**Arithmetic Without an Electronic Calculator**

someonesdad1@gmail.com 25 Aug 2012

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I've used some symbols to help you understand what sections you might want to read first. The ♥ symbol denotes the sections that I think are the most important and contain things that most folks should know to be able to reason quantitatively more easily. The symbol ♦ marks stuff that is probably important for an engineering/science type of person to know. The remaining unmarked material is for reference and can be omitted without missing the gist of the material.

# Introduction ♥

I find myself occasionally needing to perform calculations when a calculator isn't handy. As there are various ways of doing this, I thought I'd write down some these techniques. I am not advocating a return to doing arithmetic without calculators. But if you're thinking about a career in anything technical (math, science, engineering, construction, plumbing, electrical, etc.), **being able to do calculations when a calculator isn't handy can be a time-saver and make you more effective**.

Being able to do approximate calculations with paper and pencil can have some benefits:

- You can make independent checks of calculations done by computer, calculator, or other person. This makes it more likely you'll catch errors. We always make errors.
- You can do back of the envelope calculations with facility.
- Check statements for reasonableness.
- You can do approximate calculations when no calculator or computer is handy.

This paper has two types of content. One type has to do with making approximations. The other type has to do with calculational tips and tricks that can help you do calculations without a calculator handy.

There's no specific technical level for this paper, so most of these techniques should be understandable and usable by anyone interested enough to learn them.

> **Important**: If you want to use some of these techniques, you won't remember them unless you practice them before you need them. It's straightforward to learn something in your easy chair, but when you're e.g. on a job site and in a rush or in a meeting, that knowledge flies out the window unless it becomes second nature. A recommendation is to adopt one of the techniques and start using it in your work. Then, as you become comfortable with that technique, add another. Remember how much you practiced basic reading, writing, and arithmetic in grammar school.

If I think a point is worth emphasizing, I'll express it in **this font**. Key words are in **this font**.

**ppm** means "parts per million" and is one part per $10^6$. **log** stands for the base 10 logarithm and **ln** stands for the natural logarithm.

References are given at the end of the paper. [*wilson:223*] means page 223 of the indicated reference. The form [*xyz:n:m*] means the document is a PDF and the page of interest is *n* in the document's page numbering and *m* in the PDF's pagination.

## Assumptions

One prerequisite to using these techniques is you have to know your multiplication table cold. Make sure you know the 12x12 table, as you'll use it a lot (a century ago it was not uncommon for grammar school kids to know their 20x20 tables). You also need to be able to confidently add and subtract and long-hand division is sometimes needed. In other words, you need to be able to do your grammar school arithmetic well.

I'll assume you have an electronic calculator. You should use it to check the work you do manually. Don't get frustrated, as it's likely in the beginning of doing manual calculations you'll make lots of errors and feel that it's just not worth the effort. But, like many other things in life, it becomes a bit easier after you spend some time with it. So have faith and keep in mind the end result: you'll be better able to do quantitative reasoning on-the-fly when calculators or computers aren't available.

A good attitude in performing numerical calculations is that **the answer is always wrong and it's your job to know by how much**. This sounds flippant, but it contains a kernel of truth -- except for integers and rational numbers, we always use numbers that are approximations of the "correct" value. Additionally, numbers that come from measurements always have a limited resolution and an inherent stochastic uncertainty.

## Scientific notation ♥

The number 12.34 is really a short-hand notation for the rational number $12+\frac{34}{100}$. This notation and the number 0 were two of the most significant mathematical inventions ever, even though we take them for granted today. When you do a lot of calculation with pencil and paper, developing a compact notation is important because it can save time and reduce errors. A notation is an **encoding method**.

Since we do our arithmetic in base 10, we find it easy to multiply and divide by 10. The quick way of doing it on paper is to move the decimal point. For example

$$\frac{12.34}{10}=1.234 \quad \text{and} \quad 12.34\times 10=123.4$$

This is probably so obvious it shouldn't be mentioned, but it leads to some other things worth knowing.

**Scientific notation is a method of encoding big or small numbers in a compact form**. A number *x* in **scientific notation** is in the form $M\times 10^N$ where $M$ is the **significand**[1], $1 \leq M < 10$, and $N$ is a positive or negative integer. If $N$ is positive or zero, the encoded number is greater than or equal to 1; if $N$ is negative, the encoded number is less than 1. There are other definitions of scientific notation; one example is where $M$ is instead on the half-open interval [0.1, 1).

Scientific notation reduces errors and results in less work to write numbers than with other methods. The following number is cumbersome: 12340000000000000000000000000. We see the significand easily but the number's magnitude is not easy to see. Is it bigger than 1234000000000000000000000000? You can't tell without counting digits -- and this is laborious and error-prone. Here's how to convert 12340000000000000000000000000 to scientific notation: write the significand as 1.234 and count how many times you have to move the decimal place to the left to get it between the 1 and the 2. We have to move the decimal point left 28 times, so this number is $1.234\times 10^{28}$ in scientific notation. This is more compact and easy to grasp -- it encodes the identical information, but in a smaller space and with less writing effort. Numbers less than 1 are done in an analogous fashion: 0.0000000000001234 has the significand 1.234. Then count the number of decimal positions from the decimal point to between the 1 and 2 to get 13. The number in scientific notation is then $1.234\times 10^{-13}$; numbers less than 1 have negative exponents. Instead of $1.234\times 10^{28}$, I often write 1.234e28, especially in handwritten stuff, so that's an alternative method of writing scientific notation (and it's the form used in most programming languages).

**Arithmetic with scientific notation is easier because you're dealing with the arithmetic of small numbers** (the significands) between 1 and 10. Because multiplication is associative (i.e., $a\cdot(b\cdot c)=(a\cdot b)\cdot c$), we can put all the significands together and all the powers of 10 together. Here's an arithmetic problem written with the usual decimal notation:

$$\frac{1234000\,(0.0043)}{12200000}$$

1 Some people call this the mantissa, but that term is more correctly used with logarithms.

This is easier to evaluate when written in scientific notation and grouping the significands together:

$$\frac{1.234\times10^{6}\left(4.3\times10^{-3}\right)}{1.22\times10^{7}} = \frac{1.234\left(4.3\right)}{1.22}\,\frac{10^{6}10^{-3}}{10^{7}}$$

To calculate the powers of 10, add the exponents in the numerator and subtract from the result the sum of the exponents in the denominator. Thus, the right-hand term is $10^{-4}$.

You'll probably agree that the arithmetic with the significands is easier to grasp. The term $\frac{1.234}{1.22}$ is nearly 1, so we know the answer has to be about $4.3\times10^{-4}$.

Here's one final tip about dealing with tens. When you've written down a rational expression of products, you can immediately cancel paired right-most zeros (i.e., whole tens) in numerator and denominator terms (assuming you haven't yet written things with scientific notation). This is because canceling a 0 in the numerator and denominator is equivalent to dividing the numerator and the denominator separately by 10. In other words, multiplying a number by $\frac{x}{x}$ is the same as multiplying by 1; here, x is 1/10.

Because of this tip, when you see an expression like

$$\frac{1230\left(900\right)}{6400\left(20\right)}$$

you can cross out the corresponding trailing zeros on the top and bottom to quickly get the expression:

$$\frac{123\left(9\right)}{64\left(2\right)}$$

Since twice 64 is 128, you can see that the final result will be a number a little less than 9. If you had blindly started to multiply the numbers together in the original expression, you would have been in for more work.

You could also rewrite the expression in scientific notation:

$$\frac{1.23\times10^{3}\left(9\times10^{2}\right)}{6.4\times10^{2}\left(2\times10^{1}\right)} = \frac{1.23\left(9\right)}{6.4\left(2\right)}\,\frac{10^{3}10^{2}}{10^{2}10^{1}}$$

and you'd get the answer in a different fashion. But when working on paper, crossing out zeros is faster.

## Estimating the order of magnitude ♥

> **Every calculation must be checked, or it is without value.**
> **[*wilson:343*]**

**This section is by far the most important section of this paper**. I talk about a simple method of approximating the answer to an arithmetic problem. The reason it's important is that it can provide you with an independent check of a problem's answer.

Besides learning to do basic arithmetic, if there's one math skill I think everyone should exit their schooling with, it's the ability to do these order of magnitude estimations with confidence and facility. It's a skill you'll use the rest of your life.

The method is simple: replace complicated numbers by simpler integers and fractions to approximate what the final result is. You want to get the answer to about within a factor of 10 and you can often do better. Getting an answer to within about a factor of 10 is called **estimating the order of magnitude**, which is another way of saying you only care about the exponent in the power of 10 of the answer in scientific notation.

Here's an example problem: calculate

$$\frac{1234(456)}{78(0.0987)}$$

If your first instinct is to turn to your calculator and do the problem, this is a habit you want to get out of. Your first task should be to estimate the order of magnitude of the answer -- and do it in your head if you can (you'll get better at doing it in your head by practice).

> Actually, the first task should be to check that you've written the expression down correctly to ***avoid getting the right answer to the wrong problem***. It's sometimes disheartening to realize how many times we write down what's not there, transpose terms, etc.

The easiest way to estimate the answer is to just write the terms replaced by their nearest power of ten:

$$\frac{1000(100)}{100(0.1)} = \frac{10^3 \cdot 10^2}{10^2 \cdot 10^{-1}} = 10^4 = 10{,}000$$

Then this answer is used to check the calculation result: if the calculation's answer is less than around 1,000 or more than 100,000, you'd be suspicious that an error has been made.

With a little more work (keeping more significant digits), a better approximation can be gotten:

> 1234 is about 1000 and 456 is about 500, so the numerator is 500,000. The 78 in the denominator is about 100 and 0.0987 is about 0.1, so the denominator is about 10. Thus, the answer to the problem should be about 50,000.

Now, if you proceed to use your calculator, you'll get the answer 73,091.7. No surprise.

However, suppose you entered the wrong number into your calculator . If you had gotten answers on the order of 1000 or a few hundred thousand, you'd know something was wrong because the order of magnitude of the answer is wrong -- the answer should have been around 50 thousand.

This order of magnitude method doesn't protect you when you make errors in the less significant digits. For example, if in the above problem you had typed in 0.0978 into your calculator, your answer would have been in error, but the order of magnitude would have still been correct. The technique is necessary to find errors, but not sufficient.

If you write the expression to be evaluated in scientific notation, the rules for making an approximate calculation can be given simply as:

> For each term in the expression, keep only the first digit of the significand and the power of 10 for each number. If the second digit of a significand is 5 or greater[2], increase the first digit by 1.

