Generating Functions, and Distributions

1 Generating Functions

1.1 OGF (Ordinary Generating Function)

For a sequence $\{a_n\}_{n\geq 0}$:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < R.$$

Properties:

$$\begin{aligned}
&\text{Linearity: } \mathcal{G}\{c_1 a_n + c_2 b_n\} = c_1 A(x) + c_2 B(x).\\
&\text{Shift: } \sum_{n\geq 1} a_{n-1} x^n = xA(x).\\
&\text{Convolution: } A(x)B(x) \leftrightarrow c_n = \sum_{k=0}^{n} a_k b_{n-k}.
\end{aligned}$$

Example (formation & final): $a_n = 1 \Rightarrow A(x) = \dfrac{1}{1-x}$.

1.2 EGF (Exponential Generating Function)

For $\{a_n\}$:

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Product of EGFs corresponds to labelled combinatorial products. **Example:** $a_n = n! \Rightarrow E(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}$.

1.3 PGF (Probability Generating Function)

For integer-valued $X \geq 0$:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k\geq 0} P(X=k)s^k.$$

Useful identities:

$$G_X(1) = 1, \quad G'_X(1) = \mathbb{E}[X], \quad G''_X(1) = \mathbb{E}[X(X-1)].$$

Example: $X \sim \text{Poisson}(\lambda) \Rightarrow G_X(s) = \exp(\lambda(s-1))$.

1.4 MGF (Moment Generating Function) and CF (Characteristic Function)

$$M_X(t) = \mathbb{E}[e^{tX}], \qquad \varphi_X(t) = \mathbb{E}[e^{itX}].$$

Moments are obtained by derivatives at zero:

$$M_X^{(n)}(0) = \mathbb{E}[X^n], \quad \varphi_X^{(n)}(0) = i^n \mathbb{E}[X^n].$$

If MGFs exist in a neighborhood of 0, they uniquely determine distribution. Characteristic functions always exist.

2 Discrete Distributions

We present standard discrete distributions. For each: PMF, domain, PGF (if applicable), MGF, CF, mean, variance, short example.

2.1 Bernoulli(p)

PMF: $P(X = x) = p^x(1-p)^{1-x},\ x \in \{0, 1\}$.
PGF: $G(s) = 1 - p + ps$.
MGF: $M(t) = 1 - p + pe^t$.
CF: $\varphi(t) = 1 - p + pe^{it}$.
$\mathbb{E}[X] = p,\ \mathrm{Var}(X) = p(1-p)$.
Example: $p = 0.3 \Rightarrow P(X = 1) = 0.3$.

2.2 Binomial(n, p)

PMF: $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$, $k = 0, \dots, n$.
PGF: $G(s) = (1 - p + ps)^n$.
MGF: $M(t) = (1 - p + pe^t)^n$.
CF: $\varphi(t) = (1 - p + pe^{it})^n$.
$\mathbb{E}[X] = np,\ \mathrm{Var}(X) = np(1-p)$.
Example: $n = 5, p = 0.4 \Rightarrow P(X = 2) = \binom{5}{2}0.4^2 0.6^3$.

2.3 Multinomial($n, \mathbf{p}$)

PMF: $P(X_1 = k_1, \dots, X_r = k_r) = \dfrac{n!}{\prod k_i!} \prod p_i^{k_i}$ with $\sum k_i = n$.
MGF: $M(\mathbf{t}) = \left(\sum_{i=1}^r p_i e^{t_i}\right)^n$.
CF: $\varphi(\mathbf{t}) = \left(\sum p_i e^{it_i}\right)^n$.
Marginal $X_i \sim \mathrm{Binomial}(n, p_i)$.
Example: $r = 3, n = 4, \mathbf{p} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ probability of $(2, 1, 1)$: use PMF.

2.4 Geometric(p) (two parameterizations)

We use support $k = 1, 2, \dots$ (number of trials until first success).
PMF: $P(X = k) = p(1-p)^{k-1}$.
PGF: $G(s) = \dfrac{ps}{1 - (1-p)s}$ for $|s| < 1/(1-p)$.
MGF: $M(t) = \dfrac{pe^t}{1 - (1-p)e^t}$ for $t < -\ln(1-p)$.
CF: $\varphi(t) = \dfrac{pe^{it}}{1 - (1-p)e^{it}}$.
$\mathbb{E}[X] = 1/p,\ \mathrm{Var}(X) = (1-p)/p^2$.
Example: $p = 0.2 \Rightarrow P(X = 3) = 0.2(0.8)^2$.

