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An infinite plate weakened by a hole having arbitrary shape

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Abstract

Complex variable methods have been applied to derive exact expressions for Gaursat's functions for the first and second fundamental problems of the infinite plate weakened by a hole having arbitrary shape which is conformally mapped on the domain outside a unit circle by means of rational mapping function. The interesting cases when the shape of the hole takes different shapes are included.

1. Introduction

The boundary value problems for isotropic homogeneous performed infinite plates have been discussed by several authors [1, 3, 4]. It is known that [4] the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions satisfy the boundary conditions

$$k\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = f(t), \tag{1.1}$$

where $k = -1$ and $f(t)$ is a given function of stresses for the first fundamental problem, while $k = \chi = (\lambda + 3\mu)/(\lambda + \mu) > 1$ and $f = 2\mu g(t)$ is a given function of the displacement for the second fundamental problem; λ, μ are called the lame constants and t denotes the affix of a point on the boundary.

In terms of $z = cw(\xi)$, $c > 0$, $w'(\xi)$ does not vanish or becomes infinite for $|\xi| > 1$, the infinite region outside a closed contour conformally mapped outside the unit circle γ. The two complex

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functions of potentials $\phi_1(z)$ and $\psi_1(z)$ take the form

$$\phi_1(z) = -\frac{X + \mathrm{i}Y}{2\pi(1+\chi)}\ln\xi + c\Gamma\xi + \phi_0(\xi), \tag{1.2}$$

$$\psi_1(z) = \frac{\chi(X - \mathrm{i}Y)}{2\pi(1+\chi)}\ln\xi + c\Gamma^*\xi + \psi_0(\xi), \tag{1.3}$$

where X, Y are the components of the resultant vector of all external forces acting on the boundary, and Γ, Γ^* are constants. Generally, the complex functions $\phi(\xi)$, $\psi(\xi)$ are single-valued analytic functions within the region outside the unit circle and $\phi(\infty) = 0$.

It will be assumed that $X = Y = 0$ and $\Gamma = \bar{\Gamma}$ for the first fundamental problem.

Muskhelishvili [4] used the transformation $z = c(\xi + m\xi^{-1})$ for solving the problem of stretching of an infinite plate weakened by an elliptic hole.

This transformation conformally maps the infinite domain bounded internally by an ellipse onto the domain outside the unit circle $|\xi| = 1$ in the ξ-plane.

The application of the Hilbert problem is used by Muskhelishvili [4] to discuss the case of a stretched infinite plate weakened by a circular cut.

In a previous paper [3], the complex variable method is used to solve the first and second fundamental problems of the infinite plate with a curvilinear hole conformally mapped on the domain outside a unit circle by using the rational function

$$z = cw(\xi) = c\frac{\xi + m\xi^{-1}}{1 - n\xi^{-1}}, \quad |n| < 1. \tag{1.4}$$

In this paper the complex variable method has been applied to solve the first and second fundamental problems for the same previous domain of the infinite plate with a general curvilinear hole C conformally mapped on the domain outside a unit circle γ by the rational function

$$z = cw(\xi) = c\frac{\xi + m\xi^{-l}}{1 - n\xi^{-l}}, \quad |n| < 1, \tag{1.5}$$

where $c > 0$, $l = 1, 2, \ldots, p$; m and n are real parameters restricted such that $w'(\xi)$ does not vanish or become infinite outside γ. The interesting cases when the shape of the hole is an ellipse, hypotrochoidal, a crescent or a cut having the shape of a circular arc are included as special ones. Holes corresponding to certain combinations of the parameters m and n are sketched. Some applications of the first and second fundamental problems of the infinite plate with a curvilinear hole are investigated.

2. Method of solution

The expression $w(\xi^{-1})/w'(\xi)$ can be written in the form

$$\frac{w(\xi^{-1})}{w'(\xi)} = \alpha(\xi^{-1}) + \beta(\xi), \tag{2.1}$$

where

$$\alpha(\xi) = \frac{h}{\xi^l - n}, \tag{2.2}$$

$$h = \frac{(m + n^\nu)(1 - n^2)^2}{1 + (l + 1)n^2 - lmn^\nu}, \quad (\nu = 1 + 1/l), \tag{2.3}$$

and $\beta(\xi)$ is a regular function for $|\xi| > 1$.