The above rules are relevant to multiplications and divisions. But arithmetic expressions also include additions and subtractions. The rule for these is

> If two numbers don't have the same order of magnitude in an addition or subtraction, ignore the number with the smaller absolute value.

You can see why if you look at the example 20 + 2 in scientific notation: $2\times10^1 + 2\times10^0 = (2+0.2)\times10^1$. The 0.2 term is only 10% of the total sum, so we ignore it because we know it will have less effect on the answer's magnitude when compared to the 2's contribution.

Use this latter rule with common sense -- if you had a sum like

$$10+1+1+\ldots+1 \quad \text{(10 ones)}$$

you'd not ignore the ten 1 terms because they're half of the sum. Another example: a sum like 1 + 0.9 would be written in scientific notation as $1\times10^0 + 9\times10^{-1}$. If you followed the rule blindly, you'd ignore the 0.9 term -- but it should be rounded to 1. While the order of magnitude of the problem is still gotten correctly with the rule, in this case the approximate answer will be closer with little extra

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2 This is biased rounding, but that's OK -- we're making an approximation.

work.

These order of magnitude calculations can be used to estimate the answers to hard problems. An example of such a problem is the question "How many piano tuners are there in Chicago?". These problems are sometimes called Fermi problems, but the techniques were being used long before Fermi's name was associated with them (Fermi was a master at doing them). It's common to use these types of questions in technical interviews to see how a person thinks on their feet.

A use of this technique is to check that you haven't been overcharged for something. Suppose you buy 14 boxes of nails at $3.34 each and the amount on your bill is $56.76. You'd approximate the answer by 3 times 14, or 42, then add in one tenth of this amount to get 42 + 4.2 or $46.2. The correct answer is a bit higher, but not $10 higher, so the seller has made a mistake (they probably wrote down 56 when meaning 46). If you didn't catch this, you'd be adding $10 of pure profit to the sale.

There's a related technique that doesn't involve calculations: estimating the number of objects present. For example, estimate the number of birds sitting on a telephone wire by estimating the number of birds in one unit of length, estimating the number of lengths in all the birds, and doing an approximate multiplication. You get better at such things with practice, but things are difficult in the beginning. When driving on a long trip, I'll spot some feature in the distance and estimate how far away it is. I check the odometer and see how close my estimate was. Estimating the number of people in a crowd is another useful skill. You'll find all kinds of estimation tasks like this if you look around.

## Back of the envelope calculations ♥

I prefer to call these order of magnitude calculations "back of the envelope" calculations. Before email and the Internet, it was common for a person to have many opened envelopes on his or her desk -- and these were often used as scratch paper (thus, you'll also occasionally hear them called "scratch" calculations).

I'd like to suggest a still-useful technique for performing such calculations: when you're dealing with big numbers, do your arithmetic with base 10 logarithms instead. You also round to integers or keep only 1 or two decimal places. Let's look at an example by estimating the number of atoms in the observable universe. I will use approximate numbers and you may disagree with my choices -- if so, substitute your own numbers.

First, let's make a table of the base 10 logarithms to help:

| **x** | **log(x)** |
|---|---|
| 2 | 0.30 |
| 3 | 0.48 |
| 4 | 0.60 |
| 5 | 0.70 |
| 6 | 0.78 |
| 7 | 0.85 |
| 8 | 0.90 |
| 9 | 0.95 |

Suppose there are $400 \times 10^9$ stars in a galaxy. The log of this is (0.6 + 2) + 9 = 11.6. Suppose there are $100 \times 10^9$ galaxies in the observable universe; its log is 11. Thus, the log of the number of stars in the observable universe is 11.6 + 11 = 22.6. The sun's mass is $2 \times 10^{33}$ g; the log of this is 33.3. Assume the sun is all atomic hydrogen so that the atomic mass is 1 g/mol. The number of hydrogen atoms in 1 g is Avogadro's number $6 \times 10^{23}$, whose log is 23.8. The log of the number of atoms in the sun is 23.8 + 33.3 = 57.1. The log of the number of atoms in the universe is thus 22.6 + 57.1 = 79.7, which we round up to 80.

Thus, we've estimated the number of atoms in the observable universe to be $10^{80}$. You can do a web search to see how other people have estimated this number. For an entertaining example, you may want to estimate the number of intelligent civilizations in a galaxy using the Drake equation. This equation is important because it was designed to foster discussion, but it has no experimental credibility because the numbers used are speculative[3].

## Significance and roundoff error ♦

Significant figures are often taught in elementary science classes. They have two uses:

1. Minimize the labor of arithmetic calculations.
2. Approximately encode the uncertainty of a number calculated from measured values.

The first use is appropriate and is used frequently. Here's an example. Suppose I wanted to calculate the volume of a box. Two sides were measured as 1.2345 and 2.3456. The third side was measured to be 4. The volume is thus

$$1.2345 \times 2.3456 \times 4$$

If you do this with a calculator, you get 11.5826. If you were doing this calculation manually (remember, it's the subject of this paper), you'd likely round off the first two numbers before doing the calculation because the number 4 only has one significant figure. You do this to minimize the arithmetical work; the problem might be reduced to

$$1.2 \times 2.3 \times 4 = 4.8 \times 2.3 = 2.3(4 + 0.8) = 9.2 + 1.84 = 11.04$$

and you'd state the volume is about 11.

The second use of significant figures is to attempt to encode the uncertainty in a measured number and then **propagate the uncertainty** into subsequent calculations. The thinking is that you report the measured value and the uncertainty is approximately ±1 digit in the least significant digit (some people use ±0.5 in the least significant digit and others use ±2, so there's no standard). This method of propagating uncertainty is flawed, so I recommend you don't use it, at least for serious work.

I wish the elementary science teachers would show their students why the method is flawed and warn them that it's only an approximate tool. Here's the flaw. Suppose I measure the dimensions of a rug as 1.1 by 2.2 (the units aren't germane to the example). By our assumptions about significant numbers, this means the uncertainty can be stated as 1.1 ± 0.1 and 2.2 ± 0.1. Now calculate the area as $1.1 \times 2.2 = 2.42$. One rule of significant figures is that the answer can't have more significant figures than the least number of significant figures in any of the components. So we have to write the answer as 2.4 -- and the interpretation of this number is 2.4 ± 0.1.

You can estimate the propagated uncertainty with interval arithmetic[4]; here, the relevant calculation would be

$$[1.0, 1.2] \times [2.1, 2.3] = [2.1, 2.76] \approx 2.43 \pm 0.33$$

The flaw is that there is no expression using significant figures that accurately represents this uncertainty (ignore for the moment that it's not the correct uncertainty). Thus, we're forced to use 2.4, but the implied ±0.1 underestimates the correct uncertainty interval. In other cases, you'll find that the rules of significant figures cause you to overestimate the actual uncertainty interval. Professional scientists and engineers simply don't use it in serious work for uncertainty propagation; instead, they use the methods in [*GUM*]. See [*De Lury*] for a discussion.

The results of real-world measurements are stochastic variables and require the machinery of statistics to properly handle the estimation and propagation of uncertainty. This is important when making decisions from the results or where others want to use your work. If you're a scientist or engineer, I feel a good engineering statistics class is mandatory -- you'll use the material for the rest of your career.

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3 We have experimentally measured either 0 or 1 intelligent civilizations, depending on the definition of "intelligent".

4 Interval arithmetic almost always over-estimates the uncertainty.

In this document, I'll use significant figures only to ensure I'm not calculating with more information than was present in the original problem.

Note it's OK to carry one or more extra digits in a calculation beyond e.g. what would be indicated by the number of significant figures in the components. Of course, this is easy to do with electronic calculators, but more work with manual calculations. Regardless, if the result is a measurement and is a stochastic variable (which it virtually always will be), its real significance will be communicated in its stochastic uncertainty quantification, not its number of significant figures.

Another problem area for significant figures is integers that have trailing zeros, such as 123000. How many significant figures are in this number? The answer is ambiguous without other information. You may come across various notations like an overline or underline of the zeros that are significant, but this is not standardized. The best way to avoid this ambiguity is to express the number in scientific notation instead.

### *Significant numbers revisited*

After writing the previous section, I've softened my stance a bit about the use of significant numbers for approximate uncertainty propagation. Since it's an approximate method and it's easy to do, it's OK to use for approximate calculations, one of the main topics of this paper. So, use the technique if you wish, but realize the results are approximate. In this section, we'll compare the uncertainties calculated by using significant figures and the more proper linear uncertainty propagation methods. Rather than messing with the analytical formulation for uncertainty propagation, I'll use the python `uncertainties` library to do the calculations. Once you see how trivial it is to do more proper uncertainty calculations with this library, you'll have no excuse not to do proper uncertainty calculations if you're a scientist or engineer.

I'll give two examples. The first is the rug problem from above. You'll see the significant figures method gives a pretty good estimate for the rug problem. The second example shows a case where the significant figures method over-estimates the uncertainty by about a factor of 10.

For the rug area example above, the rug measures 1.1 x 2.2 units. For estimating the uncertainty with significant figures, let's assume the uncertainty is ±1 in the last significant figure. For using linear uncertainty propagation, we need to use a number that represents the standard deviation of the stochastic variable. I usually use a normal deviate of 2 or 3 standard deviations (the GUM calls this a "k-value"), depending on how I feel about the problem. For this problem, let's use both. Thus, we'll first use a standard deviation (statistical uncertainty) of 0.1/2; then repeat the calculation with 0.1/3.

The significant figure method gives the area as 2.4 square units, with the implied uncertainty of ±0.1. The calculation is easy and fast.

Linear uncertainty propagation can be done using the following python script

```
from uncertainties import ufloat
print("Area = " + str(ufloat(1.1, 0.1/2)*ufloat(2.2, 0.1/2)))
```

which prints the result

```
Area = 2.42+/-0.12
```

If we use 0.1/3 for the uncertainty, we get

```
Area = 2.42+/-0.08
```

Thus, the significant figures method adequately propagated the estimated uncertainty, at least when we remember we're making an approximation.

Think of using significant figures for uncertainty propagation as a rule of thumb that's fast and easy to apply -- you'll probably be within a factor of 2 or 3 of the correct estimate most of the time. It can be seriously wrong is when there is correlation between the random variables. If you have the python `uncertainties` library, you can perform some numerical experiments to get a feel for how well uncertainty propagation by significant figures works. It's especially worthwhile to experiment with correlated variables to see how much influence correlation can have on uncertainty propagation. Make sure to experiment with both positive and negative correlations.

