RMS Measurement

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This is intended to be a practical guide to making RMS measurements with real instruments. There is no perfect tool, so one has to learn the advantages and disadvantages of each approach. The technical level is at roughly a lower-division undergraduate engineering/science level. The math level is mostly basic calculus. Derivations are confined to an appendix.

- **Waveform catalog:** One thing I've not seen elsewhere is the proposed catalog of common waveforms. I will derive the formulas for the RMS, AC-coupled RMS, average-responding, and DC offsets associated with each waveform (these derivations don't need to be part of the document that customers see, but the derivations must be correct). Thus, for example, you'd be able to calculate the RMS value of a waveform and also calculate how much in error the average-responding meter would make if it measured the same waveform. Where appropriate, I'll include plots that show this error as a function of some waveform parameter like duty cycle.
  - Since the analytical derivations might be a pain, I could do all the work with a python script and present things to e.g. 4 significant figures. In fact, this script would make a plot of the waveform and print out the formulas in Open Office math format.
- Agilent's "RMS with DMM" comments on pg 2 that RMS is the only AC voltage measure that does not depend on the shape of the waveform.
- In the Making RMS measurements section, show a simple table of measurements of a 1 V RMS sine wave and square wave made with the HP 3435A, 3456A, and 3400A. Do this for 1) centered waves and 2) ones with a DC offset to put the bottom of the wave on the x-axis.
- **RMS with a scope**: Though I love my HP 3400A, a modern user doesn't need an RMS meter because a digital scope can do this measurement for him. If the waveform is DC coupled, you get the AC+DC measurement; if AC-coupled, you get the AC-coupled RMS. Verify this is true on the HP scope (yes, it's true). Then get out the 50 MHz function generator and verify you get the AC+DC and AC-coupled values up to 50 MHz. This will show that the scopes can measure the RMS values up to the bandwidth of the scope. At 1 kHz, a sine wave from the B&K 3003 gave 3.23 V on DC-coupled and 1.78 V on AC-coupled. Note this sine wave has a DC offset to make the lower portion 0 volts. The sine wave measured 1.79 V on the HP 3400A. The square wave measured 2.48 V on the 3400A and 2.45 V on the scope. Changing the square wave to 1.001 MHz, the 3400A read 2.42 and the scope read 2.39. Thus, we see these tools repeat to perhaps 1%, which is fine for troubleshooting. With the scope averaging 8 values, the RMS value settled down to 2.386 ± 0.002 V. Switching to 9.99 MHz, the scope read 1.756 V and the 3400A read 1.82 V. Still pretty close. The scope had a bit of a challenge to display the waveform and it was definitely not a sine wave.
- **Beware:** Beware of tutorials found on the web -- some of them have inaccurate or misleading information. I've noticed this in a number of pages where it was clear that English wasn't the first language of the writer.
- **Scope probe compensation signal:** If you have a scope, use its square wave output as a check of your voltage measurement (tip from [*hpbb*]). My HP 54601B scope uses a 0 to 5 V

square pulse (bottom at 0 V) at 1 kHz. The HP 3435A gives 2.77 V, the HP 3456A gives 2.46 V for AC-coupled and 3.48 V for RMS. The HP 3435A gives 2.52 V DC, the HP 3456A gives 2.52 V DC.

- ♦ The RMS value should be $5/\sqrt{2} = 3.54$.
- ♦ HP 3456A
  - ♦ 2.46 V for AC-coupled, 2.52 V for DC
  - ♦ AC+DC was 3.48 V
  - ♦ DC and AC-coupled in quadrature is 3.52 V
- ♦ HP 3435A
  - ♦ 2.77 V for AC, 2.52 V for DC
  - ♦ DC and AC-coupled in quadrature is 3.74 V (so it over-estimates)
- ♦ HP 54601B scope with a 150 MHz B&K probe in 10X attenuation (and properly compensated)
  - ♦ RMS value was 3.53 V
- ♦ **Conclusions**: The scope was the most accurate. Certainly it has the largest bandwidth. The average-responding 3435A was the most in error, not surprisingly.

Tool	RMS value, V	% dev from theoretical value
HP 3435A	3.74	+5.6
HP 3456A, quadrature	3.52	-0.6
HP 3456A, AC+DC	3.48	-1.7
HP 54601B scope	3.53	-0.3

♦

1. Motivation

a) Heat produced in a resistor
   i. Production of heat by DC
   ii. Production of heat by AC
   iii. Experiment is a resistor in a Styrofoam box with a silicon signal diode or thermistor to measure temperature inside the box compared to the temperature outside the box. The steady state temperature differential is measured.
   iv. The AC signal will be a square wave because it's easy to produce with a switching transistor.
   v. The experiment will demonstrate that the average values for the square wave's current and voltage do not predict the same temperature differential as the DC experiment.

b) Power dissipation in a resistor
   i. Classic voltage source & resistance diagram using the lumped parameter model. Mention this and its importance. Discuss conditions when the model is no longer correct (high frequencies, external magnetic field, E&M or nuclear radiation, etc.).
   ii. Atomic model of what's going on: inelastic collisions of electrons with material's lattice atoms.
   iii. Ohm's law is a misnomer; it really should be called the definition of resistance. Ohm's discovery was the linearity of a metallic conductor.
   iv. Lumped parameter models are approximations. For example, in a real circuit with significant Joule heating in the wires with high currents, the lumped parameter model needs to be amended.
   v. Conservation of energy and charge have fundamental circuit ramifications.

c) Motivate the need for RMS measurements: A simple example shows that the RMS amplitudes of the current and voltage that are sinusoids are the only amplitude measures that predict the same power level as in the DC case. This is the fundamental reason the RMS value is so important: it lets us use the familiar equation $P = V\,i$ for both DC and AC calculations.

2. **Practical measurement examples**
   a) These demonstrate a variety of meters and the results you get from them. These results will be examples that get analyzed later in the document when more background is given.
   b) Analog meter with simple averaging circuit
   c) Fluke 83 (older average responding DMM)
   d) Three Aneng meters (low bandwidth DMMs, AC-coupled RMS)
   e) HP 3400A (AC-coupled RMS)
   f) HP 3456A (real RMS)
   g) Adding AC-coupled RMS and DC values in quadrature
   h) Digital scope measurements
3. **RMS -- the details**
   a) The main text will just contain a few formulas; the derivations will be off in an appendix, as most users won't need them or care.
   b) Definition
   c) Different types of measurements
      i. "True RMS" = AC-coupled RMS (this is what most "RMS" meters read).
      ii. "AC+DC" = the real RMS (equal to the standard definition of RMS) = quadrature sum of AC-coupled RMS and DC offset.
      iii. Average-responding
   d) Crest factor
      i. Definition
      ii. Why it's important
      iii. What it tells you about the frequency content of a waveform.
      iv. An instrument's specification for crest factor tell you about the instrument's bandwidth and dynamic range.
      v. Examples
         A. In the study of vibration, CF tells you about impact events.
         B. In acoustics, CF tells you about pops and short transients.
         C. Why the FFT can mislead: FFT of pulse vs. noise -- they look the same.
         D. Sizing a UPS can be strongly affected by the CF of the load's power waveform.
   e) Similarity: how it helps you recognize when waveforms have the same RMS values.
   f) Piecewise estimation of the RMS value of complicated waveforms (simple method for approximating the RMS value of arbitrary waveforms).
   g) RMS properties
      i. Effect of adding a DC offset to a waveform
      ii. Addition of two arbitrary waveforms
      iii. Ordering
         A. The RMS value for an AC waveform is always greater than the value measured by an average-responding meter.
         B. Example: can lead to undersized wiring when measuring current using an average-responding meter or measuring waveforms with crest factors larger than the instrument can handle.
4. **Numerical calculations of RMS values**
   a) Use of python scripts to avoid the need of analytically deriving formulas
   b) The conversion factors can be given to e.g. 4 figures, which should be adequate for virtually all practical measurement situations
5. **Measuring RMS values**
   a) Types of instruments
      i. Digital multimeters
      ii. Specialized instruments
      iii. Oscilloscopes
      iv. Analog meters
   b) Measurement methods
      i. Thermal conversion

ii. Analog computation
iii. Digital computation

c) How to determine what type of meter you have
d) Using oscilloscope probes with your RMS meter
   i. Can work well
   ii. Can also be a source of significant measurement error if you're not aware of the issues
e) How to measure AC+DC with your true RMS meter
f) Frequency response problems
   i. Plot of measurement errors of the different types of measuring tools for a challenging measurement: determining the RMS value of a narrow pulse.
g) Practical RMS measurements
   i. Know what's being measured (e.g., true RMS or AC+DC).
   ii. Know the limits of your measurement tool.
   iii. Verify important characteristics such as wave type, crest factor, and frequency content. The usual method is to use a scope in conjunction with the RMS meter.
   iv. Where possible, verify the measurement with another measurement tool that uses a different method (e.g., use the RMS measurement of a digital scope when viewing the waveform).

6. **Examples of RMS measurements**
   a) Non-sinusoidal waveforms
      i. Noise
      ii. Acoustics
      iii. Low duty cycle pulses
      iv. Unknown voltages
   b) Measuring power
      i. A nonlinear load: a PC switching power supply. The current and voltage will be measured and the crest factor of both calculated. This gives a graphic demonstration of how both average-responding meters and even RMS meters can give incorrect results.
   c) Tracing a circuit
   d) Monitoring nonlinear loads
   e) Challenges with high crest factor waveforms
7. **Waveform compendium**
   a) Collection of the different waveforms seen in engineering practice with formulas for
      i. DC voltage
      ii. Average-responding voltmeter measurement
      iii. AC-coupled RMS voltage ("true RMS")
      iv. RMS voltage ("AC+DC")
   b) Sine
   c) Square
   d) Triangle
   e) Rectangular pulse
   f) Trapezoidal pulse
   g) Sawtooth pulse
   h) Fractional sine pulse
   i) Half sine
   j) Full-wave rectified sine
   k) DC + linear ripple
   l) Numerical factors related to RMS measurements: when making RMS measurements, you sometimes get results different than you expect. If you calculate the ratio of the measured value to the expected value, you may find it in this table and it can help you account for the discrepancy.
8. **Appendix: Derivations of the results given in the text.**

9. Appendix: simple experiment demonstrating that the RMS values of AC voltages and currents are the proper measures to predict the heating in a resistor.
   a) Equipment: DC power supply, 1.5 volt battery, 6.3 V filament transformer, two 1/4 watt resistors, small silicon signal diode, miscellaneous construction materials found around the home.
10. References

- Relate the statistical variance to the RMS for a discrete set of values. Note from wikipedia page on variance for a set of equally-likely values

$$Var(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2$$

  In other words, the variance is proportional to the sum of the squared deviations of all points from each other. **The RMS value of a waveform with no DC offset is the same as the population standard deviation of its points**.
- In `docs/crest_factor.pdf`, a simple argument shows that the FFT of a pulse and random noise are the same (i.e., you can't tell them apart from their *amplitude* spectra). Instead, the crest factor is a measure that's useful to distinguish between such things. Impact acoustic noise cause by bearing or gear tooth wear or cavitation can be found by examining the crest factor (such impact noise will have high crest factors). The paper gives two pictures, one of a machine with serious rolling element bearing wear and the other with a bit of noise, but only out of balance. The machine with serious wear has a CF of 3 vs. the CF of the out-of-balance machine of 1.6.
  - An interesting experiment would be to record acoustical signals at various points on a car. This likely has been done in depth by some engineering folks somewhere. The RMS and crest factor measurements might be valuable to help diagnose problems.
- HP Journal and app notes
  - Application note 60, *Which AC Voltmeter*, 13 Apr 1967. Appendix 2 is by Barney Oliver.
  - Measure impedance with an oscillator and voltmeter, HPJ Jul-1952. This needs to be looked at, as it could make plotting impedance as a function of frequency fairly fast, probably up to around 1 MHz or more with the HP 3400A RMS voltmeter. However, note that what's measured is |Z|; no phase information is given.
- Add an experiment that uses a function generator and a selection of voltmeters that measure average-responding (Fluke 83), AC-coupled RMS (HP 3400), and real RMS (HP 3456A). Contrast the measurements and show that the formulas given in the compendium predict these readings. Look at
  - A sinusoid with no DC offset
  - A sinusoid with a DC offset
  - A square wave with no DC offset
  - A square wave with a DC offset
- The c:/cygwin/ebooks/phys_eng/electronics/Articles/ACvoltmeter.pdf has a number of formulas

in it for situations I haven't covered yet. But it's not written all that well and the math notation is sloppy, so the formulas should be carefully checked. If the formulas are correct, it might be the basis of a useful script to calculate desired values.

- Books to get from library to check for more info:

  Beatty, Fink, *Standard Handbook for EEs* 16th ed.

  K. Gieck, R. Gieck, *Engineering Formulas*, Publication Date: June 5, 2006 | ISBN-10: 0071457747 | ISBN-13: 978-0071457743 | Edition: 8

- For the script that checks formulas, an additional way to check an RMS formula is to get the Fourier series coefficients and sum them in quadrature. This puts the burden of calculating the details on numpy.
  - Fourier's theorem is that a well-behaved bounded periodic function can be expressed as a constant plus an infinite sum of sinusoids $f(t) = A_0 + \sum_{k=1}^{\infty} \left[A_k \cos(k\,\omega\,t) + B_k \sin(k\,\omega\,t) \right]$
  - The RMS value of f is $\sqrt{A_0 + \frac{1}{2} \sum_{k=1}^{\infty} \left(A_k^2 + B_k^2 \right)}$
- Write a script that simulates the output of a DC power supply with a sawtooth ripple. Then calculate the Fourier expansion and print it out. Include a column showing the measurement error if the measuring instrument's bandwidth was cut off at that point. For an example, see the `Measuring RMS Values of Voltage & Current.pdf` document.
- http://ecmweb.com/power-quality/crest-factor-key-troubleshooting-parameter has a pretty good discussion of crest factor related to power stuff, especially the practical method of monitoring the peak voltage along with the RMS and comparing the peak to 1.41 times the RMS -- if they're significantly different, the waveform probably has lots of harmonics.
- The 1973 HP article on AC analog voltmeters by Harry Logan is excellent.

This is an article aimed at hobbyists who want to know a little bit more about RMS (root mean square) measurements. It's motivated by a need to quantify the power in an electrical waveform that changes over time. I assume you've had a few math/science courses at the college level if you want to understand the math; however, the math knowledge is not essential to get the main points of this paper.

Motivation

Let's do a simple thought experiment: hook up a resistor in series with a battery. Most folks know that the resistor will heat up. Now, connect the same resistor to an alternating voltage; say, a low-voltage sine wave. Most of us know the resistor will also heat up -- this is how the elements on our electric stove helps us cook our meals. But, if you think about it, it tells you something rather deep about electrical current flow in materials -- you'd be led to an atomic theory of charge carriers converting some their kinetic energy to thermal vibrations of the atomic lattice. You could quickly get mired down in quantum statistical mechanics trying to explain things.

We don't need to go that deep, but being quantitative beings, we want to say something about *how much* the resistor will heat up. An experiment that can give important insight is in the section *Experiment: Heating value*. It is a relatively simple experiment that can be done with basic lab equipment that demonstrates that it is the RMS values of the voltage and current through a device that result in the same power as a DC voltage and current that are numerically the same. This experiment and ones like it lead us to the equations for the time-averaged macroscopic power being dissipated in the resistor[1]:

1 We're assuming it's a pure resistance so there are no complications with reactance.

$$P_{\text{time averaged}} = V_{rms} i_{rms} = \frac{V_{rms}^2}{R} = i_{rms}^2 R \tag{1}$$

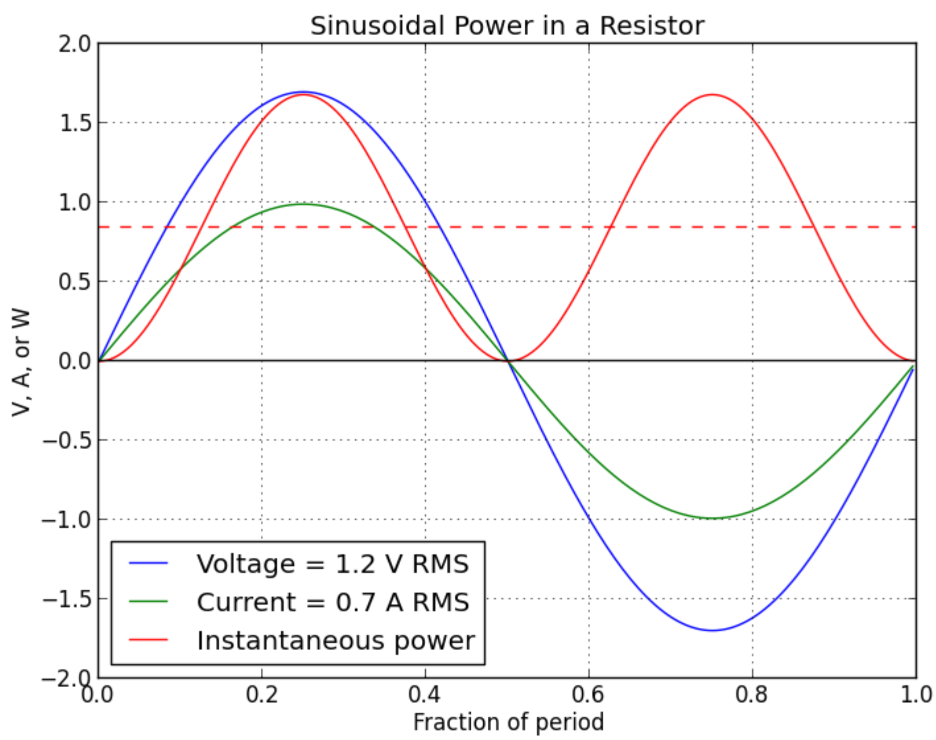
This is desirable because these equations look like the same formulas for the instantaneous power; thus, it is easier to remember.

This document will look at some of the issues with measuring RMS electrical values, which not quite as simple as measuring unchanging values like DC measurements.

Energy and power

We often are interested in comparing the energy and power involved in the use of electricity. For a resistive load R with a voltage V across the load, the power being dissipated in the load (i.e., turned into heat) is V^2/R. If the voltage is a function of time $V(t)$, then the instantaneous power is $V^2(t)/R$.