Using (2.1) in (1.1), we have

$$k\phi(\sigma) - \alpha(\sigma)\overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = f_*(\sigma), \tag{2.4}$$

where

$$\psi_*(\xi) = \psi(\xi) + \beta(\xi)\phi'(\xi), \tag{2.5a}$$

$$f_*(\xi) = F(\xi) - ck\Gamma\xi + c\overline{\Gamma^*}\xi^{-1} + N(\xi)(\alpha(\xi) + \overline{\beta(\xi)}), \tag{2.5b}$$

$$N(\xi) = c\bar{\Gamma} - \frac{X - \mathrm{i}Y}{2\pi(1 + \chi)}\xi,$$

and

$$F(\xi) = f(t). \tag{2.5c}$$

Assume that the derivatives of $F(\sigma)$ must satisfy the Hölder condition. Multiplying both sides of (2.4) by $(1/2\pi\mathrm{i})d/(\sigma - \xi)$ and integrating with respect to σ on γ, one has

$$k\phi(\xi) + \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \xi}\,\mathrm{d}\sigma = c\bar{\Gamma}^*\xi^{-1} + \frac{hN(n^{\nu-1})}{\xi - n^{\nu-1}} - A(\xi), \tag{2.6}$$

where

$$A(\xi) = -\frac{1}{2\pi\mathrm{i}}\sum_{\eta=0}^{\infty}\frac{1}{\xi^{\eta+1}}\int \sigma^\eta F(\sigma)\,\mathrm{d}\sigma, \quad |\xi| > 1.$$

Using (2.1), we have

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \xi}\,\mathrm{d}\sigma = \frac{chb}{(n^{\nu-1} - \xi)}, \tag{2.7}$$

where b is a complex constant to be determined.

Hence,

$$-k\phi(\xi) = A(\xi) - c\overline{\Gamma^*}\xi^{-1} + \frac{h}{n^{\nu-1} - \xi}(cb + N(n^{\nu-1})). \tag{2.8}$$

Differentiating (2.8) with respect to ξ, and using the result in (2.7), we obtain

$$kcb + cn^{2(\nu-1)}\Gamma^* + ch\bar{b}_0 d = -\overline{A'(n^{1-\nu})}, \tag{2.9}$$

where

$$b_0 = b + N(n^{\nu-1}), \tag{2.10a}$$

$$d = n^{2(\nu-1)}[1 - n^{2(\nu-1)}]^{-2}. \tag{2.10b}$$

Hence,

$$b = \frac{kE - hd\bar{E}}{c(k^2 - h^2 d^2)}, \tag{2.10c}$$

where

$$E = -\overline{A'(n^{1-\nu})} - c\Gamma^* n^{2(\nu-1)} - hd\overline{N(n^{\nu-1})}. \tag{2.10d}$$

Also from (2.4), $\psi(\xi)$ can be determined in the form

$$\begin{aligned}\psi(\xi) = ck\bar{\Gamma}\xi^{-1} &= \frac{w(\xi^{-1})}{w'(\xi)}\phi_*(\xi) + \frac{h\xi^l}{1 - n\xi^l}\phi_*(\xi)\\ &+ B(\xi) + \frac{ch^2 b_0}{k}\delta_1(n^{\nu-1}) + \frac{ch\overline{\Gamma^*}}{k}\delta_2(\xi) + I_0\\ &+ \frac{h(X + \mathrm{i}Y)}{2\pi(1+\chi)}\delta_3 = B,\end{aligned} \tag{2.11}$$

where

$$\phi_* = \phi'(\xi) + \overline{N(\xi)}, \tag{2.12a}$$

$$B(\xi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma - \xi}\,\mathrm{d}\sigma, \tag{2.12b}$$

$$\delta_1(\sigma) = (\sigma)^{l-2}(1 - n\sigma^l)^{-2}(\sigma - \xi)^{-2}\,[(1 - l - n\sigma^l)(\sigma - \xi) + (1 - n\sigma^l)\sigma]\,\xi, \tag{2.12c}$$