> Don't think that the existence of correlation is uncommon -- there's a surprising amount of correlation in real-world measurements, even if you can't account for the cause of the correlation.

The second example is more problematic. The ideal gas constant *R* and Boltzmann's constant *k* are related by a factor of Avogadro's number $N_a$:

$$k = \frac{R}{N_a}$$

Here are the latest values of these constants from the NIST website [http://physics.nist.gov/cgi-bin/cuu/Info/Constants/index.html](http://physics.nist.gov/cgi-bin/cuu/Info/Constants/index.html)

$$\begin{aligned} k &= 1.3806488(13)\times 10^{-23}\ \mathrm{J\ K^{-1}} \\ R &= 8.3144621(75)\ \mathrm{J\ mol^{-1} K^{-1}} \\ N_A &= 6.02214129(27)\times 10^{23}\ \mathrm{mol^{-1}} \end{aligned}$$

where the number in parentheses is the uncertainty in the corresponding least significant digits. Let's propagate the uncertainty by both methods to see if the uncertainty in *k* is reasonable.

Using significant figures where the uncertainty is ±1 in the least significant digit, we calculate the significand as 8.3144621/ 6.02214129 = 1.380648792. Since *R* is only known to six significant figures, we round the significand and report it as 1.38065(1).

> The short-hand notation 1.38065(1) is commonly used and means the uncertainty is 1 unit in the least significant digit.

Using the `uncertainties` library, the script

```
from uncertainties import ufloat, ufloat_fromstr
R  = ufloat_fromstr("8.3144621(75)")
NA = ufloat_fromstr("6.02214129(27)")
print(str(R/NA))
```

prints[5]

```
1.3806488+/-0.0000012
```

which would get reported as 1.380649(1) to compare to the significand from the significant figures method.

In this case, the significant figures method has over-estimated the uncertainty by a factor of 10. That's to be expected, as we rounded the actual uncertainties up to the nearest power of 10 to use the significant figure method. Used this way, the significant figures method should result in a conservative estimate of the uncertainty, as long as there's no significant correlation or anticorrelation between the variables. Thus, it's appropriate as an approximate technique.

However, remember that the uncertainty is usually interpreted as the standard deviation of a stochastic variable. Thus, it's used to help make decisions using statistical techniques to maximize the use of the available information. The smaller the standard deviation, the more precise of an estimate that can be made. Thus, the linear uncertainty propagation of the `uncertainties` library is the better tool than significant figures because it helps you make better decisions.

## *Roundoff error* ♦

**Roundoff error**, is important when doing calculations, especially when the number of significant figures is kept to a minimum and rounding occurs. It's an error in calculations and has nothing to do with measurement uncertainty. Suppose we have a calculator that can only do arithmetic to the second decimal place. Further suppose we want to calculate $e^2$. If we calculate $e^2$ long-hand with $2.72^2$, we get 7.3984. Since our calculator can only express to two decimal places, the answer is

5 I don't know the cause of the discrepancy between the NIST uncertainty and the uncertainty calculated here (12 versus 13); it's possible there's some correlation effect that NIST has taken into account that we haven't.

rounded to 7.40.

If we calculate the percent difference between 2.72 and the accepted value of e, we get a 0.063% difference. If we calculate the % difference between 7.40 and the accepted value of $e^2$, we get a 0.15% difference. **Note how the relative error has increased**. This is roundoff error and has come about purely because of the mechanics of finite arithmetic in our two-decimal-place calculator. With enough calculations (or loss of significance; see below), roundoff error can propagate into the most significant figures of a calculation, destroying accuracy (see the example at the end of this section which demonstrates this).

> Floating point arithmetic with finite representations has other surprises. Suppose we have 8 digits of calculation accuracy; set a = 11111113, b = -11111111, and c = 7.51111111. The associative law breaks down: calculate a + (b + c) and compare it to (a + b) + c. Let a = 20000, b = -6, and c = 6.0000003. The distributive law breaks down: calculate a*b + a*c and compare it to a*(b + c).
>
> These examples are from "Accuracy of Floating Point Arithmetic", section 4.2.2 of Knuth's *Seminumerical Algorithms*.

Lest you think that these are artificial examples, realize that the calculations done by your calculator or computer are done in **exactly** the same fashion -- their floating point number representations hold a finite and (usually) fixed number of digits. Roundoff error is, in general, going to happen.

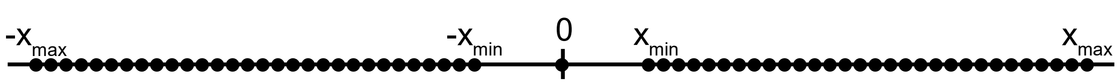
A lot has been written on roundoff error and I'll refer you to the web and numerical analysis books if you're interested.

### *Loss of significance* ♦

You should also be warned about **loss of significance**, which can happen during subtraction and division[6]. Suppose we have a calculator that is good to two decimal digits. Further suppose we have to evaluate the expression (a - b)*c. Suppose a is 2.72, b is 2.71, and c is 5. The calculated result of the expression is 0.05. If a is 2.73 (about 0.3% larger than its previous value), then the calculated result of the expression would be 0.10. A 0.3% change in the input results in a 100% difference in the output. This is loss of significance. It can require ingenuity to figure out alternate calculational methods that get rid of this loss of significance.

These problems can happen to you whether you're doing a calculation by hand, with a calculator, or a computer. An advantage of hand computation is you're more likely to notice them, whereas blindly chugging away with a computer or calculator can result in an end number whose accuracy is unknown. These examples can show you that the loss of accuracy can result in numbers that range from being accurately known to meaningless. Bitter experience will teach you that floating point calculations can be nontrivial.

Here's one thing to think about: suppose you had a set of numbers that you do calculations with that look like this plotted on the real line:




That set of numbers gives you 30 negative numbers, 30 positive numbers, and zero to calculate with. You can see easy ways to use simple arithmetic operations on two of these numbers to produce results that are not expressible in this set of numbers (i.e., the set isn't closed under arithmetic). Two examples are $x_{min}/2$ or $2x_{max}$. You'd probably worry about doing calculations with these numbers, getting wrong results, and not knowing it.

Before you dismiss this example as far-fetched, realize that the floating point numbers you calculate with on your computer or calculator are <u>exactly</u> analogous to this set of numbers. They just have substantially more numbers.

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6 Division is just a form of subtraction (like multiplication is a form of addition).

# Checking ♥

**Doing a calculation once means you don't know whether you made a mistake or not.**

The checking of calculations is **critical**, especially for scientists and engineers. The best way to check a calculation is to have someone else do it independently of your work. If that's not possible, do it again yourself, but use a different calculation method or order of terms. A common example of unchecked calculations is where students do their homework by typing their numbers into a calculator or computer and they don't approximate the answers as a check or use common sense or physical intuition to evaluate the reasonableness of their answers.

**The real world doesn't pay for wrong answers.**

Imagine you're an engineer designing a critical piece of equipment where, if an error occurs in the design, people may die or extensive property damage may result. You'd be much more careful with your calculations and checking them. This habit of checking calculations is a good thing to get into even if you don't have to do calculations with such serious consequences.

Besides checking the answer, you should also evaluate it for reasonableness:

1. Does it have the right sign?
2. Does it have the right order of magnitude?
3. Does it have the correct physical units?
4. Does it agree with previous results?
5. Does "feel right" to people experienced in the technology?
6. Does it agree with common sense?

Errors made in copying numbers or results can be more frequent than you might expect. A distracted human can easily misplace a decimal point or transpose some digits. One defense against this is to do the calculations in detail on paper and as neatly as possible, as this allows someone to check things more easily later. Many people think they don't have time to do such extensive calculation documentation -- which leads to the old sarcastic saying of ***there's never enough time to do it right, but always time to do it over***.

If you don't have someone to check your work, one of the best tools is to set the calculations aside for a while so you forget the details. Then come back and check the written results -- you'll often be able to spot errors more easily than if you had just done the calculation.

**... a calculation containing a hidden error is much worse than no calculation at all. [*wilson:332*]**

# Tips and tricks

One of the best techniques for doing manual arithmetic problems is to **turn a hard problem into two easier problems**. Some of the techniques in this section are based on this technique, which could also be called divide and conquer.

## Distributive law ♥

You probably remember being taught the distributive law, but saw no use for it other than passing an exam. I find I use it all the time for arithmetic. It is (remember, the symbols represent numbers):

$$a(b+c) = ab+ac$$

Practically, this means you can **change a hard multiplication into two easier multiplications**.

***Example***: suppose you need to calculate 87×30. First, you approximate the answer by changing it to 90×30 or 9×3×100 or 2700 -- thus, we know about what the answer should be. Let's get the exact value. Rewrite 87 as 80 + 7. Using the distributive law, the problem becomes

$$(80+7)\times 30 = 80\times 30 + 7\times 30 \tag{1}$$

We can do the two multiplications on the right pretty easily. That's 2400 + 210 or 2610. Hopefully, you can do those sums in your head. If not, practice by shifting decimal points. For example, 80×30 is 8×3 followed by two zeros, or 24 hundred and 7×30 is done similarly. Shift the decimal point two places to the left to change 2400 + 210 into 24 + 2.1. That's slightly over 24 + 2 or 26.1. Now shift the decimal point back two places to the right to get 2610.

Here's how I look at the expressions on the right-hand side: I see 8×3 as 24 and mentally recognize that the two zeros will cause the answer to be "hundred". Thus, I see "twenty four hundred". The 7×3 is 21 and the single zero is added to get 210. ▮

That's all there is to it. You'll find that you can do lots of problems by this technique without a calculator.

A refinement is to apply it twice:

$$(a+b)(c+d) = ac+bc+ad+bd$$

This can make any two digit multiplication problem fall over dead, immediately.

***Example***: 73×37. Rewrite it as (70 + 3)(30 + 7). This is 2100 + 490 + 90 + 21. This sums to 2701. ▮

You may realize that using the distributive law is similar to the multiplication algorithm you were taught in grammar school. For the previous example, the classic method of multiplication is a shorthand for (write it out as you were taught in grammar school and you'll see)

$$73\times 37 = 73(30+7) = 2190 + 511 = 2701.$$

In case you missed it, there's another way to do the calculation in equation (1). Note one term is 30, so you immediately can see that you just need to multiply 87 by 3 and move the decimal point one place to the right. You can just write down the answer as 261 and multiply it by 10 to get 2610. Note we used the same technique in the 73×37 problem in the previous paragraph. This is basically using scientific notation without writing down the actual expression.