For a typical sinusoidally-varying AC waveform, the instantaneous power isn't of much interest because it goes from zero to a maximum positive value, to zero again, then repeats. Here's a plot of the instantaneous power dissipated in a resistor given an assumed sinusoidal voltage and current:



We note:

- The instantaneous power is always greater than or equal to zero.
- The power waveform is at twice the frequency of the voltage or current.
- The power waveform is also a sinusoid, but it has a DC offset, unlike the voltage or current. This DC offset results in real power dissipation in the resistor.
- The dashed line is 0.84 W, the product of the RMS voltage and RMS current. Note how it's at the center of the power waveform.

We show elsewhere that the 0-to-peak value of a sinusoid is $\sqrt{2}$ times its RMS value, so if these voltage and current waveforms have RMS values V_0 and i_0, then their equations are (ω is the angular frequency[2])

$$V = \sqrt{2}\, V_0 \sin(\omega t)$$
$$i = \sqrt{2}\, i_0 \sin(\omega t)$$

The equation for the instantaneous power is

$$P(t) = 2 V_0 i_0 \sin^2(\omega t) = 2 V_0 i_0 \left(\frac{1}{2}[1 - \cos(2\omega t)]\right) = V_0 i_0 [1 - \cos(2\omega t)]$$

This shows that the DC offset of the power waveform is $V_0 i_0$. Since the cosine varies between 1 and -1, you can see that the power waveform varies symmetrically between 0 and $2 V_0 i_0$ about the value $V_0 i_0$.

If we were to average the instantaneous power waveform over an integer number of periods, the contribution of the cosine averages to zero. We conclude that the average power over times substantially longer than the period is the constant $V_0 i_0$. For this to be true, we must have that V_0 and i_0 are equal to $1/\sqrt{2}$ times the sinusoid's 0-to-peak amplitude -- ***and this is the only relationship that will give us these results***.

We've thus shown

> The sinusoidal voltage $\sqrt{2} V_0 \sin(\omega t)$ and current $\sqrt{2}\, i_0 \sin(\omega t)$ over times substantially longer than the period dissipate the same power as a DC voltage V_0 and DC current i_0.

Notation

All waveforms in the document are assumed to be periodic with period *T* unless otherwise stated.

t	Time
$V(t)$	Arbitrary function of time
$V_{dc} = V_{avg}$	The DC offset of a periodic waveform. It is equal to the average calculated by equation (2). It is also the constant term in a Fourier expansion of a periodic waveform.
V_{aa}	The absolute average value of a periodic waveform that is measured by average-responding meters and calculated by equation (3).
$\bar{V}$	The average of a periodic waveform over a period. Same as V_{aa}.
V_{pp}	Peak-to-peak voltage.
V_{rms}	The real RMS value of a periodic waveform as calculated by equation (4).
$\boldsymbol{V_{ac}}$	The AC-coupled RMS value of a periodic waveform. This is the RMS value of a waveform with the waveform's DC offset removed (this is usually done in instruments by a coupling capacitor). I'll try to remember to use color to remind you that it is not really the true RMS functional defined by the mathematics.
true RMS	An unfortunate marketing term that refers to the AC-coupled RMS value of a periodic waveform.
AC+DC	A marketing term that refers to the real RMS value of a periodic waveform (i.e., one that would be calculated by equation (4)). It is equal to V_{rms}.
[*xyz*]	Refers to a reference with the short-hand name *xyz* in the *References* section.
$\mathbb{Z}_0$	The set of all positive and negative integers excluding zero.

2 Angular frequency is 2π times the frequency in reciprocal time units; this lets the sine function be evaluated using radians.

$\mathbb{Z}$	The set of all positive and negative integers.
$\sum_{k\in\mathbb{Z}}$	Sum over all integers k.
$\sum_{k\in\mathbb{Z}_0}$	Sum over all non-zero integers k.

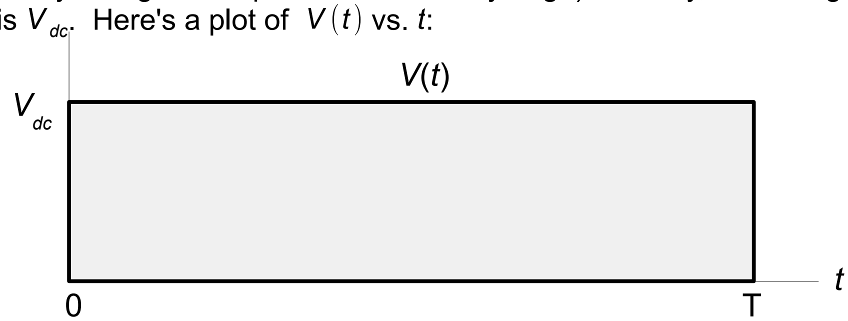
SI units will be used and I will try to adhere to strict SI syntax. Thus, I'll use "V AC" to denote an AC voltage (note the space character), where common usage is VAC -- which is incorrect SI syntax. The above symbols in the table are in italics, whereas SI units are in a non-italics font.

Root Mean Square

To use formula (1), we need to quantify the "amount" of a waveform in a single number. Thus, given a waveform[3] $V(t)$ that's a periodic function of the time *t* over the interval of 0 to *T*, we want a "thing" $\boldsymbol{F}(V(t),\ 0,\ T)$ to produce a number that characterizes how "big" the waveform is. This thing ***F*** that **maps a function to a number** is called a functional (and the study of the extrema of functionals leads to an interesting area of mathematics [*gf*]). Note it's not a function, as a function maps one number into another (your elementary math classes introduced you to a common functional, the Riemann integral, but probably didn't call it a functional). *The functional is mapping the whole set of values of a function into a single number.*

Average

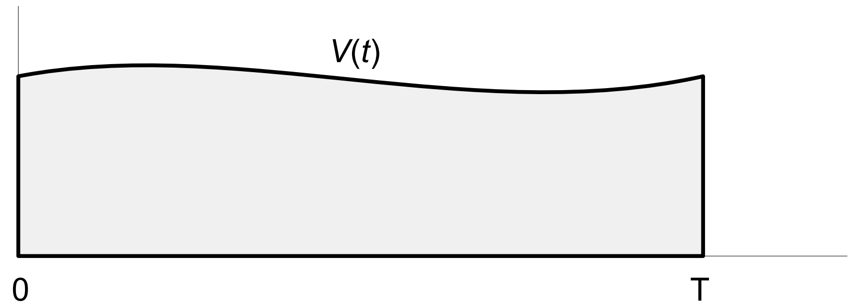
One common functional that measures a quantity $V(t)$ that varies over time *t* is the **average**. Let's motivate this with a geometrical argument. Suppose we have a DC voltage $V(t)=V_{dc}$ (a DC voltage is a periodic signal if you regard the period as arbitrarily large). Clearly, its average value over the time from 0 to *T* is V_{dc}. Here's a plot of $V(t)$ vs. *t*:



A geometrical insight is that the area of the gray rectangle under the constant curve $V(t)=V_{dc}$ is $V_{dc}T$. Thus, if we divide the area under the "curve" $V(t)$ by *T*, the width of the interval, we get the average voltage $V_{avg}=V_{dc}$.

This geometrical insight suggests how to extend the definition to non-constant functions. Let's perturb the constant waveform a bit so it's no longer constant:

3 The waveform is arbitrary, but since we're often dealing with voltage measurements, we'll use the designation *V*(*t*) for an arbitrary periodic function of time.

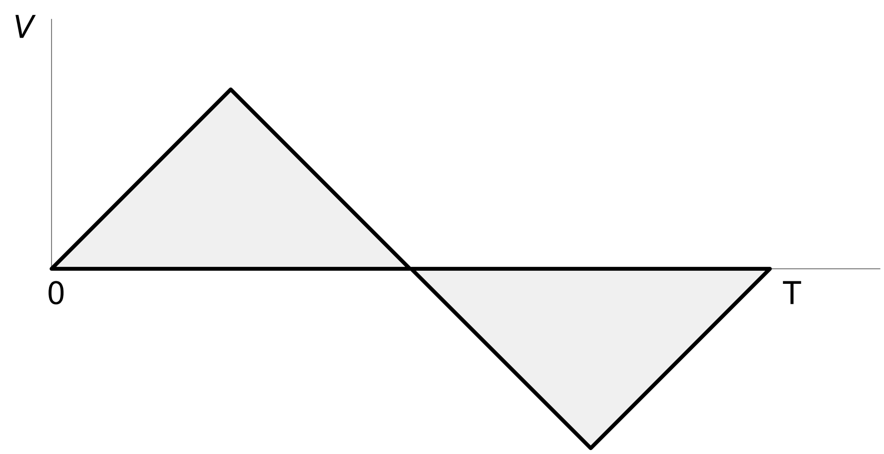


Now, we'll have to give the function's form $V(t)$ to specify the waveform. But we can still use the concept of the area under the waveform divided by the width *T* to calculate the average. This area is the functional

$$V_{avg} = \frac{1}{T}\int_0^T V(t)\,dt \tag{2}$$

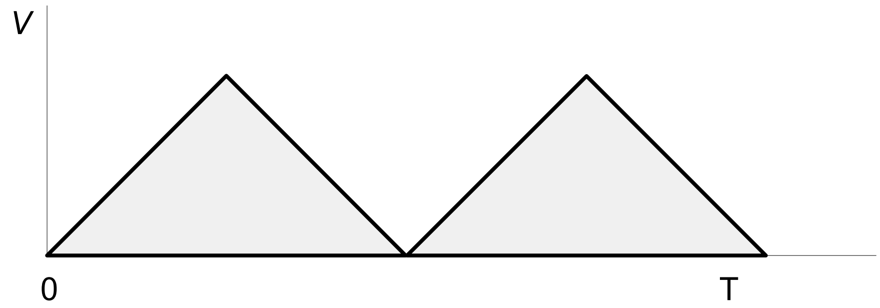
This formula reduces to the same result as the DC case when $V(t)$ is a constant. This makes it a generalization of the "average" of a DC voltage. For an analog average-reading meter, this integral is "computed" because of the inertia of the meter movement [*hp60:1A-1:8*].

While we might think this average is a good single number to quantify the "amount" in a waveform, it doesn't work for a triangle wave:



If we calculate V_{avg} for this waveform, it will come out zero because the portion below the *t* axis contributes a negative area. A little thinking shows how to fix this. We use our physical intuition to realize that the positive voltage through a load results in a positive current and, thus, a positive power being dissipated[4]. Similarly, a negative voltage results in a negative current, which also results in a positive power being dissipated. We can get the desired result by taking the absolute value of $V(t)$, which "simulates" the negative voltage being multiplied by a negative current of 1:

4 Assuming there are no sources of power in the load.



In fact, this is a perfectly acceptable definition of an average value (we call it the absolute average to distinguish it from the average calculated without using the absolute value):

$$\bar{V} = V_{aa} = \frac{1}{T}\int_0^T |V(t)|dt \tag{3}$$

I'll use the notation $\bar{V}$ to indicate the average of a waveform as defined by equation (3).

For an odd[5] periodic function such as we've shown (or, such as the sine), we could instead calculate the average of the first half of the waveform from 0 to $T/2$ and double the result -- we'd get the same answer because of the symmetry. Let's calculate the average of the sine wave with unity amplitude (just do the integration for the positive half of the waveform):

$$\bar{V}_{avg} = \frac{1}{\pi/2}\int_0^{\pi/2} \sin t\, dt = \frac{2}{\pi}[-\cos t]_0^{\pi/2} = \frac{2}{\pi}[0-(-1)] = \frac{2}{\pi} = 0.6366$$

We could stick with this definition of an average -- and that's what average-responding voltmeters measure. Their measured value is multiplied by $\pi/2\sqrt{2} = 1.1107$ to get the equivalent RMS value of a sine wave (see the integration below that shows that the RMS value of a unity amplitude sine wave is $1/\sqrt{2}$).

RMS

Unfortunately, it is known from experimental measurements that **the absolute average doesn't predict the heating power of a waveform**. Thus, the average voltage and current can't be used in equation (1) above to calculate the time-averaged power. Instead, the RMS current squared times the resistance gives the power being dissipated in a resistance due to Joule heating.

> You can do such an experiment yourself and verify this important experimental fact. See the section *Experiment: Heating value*.

This is why the RMS values of current and voltage are important -- they let us calculate the heating power for voltages and currents that change with time and, if we know the power, we can calculate the associated energy and use the conservation of energy to derive important knowledge about the system.

> The insight is that you will get the same heat generation in a resistive load with a DC voltage *V* across it as you will get if you put an AC voltage of RMS value *V* across the same load, no matter what the shape of the waveform.

This insight is for a theoretically perfect resistive load. Complications are introduced with non-zero reactance, copper losses, and things like frequency-dependent and lossy permittivities and permeabilities.

The name "root mean square" tells you how the RMS calculated -- it's the square root of the mean of

5 Odd in the mathematical sense, not "strange". An odd function f has the property that $f(t) = -f(-t)$ for all t (note that the function's value at zero must be zero).

the square of the function *f(t)*. Thus, square the function, integrate it over its period, divide the result by the period, and take the square root:

$$\text{RMS} = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2\,dt} \tag{4}$$

The RMS value is sometimes called the effective value, the heating value, or a century ago some people called it the virtual value. We'll call it the **RMS value**.

The square of a function has the nice property that it's everywhere positive or zero. Thus, the RMS value is always positive or zero -- and it's only zero if the function *f(t)* is *identically* equal to zero on the interval [0, T].

For periodic functions, the RMS value is calculated over one period -- and in the majority of cases when we're measuring things, we're interested in periodic waveforms.

Let's calculate the RMS value of a unit-amplitude unit-frequency sine wave $V(t) = \sin(2\pi t)$ where the period is $T = 1$:

$$V_{rms}^2 = \frac{1}{T}\int_0^T \sin^2 2\pi t\,dt = \frac{1}{1}\left[\frac{1}{2}t - \frac{1}{4}\sin 4\pi t\right]_0^1 = \frac{1}{2}$$

Thus, the RMS value of a unit-amplitude sine wave is $\frac{1}{\sqrt{2}} = 0.707$. This yields the useful rule **for a sine wave** that you can multiply the peak-to-peak amplitude by $\frac{1}{2\sqrt{2}} = 0.354$ to get the RMS value and multiply the RMS value by $2\sqrt{2} = 2.83$ to get the peak-to-peak amplitude.

> You likely have 240 V RMS running your stove and other heavy appliances. The peak-to-peak voltage of this waveform is $240(2\sqrt{2})$ or 679 V. This is definitely a high voltage -- you don't want to touch those two main conductors.
>
> When discussing AC voltages, it's common to utilize the RMS value. Thus, a line voltage of 240 volts AC is referring to a sine wave at line frequency with an RMS value of 240 volts. The RMS value is so common that it is generally understood that an AC voltage measurement is RMS unless otherwise stated.

For a periodic waveform, a useful calculational technique to get the RMS value is to add the squares of the averages for the negative portion and the positive portion and take the square root of the sum.

$$\text{RMS} = \sqrt{\bar{V}_{\text{neg}}^2 + \bar{V}_{\text{pos}}} \tag{5}$$

The RMS value of a discrete function is analogously defined to be

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{n} [V(t_i)]^2}{n}} \tag{6}$$

There's no requirement that the samples be evenly spaced, but in practice they usually are. Both expressions (4) and (6) are functionals. You could instead define $V(t)$ in the continuous-case definition to be a sum of suitably-weighted Dirac delta functions and the continuous definition could be used for the discrete case.

For a sampled waveform, if a signal has no DC component, the AC-coupled RMS value (we'll talk more about this below) is equal to the population standard deviation statistic (such statistics are also functionals). You can recognize this from the definition of population standard deviation if you realize that no DC offset implies the average of the sampled points is zero.

See the section *Ordering* for more functionals that represent the "amount" of waveform in a single number.

Crest factor

A crest factor less than 1.4 indicates a waveform that tends to have a flat top. Greater than that indicates a pointy waveform. A sine wave has a crest factor of $\sqrt{2}$.

In IR doc on fluorescent ballast design, it notes that CF is important in relation to the life of fluorescent tubes. Tube manufacturers recommend that tubes be run at CFs below 1.7. The challenge in such a design is the high starting impedance of a tube with the very low resistance of the plasma once the tube has been started -- and the cost must be kept down.

CF is also important for sizing current transformers.

The following google books link from a 1916 document talks about crest factor. Here's a 1914 link and the definition on pg 1794:

> **15 Crest-Factor or Peak-Factor.** The ratio of the crest or maximum value to the r.m.s. value. The crest factor of a sine-wave is $\sqrt{2}$.

This is from "Transactions of the American Institute of Electrical Engineers", Jun 25 to Dec 1914, Vol. 33, part 2.

Show some examples of high crest factors for some real loads. http://www.pge.com/includes/docs/pdfs/mybusiness/customerservice/energystatus/powerquality/harmonics.pdf shows such a waveform on page 4. Crest factor is also used in vibration studies to characterize mechanical bearing faults.

A similar statistical measure is kurtosis.

In acoustics, the crest factor is usually given in dB: $\text{dB} = 20\log(CF)$. Ambient noise has a crest factor around 10 dB (CF = 3.2); a gunshot can have a crest factor of 30 dB (CF = 31.6). In music, a processed mix (has audio level compression) can have CF around 4-8 and 8-10 for an unprocessed recording. ENT surgeons have used crest factors to help with the diagnosis and treatment of snoring problems.

DC waveforms have crest factors of 1. Sine waves have crest factors of $\sqrt{2}$. Any waveform with a crest factor larger than 1.4 might be a distorted sine wave.

Generate some waveform plots with different crest factors.

CF is quite important for power meters. Voltages and currents with high CFs can lead to much higher product waveforms like power.

The CF spec for an instrument implies the range over which it can measure things linearly.

An application area is uninterruptible power supplies (UPS): the crest factor specification describes how much current over the average levels the UPS can supply. This is important for sizing a UPS for a particular load. Such things make it important for the designers of the load (e.g., a computer's power supply) to keep the crest factor requirements of the load's power supply near that of a sine wave. If you've ever monitored the power line current of a typical computer's power supply (e.g., a switching power supply), you've seen that it's definitely not a sinusoidal current, but one with a high crest factor.

True random noise should have an infinite CF when measured, but obviously you won't see such things in practice.

