Laying out a frustum of a cone

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You may occasionally need to lay out the frustum of a cone for a home or shop project. By "frustum", we mean a right circular cone[1] that has its top cut off by a plane perpendicular to the cone's axis:

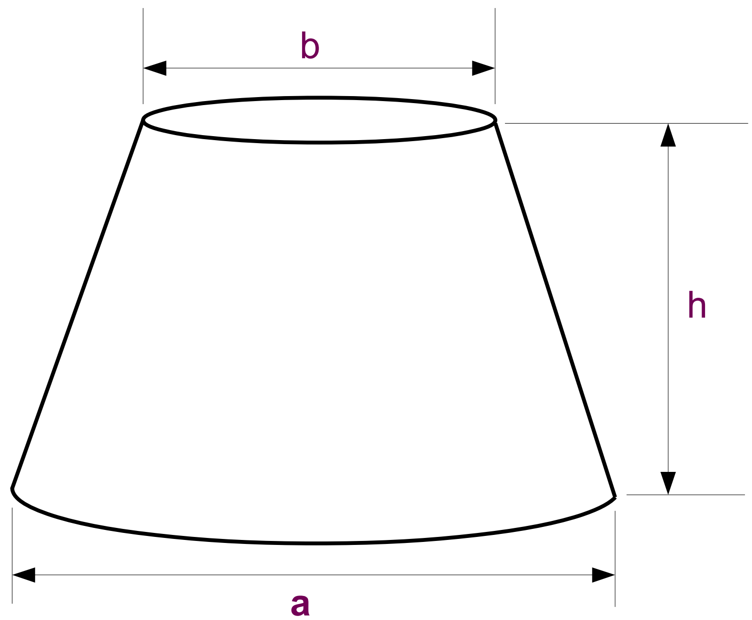


Figure 1

It looks like a common lampshade. If **b** is 0, then we have the usual cone with a pointed top. We'll make the assumption that **a** > **b**.

To lay a frustum out on flat material like sheet metal or paper, the only tools you need are a pencil, ruler, and dividers or a compass. A square to draw a right angle can save a little bit of time. The method is taken from the October 1945 issue of Popular Science, page 178 -- I've managed to expand their small box of material into 9 pages!

Here's how to make the layout -- it's a geometrical construction like you learned in high school geometry. I will give **a**, **b**, and **h** in color to remind you that these dimensions are given. For each step, the associated figure will follow the text.

Step 1 (Figure 2): lay out a horizontal line AB of distance **a** and construct its perpendicular bisector CD (use the square to construct the bisector or, if you've forgotten how to do it with dividers, see *Appendix 1: constructing a perpendicular bisector* on page 8).

1 Frustums can be from more general objects, but we'll just consider right circular cones in this document. It has been misspelled as "frustrum" for centuries.

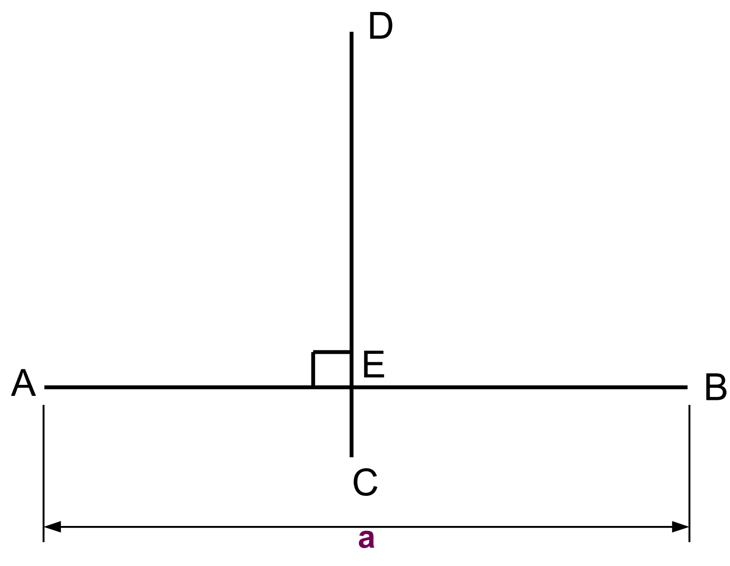


Figure 2

Step 2 (Figure 3): from point E, measure up height **h** perpendicularly to AB and draw a horizontal line FG that is parallel to AB

