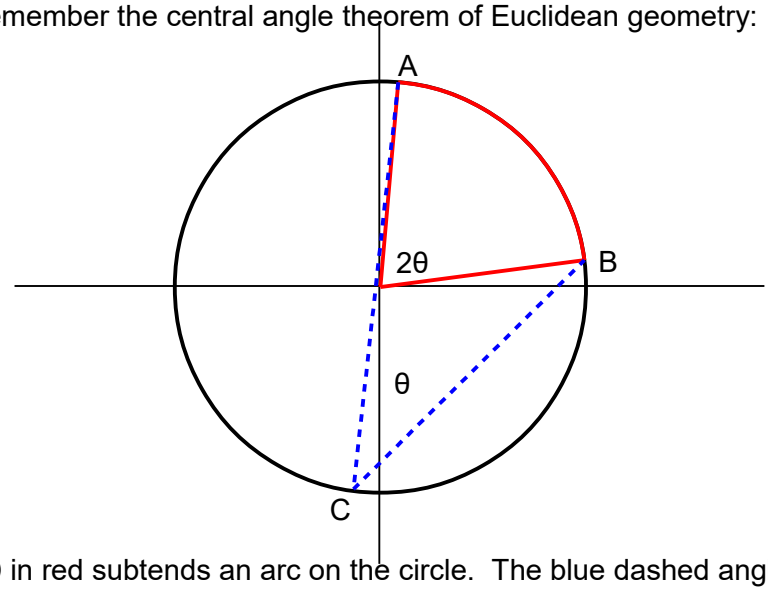
$$\tan\psi = \frac{\sin\theta}{n - 1 + \cos\theta}$$

By the half-angle formula for the tangent, this gives the proper angle only in the case n = 2; i.e.,

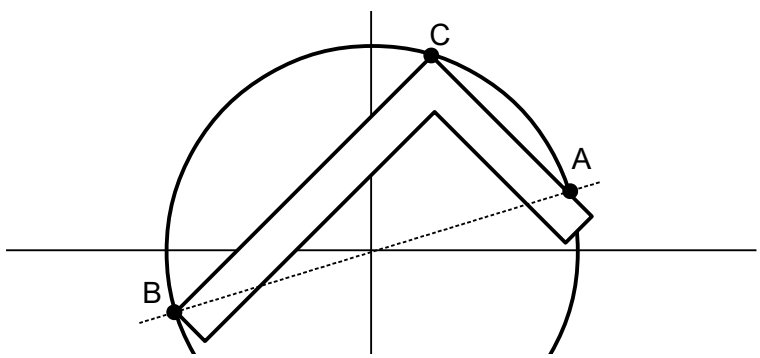
when you want to bisect the angle. When $\theta \ll 1$, this gives approximately $n\psi \approx \theta$, so dividing the chord for small angles can produce useful results.

Laying out angles

It can be useful to remember the central angle theorem of Euclidean geometry:



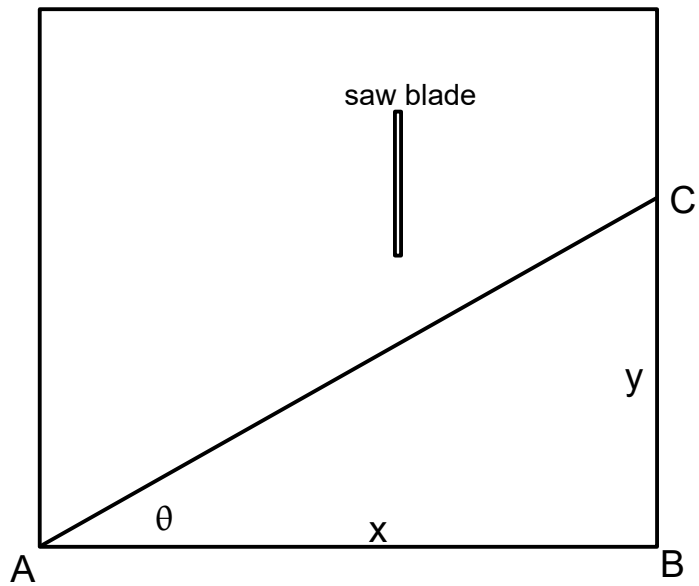
The central angle 2θ in red subtends an arc on the circle. The blue dashed angle with vertex on the circle that subtends the same arc on the circle has an angle of θ. The theorem is true for any point C on the circle as long as C is not between A and B. A corollary is Thales' theorem: an angle whose vertex is on the circle and that subtends a diameter is a right angle. This is immediately useful in the shop with a carpenter's square to find the center of a circle:



Locate line AB as shown with the sharp edge of the square at point C on the circle; the line AB will be a diameter. Locate another such line approximately at right angles to the first line and the intersection is the center of the circle.

Using tangents

Most folks are familiar with protractors, but because they are typically small, they can't be used for accurate angle layout. Drafting triangles are better, as they are typically made to close tolerances and a well-made drafting machine can be good too. My favorite method, however, is to use trigonometric relationships, typically the tangent. Here's an example of laying out an angle on the table top of a table saw (this is used e.g. to set a guide when one wants to cut a cove).



You can measure distance AB on the table. Then to get an angle θ with respect to AB, measure up a distance $y = x \tan\theta$ on BC. For accurate work, this requires that ABC is a right angle; if it isn't, you can measure it and still make the layout, although you'll have to use the cosine law.

Here's an estimate of the angle error:

$$d\theta = \frac{\alpha}{x^2}(-y\,dx + x\,dy) \quad \text{where} \quad \alpha = \frac{1}{1+\left(\frac{y}{x}\right)^2}$$

For my table saw, the distance x is 28.2". For an angle of 30°, we have y = 16.28". If we assume $dx = dy = 0.1$ inches, then $d\theta = \frac{1.333}{28.2^2}(-1.628+2.82) = 2$ mrad or $0.11°$. Reducing the dx and dy uncertainties will reduce the $d\theta$ uncertainty proportionately.

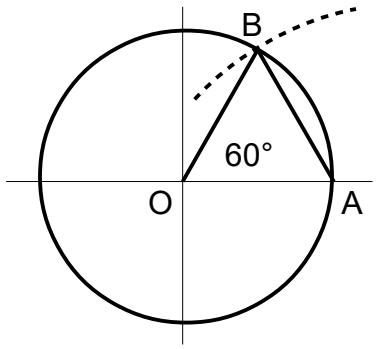
You can use this method even if your saw's table top is not square, but then you'll want to use the cosine law.

An alternative technique is to use a carpenter's square. Since these are annoyingly graduated in fractional inches in the US, one needs to use a table or a calculator.

Another technique for laying out angles is to use dividers and a rule for construction, as in the following examples.

60°

The radius of a circle is used to layout an angle of $\pi/3$ = 60°:



Set the dividers to OA, the radius of the circle, then scribe the arc through B with center A. OB is at a 60° angle to OA.

45°

Bisect a right angle. A quick template can be made from paper with square edges: fold one corner over so that its edge lays along the an opposite edge. The fold will be at 45°.

30°, 15°

Make a 60° angle and bisect it once or twice.

