

# Discrete Mathematics

## Set

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# Sets

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- A *set* is an unordered collection of objects.
  - E.g., the students in this class, the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set.
  - A set is said to *contain* its elements.
- The notation  $a \in A$  denotes that  $a$  is an element of  $A$ 
  - If  $a$  is not a member of  $A$ , write  $a \notin A$

# Defining a Set: Roster Method

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- $S = \{a, b, c, d\}$ 
  - Order does not mean any thing:  $S = \{a, b, c, d\} = \{b, c, a, d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, ..., z\}$$

# Examples

- Set of all vowels in the English alphabet:

$$V = \{a, e, i, o, u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1, 3, 5, 7, 9\}$$

- Set of all positive integers less than 100:

$$S = \{1, 2, 3, \dots, 99\}$$

- Set of all integers less than 0:

$$S = \{\dots, -3, -2, -1\}$$

# Conventional Notions of Important Sets

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$\mathbb{N}$  = *natural numbers* =  $\{1, 2, 3, \dots\}$

$\mathbb{Z}$  = *integers* =  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Z}^+$  = *positive integers* =  $\{1, 2, 3, \dots\}$

$\mathbb{R}$  = *set of real numbers*

$\mathbb{R}^+$  = *set of positive real numbers*

$\mathbb{C}$  = *set of complex numbers.*

$\mathbb{Q}$  = *set of rational numbers*

# Defining a Set with Set-Builder Notation

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- To specify the properties that all members must satisfy:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$$

- A predicate may be used:  $S = \{x \mid P(x)\}$

- Example:  $S = \{x \mid \text{Prime}(x)\}$

- Example: Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q \text{ for } p \in \mathbb{Z}^+ \text{ and } q \in \mathbb{Z}^+\}$$

# Defining a Set with Interval Notation

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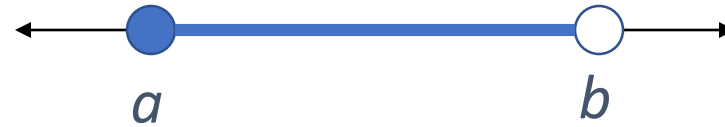
$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

E.g.,  $[a,b)$



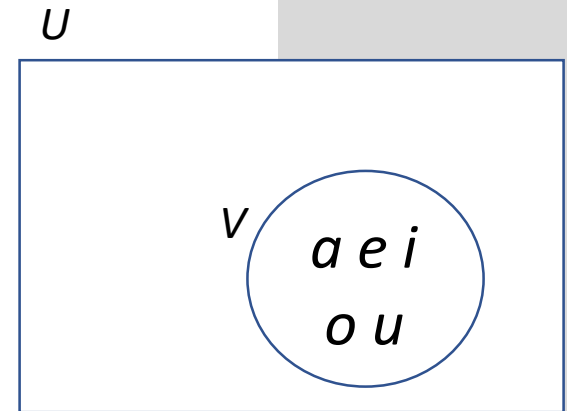
*closed interval*       $[a,b]$

*open interval*       $(a,b)$

# Universal Set and Empty Set

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- The *universal set*  $U$  is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated
  - Contents depend on the context
- The empty set is the set with no elements.
  - denoted as  $\emptyset$ , or  $\{\}$





# Some things to remember

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- Sets can be elements of sets.

$$\{\{1,2,3\}, a, \{b,c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

$$\emptyset \neq \{ \emptyset \}$$

# Set Cardinality

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**Definition:** If there are exactly  $n$  distinct elements in  $S$  where  $n$  is a non-negative integer, we say that  $S$  is *finite*. Otherwise it is *infinite*.

**Definition:** The *cardinality* of a finite set  $A$ , denoted by  $|A|$ , is the number of (distinct) elements of  $A$ .