If you're wondering how I saw 87 × 3 is 261, I multiply the digits of 87 from ***left to right*** and mentally watch where there has to be a carry. Thus, 8 times 3 gives 24 and there will be a carry of 2 from the 7 times 3, so that's 26. Then the ending digit has to be the 1 from the 21 resulting from 7 times 3. It's worth practicing because it becomes easier to do such things in your head.

The distributive law helps you square numbers more easily. For example, calculate $345^2$. Replace the hard-to-square number with an easier one:

$$345^2 = (300+45)^2 = 90000 + 2(300)45 + 45^2.$$

Replace the $45^2$ by $(50-5)^2 = 2500 - 2(5)(50) + 25 = 2500 - 500 + 25 = 2025$. So our final answer is $90000 + (300)90 + 2025 = 119025$.

## Common factors ♦

When you're dealing with expressions that have common factors, it's almost always worthwhile to remove these factors.

$$\frac{75\cdot 21}{63} = \frac{(\cancel{3}\cdot 5\cdot 5)\cdot(\cancel{3}\cdot\cancel{7})}{\cancel{3}\cdot\cancel{3}\cdot\cancel{7}} = 25$$

I used two techniques here to find these factors. First, 75 ends in 5, so I knew it was divisible by 5. My knowledge of the multiplication table let me replace the other terms with their factors.

**Use approximate common factors too**. Thus, $\frac{30}{91}$ can immediately be approximated by 1/3 by dividing numerator and denominator by 30. As you'd expect from the 1 part out of 90 change, it's about 1% off the correct answer.

Be alert for non-integer common factors. In the following example, we make use of the fact that 2.5 times 30 is 75:

$$\frac{75 \cdot 21}{60} = \frac{2.5 \cancel{30} \cdot 21}{2 \cancel{30}} = 1.25 \cdot 21 = 1.25 \cdot 20 + 1.25 = 26.25$$

You can determine factors of an integer n with the following rules [*wilson:335*]:

| **n is divisible by** | **if** |
|---|---|
| 2 | n is even |
| 3 | The sum of n's digits is divisible by 3 |
| 4 | Last two digits of n are divisible by 4 |
| 5 | Last digit of n is 0 or 5 |
| 6 | If n is even and divisible by 3 |
| 8 | Last three digits of n are divisible by 8 |
| 9 | The sum of n's digits (the digital root) is divisible by 9 |
| 10 | Last digit of n is 0 |

Another feature of factors is illustrated in the following problem that uses the commutative property of multiplication (i.e. that $a \times b = b \times a$):

$$25 \times 9 \times 5 \times 7 \times 2 \times 4$$

If you were to blindly start multiplying from left to right, you'd be in for a lot of work. It's better to look at all the numbers and look for certain pairs that might simplify things. In the present case, we'd rewrite things as

$$(25 \times 4) \times (5 \times 2) \times 9 \times 7 = 100 \times 10 \times 63 = 63000$$

Such rearranging of the factors lets us (in this problem, at least) do the whole thing quickly in our head.

## Difference of squares

When you took high school algebra, you probably saw the expression

$$a^2 - b^2 = (a+b)(a-b) \qquad (2)$$

We can use this to help with multiplication. As an example, suppose you have to calculate the product 55(65). You first think "the answer is a little more than 50(60) or 3000". Now, take a look at equation (2) above. 55 and 65 differ by 10, so they are also given by $60 \pm 5$. If we take $a = 60$ and $b = 5$ in equation (2), we can calculate the product as $60^2 - 5^2$ and this is easier to do: you just write the answer down as 3600 - 25 or 3575. And it's exact. This is an example of a problem that may be easier to do in your head (once you know the technique) than it is to type it into a calculator.

## Simplifying factors ♦

A product *ab* is unchanged by multiplying one factor by a constant μ and dividing the other factor by the same constant:

$$ab = (a\mu)\left(\frac{b}{\mu}\right)$$

You might be able to simplify an expression with this fact. In the following example, seeing the 0.25 decimal fraction and observing that 84 is divisible by 4 would let us write

$$2.25(84) = (2.25 \cdot 4)\left(\frac{84}{4}\right) = 9(21) = 189$$

Another way of doing this quickly is to see that this is the distributive law:

$$(2+\frac{1}{4})84 = 168 + 21 = 189.$$

A commonly-used form of this method is to try to cause one of the factors in a multiplication or division become a single digit -- arithmetic with a single digit is less work. For example

$$29\times 14 = (29\times 2)\left(\frac{14}{2}\right) = 58\times 7 = (50+8)7 = 350+56 = 406$$

In division problems, it's really just factoring:

$$\frac{29}{14} = \frac{29\left(\frac{1}{2}\right)}{14\left(\frac{1}{2}\right)} = \frac{14.5}{7} = 2+\frac{0.5}{7}$$

You could do long division to calculate the last term. However, if you know your reciprocals and only need a few decimal places, you know the reciprocal of 7 is 0.1428, so you divide this by 2 to get 0.0714. Hence 29/14 = 2.0714.

This last problem could have been done by writing the problem as

$$\frac{29}{14} = 2+\frac{1}{14} = 2+\frac{1}{2}\left(\frac{1}{7}\right)$$

and proceeding as in the previous calculation. Improper fractions with relatively prime numerators and denominators are best done this way.

A corollary of this method is to factor multiplications. Thus, in $18\times 19$, you can write the problem as 2(3)3(19) and you may find it easier to do the single digit multiplications quickly. I usually find an application of the distributive law is more straightforward.

## Speeding things up

Tips for faster arithmetic come from paying attention to the significands and (mostly) ignoring the power of 10. For example, suppose we have to multiply by 5. From a significand standpoint, this is the same as multiplying by 0.5. But that's the same as dividing by 2, so you develop the rule: **to multiply by 5, move the decimal point to the right one place and divide by 2**. You can see this rule immediately by writing 5 as 10/2.

Some other tips:

- Multiplying by 15 is the same as 1.5 from the significand standpoint. 1.5, in turn, is $(1+\frac{1}{2})$. Thus, divide a number by 2 and add the number to the result to multiply by 1.5. Also $15 = 3\times 5 = 3\frac{10}{2}$, which gives you another method for multiplying or dividing by 15.
- To multiply by 4, multiply by 2 twice. Many of us find multiplying and dividing by 2 an easy thing to do.
- Memorize reciprocal decimal expansions. For example, divide by 4 to get the same result as multiplying by 25 after shifting the decimal point properly.
- Memorize powers of two, particularly $2^8$, $2^{10}$, and $2^{16}$. $2^{20}$ is close to $10^6$. Knowing a few, you can work out the others when you need them.
- To divide by 6, first divide by 2, then by 3.

Using basic algebra, you can come up with many more arithmetic shortcuts.

## Adding numbers ♥

With a bit of practice, you can add columns of two digit numbers in your head (and three digit numbers with a bit more practice). Suppose you have the problem of adding the following four numbers

37
45
22
87

The grammar school method is to add the 1's column, write down the right-most digit in the answer, write the carry above the next column, etc. It works, but here's another way.

Look at the 37, then mentally add 45 to it. Think 37 + 40 + 5. You mentally say to yourself the intermediate sums[7] of 77 and 82. The next thoughts are 102 and 104. The last term gives 184 and 191. It's quite a bit faster than doing it on paper.

You can add longer numbers, but a different approach can be used. For the problem

$$\begin{array}{r} 3745 \\ \underline{2287} \end{array}$$

start at the left-most digits and start adding, ignoring the carried digit. You think 5922. Then you deal with the columns that had carries and correct things to 6032. A perversely difficult problem would have carries in nearly every column, but in practice this rarely occurs.

You could also develop another method based on the two-digit number method:

$$\begin{array}{rr} 37 & 45 \\ \underline{22} & \underline{87} \\ 59 & 132 \end{array}$$

You'd see the 100 from the first two columns and know this meant a carry for the hundred's digits column. Thus, you'd write down immediately 6032. Note **this gives you a way to check things by doing the problem in a different way**:

$$\begin{array}{rrr} 3 & 74 & 5 \\ \underline{2} & \underline{28} & \underline{7} \\ 5 & 102 & 12 \end{array}$$

You'd see the carry from the 1's column and the unit carry from the hundred's digits column and, once again, write down 6032.

## Using algebra ♦

Algebra lets you develop general statements about numbers, so you can derive further aids to calculation when needed. Suppose you were asked to hand calculate a table of the squares of each integer from 1 to 10000.

> An example might be the famous Barlow's tables of squares [*comrie*], compiled about 200 years ago. This table gave the square, cube, square root, cube root, and reciprocal for the integers from 1 to 10000. The numbers that were not integers were given to 8 significant figures. Of course, the whole table was calculated by hand back in those times.

If you didn't know algebra, then you'd probably calculate the table by laboriously hand-calculating each square. 8002 would be easy to get, but 8162 would take quite a bit more arithmetic. If you haven't done such arithmetic in a while, calculate 8162 by hand and see if you can get the correct value.

Knowing algebra, you can look for a relationship between the squares. For any integer $n$, we have that the next integer squared is $(n+1)^2 = n^2 + 2n + 1$. Subtract $n^2$ from both sides to get

$$(n+1)^2 - n^2 = 2n + 1$$

7 You should be practiced enough not to have to say, e.g., "37 plus 40 is" in your mind -- you just look at the two numbers and write down their sum.

Thus, the square for $n + 1$ is twice the previous number plus 1 added to the previous square. You'd find this easier arithmetic than calculating the exact expression from scratch, especially for numbers over 100.

A similar technique would let you do an easier calculation for the cubes. We get

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Since the $n^2$ column is already calculated, the cubes column's entries are gotten by adding $n^2$ and $n$, multiplying by 3, and adding 1. It would still be drudgery, but a drudgery of a lower level.

### *A warning*

There are numerous web sites that describe a variety of techniques for faster arithmetic. There's nothing wrong with this, but they are impractical. This is because they give lots of rules for specific cases and these rules simply can't be remembered if you only use them occasionally.

> It's analogous to learning a clever and useful knot while sitting in your easy chair. A year later, you're out in a blinding snowstorm and really need that knot, but you won't be able to remember it unless you practiced it a lot.

