Practice Problem Set

I. Discrete Distributions (Bernoulli, Binomial, Geo, Poisson, Hypergeo)

Q1. **Bernoulli/Binomial (Simple):** A fair coin is flipped 5 times. What is the probability of getting exactly 3 heads?

Q2. **Geometric (Simple):** A basketball player makes 40% of their free throws. What is the probability that their first made shot occurs on their 4th attempt?

Q3. **Poisson (Simple):** The average number of emails a professor receives during the night is 6. What is the probability they receive exactly 4 emails tonight?

Q4. **Hypergeometric (Simple):** An urn contains 8 red balls and 4 blue balls. If 3 balls are drawn without replacement, what is the probability that exactly 2 are red?

Q5. **Binomial Moments:** For $X \sim B(10, 0.2)$, find the mean $\mathbb{E}[X]$ and the variance $Var(X)$.

Q6. **Geometric Expectation:** The probability of a successful login to a server is $p = 0.85$. What is the expected number of attempts until the first successful login?

Q7. **Poisson Rate Adjustment:** Calls arrive at a call center at an average rate of 15 per hour. What is the probability of receiving exactly 1 call in a 5-minute interval?

Q8. **Bernoulli PMF:** Write the Probability Mass Function (PMF) for $X \sim Bernoulli(0.7)$.

Q9. **Binomial Cumulative:** In a batch of 50 products, 5% are defective. What is the probability that at most 2 products are defective?

Q10. **Geometric Conditional (Memoryless):** The probability of a component failing on any given day is 0.05. Given the component has lasted 10 days, what is the probability it lasts an additional 3 days?

Q11. **Poisson vs. Binomial:** If a company has a 0.01 chance of being audited each year, what is the Poisson approximation for the probability that a company is audited exactly once over 100 years?

Q12. **Hypergeometric Range:** For a population of $N = 20$, $K = 5$ successes, and a sample $n = 6$, state the possible range of values for the Hypergeometric RV X.

Q13. **Hypergeometric Expected Value:** A deck of 52 cards is shuffled. If 5 cards are drawn without replacement, what is the expected number of Aces?

Q14. **Binomial Variance:** If $X \sim B(n, p)$ has a mean of 10 and a standard deviation of 3, find the values of n and p.

Q15. **Poisson Difference (Independent):** If $X \sim Poisson(2)$ and $Y \sim Poisson(3)$ are independent, find $\mathbb{E}[X - Y]$.

Q16. **Geometric Sum:** Let $X_1, X_2, \ldots, X_{10}$ be i.i.d. Geometric RVs with $p = 0.2$. Find the expected value of the sum $S = \sum_{i=1}^{10} X_i$.

Q17. **Combined Discrete I (Independent):** A machine has two independent failure modes: Type A (Binomial, $n = 5, p = 0.1$) and Type B (Poisson, $\lambda = 1$). Find the probability that there is exactly 1 failure of Type A and 0 failures of Type B.

Q18. **Hypergeometric FPCF:** A population has $N = 1000$ with $K = 100$ successes. A sample of $n = 20$ is drawn. Calculate the Finite Population Correction Factor (FPCF).

Q19. **Binomial $\mathbf{P(X \geq n - 1)}$:** For $X \sim B(6, 0.4)$, calculate $P(X \geq 5)$.

Q20. **Finding λ from $P(X = 0)$:** A Poisson RV X satisfies $P(X = 0) = 0.1353$. Find the rate parameter λ.

Q21. **Custom PMF Mean:** A discrete RV X has PMF $p(x) = c(x^2 + 1)$ for $x = 0, 1, 2$. Find the constant c and the mean $\mathbb{E}[X]$.

Q22. **Custom PMF Variance:** Using the results from the previous problem, calculate $Var(X)$.

Q23. **Poisson $\mathbf{P(X > \mu)}$:** If $X \sim Poisson(4)$, calculate $P(X > 4)$.

Q24. **Geometric $\mathbf{P(X \leq 2)}$:** A process has $p = 0.6$ for success. Calculate the probability that the first success occurs on or before the 2nd trial.