A pragmatic check of crest factor: most instruments specify their crest factor behavior for signals at full scale. The instrument can typically measure signals with higher crest factors at smaller signal levels (this stands to reason because the dynamic range and bandwidth of the meter haven't changed). Because of this, change the range of your meter (go to a higher range) and repeat the measurement. **If it is significantly different than the first measurement, you may be looking at a signal with a crest factor beyond the capabilities of your instrument**. Remember that the meter's accuracy may be specified for full scale, so the use of the higher range may reduce the measurement accuracy.

Be aware that the input impedance of a measurement system can vary substantially with frequency. The things you connect to the meter's input can affect this strongly. For example, a passive scope probe's impedance can drop by 4 orders of magnitude (or more) from DC to its bandwidth rating.

For critical measurements where a signal's crest factor is unknown, it is wise to back up the measurement with an examination of the signal with an oscilloscope.

An important characteristic of a waveform for practical RMS measurements is the **crest factor**. This is defined to be the 0-to-peak value (positive or negative, whichever is larger) of the waveform divided by the waveform's RMS value.

> You may occasionally see definitions for the crest factor that say it's the peak divided by the AC-coupled RMS value. However, this would lead to division by zero for a DC waveform, which is obviously not what we want. Using the real RMS value on a DC waveform results in a crest factor of 1, which is what is expected. All other waveforms will have a crest factor greater than 1 -- for example, the sine wave has a crest factor of $\sqrt{2} = 1.4$. It is rare to see the crest factor specified to more than one decimal place.

The larger the crest factor is, the more difficult it is for circuits to measure the RMS value because of the increased bandwidth and dynamic range needed to capture the signal accurately.

- ♦ Less expensive instruments can handle waveforms with crest factors in the region of 3 to 5.
- ♦ More expensive lab-quality instruments can handle crest factors of 10 or higher.

Let's calculate the crest factor for a pulse waveform of period *T*, amplitude *V*, and duty cycle *D* as shown in the following diagram

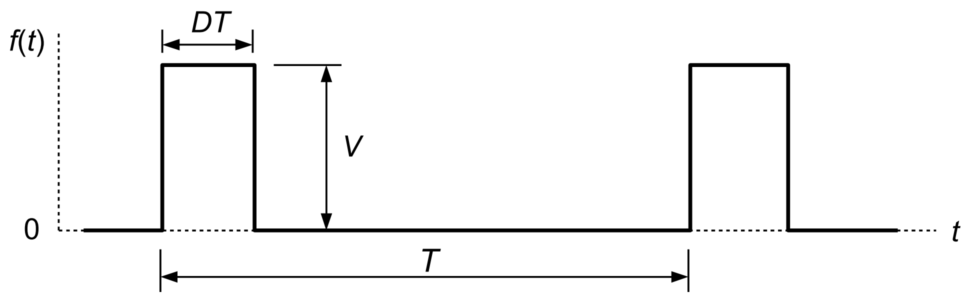


Figure 1

From the definition of RMS (equation (4)), we have, assuming the pulse starts at t = 0,

$$V_{rms} = \sqrt{\frac{V^2}{T} \int_0^{DT} dt} = V\sqrt{D}$$

Since the crest factor is defined as the ratio of the peak value to the RMS value, we've derived that the crest factor of a pulse as shown in Figure 1 is

$$\text{Crest factor of a 0-to-V peak pulse} = \frac{1}{\sqrt{D}}$$

It can be a worthwhile exercise to measure the crest factor response of your voltmeter(s) if you have a function generator that can deliver pulses. This is because the crest factor for pulses is easily calculated and, if you do not measure the set value, then the instrument is having trouble with signals at that crest factor level. Following are some example measurements.

HP 3456A voltmeter, 1 V range, AC-coupled RMS value.

Some interesting uses of crest factor (CF) measurements:

- ♦ The current CF is important in the life of fluorescent tubes (ref. P. Wood, *Fluorescent Ballast Design Using Passive P.F.C. and Crest Factor Control*, International Rectifier, no date given).

- Sizing current transformers.
- Vibration studies to characterize impact noise in mechanical bearings.
- CF in acoustics can characterize the noise (e.g., a gunshot can have a CF over 30).
- Ear-nose-throat surgeons have used CF to help with the diagnosis and treatment of snoring problems.
- Sizing uninterruptible power supplies: the CF describes how much current the UPS can supply over the average needs.
- CF in power systems are important to properly size/rate the components, especially transformers (see here).
- High CF in currents and/or voltages can result in high instantaneous powers.

Numerical calculation of RMS

Using python and numpy (see [*python*] and [*numpy*], you can calculate the RMS values of waveforms using the discrete definition (6) without having to do any integration. Here's a chunk of code that prints the RMS value of a sine wave (the RMS calculation is emphasized):

```
from numpy import *
n, period = 100, 2*pi
t = arange(0, period, period/n)          # Make an array of points
f = sin(t)                               # Get the sine at each point
print("RMS = %.3f" % sqrt(mean(f*f)))      # Square root of the mean square
print("Absolute average = %.3f" % mean(abs(f)))
```

When run, the output is

```
RMS = 0.707
Absolute average = 0.636
```

The RMS value is $1/\sqrt{2}$. The average value is $0.707/0.636 = 1.111$ lower than the RMS value. If you run the script for other waveforms, you'll see that this factor doesn't work to get the RMS value. We have to conclude **average-responding meters won't correctly display the RMS value of non-sinusoids**. Note that scripts such as these will give numerical results that approach the analytical values when you increase the number of points in the waveform sufficiently.

Let's illustrate this result with some experimental measurements. A function generator was set to produce a 100 Hz square wave (with no DC offset) with a 1.000 V RMS amplitude as measured by an HP 3456A voltmeter. When this was put into a Fluke 83 digital multimeter (an average-responding meter), it read 1.111 volts. A 45 year-old HP 3435A digital multimeter read 1.109 V, so it's another average responding meter.

You can use a script to calculate the RMS value of a pulse train. Page 5 in the document [*ag1392*] gives the example of a 1.984 Vpp pulse train with a 2% duty cycle (and calculates that the RMS value will be 280 mV). The following script calculates the RMS value of this waveform:

```
from numpy import *
n, duty_cycle = 100, 0.02
t = arange(0, 1, 1./n)
f = zeros(n)
for i in range(int(n*duty_cycle)):  # Set the non-zero points
    f[i] = 1.984
rms = sqrt(mean(f*f))
print("rms = %s  crest factor = %s" % (rms, max(f)/rms))
print("average = %s" % mean(f))
```

When run, the output is

```
rms = 0.280579970775   crest factor = 7.07106781187
average = 0.03968
```

You can see the average is only 14% of the RMS value -- an average-responding meter would be seriously in error. The crest factor of this pulse train is $1/\sqrt{0.02}$ or about 7.

This 1.984 V 0-to-peak pulse with a 2% duty cycle was set up in a function generator. The AC-coupled RMS value measured by the HP 3456A digital multimeter was 0.2793 V. The HP 3400A measured 0.278 V. The Fluke 83 measured an unstable 74 mV. This doesn't mean the Fluke 83 is a "bad" meter -- it just means it is unable to make such measurements

When you're experimenting with such scripts, you may want to increase the number of points n by a factor of 10 and rerun the script to ensure you're not seeing sampling size effects.

RMS of a Fourier expansion

Steinmetz 1916 p 14

If a waveform with period T has the Fourier expansion

$$f(t) = \sum_{k=1}^{\infty} A_n \sin\frac{k\omega t}{T} + \sum_{k=1}^{\infty} B_n \cos\frac{k\omega t}{T}$$

(note there's no DC component), then this can be substituted into equation (4) to get the RMS value

$$f_{RMS} = \sqrt{\frac{1}{2}\sum_{k=1}^{\infty}\left(A_k^2 + B_k^2\right)}$$

Example: the square wave with unity amplitude has only odd harmonics:

$$f(t) = \frac{4}{\pi}\sum_{k=1,3,5\ldots}^{\infty}\frac{1}{k}\sin\frac{2\pi k t}{T}$$

The RMS value is then

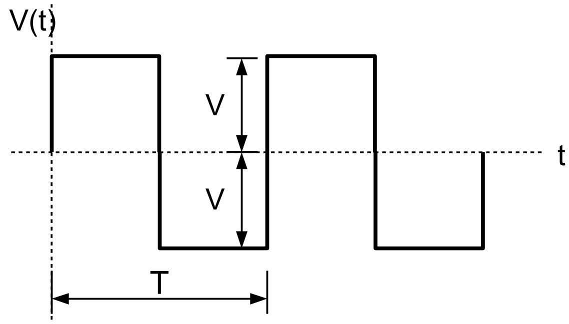
$$f_{RMS}^2 = \frac{1}{2}\sum_{k=1,3,5\ldots}^{\infty}\left(\frac{4}{\pi k}\right)^2 = \frac{8}{\pi^2}\sum_{k=1,3,5\ldots}^{\infty}\frac{1}{k^2}$$

The sum of the odd terms is $\pi^2/8$ ([*crcmt* pg 454]), so the resulting RMS value is 1, as expected.

AC-coupled RMS

Since many (if not most) of the RMS meters on the market make AC-coupled RMS measurements, let's look at AC-coupled RMS measurements in a little more detail. Let's use pulse waveforms to illustrate things, as they are easy to analyze.

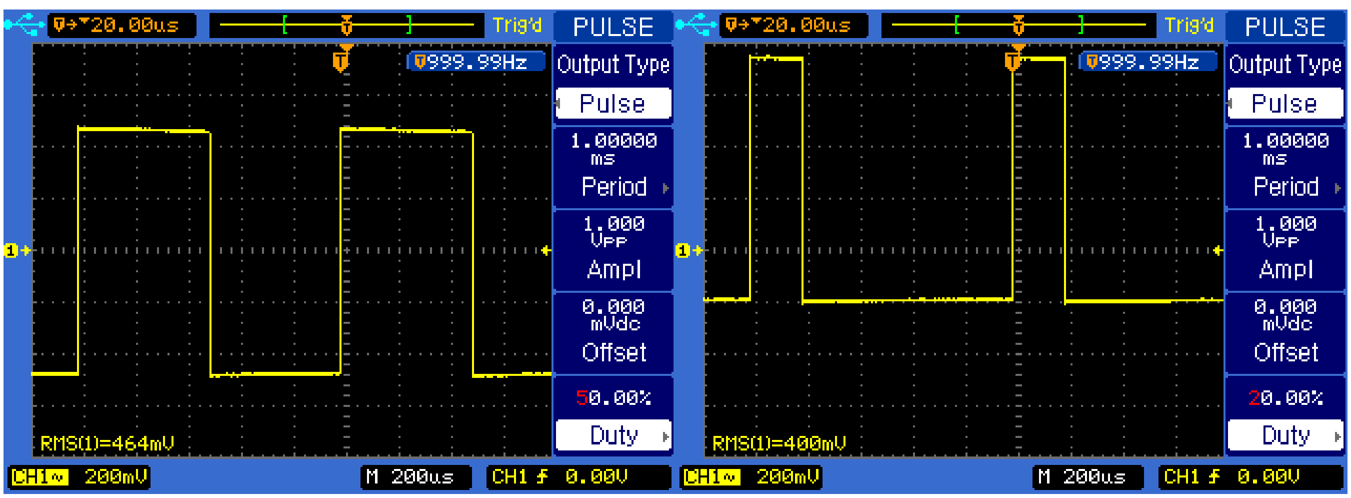
First, assume we have set a pulse generator to output a square wave that has equal excursions above and below zero volts (this means the average functional is zero and that there's no DC offset):



If you use a digital oscilloscope that is DC-coupled and set to display the RMS value of the waveform, you'll measure *V* for the RMS value of this waveform. Perhaps a little more surprising, you can vary the duty cycle and the scope will continue to measure the same value *V* for the RMS value of the square wave.

> It's not hard to see why this is true. Square the waveform and it will be a constant value of V^2 everywhere. Thus, the integral in the definition of RMS voltage will be *T* and the RMS value will always be *V*.

Now, set the scope to AC-coupled. For the square wave, you'll get the same measured value of *V* for the RMS value. However, when you start varying the duty cycle, you'll get RMS voltages less than *V* for duty cycles above and below 50%. If you watch these non-50% duty cycle waveforms on the scope's screen, you'll see them move up and down as the duty cycle is varied. The conclusion is that **non-50% duty cycle pulses have a DC offset**. Here's a picture of this happening on a B&K 2542B-GEN scope/function generator with a 1 kHz square wave:



The square wave in the left trace is centered vertically around 0 volts, but when the duty cycle is changed to 20% (and nothing else is changed), you can see the waveform moves as shown in the right-hand trace (remember the scope is AC-coupled).

Compare the areas of the waveform above and below the 0 volts line.

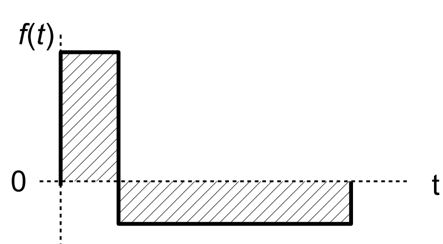


Figure 2

You see these areas are nearly equal (about 4 grid squares). This DC offset is the zeroth-order term of the Fourier expansion of the waveform *f*(*t*):

$$DC = \frac{1}{T}\int_0^T f(t)\,dt$$

In other words, it's the average. Since the AC-coupled measurement blocks this DC, the waveform displayed on the screen is offset by an amount such that the integral of the waveform is zero (i.e., the areas above and below 0 volts are equal). We can calculate this DC offset in terms of the duty cycle for these pulses. Suppose the pulse as a function of time is

$$V(t) = \begin{cases} V + V_{dc} & 0 \le t \le DT \\ V_{dc} & DT < t \le T \end{cases}$$

Requiring the integral to be zero means

$$\int_0^{DT} (V + V_{dc})\,dt + \int_{DT}^{T} V_{dc}\,dt = (V + V_{dc})\,DT + V_{dc}(T - DT) = 0$$

and solving yields

$$V_{dc} = -V\,D \qquad (7)$$

This lets us predict the AC-coupled RMS measured value for a pulse waveform. Start with a pulse of amplitude *V* and duty cycle *D* as shown in Figure 1 above. We know its RMS value is $V\sqrt{D}$. Subtracting the DC component in quadrature yields

$$\boldsymbol{V_{ac}} = \sqrt{V^2 D - (-V\,D)^2} = V\sqrt{D(1-D)} = \beta V$$

You can calculate what duty cycle is needed to get a particular value of β:

$$D = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\beta^2}\right)$$

where $\beta \le 1/2$. Note there are two distinct values; you'll see this behavior in the following measurements of the multiplier β:

D, %	β	Measured	Δ%
0.01	0.0100	0.0031	-69
0.1	0.0316	0.0323	2.2
1	0.0995	0.0998	0.30
10	0.3000	0.2996	-0.13
20	0.4000	0.4000	0.00
30	0.4583	0.4585	0.05
40	0.4899	0.4901	0.04
50	0.5000	0.5000	0.00
90	0.3000	0.2993	-0.25
99	0.0995	0.0998	0.30
99.9	0.0316	0.0332	5.0
99.99	0.0100	0.0033	-67

The measured values of β were gotten with an HP 3456A voltmeter making an AC-coupled RMS measurement of a 1 kHz pulse from a B&K 2542B-GEN function generator. The square wave's amplitude was set to read 0.5 volts at a 50% duty cycle, then the duty cycle was varied. Excluding the two duty cycles of 0.01% and 99.99% (these measurements have crest factors beyond the voltmeter's abilities), you can see that the formula predicts the measured value well.

The crest factor for the AC-coupled case is different. The crest factor for the pulse shown in Figure 1 is

$$\mathrm{CF} = \frac{1}{\sqrt{D}}$$

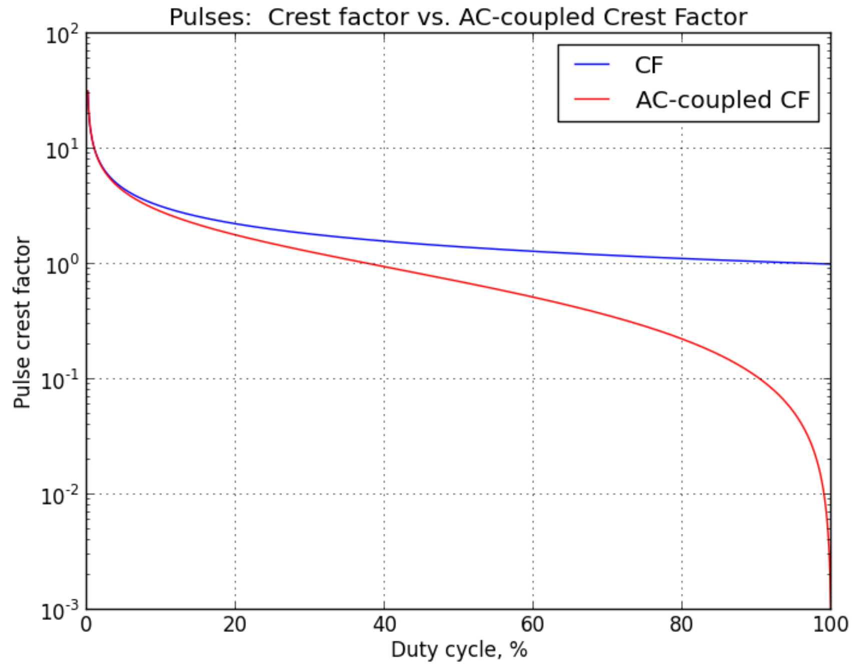
because the 0-to-peak value is V and the RMS value is $V\sqrt{D}$. For an AC-coupled measurement, the crest factor will be different because of the DC offset:

$$\mathbf{CF}_{ac} = \frac{V - V_{dc}}{V\sqrt{D}} = \frac{V - (-V\,D)}{V\sqrt{D}} = \frac{1 - D}{\sqrt{D}} = \mathrm{CF} - \sqrt{D}$$

or

$$\mathbf{CF}_{ac} = \mathrm{CF} - \sqrt{D} \qquad (8)$$

Here's a plot of this crest factor behavior **for pulses**



Instrumentation

In this section, we look at some of the instruments that can be had to make RMS measurements. Since RMS measurements are more complicated than average measurements, you should not be surprised that RMS-capable instruments are more complicated than averaging meters -- and, thus, more expensive.