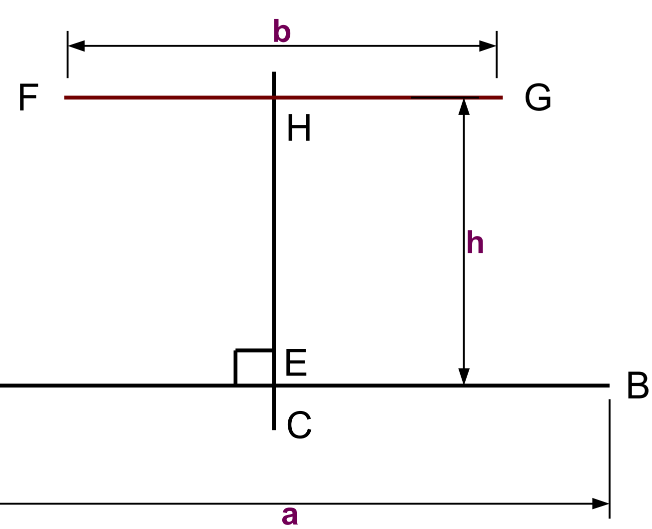


Figure 3

Step 3 (Figure 4): extend lines AF and BG to meet on line CD at the apex of the cone (point D is the apex of the cone):

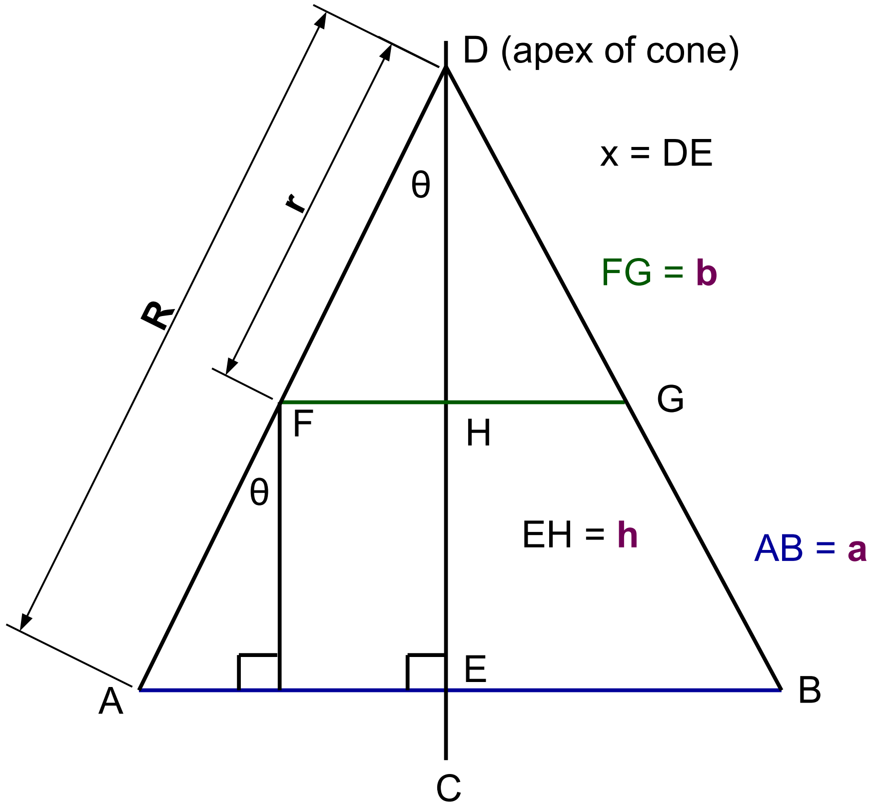


Figure 4

ABGF is the frustum in side view and 2θ is the angle ADB subtended by the full cone. The dimensions *r* and *R* will be used in the next step.