Q25. **Hypergeometric Inequality:** In a group of 12 students, 7 are male. If 5 students are randomly selected, what is the probability that at least 3 are male?

Q26. **Binomial Mode:** For $X \sim B(10, 0.5)$, what is the mode (the most likely value) of X?

Q27. **Poisson Conditional:** Given $X \sim Poisson(3)$, what is $P(X = 2|X \leq 2)$?

Q28. **Geometric $P(X = x + 1)$ relation:** If $X \sim Geo(p)$, what is the ratio $\frac{P(X=x+1)}{P(X=x)}$?

Q29. **Hypergeometric $\mathbf{P(X = 0)}$:** Calculate $P(X = 0)$ for $N = 10, K = 3, n = 4$.

Q30. **Binomial Trial Count:** If $p = 0.2$ and the expected number of successes is $\mathbb{E}[X] = 8$, how many trials n are in the experiment?

Q31. **Negative Binomial to Geometric:** A sequence of Bernoulli trials is run with $p = 0.4$. Let X be the number of trials until the 5th success. Find $\mathbb{E}[X]$.

Q32. **Conditional Binomial Mean:** If $X \sim B(10, 0.5)$, find the conditional expectation $\mathbb{E}[X|X \geq 9]$.

Q33. **Poisson Time Interval:** Customers arrive at a store following a Poisson process at a rate of 10 per hour. What is the probability that no customers arrive in a 15-minute period?

Q34. **Joint Discrete Marginal:** A joint PMF is given by $P(X = x, Y = y) = c(x + y)$ for $x = 1, 2$ and $y = 1$. Find the marginal PMF for X, $P_X(x)$.

Q35. **Compound Discrete Problem:** The number of defects N in a product follows a Poisson distribution with $\lambda = 2$. If each defect has a probability $p = 0.1$ of being critical, independently of others, what is the expected number of critical defects?

II. Continuous Distributions (Uniform, Exponential, Normal)

Q36. **Uniform Probability:** If $X \sim U(5, 15)$, find $P(X > 12)$.

Q37. **Exponential Mean/Rate:** An Exponential RV has a mean $\mathbb{E}[X] = 4$. What is the rate parameter λ?

Q38. **Uniform PDF:** Write the PDF $f(x)$ for a Uniform distribution on $[-2, 4]$.

Q39. **Exponential Survival:** If $X \sim Exp(0.2)$, calculate the probability that X is greater than 5.

Q40. **Normal Z-score:** For $X \sim N(100, 16)$, find the Z-score corresponding to $X = 104$.

Q41. **Normal Empirical Rule:** For $X \sim N(50, 25)$, what percentage of values fall between 40 and 60?

Q42. **Exponential CDF:** Write the Cumulative Distribution Function (CDF) $F(x)$ for $X \sim Exp(3)$.

Q43. **Uniform Variance:** Calculate the variance $Var(X)$ for $X \sim U(-1, 9)$.

Q44. **Normal Standardization:** Use Z-scores to express $P(X < 65)$ for $X \sim N(70, 100)$.

Q45. **Exponential Probability Interval:** For $X \sim Exp(0.1)$, find $P(5 < X < 10)$.

Q46. **Uniform Density Integration:** A CRV X has PDF $f(x) = cx$ for $0 \leq x \leq 2$. Find c.

Q47. **Normal Percentile (Inverse Z):** A score is Normally distributed with $\mu = 60$ and $\sigma = 8$. If a student's score is at the 97.72nd percentile (use $Z = 2.0$), what is their score?

Q48. **Exponential Memoryless Property:** The lifetime of a device is $Exp(\lambda)$. If $P(X > 10) = 0.3$, what is $P(X > 20|X > 10)$?

Q49. **Uniform $\mathbf{E[X^2]}$:** Find $\mathbb{E}[X^2]$ for $X \sim U(0, 5)$.

Q50. **Custom PDF Expectation:** A CRV X has PDF $f(x) = \frac{1}{2}e^{-x/2}$ for $x \geq 0$. Find the expected value $\mathbb{E}[X]$.