## Examples:

1.  $|\emptyset| = 0$
2. Let  $S$  be the letters of the English alphabet. Then  $|S| = 26$
3.  $|\{1,2,3\}| = 3$
4.  $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

# Russell's Paradox

- Let  $R$  be the set of all sets each of which is not a member of itself.

- $R = \{S \mid S \notin S\}$

- A paradox results from trying to answer the question “Is  $R$  a member of itself?”

- Related Paradox:

- Henry is a barber who shaves every man if and only if the man does not shave himself. A paradox results from trying to answer the question “Does Henry shave himself?”



Bertrand  
Russell  
(1872-1970)  
Cambridge,  
UK  
Nobel Prize  
Winner

# Set Equality

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**Definition:** Two sets are *equal* if and only if they have the same elements.

- Therefore if  $A$  and  $B$  are sets, then  $A$  and  $B$  are equal if and only if

$$\forall x (x \in A \leftrightarrow x \in B)$$

- We write  $A = B$  if  $A$  and  $B$  are equal sets.

$$\{1, 3, 5\} = \{3, 5, 1\}$$

$$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$$

# Another look at Equality of Sets

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- Recall that two sets  $A$  and  $B$  are *equal*, denoted by  $A = B$  iff  
 $\forall x (x \in A \leftrightarrow x \in B)$
- Using logical equivalences we have that  $A = B$  iff  
 $\forall x [(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
- This is equivalent to  
$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

**Definition:** The set  $A$  is a *subset* of  $B$ , if and only if every element of  $A$  is also an element of  $B$ .

- The notation  $A \subseteq B$  is used to indicate that  $A$  is a subset of the set  $B$ .
- $A \subseteq B$  holds if and only if  $\forall x(x \in A \rightarrow x \in B)$  is true.
  1. Because  $a \in \emptyset$  is always false,  $\emptyset \subseteq S$ , for every set  $S$ .
  2. Because  $a \in S \rightarrow a \in S$ ,  $S \subseteq S$ , for every set  $S$ .

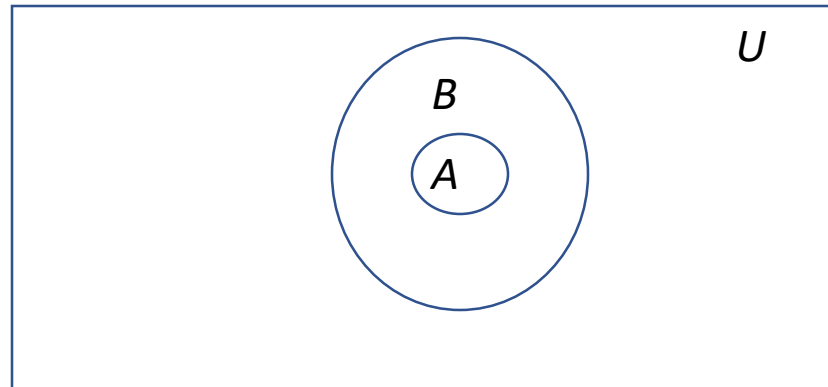
# Proper Subsets

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**Definition:** If  $A \subseteq B$ , but  $A \neq B$ , then we say  $A$  is a *proper subset* of  $B$ , denoted by  $A \subset B$  or  $A \subseteq B$

If  $A \subset B$ , then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



# Power Sets

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**Definition:** The set of all subsets of a set  $A$ , denoted  $P(A)$ , is called the *power set* of  $A$ .

**Example:** If  $A = \{a, b\}$  then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- If a set has  $n$  elements, then the cardinality of the power set is  $2^n$ .  
(In Chapters 5 and 6, we will discuss different ways to show this.)



# Cartesian Product



René Descartes  
(1596-1650)

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**Definition:** The *Cartesian Product* of two sets  $A$  and  $B$ , denoted by  $A \times B$  is the set of ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ .

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

**Example:**

$$A = \{a, b\} \quad B = \{1, 2, 3\}$$

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

- **c.f. Relation**

- A subset  $R$  of the Cartesian product  $A \times B$  is called a *relation* from the set  $A$  to the set  $B$ .
- Will be covered in depth in Chapter 9.

# Cartesian Product

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**Definition:** The cartesian products of the sets  $A_1, A_2, \dots, A_n$ , denoted by  $A_1 \times A_2 \times \dots \times A_n$ , is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_i$  belongs to  $A_i$  for  $i = 1, \dots, n$ .

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

**Example:** What is  $A \times B \times C$  where  $A = \{0, 1\}$ ,  $B = \{1, 2\}$  and  $C = \{0, 1, 2\}$

**Solution:**  $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

# Tuples

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- The *n-tuple*  $(a_1, a_2, \dots, a_n)$  is the ordered collection of objects, which has  $a_1$  as its first element and  $a_2$  as its second element and so on until  $a_n$  as its last element.
  - Two *n-tuples* are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
  - The ordered pairs  $(a, b)$  and  $(c, d)$  are equal if and only if  $a = c$  and  $b = d$ .

# Union

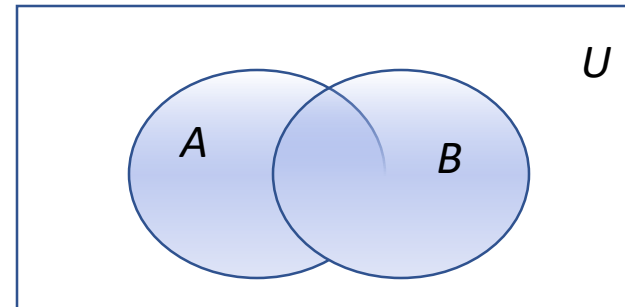
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- **Definition:** Let  $A$  and  $B$  be sets. The *union* of the sets  $A$  and  $B$ , denoted by  $A \cup B$ , is the set:  $\{x | x \in A \vee x \in B\}$

- **Example:** What is  $\{1, 2, 3\} \cup \{3, 4, 5\}$ ?

**Solution:**  $\{1, 2, 3, 4, 5\}$

Venn Diagram for  $A \cup B$



# Intersection

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- **Definition:** The *intersection* of sets  $A$  and  $B$ , denoted by  $A \cap B$ , is

$$\{x | x \in A \wedge x \in B\}$$

- Note if the intersection is empty, then  $A$  and  $B$  are said to be *disjoint*.

- **Example:** What is?  $\{1,2,3\} \cap \{3,4,5\}$  ?

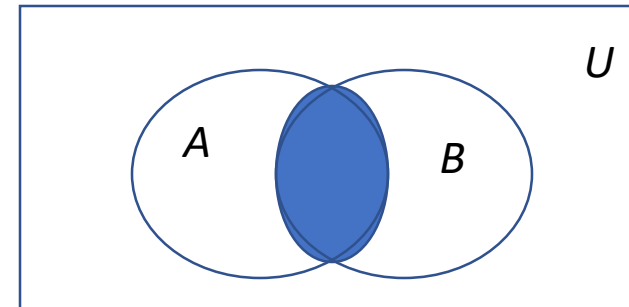
**Solution:**  $\{3\}$

- **Example:** What is?

$$\{1,2,3\} \cap \{4,5,6\} ?$$

**Solution:**  $\emptyset$

Venn Diagram for  $A \cap B$



# Complement

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**Definition:** If  $A$  is a set, then the complement of the  $A$  (with respect to  $U$ ), denoted by  $\bar{A}$  is the set  $U - A$

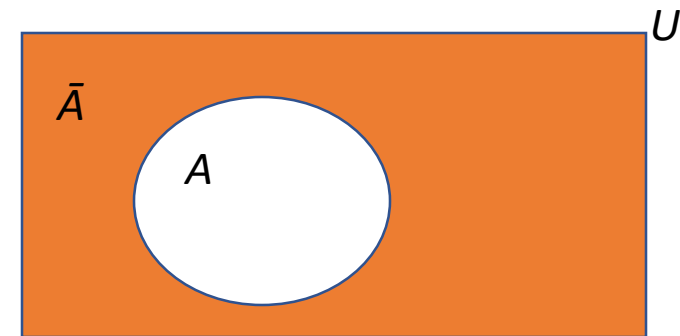
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of  $A$  is sometimes denoted by  $A^c$ .)