Step 4 (Figure 5): draw two arcs with radii *r* = DF and *R* = DA centered on D (these are the edges of the base and top of the frustum after it is rolled into shape):

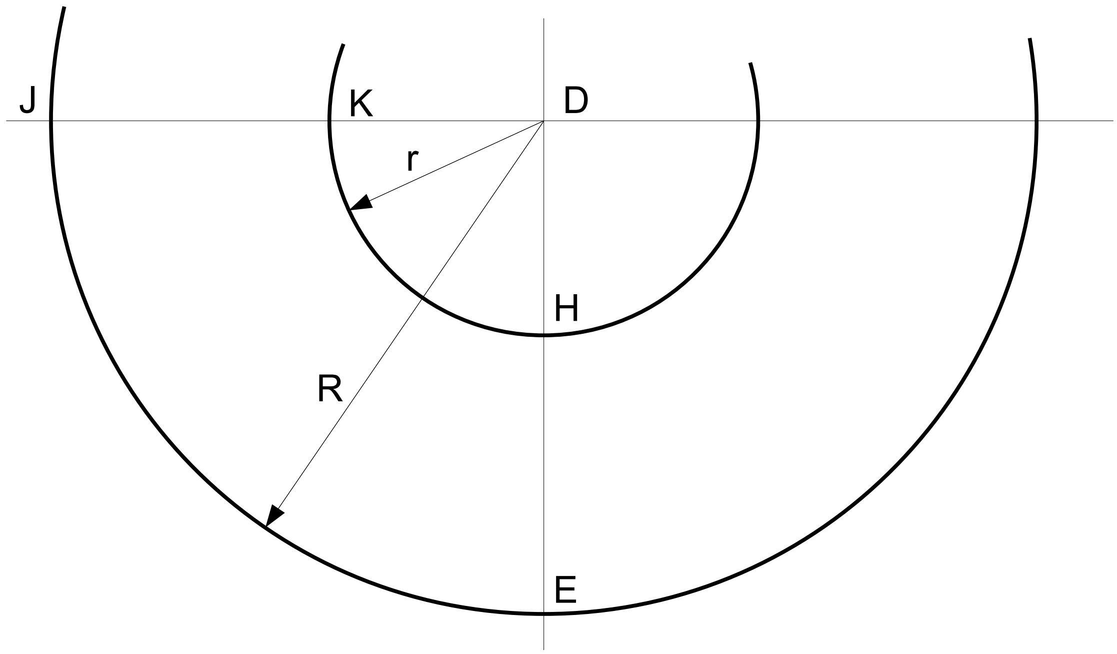


Figure 5

Step 5 (Figure 6): draw the line JK, then set the dividers to 1/15th to 1/20th of π**a**. Using this constant setting, step off the circumference of the base of the cone (π**a** from Figure 1) starting from point J along the circle of radius *R*. The ending point is L; draw the line LM along the radius from D to complete the pattern (the pattern is colored red):