I think a better approach is to know enough algebra so you can derive things when needed, then use a few core techniques. In my case, I use the following techniques over and over again:

- Rewriting the problem in scientific notation
- Estimating the answer by using 1 digit in each term
- Distributive, commutative, and associative laws
- Approximate factoring
- Special multiplication/division replacements (for example, multiplying by 5 by dividing by 2 and shifting the decimal point).
- Knowing the reciprocals of 1 to 12 and the low powers of 2.
- Approximations using the binomial series.

Each person will work out what's most useful for himself. The one absolute I can confidently state is that these techniques won't be useful unless you practice and use them on a regular basis.

## Tables

Before hand-held calculators, it was common to use various numerical tables to help with calculations. Such tables were common in books, but the calculator was so useful that the tables were phased out because they took up too much precious paper. Yet they can still be useful -- and you can make them with a spreadsheet or other computer tools. I keep a little booklet in my pocket (the pages are 70x108 mm) I made printing on both sides of a sheet of paper that has the following math tables (the number in square brackets is the number of significant figures in the table):

- Base 10 logarithms [4]
- Square roots (2 pages) [5]
- Reciprocals [5]
- Squares[4]
- Cosines of angles with a given slope [5]
- Secants of angles with a given slope [5]
- Degree to radian conversions, various rational approximations
- Sine, cosine, and tangent functions [4]
- Slope to degree angle conversions [4]

I've found it's quite practical to work with slopes in the shop and on DIY projects. For example, I make a sawhorse's legs slope at 1 in 4, an easy to remember number. The slope is also the tangent of an angle and I find tangents most useful for practical stuff. You can easily derive that

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

Thus, if you know the slope, getting the other trigonometric functions isn't terribly onerous by a hand calculation when necessary.

The secants table is useful for right triangles where you know the base of the triangle and the adjacent angle. To get the length of the hypotenuse, multiply the base by the secant of the angle with the given slope.

**Example**: my home's roof has a slope of about a 4 inch drop over 12 inches or $m = 1/3$. If I want to cut a rafter to span a distance of 2.7 m, I look up in the secant table the value opposite a slope of 0.33 to get 1.053. The multiplication yields 2.843 m. Since the secant is the reciprocal of the cosine, this could be checked with

$$\sec\theta = \sqrt{1+m^2} = \sqrt{1+\frac{1}{9}} = \sqrt{10}/3$$

We can approximate the square root of 10:

$$\sqrt{10} = (9+1)^{1/2} = 3\left(1+\frac{1}{9}\right)^{1/2} \approx 3\left(1+\frac{1}{2}0.111\right)^{1/2} = 3(1+0.0555)$$

Thus, the desired secant is 1.0555. A calculator gives 1.054 for the actual secant; the discrepancy is due to approximating 1/3 by 0.33.

A calculator gives the rafter length as 2.846, so we got an answer to 3 parts out of 2800 or about 1 in 900 -- plenty good for DIY stuff.

## Area of a circle

In practical work around the home and in the shop, the area of a circle is often needed. Here's a specialized method that's kind of fun to use, as it only needs two multiplications (actually, three, but one of them is doubling a number, which is easy). The method comes from [*cs:50*] and relies on you memorizing that $\pi/4 = 0.7854$, something that quickly gets remembered after use.

Given a circle diameter *d*, calculate $d^2$ and multiply by 0.7854. But there's a trick in doing this; let's find the area of a circle 37 units in diameter to demonstrate.

$$(30+7)37 = 1110+259 = 1369$$

and the needed multiplication is

```
1 3 6 9
7 8 5 4
-------
9 5 8 3
  9 5 8 3
  1 9 1 6 6
    1 9 1 6 6
-------------
1 0 7 5 1 2 6
```

The trick is the first multiplication by 7, giving the first row. Copy this down to the next row, **shifted to the right one**. Double this number and write it on the next line, not shifted; then write the same number on the next line shifted one. The proof of this calculation is given by the sum

```
7
  7
  1 4
    1 4
-------
7 8 5 4
```

Note the first 14 is under the 7!

Memorize the proof and you'll remember the method.

[*cs:45*] also points out that the following factorization is useful for calculations with π

$$31416 = 2\times3\times4\times7\times11\times17$$

This will give better than 5 figures.

> **Example**: Calculate π/12:
>
> $$\frac{\pi}{12} \approx \frac{2\times3\times4\times7\times11\times17}{3\times4\times100\times100} = \frac{2\times7\times11\times17}{100\times100} = \frac{77\times34}{10^4} = \frac{(70+7)\times34}{10^4} = \frac{2380+238}{10^4} = 0.2618$$
>
> which is correct within 2 parts per million.

## Logarithms

Before the age of electronic calculators, base 10 logarithms helped with multiplication, division, and powers. The common practice was to use a 4-place log table, which would fit on two facing pages of a book with the linear interpolation digits included (i.e., the "proportional parts"). This would allow calculations to around 0.1%.

Today, few books show log tables anymore, but you may be surprised to learn that it's not too hard to construct a rudimentary one if you memorize some "atomic" facts:

- log(2) = 301
- log(3) = 477
- log(7) = 845
- log(11) = 1041
- You can't linearly interpolate to 1% for the log of numbers between 1 and 3

These integers are 1000 times the actual logarithm.

First, let's construct the logs of the integers:

| n | Construction | Value |
|---|---|---|
| 2 | -- | 301 |
| 3 | -- | 477 |
| 4 | 2*log(2) | 602 |
| 5 | log(10) - log(2) | 699 |
| 6 | log(2) + log(3) | 778 |
| 7 | -- | 845 |
| 8 | log(4) + log(2) | 903 |
| 9 | 2*log(3) | 954 |
| 10 | -- | 1000 |
| 11 | -- | 1041 |

Next, you'd use these values to construct the logarithms of the integers from 12 to 30. This covers the part of the table where linear interpolation is inadequate to have less than 1% error in the computed logarithm.

I won't construct the table here, but I'll show the steps. Let's start with getting the logarithm of 13. We'd calculate this with log(6.5) + log(2). Get log(6.5) by averaging log(6) and log(7) from above to get (778 + 845)/2 = 812. Therefore log(13) is 812 + 301 or 1113. The actual value is 1114, so we're off by 1.

The logs of the primes up to 30 are gotten similarly. The logs of the other integers are gotten by adding the logs of the factors. This assumes you fill the table values sequentially so that log(n//2) is already in place.

When you're doing a calculation, you only calculate the logs you need, so you don't have to work with a full table.

> **Example**: Suppose you needed to calculate $7.5^{18}$ and you only have pencil and paper. The logarithm of the answer is 18(log(75) - 1), so we need to find the logarithm of 75 and the antilogarithm of the result. Since 75 is 3(5)5, the logarithm of 75 is
>
> $$\log(3) + 2\log(5) = 477 + 2(699) = 1875$$
>
> Therefore the logarithm of the answer is 18(0.875). This is $18(7/8) = 63/4 = 15.75$. Now we need the antilog of 0.75. From the above table, we see it's a number between 5 and 6; 778 - 699 = 101 and 750 - 699 = 51, so the inverse interpolation fraction is 51/101, which we'll take as 0.5.
>
> To get the second digit, we need to estimate it; start with
>
> $$\log(55) = \log(5) + \log(11) = 699 + 1041 = 1740$$
>
> This is a little low of the 1750 we're looking for, so try 56
>
> $$\log(56) = 3\log(2) + \log(7) = 903 + 845 = 1748$$
>
> and this is close enough. Therefore, the answer is $5.6 \times 10^{15}$.

Rather than memorizing the logarithms, it would make more sense to make a small card of logarithms for when you need such things:

| | .0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | **log** | **041** | **079** | **114** | **146** | **176** | **204** | **230** | **255** | **279** |
| **2** | **301** | **322** | **342** | **362** | **380** | **398** | **415** | **431** | **447** | **462** |
| **3** | **477** | **491** | **505** | **519** | **531** | **544** | **556** | **568** | **580** | **591** |
| **4** | **602** | **613** | **623** | **633** | **643** | **653** | **663** | **672** | **681** | **690** |
| **5** | **699** | **708** | **716** | **724** | **732** | **740** | **748** | **756** | **763** | **771** |
| **6** | **778** | **785** | **792** | **799** | **806** | **813** | **820** | **826** | **833** | **839** |
| **7** | **845** | **851** | **857** | **863** | **869** | **875** | **881** | **886** | **892** | **898** |
| **8** | **903** | **908** | **914** | **919** | **924** | **929** | **934** | **940** | **944** | **949** |
| **9** | **954** | **959** | **964** | **968** | **973** | **978** | **982** | **987** | **991** | **996** |

Here's a script which shows how close a calculated ("Calc") table constructed by these means is to the actual ("Act.") values:

```
  n Calc Act. Diff
  2  301  301                35 1544 1544                68 1831 1833 -2
  3  477  477                36 1556 1556                69 1838 1839 -1
  4  602  602                37 1567 1568 -1             70 1845 1845
  5  699  699                38 1579 1580 -1             71 1851 1851
  6  778  778                39 1589 1591 -2             72 1857 1857
  7  845  845                40 1602 1602                73 1862 1863 -1
  8  903  903                41 1612 1613 -1             74 1868 1869 -1
  9  954  954                42 1623 1623                75 1875 1875
 10 1000 1000                43 1633 1633                76 1880 1881 -1
 11 1041 1041                44 1643 1643                77 1886 1886
 12 1079 1079                45 1653 1653                78 1890 1892 -2
 13 1112 1114 -2             46 1662 1663 -1             79 1896 1898 -2
 14 1146 1146                47 1671 1672 -1             80 1903 1903
 15 1176 1176                48 1681 1681                81 1908 1908
 16 1204 1204                49 1690 1690                82 1913 1914 -1
```

```
17 1229 1230 -1
18 1255 1255
19 1278 1279 -1
20 1301 1301
21 1322 1322
22 1342 1342
23 1361 1362 -1
24 1380 1380
25 1398 1398
26 1413 1415 -2
27 1431 1431
28 1447 1447
29 1462 1462
30 1477 1477
31 1491 1491
32 1505 1505
33 1518 1519 -1
34 1530 1531 -1
50 1699 1699
51 1706 1708 -2
52 1714 1716 -2
53 1723 1724 -1
54 1732 1732
55 1740 1740
56 1748 1748
57 1755 1756 -1
58 1763 1763
59 1770 1771 -1
60 1778 1778
61 1785 1785
62 1792 1792
63 1799 1799
64 1806 1806
65 1811 1813 -2
66 1819 1820 -1
67 1825 1826 -1
83 1918 1919 -1
84 1924 1924
85 1928 1929 -1
86 1934 1934
87 1939 1940 -1
88 1944 1944
89 1949 1949
90 1954 1954
91 1957 1959 -2
92 1963 1964 -1
93 1968 1968
94 1972 1973 -1
95 1977 1978 -1
96 1982 1982
97 1986 1987 -1
98 1991 1991
99 1995 1996 -1

Difference counts (total = 98):
  -2:  10
  -1:  28
   0:  60
```

The `Diff` column shows how far the calculated value is away from the correct value.