Q51. **Normal Combined Intervals:** For $Z \sim N(0, 1)$, calculate $P(Z < -1.5 \text{ or } Z > 2.5)$.

Q52. **Custom PDF CDF:** Find the CDF $F(x)$ for the PDF $f(x) = 2x$ for $0 \leq x \leq 1$.

Q53. **Exponential Mean Calculation:** If $P(X > 1) = 0.5$ for $X \sim Exp(\lambda)$, find the mean $\mathbb{E}[X]$.

Q54. **Uniform Conditional Probability:** For $X \sim U(0, 10)$, calculate $P(X > 7|X > 3)$.

Q55. **Exponential Quantile:** For $X \sim Exp(\lambda = 0.5)$, find the value x_0 such that $P(X \leq x_0) = 0.95$.

Q56. **Median of Custom PDF:** A CRV has PDF $f(x) = 3x^2$ for $0 \leq x \leq 1$. Find the median of the distribution.

Q57. **Normal and Uniform Combined:** If $X \sim N(5, 1)$ and $Y \sim U(0, 1)$ are independent, find $\mathbb{E}[X + Y]$ and $Var(X + Y)$.

Q58. **Transformation of Variables Var($\mathbf{X^2}$):** If $X \sim U(0, 1)$, find the variance of the transformed variable $Y = X^2$.

Q59. **Finding PDF from CDF:** Given the CDF $F(x) = 1 - e^{-(x/5)^2}$ for $x \geq 0$, find the corresponding PDF $f(x)$.

Q60. **Conditional Exponential Density:** The time until failure T is $Exp(\lambda)$. Find the conditional density function $f(t|T > t_0)$.

III. Normal Approximation, CLT, and Complex Combined Theory

Q61. **Binomial Approximation Check:** For $X \sim B(50, 0.05)$, should the Normal approximation be used? Explain why or why not using the conditions.

Q62. **Binomial Approximation Mean/SD:** For $X \sim B(100, 0.64)$, find the mean and standard deviation of the approximating Normal distribution.

Q63. **Continuity Correction (Single Point):** Approximate $P(X = 20)$ for $X \sim B(50, 0.5)$ using the Continuity Correction Factor (CCF).

Q64. **Continuity Correction (Interval):** Approximate $P(10 \leq X \leq 25)$ for a Binomial distribution using the CCF.

Q65. **Binomial Approximation Calculation:** A survey of 200 people finds $p = 0.4$ favor a product. Use the Normal approximation (with $\mu = 80, \sigma = \sqrt{48} \approx 6.93$) to estimate the probability that less than 75 people favor the product.

Q66. **CLT Standard Error:** A population has $\mu = 20$ and $\sigma = 4$. If a sample of $n = 64$ is taken, calculate the standard error of the sample mean $\overline{X}$.

Q67. **CLT Z-score:** For the sample mean $\overline{X}$ from the previous problem, find the Z-score for $\overline{X} = 20.5$.

Q68. **CLT Probability:** The average amount spent on lunch is $\mu = \$12$ with $\sigma = \$3$. For a random sample of $n = 36$ employees, what is the probability that the sample mean spending $\overline{X}$ is less than \$11.50?

Q69. **CLT Minimum n:** A population has $\sigma = 10$. What is the minimum sample size n required for the standard error of the mean to be no more than 1.0?

Q70. **CLT and Non-Normal Pop:** If the population is highly skewed, what sample size n is generally considered large enough to apply the CLT?

Q71. **CLT Interval:** For a sample of $n = 49$ from a population with $\mu = 50, \sigma = 7$, find the probability that the sample mean is between 48 and 52.

Q72. **Binomial/Geometric Relation:** Let X be the number of successes in 10 Bernoulli trials with $p = 0.1$. Let Y be the number of trials until the first success. Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Q73. **Poisson/Exponential Relation:** A Poisson process has a rate $\lambda = 2$ events per hour. What is the expected time (in hours) between events?

Q74. **Uniform/Exponential Comparison:** Compare the expected value and variance of $X \sim U(0, 2)$ and $Y \sim Exp(1)$.