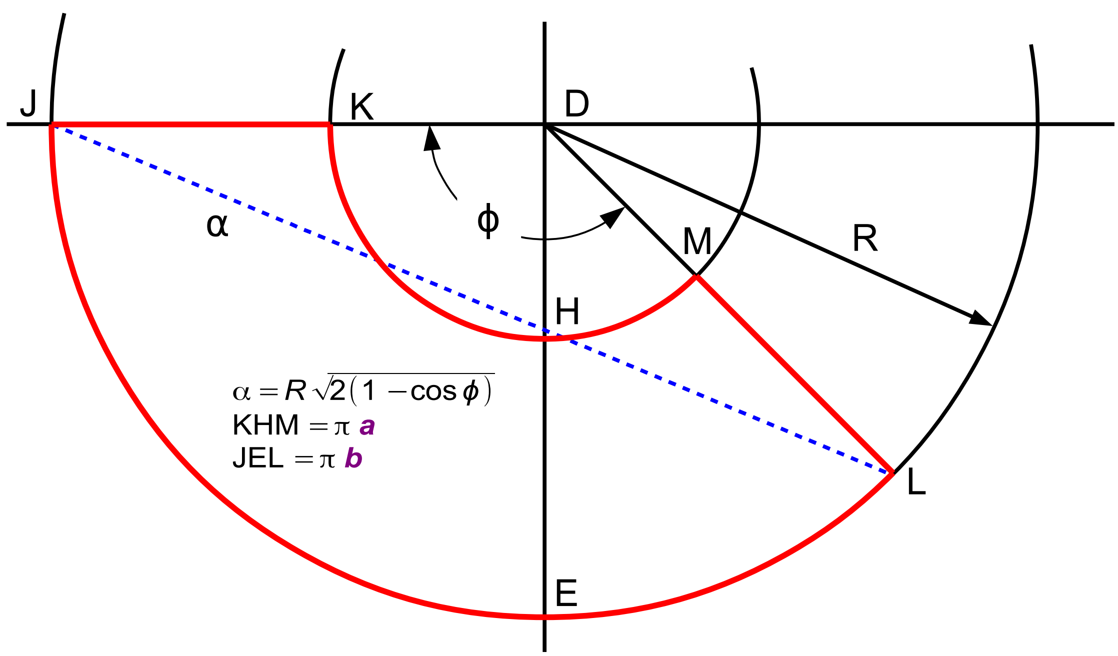


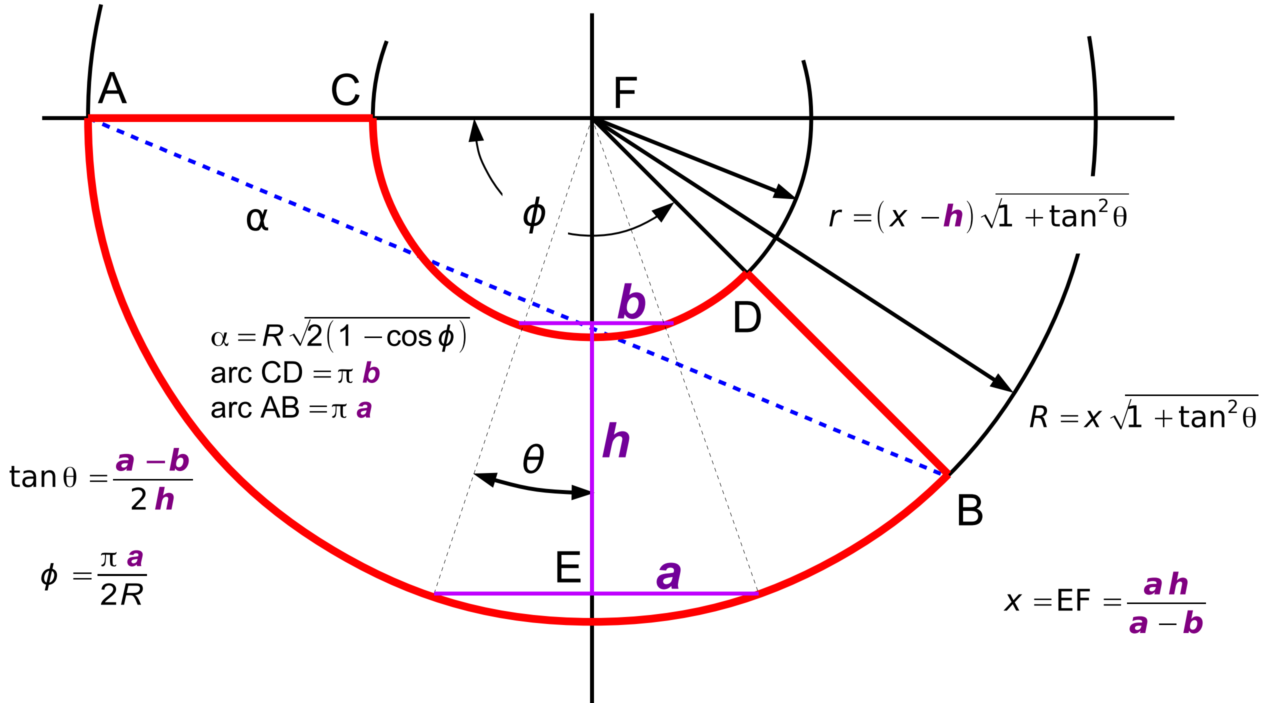
Figure 6

Instead of stepping off with the divider, you can just measure distance α to find the point L; see equation (3) below.