# Approximations

## Binomial theorem ♥

This section discusses some approximations that are used everywhere by technical people. First is

$$\frac{1}{1 \pm x} \approx 1 \mp x \text{ for } x \ll 1 \tag{3}$$

The symbol $\ll$ means "much less than" or "small with respect to". **This approximation replaces a division by an addition** and gets better the smaller $x$ is with respect to 1. Try a few examples with your calculator to convince yourself it's true.

This approximation comes from the series

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \ldots \qquad \text{where } |x| < 1$$

The error of the approximation is less than $x^2$. When you're doing a hand calculation, you can estimate the error as needed and, if it's too large, include extra terms. Then the error estimate is the first term that you left out.

> ***Example***: $1/(1+0.012) \approx 1 - 0.012 = 0.988$. The error is less than $0.012^2$, which we calculate as $12^2 \times (10^{-3})^2 = 144 \times 10^{-6} = 0.000144$. If you use a calculator, you'll find the actual error is 0.000142. Note that if we added the $x^2$ term in, we'd have $1 - 0.012 + 0.012^2 = 0.988 + 0.000142 = 0.988142$ and we'd have six significant figures. For hand calculations, using these approximations can be quite worthwhile when x is small. ∎

Many expressions can be converted to use the form of equation (3). For example, $\dfrac{a}{b \pm c}$ doesn't look like this form, but some algebra converts it into the form

$$\frac{a}{b}\left(\frac{1}{1\pm\frac{c}{b}}\right)$$

If $c/b$ was a small number with respect to 1, then we could use the approximation to get

$$\frac{a}{b\pm c}\approx\frac{a}{b}\left(1\mp\frac{c}{b}\right)$$

The second approximation is

$$(1\pm\delta)(1\pm\epsilon)\approx 1\pm\delta\pm\epsilon \quad \text{where } \delta\ll 1 \text{ and } \epsilon\ll 1 \tag{4}$$

This is a good approximation because the $\delta\epsilon$ term is smaller than the other terms, so we ignore it. For example,

$$1.001(0.995)=(1+0.001)(1-0.005)\approx 1+0.001-0.005=1-0.004=0.996.$$

This is 5 ppm off the correct answer. Remember, the reason for doing it is because you're doing the calculation by hand and two simple additions are less work than multiplying a 3 and 4 digit number.

Another useful approximation is

$$(1+x)^a\approx 1+a\,x+\frac{a(a-1)}{2}x^2 \quad \text{where } x\ll 1 \tag{5}$$

Here, $a$ is a real number[8]. Usually only the first two terms are used; the third helps you estimate the error. Approximation (5) contains (3) when $a$ is set equal to -1. If you're interested in where this approximation comes from, look up the generalized binomial theorem. Check this formula with your calculator with some sample numbers to get a feel for its validity.

***Example***: we can use this approximation to get a square root of a number $b$ if it is fairly close to the known square root of a number $a$. This is because we can write

$$\sqrt{b}=\sqrt{a\,(1\pm x)}\approx\sqrt{a}\,(1\pm\frac{1}{2}\,x)$$

Suppose we need the square root of 1.23. Because we know our multiplication table, we know $11^2=121$ or $1.1^2=1.21$. We thus write

$$\sqrt{1.23}=\sqrt{1.21\left(1+\frac{0.02}{1.21}\right)}\approx 1.1\left(1+\frac{1}{2}\,\frac{0.02}{1.21}\right)=1.1+\frac{1.1\,(0.01)}{1.21}=1.1+\frac{0.01}{1.1}$$

Shifting decimal points in the right-hand term gives 1/110. Using the approximation again, we rewrite this as

$$\frac{1}{110}=\frac{1}{100\,(1+0.1)}\approx 0.01\,(1-0.1)=0.009$$

We thus get the result that $\sqrt{1.23}=1.109$. ▌

Here's another example:

***Example***: in the *Common factors* ♦ section, we calculated 30/91 by the approximation $30/90=0.3$. Because we decreased the denominator, we know our estimate is a little high. We can use the approximations of this section to get a bit closer:

$$\frac{30}{91}=\frac{30}{90\,(1+\frac{1}{90})}\approx 0.3333\,(1-\frac{1}{90})=0.3333-\frac{0.3333}{90}$$

You could do the last term by long division, but it's easier to write 90 as $100(1-0.1)$ and

8 Both $a$ and δ can be complex and this formula is still true as long as the magnitude of δ is small with respect to 1.

use the approximation again to get

$$\frac{0.3333}{90} \approx \frac{0.3333}{100}(1+0.1) = 0.003333 + 0.0003333 = 0.0037$$

to the fourth decimal place. Thus, 30/91 is about 0.3333 - 0.0037 = 0.3296. You can check how close that is with a calculator. It's possible that a long division in this case might have been just as fast, depending on how much you had to write down on paper to use the approximations. ▌

## Approximate conversion factors

Besides the general mathematical approximations just discussed, there's another kind of approximation I find I use a lot: approximate conversion factors. In my shop, my US machine tools are graduated in inches. Since I prefer the SI system for general technical work, I find myself needing to convert between inches and mm frequently.

Converting inches to mm is relatively easy by multiplying the inches number by 25.4. This is first done by multiplying by 100/4 (i.e., divide by 2 twice and shift the decimal point two places to the right. Then multiply the original number by 2 twice and shift the decimal point to the left one place and add to the first term.

> ***Example***: Convert 2.33 inches to mm. Divide by 2 twice to get 1.165 and 0.5825; shift the decimal point right 2 places to get 58.25. Multiply 2.33 twice by 2 to get 9.32 and shift the decimal point left one place to get 0.932. Sum to get 58.25 + 0.932 = 59.182 mm.

Converting mm to inches is a bit harder; the reciprocal of 25.4 is about 0.03937. I do this approximately by calling it 0.04 and reducing the answer by 1.5%. It's approximate, but a pretty good approximation (the more correct factor is 1.6%).

> ***Example***: I need to drill a 6 mm hole, but I only have the screwball US drill sizes on hand. What drill size should I use? Since 1 mm is about 0.04 inches, we get 0.24 inches. Correct this down by 1.6% to get the proper answer. I usually use a 1.5% correction because it's easier to do. 1% of 0.24 inches is 2.4 mils; add half of that (1.2 mils) to get 3.6 mils. Subtract 3.6 mils from 240 mils to get 236.4 mils or 0.2364 inches. The exact answer is 0.2362 inches, so the approximation is good enough for all but the most exacting work. I'd use either a 15/63 inch drill or a size B drill. ▌

## Trigonometric approximations ♦

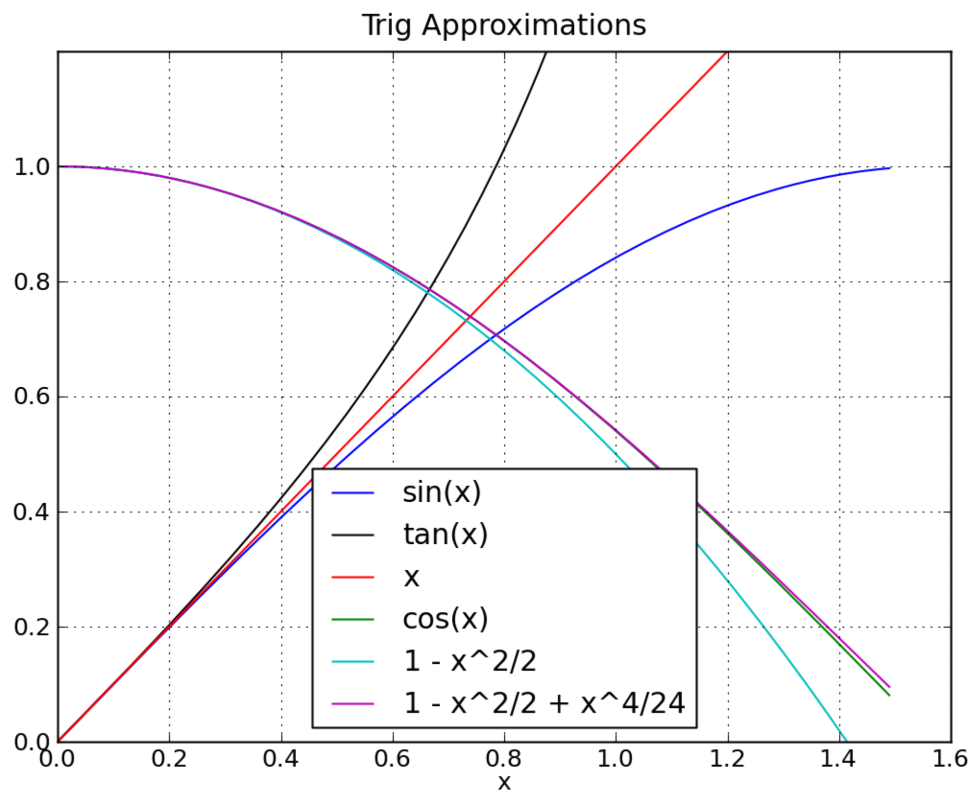
There are some rules and approximations useful in dealing with trigonometric functions. These rules are formulated roughly at the 0.1% to 1% precision level[9].

1. The sine and tangent are nominally equal for angles with absolute values less than about 6° (0.1 radians).
2. $\sin(x) \approx x$ and $\tan(x) \approx x$ for $|x| < 0.1$ radians (x must be in radians); hence the previous statement.
3. tan(x): if $\frac{\pi}{4} < x < \frac{\pi}{2}$, this can be calculated from $\tan x = \frac{1}{\tan\left(\frac{\pi}{2} - x\right)} = \cot\left(\frac{\pi}{2} - x\right)$. This is an identity, not an approximation.
4. For x near zero, $\cos(x) \approx 1 - \frac{x^2}{2}$ (x must be in radians).
5. For x near $\frac{\pi}{2}$, $\sin(x) \approx 1$ and $\cos(x) \approx 0$.
6. For small angles in degrees (say, less than about 6°), you can calculate the sine and tangent by dividing the number of degrees by $57.3 = 180/\pi$ (this is just a restatement of a

9 These rules were well-known to slide rule users.

previous rule).

