Problem Set

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Part I: Fundamentals, Classification, and Basic First-Order ODEs (60 Problems)

A. Introduction, Classification, and Derivation (Q1 - Q15)

Q1: [S]Determine the order and degree of the differential equation: $\left(\frac{d^3y}{dx^3}\right)^2 + x\left(\frac{dy}{dx}\right)^4 - y = 0$.

Q2: [S]Classify the following DE as Ordinary or Partial, and Linear or Non-linear: $\frac{d^2y}{dx^2} + x^2y = \cos(x)$.

Q3: [S]Verify that $y = Ce^{-2x}$ is a solution to the DE $\frac{dy}{dx} + 2y = 0$.

Q4: [S]Determine the order and degree of the DE: $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$. (Hint: Rationalize)

Q5: [S]Check if the function $y = x^2$ is a solution to $x\frac{dy}{dx} - 2y = 0$.

Q6: [S]Derive the DE that has the general solution $y = Cx^2$. (Eliminate one constant).

Q7: [I]Derive the DE for the family of straight lines $y = mx + b$, where m and b are arbitrary constants.

Q8: [I]Determine the interval on which the solution to the Initial Value Problem (IVP) $\frac{dy}{dx} = y^{1/3}$, $y(0) = 0$ exists and is unique.

Q9: [I]Show that the DE $y' = x^2y - y^2$ is Non-linear and state its order.

Q10: [I]Derive the DE for the family of circles centered at the origin, $x^2 + y^2 = C^2$.

Q11: [I]Given the solution $y = c_1\sin(x) + c_2\cos(x)$, derive the corresponding second-order linear homogeneous DE.

Q12: [C]Derive the third-order DE from the general solution $y = C_1e^x + C_2e^{2x} + C_3e^{3x}$.

Q13: [C]Find the differential equation for the family of all parabolas with axis parallel to the y-axis.

Q14: [C]Given the DE $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$. If a solution passes through $(1, 0)$, can you find the particular solution implicitly? Justify your steps.

Q15: [C]Provide a DE that is both first-order and non-linear, but where the substitution $v = y/x$ makes it separable.

B. Variable Separable Equations (Q16 - Q30)

Q16: [S]Solve: $\frac{dy}{dx} = \frac{x^2}{y^3}$.

Q17: [S]Find the general solution to $\frac{dy}{dx} = e^{3x-2y}$.

Q18: [S]Solve the IVP: $x\frac{dy}{dx} = y$, with $y(1) = 3$.

Q19: [S]Solve: $\frac{dy}{dx} = y\sin(x)$.

Q20: [S]Find the general solution to $y' = 1 + x^2 + y^2 + x^2y^2$.

Q21: [S]Solve: $\frac{dy}{dx} = \frac{x}{y}\sqrt{1-x^2}$.

Q22: [I]Solve the IVP: $\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2}$, with $y(0) = \pi/4$.

Q23: [I]Find the implicit solution to $(1+e^x)\frac{dy}{dx} = e^x \sin(y)$.

Q24: [I]Solve: $y\ln(x)\frac{dx}{dy} = \left(\frac{y^2+1}{y}\right)$.

Q25: [I]A population P grows at a rate proportional to its current size. Formulate the DE and find the general solution.

Q26: [I]Find the solution to $\frac{dy}{dx} = \frac{xy^2+x}{yx^2+y}$.

Q27: [C]Solve the IVP: $\frac{dy}{dx} = \frac{xy+y}{xy-x}$, with $y(1) = 1$. Carefully consider the singularity.

Q28: [C]Find the solution to $\frac{dy}{dx} = e^{x+y} + e^{x-y}$ and express y explicitly as a function of x.

Q29: [C]An object at $100°C$ cools in a $30°C$ room. The cooling rate is proportional to the temperature difference. Formulate the DE and find the solution for the temperature $T(t)$.

Q30: [C]Solve the DE $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ using the substitution $v = x + y$.

C. Homogeneous Equations (Q31 - Q45)

Q31: [S]Determine if the function $f(x,y) = x^2 + 2xy$ is homogeneous. If so, state its degree.

Q32: [S]Solve: $\frac{dy}{dx} = \frac{x+y}{x}$.