The pattern to cut out is EJKHMLE, shown in red in Figure 6. When rolled into shape and the ends JK and LM are connected together, your frustum will be complete. If you wish, make the edges overlap a bit so they can be connected by riveting, soldering, etc.

To help visualize what's going on with the above development, imagine that you cut the frustum with a vertical cut in Figure 4 through line CD and roll it out until it lies in the plane of the paper. The four points A, B, F, and G are fixed; what you wind up with is the development shown in Figure 6.

Here's everything in one (busy) illustration:



Doing it via a calculation

You may instead wish to calculate the numbers directly. Remember, we are given **a**, **b**, and **h**.

From similar triangles and the Pythagorean theorem in Figure 4 we can write

$$r = \sqrt{\frac{\mathbf{b}^2}{4} + \left(\frac{\mathbf{bh}}{\mathbf{a}-\mathbf{b}}\right)^2} \quad \text{and} \quad R = \sqrt{\frac{\mathbf{a}^2}{4} + \left(\frac{\mathbf{ah}}{\mathbf{a}-\mathbf{b}}\right)^2} \tag{1}$$

The arc JEL has a length of $\frac{\pi\, \mathbf{a}}{2}$, as it is the circumference of a circle with diameter **a**.

For direct calculation, let's call distance DE in Figure 4 x. Then we have

$$\begin{aligned} \tan\theta &= \frac{\mathbf{a}-\mathbf{b}}{2\mathbf{h}} \\ x &= \frac{\mathbf{a}\,\mathbf{h}}{\mathbf{a}-\mathbf{b}} \\ R &= \frac{x}{\cos\theta} = x\sqrt{1+\tan^2\theta} \\ r &= \frac{x-\mathbf{h}}{\cos\theta} = (x-\mathbf{h})\sqrt{1+\tan^2\theta} \end{aligned} \tag{2}$$

The relation for x is gotten from similar triangles by writing

$$\tan\theta = \frac{\mathbf{a}/2}{x} = \frac{\mathbf{a}-\mathbf{b}}{2\mathbf{h}}$$

The angle ϕ in Figure 6 is then this arc length divided by the radius R

$$\phi = \frac{\pi\, \mathbf{a}}{2R}$$

Instead of stepping off the arc JEL, you can measure the line from JL = α to mark the point L. The distance α is gotten from the cosine law using the triangle's legs JD and DL, which are both R:

$$\alpha^2 = R^2 + R^2 - 2R^2 \cos\phi$$

or

$$\alpha = R\sqrt{2(1-\cos\phi)} \tag{3}$$

Note the formula works for any angle ϕ.

Example

Before relying on formulas and methods, it's a good idea to construct at least one example. If the example doesn't work, then the formulas/methods are flawed. Here's a small frustum you can lay out on a sheet of A4 or ANSI A size paper and verify the frustum’s dimensions are as predicted.

Choose the following dimensions:

$$\begin{aligned} \mathbf{a} &= 3 \text{ inches} = 76.2 \text{ mm} \\ \mathbf{b} &= 1 \text{ inches} = 25.4 \text{ mm} \\ \mathbf{h} &= 2 \text{ inches} = 50.8 \text{ mm} \end{aligned}$$

Then the calculated radii of the layout circles from equations (1) are

$$r = \sqrt{\frac{\mathbf{b}^2}{4} + \left(\frac{\mathbf{bh}}{\mathbf{a}-\mathbf{b}}\right)^2} = \sqrt{\frac{1}{4} + \left(\frac{2}{2}\right)^2} = \sqrt{1.25} = 1.12 \text{ inches} = 28.45 \text{ mm}$$

$$R = \sqrt{\frac{\mathbf{a}^2}{4} + \left(\frac{\mathbf{ah}}{\mathbf{a}-\mathbf{b}}\right)^2} = \sqrt{\frac{9}{4} + \left(\frac{6}{2}\right)^2} = \sqrt{11.25} = 3.35 \text{ inches} = 85.09 \text{ mm}$$