Q75. **CLT $\overline{\mathbf{X}}$ vs $\sum X_i$:** If $X_i \sim N(5, 4)$ are i.i.d., what is the distribution of the sum $S = \sum_{i=1}^{16} X_i$?

Q76. **Normal Approximation Boundary:** If $X \sim B(n, p)$ is approximated by $Y \sim N(\mu, \sigma^2)$, what is the value of $P(X > x)$ using the CCF?

Q77. **CLT Sum of Uniforms:** Let $X_1, \ldots, X_{100}$ be i.i.d. $U(0, 1)$ random variables. Use the CLT to estimate $P(\sum_{i=1}^{100} X_i < 45)$. (Recall $\mu = 0.5, \sigma^2 = 1/12$ for $U(0, 1)$).

Q78. **CLT Two-Sample Mean (Concept):** Two independent samples of size n_1 and n_2 are taken. Write the mean and variance of the difference between the sample means $\overline{X}_1 - \overline{X}_2$.

Q79. **Poisson and Large Numbers:** If $X \sim Poisson(\lambda = 100)$, use the Normal approximation to estimate $P(90 < X < 110)$.

Q80. **Exponential/Poisson Connection (Advanced):** Let T_1 be the time until the first event in a Poisson process with rate $\lambda = 0.5$. Find $P(T_1 \leq 3)$.

Q81. **Normal Distribution of Sample Variance (Concept):** If X_i are i.i.d. $N(\mu, \sigma^2)$, what distribution is related to the sample variance S^2? (Conceptual hint: $\sum(X_i - \overline{X})^2/\sigma^2$).

Q82. **Geometric and Sum of Exponentials:** Let $X_1, X_2, \ldots, X_k$ be i.i.d. Exponential RVs. What distribution does their sum $\sum X_i$ follow?

Q83. **CLT Sample Size Calculation (Probability):** The height of a certain population has $\sigma = 4$ cm. Find the minimum sample size n required so that the probability of the sample mean $\overline{X}$ being within 1 cm of the population mean μ is at least 0.95.

Q84. **Combined Discrete/Continuous (Advanced):** The number of cars passing a point in an hour is $X \sim Poisson(5)$. The speed of each car Y_i is $U(40, 60)$. What is the expected total speed of all cars that pass in an hour, $\mathbb{E}[\sum_{i=1}^{X} Y_i]$?

Q85. **Law of Large Numbers (LLN) Concept:** Explain the difference between the Central Limit Theorem and the Weak Law of Large Numbers (WLLN) in terms of what they describe about the sample mean $\overline{X}$.

IV. MGFs, Chebyshev's Inequality, and Joint Distributions

Q86. **MGF Definition (CRV):** Write the general integral definition of the MGF $M_X(t)$ for a Continuous Random Variable (CRV) X with PDF $f(x)$.

Q87. **MGF Bernoulli Moments:** A Bernoulli RV has MGF $M_X(t) = 0.3e^t + 0.7$. Find the mean $\mathbb{E}[X]$ using the MGF.

Q88. **MGF Derivation:** Given $M_X(t) = (1 - 2t)^{-1}$, find the second moment $\mathbb{E}[X^2]$.

Q89. **MGF of Sum (Independence):** If X and Y are independent with $M_X(t)$ and $M_Y(t)$, write the MGF for $Z = X + Y$.

Q90. **MGF Transformation:** If $X \sim Poisson(\lambda = 3)$, which has $M_X(t) = e^{3(e^t - 1)}$, find the MGF of $Y = 2X + 1$.

Q91. **MGF Identification:** A RV X has MGF $M_X(t) = \frac{0.8}{1 - 0.2e^t}$. What distribution does X follow?

Q92. **MGF for Uniform $U(a, b)$:** The MGF for $U(a, b)$ is $\frac{e^{bt} - e^{at}}{t(b-a)}$. Using this, what is the MGF of $U(0, 1)$?

Q93. **MGF and Variance:** If $M'_X(0) = 5$ and $M''_X(0) = 30$, calculate the variance $Var(X)$.

Q94. **MGF of $Z = X - Y$:** If X and Y are independent, write the MGF for $Z = X - Y$.

