GRAPH SIGNAL PROCESSING

GSP

Research in Graph Signal Processing (GSP) aims to develop tools for processing data defined on irregular graph domains.

PURPOSE

The internet of things by definition is multi-device/thing networks, where the devices (things) of a network collect, process and exchange data in a continuous fashion.

The ability to exchange data between the network devices facilitates complementary and competitive collaboration among the devices of IOT networks.

In many cases, the data captured by the network in totality provide an insightful view at the IOT network level.

Such view should answer questions such as: which ”thing” is more tightly connected to what other thing?, how active is the network? How much connected is the network? How the data collected by the network vary in time, at the individual “thing" level as well as at a multi-thing level, and the network level.

SIGNAL PROCESSING TOOLS

The domain of signal processing provide a suite of signal processing tools, e.g., FFT, etc, that are quite effective in analyzing time (space) varying signals.

These tools have been well utilized in analyzing single variable and regular multi-variable time (space) signals.

The question is how one can adopt these tools in irregular multi-variable situations, such as IOT networks.

GRAPH REPRESENTATION OF IOT

Graphs are a powerful tool for capturing an IOT network structure and the interconnectivity among its "things", to the extent it can be used to facilitate a wide range of analysis of the network and its "things".

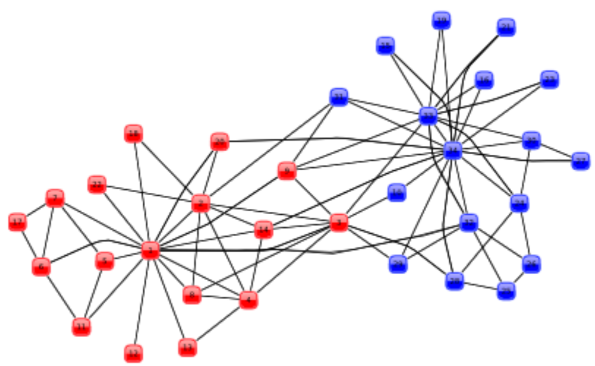
It is a powerful abstraction that offers a wide range of Graph representation and manipulation tools, including the recent development of Graph Signal Processing.

GRAPH SIGNAL PROCESSING

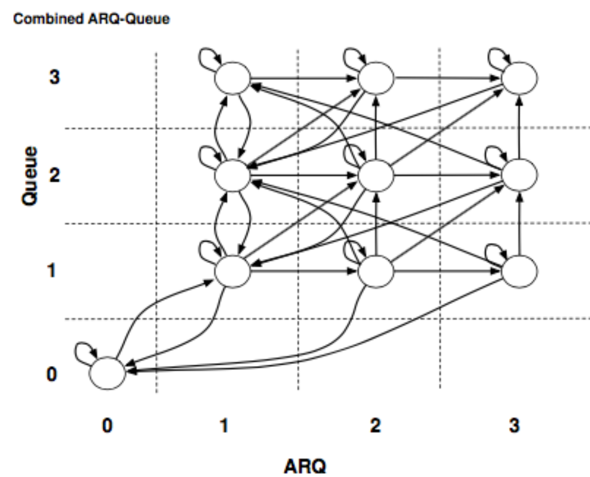
Graph Signal Processing (GSP) aims at developing tools for processing data defined on irregular graph domains.

It is concerned with a wide variety of operations on graphs, ranging from simple ones like frequency analysis and filtering to advanced ones like interpolation or graph learning.

- Examples in non-Euclidean settings



(a)

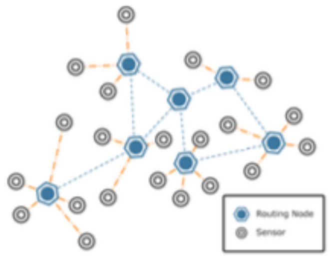


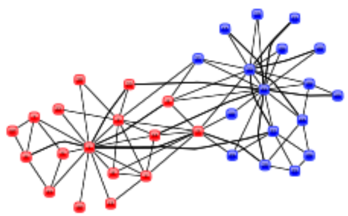
(b)

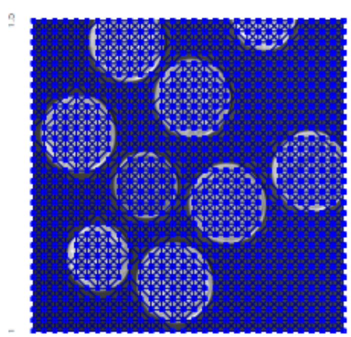
(a) Social Networks [4], (b) Finite State Machines(FSM)

Graphs can capture complex relational characteristics (e.g., spatial, topological).

EXAMPLES

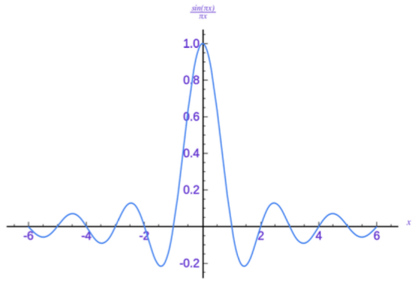


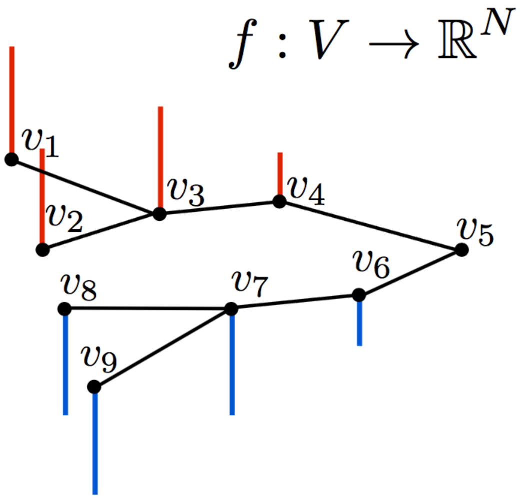




- Sensor network
  - Relative positions of sensors (kNN), temperature
  - does temperature vary smoothly?
- Social network
  - friendship relationship, age
  - are friends of similar age?
- Images
  - pixel positions and similarity, pixel values
  - discontinuities and smoothness

REGULAR VS IRREGULAR

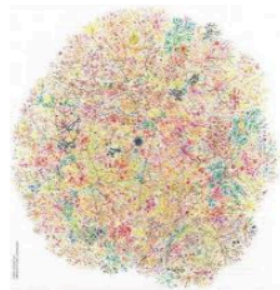




How to generalize classical signal processing tools on irregular domains such as graphs?

Online social media

Internet

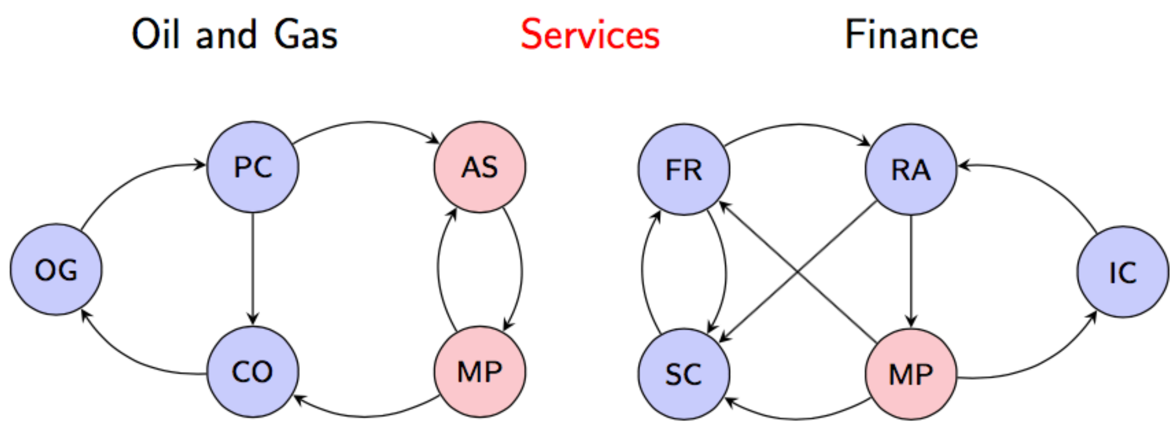


Clean energy and grid analytics

- Desiderata: Process, analyze and learn from network data [Kolaczyk'09]
- Network as graph $G = (\mathcal{V}, \mathcal{E})$: encode pairwise relationships
- Interest here not in G itself, but in data associated with nodes in $\mathcal{V}$
  $\Rightarrow$ Object of study is a graph signal $\mathbf{x}$
- Q: Graph signals common and interesting as networks are?