**Example:** If  $U$  is the positive integers less than 100, what is the complement of  $\{x \mid x > 70\}$

Solution:  $\{x \mid x \leq 70\}$

Venn Diagram for Complement

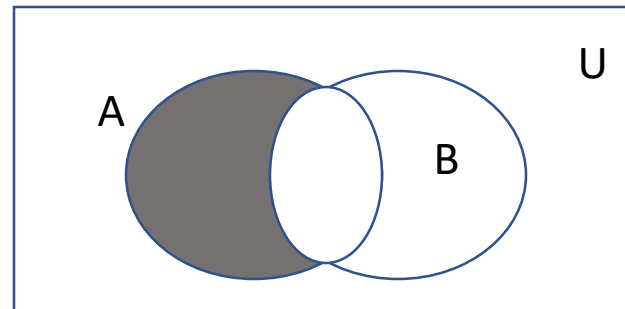


# Difference

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- **Definition:** Let  $A$  and  $B$  be sets. The *difference* of  $A$  and  $B$ , denoted by  $A - B$ , is the set containing the elements of  $A$  that are not in  $B$ . The difference of  $A$  and  $B$  is also called the complement of  $B$  with respect to  $A$ .

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$$

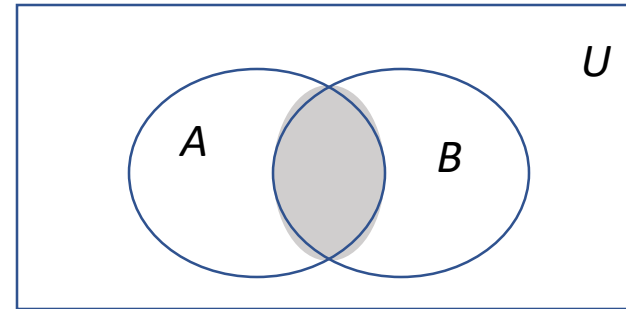


Venn Diagram for  $A - B$

# The Cardinality of the Union of Two Sets

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- Inclusion-Exclusion  
 $|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for  $A$ ,  $B$ ,  $A \cap B$ ,  $A \cup B$

- **Example:** Let  $A$  be the math majors in your class and  $B$  be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of  $n$  sets, where  $n$  is a positive integer.



# Symmetric Difference (*optional*)

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**Definition:** The *symmetric difference* of **A** and **B**, denoted by  $A \oplus B$  is the set

$$A \oplus B$$

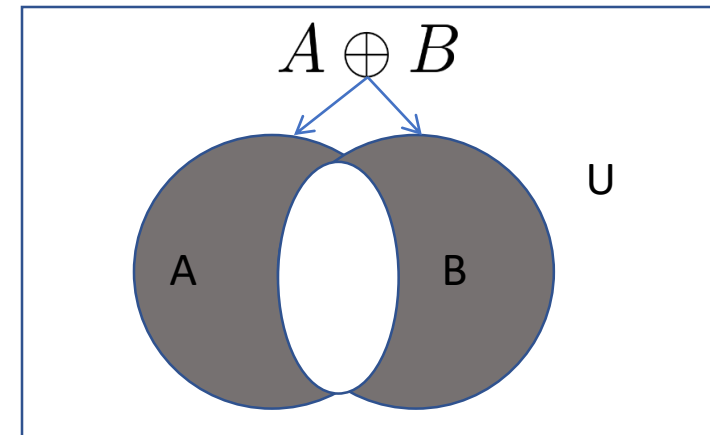
**Example:**

$$U = \{0,1,2,3,4,5,6,7,8,9,10\} \quad (A - B) \cup (B - A)$$

$$A = \{1,2,3,4,5\} \quad B = \{4,5,6,7,8\}$$

What is:

- **Solution:**  $\{1,2,3,6,7,8\}$



Venn Diagram

# Set Identities

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- Identity laws

$$A \cup \emptyset = A \quad A \cap U = A$$

- Domination laws

$$A \cup U = U \quad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \quad A \cap A = A$$

- Complementation law

$$\overline{\overline{A}} = A$$

*Continued on next slide →*

# Set Identities

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• Commutative laws  $A \cup B = B \cup A$        $A \cap B = B \cap A$

• Associative laws  $A \cup (B \cup C) = (A \cup B) \cup C$   
 $A \cap (B \cap C) = (A \cap B) \cap C$

• Distributive laws  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$   
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

*Continued on next slide →*

# Set Identities

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- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

# Proving Set Identities

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- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity.  
Use 1 to indicate it is in the set and a 0 to indicate that it is not.

# Proof of Second De Morgan Law

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**Example:** Prove that  $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**Solution:** We prove this identity by showing that:

Lemma 1      $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$      and

Lemma 2      $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

*Continued on next slide →*

# Proof of Second De Morgan Law

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Lemma 1:  $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

1.  $\forall x (\neg(x \in \overline{A \cap B}) \vee x \in (\bar{A} \cup \bar{B}))$  By def.
2.  $\forall x (\neg(x \in \overline{A \cap B}) \vee x \in \bar{A} \vee x \in \bar{B})$  By def.  $\cup$
3.  $\forall x ((x \in A \wedge x \in B) \vee x \in \bar{A} \vee x \in \bar{B})$  By def. comp.
4.  $\forall x ((x \in A \vee x \in \bar{A} \vee x \in \bar{B}) \wedge (x \in B \vee x \in \bar{A} \vee x \in \bar{B}))$  By distrib. law
5.  $\forall x \top$  By negation law

# Set-Builder Notation: Second De Morgan Law

$$\begin{aligned}
 \overline{A \cap B} &= \{x | x \notin A \cap B\} && \text{by defn. of complement} \\
 &= \{x | \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
 &= \{x | \neg(x \in A \wedge x \in B)\} && \text{by defn. of intersection} \\
 &= \{x | \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law} \\
 &&& \text{for Prop Logic} \\
 &= \{x | x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
 &= \{x | x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
 &= \{x | x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
 &= \overline{A} \cup \overline{B} && \text{by meaning of notation}
 \end{aligned}$$





# Generalized Unions and Intersections

- Let  $A_1, A_2, \dots, A_n$  be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n \qquad \bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

These are well defined, since union and intersection are associative.

- For  $i = 1, 2, \dots$ , let  $A_i = \{i, i + 1, i + 2, \dots\}$ . Then,

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{1, 2, 3, \dots\}$$

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{n, n + 1, n + 2, \dots\} = A_n$$

# Representing Sets in Computer

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- List of elements
- Sorted list of elements
- Bit string (bit vector)
  - For a universal set  $U=\{a_1, a_2, a_3, \dots, a_n\}$ , a set is represented by a vector  $v$  with  $n$  Boolean elements such that  $v$ 's  $i$ -th element is true iff  $a_i$  is a member of the set.
  - Example:  $U=\{1, 2, 3, 4, \dots, 10\}$ ,  $S_1=\{1, 3, 5, 7, 9\}$ ,  $S_2=\{1, 2, 3, 4, 5\}$   
 $v_{S_1} = \langle 1 0 1 0 1 0 1 0 1 0 \rangle$   
 $v_{S_2} = \langle 1 1 1 1 1 0 0 0 0 0 \rangle$