Unfortunate marketing terminology

You'll come across the term true RMS when looking at marketing materials for voltmeters. Most of us would assume that this term means that the measured value is per equation (4), but you'd be wrong -- it means the AC-coupled RMS value. In other words, it's the RMS value of the waveform with the DC component removed. If you've used an oscilloscope, this concept of AC-coupling will be familiar. If I use RMS in a plain font, it means a measurement per equation (4).

If you have an AC-coupled RMS meter, you'll need to first measure the DC value of the waveform and combine it in quadrature with the AC-coupled RMS value to get the real RMS value of the waveform[6]:

$$\mathrm{RMS} = \sqrt{V_{dc}^2 + \boldsymbol{V}_{AC-coupled}{}^2} \tag{9}$$

To indicate a real RMS measurement, the marketers have decided on the term AC+DC. This is equivalent to the definition in equation (9).

Users would likely be less confused if the industry standardized on AC-coupled RMS and RMS, but we're probably stuck with the unfortunate terms true RMS and AC+DC.

The reasons for this go back to at least the introduction of the HP 3403A voltmeter in the early 1970's, which is where I've seen the first use of AC+DC (see page 63 of the 1972 HP catalog). HP introduced the 3400A RMS voltmeter in 1964 and called it a true RMS meter in the 1960's catalogs.

Agilent notes in [*ag1392*] that the term "true RMS" was used to distinguish between meters that actually read a real RMS value and those that calculated the RMS value based on a peak or

6 See *Adding a DC offset* to see why this is true.

average-responding measurement. Most measurement equipment fell into the latter category, such as the typically available VOMs and VTVMs. The situation is still roughly the same, as a digital multimeter with the ability to make a real RMS measurement is more expensive than the average-responding types, all other things being equal.

Instruments

The RMS-measuring instruments are divided into two categories: the lightweights and the heavyweights. These terms not only refer to the approximate mass of the instrument, but also to their capabilities. The lightweights are the numerous digital multimeters sold as true RMS instruments. The heavyweights are the more expensive lab-quality meters; these usually have higher bandwidths and crest factor specs and better accuracies -- and they often let you measure either true RMS or AC+DC.

I have only used a few of the instruments on this list and I emphasize that it isn't intended to be complete. There are likely other instruments out there I don't know about. These lists contain what I could track down with a reasonable amount of effort.

First are the heavyweights. These instruments tend to be able to measure waveforms with crest factors of 10 or greater and the meters have wide bandwidths. They are the lab-quality instruments that are intended to be used for serious scientific and engineering use -- and the new instruments are priced way beyond what the typical hobbyist can afford. However, a hobbyist can sometimes find older lab-quality instruments used on places like ebay. While there is some risk that you'll buy a broken instrument, the benefit is that you can get a working instrument for a fraction of what a new one would cost.

I got my HP 3400 voltmeter used a few decades ago for $65 from an old engineer who had reconditioned it and it has worked flawlessly for me for over two decades. In Nov 2020, I got it out to use it after it had been sitting on the shelf for a while. It operated erratically on various scales and OK on others. I suspected it was the electrical contacts of the rotary switch. I sprayed the switch contacts with DeoxIT[7] fluid, worked the switch through its ranges while wet, and the voltmeter worked perfectly again. This 53 year old instrument keeps on impressing me with its abilities.

The models with a gray background are obsolete, but can be found used. I used to include price estimates, but the used items are getting rarer and prices continue to go up.

Manufacturer	Model	Comments
Agilent	34401A	Digital, 10 Hz to 20 kHz (up to 100 kHz at reduced accuracy)
Ballantine	323	Analog, 2 Hz to 25 MHz
Boonton	93A	Analog, 10 Hz to 20 MHz, crest factor 6 at full scale, accuracy 1.5%.
Fluke	8920A	Digital, 10 Hz to 20 MHz, adjustable dBm reference resistance
HP	3456A	Digital, 20 Hz to 100 kHz, up to 250 kHz at 5% accuracy
HP	3403A	Digital, DC to 100 MHz
HP	3400A/B	Analog, 10 Hz to 10 MHz (B model was 20 MHz)
Rohde & Schwarz	URE3	Digital, 0.01 Hz to 30 MHz. Also reads DC and peak values.

The Agilent (now Keysight) 34401A (and its follow-on models) is a descendant of the HP 3456A; both instruments are general purpose digital multimeters. The other instruments are (primarily) dedicated RMS instruments. I've left out the HP 3455, 3457, and 3458 digital multimeters, although

7 A sprayed-on liquid used to recondition electrical contacts.

they could also be in this list (Agilent (now Keysight) still sells the 3458).

Here are a few lightweights. In general, the crest factor abilities are in the neighborhood of 3 to 5. DMM is "digital multimeter". Note that accuracy is generally specified in multiple parts with different values in different frequency bands, so quoting a single number is meant to give you a flavor of the order of magnitude. Prices gotten from web in July 2012; street prices may be cheaper.

Manufacturer	Model	Price, $	Comments
Agilent	U1251B	390	DMM, 1% accuracy over 45 Hz to 5 kHz (up to 30 kHz at reduced accuracy), crest factor < 3 at full scale, < 5 at half scale.
B&K Precision	2709B	105	DMM, 50-500 Hz, accuracy probably in the 1.5% to 2% range, crest factor < 3. It can measure both true RMS and AC+DC values.
Fluke	87V	400	DMM, 50 Hz to 20 kHz, 2% accuracy over this range. Crest factor up to 3 at full scale; up to 6 at half scale. The specs say add (2% of reading + 2% of full scale) for non-sinusoidal waveforms, so it's at best around 5% measurement for non-sinusoids.

Making RMS measurements

In practical work these are the types of measurements you'll run into:

1. Average
2. AC-coupled RMS (true RMS)
3. Real RMS (AC+DC)
4. Peak

I won't discuss peak measurements (see [*hpbb*] for some thoughts).

For DC and sinusoids, the average-responding meter works just fine and measures, not surprisingly, the absolute average V_{aa} (then the result is multiplied by $\pi/\sqrt{8} = 1.1107$ to get the equivalent V_{rms} value). If you're just measuring "normal" line voltages and currents and DC values, this is all the meter you need. I used quotes around "normal" to alert you that all may not be well. When you get around non-sinusoidal waveforms -- e.g., loads with reactive components, pulses, loads with switching going on (e.g., a triac or SCR), or lots of noise, an average-responding meter will give you readings less than the real RMS values -- and you won't be aware of this fact. In a previous section you saw why -- for a high crest factor waveform like a pulse, the average can be substantially less than the RMS value (and for narrow pulses, most of the energy content is in the harmonics, not the fundamental). If you're calculating the power, a non-unity power factor can also mess up your calculations if you don't take it into account, even if you're using RMS values for voltage and current.

> If you're measuring power, you really need to see the voltage and current waveforms and their phase relationship to be sure of calculating the right value. This usually means using an oscilloscope unless you know the actual waveforms. You also need to be cognizant of the measuring bandwidths of the tools you're using and the limitations of connections, meters, instruments, probes, etc.

As we've discussed above, there are two types of RMS measurements: AC-coupled RMS and a real RMS measurement. You need to know which of these your meter measures.

Fortunately, it's pretty easy to tell. First, the meter's manual may tell you. If you can't figure it out from the documentation, a quick measurement is in order. Connect your RMS meter to a source of DC voltage. If the meter reads the DC voltage level properly when you have it set to measure AC voltage, it's an AC+DC type of RMS meter. Otherwise, it's likely an AC-coupled RMS meter and probably reads the DC voltage as 0 volts. You can check this with a function generator -- set the function generator to a 2.82 volt peak-to-peak sine wave at line frequency. Your RMS meter should read 1 volt RMS. If you then put a DC offset on the output of the function generator and the RMS meter's reading doesn't change, you have an AC-coupled RMS (i.e., what the marketers call a true RMS) meter.

If you have a meter that can read both RMS and AC-coupled RMS values, beware that it can be easy to forget which you have selected and thus make an incorrect measurement. Any AC waveform that is not symmetrical about the 0 volts axis has a non-zero DC offset and will measure differently with the RMS and AC-coupled RMS measurements. This makes for an easy check of the two different measurement modes: input a waveform with no DC offset and you should get the same value with either mode of measurement; include a DC offset and the two values should differ.

I'd hazard a guess that most folks find the AC-coupled RMS measurement the most useful. Example uses are measuring power supply ripple, measuring noise, tracing an AC signal through an amplifier, and measuring AC line voltage and current. Using a modern DMM, you can measure the DC voltage and the AC-coupled RMS voltage and get the real RMS value of the waveform if you wish (what the marketers call the AC+DC measurement) by adding them in quadrature (see below).

The AC+DC measurement is probably of most use to calculate the real power. An example is measuring the output voltage from an unfiltered DC power supply. As mentioned, however, if there's a phase difference between the current and voltage, you'll also need to know this phase angle to calculate the power correctly.

Calculating the real RMS value

Most instruments on the market measure the AC-coupled RMS value, usually denoted as true RMS. Unless your meter also includes the AC+DC RMS measurement feature, you'll have to make an additional measurement and calculation to get the real RMS value. The procedure is:

1. Measure the AC-coupled RMS value $\boldsymbol{V_{ac}}$.
2. Measure the DC value of the waveform V_{dc}.
3. Calculate the real RMS value $V_{rms} = \sqrt{\boldsymbol{V_{ac}}^2 + V_{dc}^2}$.

It's important that you use the AC-coupled RMS value in this calculation -- you can't use a previously-measured real RMS value V_{rms} unless you're sure it has no DC offset. If V_{rms} includes a DC offset, you must use an additional term under the square root -- see the section *Adding a DC offset*.

Termination

One thing that can surprise users making measurements is when they use coaxial cables with BNC connections to things like function generators. Most modern function generators have a 50 Ω output impedance. This is done to match the characteristic impedance of the coaxial cables, which is usually 50 Ω. This impedance matching is done to minimize reflections and, thus, spurious features on the generated waveform.

> If you don't know the output impedance of your function generator, set it to a low output (say, 100 mV) sine wave at 100 Hz. Then apply this voltage to a resistance box, measuring the voltage across the resistance box. Adjust the resistance until the voltage is half of the generator's open circuit voltage. The resistance box's value then equals the output impedance of the generator. The generator drops half of its output voltage across the Thevenin equivalent resistance of the generator (i.e., the output impedance) and half across the resistance box.

For measurements with low frequency signals, you can operate a function generator into a high impedance like a scope input or a voltmeter and you should get useful results. However, when the frequency increases, the impedance of the coaxial cable may become important because it acts as a transmission line.

The usual way a user finds out about this is they use a scope to examine a waveform like pulses or a square wave. The sharp edges on the waveform mean that there's lots of high frequency harmonic content in the signal. They may wonder why they see spikes and ringing on the scope display.

The reason for these anomalies is that they're getting reflections from the open or high-impedance end of the coaxial cable. The fix is to terminate the cable at the high/open end in its characteristic impedance. There are two common tools for this: a pass-through termination and a regular termination:

The pass-through termination is on the left. It allows you to connect the termination in-line with the cable, then connect the cable to e.g. a scope input (some scopes let you configure their inputs to have a 50 Ω input impedance, removing the need for the pass-through termination). The termination on the right needs to be used e.g. with a BNC tee to terminate the end of a cable. As soon as you use one of these, you should see those spikes on the waveform disappear.

However, there's another problem with making measurements with terminated cables: the voltage will be cut in half when you terminate the cable (the generator's output impedance and the termination's impedance form a voltage divider). This is fine if you're aware of it, but it can lead users to factor of 2 mistakes if they don't take it into account. If you see a measurement that is off by a factor of 2 and you're using 50 Ω cables, you should suspect a termination issue.

Some function generators let you choose the output impedance that the generator is operating into to help you set the correct amplitude. The generator still has a 50 Ω output impedance, but the generator's stated amplitude will be doubled if you choose the 50 Ω load setting. This helps cut down on mistakes for careful users, but unaware users can still be caught.

For low frequency measurements, a 50 ohm carbon film resistor can work as a termination resistor. As the frequency increases, however, it won't work as well as the properly-designed coaxial terminators, so it's probably worth your money to get one or two of them if you'll be making lots of measurements using coaxial cable.

An alternative: a scope

If you're a hobbyist on a budget, an alternative to an RMS meter is a digital oscilloscope. Modern digital oscilloscopes typically have the ability to measure the RMS value of a waveform. The ones I have used measure the real RMS value when they are DC coupled and the AC-coupled RMS value when they are AC-coupled. Thus, these scopes can give you either the true RMS or AC+DC value of a waveform. In addition, these scopes usually give you many more measurement functionals associated with the waveform.

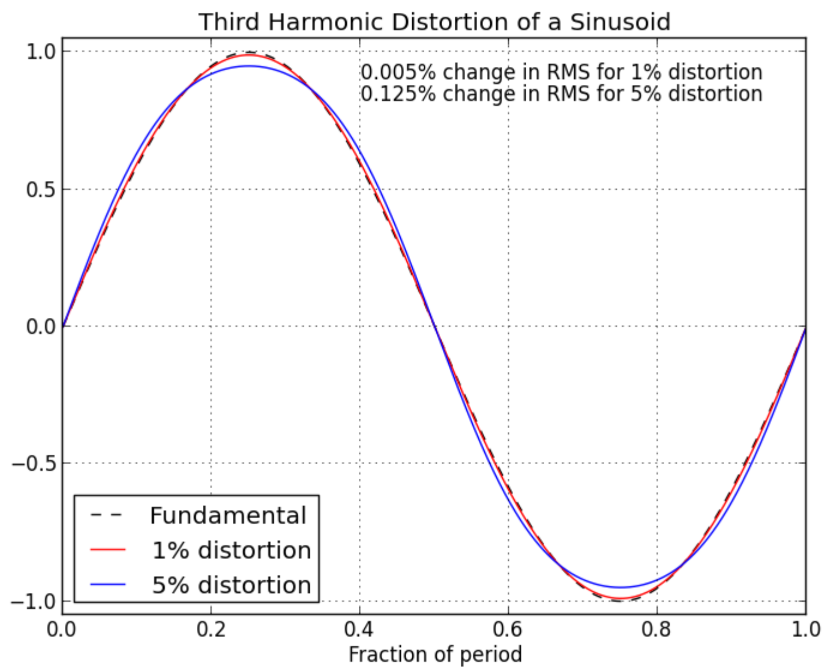
Besides giving the measurements, a modern dual channel digital scope is a powerful general purpose tool. You can do things like display the FFT of a waveform and multiply the two channels -- thereby getting the instantaneous power waveform if you have the current and voltage waveform.

Since oscilloscope bandwidths can go into the GHz region, you can make these RMS measurements on waveforms beyond the capabilities of the typical RMS meter. Thus, for a hobbyist, in general I would say a scope is a better investment than an RMS meter. Of course, a scope may be more money, but (next to a digital multimeter) it's probably the most useful general-purpose electrical measurement tool.

The typical low-cost digital scope has 8 bits of voltage resolution, so if you need RMS measurements with better resolution, you'll have to look for a scope with higher resolution (or get a dedicated RMS meter). 8 bits gives you roughly 0.4% resolution, which is better than the typical 3% to 5% resolution you can get from the scope's screen. An analog meter like the HP 3400A has roughly 0.5% resolution when measuring full scale voltages, less at fractions of full scale.

RMS and distortion

For measuring the distortion of sine waves, the RMS-responding meter isn't useful by itself unless you have some way of accurately subtracting out the fundamental. Here's a plot showing the effects of third harmonic distortion on a sine wave -- the visible effects on an oscilloscope are small:



The 1% distortion wave was calculated by creating the fundamental and adding to it 1% of another waveform of three times the frequency. It's clear that the RMS-responding meter wouldn't be terribly helpful in quantifying the distortion of a waveform.

Crest factor of instruments

It's useful to supply a pulse waveform to your RMS instruments and measure the instruments' response as a function of duty cycle. Since the crest factor of a pulse is $1/\sqrt{D}$ where D is the duty cycle, it's easy to measure this response if your function generator can be directly adjusted in duty cycle.

Here's the procedure I used. Choose the scale on the voltmeter you wish to characterize. Set the function generator to a pulse waveform with a 50% duty cycle. Adjust the amplitude of the waveform so that the voltmeter reads full scale. Then adjust the duty cycle to produce pairs of duty cycle and RMS voltages. For an AC-coupled RMS voltmeter, the measured value should be

$$\boldsymbol{V_{ac}} = VD\sqrt{\frac{1}{D}-1}$$

where V is the pulse height. For a 50% duty cycle, V is twice the measured $\boldsymbol{V_{ac}}$ value.

This section looks at some RMS measurements with two of the RMS-capable tools that I have.

HP3400A RMS voltmeter

Starting with a 50% duty cycle pulse at 1 V peak-to-peak from a B&K 4052 function generator, I adjusted the amplitude to read 0.5 V RMS on the 3400A (an RMS meter that measures the AC-coupled RMS value) and read the voltages on the 1 V scale. A 50 Ω termination was used and the 4052 was set to read for an output impedance of 50 Ω. The measurements were (D is duty cycle):

	V_{ac}, volts			Crest
D, %	Measured	Calculated	Δ%	factor
50	0.500	0.500	0.0	1.4
25	0.432	0.433	0.2	2.0
10	0.301	0.300	-0.3	3.2
5	0.222	0.218	-1.9	4.5
1	0.0990	0.0995	0.5	10.0
0.5	0.0700	0.0705	0.8	14.1
0.1	0.0313	0.0316	1.0	31.6

The values at 1% duty cycle and below were measured on the 0.1 V scale (the readings with the gray background). The 3400A is specified to have a crest factor of 10 at full scale, 20 at half-scale, and 100 at tenth-scale. **Thus, low-duty cycle pulses should be measured at 10-20% of full scale**, which is opposite to the usual way of using an analog meter, which is to measure as close to full-scale deflection as possible to minimize the relative measurement error.