NETWORK OF ECONOMIC SECTORS USA

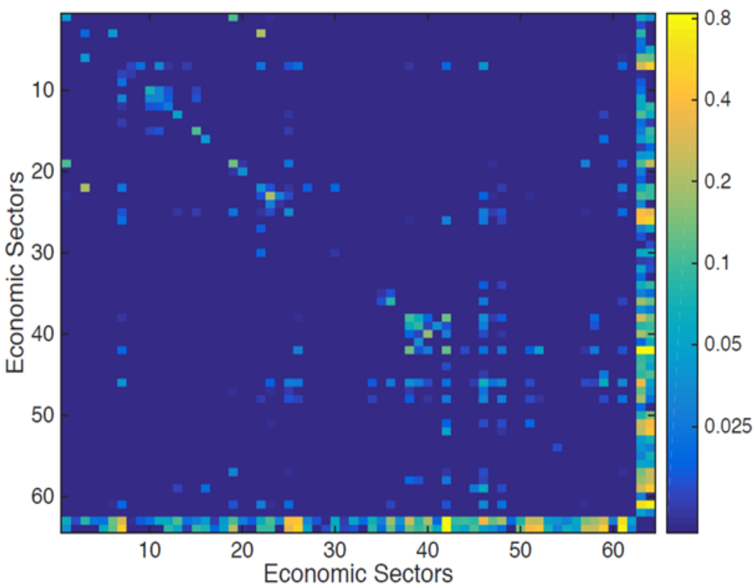
- Bureau of Economic Analysis of the U.S. Department of Commerce
- $\mathcal{E}$ = Output of sector i is an input to sector j (62 sectors in $\mathcal{V}$)



- Oil extraction (OG), Petroleum and coal products (PC), Construction (CO)
- Administrative services (AS), Professional services (MP)
- Credit intermediation (FR), Securities (SC), Real state (RA), Insurance (IC)
- Only interactions stronger than a threshold are shown

US NETWORK OF ECONOMICS SECTORS

- Bureau of Economic Analysis of the U.S. Department of Commerce
- $\mathcal{E}$ = Output of sector i is an input to sector j (62 sectors in $\mathcal{V}$)



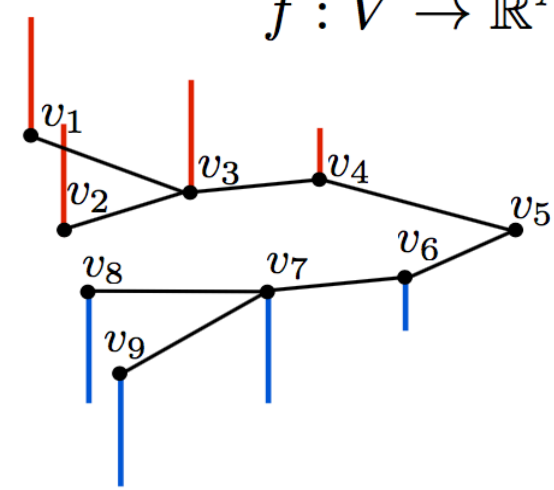
- A few sectors have widespread strong influence (services, finance, energy)
- Some sectors have strong indirect influences (oil)
- The heavy last row is final consumption

- This is an interesting network $\Rightarrow$ Signals on this graph are as well

Graph signal processing

- Graph signals provide a nice compact format to encode structure within data
- Generalization of classical signal processing tools can greatly benefit analysis of such data
- Numerous applications: Transportation, biomedical, IOT Networks social network analysis, etc.
- An increasingly rich literature
  - classical signal processing
  - algebraic and spectral graph theory
  - computational harmonic analysis

$$f : V \to \mathbb{R}^N$$



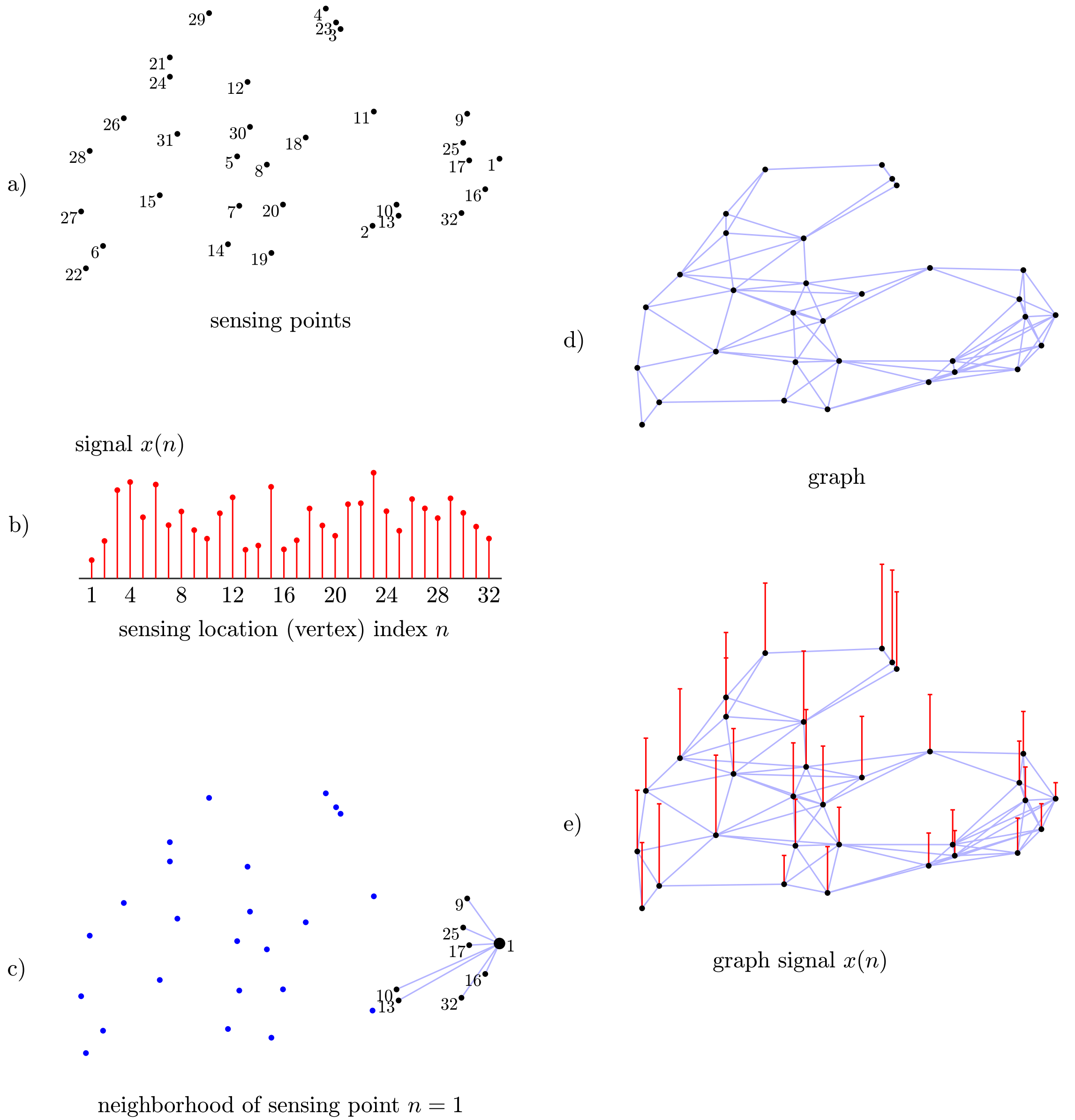
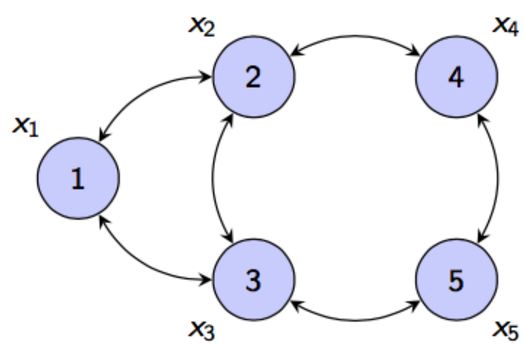


Fig. 1 Graph and a signal on the graph illustration.

GRAPH SIGNAL PROCESSING

- Graph SP: broaden classical SP to graph signals [Shuman et al.'13]
  $\Rightarrow$ Our view: GSP well suited to study network (diffusion) processes

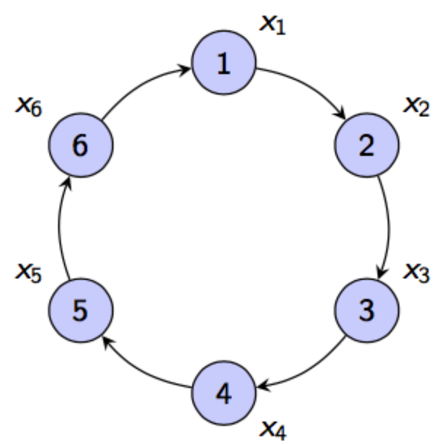


- **As.:** Signal properties related to topology of G (locality, smoothness)
  $\Rightarrow$ Algorithms that fruitfully leverage this relational structure
- Q: Why do we expect the graph structure to be useful in processing $\mathbf{x}$?