$$\delta_2(\xi) = \begin{cases} n + \xi^{-1}, & l = 1,\\ 1, & l = 2,\\ 0, & l = 3, 4, \ldots,\end{cases} \tag{2.12d}$$

$$I_0 = \frac{h}{2\pi\mathrm{i}k}\int_\gamma \delta_1(\sigma)F(\sigma)\,\mathrm{d}\sigma,$$

$$\delta_3 = \begin{cases} 1, & l = 1,\\ 0, & \text{otherwise},\end{cases} \tag{2.12e}$$

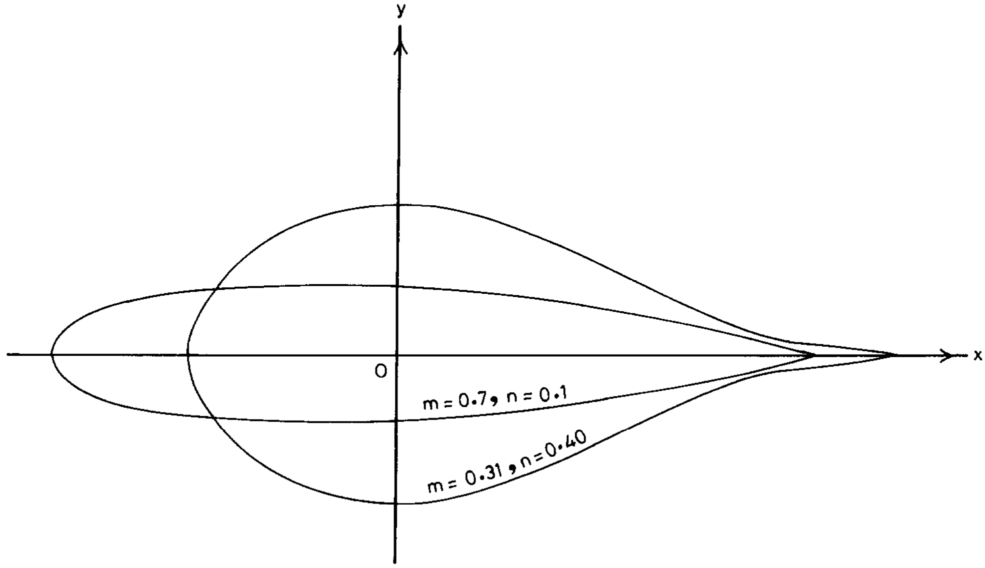

Fig. 1. $z = c(\xi + m\,\xi^{-1})/(1 - n\,\xi^{-1})$.

and

$$B = \frac{1}{2\pi i}\int_\gamma \overline{F(\sigma)}\,\frac{d\sigma}{\sigma}. \tag{2.12f}$$

3. Special cases

(i) For $l = 1$, we have the mapping $z = c(\xi + m\xi^{-1})/(1 - n\xi^{-1})$, and (2.8), (2.11) are in agreement with (2.9), (2.12) in [3] (see Fig. 1).

(ii) For $l = 2, 3, \ldots, p$ we have new cases for the rational mapping function, and the two complex functions $\phi(\xi)$, $\psi(\xi)$ are quickly determined from (2.8) and (2.11).

(iii) For $n = 0$, we have $z = c(\xi + m\xi^{-l})$ $(0 \leqslant m < 1/l)$. The main reason of interest in this mapping is that the general shapes of the hypotrochoids are curvilinear polygons; for $l = 1$, our basic functions agree with (82.4′), (82.5′); (83.10) and (83.11) of Muskhelishvili's result obtained for the elliptic hole [4].

For $l = 2$, we have a curvilinear triangle, for $l = 3$ a curvilinear square, and hence approximate regions of physical interest (see [4]).