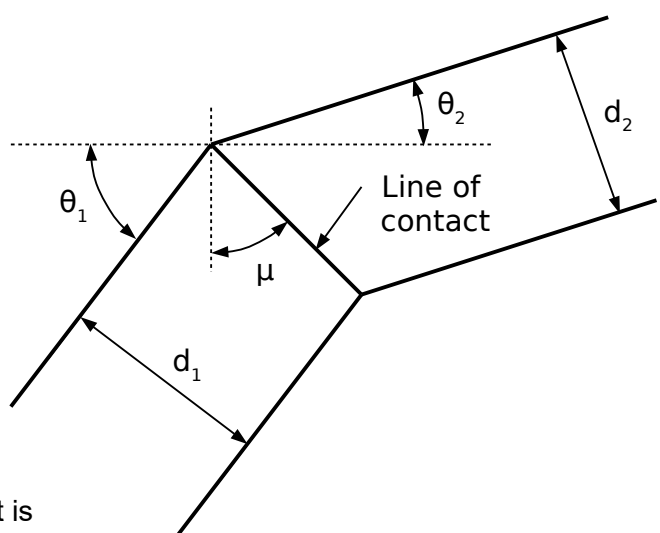
Other special angles

Here's an example of making a 40° angle. Other special angles can be made through combinations of the techniques shown here. In general, however, the method of using the tangent above will probably be found to be the quickest as long as one has a calculator or table of tangents handy. Sometimes these special methods can be used when no tables or calculator are handy, so they're useful to have in your back pocket.

Make a 45° angle by bisecting a right angle and make a 30° angle by bisecting a 60° angle. Now divide the 15° angle into three equal angles and the needed 40° angle will be 5° less than the 45° angle.

Miter joints

Given the following angles at the joint of two beams:



The angle of the line of contact is

$$\tan\mu = \frac{d_1\cos\theta_2 - d_2\cos\theta_1}{d_2\sin\theta_1 - d_1\sin\theta_2}$$

Triangles

Inscribed circle

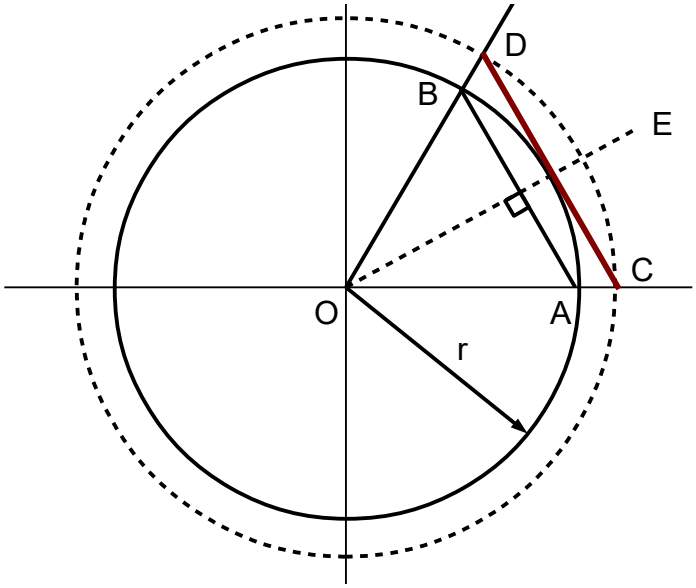
The bisectors of the internal angles of a triangle intersect at the center of the inscribed circle.

Circumscribed circle

The perpendicular bisectors of the sides of a triangle intersect at the center of the circumscribed circle.

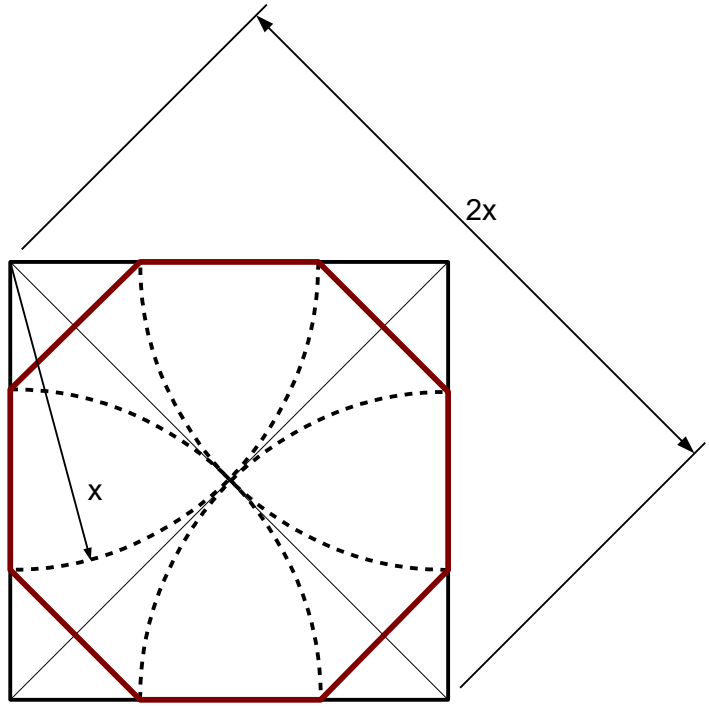
Hexagon

To layout the inscribed hexagon within a circle, lay out the radius along the circumference. For the circumscribed hexagon,



lay out radius r as AB and draw its perpendicular bisector OE. Construct tangent CD that intersects line OB. Then lay out radius CD around the dashed circle as in the inscribed circle case.

Octagon



From each corner of a square, scribe the dashed circles with a radius of one-half of the square's diagonal. Then the required octagon is at the indicated intersections of these arcs with the square.

