November 2, 2018

Dear Tobias,

This is my explanation of why what you wrote about measurability in $\mathbf{NRep}(C(S^1)^{**})$ in Example 4.2 is wrong.

First I will recap your example of a measurable family, so there is no doubt as to what I am referring to. For each $x \in X$, we define the representation $\pi_x : C(S^1) \to B(\mathbb{C})$, where $\mathbb{C}$ has its usual Hilbert space structure, by

$$\pi_x(a)(\psi) = a(x)\psi,$$

where $a \in C(S^1)$ and $\psi \in \mathbb{C}$. I do not dispute the fact that $(\pi_x)_{x \in S^1}$ is a measurable family of representations of $C(S^1)$ (with S^1 being given either the Borel σ-algebra or Lebesgue measure σ-algebra when it is considered as the parameter space of the family).

We then extend each π_x to a normal representation $\tilde{\pi}_x$ of $C(S^1)^{**}$ in the canonical way. When doing this, it helps to have the following definition. We define, for all $x \in S^1$, $\delta_x \in C(S^1)^*$:

$$\delta_x(a) = a(x),$$

where $a \in C(S^1)$. So $\pi_x(a)(\psi) = \delta_x(a)\psi$. The extension $\tilde{\pi}_x$ can be defined by

$$\tilde{\pi}_x(\Phi)(\psi) = \Phi(\delta_x)\psi,$$

where $\Phi \in C(S^1)^{**}$ and $\psi \in \mathbb{C}$. It is clear that this agrees with π_x on the double-dual embedding $C(S^1) \to C(S^1)^{**}$, and it is a normal representation because for every trace-class operator $\rho \in \mathcal{TC}(\mathcal{H})$, $\Phi \mapsto \mathrm{tr}(\Phi(\delta_x)\rho) = \Phi(\delta_x)\rho = \Phi(\rho\delta_x)$ is a normal linear functional on $C(S)^{**}$ (arising from evaluating at $\rho\delta_x \in C(S^1)^*$). Since the image of $C(S^1) \to C(S^1)^{**}$ is $\sigma(C(S^1)^{**}, C(S^1)^*)$-dense in $C(S^1)^{**}$, this proves that $\tilde{\pi}_x$, as defined above, is the unique normal extension of π_x to $C(S^1)^{**}$.

You claim that the maps $x \mapsto \langle \psi, \tilde{\pi}_x(\Phi)(\psi) \rangle$ are measurable for all $\psi \in \mathbb{C}$, and $\Phi \in C(S^1)^{**}$. Specializing to $\psi = 1$, this would imply that $x \mapsto \Phi(\delta_x)$ is measurable for all $\Phi \in C(S^1)^{**}$. This is what I shall refute in the next section.

1 Unmeasurability Proof

To show that these maps are not all measurable, we will need some background bits and pieces. Let $M(S^1)$ be the space of finite complex (Borel) measures on

S^1, with its usual topology. Then the Riesz representation theorem states that the map $i : M(S^1) \to C(S^1)^*$

$$i(\mu)(a) = \int_{S^1} a \,\mathrm{d}\mu,$$

where $a \in C(S^1)$, is a positive linear isomorphism. We use $j : C(S^1)^* \to M(S^1)$ for its inverse. We can therefore define a map $k : \mathcal{L}^\infty(S^1) \to C(S^1)^{**}$ as follows:

$$k(a)(\phi) = \int_{S^1} a \,\mathrm{d}j(\phi),$$

where $a \in C(S^1)$, $\phi \in C(S^1)^*$ and therefore $j(\phi) \in M(S^1)$.

To prove that k is a *-homomorphism, we need the definition of the -* and multiplication in A^{**}, where A is a C*-algebra. This can be found in [1, §3.1]. For $a, b \in A$, $\phi \in A^*$ and $\Phi, \Psi \in A^{**}$, we define

$$\begin{aligned}(\phi \cdot a)(b) &= \phi(ab)\\ (\Phi \cdot \phi)(a) &= \Phi(\phi \cdot a)\\ (\Phi \cdot \Psi)(\phi) &= \Phi(\Psi \cdot \phi).\end{aligned}$$