(iv) For $m = 0$, we have the transformation

$$z = \frac{c\xi}{1 - n\xi^{-l}}, \quad c > 0 \left(|n| \leqslant \frac{1}{l+1}\right),$$

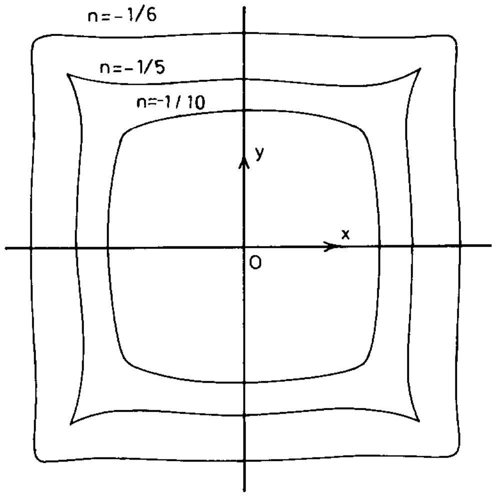

Fig. 2. $z = c\xi/(1 - n\xi^{-4})$.

which leads to a certain regular curvilinear polygon with l sides and l round vertices which become cusps when $|n| = 1/(l+1)$ (see Figs. 2 and 3).

4. Examples

(i) For $k = -1, \Gamma = \frac{1}{4}P, \Gamma^* = -\frac{1}{2}P\mathrm{e}^{-2i\theta}$ and $X = Y = f = 0$, we have the infinite plate weakened by the curvilinear hole C which is free from stresses and the plate stretched at infinity by the application of a uniform tensile stress of intensity P, making an angle θ with the x-axis.

The two complex functions can be represented by the formulae

$$\phi_1(z) - \tfrac{1}{4}cP\,\xi = \phi(\xi) = \tfrac{1}{2}cP\,\mathrm{e}^{2i\theta}\,\xi^{-1} + \tfrac{1}{4}cPh\,(\xi - n^{\nu-1})^{-1}\left[\frac{2n^{2(\nu-1)}\cos 2\theta - 1}{1 - hd} - \frac{2in^{2(\nu-1)}\sin 2\theta}{1 + hd}\right], \tag{4.1}$$

$$\psi_1(z) + \tfrac{1}{2}\,cP\mathrm{e}^{-2i\theta} = \psi(\xi) = -\tfrac{1}{4}cP\,\xi^{-1} - \frac{w(\xi^{-1})}{w'(\xi)}\,\phi_*(\xi) + \frac{h\xi^l}{1 - n\xi l}\,\phi_*(\xi) - ch^2\,b_0\,\delta_1(n^{\nu-1}) + \tfrac{1}{2}chP\,\mathrm{e}^{2i\theta}\,\delta_2(\xi). \tag{4.2}$$

The previous result agrees with [3], when $l = 1$.

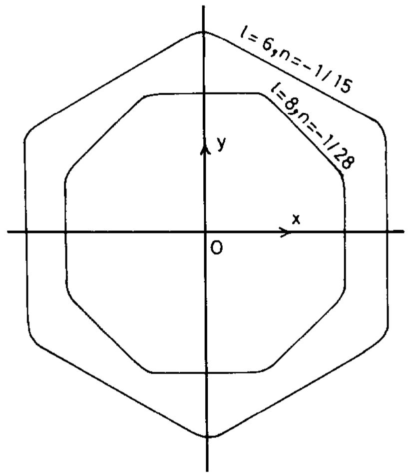

Fig. 3. $z = c\xi/(1 - n\xi^{-l})$ $(l = 6, 8)$.

(ii) For $k = -1$, $X = Y = \Gamma = \Gamma^* = 0$ and $f = Pt$ where P is a real constant, we have

$$\phi(\xi) = cP(m + n^{\nu})(n^{\nu-1} - \xi)^{-1}(1 + J_1), \tag{4.3}$$

$$\psi(\xi) = -\frac{w(\xi^{-1})}{w'(\xi)}\phi'(\xi) + \frac{h\xi^{l}}{1 - n\xi^{l}}\phi'(\xi) - \frac{cP}{\xi} - B_1$$
$$- cPh(m + n^{\nu})(1 + J_1)\delta_1(n^{\nu-1}), \tag{4.4}$$

where

$$J_1 = \frac{n^{2(\nu-1)}(1 - n^2)^2(m + n^{\nu})(1 - n^{2(\nu-1)})^{-2}}{1 - lmn^{\nu} - (1 + l + m + n^{\nu})n^2}, \tag{4.5a}$$

and

$$B_1 = \begin{cases} cPn & \text{for } l = 1, \\ 0 & l = 2, 3, \ldots. \end{cases} \tag{4.5b}$$

Thus (4.4) and (4.5) give the solution of the first fundamental problem for an isotropic infinite plate with a curvilinear hole when there is no external force and the edge of the hole is subject to a uniform pressure P.