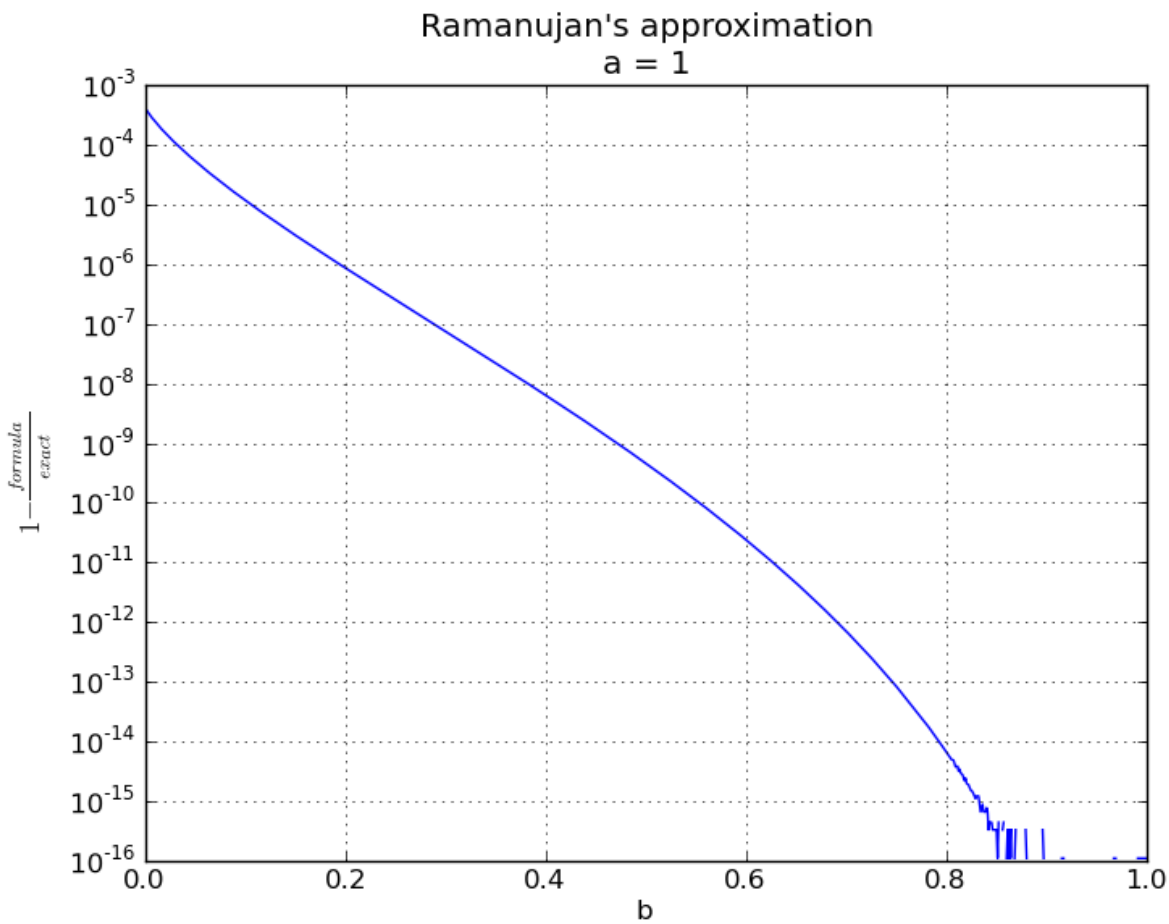
Ellipse

In the following section on ellipses, $2a$ is the major diameter of the ellipse and $2b$ is the minor diameter.

The area of an ellipse is $\pi a b$ and the perimeter is $4aE$ where E is the complete elliptic integral of the second kind with $k = \sqrt{a^2 - b^2}/a$. Ramanujan's rather amazing 1914 approximation for the perimeter is

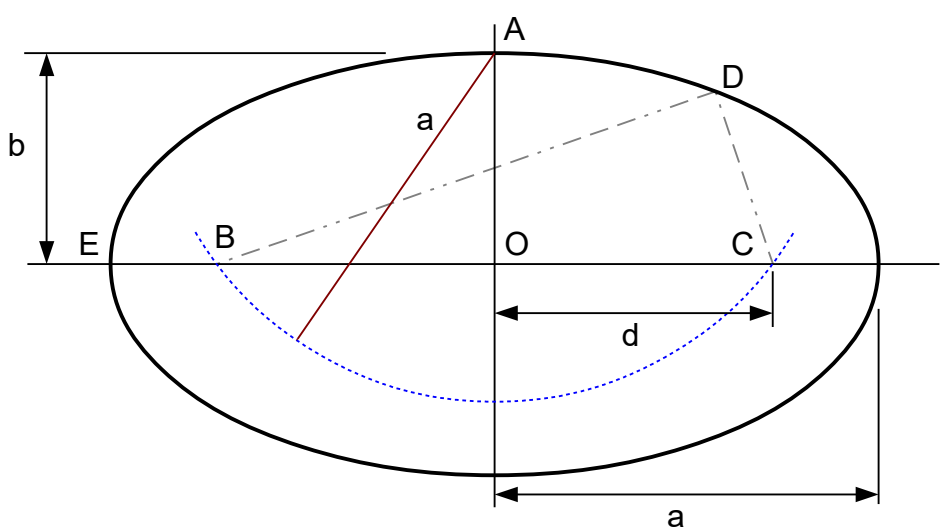
$$p \approx \pi(a+b)\left[1+\frac{\xi}{10+\sqrt{4-\xi}}\right], \quad \text{where } \xi = 3\left(\frac{a-b}{a+b}\right)^2$$

Here's a plot of the error of this approximation



Piece of string and two foci

Set the dividers to *a* and scribe the arc from A to intersect the major diameter at points B and C. Points B and C are the foci. The foci have the property that any beam of light BDC emitted from one foci will arrive at the other foci, assuming the interior of the ellipse is perfectly reflecting. The abscissas of the foci are $d = OC = \sqrt{a^2 - b^2}$.



The ellipse has the property that any point D on the ellipse has a constant sum of distances from the foci (i.e., BD + CD = 2*a*).

To construct the ellipse, push two pins or nails in at the foci. Then tie a string such that the loop causes the pencil point, when stretching the loop taut, to mark at point A. Then you should be able to trace the ellipse by keeping the pencil point taut in the string loop. The loop length needs to be BD + DC + BC which is equal to

$$2(a+d) = 2(a+\sqrt{a^2-b^2}).$$

While the method is exact, it can be difficult to keep the pencil perpendicular and the string stretched a uniform amount. Thus, don't expect perfection. If you're using a wooden pencil, file a small groove near the tip in the wood to help keep the string in consistent contact with the pencil point.

Calculated points

Since the equation of the ellipse is

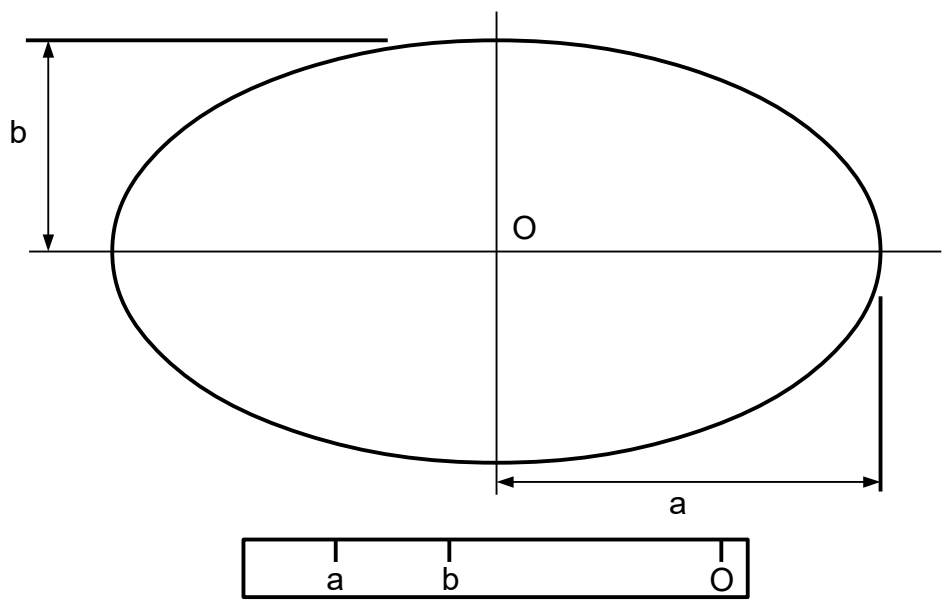
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

the required points can be calculated and plotted. A polygonal approximation to the ellipse can be gotten, good enough for many shop tasks, especially if a little smoothing is done with a file or sander.