From equations (2), we get as checks

$$\tan\theta = \frac{3-1}{4} = \frac{1}{2}$$

$$x = \frac{3(2)}{3-1} = 3$$

$$R = 3\sqrt{1+\frac{1}{4}} = 3.354$$

$$r = 3 - 2\sqrt{1+\frac{1}{4}} = 1.118$$

The subtended angle ϕ of the arc length π**a** results in 3π/3.35 = 2.81 radians = 161°. If you use a protractor and draw the angle as 165°, you'll have a small tab of material to overlap to make it easier to apply a piece of tape to hold the frustum together. Mark a line at 161° to show you where to stop the overlap, then tape the frustum. Or, lay out the angle using the distance α = 6.62 inches.

You should wind up with a frustum of the required size.

Burner flare

Here's an example of a real calculation. I wanted a flare to put on a propane burner I made from some 1 inch diameter steel tubing. I decided the slope of the included angle of the flare would be 1/10, which translates into a half-angle θ of 2.86°. Since **b** was 1 inch and h was **2** inches, this resulted in **a** being $1 + 2\tan\theta = 1.10$.

The relevant numbers were thus

$$\begin{aligned} \mathbf{a} &= 1.10 \text{ inches} \\ \mathbf{b} &= 1 \text{ inch} \\ \mathbf{h} &= 2 \text{ inches} \end{aligned}$$

Plugging in the numbers into the formulas, I got

$$\tan\theta = \frac{0.1}{2(2)} = 0.025$$
$$x = \frac{1.1(2)}{0.1} = 22 \text{ in}$$
$$r = (22-2)\sqrt{1+0.025^2} = 20$$
$$R = 22\sqrt{1+0.025^2} = 22$$

where I've rounded things off for easy layout with a rule.

Laying out an arc with dividers

You might wonder how good this method of laying out an arc with dividers is. The relevant figure is:

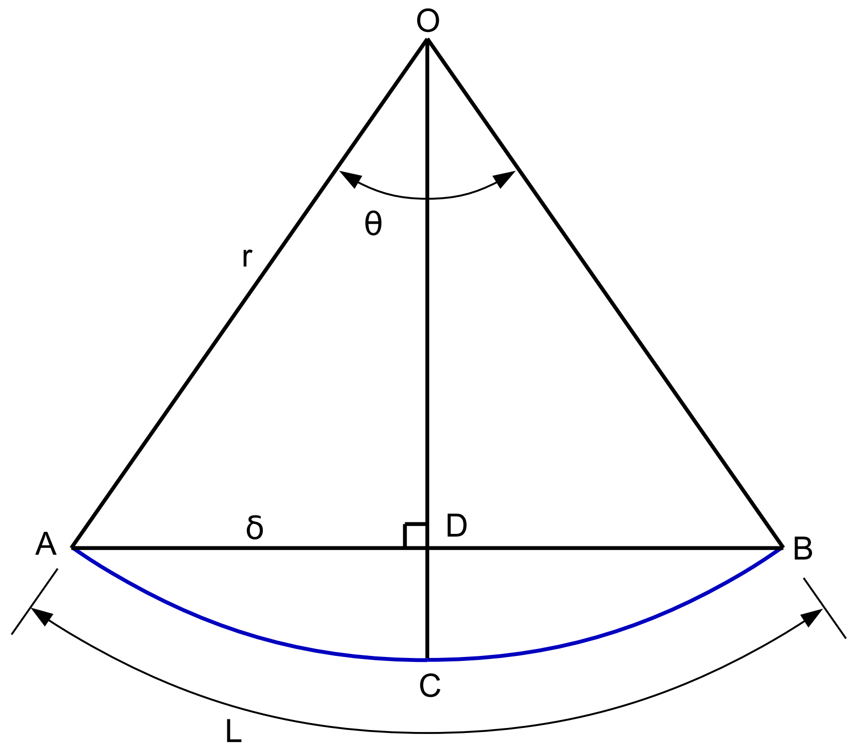


Figure 7

The method depends on approximating the arc ACB with the chord ADB. The smaller the angle θ is, the better the approximation.