Starting with a 50% duty cycle pulse at 50 mV RMS and the 3400A on the 100 mV range, the following data were taken (the values at 1% duty cycle and below were measured on the 10 mV range (the gray background)):

	V_{ac}, mV			Crest
D, %	Measured	Calculated	Δ%	factor
50	50.0	50.0	0.0	1.4
25	43.2	43.3	0.2	2.0
10	30.1	30.0	-0.3	3.2
5	22.0	21.8	-0.7	4.5
1	10.0	9.95	-0.5	10.0
0.5	7.08	7.05	-0.4	14.1
0.1	3.18	3.16	-0.5	31.6

Starting with a 50% duty cycle pulse at 5 mV RMS and the 3400A on the 10 mV range, the following data were taken (the values at 1% duty cycle and below were measured on the 1 mV range (the gray background)):

	V_{ac}, mV			Crest
D, %	Measured	Calculated	Δ%	factor
50	5.00	5.00	0.0	1.4
25	4.32	4.33	0.2	2.0
10	3.01	3.00	-0.3	3.2
5	2.20	2.18	-0.7	4.5
1	1.00	0.99	-0.5	10.0
0.5	0.709	0.705	-0.5	14.1
0.1	0.322	0.316	-1.9	31.6

To check the crest factor specification of 20 at full scale, I set the 3400A to the 100 mV range. For a crest factor of 20, the duty cycle must be 0.25%. To get a 50 mV reading on the 3400A, the voltage then must be 1001 mV. Since the 4052 can only be set in 0.1% duty cycle increments, the 3400A reading at 0.2% duty cycle was 45.0 mV and at 0.3% duty cycle was 55.1 mV; these average out to essentially 50 mV, so the 3400A meets its specification. Not bad for a vacuum-tube instrument made over 50 years ago.

Measuring the 100 crest factor at tenth scale is harder because this means a duty cycle of 0.01%. I used a Continental Specialties 4001 pulse generator to generate a 100 Hz pulse. The pulse width needed to be (10 ms)10^{-4} or 1 μs. I set the 3400A to the 100 mV scale and adjusted the pulse generator's amplitude to get a reading of one-tenth of full scale or 10 mV. A B&K 2542B-GEN scope measured the 0-to-peak amplitude of the pulse to be 0.94 V. This is about 6% off the expected value of 1 V peak. There are few instruments besides a scope that could measure RMS amplitudes at such crest factors -- the fundamental is at 100 Hz, but there are harmonics significantly above 1 MHz in this 0.01% duty cycle pulse, which are beyond even the 3400A's 10 MHz bandwidth.

HP 3456A voltmeter

Starting with a 50% duty cycle pulse at 1 V peak-to-peak from a B&K 4052 function generator, I adjusted the amplitude to read 0.50005 V RMS on the 3456A (100 PLC, Filter on). Then the AC-coupled RMS measurements as a function of duty cycle were:

	V_{ac}, volts			Crest
D, %	**Measured**	**Calculated**	**Δ%**	**factor**
50	0.50005	0.50005	0.0	1.4
25	0.4329	0.4331	0.0	2.0
10	0.2995	0.3000	0.2	3.2
5	0.2172	0.2180	0.4	4.5
1	0.0986	0.0995	0.9	10.0
0.5	0.06979	0.0705	1.1	14.1
0.1	0.03189	0.0316	-0.9	31.6

The 3456A has a specified crest factor of > 7 at full scale. All of these measurements were made on the 1 V scale.

RMS Properties

This section gives some of the properties of RMS measurements.

Negation

Given that a waveform $f(t)$ has an RMS value of V_{rms}, the waveform $-f(t)$ does also because

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2 dt} = \sqrt{\frac{1}{T}\int_0^T [-f(t)]^2 dt}$$

AC-coupled value from real RMS value

Suppose the real RMS (AC+DC) value is V_{rms} and the DC offset is V_{dc} (i.e., the average value). Then the AC-coupled RMS value $\boldsymbol{V_{ac}}$ is gotten by subtracting in quadrature:

$$\boldsymbol{V_{ac}} = \sqrt{V_{rms}^2 - V_{dc}^2}$$

This is how users of digital multimeters that measure only the AC-coupled RMS value can measure the true RMS value: add the voltage's DC value in quadrature to the AC-coupled RMS value.

Adding a DC offset

Suppose we have a waveform $V(t)$ with RMS value V_{rms}. Add a DC offset of V_0 to it and calculate the RMS value V_{RMS} of the resulting waveform:

$$V_{RMS}^2 = \frac{1}{T}\int_0^T \left[V(t)+V_0\right]^2 dt = \frac{1}{T}\int_0^T \left[V^2+2V_0 V+V_0^2\right] dt$$
$$= \frac{1}{T}\int_0^T \left[V^2+V_0^2\right] dt + 2V_0 \frac{1}{T}\int_0^T V\,dt$$

The first term is $V_{rms}^2+V_0^2$ and the last term is $2V_0 V_{dc}$ where V_{dc} is the $V(t)$ waveform's DC offset (i.e., average). Thus, we have derived the general expression

$$\begin{aligned} V_{RMS} &= \sqrt{V_{rms}^2+V_0^2+2V_0 V_{dc}} \\ &= \sqrt{\left(\boldsymbol{V_{ac}}^2+V_{dc}^2\right)+V_0^2+2V_0 V_{dc}} \\ &= \sqrt{\boldsymbol{V_{ac}}^2+\left(V_{dc}+V_0\right)^2} \end{aligned} \tag{10}$$

where the last form shows that it's really just adding the two DC offsets, then combining the result in quadrature with the AC-coupled RMS value.

For waveforms like the sine, cosine, triangle, etc. that are symmetrical above and below zero, the waveform's average V_{avg} (its DC offset) is zero. Thus, for these waveforms, the DC and RMS components add in quadrature. However, **you need to be careful with waveforms that already have a DC offset when you add another DC signal to it** -- that additional cross term needs to be accounted for.

> **Example**: A function generator provided a 10% duty cycle 100 Hz pulse with a 1 volt baseline and 2 V peak. A bench digital multimeter measured
>
> **Measurement 1**
>
> | **Parameter** | **Measured** | **Calculated** |
> |---|---|---|
> | DC | 1.1032 | 1.1000 |
> | **AC-coupled RMS** | 0.29508 | |
> | **AC+DC** | 1.14149 | 1.14198 |
>
> The DC offset of a pulse with a 0-to-peak value of V is $V D$ (see the section Pulse) where D is

the duty cycle. Thus, the calculated value of the DC offset is 1.1 V, as shown in the table. Then the DC offset was increased by 0.5 V. The new measurements were

Measurement 2

Parameter	Measured	Calculated
DC	1.5993	1.6000
AC-coupled RMS	0.29511	
AC+DC	1.62380	1.6263

Equation (10) lets us calculate the measured true RMS value as:

$$\sqrt{0.29508^2+(1.1032+0.5)^2}=1.630\text{ V.}$$

which is close to the measured value 1.6263 V.

Since an AC-coupled RMS voltmeter will measure a waveform's RMS value with the waveform's DC offset effectively removed, the resulting V_{RMS} value will always be the quadrature sum of the AC-coupled RMS value and the DC offset because the average voltage of the AC-coupled waveform is zero.

Note, however, that if you have a waveform that already has a DC offset V_{dc} and you add another DC offset V_0 to it, you need to include the cross-term $2V_{dc}V_0$ to get the correct real RMS value (the AC-coupled RMS value will be unchanged).

Average and RMS

While the ordering relationship for various means is given in the section *Ordering* below, it is interesting to see the following proof for the discrete case, which is based on the proof for Chebyshev's sum inequality. Suppose we have a set of n sampled values $\{x_i\}$ with RMS value x_{rms} and mean $\bar{x}$. In the sum

$$\sigma=\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$$

we must have $\sigma\geq 0$ because each term in the right-hand sum is positive or zero. Expand the sum and set $S_1=\sum_{i=1}^{n}x_i=n\bar{x}$ and $S_2=\sum_{i=1}^{n}x_i^2$

$$\begin{aligned}\sum_i\sum_j(x_i-x_j)^2 &= \sum_i\sum_j(x_i^2+x_j^2-2x_ix_j)\\ &= \sum_j(x_1^2+x_j^2-2x_1x_j)+\ldots+\sum_j(x_n^2+x_j^2-2x_nx_j)\\ &= (nx_1^2+S_2-2x_1S_1)+\ldots+(nx_n^2+S_2-2x_nS_1)\\ &= \underbrace{(nx_1^2+\ldots+nx_n^2)}_{nS_2}+nS_2-2S_1S_1\end{aligned}$$

or

$$2n\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j\geq 0$$

Dividing by $2n^2$, rearranging, and taking the positive square root, we get

$$\underbrace{\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}}_{x_{rms}}\geq\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)}_{\bar{x}}$$

or

$$x_{rms} \geq \bar{x}$$

In other words, **the RMS value is always greater than the average**. The equality holds only in the DC case.

Addition of two waveforms

Suppose we have two periodic waveforms $f(t)$ and $g(t)$, both with period T and they have RMS voltage values of V_f and V_g, respectively. What's the expression for the RMS value of their sum?

We have

$$V_{f+g}^2 = \frac{1}{T}\int_0^T (f+g)^2\,dt = V_f^2 + V_g^2 + \frac{2}{T}\int_0^T f\,g\,dt$$

If the product waveform $f\,g$ is symmetrical about zero, then the last integral is zero and the resulting RMS voltage is the quadrature sum of the two voltages.

Multiplication of two waveforms

Suppose we have two periodic waveforms $f(t)$ and $g(t)$, both with period T and they have RMS voltage values of V_f and V_g, respectively. What's the expression for the RMS value of their product?

We have

$$V_{fg}^2 = \frac{1}{T}\int_0^T (f\,g)^2\,dt$$

and we see in general there's no simplification that we can do to express the result in terms of the RMS values of the individual waveforms.

Ordering

A theorem called the power-mean inequality or generalized mean inequality (see [*gmwp*]) provides insight into the numerical magnitude of the RMS value. Given $p, q \in \mathbb{R}$, if $p < q$ then $M_p(x_{1,}\ldots, x_n) \leq M_q(x_{1,}\ldots, x_n)$ where

$$M_p(x_{1,}\ldots, x_n) \stackrel{\text{def}}{=} \left[\frac{1}{n}\sum_{i=1} n\, x_i^p\right]^{\frac{1}{p}}$$

and

$$M_0 \stackrel{\text{def}}{=} \sqrt[n]{\prod_{i=1}^{n} x_i} = \text{geometric mean}$$

This applies to any set of values $\{x_i\}$ and we can get arbitrarily close to the case for an integral by letting i grow large.

Some general means are

Mean	Discrete definition
$M_{-\infty}$	Minimum of set = $\min\{x_i\}$
M_{-1}	Harmonic mean = HM = $\dfrac{n}{\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}}$
M_0	Geometric mean = GM = $\sqrt[n]{x_1 \cdot \ldots \cdot x_n}$
M_1	Arithmetic mean = Average = AM = $\dfrac{x_1+\ldots+x_n}{n}$
M_2	RMS = $\sqrt{\dfrac{x_1^2+\ldots+x_n^2}{n}}$
M_{∞}	Maximum of set = $\max\{x_i\}$

One can also define continuous functionals that are analogs of these discrete definitions (see page 128 in [*steele*]):

$$M_p \stackrel{\text{def}}{=} \left[\int_D (f(t))^p \, dt \right]^{1/p}$$

where D is a proper subset of the real line, $f: D \to [0,\infty)$, and $p \in (-\infty,0) \cup (0,\infty)$. For the case where $p = 0$, we define

$$\ln M_0 \stackrel{\text{def}}{=} \int_D \ln f(t) \, dt$$

There's also the continuous analog of the power mean inequality:

$$M_p \le M_q \text{ for all } -\infty < p < q < \infty.$$

For continuous f, equality holds only if f is a constant on D. The graph of M_p as a function of p looks qualitatively similar to the inverse tangent function.

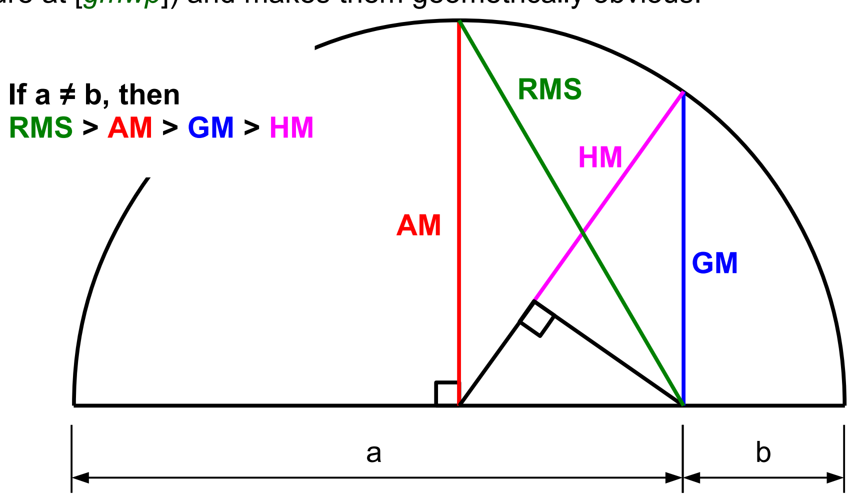
Thus, we have

$$\min\{x_i\} \le \text{HM} \le \text{GM} \le \text{AM} \le \text{RMS} \le \max\{x_i\}$$

and equality only holds in the DC case.

The average value of a time-varying waveform is always less than the RMS value. Thus, **average-responding meters will always under-measure a non-sinusoidal waveform**.

For the case of two non-equal numbers a and b, a geometric picture illustrates these inequalities (copy of picture at [*gmwp*]) and makes them geometrically obvious:



For this case of two numbers, note that $\text{GM}^2 = \text{AM} \cdot \text{HM}$.

If we have a waveform with amplitude V_{peak} that is symmetrical with respect to the time axis (i.e., no DC offset), then we have the ordering

$$V_{average} \le V_{RMS} \le V_{peak} \le V_{pp}$$

where $V_{pp} = 2V_{peak}$. Again, equality only holds in the DC case.

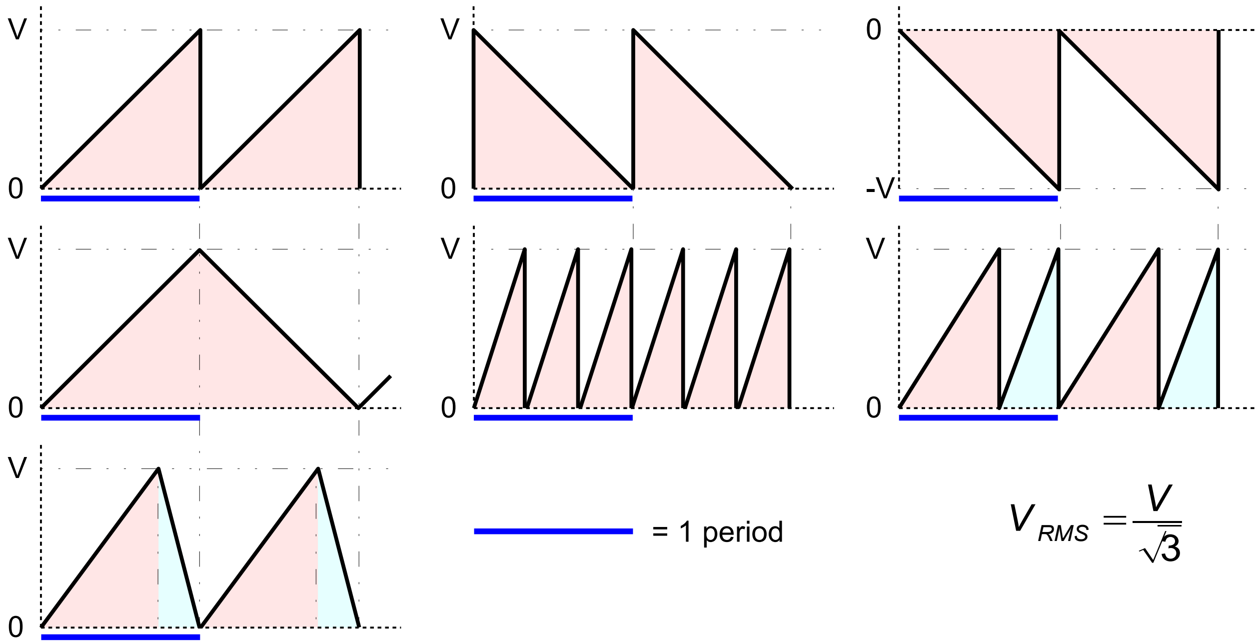
Harmonic mean	Given a set of *n* arbitrary resistors in parallel with equivalent resistance *R*, the harmonic mean *H* of these resistances is such that *n* resistors of value *H* in parallel have an equivalent resistance of *R*.
Geometric mean	The geometric mean asks an analogous question, except it's for the product of the numbers: "If all the numbers had the same value, what value gives the same product?". For two positive numbers representing the sides of a rectangle, their geometric mean gives the side of a square with the same area.
Arithmetic mean	The arithmetic mean is relevant when summing a set of numbers and is the answer to the question "If all the numbers had the same value, what value gives the same sum?".
RMS	The RMS is relevant when summing the squares of a set of numbers. It's the answer to the question "If all the numbers had the same value, what value gives the same sum of squares?".

Some properties for $c \in \mathbb{R}$:

$$\begin{array}{ll}
\text{AM}(x_1+c, x_2+c, \ldots, x_n+c) = c + \text{AM}(x_1, x_2, \ldots, x_n) & \\
\text{GM}(x_1+c, x_2+c, \ldots, x_n+c) > c + \text{GM}(x_1, x_2, \ldots, x_n) & \text{if } c > 0 \\
\text{HM}(x_1+c, x_2+c, \ldots, x_n+c) > c + \text{HM}(x_1, x_2, \ldots, x_n) & \text{if } c > 0 \\
\text{RMS}(x_1+c, x_2+c, \ldots, x_n+c) < c + \text{RMS}(x_1, x_2, \ldots, x_n) & \text{if } c > 0
\end{array}$$

Similarity

Because of the geometrical similarities in the following figures, they all have the same RMS value.



$$V_{RMS} = \frac{V}{\sqrt{3}}$$

A practical aspect of this is that your RMS voltmeter will indicate the same RMS voltage as you vary a function generator's frequency for the same waveform as long as the generator has flat output and you're within the voltmeter's bandwidth.