SIGNAL STRUCTURE

- Signal and Information Processing is about exploiting signal structure

- Discrete time described by cyclic graph
  - $\Rightarrow$ Time n follows time $n-1$
  - $\Rightarrow$ Signal value x_n similar to x_{n-1}
- Formalized with the notion of frequency

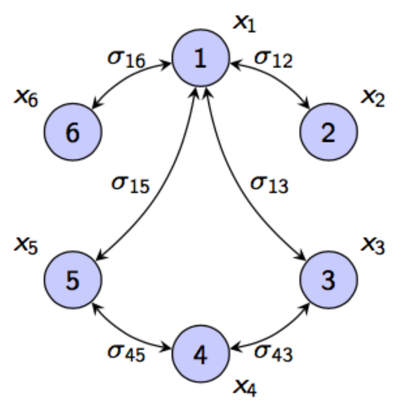


- Cyclic structure $\Rightarrow$ Fourier transform $\Rightarrow \tilde{\mathbf{x}} = \mathbf{F}^H \mathbf{x}$ $\left(F_{kn} = \frac{e^{j2\pi kn/N}}{\sqrt{N}} \right)$

- Fourier transform $\Rightarrow$ Projection on eigenvector space of cycle

COVARIANCE AND PRINCIPLE COMPONENTS

- Random signal with mean $\mathbb{E}[\mathbf{x}] = 0$ and covariance $\mathbf{C}_x = \mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right]$
  - $\Rightarrow$ Eigenvector decomposition $\mathbf{C}_x = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$

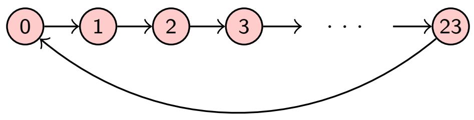


- Covariance matrix $\mathbf{C}_x$ is a graph
  - $\Rightarrow$ Not a very good graph, but still
- Precision matrix $\mathbf{C}_x^{-1}$ a common graph too
  - $\Rightarrow$ Conditional dependencies of Gaussian $\mathbf{x}$
- Covariance matrix structure $\Rightarrow$ Principal components (PCA) $\Rightarrow \tilde{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$
- PCA transform $\Rightarrow$ Projection on eigenvector space of (inverse) covariance
- Q: Can we extend these principles to general graphs and signals?

GRAPHS

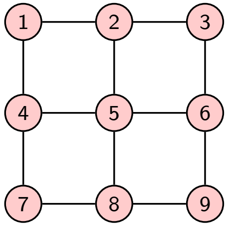
- Formally, a graph G (or a network) is a triplet $(\mathcal{V}, \mathcal{E}, W)$
- $\mathcal{V} = \{1, 2, \ldots, N\}$ is a finite set of N nodes or vertices
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges defined as ordered pairs (n, m)
  - Write $\mathcal{N}(n) = \{m \in \mathcal{V} : (m, n) \in \mathcal{E}\}$ as the in-neighbors of n
- $W : \mathcal{E} \to \mathbb{R}$ is a map from the set of edges to scalar values w_{nm}
  - Represents the level of relationship from n to m
  - Often weights are strictly positive, $W : \mathcal{E} \to \mathbb{R}_{++}$
- Unweighted graphs $\Rightarrow w_{nm} \in \{0, 1\}$, for all $(n, m) \in \mathcal{E}$
- Undirected graphs $\Rightarrow (n, m) \in \mathcal{E}$ if and only if $(m, n) \in \mathcal{E}$ and $w_{nm} = w_{mn}$, for all $(n, m) \in \mathcal{E}$

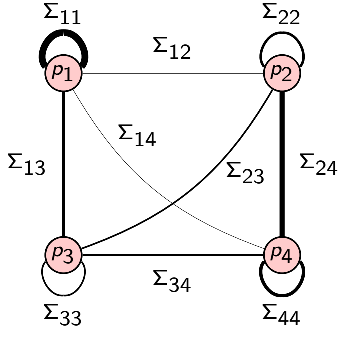
GRAPH EXAMPLES



- Unweighted and directed graphs (e.g., time)
  - $\mathcal{V} = \{0, 1, \ldots, 23\}$
  - $\mathcal{E} = \{(0,1), (1,2), \ldots, (22,23), (23,0)\}$
  - $W : (n,m) \mapsto 1$, for all $(n,m) \in \mathcal{E}$

- Unweighted and undirected graphs (e.g., image)
  - $\mathcal{V} = \{1, 2, 3, \ldots, 9\}$
  - $\mathcal{E} = \{(1,2), (2,3), \ldots, (8,9), (1,4), \ldots, (6,9)\}$
  - $W : (n,m) \mapsto 1$, for all $(n,m) \in \mathcal{E}$





- Weighted and undirected graphs (e.g., covariance)
  - $\mathcal{V} = \{1, 2, 3, 4\}$
  - $\mathcal{E} = \{(1,1), (1,2), \ldots, (4,4)\} = \mathcal{V} \times \mathcal{V}$
  - $W : (n,m) \mapsto \sigma_{nm} = \sigma_{mn}$, for all (n,m)

ADJACENCY MATRIX

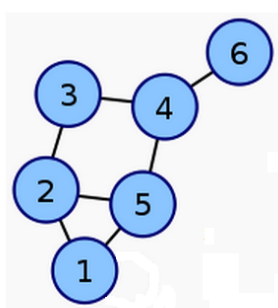
- Algebraic graph theory: matrices associated with a graph G
  - $\Rightarrow$ Adjacency $\mathbf{A}$ and Laplacian $\mathbf{L}$ matrices
  - $\Rightarrow$ Spectral graph theory: properties of G using spectrum of $\mathbf{A}$ or $\mathbf{L}$
- Given $G = (\mathcal{V}, \mathcal{E}, W)$, the adjacency matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ is

$$A_{nm} = \begin{cases} w_{nm}, & \text{if } (n, m) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases}$$

- Matrix representation incorporating all information about G
  - $\Rightarrow$ For unweighted graphs, positive entries represent connected pairs
  - $\Rightarrow$ For weighted graphs, also denote proximities between pairs

DEGREE AND K-HOP NEIGHBOURS

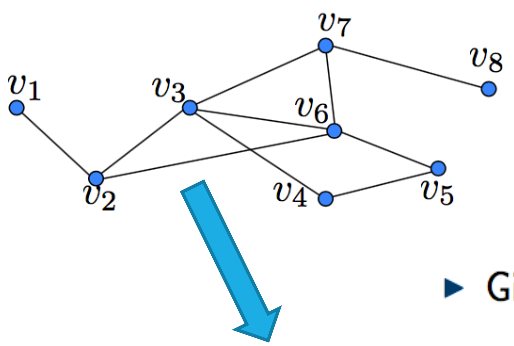
- If G is unweighted and undirected, the degree of node i is $|\mathcal{N}(i)|$
  - $\Rightarrow$ In directed graphs, have out-degree and an in-degree
- Using the adjacency matrix in the undirected case
  - $\Rightarrow$ For node i: $\deg(i) = \sum_{j \in \mathcal{N}(i)} A_{ij} = \sum_j A_{ij}$
  - $\Rightarrow$ For all N nodes: $\mathbf{d} = \mathbf{A1} \rightarrow$ Degree matrix: $\mathbf{D} := \mathrm{diag}(\mathbf{d})$
- Q: Can this be extended to k-hop neighbors? $\rightarrow$ Powers of $\mathbf{A}$
  - $\Rightarrow$ $[\mathbf{A}^k]_{ij}$ non-zero only if there exists a path of length k from i to j
  - $\Rightarrow$ Support of $\mathbf{A}^k$: pairs that can be reached in k hops



$[A^2]_{3,5}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Graph Laplacian



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A

Weighted and undirected graph:

$G = \{V, E\}$

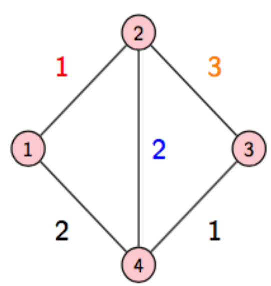
- Given undirected G with $\mathbf{A}$ and $\mathbf{D}$, the Laplacian matrix $\mathbf{L} \in \mathbb{R}^{N \times N}$ is

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

$\Rightarrow$ Equivalently, $\mathbf{L}$ can be defined element-wise as

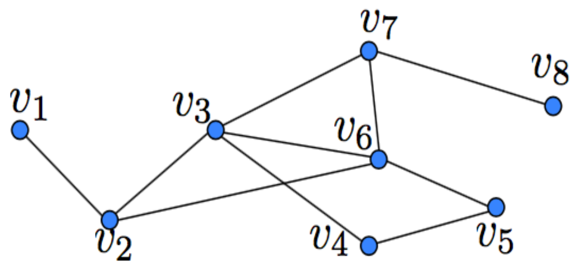
$$L_{ij} = \begin{cases} \deg(i), & \text{if } i = j \\ -w_{ij}, & \text{if } (i,j) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases}$$

- Normalized Laplacian: $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ (we will focus on $\mathbf{L}$)



$$\mathbf{L} = \begin{bmatrix} 3 & -1 & 0 & -2 \\ -1 & 6 & -3 & -2 \\ 0 & -3 & 4 & -1 \\ -2 & -2 & -1 & 5 \end{bmatrix}$$