As we work in the commutative case, it actually does not matter which side we put the multiplication by a on. We have $\phi \cdot a \in A^*$, $\Phi \cdot \phi \in A^*$ and $\Phi \cdot \Psi \in A^{**}$. Similarly, the -* is defined by

$$\begin{aligned}\phi^*(a) &= \overline{\phi(a^*)}\\ \Phi^*(\phi) &= \overline{\Phi(\phi^*)}.\end{aligned}$$

Recall that given a measure $\mu \in M(S^1)$ and a function $a \in \mathcal{L}^1(\mu)$, we can define a measure $a \cdot \mu$ by

$$(a \cdot \mu)(T) = \int_{S^1} \chi_T a \,\mathrm{d}\mu$$

We will use the fact that for any finite complex measure μ, $\mathcal{L}^\infty(S^1) \subseteq \mathcal{L}^1(\mu)$.

Lemma 1.1. *Let $a, b \in \mathcal{L}^\infty(S^1)$ and $\mu \in M(S^1)$. Then*

$$\int_{S^1} b \,\mathrm{d}(a \cdot \mu) = \int_{S^1} ba \,\mathrm{d}\mu.$$

This implies that if $a \in C(S^1)$,

$$i(a \cdot \mu) = i(\mu) \cdot a.$$

Therefore for $a \in C(S^1)$ and $\phi \in C(S^1)^*$

$$j(\phi \cdot a) = a \cdot j(\phi).$$

Proof. We first prove that for all $a, b \in \mathcal{L}^\infty(S^1)$,

$$\int_{S^1} b \, \mathrm{d}(a \cdot \mu) = \int_{S^1} ba \, \mathrm{d}\mu.$$

It is true for $b = \chi_T$ for a Borel set T by definition. By linearity it is true for simple functions. We can then take a sequence of simple functions $(b_i)_{i \in \mathbb{N}}$ approximating any given $b \in \mathcal{L}^\infty(S^1)$ pointwise and deduce that it is true for $b \in \mathcal{L}^\infty(S^1)$ by the dominated convergence theorem. Since continuous functions are Borel measurable, for all $a, b \in C(S^1)$ we have

$$i(a \cdot \mu)(b) = \int_{S^1} b \, \mathrm{d}(a \cdot \mu) = \int_{S^1} ba \, \mathrm{d}\mu = i(\mu)(ba) = i(\mu)(ab) = (i(\mu) \cdot a)(b)$$

This proves the second statement. Now, let $a \in C(S^1)$ and $\phi \in C(S^1)^*$. Then

$$i(a \cdot j(\phi)) = i(j(\phi)) \cdot a = \phi \cdot a = i(j(\phi \cdot a)),$$

so, as i is an isomorphism, $a \cdot j(\phi) = j(\phi \cdot a)$. $\square$

Lemma 1.2. *k is a *-homomorphism.*

Proof. By linearity of integration, k is linear. Let $a, b \in \mathcal{L}^\infty(S^1)$. Then for all $\phi \in C(S^1)^*$:

$$(k(a) \cdot k(b))(\phi) = k(a)(k(b) \cdot \phi) = \int_{S^1} a \, \mathrm{d}j(k(b) \cdot \phi)$$

For all $c \in C(S^1)$, we have

$$\begin{aligned}
(k(b) \cdot \phi)(c) &= k(b)(\phi \cdot c) \\
&= \int_{S^1} b \, \mathrm{d}j(\phi \cdot c) \\
&= \int_{S^1} b \, \mathrm{d}(c \cdot j(\phi)) && \text{Lemma 1.1} \\
&= \int_{S^1} bc \, \mathrm{d}j(\phi) && \text{Lemma 1.1} \\
&= \int_{S^1} cb \, \mathrm{d}j(\phi) \\
&= \int_{S^1} c \, \mathrm{d}(b \cdot j(\phi)).
\end{aligned}$$

Therefore $k(b) \cdot \phi = i(b \cdot j(\phi))$, so $j(k(b) \cdot \phi) = b \cdot j(\phi)$. So

$$\begin{aligned}
(k(a) \cdot k(b))(\phi) &= k(a)(k(b) \cdot \phi) \\
&= \int_{S^1} a \, \mathrm{d}j(k(b) \cdot \phi) \\
&= \int_{S^1} a \, \mathrm{d}(b \cdot j(\phi)) \\
&= \int_{S^1} ab \, \mathrm{d}j(\phi) && \text{Lemma 1.1} \\
&= k(ab).
\end{aligned}$$