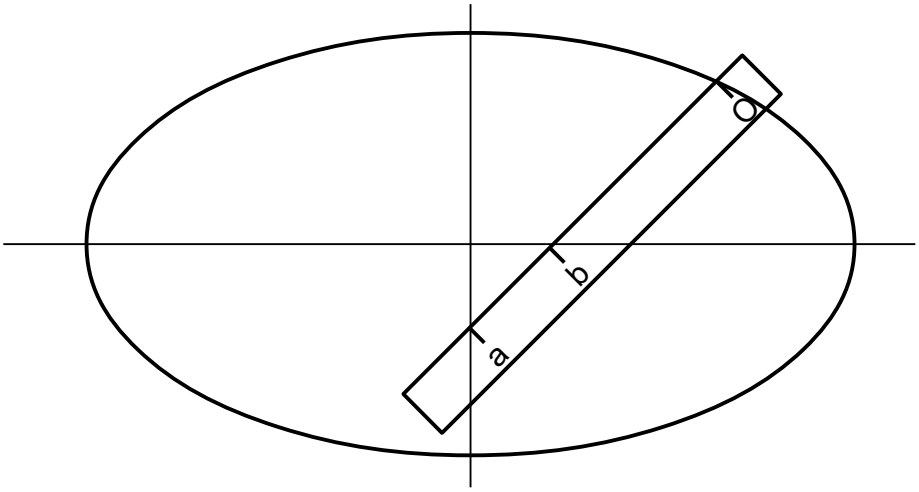
This method is relatively fast if a template is made -- and only a quarter of the ellipse's form has to be generated, as the other three quarters can be gotten by reflection (turning the template over) and rotation.

Using a marked stick

Using the following ellipse, we mark a stick or card with the dimensions *a* and *b*.



Once the stick is marked, one can lay out the ellipse by keeping the points marked a and b on the horizontal and vertical axes, respectively, and marking points on the ellipse at the mark O.



By moving the stick, one can mark out different points on the ellipse at point O, as long as the points marked *a* and *b* are in contact with the major and minor diameters.

If you're doing this on a surface that you can attach a framing square to, then instead of using marks, you can drive nails through the stick at the a and b marks (file their tips off a bit so they're not so sharp after driving them through the wood). Drill a hole at O large enough to hold a pencil and you can draw the ellipse by keeping the nails against the square (the square's right angle is used to mark the coordinate axes). Use double-sided tape or clamp the square to the work.

Approximate Ellipse with circles

[*blandford:107*]

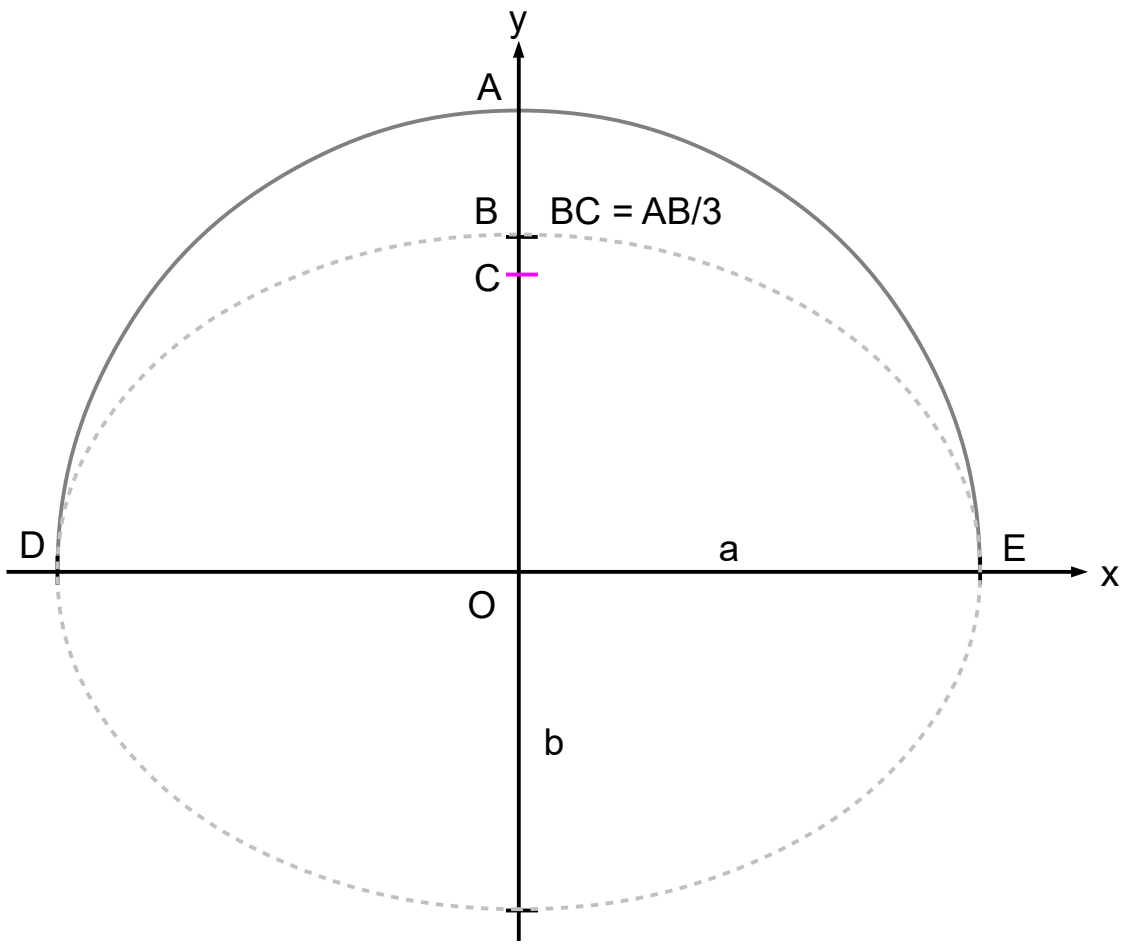


Figure 5

In Figure 5, construct the ellipse's axes and mark off the semimajor axis a (OE) and semiminor axis b (OB). Mark arc DAE with center at O and radius of a. Mark BC to be a length of 1/3 of AB.

In the following diagram, we'll only show the constructions in one quadrant, but the extensions to the other quadrants is obvious. The dotted line shows the true ellipse.