Graph Laplacian



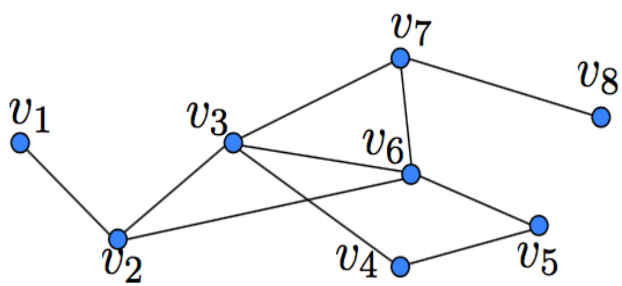
Weighted and undirected graph:

$G = \{V, E\}$

$D = \mathrm{diag}(\mathrm{degree}(v_1) \quad \ldots \quad \mathrm{degree}(v_N))$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{D} \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{A}$$

Graph Laplacian



Weighted and undirected graph:

$G = \{V, E\}$

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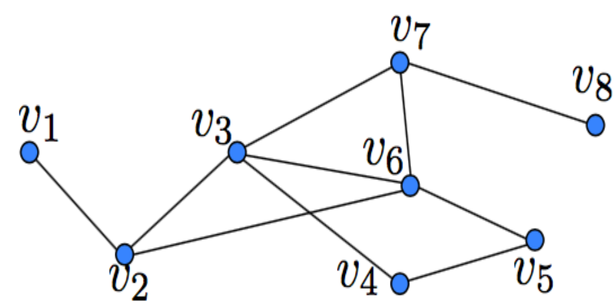
$L = D - A$ **Equivalent to G!**

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{D} - \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{L}$$

Graph Laplacian



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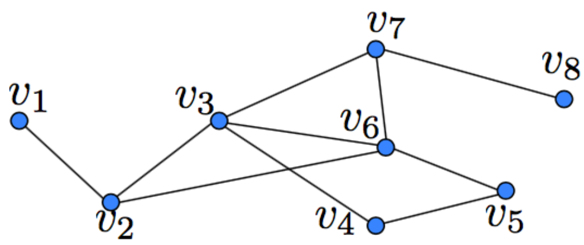
- Symmetric
- Off-diagonal entries non-positive
- Rows sum up to zero

Graph Laplacian



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$L = D - A$ **Equivalent to G!**

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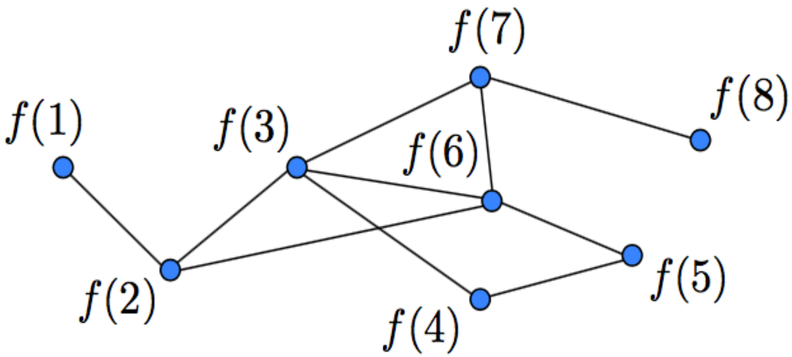
- Symmetric
- Off-diagonal entries non-positive
- Rows sum up to zero

Why graph Laplacian?

- standard stencil approximation of the Laplace operator
- leads to a Fourier-like transform

Graph Laplacian

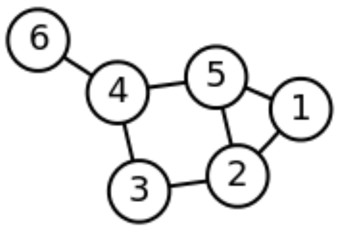


Graph signal $f : V \to \mathbb{R}^N$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$L$$

Modeling the graph

Undirected graph	Degree matrix: **D**	Adjacency matrix: **A**	Laplacian matrix: **L**
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

The **adjacency matrix** is a matrix, $\mathbf{A}$, such that $\mathbf{A}_{ij} = w_{ij}$.

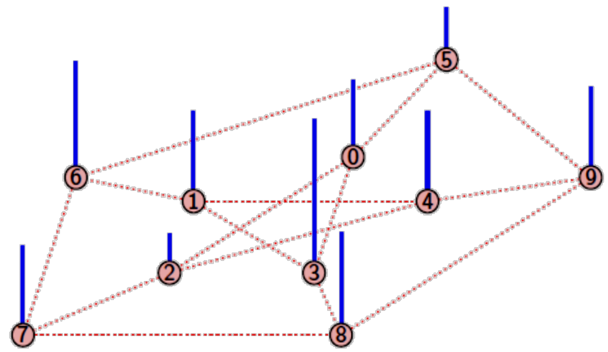
if the graph is undirected, $w_{ij} = w_{ji}$, and $\mathbf{A}$ is symmetric

The **degree matrix** of G is a diagonal matrix, $\mathbf{D}$, with entries $(\mathbf{D})_{ii} = \sum_{j=1}^{N} \mathbf{A}_{ij}$ and $(\mathbf{D})_{ij} = 0$ for $i \neq j$,

The **combinatorial graph Laplacian** defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, and the **symmetric normalized Laplacian** $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$.

Signals on Graphs

Graph signal f in R^N, where $|V|=N$



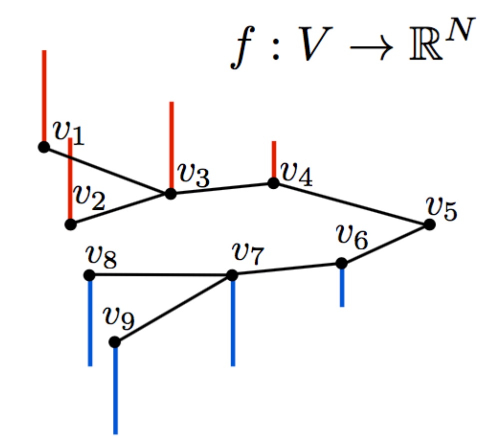
$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_9 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.7 \\ 0.3 \\ \vdots \\ 0.7 \end{bmatrix}$$

Graph Laplacian $\mathcal{L} := \mathrm{D} - \mathrm{W}$, D: diagonal with sums of weights

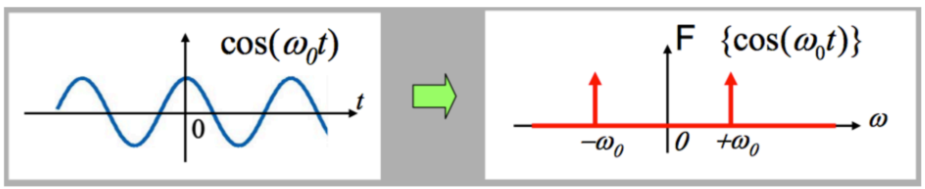
W: weight matrix

Normalized Graph Laplacian $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$

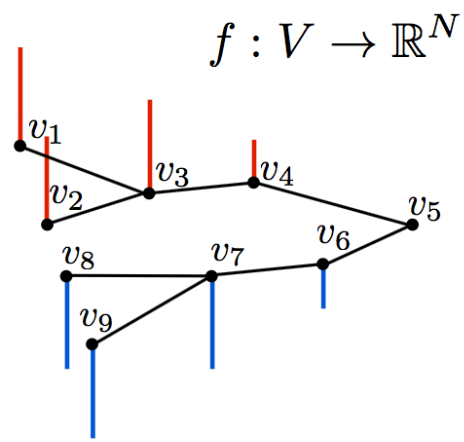
- The main approaches can be categorized into two families:
  - Vertex (spatial) domain designs
  - Frequency (graph spectral) domain designs



- Classical Fourier transform provides the frequency domain representation of the signals



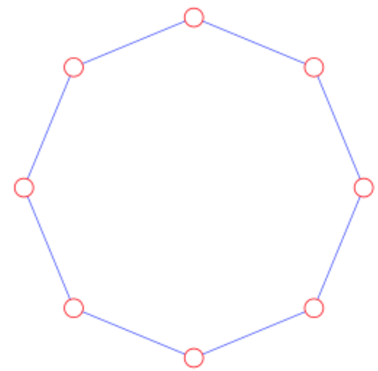
Source: http://www.physik.uni-kl.de



A notion of frequency for graph signals:

We need the graph Laplacian matrix

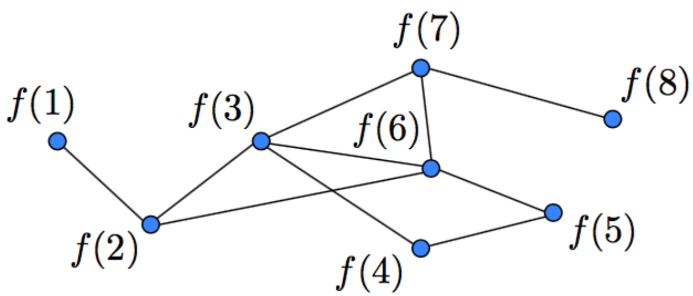
TRANSFORMATIONS ON GRAPHS



$$\mathbf{L} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- $\mathbf{A}$ and $\mathbf{D}$: adjacency and degree matrices, $\mathbf{L} = \mathbf{D} - \mathbf{A}$: graph Laplacian
- $\mathbf{L}$ can be interpreted as a local (high-pass) operation on this graph
- Circulant matrix – Eigenvectors: DFT

Graph Laplacian



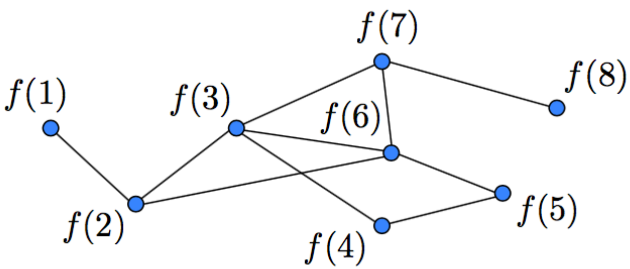
Graph signal $f : V \to \mathbb{R}^N$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}}_{f}$$

A difference operator:

$$Lf = \sum_{i,j=1}^{N} A_{ij} \left(f(i) - f(j) \right)$$

Graph Laplacian



Graph signal $f : V \to \mathbb{R}^N$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}}_{f}$$

A difference operator:

$$Lf = \sum_{i,j=1}^{N} A_{ij} \left(f(i) - f(j) \right)$$

Laplacian quadratic form:

$$f^T L f = \frac{1}{2} \sum_{i,j=1}^{N} A_{ij} \left(f(i) - f(j) \right)^2$$

A measure of "smoothness" [Zhou04]

SPECTRAL PROPERTIES OF THE LAPLACIAN

- Denote by λ_i and $\mathbf{v}_i$ the eigenvalues and eigenvectors of $\mathbf{L}$
- $\mathbf{L}$ is positive semi-definite
  $\Rightarrow \mathbf{x}^T\mathbf{L}\mathbf{x} = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{ij}(x_i - x_j)^2 \geq 0$, for all $\mathbf{x}$
  $\Rightarrow$ All eigenvalues are nonnegative, i.e. $\lambda_i \geq 0$ for all i
- A constant vector $\mathbf{1}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0

$$[\mathbf{L}\mathbf{1}]_i = \sum_{j\in\mathcal{N}(i)} w_{ij}(1-1) = 0$$

  $\Rightarrow$ Thus, $\lambda_1 = 0$ and $\mathbf{v}_1 = (1/\sqrt{N})\ \mathbf{1}$
- In connected graphs, it holds that $\lambda_i > 0$ for $i = 2, \ldots, N$
  $\Rightarrow$ Multiplicity$\{\lambda = 0\}$ = number of connected components

The eigenvalues of a graph characterize the topological structure of the graph

Examples :

(1) if $\lambda_1(G) = -\lambda_n(G)$, then G is bipartite;

(2) if $\lambda_2(G) = 0$, then G is complete multi-partite;

(3) if $\lambda_2(G) = -1$, then G is a complete graph;

(4) ...

In geographic studies, the eigenvalues and eigenvectors of a transportation network provide information about its connectedness. It is proven that the more highly connected in a transportation network G is, the larger is the largest eigenvalue $\lambda_1(G)$. [Tinkler, 72], [Roberts, 78].

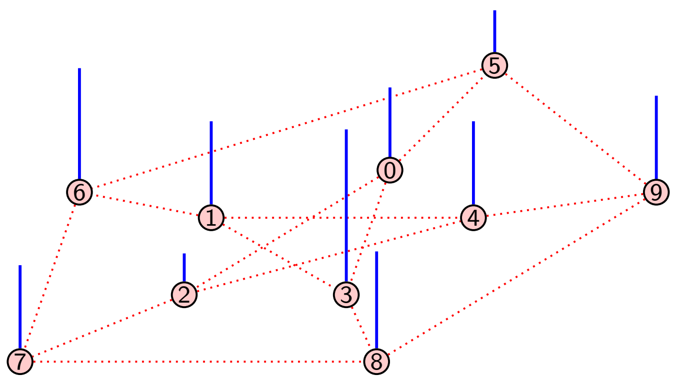
Given the numbers of vertices and edges, how to design a graph with larger $\lambda_1(G)$? – very interesting

GRAPH SIGNALS

- Consider graph $G = (\mathcal{V}, \mathcal{E}, W)$. Graph signals are mappings $x : \mathcal{V} \to \mathbb{R}$
  - $\Rightarrow$ Defined on the vertices of the graph (data tied to nodes)

  Ex: Opinion profile, buffer congestion levels, neural activity, epidemic

- May be represented as a vector $\mathbf{x} \in \mathbb{R}^N$
  - $\Rightarrow$ x_n denotes the signal value at the n-th vertex in $\mathcal{V}$
  - $\Rightarrow$ Implicit ordering of vertices (same as in $\mathbf{A}$ or $\mathbf{L}$)



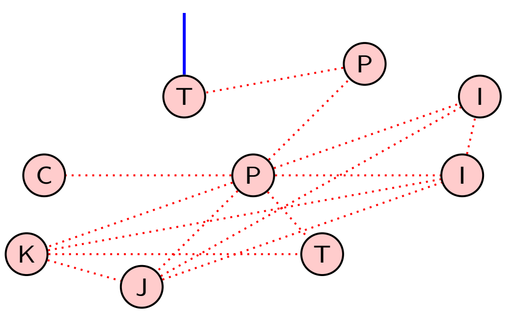
$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_9 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.7 \\ 0.3 \\ \vdots \\ 0.7 \end{bmatrix}$$

- Data associated with links of G $\Rightarrow$ Use line graph of G

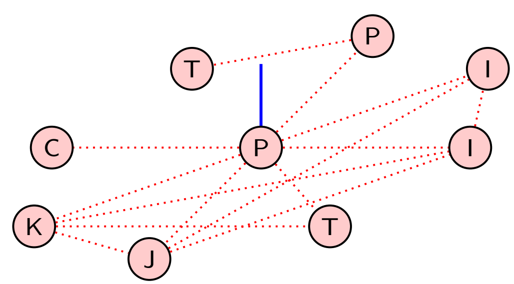
GRAPH EXAMPLE- GENE PROFILE

- Graphs representing gene-gene interactions
  - $\Rightarrow$ Each node denotes a single gene (loosely speaking)
  - $\Rightarrow$ Connected if their coded proteins participate in same metabolism
- Genetic profiles for each patient can be considered as a graph signal
  - $\Rightarrow$ Signal on each node is 1 if mutated and 0 otherwise

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



Sample patient 1 with subtype 1



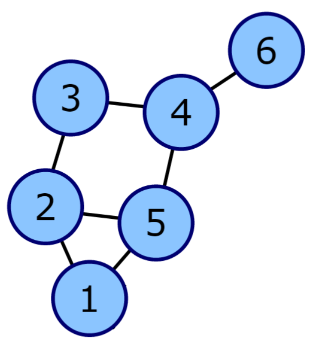
Sample patient 2 with subtype 1

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- To understand a graph signal, the structure of G must be considered

GRAPH-SHIFT OPERATOR

- To understand and analyze $\mathbf{x}$, useful to account for G's structure
- Associated with G is the graph-shift operator $\mathbf{S} \in \mathbb{R}^{N \times N}$
  $\Rightarrow S_{ij} = 0$ for $i \neq j$ and $(i, j) \notin \mathcal{E}$ (captures local structure in G)
- $\mathbf{S}$ can take nonzero values in the edges of G or in its diagonal



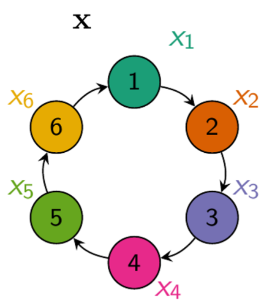
$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{23} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & 0 & S_{64} & 0 & S_{66} \end{pmatrix}$$

- Ex: Adjacency $\mathbf{A}$, degree $\mathbf{D}$, and Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ matrices

RELEVANCE OF THE GRAPH SHIFT OPERATOR

- Q: Why is **S** called shift? A: Resemblance to time shifts

Directed Cycle

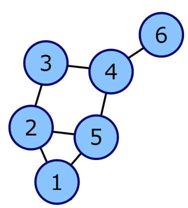


$$\Rightarrow \text{Set } \mathbf{S} = \mathbf{A_{dc}} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

LOCAL STRUCTURE OF THE GRAPH-SHIFT OPERATOR

S *represents a linear transformation that can be computed locally at the nodes of the graph. More rigorously, if* **y** *is defined as* $\mathbf{y} = \mathbf{S}\mathbf{x}$*, then node* i *can compute* y_i *if it has access to* x_j *at* $j \in \mathcal{N}(i)$.

