Math 512
Homework Report 3

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Question 1.

Consider the codes given in $1.9 BinomialTreeUsingClasses$. Let us add more features.

1.a.

Consider an option with a payoff function of the strangle option with $K1 \leq K2$. Write a derived class of Payoff with an appropriate constructor to model this type of options.

$$\Phi(S_T; K_1, K_2)\begin{cases} K_1 - S_T, 0 \leq S_T \leq K_1 \\ S_T - K_2, S_T \geq K_2 \\ 0, K_1 \leq S_T \leq K_2 \end{cases}$$

Use assert to check the condition $K1 \leq K2$.

Steps:
1. Use assert to check if $K1 \leq K2$.
2. To write an appropriate constructor.
3. Use if-else statement to construct the strangle option such that satisfies the requirements of the question.

Details:
For the second step, the constructor must include strike prices K1 and K2 instead of just one strike price K. In the third steps, if spot price smaller or equal to K1, return $K1 -$ spot price, else if spot price larger than K2, return spot price $- K2$, otherwise, return 0.

1.b.

Implement the derived class VanillaOption::AmericanOption to handle the pricing of American options. By definition, an American option can be exercised anytime on or before the maturity date T. The no-arbitrage value of this option can be obtained by backward induction using the formula

$$V_t(k) = max\big\{\Phi(S_t(k)), \tfrac{1}{1+R}(q_u V_k + q_d V_k)\big\}$$

(Here we are using the notations in 1.4 Binomial tree.) Mathematically this computes the so-called Snell envelope. Implement this algorithm in the member function AmericanOption::price Snell. Illustrate your codes in main() using call, put and strangle options.

Steps:
1.Set up the binomial model: (T, u, d, R, s).
2.Compute risk-neutral probabilities q_u, q_d.
3.Set up the boundary condition $V_T(k) = \Phi(su^k d^{T-k})$.
4.Perform the backward induction

$$V_t(k) = max\big\{\Phi(S_t(k)), \tfrac{1}{1+R}(q_u V_k + q_d V_k)\big\}.$$

Details:
First three steps are similar to sample European::vanilla, but in the fourth step I use if-else statement to compare $\Phi(S_t(k))$ and $\frac{1}{1+R}(q_u V_k + q_d V_k)$ and choose the larger one as this grid. In the main function, I design the binomial model with u=1.1, d=1/u, r=0.03, spot price= \$30.0 and number of steps=5. I choose call option with strike price \$32.0, put option with strike price

\$30.0 and strangle option with strike prices K1=\$30.0, K2=\$32.0. Implement American snell with these options

Question 2.

Consider the following variant of the Box-Buller Method:

- Generate independent $Y_1, Y_2 \sim Exp(1)$ until $Y_2 > (1 - Y_1)^2/2$
- Generate $U \sim Uni(0,1)$ and take

$$Z = \begin{cases} Y_1, U < 0.5 \\ -Y_1, U > 0.5 \end{cases}$$

Then $Z \sim N(0,1)$.

2.a.

Prove mathematically that this method indeed produces normally distributed random variables.

Proof: Need to prove that $Z \sim N(0,1)$. The cumulative distribution function of Z should be

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y_1^2}{2}} dy_1$$

$$\begin{aligned} LHS &= \mathbb{P}(Y_1 I_{(U<1/2)} - Y_1 I_{(U>1/2)} \leq z | Y_2 > (1-Y_1)^2/2) \text{ (I is indicator function)} \\ &= \frac{\Big\{\mathbb{P}\big(\{Y_1 \leq z\} \cap \{Y_2 > (1-Y_1)^2/2\}\big) + \mathbb{P}\big(\{Y_1 \leq -z\} \cap \{Y_2 > (1-Y_1)^2/2\}\big)\Big\}/2}{\mathbb{P}(Y_2 > (1-Y_1)^2/2)} \end{aligned}$$

where

$$\begin{aligned} denominator &= \int_0^{\infty} e^{-y_1} \big(\int_{\frac{(1-y_1)^2}{2}}^{\infty} e^{-y_2} dy_2 \big) dy_1 \\ &= \frac{\sqrt{2\pi}}{2\sqrt{e}} \end{aligned}$$

since the CDF of $Y \sim exp(1)$ is $F(y) = \int_0^{\infty} e^{-y} dy$, and similarly, we can get

$$numerator = \frac{1}{2\sqrt{e}} \int_{-\infty}^{z} e^{-\frac{y_1^2}{2}} dy_1$$

Thus,

$$\frac{numerator}{denominator} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y_1^2}{2}} dy_1 = RHS \Rightarrow Z \sim N(0,1).$$

2.b.

This question requires us to implement this algorithm in C++, assuming we can only simulate from the uniform distribution U(0,1) (and then do appropriate transformations).

Steps:
1. Generate U(0,1).
2. Transform uniform distribution to exponential distribution with $\lambda = 1$.
3. Use recursion to generate Y1 such that $Y_2 > (1 - Y_1)^2/2$. 4. Use U(0,1) to get a random number U, if $U < 0.5, Z = Y_1$, else $Z = -Y_1$.

Details:
In the second step, use the formula $Y = -ln(U)/\lambda$.
In addition, in the test part, I use abstract class to test random distribution generator cases.

2.c.

Consider a random walk

$$\text{St} = \text{X1} + \cdot\ \cdot\ \cdot + \text{Xt},$$

where X1,...,Xt are i.i.d. samples of the N(0,1) distribution. Define the stopping time

$$\tau = inf\{t \geq 0 : S_t \geq 5 \text{ or } S_t \leq 10\}\ .$$

Simulate paths of the random walk to estimate of E and Var(). To be specific, simulate 10000 paths and give approximate 95% confidence intervals of the two quantities. For this part you may use the generator constructed in (b) or use normal distribution in <random>.

Steps:
1. Design for loop to generate 10000 paths to get an array for τ.
2. Write functions to calculate mean, standard deviation and also 95% interval.

Details:
In the first step, I used both my generator constructed in (b) and also normal distribution in <random>. Their results are similar. In the second step, the 95% confidence interval is $\mu \pm 1.96 * \sigma$.

PS: The result pictures are attached in another pdf file.