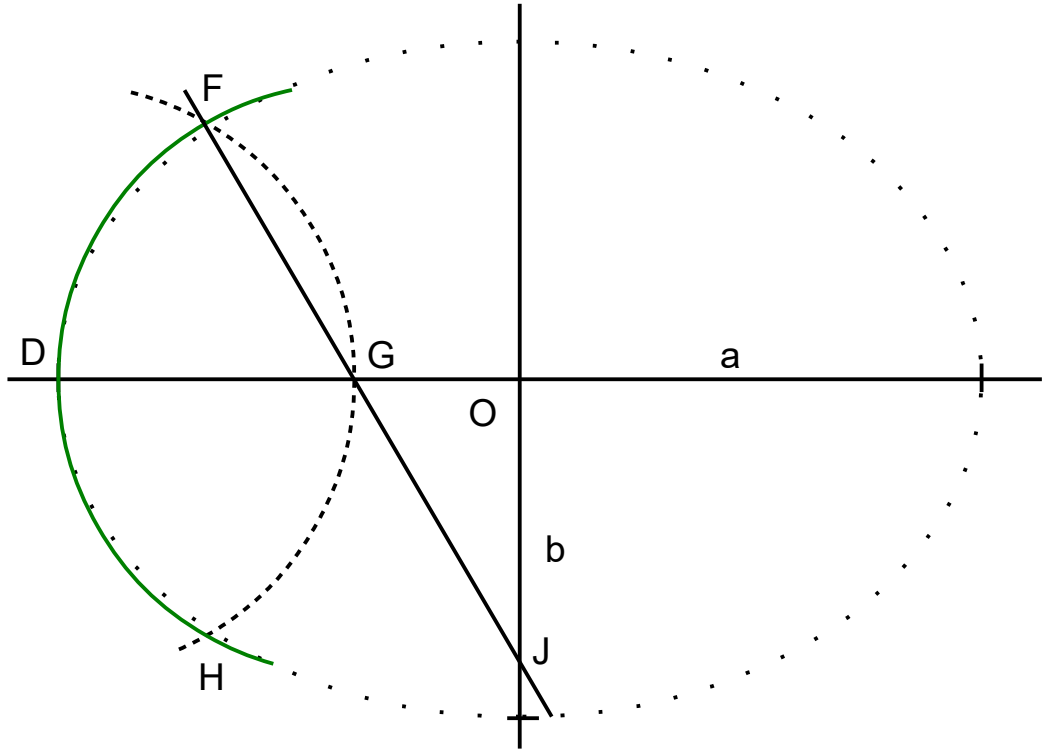


Figure 6

In Figure 6, set the dividers to distance OC in Figure 5 and mark the arc FGH with center at D. Without changing the dividers, mark the arc FDH with center at G. Mark point J from the intersection of the line FG with the y axis.

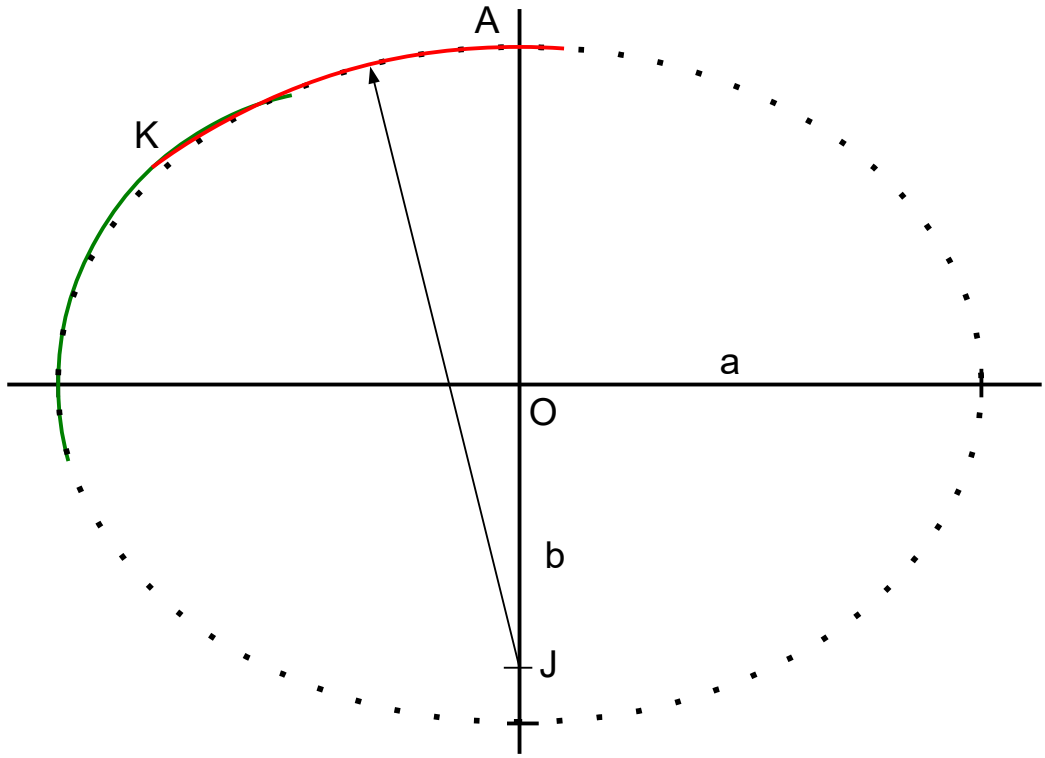
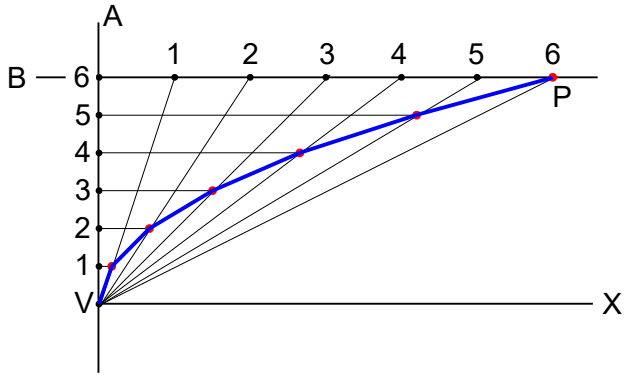


Figure 7

In Figure 7, draw arc AK with center at J, blending in with the green arc. The dots show the desired ellipse and the red and green arcs show the approximation.

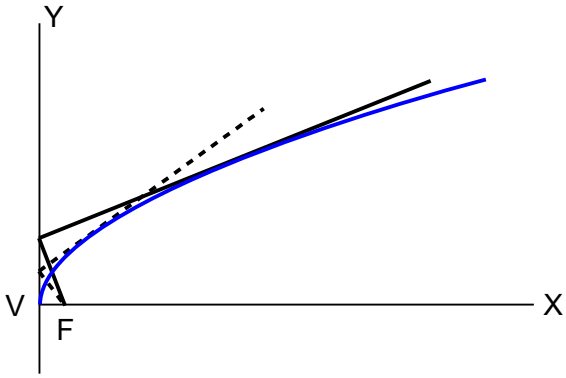
Parabola

Given the vertex V and a point P on the parabola.

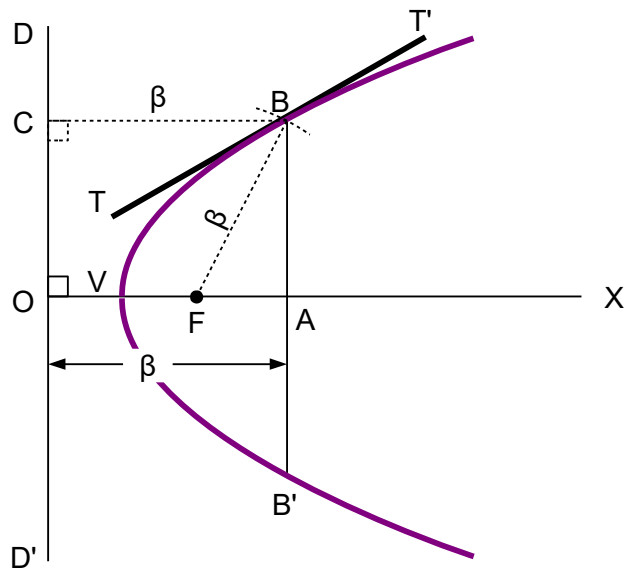


Assume the x axis extends from V to the right along VX. Draw line BP parallel to VX through P. Then construct the line AV perpendicular to VX. Mark *n* equally-spaced points (*n* = 6 shown in diagram) along VA with the nth point ending on BP. Mark n equally spaced points between the intersection of BP and AV and point P. Draw horizontal lines through the points on VA and lines from V to the points on BP. The points at the intersection of lines with the same number are on the parabola.