To prove that $k(a^*) = k(a)^*$, we use the fact that Φ^* is characterized by the fact that $\Phi^*(\phi) = \overline{\Phi(\phi)}$ for all self-adjoint ϕ. Let ϕ be a self-adjoint element of $C(S^1)^*$, so $j(\phi)$ is a real signed measure. Then for all $a \in C(S^1)$,

$$\begin{aligned}
k(a^*)(\phi) &= \int_{S^1} a^* \, \mathrm{d}j(\phi) \\
&= \int_{S^1} \Re(a^*) + i\Im(a^*) \, \mathrm{d}j(\phi) \\
&= \int_{S^1} \Re(a) - i\Im(a) \, \mathrm{d}j(\phi) \\
&= \int_{S^1} \Re(a) \, \mathrm{d}j(\phi) - i \int_{S^1} \Im(a) \, \mathrm{d}j(\phi) \\
&= \overline{\int_{S^1} \Re(a) \, \mathrm{d}j(\phi) + i \int_{S^1} \Im(a) \, \mathrm{d}j(\phi)} \\
&= \overline{\int_{S^1} a \, \mathrm{d}j(\phi)} \\
&= \overline{k(a)(\phi)} \\
&= k(a)^*(\phi^*) \\
&= k(a)^*(\phi),
\end{aligned}$$

because ϕ is self-adjoint. □

Using the map k, we can construct a family of projections in $C(S^1)^{**}$ corresponding to points in S^1. We define, for all $x \in S^1$,

$$p_x = k(\chi_{\{x\}}).$$

As k is a *-homomorphism, p_x is a projection in $C(S^1)^{**}$, and $(p_x)_{x \in S^1}$ is an orthogonal family of projections.

Theorem 1.3. *There exist elements $\Phi \in C(S^1)^{**}$ such that $x \mapsto \Phi(\delta_x)$ is not measurable. Therefore, for such Φ, $x \mapsto \langle 1, \pi_x(\Phi)(1) \rangle$ is not measurable.*

Proof. Let $T \subseteq S^1$ be an unmeasurable set. As $C(S^1)^{**}$ is a W*-algebra, its projection lattice is complete, so we can define

$$\Phi = \bigvee_{x \in T} p_x$$

By [2, §30 Theorem 1], sums of orthogonal families of projections on a Hilbert space converge to their joins in the strong operator topology (and *a fortiori* in the weak operator topology). By taking a faithful normal representation of $C(S^1)^{**}$ on a Hilbert space, and using the fact that the weak operator topology and the ultraweak topology agree on norm-bounded sets, $\sum_{x \in T} p_x = \Phi$ in the ultraweak topology, $\sigma(C(S^1)^{**}, C(S^1)^*)$. So $\Phi(\phi) = \sum_{x \in T} p_x(\phi)$ for all $\phi \in C(S^1)^*$.

For $x, y \in S^1$, we have

$$p_x(\delta_y) = k(\chi_{\{x\}})(\delta_y) = \int_{S^1} \chi_{\{x\}} \, \mathrm{d}j(\delta_y) = j(\delta_y)(\{x\}),$$

which is 1 if $x = y$ and 0 if $x \neq y$. So $\Phi(\delta_x)$ is 1 if $x \in T$ and 0 otherwise, and so $(x \mapsto \Phi(\delta_x)) = \chi_T$, which is not measurable. $\square$

Your argument refers to a "canonical morphism" $C(S^1)^{**} \to \mathcal{L}^\infty(S^1)$, but there is no such thing. Perhaps you are under the illusion that $\mathcal{L}^\infty(S^1)$ is a W*-algebra.

To some extent the counterexample above can be viewed as arising from the map $C(S^1)^{**} \to \ell^\infty(S^1)$, the adjoint of the inclusion $\ell^1(S^1) \to C(S^1)^*$. This is because $\ell^\infty(S^1)$ is thereby expressed as a direct summand of $C(S^1)^{**}$.

Best wishes, Robert

References

[1] John Dauns. Categorical W^*-Tensor Product. *Transactions of the American Mathematical Society*, 166:pp. 439–456, 1972. 2

[2] Paul R. Halmos. *Introduction to Hilbert Space and the Theory of Spectral Multiplicity.* Chelsea Publishing Company, 1951. 5