A plot of the trig functions shows the first few approximations:




You can see that the straight line nicely approximates the sine and tangent for small x and that the $1-x^2/2$ for the cosine is good over a somewhat larger range. I added in another term for the cosine to show how the fourth power inclusion improves things.

There are analogous rules that relate to the cofunctions. For example, the cosine and cotangent are nominally equal for angles within 6° of 90°.

I haven't done this, but it might be worthwhile to memorize the values of some sines and tangents (decimal points removed):

| | sin | tan |
|---|---|---|
| 10° | 174 | 176 |
| 20° | 342 | 364 |
| 30° | **500** | 577 |
| 40° | 643 | 839 |
| 50° | 766 | 1192 |
| 60° | **866** | **1732** |
| 70° | 940 | 2747 |
| 80° | 985 | 5671 |

For 60°, the tangent is twice the sine or $\sqrt{3}$. For 30°, the tangent is $1/\sqrt{3}$.

Many people already know the emphasized values from doing them many times in elementary science/math classes (plus you know the sine of 45° as $1/\sqrt{2}$). This means there are only 6 values to memorize for the sine and 7 for the tangent. If needed, you'd get the cosine values by dividing the sine by the tangent.

Linear interpolation with these values can give you better than half a percent accuracy. A quick check shows you it's better to memorize the tangents than to try to figure them out because of the need to calculate the square root to get the cosine. Steinmetz in his 1911 book *Engineering Mathematics* recommended doing this memorization at the bottom of page 133 after showing an extended hand calculation of a Fourier series.

## Square roots

In the old days, most students were taught an algorithm to find square roots to arbitrary precision (I've long since forgotten it). But we should be able to get an approximation when we need one without using a calculator. Let's look at a method that's appropriate for 1% to 10% type of stuff that you can do with pencil and paper.

Suppose we want $\sqrt{x}$. The first step is to write the number $x$ as a number α between 1 and 100 and an even exponent $2n$ of 10 so that $x = \alpha \times 10^{2n}$. The square root of $10^{2n}$ is $10^n$, so that part of getting the square root is easy.

If you have a number α between 1 and 100, you know its square root lies between 1 and 10. You can estimate $\sqrt{\alpha}$ by first writing down an integer $y$ that is the integer part of the square root. This is easy, as it comes from the multiplication table.

The second step is to do a linear interpolation for the next digit. To do this, look at $y^2$ and $(y + 1)^2$. Gauge about how much α is in between these two numbers. Pick a decimal $z$ between 0.0 and 0.99 that is about in the same ratio (or calculate it). Then the desired square root is $y + z$.

> ***Example***: estimate $\sqrt{77500}$. Rewrite the number as $7.75\times10^4$. Then we have $\sqrt{77500} = \sqrt{7.75}\times10^2$ and here, α is 7.75. $2^2$ is 4 and $3^2$ is 9, so we know the square root of 7.75 is between 2 and 3. The difference between 9 and 4 is 5. 7.75 is a fraction $\frac{7.75-4}{5} = \frac{3.75}{5} = 0.67$ of the distance between 22 and 32,. We round this to 0.7 and pick 2.7 as the square root of 7.75. We thus get the answer $270^2 \approx 77500$ and we're off by 3%. The correct answer, by calculator, is 278.4. ▌
>
> ***Example***: estimate $\sqrt{137}$. If we follow our rule, we'd rewrite this as $\sqrt{1.37\times10^2}$. But it's easier to work with the number as is. We know $11^2 = 121$ and $12^2 = 144$, which differ by 23, so we know the answer is between 11 and 12. 137 is a fraction $\frac{137-121}{23} = \frac{16}{23}$ between the two integer squares. I can't do 16/23 in my head, so I change it to 16/24, which reduces to 2/3 = 0.67. We round to 0.7 (round up because we're decreasing the denominator by 1) and thus get 11.7 as our square root estimate. $11.7^2 = 136.89$. ▌

You can refine your estimate with a little more work. Let's refine the answer to the first example. If we manually square 2.7, we get 7.29. This is 6% lower than the significand we want, so we could increase 2.7 by half of this, 3%, and try again: 2.7(1.03) = 2.7(1 + 0.03) = 2.7 + 0.081 = 2.78. This is 0.3% less than the correct value, so we've improved things by an easy calculation that can mostly be done in your head.

Here's another approximation that gets pretty easy to use with practice. Suppose we want $\sqrt{a}$. Rewrite this as above and find an integer $b$ such that $b^2 < a$ but is as close as possible to $a$. Then the approximation is

$$\sqrt{a} \approx b + \frac{a^2 - b^2}{2b}$$

> **Example**: estimate $\sqrt{75}$. We pick $b$ to be 8 and we write down the square root as
>
> $$8 + \frac{75-64}{2(8)} = 8 + \frac{11}{16} = 8.68$$
>
> where I get the decimal fraction because I know 1/16 of an inch is a bit over 0.06 inches. 11/16 is 5/8 + 1/16; since I know 5/8 is 0.62, we add 0.06 to get the answer.

> **Example**: estimate $\sqrt{955}$. The relevant significand is $\sqrt{9.55}$. Since *b* is 3, we write down
>
> $$3+\frac{0.55}{6}\approx 3.091$$
>
> where we mentally use long division for the fractional part. Thus, $\sqrt{955}\approx 30.91$ and we got nearly 4 figures.

### *Iteration for the square root*

Once you have a square root to two or three significant figures, you can refine it using iteration if needed.

Suppose you want the number $\sqrt{a}$. Make an initial guess of *x* for the value of this square root using the method of the previous section, then improve the estimate with the formula

$$x_{new}=\frac{1}{2}\left(x+\frac{a}{x}\right) \tag{6}$$

The work is that you've got to do one long division of $a/x$ for each step of the iteration. The method converges quickly. It is sometimes called the Babylonian method (it's easily derived from the square root function with Newton's root-finding formula).

You can get a feel for how this method works as follows. We start with a guess of *x* for the square root of *a*. Then $\frac{a}{x}$ will be greater than $\sqrt{a}$ if *x* underestimates the root and, when averaged with *x*, will tend to pull the next estimate upwards. Conversely if *x* overestimates the root, the average pulls the estimate down.

Suppose we want to refine the estimate of $\sqrt{7.75}$ above. Starting with the 2.7 estimate, we get an excellent estimate in a few iterations:

```
2.7000000000000000
2.7851851851851852
2.7838824862096141
2.7838821814150276
2.7838821814150112
```

where the non-significant digits are gray. The number of significant digits in the answer approximately double each step (quadratic convergence). In this example, two long divisions give us more than 6 significant figures, which should be suitable for virtually any hand calculation.

## Reciprocal tricks ♥

Once you start working with numbers a bit, you start to learn the reciprocals of some numbers (the underlined digits are repeating):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1/2 | 0.5 | 1/7 | $0.\underline{142857}$ | 1/13 | $0.0\underline{769230}$ | 1/18 | $0.05\underline{5}$ |
| 1/3 | $0.3\underline{3}$ | 1/8 | 0.125 | 1/14 | $0.0\underline{714285}$ | 1/19 | 0.052631 |
| 1/4 | 0.25 | 1/9 | $0.11\underline{1}$ | 1/15 | $0.066\underline{6}$ | 1/20 | 0.05 |
| 1/5 | 0.2 | 1/11 | $0.09\underline{09}$ | 1/16 | 0.0625 | 1/25 | 0.04 |
| 1/6 | $0.16\underline{6}$ | 1/12 | $0.083\underline{3}$ | 1/17 | 0.058823 | 1/75 | $0.013\underline{3}$ |

If you work with fractions of an inch, you'll probably learn the decimal equivalents of some of the powers of 2 (the third column is rounded to the nearest thousandth of an inch, which is handy in shop work in inches):

| | | |
|---|---|---|
| 1/2 | 0.5 | 0.500 |
| 1/4 | 0.25 | 0.250 |
| 1/8 | 0.125 | 0.125 |
| 1/16 | 0.0625 | 0.062 |
| 1/32 | 0.03125 | 0.031 |
| 1/64 | 0.015625 | 0.016 |

Here's how reciprocals can come in handy: look at the significands on both sides (i.e., numbers and reciprocals) and you may be able to substitute one for the other -- and then be able to approximately factor some term out of a numerical expression. Let's look at some example approximations that come from the reciprocals. The "You see" column is the significand with no decimal point. The shaded ones are approximate.

| **You see** | **You think** | |
|---|---|---|
| 25 | $100/[2(2)]$ | |
| 125 | 1000/8 | |
| 16 | 1/64 | (approximate) |
| 625 | 5/8, 1/16 | |
| 3125 | 5/16, 1/32 | |
| 1/5 | 10/2 | |
| 373 | 3/8 | |
| 33, 333, ... | 1/3 | |
| 167, 1667, ... | 1/6 | |
| 14, 143, 1428, ... | 1/7 | (approximate) |

You probably see the patterns. When you're doing approximate hand calculations, it can be useful to have a small table of reciprocals and decimal equivalents in front of you.

Here's one other tip when you're dealing with a reciprocal. Suppose we need to calculate the decimal fraction of 3/17. Now, I certainly don't remember the decimal equivalent of 1/17 because I never use it. Use an approximation technique instead:

$$\frac{3}{17}=\frac{3}{17}\frac{6}{6}=\frac{18}{102}=\frac{18}{100(1+0.02)}\approx 0.18(1-0.02)=0.18-0.0036=0.1764$$

The key was **recognizing that we could multiply 17 by an integer and get close to 100**. If 100 doesn't work, try for other easy-to-use numbers like 200, 500, 1000, etc. Don't be afraid of using bigger numbers, as that means the approximation has a smaller term in it, meaning your final answer will be more correct.

Here's another example. Suppose a calculation resulted in the fraction 71/311 and you needed to know it to about 0.1% (i.e., three significant figures). How do you get the answer with pencil and paper? If you fiddle with this problem, you might approximate things as follows:

$$\frac{71}{311}\frac{3}{3}=\frac{213}{922}\frac{1.08}{1.08}\approx\frac{230}{995} \text{ or about } 0.231$$

I multiplied by 3/3 because I could see that would get the denominator near 1000. It was 77 out of 922 parts short of 1000, or about 8%; hence the multiplication by 1.08. Finally, the 995 was 5 parts out of 1000 (or one out of 200) low, so I increased the numerator by this amount too (1 unit). The correct answer is 0.2283, so we're off by 1.2% -- and the approximations I used aren't sufficient.