2.5 Negative Binomial(r, p) (count of trials to achieve r successes)

PMF: $P(X = k) = \binom{k-1}{r-1}p^r(1-p)^{k-r},\ k = r, r+1, \dots$
PGF: $G(s) = \left(\dfrac{ps}{1 - (1-p)s}\right)^r$.
MGF: $M(t) = \left(\dfrac{pe^t}{1 - (1-p)e^t}\right)^r$.
CF: $\varphi(t) = \left(\dfrac{pe^{it}}{1 - (1-p)e^{it}}\right)^r$.
$\mathbb{E}[X] = \dfrac{r}{p},\ \mathrm{Var}(X) = \dfrac{r(1-p)}{p^2}$.
Example: $r = 2, p = 0.5 \Rightarrow P(X = 3) = \binom{2}{1}0.5^2 0.5$.

2.6 Poisson(λ)

PMF: $P(X=k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, $k \geq 0$.
PGF: $G(s) = \exp(\lambda(s-1))$.
MGF: $M(t) = \exp(\lambda(e^t - 1))$.
CF: $\varphi(t) = \exp(\lambda(e^{it} - 1))$.
$\mathbb{E}[X] = \lambda,\ \operatorname{Var}(X) = \lambda$.
Example: $\lambda = 3 \Rightarrow P(X=2) = e^{-3}3^2/2!$.

2.7 Hypergeometric(N, K, n)

PMF: $P(X=k) = \dfrac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$.
Support: $\max(0, n-N+K) \leq k \leq \min(K, n)$.
No simple PGF; use hypergeometric functions for generating functions.
$\mathbb{E}[X] = n\frac{K}{N},\ \operatorname{Var}(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$.
Example: $N = 20, K = 5, n = 4, k = 2 \Rightarrow$ use PMF.

2.8 Multivariate Hypergeometric

Omitted detailed PGF; marginals hypergeometric. Example analogous to urn draws.

3 Continuous Distributions

3.1 Uniform(a, b)

PDF: $f(x) = \dfrac{1}{b-a}$ for $a \leq x \leq b$.
MGF: $M(t) = \dfrac{e^{bt} - e^{at}}{t(b-a)}$ for $t \neq 0$.
CF: $\varphi(t) = \dfrac{e^{ibt} - e^{iat}}{it(b-a)}$.
$\mathbb{E}[X] = \dfrac{a+b}{2},\ \operatorname{Var}(X) = \dfrac{(b-a)^2}{12}$.
Example: $U(0,1)$ gives $P(0.2 < X < 0.4) = 0.2$.

3.2 Exponential(λ)

PDF: $f(x) = \lambda e^{-\lambda x},\ x \geq 0$.
MGF: $M(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$.
CF: $\varphi(t) = \dfrac{\lambda}{\lambda - it}$.
$\mathbb{E}[X] = 1/\lambda,\ \operatorname{Var}(X) = 1/\lambda^2$.
Example: $\lambda = 1 \Rightarrow P(X > 2) = e^{-2}$.

3.3 Gamma(α, β) (shape α, scale β)

PDF: $f(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ x > 0$.
MGF: $M(t) = (1 - \beta t)^{-\alpha}$ for $t < 1/\beta$.
CF: $\varphi(t) = (1 - i\beta t)^{-\alpha}$.

$\mathbb{E}[X] = \alpha\beta,\ \mathrm{Var}(X) = \alpha\beta^2$.
Example: Erlang is integer α special case; $\alpha = 2, \beta = 1 \Rightarrow \mathbb{E} = 2$.

3.4 Chi-square(ν)

PDF: special Gamma with $\alpha = \nu/2,\ \beta = 2$.
MGF: $M(t) = (1-2t)^{-\nu/2}$ for $t < 1/2$.
CF: $\varphi(t) = (1-2it)^{-\nu/2}$.
$\mathbb{E}[X] = \nu,\ \mathrm{Var}(X) = 2\nu$.
Example: $\nu = 3 \Rightarrow M(t) = (1-2t)^{-3/2}$.

3.5 Normal(μ, σ^2)

PDF: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$.
MGF: $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
CF: $\varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.
$\mathbb{E}[X] = \mu,\ \mathrm{Var}(X) = \sigma^2$.
Example: $Z \sim N(0,1) \Rightarrow \varphi_Z(t) = e^{-t^2/2}$.