(iii) For $k = -1$, $X = Y = \Gamma = \Gamma^* = 0$ and $f(t) = -\mathrm{i}Tt$, we have the case of the first fundamental problem, when the edge of the hole is subject to uniform tangential stress T, and the result is obtained directly from (4.4) and (4.5) by putting $-\mathrm{i}T$ instead of P. Also this result agrees with [3] when $l = 1$.

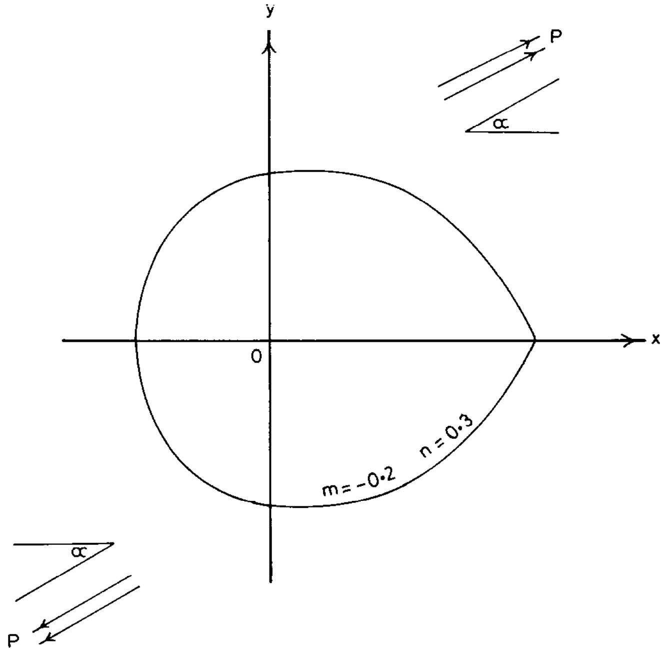

Fig. 4. $z = c(\xi + m\xi^{-1})/(1 - n\xi^{-1})$.

(iv) For $k = \chi$, $\Gamma = \frac{1}{4}P$, $\Gamma^* = -\frac{1}{2}P e^{-2i\theta}$, $X = Y = 0$ and $f = 2\mu g(t)$, we have

$$-\chi\phi(\xi) = \tfrac{1}{2}cP\,e^{2i\theta}\xi^{-1} + cb^*(n^{\nu-1} - \xi)^{-1}, \tag{4.6}$$

$$\psi(\xi) = -\left(-\frac{w(\xi^{-1})}{w'(\xi)} + \frac{h\xi^l}{1 - n\xi^l}\right)\phi_*(\xi) + \frac{chb_0^*}{\chi}\delta_1(n^{\nu-1}) + c(\tfrac{1}{4}\chi_P + 2i\mu\varepsilon)\xi^{-1} - \delta_4 - \frac{cPhe^{2i\theta}}{2\chi}\delta_2(\xi), \tag{4.7}$$

where

$$b_0^* = \frac{Ph(\chi + 2n^{2(\nu-1)}\cos 2\theta)}{4(\chi + hd)} + i\,\frac{4\mu\varepsilon\chi(m + n^\nu) - Phn^{2(\nu-1)}\sin 2\theta}{2(\chi - hd)}, \tag{4.8a}$$

$$\delta_4 = \begin{cases} -2i\,cn\mu\varepsilon & \text{for } l = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8b}$$

and

$$\phi_*(\xi) = \phi'(\xi) + \tfrac{1}{4}cP. \tag{4.8c}$$

In the previous example, we have the case of unidirectional tension of an infinite plate with a rigid curvilinear centre (see Figs. 4 and 5).

