Multiplication and Division to a Specified Number of Significant Figures

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I happened to be perusing the math section of the 1916 edition of Marks' *Mechanical Engineers' Handbook* and I came across two rather simple methods for doing multiplications and divisions when you only want to keep a stated number of significant figures. I wish I had been taught these methods when I was young.

These are methods of hand-calculating multiplications and divisions to a stated number of significant figures. Obviously, most of us will have calculators handy when we want to do calculations, but occasionally we need to calculate when the only things available are pencil and paper. These two methods are for this situation. I also give a short discussion on doing columnar additions.

Multiplication

Suppose we want to multiply 4956 by 8372 and get four significant figures in the answer. We should then carry the calculation to 5 significant figures and round at the end. Here's a non-traditional way to do this multiplication. Remove the decimal points and fix the answer's decimal point with the usual order of magnitude estimation: 5000 times 8000 will give 40 million, so we know the size of the answer. The calculation is (note we **multiply left to right**):

		4	9	5	6				
		8	3	7	2				
	3	9	6	4	8				A
		1	4	8	6	***8***			B
			3	4	6	***9***	***2***		C
					9	***9***	***1***	***2***	D
	4	1	4	9	2	***x***	***x***	***x***	
Col	8	7	6	5	4	3	2	1	

The rows are labeled by letters and the columns by numbers. Draw a vertical line to indicate which digits will be significant -- the significant ones will go to the left of this line and the insignificant ones to the right.

The first row A is gotten by multiplying the 8 in the multiplier by 4956. The answer is written down from right to left as is usual, starting in column 4. Row B is gotten by multiplying by the 3 in the multiplier and its result is **shifted right** one column so it starts in column 3. Rows C and D are gotten analogously. The x's indicate digits we don't really care about.

Next, we inspect column 3 and 2 to see about what column 3 will add up to. We see it will result in a 6 and a carry of 2. Then we add the 2 to column 4's numbers and get 31. We write down 2 because the 6 will cause us to round up. Then add up columns 5 through 8 in the usual way. The answer, rounded to 4 significant figures, is 4149. Thus, the multiplication's answer is 41,490,000.

If the original numbers had decimals points in them, you could have written the power of 10 to the right of the original numbers in, say, column 1 that would place the decimal point in the right location. Suppose the two numbers being multiplied were 4.956 and 0.08372 (so the answer should be about 0.5). Then we would have written -3 for the exponent of the multiplicand and -5 for the multiplier's exponent. In other words, $8372\times10^{-5} = 0.08372$. These exponents add together to get -8, which tells us we must shift the decimal point to the left 8 times in the answer. The result would be 0.4149. This method is somewhat like the slide rule, as you need to fix the decimal point correctly. The order of magnitude calculation is critical for this.

You can see that this method would be a bit faster than the traditional method, especially if you only wanted a few significant figures. You'd probably only write down numbers one column to the right of the vertical line. If you're willing to be off a digit or two, you'll probably learn to shorten things a bit more and not write any of the numbers in bold italics down.

Division

Suppose we want to calculate π/ln(10) = 3.1416/2.3026 to 5 significant figures, implied by the way we've written the problem down. The decimal points are dropped so we can calculate with integers; do an approximate calculation to be able to fix the decimal point in the final answer. Obviously, here it will be about 1.5. Here's the calculation -- then we'll explain the method.

```
23026 ) 3 1 4 1 6 (1    A
       -2 3 0 2 6       B
 2303 )   8 3 9 0 (3    C
       -  6 9 0 9       D
  230 )   1 4 8 1 (6    E
       -  1 3 8 0       F
   23 )     1 0 1 (4    G
       -      9 2       H
    2 )         9 (4    J
```

The quotient is thus 13644 (without a decimal point). Here's how it's done.

The divisor 23026 is written on the left in the first column on row A. Then the dividend 31416 is written down; so far this looks like a traditional division problem. The 1 at the right in row A is how many times 23026 goes into 31416. Then the two numbers are subtracted to give 8390, which is written down with its least significant digit underneath the least significant digit of the dividend. Here's the key step: **the next divisor brought down has the first right-hand digit of the original divisor chopped off**. Then the number of times it can go into 8390 is written down as the 3 after the left parenthesis. Subtract and bring down the divisor again, chopping off the right hand digit.

We inspect the original problem to get the decimal point and thus write the answer as 1.3644.

I put in the parentheses and the - signs as clues to what's going on. However, when you write the problem on paper, you'll probably do it as follows:

```
23026|3 1 4 1 6|1
     |2 3 0 2 6|
 2303|  8 3 9 0|3
     |  6 9 0 9|
  230|  1 4 8 1|6
     |  1 3 8 0|
   23|    1 0 1|4
     |      9 2|
    2|        9|4
```

```
           1 3 6 4 3
23026|3 1 4 1 6 0 0 0 0
      2 3 0 2 6
        8 3 9 0 0
        6 9 0 7 8
        1 4 8 2 2 0
        1 3 8 1 5 6
          1 0 0 6 4 0
            9 2 1 0 4
              8 5 3 6 0
              6 9 0 7 8
              1 6 2 8 2
```

The calculation to the right of the line is the traditional method of division, showing that the calculation needs to be done with 5 and 6 digit numbers. Of course, the long division method has the advantage of giving the answer to any number of figures desired and giving a remainder -- but in practical work, it's rare to need more than 4 significant digits or the remainder.

Here's the same division when only 3 figures are wanted:

230	3 1 4	1
	2 3 0	
23	8 4	3
	6 9	
2	1 4	6
	1 3	

I like how the arithmetic gets simpler in the approximate method as you get successive digits in the answer: that means less work -- and that can translate into fewer errors.

Addition

With a bit of practice, you can add columns of two digit numbers in your head. Suppose you have to calculate the sum

37
45
22
87

The grammar school method is to add the 1's column, write down the right-most digit in the answer, write the carry above the next column, etc. It works, but here's another way.

Look at the 37, then mentally add 45 to it. Think **37 + 40 + 5**. You mentally say to yourself the intermediate sums[1] of 77 and 82. The next thoughts are 102 and 104. The last term gives 184 and 191. It's a lot faster than doing it on paper.

You can add longer numbers, but a different approach can be used. For the problem

3745
2287

start at the left-most digits and start adding. You think 5922. Then you deal with the columns that had carries and correct things to 6032. A perversely difficult problem would have carries in nearly every column, but in practice this rarely occurs.

You could also develop another method based on the two-digit number method:

37	45
22	87
59	132

You'd see the 100 from the first two columns and know this meant a carry for the hundred's digits column (or instead of writing 132, write the carried 1 over the hundred's digits' column). Thus, you'd write down immediately 6032. Note this problem decomposition gives you a way to check things by doing the problem in a different way:

3	74	5
2	28	7
5	102	12

You'd see the carry from the 1's column and the unit carry from the hundred's digits column and, once again, write down 6032.

Clearly, you can extend the method to arbitrary sums of digits. For longer sums of numbers with a variety of lengths, it may make sense to write things down on graph paper or columnar paper to help keep things aligned.

1 You should be practiced enough not to have to say, e.g., "37 plus 40 is" in your mind -- you just think the two numbers and write down their sum.