Q33: [S]Find the general solution to $x\frac{dy}{dx} = y - xe^{y/x}$.

Q34: [S]Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$.

Q35: [S]Find the solution to the IVP: $(x^2+y^2)dx + 2xydy = 0$, with $y(1) = 1$.

Q36: [S]Solve the homogeneous equation: $\frac{dy}{dx} = \frac{y^3}{x^3} + \frac{y}{x}$.

Q37: [I]Solve: $(y^2 - x^2)\frac{dy}{dx} = 2xy$.

Q38: [I]Find the general solution to $\left(x\sec\left(\frac{y}{x}\right) + y\right)dx - xdy = 0$.

Q39: [I]Solve the DE: $\frac{dy}{dx} = \frac{2x+3y}{x-y}$ and leave the solution in implicit form.

Q40: [I]Find the particular solution to $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$, with $y(1) = \pi/2$.

Q41: [I]Show that the substitution $x = X + h, y = Y + k$ can reduce $\frac{dy}{dx} = \frac{x-y-1}{x+y-5}$ to a homogeneous equation, and find the corresponding values of h and k.

Q42: [C]Find the general solution to $\left(x^2 - xy + y^2\right)dx - xydy = 0$. (Implicit solution is acceptable).

Q43: [C]Derive the condition for a DE of the form $\frac{dy}{dx} = f(x,y)$ to be homogeneous.

Q44: [C]Find the orthogonal trajectories of the family of curves $y = Cx^2$. (Requires combining DE formulation with homogeneous method).

Q45: [C]Solve the IVP: $\left(y^2 + x^2\right) dy + xydx = 0$, with $y(0) = 2$. Express y explicitly as a function of x.

D. First-Order Linear and Bernoulli Equations (Q46 - Q60)

Q46: [S]Find the general solution to the first-order linear DE: $\frac{dy}{dx} + 2xy = 4x$.

Q47: [S]Solve: $x\frac{dy}{dx} + y = x^2$.

Q48: [S]Identify the integrating factor (IF) for the linear DE: $\frac{dy}{dx} + \frac{1}{x}y = \cos(x)$.

Q49: [S]Solve the Bernoulli equation: $\frac{dy}{dx} + y = y^2$.

Q50: [S]Find the general solution to $\frac{dy}{dx} + y = e^{-x}$.

Q51: [S]Solve the IVP: $\frac{dy}{dx} - y = 2$, with $y(0) = 0$.

Q52: [I]Solve the Bernoulli equation: $\frac{dy}{dx} + \frac{1}{x}y = x^2y^3$.

Q53: [I]A $100L$ tank is initially filled with pure water. A solution containing 0.1 kg of salt per liter enters at $5L$/min. The well-mixed solution leaves at the same rate. Formulate the IVP for the amount of salt $A(t)$ and solve for $A(t)$.

Q54: [I]Solve: $\frac{dy}{dx} + y\cot(x) = 2\cos(x)$.

Q55: [I]Find the solution to $\frac{dy}{dx} + y^2 = \frac{y}{x}$ by converting it to a linear equation in $z = 1/y$.

Q56: [I]Solve the non-homogeneous linear DE: $x\frac{dy}{dx} + (1-x)y = e^{2x}$.

Q57: [I]Determine the unique value of $y(1)$ for the solution to $\frac{dy}{dx} - 3y = 6$ to remain finite as $x \to \infty$.

Q58: [C]Derive the general formula for the integrating factor of the first-order linear equation $\frac{dy}{dx} + P(x)y = Q(x)$.

Q59: [C]Find the solution to $\frac{dy}{dx} = \frac{1}{x\sin(y)+2\sin(2y)}$. (Hint: Re-examine the dependent and independent variables).

Q60: [C]Solve the IVP for the Bernoulli equation: $x\frac{dy}{dx} + y = xy^2\ln(x)$, with $y(1) = 1$.

Part II: Exact Equations and Integrating Factors (30 Problems)

A. Exact Equations (Q61 - Q75)

Q61: [S]State the necessary and sufficient condition for the DE $M(x,y)dx + N(x,y)dy = 0$ to be exact.