Numerical factors

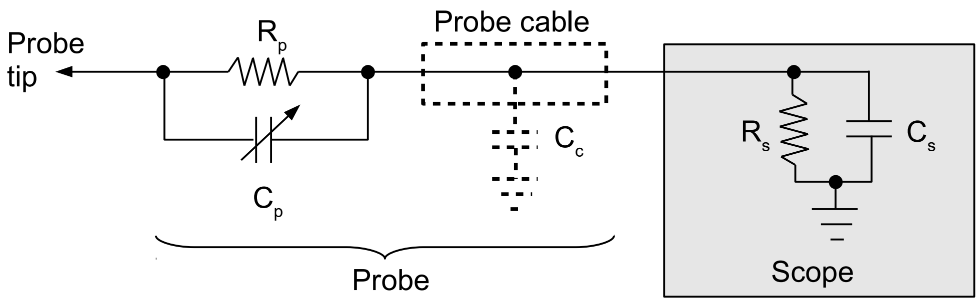
The following table lists some numerical factors related to RMS and average measurements. It may prove useful when you make a measurement and don't get the value you expect. If the ratio of the measured value to the expected value (or its reciprocal) is close to a value in this table, it may give you a clue about the cause of the discrepancy.

Value	Exact	Purpose
0.318	$\frac{1}{\pi}$	Ratio of the absolute average value of a sine wave to the peak-to-peak value. An average-responding meter will be calibrated to read RMS of a sine wave, so it will multiply this value by 1.111.
0.354	$\frac{1}{2\sqrt{2}}$	Ratio of the RMS value of a sine wave to the peak-to-peak value.
0.500	1/2	Factor to convert a open-circuit voltage to what will be measured with a 50 Ω source impedance terminated into 50 Ω. It also is the ratio of the peak to peak-to-peak value of any waveform that is symmetrical about the time axis (i.e., no DC offset).
0.637	$\frac{2}{\pi}$	Ratio of the peak value of a sine wave to the absolute average.
0.707	$\frac{1}{\sqrt{2}}$	Factor to calculate RMS voltage of a sine wave with no DC offset from the sine wave's amplitude.
0.900	$\frac{\sqrt{8}}{\pi}$	Ratio of RMS to absolute average for a sine wave.
1.11	$\frac{\pi}{\sqrt{8}}$	Ratio of absolute average of a sine wave to the RMS value. This is the voltage an average responding meter will read when given a square wave of 1 volt amplitude with no DC offset. Set a 1 volt amplitude square wave with no DC offset on a function generator and read it with a voltmeter. If the measured value is 1 volt, the voltmeter is an RMS-responding meter (either AC-coupled RMS or AC+DC); if the

Value	Exact	Purpose
		measured value is 1.111 volts, the voltmeter is an average-responding type.
1.41	$\sqrt{2}$	Ratio of peak to RMS for a sine wave.
1.57	$\frac{\pi}{2}$	Ratio of peak to absolute average for a sine wave.
2	2	Factor to convert a 50 Ω terminated voltage from a 50 Ω source to an open-circuit voltage. It also is the ratio of the peak-to-peak to peak value of any waveform that is symmetrical about the time axis (i.e., no DC offset).
2.83	$2\sqrt{2}$	Ratio of peak-to-peak to RMS for a sine wave.
3.14	π	Ratio of peak-to-peak to absolute average for a sine wave.

Using scope probes with your RMS meter

You can use your scope probes with your RMS meter -- **however, you may be in for measurement mistakes unless you're careful**. To understand why, let's look at a typical 10X scope probe (this is a lumped-parameter model):



The probe cable's distributed capacitance is shown as a dashed lumped component C_c.

Most scope inputs have an input resistance R_s of 1 MΩ. Because of this, the value of the resistor R_p is typically 9 MΩ for 10X probes[8]. This is done so that the effective loading on the circuit being probed is 10 MΩ at and near DC. For low frequencies, erase the capacitors and you'll see R_p and R_s are in series, comprising a voltage divider. At higher frequencies, the capacitances work as a capacitive divider to keep the attenuation factor at 10X.

> **Important**: While a probe/scope's impedance magnitude may be 9 MΩ at DC and very low frequencies, it can be 4 orders of magnitude (or more) smaller at the probe's bandwidth frequency (this impedance drop is caused by the capacitances).

The probe's 10X attenuation comes from the fact that R_p and R_s form a voltage divider at low frequencies where we can ignore the capacitances; the attenuation is

$$\frac{R_s}{R_s+R_p}=\frac{1}{9+1}$$

The typical DMM or RMS voltmeter has a 10 MΩ input resistance. Thus, the voltage divider's attenuation factor is now

$$\frac{R_s}{R_s+R_p}=\frac{10}{9+10}=0.526$$

You'll make measurement errors if you don't take this into account.

> ***Example***: A function generator output a 1 V RMS sine wave at 100 Hz. A 10X/1X scope probe

8 Use your DMM to measure the DC resistance of the center conductor to verify this. Some probes use a special resistance wire rather than a discrete resistor, but the effect is the same at DC.

> was used to connect the generator's output to an RMS voltmeter with a 10 MΩ input resistance and the amplitude was adjusted to get the voltmeter to read 1.00 V RMS when the probe's switch was in the 1X attenuation position. When the probe was switched to the 10X attenuation, the voltmeter's reading was 0.501 V RMS, about 5% below the calculated 0.526 V. An average-responding digital multimeter with a 10 MΩ input resistance was set to read 1.000 V with the probe's switch in the 1X attenuation position. When the probe was switched to the 10X attenuation, the voltmeter's reading was 0.504 V RMS, about 4% below the calculated 0.526 V.

Rule: for sine waves and a probe on the 10X setting, multiply the reading by 1.9 to get the correct voltage measurement. For non-sinusoidal waveforms, no generalization can be made because of the higher frequency components and the bandwidth of the measuring equipment.

From the above, it would seem to be better to make your RMS measurements with a 1X scope probe. For low frequencies, this works well, but the problem with a 1X probe is the limited bandwidth.

For example, I used a 150 MHz 10X/1X scope probe with the 10X attenuation setting with my HP 3400A AC-coupled RMS meter and it worked fine for sine wave measurements up to the meter's 10 MHz bandwidth (and being cognizant of the above 0.526 factor). However, when it was in the 1X position, the output dropped by 1% from the 100 Hz sine wave response when the frequency reached 450 kHz. Thus, you need to be aware of these effects -- if you don't take them into account, you'll make measurement mistakes.

If you're making RMS measurements with a 1X probe of distinctly non-sinusoidal waveforms, realize that even if the fundamental is well below the upper frequency limit of your 1X probe, some of the harmonics may exceed this frequency limit and not be measured properly.

Another complication can be when you use your scope probes to measure DC values. The input impedance of the voltmeter may vary with the range. For example, some voltmeters have input resistances 10 GΩ or larger for the lower ranges, but then drop to 10 MΩ for the higher ranges. For the 10 GΩ range, a 10X probe and 1X probe will read the same DC values.

> ***Example***: a 10X/1X probe was used to measure the output voltage of a DC power supply. When the supply was set to 12 V, an HP3456A voltmeter read 12.031 V with the probe in the 1X position and 6.329 V with the probe in the 10X position. With the power supply set to 1 V, the voltmeter read 1.01265 V in both the 1X and 10X attenuation switch positions.

Adjust the input impedance

One approach to "fixing" the 10X probe's impedance so that you can get accurate RMS measurements with a 10 MΩ input resistance voltmeter is to reduce the input impedance of the voltmeter to 1 MΩ by putting a resistor in parallel with the input. The parallel resistor R needed to do this is in MΩ

$$1 = \frac{1}{R} + \frac{1}{10}$$

so R needs to be 1.11 MΩ.

A disadvantage of this adjustment of the input impedance is that **it may not work well at low frequencies** -- if you change the frequency, the voltmeter's reading will change. You'll have to test your 10X probe to see if this is true or not. You'll probably find that there's a frequency above which the 10X probe makes frequency-independent measurements. For the 10X/1X switchable probes I use, this frequency is around 10 kHz. For RMS measurements below 10 kHz, I use the 1X attenuation and above that the 10X attenuation.

The rules you make for your probe and instrument(s) will be probably be different, so measurements are in order.

Waveforms

Other waveforms to add: Gaussian pulse, Lorentz pulse

This section looks at some of the properties of various waveforms. The notation is

V_{dc} DC offset of a waveform = $\frac{1}{T}\int_0^T f(t)\,dt$. This is the voltage a DC voltmeter will measure. A function that is symmetrical above and below 0 volts will have a V_{dc} of zero.

V_{aa} Absolute average value = $\frac{1}{T}\int_0^T |f(t)|dt$. This is the voltage that an average-responding AC voltmeter will read **with no correction for a sine wave**. Since these meters are typically scaled by the factor $\pi/\sqrt{8}$ so that they will read the correct RMS value of a sine wave, you must multiply the voltmeter's reading by $\sqrt{8}/\pi = 0.9$ to recover the V_{aa} value given in the following section. A function that is symmetrical above and below 0 volts will have a V_{aa} that is not zero.

$\boldsymbol{V_{ar}}$ Because average-responding voltmeters are so common, this value is what an average-responding voltmeter would read for the given waveform. $\boldsymbol{V_{ar}} = \frac{\pi}{\sqrt{8}} V_{aa}$

V_{rms} RMS value = $\sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt} = \sqrt{\boldsymbol{V_{ac}^2} + V_{dc}^2}$. The marketers call this the AC+DC RMS value.

$\boldsymbol{V_{ac}}$ AC-coupled RMS value = $\sqrt{V_{rms}^2 - V_{dc}^2}$. The marketers call this the true RMS value.

C_n nth term of waveform's Fourier series, $n > 0$

CF Crest factor

In the graphs, the horizontal axis is time, the vertical axis is what's measured by the waveform, and the horizontal axis is a vertical value of zero. The sinc function is also used:

$$\text{sinc}\, x = \frac{\sin x}{x}$$

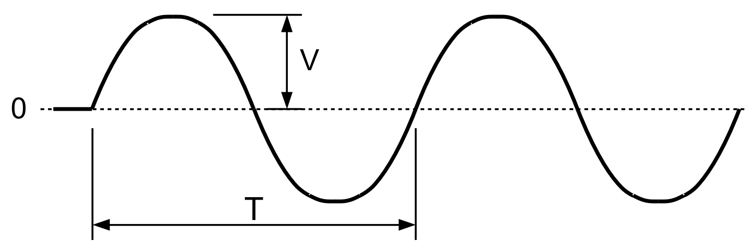
This information came from a variety of references (or are things I may have derived myself); some are [*crcmt*], [*rdre*, pg 282], and [*dudzik*].

Sine wave

Here's a table giving the relationships between the absolute average, RMS, peak, and peak-to-peak values **of a sine wave with no DC offset** (the relationships will be different for other waveforms). To convert from the measurement in the indicated row, multiply the measured value by the factor in the desired column:

	Want ↓			
Have ↓	**Average**	**RMS**	**Peak**	**Peak-to-peak**
Absolute average $\boldsymbol{V_{aa}}$	1	$\pi/\sqrt{8} = 1.111$	$\pi/2 = 1.571$	$\pi = 3.142$
RMS $\boldsymbol{V_{rms}} = \boldsymbol{V_{ac}}$	$\sqrt{8}/\pi = 0.9003$	1	$\sqrt{2} = 1.414$	$2\sqrt{2} = 2.828$
Peak $\boldsymbol{V_p}$	$2/\pi = 0.6366$	$1/\sqrt{2} = 0.7071$	1	2
Peak-to-peak $\boldsymbol{V_{pp}}$	$1/\pi = 0.3183$	$1/(2\sqrt{2}) = 0.3536$	1/2	1

> **Example**: you measure the peak-to-peak value of a sine wave on an oscilloscope as 1.23 volts and you want the RMS value. Reading from the fourth row under the column RMS, we get the multiplicative conversion factor to convert peak-to-peak voltage to RMS voltage. The RMS value of this voltage is $1.23/(2\sqrt{2}) = 1.23(0.356) = 435\text{ mV}$.



$$V_{dc} = 0$$

$$V_{aa} = \frac{2}{\pi} V = 0.6366\, V$$

$$\boldsymbol{V_{ar}} = \frac{V}{\sqrt{2}} = V_{rms}$$

$$V_{rms} = \boldsymbol{V_{ac}} = \frac{V}{\sqrt{2}} = 0.7071\, V$$

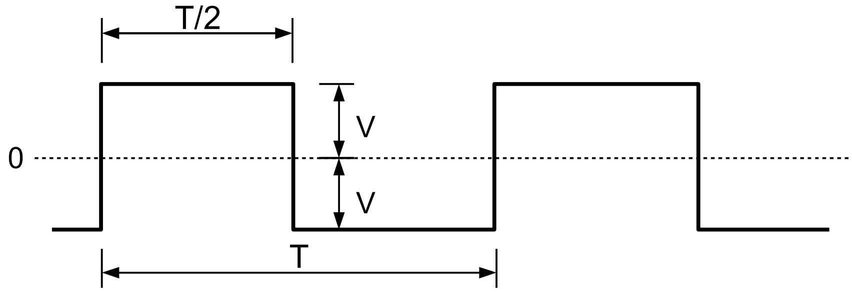
$$\text{CF} = \frac{V}{V_{rms}} = \sqrt{2}$$

$$C_n = \begin{cases} = 0 \text{ if } n \neq 1 \\ = V \text{ if } n = 1 \end{cases}$$

Vdc is zero because waveform is symmetrical about time axis.

$$V_{aa} = \int_0^{\pi/2} \sin t \, dt$$

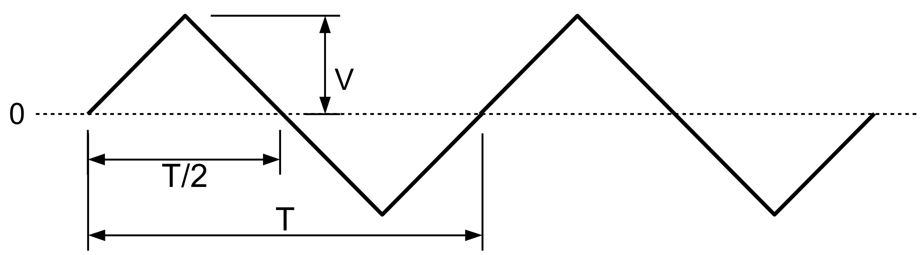
Square wave



$$V_{dc} = 0$$
$$V_{aa} = \frac{\pi}{\sqrt{8}} V = 1.111\, V$$
$$\boldsymbol{V_{ar}} = ?$$
$$V_{rms} = \boldsymbol{V_{ac}} = V$$
$$\mathrm{CF} = \frac{V}{V_{rms}} = 1$$
$$C_n = \begin{cases} = 0 \text{ if } n \text{ is even} \\ = \dfrac{4V}{n\pi} \sin \dfrac{2n\pi t}{T} \text{ if } n \text{ is odd} \end{cases}$$

Experimental check (4052 generator, 3456A voltmeter, 3435A for V_{aa}), 1 kHz square wave. Set low level to -1 and high level to 1: $\boldsymbol{V_{ac}}=0.9995$, $V_{aa} = 1.106$, $V_{dc} = 30.5$ mV, $V_{rms} = 0.9970$. Fourier coefficient from pg 455 [*crcmt*].

Triangle wave



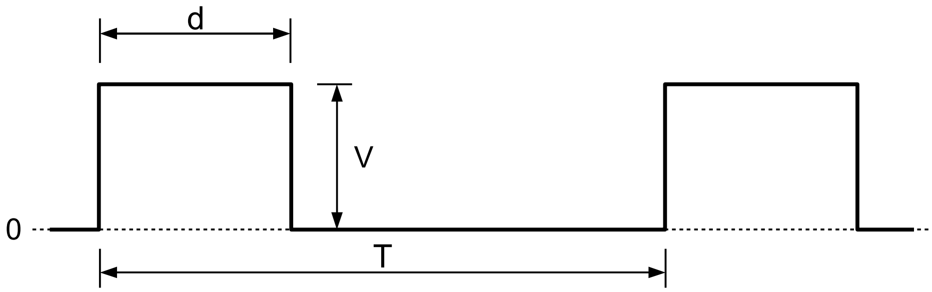
$$V_{dc} = 0$$
$$V_{aa} = \frac{V}{2}$$
$$\boldsymbol{V_{ar}} = ?$$
$$V_{rms} = \boldsymbol{V_{ac}} = \frac{V}{\sqrt{3}}$$
$$\mathrm{CF} = \frac{V}{V_{rms}} = \sqrt{3}$$
$$C_n = \begin{cases} = 0 \text{ if } n \text{ is even} \\ = \dfrac{8V}{(n\pi)^2} (-1)^{(n-1)/2} \sin \dfrac{n\pi t}{T} \text{ if n} = 1, 3, 5, 7, \ldots \end{cases}$$

Experimental check (4052 generator, 3456A voltmeter, 3435A for V_{aa}), 1 kHz triangle wave. Set low level to -1 and high level to 1: $\boldsymbol{V_{ac}}=0.5772$, $V_{aa} = 0.553$, $V_{dc} = 5.5$ mV, $V_{rms} = 0.5773$. Fourier coefficient from pg 455 [*crcmt*]. Note that the 3435A reading needs to be corrected down by the factor 1.111 because it's calibrated for a sine wave; hence, the measured V_{aa} was 0.497 with the theoretical value of 0.5.