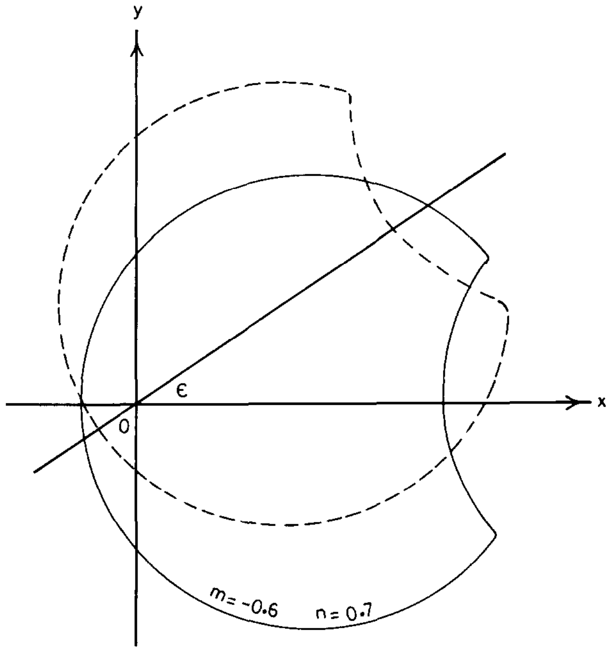

Fig. 5. $z = c(\xi + m\xi^{-1})/(1 - n\xi^{-1})$.

It remains to determine the angle ε from the condition that the resultant moment of the forces, acting on the curvilinear centre from the surrounding material, must vanish, i.e.

$$M = \operatorname{Re}\left\{ \int |\psi(\xi) - \tfrac{1}{2} cP\,\mathrm{e}^{-2\mathrm{i}\theta}\xi|\, w'(\xi)\,\mathrm{d}\xi \right\} = 0. \tag{4.9}$$

Hence, we have

$$\varepsilon = \frac{P(m+n^2)(1-(m+2)n^2)(1+\chi)\sin 2\theta}{4\mu[m(m+n^2)+\chi(1-n^2(m+2))]} \quad \text{for } l = 1, \tag{4.10}$$

$$\varepsilon = \frac{P(1-n^2)^2 \sin 2\theta}{4\mu\chi(1-2mn^{3/2}-3n^2)} \times \left\{ n + \frac{[n(m+n^{3/2})(1+n)^2 + (2mn^{3/2}+3n^2-1)\chi]J_2}{2n^2(m+n^{3/2})^2(-n^4+2n^2+7)+(1-2mn^{3/2}-3n^2)(2m/\bar{n}+n+n^3)} \right\}$$
$$\text{for } l = 2, \tag{4.11}$$

where

$$J_2 = 2n^2(n^2-3)(m+n^{3/2}) + n(2mn^{3/2}+3n^2-1)(m+n^{3/2})^{-1} + (4/\bar{n}(2mn^{3/2}+3n^2-1)\chi.$$

It is found that the rotation is zero in the case of a circular centre ($m = n = 0$ or $m = -n^2$ when $l = 1$).

Case 1: *Bi-axial tension with* $k = \chi$, $X = Y = 0$, $\Gamma = \bar{\Gamma} = \frac{1}{2}P$, $\Gamma^* = 0$, $l = 1$ *and* $f(t) = 2\mu g(t)$. Under the same condition of example (iv), one obviously will have $\varepsilon = 0$ and the two complex functions are

$$\phi(\xi) = \frac{c\,Ph}{2(\chi + hd)(\xi - n)}, \tag{4.12}$$

$$\psi(\xi) = \tfrac{1}{2}c\chi P\,\xi^{-1} - \frac{w(\xi^{-1})}{w'(\xi)}\,\phi_*(\xi) + \frac{h\xi}{1 - n\xi}\,\phi_*(n^{-1}), \tag{4.13}$$

where

$$\phi_*(\xi) = \phi'(\xi) + \tfrac{1}{2}cP.$$

Case 2: *Curvilinear centre not allowed to rotate.* Under the condition of example (iv), when $l = 1$, the rigid curvilinear kernel is restrained in its original position by a couple which is not sufficient to rotate the kernel, then $\varepsilon = 0$.