[*huntington:139:146*] Given the vertex V and the focus F: draw VY perpendicular to VF and slide the vertex of a right angle along VY so that one side always passes through F; then the other side will be tangent to the parabola.



[*morris:259:268*] Given the focus F and directrix line DD'. Construct OX perpendicular to DD' and passing through F. The vertex V bisects line OF. For an arbitrary point A on OX a distance β from point O, mark an arc with radius β with center at F intersecting the line BB'. Points B and B' are on the parabola.



The parabola is such at BC = BF = β.

To construct tangent TT' to the parabola at point B: draw BC perpendicular to DD'. The tangent line TT' is the bisector of angle FBC.

Regular polygons

Let the polygon have *n* sides. The geometry is

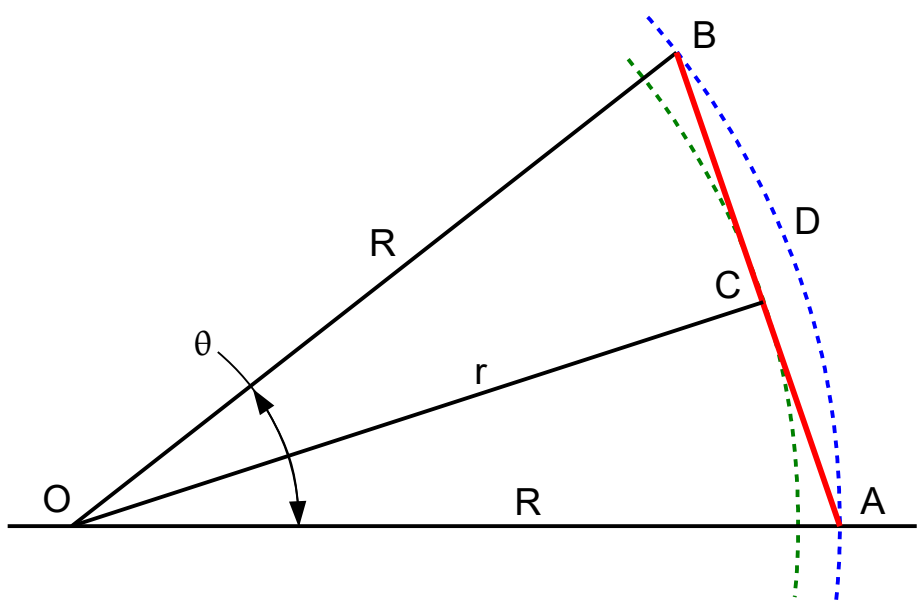


Figure 8

The blue dashed outside circle is the circumscribed circle and the green dashed inside circle is the inscribed circle. The central angle is $\theta = 2\pi/n$. The relationship between *R* and *r* is

$$r = R\cos\frac{\theta}{2} = R\cos\frac{\pi}{n}$$

The easiest way to lay out the polygon is to scribe the circumscribed circle ADB, then step off consecutive points with the dividers set to distance AB:

$$AB = 2R\sin\frac{\theta}{2} = 2R\sin\frac{\pi}{n}$$

Any setting error is cumulative, so a trial run should be done to ensure you wind up back at the starting position. You can avoid the cumulative error by calculating the divider setting for each point using the cosine law and always having one leg of the divider at point A:

$$\text{divider setting } = R\sqrt{2(1-\cos\theta_i)} \text{ where } \theta_i = \frac{2\pi}{i}$$

Accuracy of this method, however, is less when θ is near 180° because of loss of significance. In addition, it is more tedious and, thus, more prone to setting errors. However, if *n* is even, then you can scribe two points per setting from A and two more points from the point corresponding to A at the other end of the circumscribed circle's diameter.

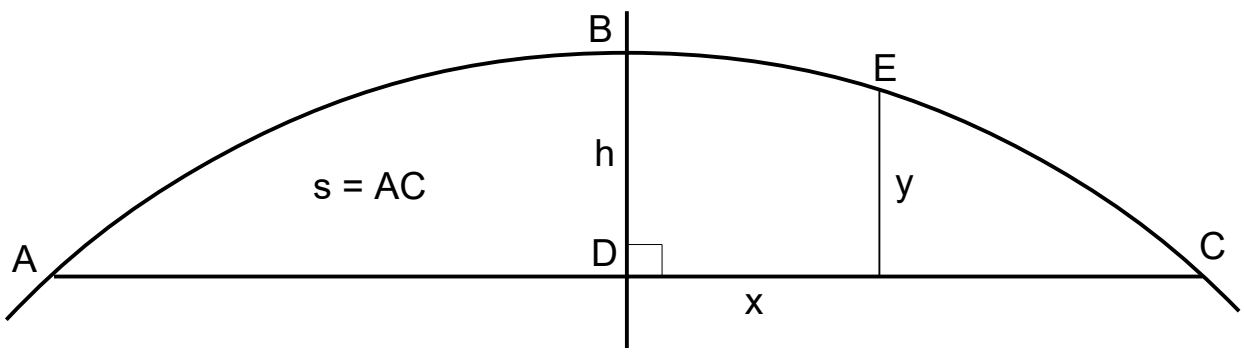
Another method which avoids the cumulative error but is also more tedious is to use the Cartesian coordinates of each point. These are gotten through the usual polar conversions:

$$x_i = R\cos\theta_i = R\cos\frac{2\pi}{i}$$
$$y_i = R\sin\theta_i = R\sin\frac{2\pi}{i}$$

where i = 0, 1, 2, ..., n -1 and $\theta_i = \frac{2\pi}{i}$. Machinist's handbooks [*mh71*] give tables of the coordinates, as toolmakers used to use them on jig boring machines.

Arcs

Suppose it is necessary to lay out the circular arc ABC as shown in the figure, but the center of circle is either inaccessible (e.g., inside a wall) or inconveniently far away. Use Cartesian coordinates.



In the diagram, suppose we know *h* and the radius *r* of the arc ABC. Then the coordinates (*x*, *y*) of the point E on the arc with the origin at point D has the ordinate *y*

$$y = \sqrt{r^2 - x^2} - r + h$$

If you have the arc ABC and know the points A and B, but don't know the radius *r* of the circle, you can calculate it from measuring *s* = AC and *h* = BD:

$$r = \frac{s^2 + 4h^2}{8h}$$

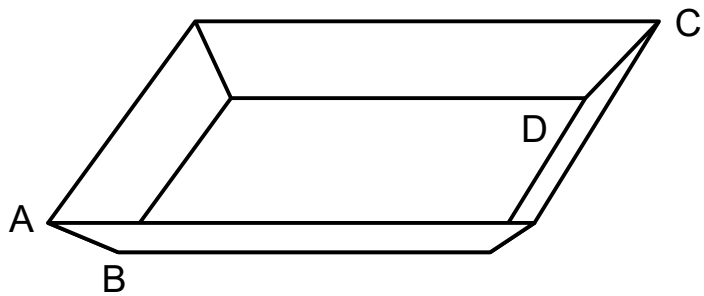
It should be evident that if *r* is large and *h* is small, small measurement errors can result in large uncertainties in *r*.