- Straightforward because $[\mathbf{S}]_{ij} \neq 0$ only if $i = j$ or $(j, i) \in \mathcal{E}$



$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & 0 & S_{64} & 0 & S_{66} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

- What if $\mathbf{y} = \mathbf{S}^2\mathbf{x}$?

  $\Rightarrow$ Like powers of **A**: neighborhoods

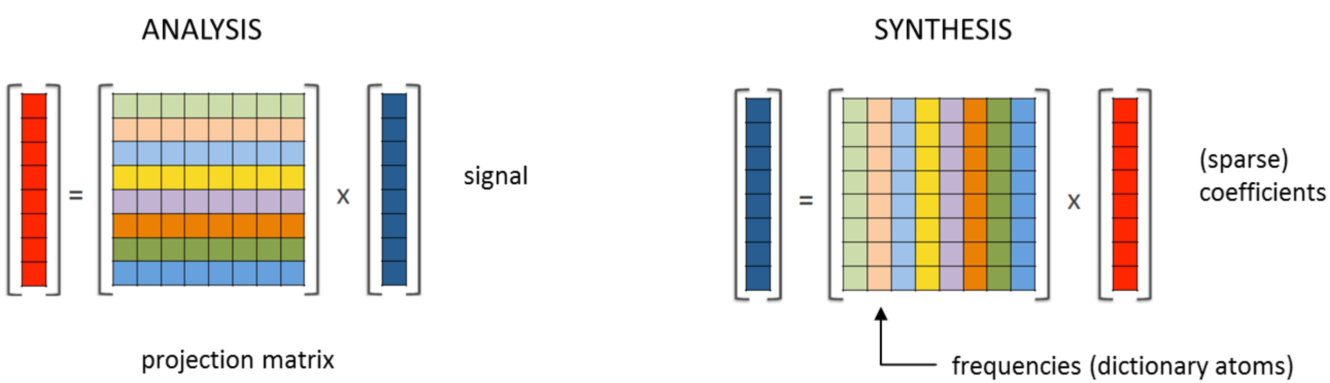
  $\Rightarrow$ y_i found using values within 2-hops

$$[\mathbf{S}^2]_{3,5} = S_{3,2}S_{2,5} + S_{3,4}S_{4,5}$$

$$\mathbf{S}^2 = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & 0 & S_{64} & 0 & S_{66} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & 0 & S_{64} & 0 & S_{66} \end{pmatrix}$$

DFT ON GRAPHS

- Let $\mathbf{x}$ be a temporal signal, its DFT is $\tilde{\mathbf{x}} = \mathbf{F}^H\mathbf{x}$, with $F_{kn} = \frac{1}{\sqrt{N}} e^{+j\frac{2\pi}{N}kn}$
  - $\Rightarrow$ Equivalent description, provides insights
  - $\Rightarrow$ Oftentimes, more parsimonious (bandlimited)
  - $\Rightarrow$ Facilitates the design of SP algorithms: e.g., filters
- Many other transformations (orthogonal dictionaries) exist



- Q: What transformation is suitable for graph signals?

DFT ON GRAPHS

- Useful transformation? $\Rightarrow$ $\mathbf{S}$ involved in generation/description of $\mathbf{x}$
  $\Rightarrow$ Let $\mathbf{S} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ be the shift associated with G
- The Graph Fourier Transform (GFT) of $\mathbf{x}$ is defined as

$$\tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}$$

- While the inverse GFT (iGFT) of $\tilde{\mathbf{x}}$ is defined as

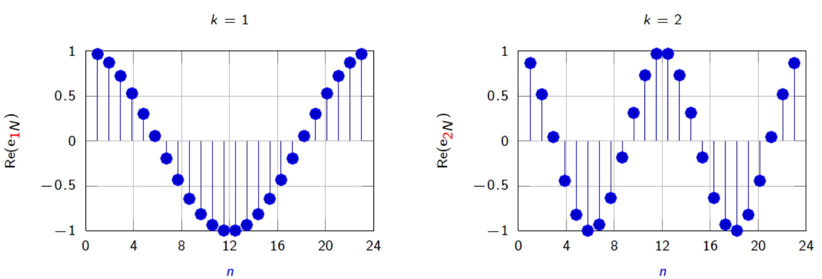
$$\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$$

$\Rightarrow$ Eigenvectors $\mathbf{V} = [\mathbf{v}_1, ..., \mathbf{v}_N]$ are the frequency basis (atoms)

- Additional structure
  $\Rightarrow$ If $\mathbf{S}$ is normal, then $\mathbf{V}^{-1} = \mathbf{V}^H$ and $\tilde{x}_k = \mathbf{v}_k^H\mathbf{x} =< \mathbf{v}_k, \mathbf{x} >$
  $\Rightarrow$ Parseval holds, $\|\mathbf{x}\|^2 = \|\tilde{\mathbf{x}}\|^2$
- GFT $\Rightarrow$ Projection on eigenvector space of shift operator $\mathbf{S}$

EIGEN VALUES AS FREQUENCIES

- Columns of $\mathbf{V}$ are the frequency atoms: $\mathbf{x} = \sum_k \tilde{x}_k \mathbf{v}_k$
- Q: What about the eigenvalues $\lambda_k = \Lambda_{kk}$
  $\Rightarrow$ When $\mathbf{S} = \mathbf{A}_{dc}$, we get $\lambda_k = e^{-j\frac{2\pi}{N}k}$
  $\Rightarrow$ λ_k can be viewed as frequencies!!
- In time, well-defined relation between frequency and variation
  $\Rightarrow$ Higher k $\Rightarrow$ higher oscillations
  $\Rightarrow$ Bounds on total-variation: $TV(\mathbf{x}) = \sum_n (x_n - x_{n-1})^2$



- Q: Does this carry over for graph signals?
  $\Rightarrow$ No in general, but if $\mathbf{S} = \mathbf{L}$ there are interpretations for λ_k
  $\Rightarrow$ $\{\lambda_k\}_{k=1}^N$ will be very important when analyzing graph filters

INTERPRETATION OF THE LAPLACIAN

- Consider a graph G, let $\mathbf{x}$ be a signal on G, and set $\mathbf{S} = \mathbf{L}$
  - $\Rightarrow$ $\mathbf{y} = \mathbf{S}\mathbf{x}$ is now $\mathbf{y} = \mathbf{L}\mathbf{x}$ $\Rightarrow$ $y_i = \sum_{j \in \mathcal{N}(i)} w_{ij}(x_i - x_j)$
  - $\Rightarrow$ j-th term is large if x_j is very different from neighboring x_i
  - $\Rightarrow$ y_i measures difference of x_i relative to its neighborhood
- We can also define the quadratic form $\mathbf{x}^T\mathbf{S}\mathbf{x}$

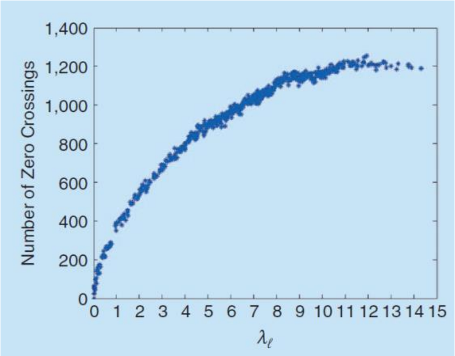
$$\mathbf{x}^T\mathbf{L}\mathbf{x} = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{ij}(x_i - x_j)^2$$

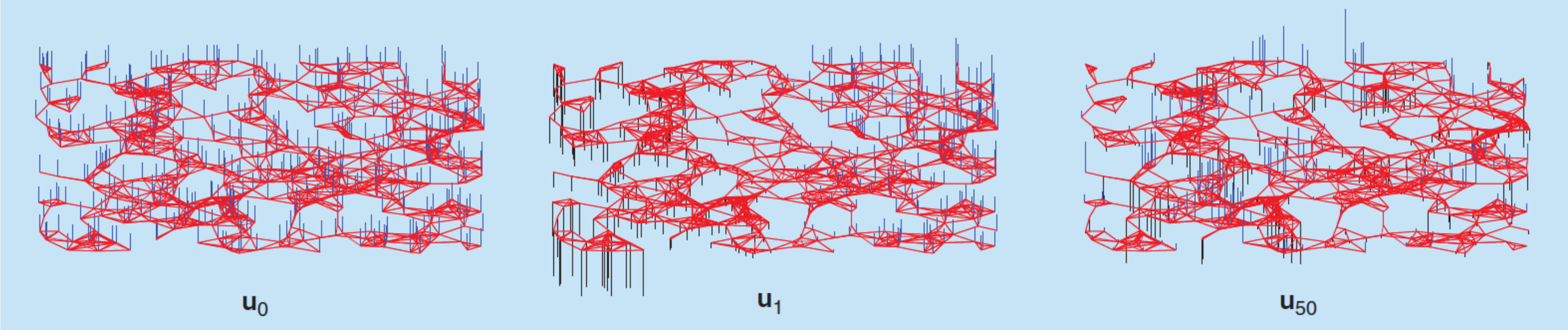
  - $\Rightarrow$ $\mathbf{x}^T\mathbf{L}\mathbf{x}$ quantifies the (aggregated) local variation of signal $\mathbf{x}$
  - $\Rightarrow$ Natural measure of signal smoothness w.r.t. G
- Q: Interpretation of frequencies $\{\lambda_k\}_{k=1}^N$ when $\mathbf{S} = \mathbf{L}$?
  - $\Rightarrow$ If $\mathbf{x} = \mathbf{v}_k$, we get $\mathbf{x}^T\mathbf{L}\mathbf{x} = \lambda_k$ $\Rightarrow$ local variation of $\mathbf{v}_k$
  - $\Rightarrow$ Frequencies account for local variation, they can be ordered
  - $\Rightarrow$ Eigenvector associated with eigenvalue 0 is constant