For V_{aa}, we only need to look at the quarter period sawtooth. $V(t) = Vt$ because the period is 1 and therefore

$$V_{aa} = \frac{1}{T}\int_0^T V\,t\,dt = \frac{V}{2}t^2\Big]_0^1 = \frac{V}{2}$$

Pulse



$$D = \frac{d}{T} = \text{duty cycle}$$

$$V_{dc} = V\frac{d}{T} = DV$$

$$V_{aa} = ?$$

$$V_{ar} = ?$$

$$V_{rms} = V\sqrt{\frac{d}{T}} = V\sqrt{D}$$

$$V_{ac} = VD\sqrt{\frac{1}{D}-1} = V_{dc}\sqrt{\frac{1}{D}-1}$$

$$\text{CF} = \frac{V}{V_{rms}} = \frac{1}{\sqrt{D}}$$

$$C_n = 2V_{dc}\,\text{sinc}\,x_n$$

$$\text{where }\; x_n = \frac{n\pi d}{T} = n\pi D$$

V_{dc}	V_{ar}	V_{rms}	V_{ac}	CF
$V\frac{d}{T} = DV$	?	$V\sqrt{D}$	$V\sqrt{D(1-D)}$ $= V_{rms}\sqrt{1-D}$	$\frac{1}{\sqrt{D}}$

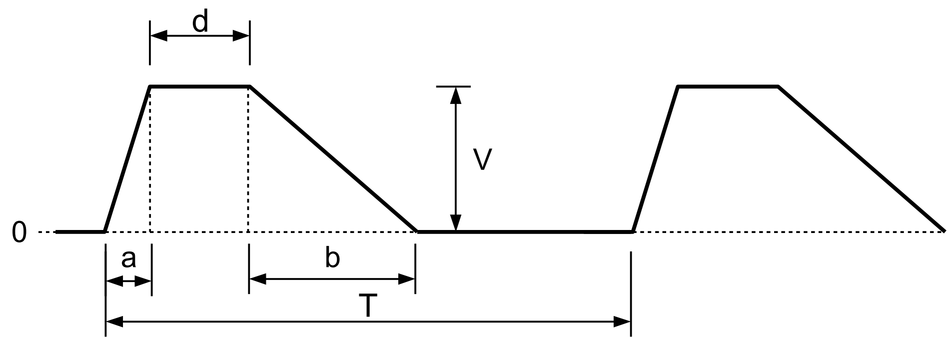
Experimental check (4052 generator, 3456A voltmeter), 100 Hz pulse. Set low level to 0 and high level V = 1.

Duty cycle, %	V_{dc} Meas.	Formula	Δ%	V_{ac} Meas.	Formula	Δ%	V_{rms} Meas.	Formula	Δ%
90	0.8987	0.9	-0.1	0.2970	0.3	-1.0	0.9465	0.9487	-0.2
50	0.5018	0.5	0.4	0.4962	0.5	-0.8	0.7055	0.7071	-0.2
10	0.1048	0.1	4.8	0.2971	0.3	-1.0	0.3148	0.3162	-0.4
1	0.0156	0.01	56	0.0977	0.0995	-2	0.0990	0.1	-1

Thus, the formulas check.

Has check in check_rms.py.

Trapezoid pulse



$$V_{dc} = \frac{V}{T}\left(\frac{a}{2} + \frac{b}{2} + d\right)$$

$$V_{aa} = ?$$

$$V_{ar} = ?$$

$$V_{rms} = V\sqrt{\frac{a + b + 3d}{3}T}$$

$$V_{ac} = ?$$

$$\mathrm{CF} = \frac{V}{V_{rms}} = \sqrt{\frac{3T}{a + b + 3d}}$$

$$C_n = 2V_{avg}\,\mathrm{sinc}\,\alpha_n\ \mathrm{sinc}\,\beta_n\ \mathrm{sinc}\,\gamma_n$$

$$\text{where } \alpha_n = \frac{n\pi a}{T},\ \beta_n = \frac{n\pi(a+d)}{T},\ \gamma_n = \frac{n\pi(b+a)}{T}$$

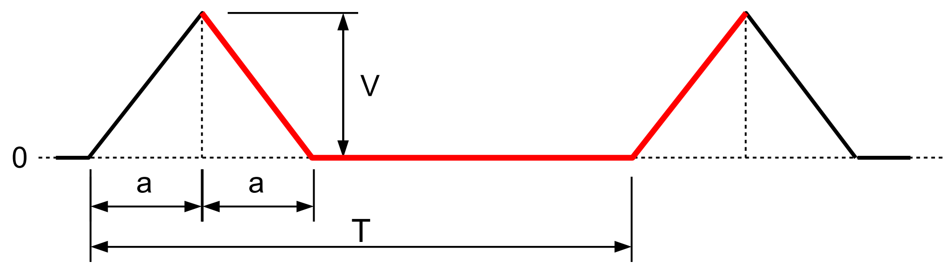
Experimental check (4052 generator, 3456A voltmeter), 1 kHz trapezia waveform in arb. Set low level to 0 and high level to 1. Waveform parameters were a = b = d = 1/4, T = 1, V = 1. Measured Vaa with 3435A as 0.410 V.

	Calc	**Meas.**	**Δ%**
Vdc	0.5000	0.5014	0.28
Vrms	0.6455	0.6434	-0.32
Vac	0.4082	0.4036	-1.14

Has check in check_rms.py.

Crest factor: set a = b = 0, get CF for pulse. Set a = b, d = 0, get CF for triangle pulse.

Triangle pulse



$$D = \frac{2a}{T} = \text{duty cycle}$$

$$V_{dc} = \frac{Va}{T} = \frac{1}{2} VD$$

$$V_{aa} = ?$$

$$V_{ar} = ?$$

$$V_{rms} = V\sqrt{\frac{2a}{3T}} = V\sqrt{\frac{D}{3}}$$

$$V_{ac} = \frac{V}{2}\sqrt{\frac{D}{3}} = \frac{1}{2} V_{rms}$$

$$\text{CF} = \frac{V}{V_{rms}} = \sqrt{\frac{3T}{2a}}$$

$$C_n = 2V_{avg}\,\text{sinc}^2\alpha_n$$

$$\text{where } \alpha_n = \frac{n\pi a}{T}$$

Note: the Fourier series

$$\sum_{n=1}^{\infty} C_n \cos\frac{2n\pi t}{T}$$

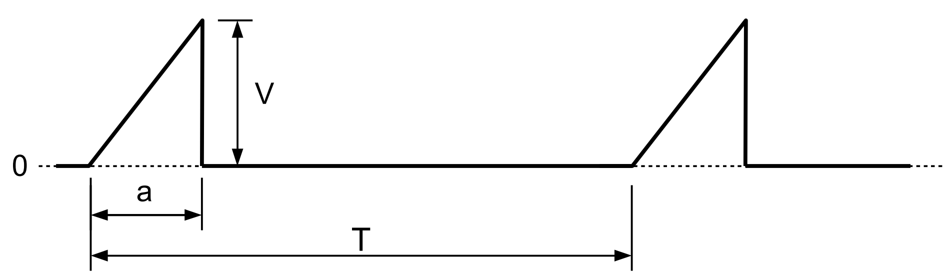
converges to the waveform shown in color.

This isn't quite true, as the plot in fourier.py shows a negative DC offset. Note the Cn are probably the coefficients in the complex form of the Fourier series.

Has checks in check_rms.py and fourier.py.

CF: set a = T/2, get CF for triangle wave.

Sawtooth pulse



$$V_{dc} = \frac{V\,a}{2\,T}$$

$$V_{aa} = ?$$

$$\boldsymbol{V_{ar}} = ?$$

$$V_{rms} = V\sqrt{\frac{a}{3}\,T}$$

$$C_n = \frac{V\,T}{2\,(\pi\,n)^2\,a}\sqrt{2\left[1-\cos 2\,\alpha_n\right]+4\,\alpha_n\left[\alpha_n-\sin 2\,\alpha_n\right]}$$

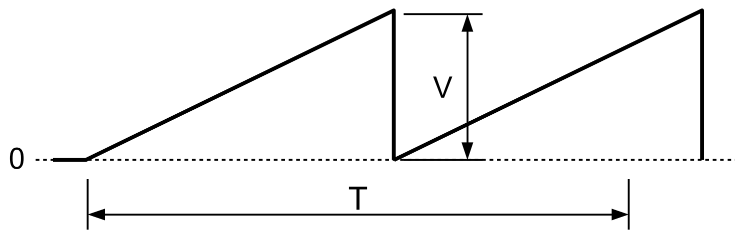
where $\alpha_n = \frac{\pi\,n\,a}{T}$.

If a is small

$$C_n = \frac{2\,V_{avg}}{\alpha_n}\left(\mathrm{sinc}\,\alpha_n - 1\right)$$

Has check in check_rms.py.

Sawtooth



$$V_{dc} = \frac{V}{2}$$

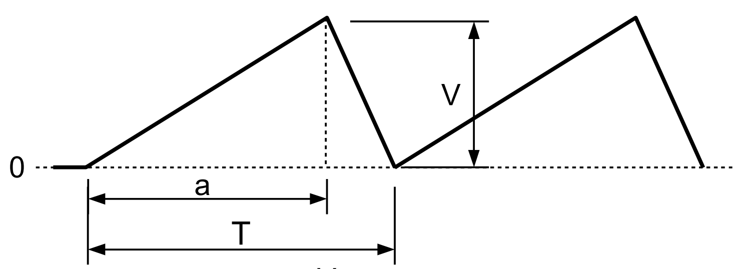
$$V_{aa} = ?$$

$$\boldsymbol{V_{ar}} = ?$$

$$V_{rms} = \frac{V}{\sqrt{3}}$$

$$C_n = \frac{2\,V_{avg}}{\pi\,n}\cos\pi\,n$$

Asymmetrical sawtooth



$$V_{dc} = \frac{V}{2}$$

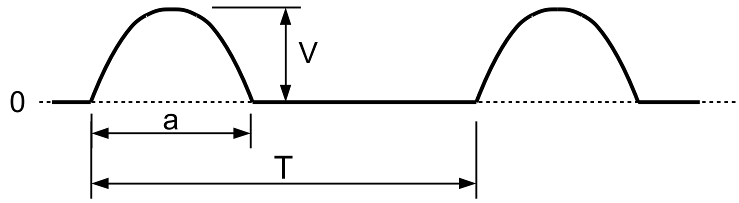
$$V_{aa} = ?$$

$$\boldsymbol{V_{ar}} = ?$$

$$V_{rms} = \frac{V}{\sqrt{3}}$$

$$C_n = \frac{2\,V_{avg}\,T}{(\pi\,n)^2 a\left(1-\frac{a}{T}\right)}\sin\frac{\pi\,a}{T}$$

Fractional sine



$$V_{dc} = \frac{V}{\pi}\left[\frac{\sin\mu - \mu\cos\mu}{1-\cos\mu}\right]$$

$$V_{aa} = ?$$

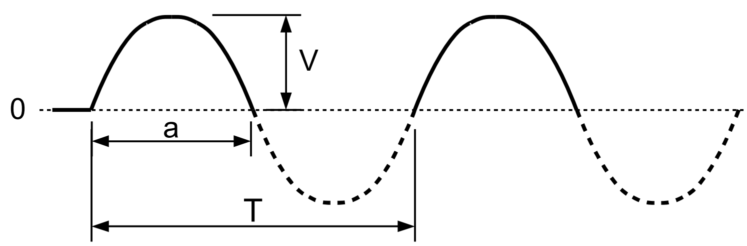
$$V_{ar} = ?$$

$$V_{rms} = \frac{V}{1-\cos\mu}\sqrt{\frac{1}{\pi}\left(\mu - \frac{3}{4}\sin 2\mu + \mu\cos^2\mu\right)}$$

$$C_n = \frac{V_{avg}\,\mu}{n(\sin\mu - \mu\cos\mu)}\left[\operatorname{sinc}(n-1)\mu - \operatorname{sinc}(n+1)\mu\right]$$

where $\mu = \frac{\pi a}{T}$

Half sine



$$V_{dc} = \frac{2Va}{\pi T}$$

$$V_{aa} = ?$$

$$V_{ar} = ?$$

$$V_{rms} = V\sqrt{\frac{a}{2T}}$$

$$C_n = \frac{\pi}{2}\left(sinc\,x + sinc\,y\right)$$

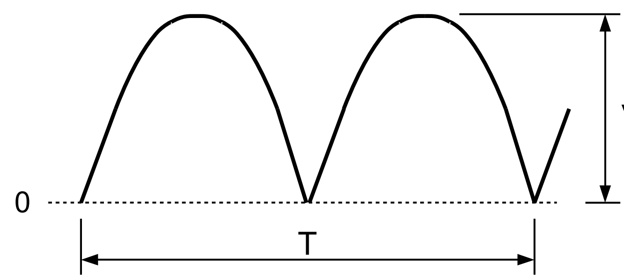
where

$$x = \frac{\pi}{2}(1-\alpha)$$

$$y = \frac{\pi}{2}(1+\alpha)$$

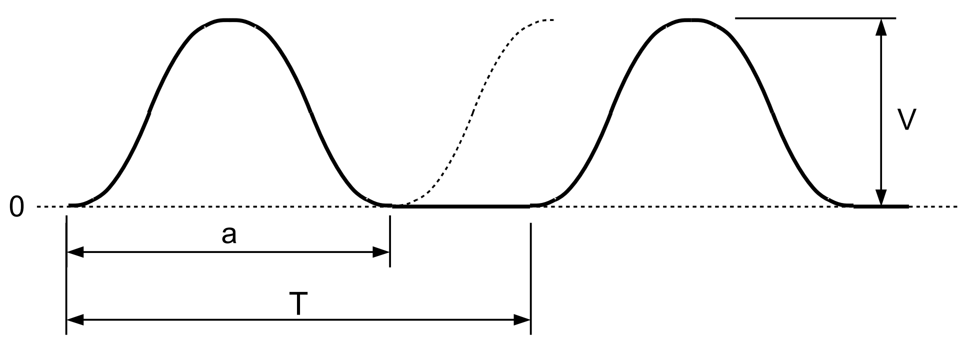
$$\alpha = \frac{2na}{T}$$

Full-wave rectified sine



$$V_{dc} = \frac{2V}{\pi}$$
$$V_{aa} = ?$$
$$V_{ar} = ?$$
$$V_{rms} = \frac{V}{\sqrt{2}} \quad \text{(i.e., same as sine wave)}$$
$$C_n = \frac{\pi V_{avg}}{2}\left[\operatorname{sinc}\frac{\pi}{2}(1-n) + \operatorname{sinc}\frac{\pi}{2}(1+n)\right]$$

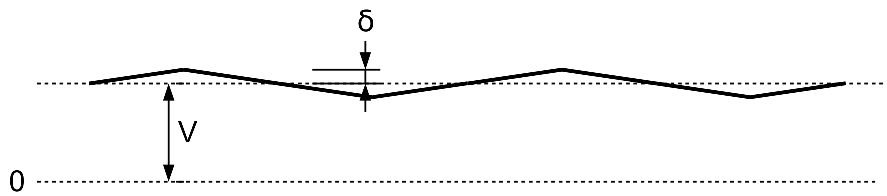
Positive sine



$$V_{dc} = \frac{Va}{2T}$$
$$V_{aa} = ?$$
$$V_{ar} = ?$$
$$V_{rms} = \frac{V}{2}\sqrt{\frac{3a}{2T}}$$
$$C_n = V_{avg}\left(2\,sinc\,x + sinc\,y + sinc\,z\right)$$

where $x = \pi\alpha$, $y = \pi(1-\alpha)$, $z = \pi(1+\alpha)$, and $\alpha = \frac{na}{T}$

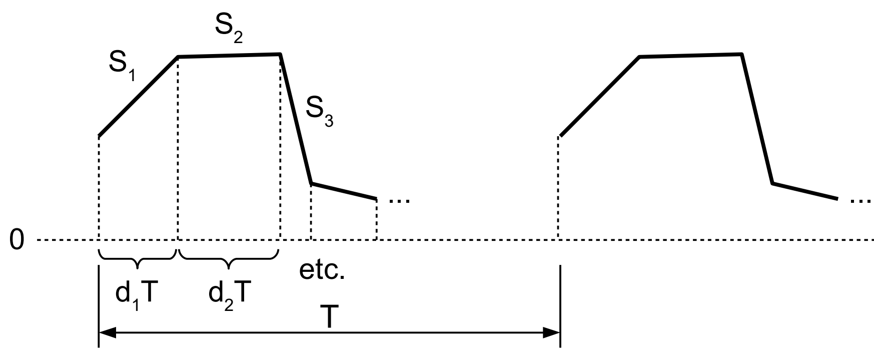
DC + linear ripple



$$V_{dc} = V$$
$$V_{aa} = ?$$
$$\mathbf{V_{ar}} = ?$$
$$V_{rms} = V\sqrt{1+\frac{1}{3}\left(\frac{\delta}{V}\right)^2}$$

Piecewise approximation

Suppose we have a periodic waveform of period T that is composed of a number of different sections S_i:



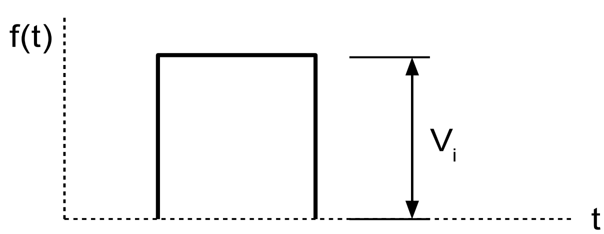
where the fractions d_i ($0 \le d_i \le 1$) represent the time-width of the section i as a portion of the period T and where

$$\sum_{i=1}^{n} d_i = 1$$

The RMS value of this waveform can be calculated by summing the RMS contributions from each piece:

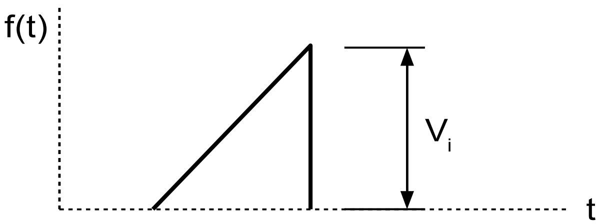
$$V_{rms}^2 = \sum_{i=1}^{n} d_i s_i$$

where s_i is the RMS contribution from the ith segment. These contributions are



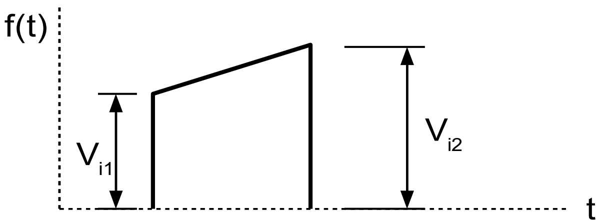
Constant segment

$$s_i = V_i^2$$



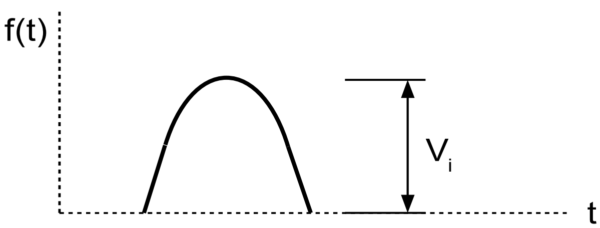
Triangular segment

$$s_i = \frac{1}{3}V_i^2$$



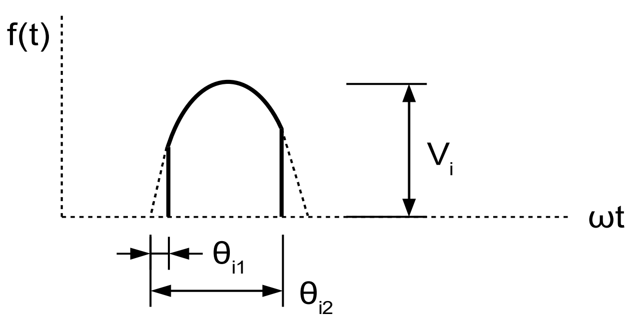
Trapezoidal segment

$$s_i = \frac{1}{3}\left(V_{i1}^2 + V_{i1}V_{i2} + V_{i2}^2\right)$$



Sinusoidal segment

$$s_i = \frac{1}{2}V_i^2$$



Sinusoidal segment (partial period)

$$s_i = \frac{1}{2}V^2\left[1 - \frac{\sin\left(\theta_{i2} - \theta_{i1}\right)\cos\left(\theta_{i2} + \theta_{i1}\right)}{\theta_{i2} - \theta_{i1}}\right]$$

Questions

Can a waveform have a negative crest factor?