**Sometimes the easiest approach is to just use long division**, as you'd get the 0.2283 result in short order, as long as you don't make a calculation error.

Of course, since a calculator isn't handy, you won't know the magnitude of the error unless you calculate it a second way.

## Approximate multiplication and division

Grammar school students are taught one way to multiply: the traditional multiplicand over the multiplier with the rows beneath the multiplier for each digit in the multiplier. Here's a method from [*hunt:89*] that lets you control the number of significant figures in the answer.

Suppose we want to multiply 4956 by 8372 and get four significant figures in the answer. We should then carry the calculation to 5 significant figures and round at the end. Here's a non-traditional way to do this multiplication. Remove the decimal points and fix the answer's decimal point with the usual order of magnitude estimation. Here, 5000 times 8000 will give 40 million, so we know the size of the answer. The calculation is (note we **multiply the single digits from left to right**):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 4 | 9 | 5 | 6 | | | | |
| | | 8 | 3 | 7 | 2 | | | | |
| | 3 | 9 | 6 | 4 | 8 | | | | A |
| | | 1 | 4 | 8 | 6 | ***8*** | | | B |
| | | | 3 | 4 | 6 | ***9*** | ***2*** | | C |
| | | | | | 9 | ***9*** | ***1*** | ***2*** | D |
| | 4 | 1 | 4 | 9 | 2 | ***x*** | ***x*** | ***x*** | |
| Col | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | |

The rows are labeled by letters and the columns by numbers. Draw a vertical line to indicate which digits will be significant -- the significant ones will go to the left of this line and the insignificant ones to the right.

The first row A is gotten by multiplying the 8 in the multiplier by 4956. The answer is written down from right to left as is usual, starting in column 4. Row B is gotten by multiplying by the 3 in the multiplier and its result is **shifted right** one column so it starts in column 3. Rows C and D are gotten analogously. The x's indicate digits we don't really care about.

Next, we inspect column 3 and 2 to see about what column 3 will add up to. We see it will result in a 6 and a carry of 2. Then we add the 2 to column 4's numbers and get 31. We write down 2 because the 6 will cause us to round up. Then add up columns 5 through 8 in the usual way. The answer, rounded to 4 significant figures, is 4149. Thus, the multiplication's answer is 41,490,000.

If the original numbers had decimals points in them, you could have written the power of 10 to the right of the original numbers in, say, column 1 that would place the decimal point in the right location. Suppose the two numbers being multiplied were 4.956 and 0.08372 (so the answer should be about 0.5). Then we would have written -3 for the exponent of the multiplicand and -5 for the multiplier's exponent. In other words, $8372\times10^{-5}=0.08372$. These exponents add together to get -8, which tells us we must shift the decimal point to the left 8 times in the answer. The result would be 0.4149. This method is somewhat like the slide rule, as you need to fix the decimal point correctly. The order of magnitude calculation is critical for this.

You can see that this method would be a bit faster than the traditional method, especially if you only wanted a few significant figures. You'd probably only write down numbers one column to the right of the vertical line. If you're willing to be off a digit or two, you'll probably learn to shorten things a bit more and not write any of the numbers in bold italics down. By the way, this method of multiplication is due to Oughtred (see [*claudel:60:77*]); he is generally considered the inventor of the slide rule.

There's a method for long division where you only want a certain number of significant figures. Here's the example [*hunt*] gives for $\pi/\ln(10)=3.1416/2.3026$. Suppose we want 5 significant figures in the answer, which is implied by the problem 31416/23026 (again, decimal points are dropped and calculations done with integers, so the first step, as usual, is to approximate the answer). Here's the calculation:

```
                            Row
23026 ) 3 1 4 1 6 (1    A
       - 2 3 0 2 6      B
 2303 )   8 3 9 0 (3    C
         - 6 9 0 9      D
  230 )   1 4 8 1 (6    E
         - 1 3 8 0      F
   23 )     1 0 1 (4    G
             - 9 2      H
    2 )           9 (4  J
```

The quotient is thus 13644 (without a decimal point). Let's see how it's done.

The divisor 23026 is written on the left in the first column on row A. Then the dividend 31416 is written down; so far this looks like a traditional division problem. The 1 at the right in row A is how many times 23026 goes into 31416. Then the two numbers are subtracted to give 8390, which is written down with its least significant digit in the least significant digit of the dividend. Here's the key step: **the next divisor brought down has the first right-hand digit of the original divisor chopped off**. Then the number of times it can go into 8390 is written down as the 3 after the left parenthesis. Subtract and bring down the divisor again, chopping off the right hand digit.

We inspect the original problem to get the decimal point and thus write the answer as 1.3644.

I put in the parentheses and the - signs as clues to what's going on. However, when you write the problem on paper, you'll probably do it like so:

```
23026|3 1 4 1 6|1.              1. 3 6 4 3
     |2 3 0 2 6|          23026|3 1 4 1 6 0 0 0 0
 2303|  8 3 9 0|3               2 3 0 2 6
     |  6 9 0 9|                  8 3 9 0 0
  230|  1 4 8 1|6                 6 9 0 7 8
     |  1 3 8 0|                  1 4 8 2 2 0
   23|    1 0 1|4                 1 3 8 1 5 6
     |      9 2|                    1 0 0 6 4 0
    2|        9|4                     9 2 1 0 4
                                        8 5 3 6 0
                                         6 9 0 7 8
                                        1 6 2 8 2
```

The right-hand calculation is the traditional calculation, showing that the calculation needs to be done with 5 and 6 digit numbers. Of course, the long division method has the advantage of giving the answer to any number of figures desired -- but in practical work, it's rare to need more than 3 or 4 significant digits.

Here's the same division when only 3 figures are wanted:

```
230|3 1 4|1
   |2 3 0|
 23|  8 4|3
   |  6 9|
  2|  1 4|6
   |  1 2|
```

I like how the arithmetic gets simpler in the approximate method as you get successive digits in the answer: that means less work -- and that can translate into fewer errors.

Compare that last calculation to one using significant figures. Since we want three significant figures, the problem is rounded to

$$\frac{3.14}{2.30} = \frac{31.4}{23}$$

and you may find it simplest to do the last division long-hand -- you'll get 1.36 as you expect.

## Logarithms

Sometimes you need base 10 logarithms to work out answers to scientific or engineering problems. If you don't have a log table handy, you can get a low precision answer by memorizing three key logarithms:

| **x** | **log(x)** | **Δ%** |
|---|---|---|
| 2 | 0.301 | -0.010 |
| 3 | 0.477 | -0.025 |
| 7 | 0.845 | 0.011 |

The Δ% column is the percentage difference from the given logarithm and the exact value, so you can see that the approximations are pretty good.

The reason you need only these three logs is that you can calculate the others from them:

| **x** | **log(x)** | **Δ%** |
|---|---|---|
| 4 | log(2) + log(2) = 0.602 | -0.010 |
| 5 | log(10) - log(2) = 0.699 | 0.0043 |
| 6 | log(2) + log(3) = 0.778 | -0.020 |
| 8 | log(2) + log(4) = 0.903 | -0.010 |
| 9 | 2 log(3) = 0.954 | -0.025 |

You may be able to approximately "flesh out" logarithms of second digits by doing some factor manipulation:

$$\log(2.7) = \log(33) + \log(0.1) = 3(0.477) - 1 = 0.431$$

Another way is to get the log of the first digit, here 2, as 0.301. The log of the next digit is 0.477, so linearly interpolate for the 0.7 and increase a small amount:
$0.7(0.477 - 0.301) = 0.7(.176) = 0.123$; call it 0.13. Thus we get 0.301 + 0.13 or 0.431. I got somewhat lucky in my guess for the increase; this will take some practice to get good at it.

# Examples

## Working on a fence

I need to cut a piece of wire to make a support for a corner fencepost; the wire went from the bottom of a post to the top of another. I measured the two legs of the right triangle and needed to cut a piece that was $\sqrt{109^2+64^2}$ inches long and the answer is needed to three significant figures. This can be solved this with pencil and paper in the following fashion. First, I rewrote the term under the square root in scientific notation as

$$\left(1.09\times10^2\right)^2 + \left(6.4\times10^1\right)^2 = 1.09^2\times10^4 + 6.4^2\times10^2$$

and I needed to evaluate the two squares.

I evaluated $1.09^2$ by rewriting it as $(1 + 0.09)^2$. This is 1.18, ignoring the last term.

$6.4^2$ could be done long-hand or by using $(6 + 0.4)^2$, but we recognize that 64 is a power of 2 -- it's $2^6$. So $(2^6)^2$ is $2^{12}$. Since $2^{10}$ is 1024, we get $2^{12}$ is 4 times this or 4096. Thus, $6.4^2 = 40.96$.

The required sum is $1.18\times10^4 + 40.96\times10^2$ or 11800 + 4096 = 15900 rounded to three significant figures.

To complete the calculation, I needed

$$\sqrt{15900} = \sqrt{1.59 \times 10^4} = 100\sqrt{1.59}$$

We know $12^2$ is 144 and $13^2$ is 169. Thus, the desired square root is between 1.2 and 1.3. Inverse linear interpolation can get the answer. 169 - 144 is 25 and 159 - 144 is 15. Thus, 159 is 15/25 = 3/5 = 0.6 of the way between 144 and 169. So $12.6^2$ is about 159; this gives us $\sqrt{1.59} = 1.26$.

Thus, the answer is 126. A calculator gives 126.4 to 4 figures.

> I have to tell the truth -- I didn't calculate this number while I was working on the fence because all I had to write on was a small space on a crumpled index card and it was well below freezing while standing in the snow and wind. I had some short pieces of salvaged wire in coils and I picked one that looked about right -- and it was perfect for the task. I did the calculation later while sitting at my desk with a pen and paper and of course checked the final answer with my calculator.

Some of the lessons illustrated in this problem are:

- Use scientific notation so you can do arithmetic with numbers between 1 and 10.
- Turn a hard problem into two easier problems.
- Since I only needed 3 significant figures in the answer, I rounded intermediate answers to 3 significant figures.
- I used numerical facts I already know (multiplication table and a power of 2).

# References

Table entries with a * after them can be downloaded from Google books, at least in the US.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[*sphere*] http://www.sphere.bc.ca/test/whatsinaname.html

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