3.6 Lognormal(μ, σ^2)

If $Y \sim N(\mu, \sigma^2)$ and $X = e^Y$. PDF is $f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$ for $x > 0$.
MGF does not exist (diverges). CF given via characteristic function of log; no closed-form simple.
$\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2},\ \mathrm{Var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.
Example: $\mu = 0, \sigma^2 = 1 \Rightarrow \mathbb{E} = e^{1/2}$.

3.7 Weibull(k, λ)

PDF: $f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k},\ x \geq 0$.
MGF: no simple closed form for general k. Moments: $\mathbb{E}[X] = \lambda\Gamma(1 + 1/k)$.
Example: $k = 1$ reduces to Exponential($1/\lambda$).

3.8 Pareto(x_m, α)

PDF: $f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}},\ x \geq x_m > 0$.
Moments exist for $\alpha > 1$ (mean) and $\alpha > 2$ (variance):

$$\mathbb{E}[X] = \frac{\alpha x_m}{\alpha - 1}, \quad \mathrm{Var}(X) = \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}.$$

Example: heavy-tailed modeling.

3.9 Cauchy(x_0, γ)

PDF: $f(x) = \frac{1}{\pi\gamma} \frac{\gamma^2}{(x-x_0)^2 + \gamma^2}$.
MGF does not exist. CF: $\varphi(t) = e^{ix_0 t - \gamma|t|}$.
Mean and variance undefined.
Example: standard Cauchy has $x_0 = 0, \gamma = 1$.

3.10 Student's t with ν d.f.

PDF: $f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}$. Mean 0 for $\nu > 1$, variance $\nu/(\nu-2)$ for $\nu > 2$.
CF in terms of Bessel functions.
Example: $\nu = 3$ finite mean 0, variance 3.

3.11 F-distribution(d_1, d_2)

PDF: $f(x) = \frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1+d_2}}}}{xB\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$ for $x > 0$. Mean $d_2/(d_2-2)$ if $d_2 > 2$. Variance finite if $d_2 > 4$.
Derived from ratio of scaled chi-squares.

4 Connections and Theorems (MGF/PGF use)

4.1 Additivity and Convolution

If X and Y independent,

$$M_{X+Y}(t) = M_X(t)M_Y(t), \qquad G_{X+Y}(s) = G_X(s)G_Y(s).$$

Example: Poisson(λ_1)+Poisson(λ_2) = Poisson$(\lambda_1 + \lambda_2)$.

4.2 Moment extraction

For MGF $M(t)$,

$$\mathbb{E}[X] = M'(0), \quad \mathbb{E}[X^2] = M''(0), \quad \mathrm{Var}(X) = M''(0) - [M'(0)]^2.$$

Example: $M(t) = (1-2t)^{-3} \Rightarrow \mathbb{E} = 6,\ \mathrm{Var} = 12$.

4.3 Characteristic functions and inversion

Characteristic functions $\varphi_X(t)$ uniquely identify distributions. Inversion formulas exist (not reproduced here).