A construction method to get the radius and the center of the arc is to construct the perpendicular bisectors to two chords of the circle. If the center is not accessible, then the line BD that it is on

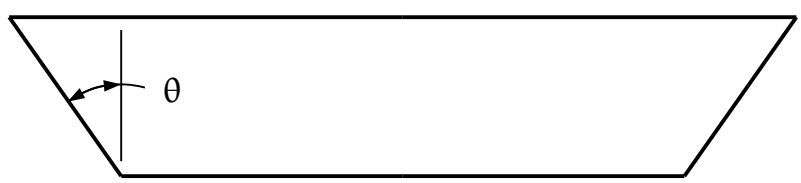
can be gotten by constructing the perpendicular bisector of the chord AC.

Tapered boxes

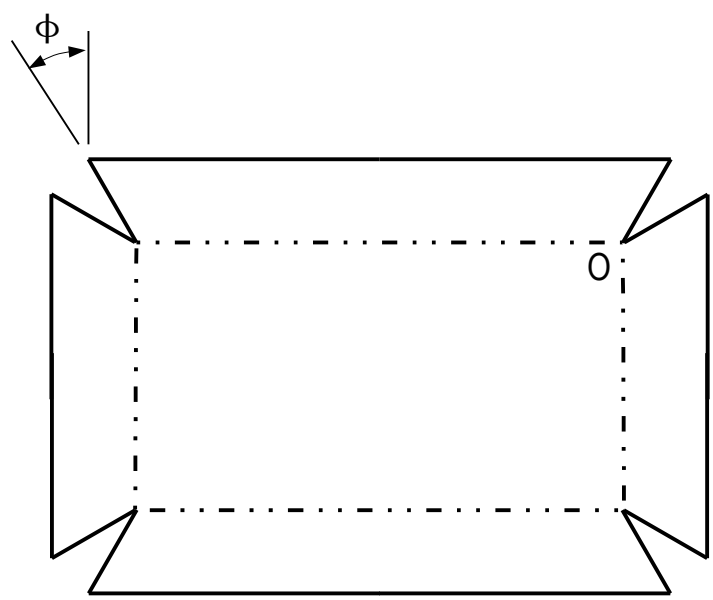
Suppose we want to make a symmetrical tapered box from some thin material. The box is as follows:



In the vertical plane containing the points A, B, C, and D, we have the following view:



In the developed pattern for the box, the question is: given θ, what is the angle ϕ?



$$\cos\phi = \frac{1}{\sqrt{1+\sin^2\theta}}$$

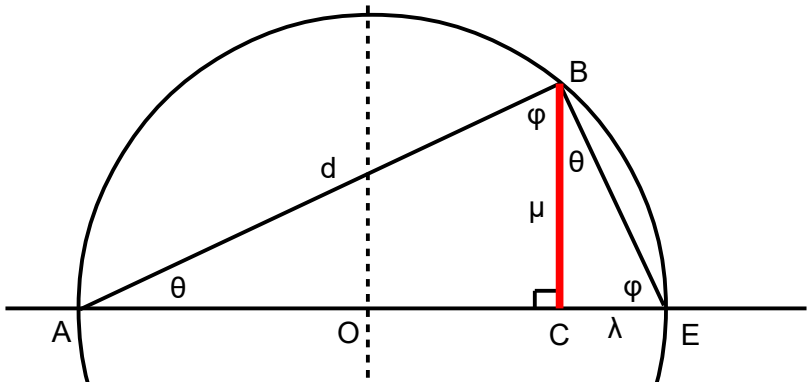
This was derived by Jerry at Dr. Math.

Rattle measurements

Rattle measurements are a technique for measuring a bore diameter when you don't have fancy

tools available. Basically, the idea is that you insert a rod of known length into a bore and measure how much it moves side-to-side (i.e., "rattles"). Knowing the rod's length and the side-to-side measurement lets you calculate the bore diameter. It's a simple and accurate technique.

Let d be the length of the rod AB and let it rattle a distance 2μ:



The diameter D of the circle is $D = \mathrm{AC} + \lambda$. We get AC from the Pythagorean Theorem: $\mathrm{AC} = \sqrt{d^2 - \mu^2}$. Thus,

$$D = \sqrt{d^2 - \mu^2} + \lambda$$

Now, we just need to express λ in terms of d and μ. Because of Thales' theorem, angle ABE is a right angle. It's then not hard to see that triangle BCE is similar to triangle ABC -- and angle CBE is equal to θ. Therefore

$$\lambda = \mu \tan\theta$$

But

$$\tan\theta = \frac{\mu}{AC} = \frac{\mu}{\sqrt{d^2 - \mu^2}}$$

so

$$\lambda = \frac{\mu^2}{\sqrt{d^2 - \mu^2}}$$

Thus

$$D = \sqrt{d^2 - \mu^2} + \frac{\mu^2}{\sqrt{d^2 - \mu^2}}$$

or

$$D = \frac{d^2}{\sqrt{d^2 - \mu^2}}$$

Finally, if we let μ be designated as a fraction of the length d: $\mu = \alpha d$, we have

$$\boxed{D = \frac{d}{\sqrt{1 - \alpha^2}}}$$

Note that if μ is zero, then α is zero and $D = d$.

This "rattle tool" can be useful when you can measure μ reasonably accurately. Since this means you probably have to have a rule in the hole, it's not practical for smaller diameters. However, you

can use some inside calipers to transfer the rattle distance 2μ to a rule -- and probably know it within, say, about 1/2 mm or less. Keep this method in your back pocket when e.g. an inside micrometer isn't available.

How good is the method? Let's look at an uncertainty calculation with with the uncertainties package of python[3]:

```
from uncertainties import ufloat
from uncertainties.umath import sqrt

d = ufloat((200, 0.01))     # Rattle rod length
mu = ufloat((10, 0.5))      # Half the rattle distance
a = mu/d
D = d/sqrt(1 - a*a)
print("Diameter = " + str(D))
```

We assume we're measuring a bore slightly larger than 200 mm. We know the length of the rattle rod accurately (200 mm with a standard deviation of 0.01 mm). The resulting bore diameter is

```
Diameter = 200.250+/-0.027
```

which we'd probably state as 200.250(27) mm. The 0.027 mm uncertainty is interpreted as a standard deviation. The bottom-line conclusion is that this method can measure the diameter to within better than 0.1 mm.

Useful tips

If you have a small angle measured in degrees such that the angle is less than about 5° (i.e, roughly 1/10th of a radian), the sine and tangent of the angle can be calculated by dividing the angle by 57.3 (i.e., 180/π); this is suitable for hand calculations to roughly 0.1%. (This approximation was well-known to slide rule makers for the ST scale and comes from the first term in the Taylor series for the sine.)

> Suppose you want this approximation to be within 0.1% for hand calculations. Then the value of x in radians for which this is true is the solution of the transcendental equation $\sin(x) - x = 0.001$. Using the Taylor series, we get the approximate equation $x^3/6 = 0.001$ which gives us x = 0.18 radians (10°).
>
> For the tangent, you'll find $\tan(x) - x = 0.001$ for x = 0.1438 radians = 8.2°.