What's the crest factor of a DC voltage of zero?

Can a waveform have a negative RMS value?

Glossary

absolute average

$V_{aa} = \frac{1}{T}\int_0^T |V(t)|\,dt$. This is the voltage that an average-responding meter measures. Note such meters typically have their measured values multiplied by $\pi/\sqrt{8} = \pi/\left(2\sqrt{2}\right)$ to make the meter read the correct RMS voltage of a sine wave.

AC-coupled RMS	The measurement that the majority of RMS meters measure. This is an RMS measurement of a waveform with its DC component removed. The marketing term for this is the misleading true RMS. The symbol is $\boldsymbol{V_{ac}}$ in the document text. $V_{ac} = \sqrt{V_{rms}^2 - V_{dc}^2}$
AC+DC	The real RMS value of a waveform; it's the same as V_{rms}.
average	For a periodic voltage waveform $V(t)$ with period T, it's the mathematical average of the waveform: $V_{dc} = \frac{1}{T}\int_0^T V(t)\,dt$. The discussion in section *Average* explains why the average is not a good functional for quantifying the "amount" in a waveform.
average-responding	A voltmeter that rectifies an AC waveform and measures the average value of the resulting waveform. The result is typically multiplied by $\pi/(2\sqrt{2})$ to make the voltmeter read the correct RMS voltage for a sine wave (it will be in error for all other waveforms).
crest factor	
DC offset = V_{dc}	For a periodic voltage waveform $V(t)$ with period T, it's the mathematical average of the waveform: $V_{dc} = \frac{1}{T}\int_0^T V(t)\,dt$. It will be zero for a waveform that is symmetrical about the time axis.
functional	A mathematical object that takes a function or set of points as an argument (with perhaps other arguments) and returns a number from some field (typically the real or complex numbers).
geometric mean	
harmonic mean	
instantaneous power	For a voltage *V(t)* and current *i(t)*, the instantaneous power is $P(t) = V(t)i(t)$. It is also a function of time and is often converted to a single number by using the RMS value. It can be a combination of real and reactive power. Note the power waveform for a sinusoidal voltage and current is twice the frequency of the voltage or current.
RMS	Root mean square functional. Without further qualification, it means $V_{rms} = \sqrt{\frac{1}{T}\int_0^T V^2(t)\,dt}$. Occasionally called "real RMS" in the text to distinguish it from true RMS, a marketing term for the AC-coupled RMS voltage. The marketing term AC+DC is used for this real RMS measurement.
time-averaged power	The instantaneous power waveform averaged over some time, usually the period of the waveform. It can be a combination of real and reactive power.
true RMS	See AC-coupled RMS.
V_{ar}	The voltage that an average-responding voltmeter measures. Such meters typically have their output scaled by the factor $\pi/\sqrt{8}$ to allow them to read

the real RMS value of a sine wave.

Experiment: A light bulb and RMS

This is would be a good experiment if a small light bulb can be found that can output useful light with a function generator that only puts out 50-100 mA. The best I've found are 2.47 V bulbs that run at 300 mA. This is likely still too high. It probably makes the resistor-diode experiment more attractive.

Objective: use a photodiode to measure the light output from a light bulb. First, power the bulb from an AC source such as two 6.3 V RMS filament transformers in series. Measure the voltage across the bulb and current through it; call these V_0 and i_0. Without moving the bulb in relation to the photodiode, measure the light output while applying power to the bulb from a DC power supply; call the new values V_1 and i_1. Verify that $V_0 = V_1$ and $i_0 = i_1$ when the AC parameters are measured as RMS values.

T1141 light bulb measurements (RMS values):

V, V	i, A	
0.06	0.05	
0.12	0.1	
0.22	0.2	
0.53	0.35	
1.36	0.5	Just barely red in middle
2.56	0.65	
3.56	0.75	
5.15	0.9	
6.41	1	
8.46	1.15	
9.81	1.25	
11.46	1.35	
13.97	1.5	

Experiment: Heating value

This is an excellent experiment to do because it demonstrates the fundamental principle that **the RMS values of the current and voltage predict the same dissipated power as the equivalent DC values**.

> **Experiment**: use a 1/4 W 500 Ω resistor next to a small signal diode like a 1N4148. Enclose these in an insulated container (use Styrofoam insulation) and use small-diameter wire for the leads to thermally isolate the resistor/diode from the outside world. Apply current to the resistor and measure the temperature rise of the air in the isolated chamber by the diode's voltage drop. Demonstrate that the heating from a 6.3 V RMS filament transformer is the same as the heating from a 6.3 V DC voltage. This runs the resistor at around 12 mA, so the power level is 80 mW.

Needed equipment and materials:

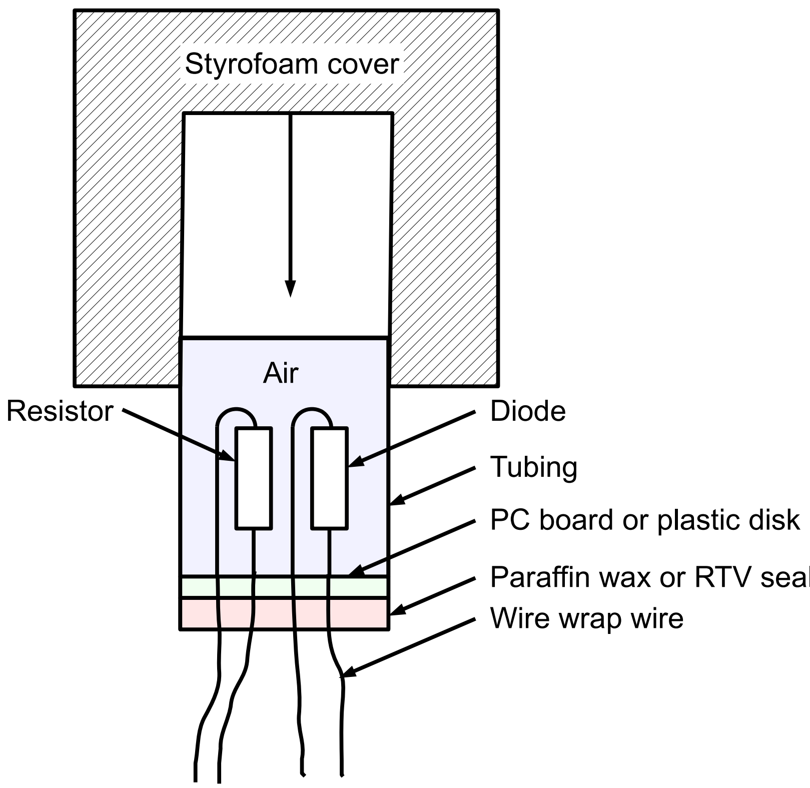
- 1/4 W resistor around 500 ohms
- Two small signal silicon diodes (e.g., a 1N4148)
- Styrofoam
- Constant voltage power supply

- 6.3 V filament transformer or suitable function generator (the typical function generator is capable of supplying 10 Vpp into 50 ohms or 250 mW)
- 3.5 digit multimeter (a 4.5 digit meter is better)

A good additional task is to use a function generator to supply a square wave and demonstrate that the RMS value of the square wave is the relevant measure to match the DC power.

Once you have the experimental setup working, you can use it with a function generator to experimentally determine the RMS values of various waveforms.

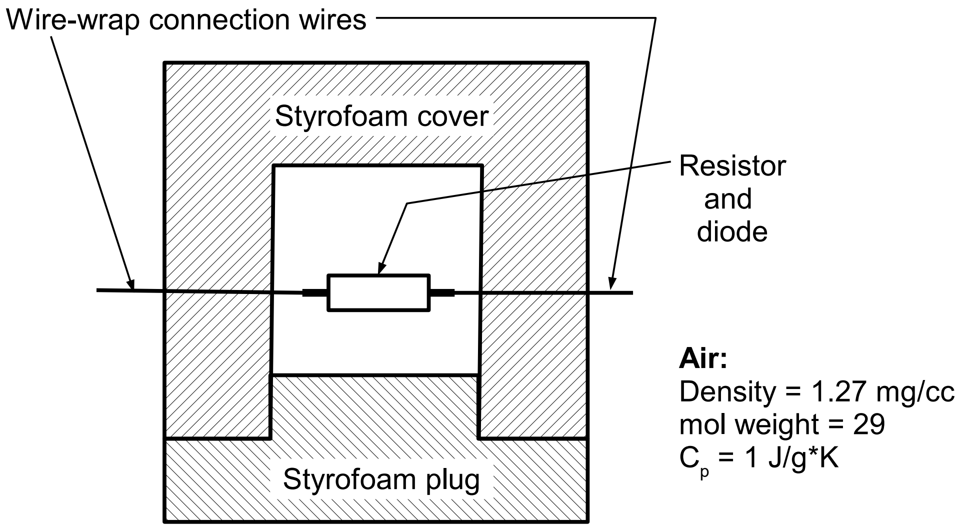
Use a 1N4148 silicon diode's forward voltage drop to measure temperature: 638 mV to 550 mV over the cold to hot temperature range of our house water. The experiment might have a change of 30-40 mV tops, so the best voltage-measuring tool would be a 4.5 digit voltmeter. However, it still can be done with the 3.5 digit voltmeter. It would be better to build a small op amp circuit to DC shift and amplify this voltage. With a constant current through a diode, the voltage drop will change by roughly -2 mV/°C. See http://en.wikipedia.org/wiki/Silicon_bandgap_temperature_sensor. The voltage across the diode for a constant current is $V = \alpha T$ where α is -2.1 mV/K for Si and T is junction temperature in K. This lets you calibrate the sensor if you wish in boiling water[9] and ice water, but it's not necessary for this experiment, as we're comparing the two heating rates.



The current source for the diode can be pretty simple -- just a resistor with a constant voltage. The current can be on the order of 100 μA to 10 mA.

Another idea: use a piece of Styrofoam like the above cover. Run wire-wrap wire soldered to the short leads of a diode & resistor *and let them hang from the "roof" of the enclosure.* Insert a plug into the bottom to perform measurements. This lets the diode and resistor be thermally well-insulated from the wires so that the heat goes to mostly raising the air temperature.

9 Remember to correct the boiling point for your altitude and barometric pressure.



You can make a small Styrofoam box by using some sheet material and cutting out the sides, joining them together with transparent packing tape. With thicker material, it's probably easiest to hollow out the cavity and make a mating plug.

The resistor needs to be 10 mm long. This means there should be about 4 mm on either side of each end, leading to a chamber diameter of 18 mm. The overall height should be about 10 to 15 mm; assume it's 10 mm. Then the chamber volume is 2.5 cm^3. The density of air is 1.27 mg/cc, so this is 3.18 mg of air. Thus, we'll need 3.18 mJ to raise the air temperature 1 K. Suppose we want the temperature to rise 1 K every 3.18 s. Then we need to dissipate 1 mW in the resistor. This lets us calculate the needed resistor for 6.3 V RMS: $R = V^2/P$, so for 1 mW we need 6.3^2 kΩ or 40 kΩ. The current will be 158 μA. Note we don't have to monitor the current; we just need to match the DC voltage to that which gives the same heating profile as the 6.3 V transformer.

Example: energy from tidal movement

Reference: Royal Academy of Engineering, *The Study of Root Mean Square (RMS) Value*, no date given, http://www.raeng.org.uk/education/diploma/maths/pdf/exemplars_engineering/8_RMS.pdf, accessed 19 Sep 2013.

The tidal height *H* varies as a function of time *t* as

$$H(t) = (4 + 3.5\cos\frac{\pi t}{7})\cos 4\pi t$$

where *t* is the time in days. An engineer wants to assess the use of the tide at this location to generate electrical power. To do this, he wants the RMS value of the tide's movement. Plot the function $H(t)$ and find its RMS value over a 14 day period.

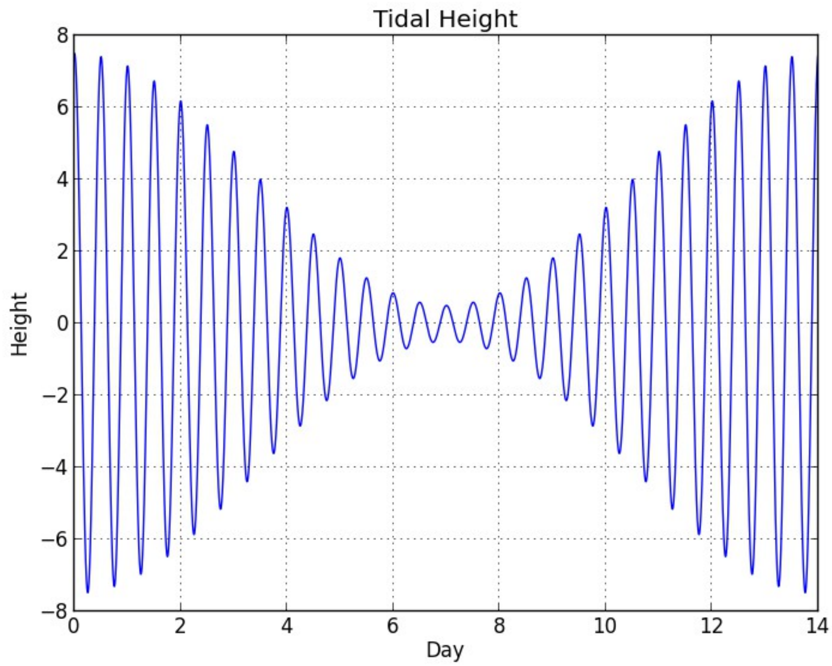
A python script using the `numpy` library can do this calculation numerically and plot the results:

```
from pylab import *
n = 10000
P = 14
t = arange(0, P, P/n)
H = (4 + 3.5*cos(pi*t/7))*cos(4*pi*t)
print("RMS value = %.3f" % sqrt(mean(H*H)))
plot(t, H)
grid()
title("Tidal Height")
xlabel("Day")
ylabel("Height")
show()
```

The result printed is

```
RMS value = 3.326
```

and the plot is



Experiment: Crest Factor

It is worthwhile to make some experimental measurements showing the effect of crest factor on an RMS voltmeter. This is a simple experiment to perform if you have a pulse generator that lets you adjust the duty cycle.

A B&K 2542B-GEN oscilloscope (it has a built-in function generator) was used to generate a 0 to 5 V amplitude pulse at 50% duty cycle. The period was set to 20 ms, meaning this is a 50 Hz waveform. The signal was run to the scope, an HP 3456A digital voltmeter, and an HP 3400A RMS voltmeter. At the input to the 3400A, a 50 Ω pass-through termination was used. Both voltmeters were set to measure AC-coupled RMS values, so the DC offset of the pulse was left at zero.

Next, the amplitude was adjusted downwards so that the 3400A read 1.0 V RMS full scale on the 1 V range. The 3456A read 0.988 V RMS.

Then the duty cycle *D* of the pulse was adjusted and the voltmeter measurements recorded:

	3400A		3456A		Mean		Crest
D, %	Trial 1	Trial 2	Trial 1	Trial 2	3400A	3456A	factor
50	1.000	1.000	0.988	0.987	1.000	0.988	1.4
25	0.848	0.864	0.854	0.855	0.856	0.855	2.0
10	0.588	0.599	0.590	0.591	0.594	0.591	3.2
5	0.429	0.436	0.427	0.428	0.433	0.428	4.5
2.5	0.307	0.307	0.304	0.305	0.307	0.305	6.3
1	0.195	0.195	0.192	0.193	0.195	0.193	10
0.5	0.138	0.138	0.135	0.135	0.138	0.135	14
0.25	0.0718	0.0721	0.0947	0.0948	0.072	0.095	20
0.1	0.0448	0.0449	0.0586	0.0587	0.045	0.059	32
0.05	--	--	0.0401	0.0401	--	0.040	45
0.02	--	--	0.0220	0.0220	--	0.022	71
0.01	--	--	0.0128	0.0128	--	0.013	100

The -- in the 3400A measurements indicate where the readings were unstable. The measurements with a gray background indicate where the two voltmeters agreed well. The cells at the lower right had crest factors significantly beyond the instruments' specified capabilities. The crest factor is calculated from the duty cycle.

The 3400A is specified to measure crest factors up to 10 at full scale and inversely proportional to meter deflection (e.g., 20 at half-scale and 100 at tenth-scale). The 3456A is specified to measure with crest factors > 7 at full scale. For these measurements, it was important that the Filter be turned on for the 3456A measurements, as it reduced measurement noise significantly.

Appendix: python and tools

The python programming language can be downloaded for a variety of platforms from http://www.python.org/. Numerical calculations are made substantially easier by using the numpy library at http://www.numpy.org/. The plotting library used in the document is called matplotlib and is available at http://matplotlib.org/.

References

One of the problems of the web is that a year or two is a long time. Many referenced URLs can go defunct after a few years, so I have moved defunct references to a separate table below.

These URLs were last checked in Nov 2020.

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hp124 Hewlett-Packard, *True RMS Measurements*, Application Note 124, 1970.

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