Graph Laplacian

Spectral properties $\mathcal{L}\mathbf{u}_\ell = \lambda_\ell \mathbf{u}_\ell$

- Laplacian is Positive Semi-definite matrix
- Eigenvalues: $0=\lambda_1(L) \leq \lambda_2(L) \leq \ \dots \leq \lambda_{N-1}(L)$
- Eigen-pair system $\{\lambda_k, \mathbf{u}_k\}$ provides Fourier-like interpretation (GFT)



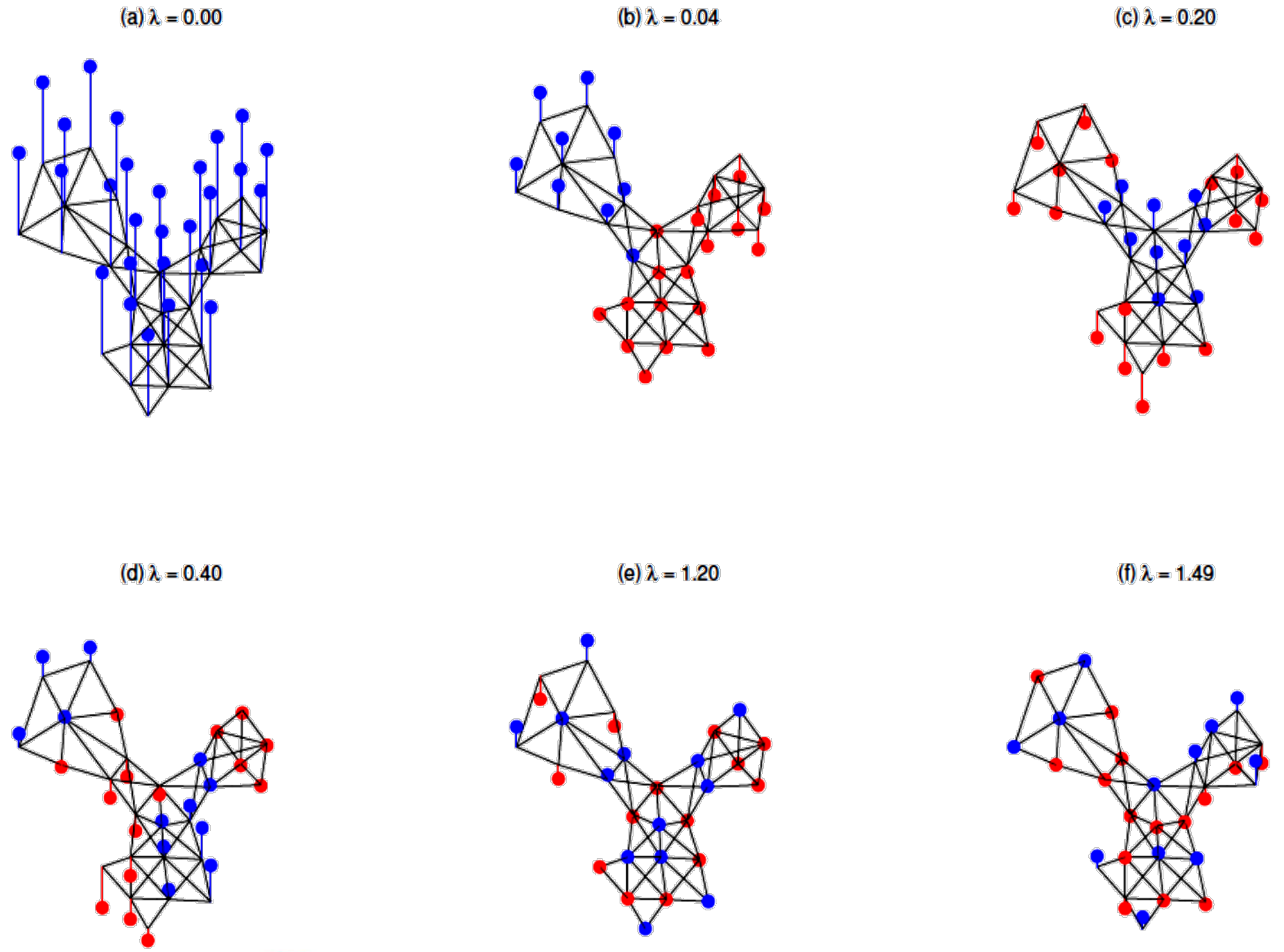


$L = \chi \Lambda \chi^T$

Low frequency → High frequency

$\chi_0^T L \chi_0 = \lambda_0 = 0$ $\qquad$ $\chi_{50}^T L \chi_{50} = \lambda_{50}$

Eigenvectors of Graph Laplacian



Importance

For connected graphs, the Laplacian eigenvector $\mathbf{u}_0$ associated with the eigenvalue 0 is constant and equal to $\frac{1}{\sqrt{N}}$ at each vertex.

The graph Laplacian eigenvectors associated with low frequencies vary slowly across the graph.

If two vertices are connected by an edge with a large weight, the values of the eigenvector at those locations are similar. The eigenvectors associated with larger eigenvalues oscillate more rapidly and are more likely to have dissimilar values on vertices connected by an edge with high weight.

Example

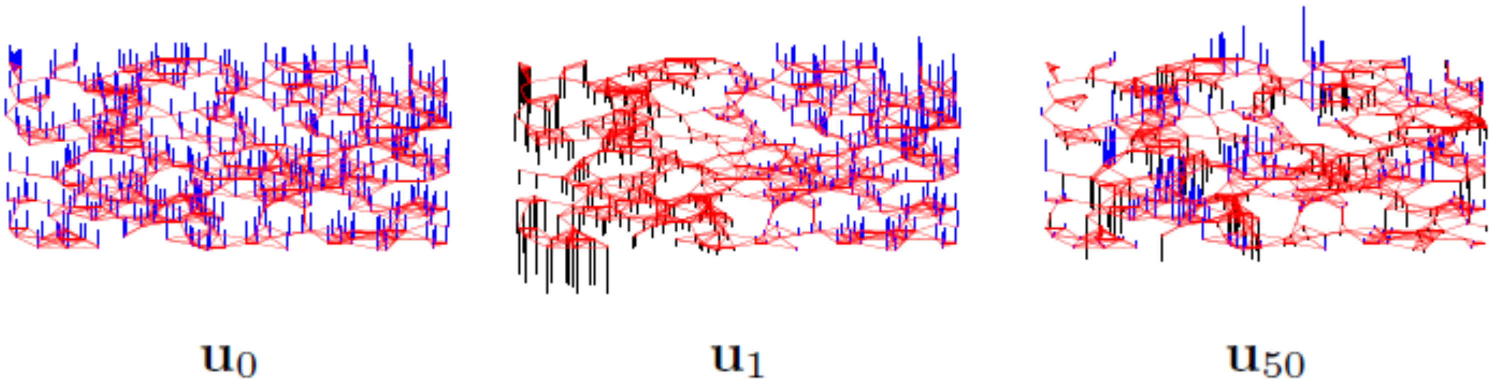


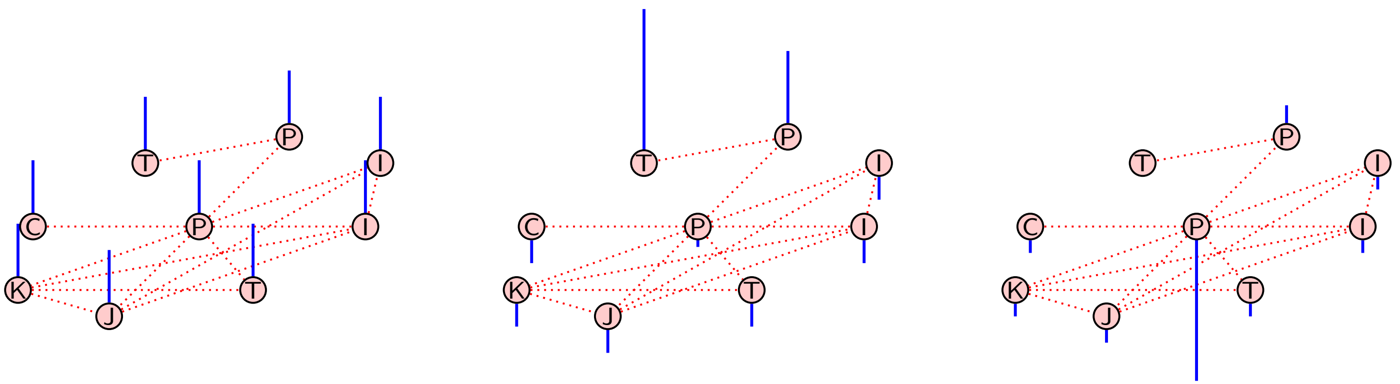
Fig. 2. Three graph Laplacian eigenvectors of a random sensor network graph. The signals' component values are represented by the blue (positive) and black (negative) bars coming out of the vertices. Note that $\mathbf{u}_{50}$ contains many more zero crossings than the constant eigenvector $\mathbf{u}_0$ and the smooth *Fiedler vector* $\mathbf{u}_1$.