Q95. **MGF Uniform $U(0, 1)$ First Moment Check:** Use the MGF of $U(0, 1)$ from Q92 and L'Hôpital's Rule to confirm $\mathbb{E}[X] = 1/2$.

Q96. **MGF of Sample Mean:** If $X_i \sim Exp(\lambda)$ are i.i.d., write the MGF for the sample mean $\overline{X}_n$.

Q97. **Chebyshev's Lower Bound:** For any RV with $\mu = 10$ and $\sigma = 2$, find the minimum probability that X is within the interval $[4, 16]$.

Q98. **Chebyshev's k Value:** If $\mu = 50$ and $\sigma = 5$, what value of k must be used to find a lower bound for $P(40 < X < 60)$?

Q99. **Chebyshev's Upper Bound:** A stock's average daily return is $\mu = 0.01$ with $\sigma = 0.03$. What is the maximum probability that the return is less than -0.05 or greater than 0.07?

Q100. **Chebyshev vs. Normal:** For $X \sim N(0, 1)$, what is the actual probability that X is within 2 standard deviations of the mean, and how does it compare to the Chebyshev lower bound?

Q101. **Chebyshev's $\mathbf{1/k^2}$:** Find the value of k such that Chebyshev's inequality guarantees $P(|X - \mu| \geq k\sigma) \leq 0.04$.

Q102. **Chebyshev and Sample Mean:** A population has $\sigma = 5$. Use Chebyshev's inequality to find the minimum sample size n such that $P(|\overline{X} - \mu| < 1) \geq 0.95$.

Q103. **Chebyshev vs. Binomial:** For $X \sim B(100, 0.5)$, use Chebyshev's inequality to find a lower bound for $P(40 \leq X \leq 60)$.

Q104. **Independence Condition (Discrete):** State the condition required for two discrete RVs X and Y to be statistically independent.

Q105. **Independence $\mathbb{E}[XY]$:** If X and Y are independent with $\mathbb{E}[X] = 3$ and $\mathbb{E}[Y] = 5$, find $\mathbb{E}[XY]$.

Q106. **Independence and Covariance:** If $Cov(X, Y) = 0$, does this imply X and Y are independent? (Conceptual question)

Q107. **Variance of Sum (Independent):** If X and Y are independent, $Var(X) = 4$, and $Var(Y) = 9$, calculate $Var(2X - Y)$.

Q108. **Joint Discrete Independence Test I:** A joint PMF has $f(1, 1) = 0.1$, $f_X(1) = 0.3$, $f_Y(1) = 0.5$. Are X and Y independent?

Q109. **Joint Continuous Independence Test II:** The joint PDF is $f(x, y) = c(x+y)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Find the marginal PDF $f_X(x)$ and determine if X and Y are independent.

Q110. **Joint Discrete Expectation:** A joint PMF has $\mathbb{E}[X] = 1, \mathbb{E}[Y] = 2$, and $\mathbb{E}[XY] = 3$. Calculate $Cov(X, Y)$.

Q111. **Expected Value of Linear Combination:** If $\mathbb{E}[X] = 10, \mathbb{E}[Y] = -2$, find $\mathbb{E}[3X - 5Y + 7]$.

Q112. **Variance of Linear Combination (Non-Independent):** Given $Var(X) = 5, Var(Y) = 2$, and $Cov(X, Y) = -1$. Find $Var(X + Y)$.

Q113. **Covariance Calculation (Uniform):** If $X \sim U(0, 1)$ and $Y = X^2$, calculate $Cov(X, Y)$.

Q114. **Conditional Expectation (Joint Discrete):** For the joint PMF given by $P(X = 1, Y = 1) = 0.2$, $P(X = 1, Y = 2) = 0.3$, $P(X = 2, Y = 1) = 0.1$, $P(X = 2, Y = 2) = 0.4$. Find the conditional expectation $\mathbb{E}[Y|X = 1]$.

Q115. **Joint Probability (Region):** A joint PDF is $f(x, y) = 6x$ for $0 \leq x \leq 1, 0 \leq y \leq 1-x$. Find $P(X+Y \leq 0.5)$.