Suppose we have an arc length *L* along a circle of radius *r* which we want to approximate by *n* chords with a divider. Then each chord ADB is approximately of length *L*/*n*. In Figure 7, we have

$$\delta = \frac{1}{2}\frac{L}{n}$$

We have

$$\sin\frac{\theta}{2} = \frac{\delta}{r}$$

so

$$\theta = 2\sin^{-1}\frac{\delta}{r}$$

The actual angle stepped off with the dividers is $n\theta$. The desired angle was $x = L/r$. The difference is $D = n\theta - x$ or

$$\frac{D}{x} = 2n\sin^{-1}\frac{\delta}{r} - x = 2n\sin^{-1}\frac{x}{2n} - x$$

Here's a plot of the fractional difference $\frac{D}{x} = \frac{2n}{x}\sin^{-1}\frac{x}{2n} - 1$ converted to a percentage:

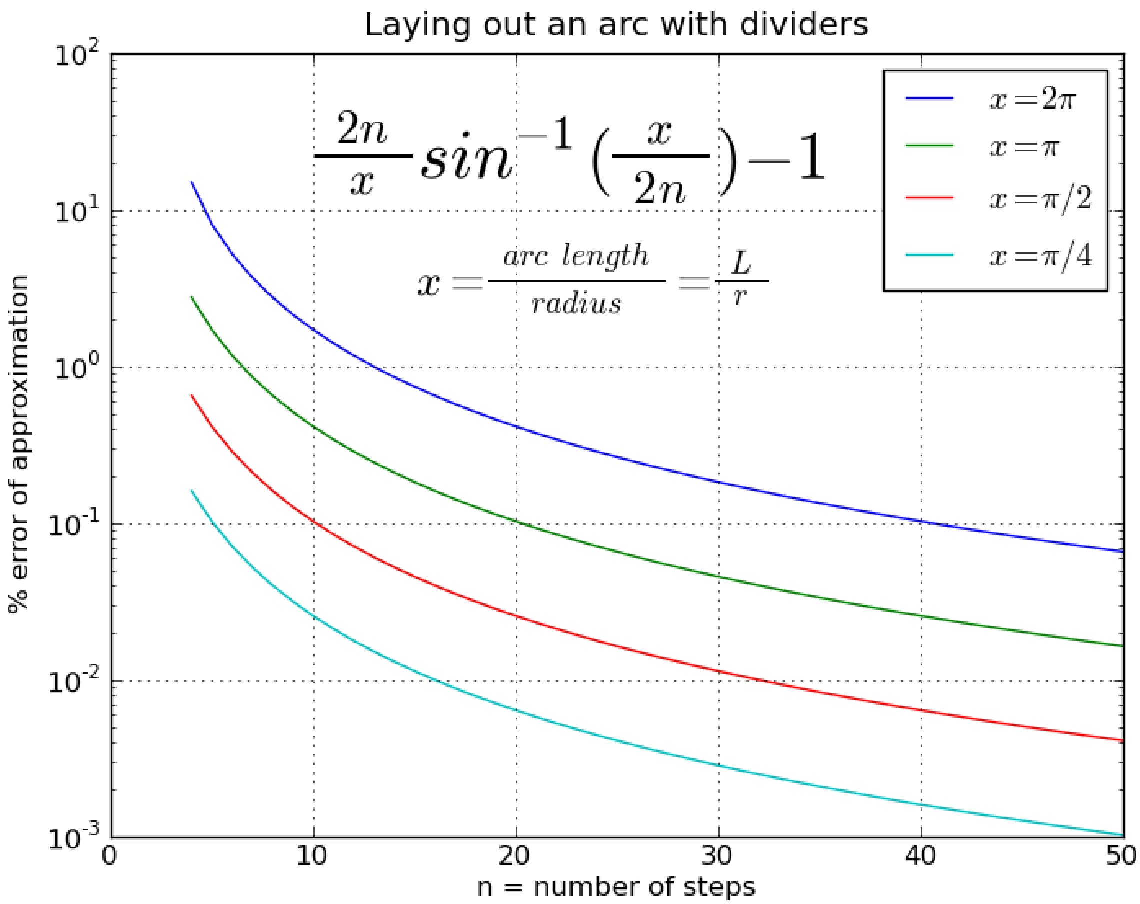


Figure 8

The approximate method always slightly overestimates the desired arc. This is not a big deal, as often you want the edges of the frustum to overlap.