For hand calculations, some useful rational approximations are:

$$
\begin{aligned}
&\pi \approx 22/7 \quad (0.04\,\%)\\
&\pi \approx 355/113 \quad (0.000008\,\%)\\
&\sqrt{2} \approx 99/70 \quad (0.005\,\%)\\
&\sqrt{3} \approx 97/56 \quad (0.005\,\%)\\
&\ln(10) \approx 175/76 \quad (0.002\,\%)\\
&\log(e) \approx 76/175 \quad (-0.002\,\%)\\
&\log(\pi) \approx 87/175 \quad (-0.001\,\%)
\end{aligned}
$$

The percentages in parentheses are the differences from the actual values.

The approximation to $\sqrt{2}$ can be written $\frac{10}{7}(1-0.01)$. This makes it easy to use in a hand calculation because in a multiplication by $\sqrt{2}$, you just shift the decimal point, divide by 7 to get the result R, then subtract R/100 (shift the decimal point) from R to give the result.

This is a general technique in hand calculation because of the approximation from the generalized binomial theorem:

3 This python library utilizes linear uncertainty propagation.

$$(1\pm\delta)^a(1\pm\epsilon)^b \approx 1\pm a\delta\pm b\epsilon \quad \text{if } \delta \ll 1 \text{ and } \epsilon \ll 1$$

For example, to calculate

$$\frac{106\times 84}{37}$$

we write

$$\frac{100(1+0.06)\times 80(1+0.05)}{40(1-0.075)} \approx \frac{100\times 80}{40}(1+0.06)(1+0.05)(1+0.075)$$

Using the approximation again results in

$$200(1+0.06+0.05+0.075)=200(1.185) = 237$$

We're low by 1.5%.

If you have to calculate the tangent of an angle near a multiple of $\pi/2$, you can use the relation

$$|\tan(x)| = \left|\frac{1}{\tan(x-\frac{\pi}{2})}\right| = \left|\cot(x-\frac{\pi}{2})\right|$$

References

Many of the older books can be downloaded from Google books.

albert Albert, A., *Solid Analytic Geometry*, McGraw-Hill, 1949.

anderson H. Anderson, ed., *Physics Vade Mecum*, American Institute of Physics, 1981.

as M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, 5th printing, 1968.

bartsch Bartsch, H., *Handbook of Mathematical Formulas*, (translation of 9th German ed.), Academic Press, 1974.

blandford Blandford, P., *The Master Handbook of Sheetmetalwork*, TAB Books Inc., 1981

carroll J. Carroll, *Practical Geometry for Art Students*, Burns & Gates, London, 1880.

casey J. Casey, *A Treatise on Spherical Trigonometry*, 1889.

coffin Coffin, J., *Vector Analysis*, Wiley, 1911.

corral Corral, Michael, *Vector Calculus*, published by author on web, 2008, http://www.mecmath.net/.

corral M. Corral, *Trigonometry*, http://www.mecmath.net/trig), 2009.

crc Beyer, W., *CRC Standard Mathematical Tables*, 26th ed., CRC Press, 1981.

drmath http://mathforum.org/dr.math/faq/formulas/faq.ag3.html#threeplanes Used with permission. Dr. Math is a great web resource and I encourage you to visit the site -- you'll find all kinds of useful stuff plus great volunteers to help you with math questions.

eucl C. Mansford, *A School Euclid*, 1874. Books 1 and 2 of Euclid's *Elements*. Based on Robert Simson's work.

feynman R. Feynman, *The Feynman Lectures on Physics*, Addison-Wesley, 1963. In particular, see chapter 22 in volume 1, as it discusses how logarithms are calculated; the discussion ends with Euler's jewel.

gf I. Gelfand and S. Fomin, *Calculus of Variations*, translated by R. Silverman, Prentice-Hall, 1963.

gr I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed., Academic Press, 1965.

haswell C. Haswell, *Mechanics' and Engineers' Pocket-Book*, Harper & Bros., 78th ed., 1920.

hawkes Hawkes, H., Luby, W., Touton, F., *Solid Geometry*, Ginn & Co., 1922.

hudm R. Hudson and J. Lipka, *A Manual of Mathematics*, Wiley, 1917. This is the math sections and tables from [*hudson*].

hudson R. Hudson, J. Lipka, H. Luther, and D. Peabody, *A The Engineer's Manual*, Wiley, 1917. While some of the engineering material is obsolete, this is a superb book.

huntington E. Huntington and L. Fischer, *Mathematical Tables*, which was taken from section 1 of *Mark's Mechanical Engineers' Handbook*, McGraw-Hill, 1916. This book has excellent tables for doing hand calculations. You can download this as a book from Google books (it's only 5 MB compared to the whole 100 MB edition (1916) of Marks).

ilyin V. Ilyin and E. Poznysk, *Analytic Geometry*, MIR Publishers, 1981.

kent W. Kent, *The Mechanical Engineers' Pocket-Book*, 9th ed., Wiley, 1916.

lawrence J.D. Lawrence, *A Catalog of Special Plane Curves*, Dover, 1972, 0-486-60288-5.

maor E. Maor, *Trigonometric Delights*, Princeton University Press, 1998. Many interesting things about trigonometry.

marks L. Marks and T. Baumeister, *Standard Handbook for Mechanical Engineers*, 7th ed., McGraw-Hill, 1967.

marks1916 L. Marks, *Standard Handbook for Mechanical Engineers*, McGraw-Hill, 1916. Much of the content of later editions is in this older edition.

mh04 E. Oberg, F. Jones, H. Horton, H. Ryffel, *Machinery's Handbook*, Industrial Press, 27th ed., 2004.

mh71 E. Oberg and F.D. Jones, *Machinery's Handbook*, Industrial Press, 19th ed., 1971.

morris I. Morris, *Practical Plane and Solid Geometry*, 6th ed., Longmans, Green and Co., London, 1897. The book's focus is on solving specific construction problems with straightedge and compass. Also see [carroll].

osgood Osgood, W. and Graustein, W., *Plane and Solid Analytic Geometry*, Macmillan, 1921.

perry R. Perry and C. Chilton, *Chemical Engineer's Handbook*, 5th ed., McGraw-Hill, 1973.

salmon Salmon, G., *A treatise on the analytic geometry of three dimensions*, 4th ed., Hodges, Figgis & Co., Dublin, 1882.

schmall Schmall, C., *A First Course in Analytical Geometry*, 2nd ed., van Nostrand, 1921.

snyder Snyder, V. and Sisam, C., *Analytic Geometry of Space*, Henry Holt and Co., 1914.

steinmetz C. Steinmetz, *Engineering Mathematics*, McGraw-Hill, 1911.

vo A. van Oosterom and J. Strackee, IEEE Transactions on Biomedical Engineering, vol. BME-30, No. 2, pg 125, Feb 1983.

wilson Wilson, E., *Vector Analysis*, Yale University Press, 1922. Based on the lectures of Gibbs.

wp A wikipedia page on the relevant topic.

wynne W. Wynne and W. Sparagen, *Handbook of Engineering Mathematics*, van Nostrand, 1920.

2945