FREQUENCIES OF THE LAPLACIAN

- Laplacian eigenvalue λ_k accounts for the local variation of $\mathbf{v}_k$

  $\Rightarrow$ Let us plot some of the eigenvectors of $\mathbf{L}$ (also graph signals)

- Ex: gene network, $N = 10$, $k = 1$, $k = 2$, $k = 9$



EXAMPLE

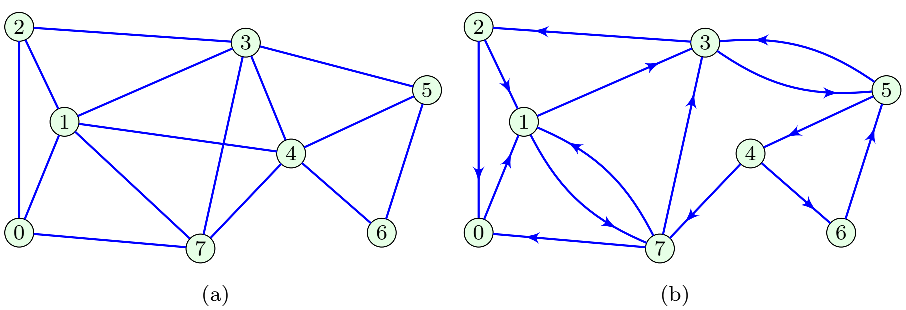


Fig. 2 Examples of: (a) Undirected graph and (b) Directed graph.

The adjacency matrices for the graphs from Fig. 2(a) and (b) are

$$\mathbf{A} = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7\\ \end{array}\!\!\begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \end{array}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1)$$

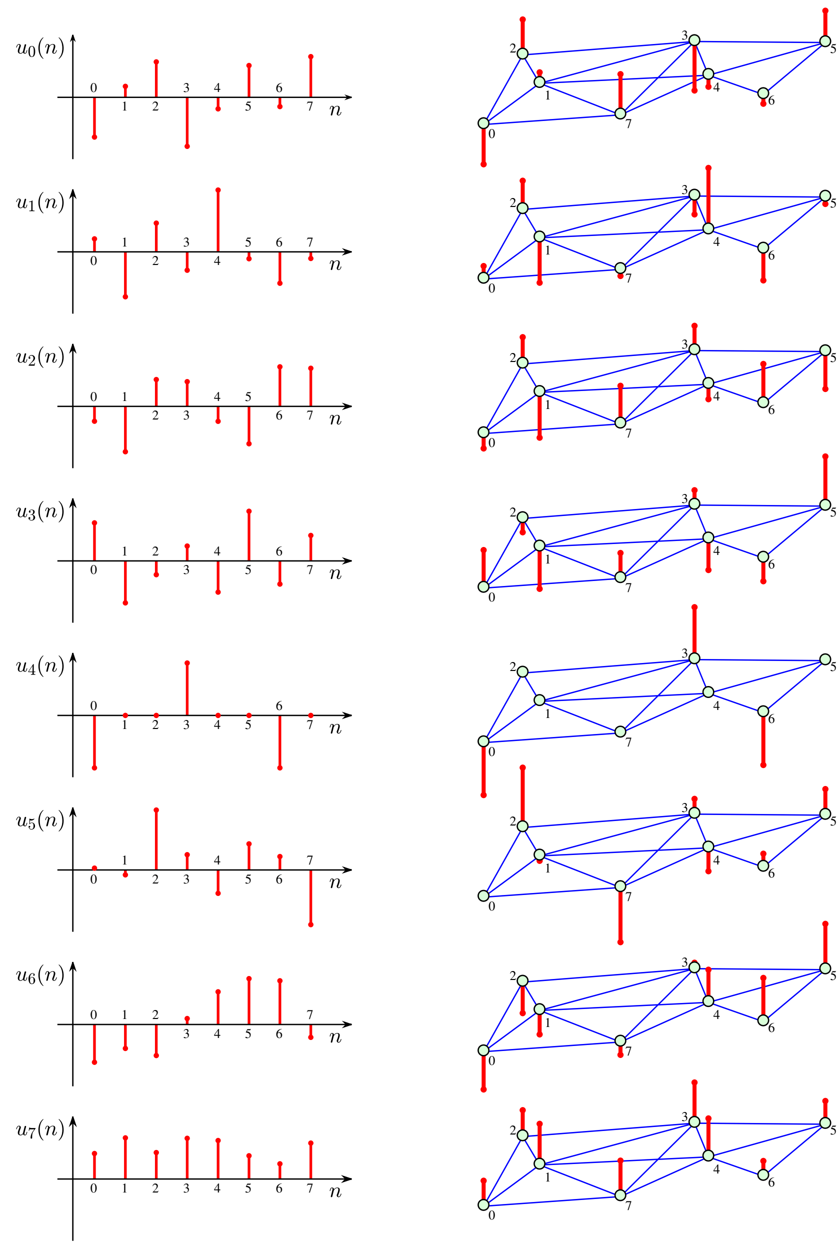
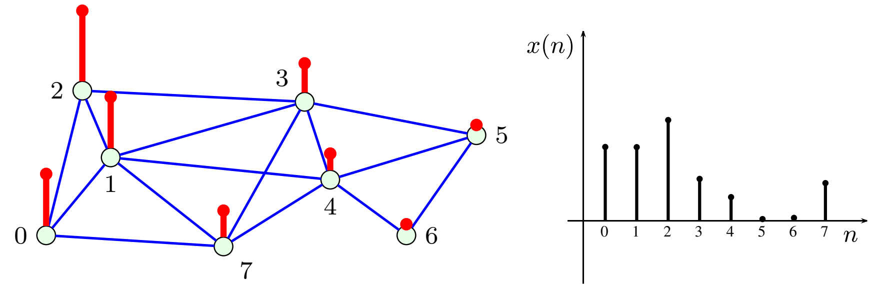
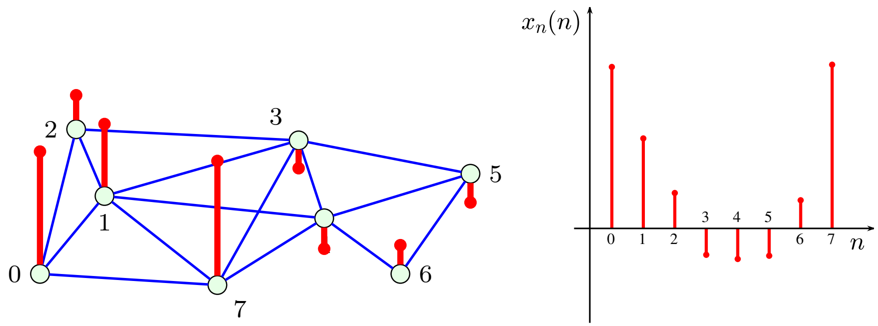


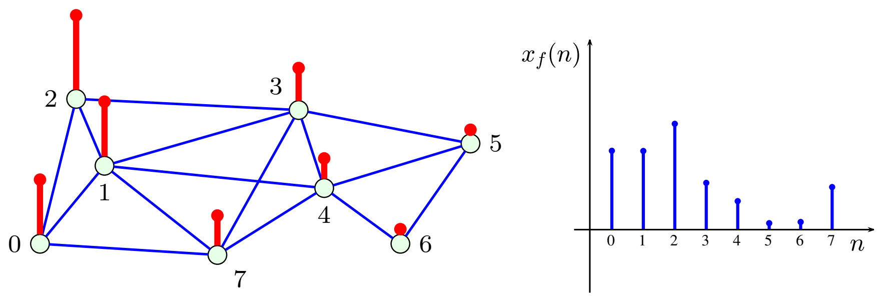
Fig. 10 Eigenvalues λ_k and corresponding eigenvectors $u_k(n)$ for the adjacency matrix of the graph presented in Fig. 2(a). The eigenvectors are shown on the vertex index line (left) and on the graph (right).



(a) original signal



(b) noisy signal



(c) filtered signal

Fig. 26 Signal filtering example. Original signal (a), noisy signal (b) and filtered signal (c). Low pass filtering with two largest eigenvalues is applied.