5 Summary Table (PMF/PDF, PGF, MGF, CF, Mean, Variance)

Distribution	PMF / PDF (domain)	PGF	MGF (domain)	Characteristic function	Mean	Variance
Bernoulli(p)	$P(X=x)=p^x(1-p)^{1-x},\ x\in\{0,1\}$	$1-p+ps$	$1-p+pe^t$	$1-p+pe^{it}$	p	$p(1-p)$
Binomial(n,p)	$\binom{n}{k}p^k(1-p)^{n-k},\ k=0,\dots,n$	$(1-p+ps)^n$	$(1-p+pe^t)^n$	$(1-p+pe^{it})^n$	np	$np(1-p)$
Multinomial$(n,\mathbf{p})$	$\frac{n!}{\prod k_i!}\prod p_i^{k_i}$	— (vector PGF: $(\sum p_i s_i)^n$)	$M(\mathbf{t})=(\sum p_i e^{t_i})^n$	$\varphi(\mathbf{t})=\left(\sum p_i e^{it_i}\right)^n$	$m_i = np_i$ (marginal)	$np_i(1-p_i)$ (marginal)
Geometric(p) $(k\geq 1)$	$p(1-p)^{k-1}$	$\frac{ps}{1-(1-p)s}$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$	$1/p$	$(1-p)/p^2$
NegativeBinomial(r,p)	$\binom{k-1}{r-1}p^r(1-p)^{k-r},\ k\geq r$	$\left(\frac{ps}{1-(1-p)s}\right)^r$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$	$\left(\frac{pe^{it}}{1-(1-p)e^{it}}\right)^r$	r/p	$r(1-p)/p^2$
Poisson(λ)	$e^{-\lambda}\lambda^k/k!$	$e^{\lambda(s-1)}$	$e^{\lambda(e^t-1)}$	$e^{\lambda(e^{it}-1)}$	λ	λ
Hypergeometric(N,K,n)	$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$	—	—	—	$n\frac{K}{N}$	$n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$
Uniform(a,b)	$\frac{1}{b-a}$ on $[a,b]$	—	$\frac{e^{bt}-e^{at}}{t(b-a)}$	$\frac{e^{ibt}-e^{iat}}{it(b-a)}$	$(a+b)/2$	$(b-a)^2/12$
Exponential(λ)	$\lambda e^{-\lambda x},\ x\geq 0$	—	$\frac{\lambda}{\lambda-t},\ t<\lambda$	$\frac{\lambda}{\lambda-it}$	$1/\lambda$	$1/\lambda^2$
Gamma(α,β)	$\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha},\ x>0$	—	$(1-\beta t)^{-\alpha},\ t<1/\beta$	$(1-i\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
Chi-square(ν)	Gamma special: $\alpha=\nu/2,\beta=2$	—	$(1-2t)^{-\nu/2},\ t<1/2$	$(1-2it)^{-\nu/2}$	ν	2ν
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/(2\sigma^2)}$	—	$e^{\mu t+\frac{1}{2}\sigma^2t^2},\ \forall t$	$e^{i\mu t-\frac{1}{2}\sigma^2t^2}$	μ	σ^2
Lognormal(μ,σ^2)	$\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\ln x-\mu)^2/(2\sigma^2)},\ x>0$	—	MGF does not exist (diverges)	CF via integral	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$
Weibull(k,λ)	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}$	—	no simple closed MGF in general	CF via transform	$\lambda\Gamma(1+1/k)$	$\lambda^2\left[\Gamma(1+2/k)-\Gamma(1+1/k)^2\right]$
Pareto(x_m,α)	$\frac{\alpha x_m^\alpha}{x^{\alpha+1}},\ x\geq x_m$	—	MGF does not exist for many α	CF integral form	$\frac{\alpha x_m}{\alpha-1}$ $(\alpha>1)$	$\frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$ $(\alpha>2)$
Cauchy(x_0,γ)	$\frac{1}{\pi\gamma}\frac{\gamma^2}{(x-x_0)^2+\gamma^2}$	—	MGF undefined	$e^{ix_0t-\gamma\|t\|}$	undefined	undefined

Distribution	**PMF / PDF (domain)**	**PGF**	**MGF (domain)**	**Characteristic function**	**Mean**	**Variance**
Student's $t(\nu)$	see text	—	MGF undefined in general	CF expressed with Bessel functions	0 (if $\nu > 1$)	$\frac{\nu}{\nu-2}$ (if $\nu > 2$)
$F(d_1, d_2)$	see text	—	MGF not elementary	CF not elementary	$\frac{d_2}{d_2-2}$ $(d_2 > 2)$	$\frac{2d_2^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)}$ (if $d_2 > 4$)
Erlang(k, λ)	Gamma with integer shape k	—	$\left(\frac{\lambda}{\lambda-t}\right)^k$	$(\frac{\lambda}{\lambda-it})^k$	k/λ	k/λ^2

Table 1: Expanded distribution summary. PGF shown only where meaningful (discrete). Some MGFs/CFs omitted when no simple closed form exists.

6 Examples

6.1 Using PGF: Compound Poisson

If $N \sim \text{Poisson}(\lambda)$ and $X_i \overset{iid}{\sim} \text{Bernoulli}(q)$, $S = \sum_{i=1}^{N} X_i$. Formulation:

$$G_S(s) = \mathbb{E}[G_{S|N}(s)] = \sum_{n\geq 0} e^{-\lambda}\frac{\lambda^n}{n!}(1-q+qs)^n = e^{\lambda((1-q+qs)-1)} = e^{\lambda q(s-1)}.$$

Conclusion: $S \sim \text{Poisson}(\lambda q)$.

6.2 Using MGF: Sum of independent normals

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ independent. Then

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \exp((\mu_1+\mu_2)t + \tfrac{1}{2}(\sigma_1^2+\sigma_2^2)t^2).$$

Conclusion: $X + Y \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$.

6.3 Using characteristic functions: Cauchy heavy-tail

If X_i iid standard Cauchy, $\varphi_X(t) = e^{-|t|}$. Sum of independent Cauchy remains Cauchy (stable) with scale parameters adding; formation via product of CFs.