The complex function can be obtained in the form

$$-\chi\phi(\xi) = \tfrac{1}{2}\,cP\,e^{2i\theta}\,\xi^{-1} + cb_0(n - \xi)^{-1}, \tag{4.14}$$

$$\psi(\xi) = \tfrac{1}{4}\,cP\chi\,\xi^{-1} - \frac{w(\xi^{-1})}{w'(\xi)}\,\phi_*(\xi) + \frac{h\xi}{1 - n\xi}\,\phi_*(n^{-1}), \tag{4.15}$$

where

$$b_0 = \frac{Ph(\chi + 2n^2\cos 2\theta)}{4(\chi + hd)} - \frac{i\,Ph\,n^2\sin 2\theta}{2(\chi - hd)}, \tag{4.16a}$$

$$d = \frac{n^2}{(1 - n^2)^2} \tag{4.16b}$$

and

$$\phi_*(\xi) = \phi'(\xi) + \tfrac{1}{4}\,cP. \tag{4.16c}$$

The resultant moment is given by

$$M = c\pi P(m + n^2)\left(\frac{\chi + 1}{\chi - hd}\right)\sin 2\theta. \tag{4.17}$$

Case 3: *Couple with a given moment acting on the curvilinear hole and the stresses vanishing at infinity.* We obtain the complex functions, when $l = 1$, in the form

$$\phi(\xi) = \frac{2\mu c\varepsilon(m + n^2)i}{(\chi - hd)(\xi - n)}, \tag{4.18}$$

$$\psi(\xi) = 2\mu c\varepsilon i(n + \xi^{-1}) - \frac{w(\xi^{-1})}{w'(\xi)}\,\phi'(\xi) + \frac{h\xi}{1 - n\xi}\,\phi'(n^{-1}), \tag{4.19}$$

where

$$\varepsilon = \frac{M[n^2(m+n^2) - \chi(1 - n^2(m+2))]}{4\pi c\mu[m(m+n^2) + \chi(1 - n^2(m+2))]}. \tag{4.20}$$

(v) When the force acts on the centre of the curvilinear kernel and the stresses vanish at infinity, it is easily seen that the kernel does not rotate.

In general, the kernel remains in its original position. Hence one assumes

$\Gamma = \Gamma^* = f = 0, \quad l = 1 \quad \text{and} \quad k = \chi.$

The Gaursat's functions are

$$\phi_1(z) + \frac{X + \mathrm{i}Y}{2\pi(1+\chi)} \ln \xi = \phi(\xi)$$
$$= \frac{hn}{2\pi\chi(1+\chi)(\xi - n)} \left[\frac{\chi h d(X + \mathrm{i}Y)}{(\chi^2 - h^2 d^2)} - \left\{ \frac{h^2 d^2}{(\chi^2 - h^2 d^2)} + 1 \right\} (X - \mathrm{i}Y) \right], \tag{4.21}$$

$$\psi_1(z) - \frac{\chi(X - \mathrm{i}Y)}{2\pi(1+\chi)} \ln \xi = \psi(\xi)$$
$$= \frac{h\xi}{1 - n\xi} \phi_*(n^{-1}) - \frac{w(\xi^{-1})}{w'(\xi)} \phi_*(\xi), \tag{4.22}$$

where

$$\phi_*(\xi) = \phi'(\xi) - \frac{X + \mathrm{i}Y}{2\pi(1+\chi)\xi}. \tag{4.23}$$

Therefore, we have the solution of the second fundamental problem in the case when a force (X, Y) acts on the centre of the curvilinear kernel.

References

[1] W.A. Bassali and R.H. Dawoud, *Proc. Cambridge Philos. Soc.* **53** (1957) 755.

[2] W.R. Dean, *Proc. Cambridge Philos. Soc.* **50** (1954) 623.

[3] I.H. El-Sirafy and M.A Abdou, First and second fundamental problems of infinite plate with a curvilinear hole, *J. Math. Phys. Sci.* **18** (1984).

[4] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity* (Moscow, 1949).