You can see that the approximation is quite good -- the error is below 1% for *n* above 13 or so; and that's for the worst case where the arc is a full circle. You can thus make the rule of thumb that if you choose $n \geq 15$, you'll have a good approximation for any practical work you do.

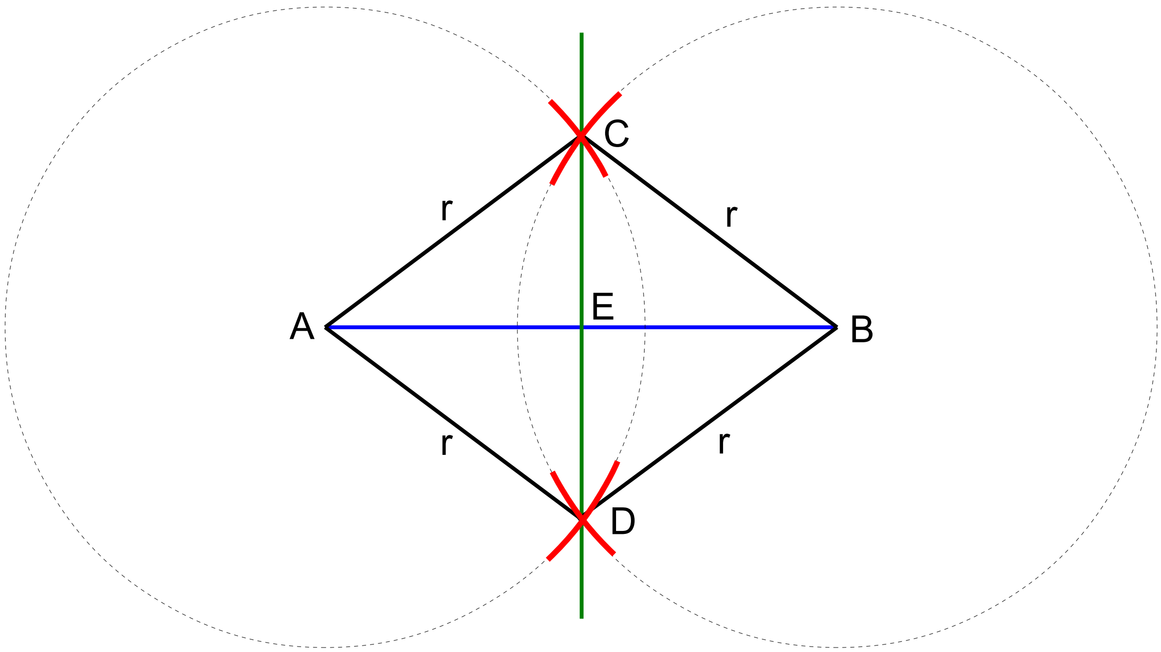
If there's a calculator handy, then the divider setting to get the exact arc desired is

$$2r\sin\frac{L}{2nr}$$

You can see this gives the right answer for the case where $n = 1$ and the angle L/r is θ. Then the distance is just the diameter of the circle, as you'd expect. Another obvious case is if we want to step off a full circle in 6 steps -- we get $2r\sin\frac{\pi}{6}$, which is just *r*.

Appendix 1: constructing a perpendicular bisector

In elementary geometry, it's proven in the following figure that if *r* is greater than half the width of the line AB, then line CD drawn between the intersection points of the two circles of radius *r* bisects the line AB perpendicularly:



The centers of the circles are the ends of the line segment AB. In practice, you just set your dividers or compass to a distance of perhaps 75% of AB, then scribe the red arcs intersecting near the points C and D.

ACBD is a rhombus with sides *r*; the diagonals of a rhombus are perpendicular and bisect each other. You may want to see the applet at http://www.mathopenref.com/rhombus.html to help visualize this.