Q62: [S]Determine if the equation $(2x + y)dx + (x + 2y)dy = 0$ is exact. If so, find the general solution.

Q63: [S]Solve the exact equation: $(ye^{xy} + 4x)dx + (xe^{xy} + \cos(y))dy = 0$.

Q64: [S]Find the implicit general solution to $\left(3x^2y + 2\right) dx + x^3dy = 0$.

Q65: [S]Determine if $(x + y)dx + (x - y)dy = 0$ is exact.

Q66: [S]Solve the IVP: $(2x \cos(y))dx - (x^2 \sin(y))dy = 0$, with $y(1) = 0$.

Q67: [I]Find the value of the constant k that makes the DE $\left(y^2 + kxy\right) dx + (x^2 + 2xy)dy = 0$ exact.

Q68: [I]Solve the exact equation: $\left(\frac{1}{x} + \frac{y}{x^2} - \frac{y^2}{x^3}\right) dx + \left(\frac{1}{x} + \frac{y}{x^2}\right) dy = 0$.

Q69: [I]Find the general solution to $(y \cosh(x) - \sinh(x))\, dx + \cosh(x)dy = 0$.

Q70: [I]Solve the IVP: $(\sin(x) \tan(y) + 1)\, dx + \left(\cos(x) \sec^2(y) - y\right) dy = 0$, with $y(\pi/2) = 0$.

Q71: [I]Given that $f(x, y) = C$ is the general solution of $Mdx + Ndy = 0$. Show that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ implies the existence of a function $\phi(x, y)$ such that $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$.

Q72: [C]Find the implicit solution to $(x \cos(x + y) + \sin(x + y))dx + (x \cos(x + y))dy = 0$.

Q73: [C]Show that any separable equation $g(x)dx + h(y)dy = 0$ is always exact.

Q74: [C]Find the curve passing through $(1, 1)$ for which the area of the region bounded by the curve, the x-axis, and the ordinate is equal to $\frac{1}{2}xy$. (Requires DE formulation).

Q75: [C]The DE $Mdx + Ndy = 0$ is exact. Show that the general solution can be written as $\int Mdx + \int \left(N - \frac{\partial}{\partial y} \int Mdx\right) dy = C$.

B. Non-Exact Equations and Integrating Factors (Q76 - Q90)

Q76: [S]Determine the integrating factor (IF) for $\left(y^2 - x^2\right) dx + 2xydy = 0$, using the rule IF depends only on x.

Q77: [S]Solve the DE: $ydx - xdy = 0$, using the integrating factor $\mu = 1/x^2$.

Q78: [S]Find the integrating factor for $\left(x^2 + y^2 + x\right) dx + xdy = 0$, assuming the IF is a function of x only.

Q79: [S]Solve: $(3x^2y)dx + (x^3 + y)dy = 0$ after finding the IF.

Q80: [S]Find the IF for $\left(3x^2y + 2xy + y^3\right) dx + \left(x^2 + y^2\right) dy = 0$, assuming the IF is a function of y only.

Q81: [S]Solve: $ydx+(x-x^3y^3)dy = 0$. (Hint: $1/(xy)^3$ might be helpful, or reduction to Bernoulli).

Q82: [I]Find the general solution to $\left(x^2 + y^2 + 1\right) dx + 2xydy = 0$ by identifying the correct integrating factor.

Q83: [I]Solve: $(2y)dx + (x - \sin(y))dy = 0$.

Q84: [I]Show that $x^k y^l$ is an integrating factor for the homogeneous DE $Mdx + Ndy = 0$ if k and l satisfy a certain condition (Euler's condition).

Q85: [I]Solve the IVP: $\left(x^3\cos(y) - 2y\sin(x)\right)dx + \left(x^4\sin(y) + 2\cos(x)\right)dy = 0$, with $y(0) = 0$, after finding the suitable IF.

Q86: [I]Use the property of homogeneous DEs to show that $\mu = 1/(Mx + Ny)$ is an IF, provided $Mx + Ny \neq 0$.

Q87: [C]Find the integrating factor for $y^2dx + (xy + 2x^2y^4)dy = 0$. (Hint: The form $\mu = x^a y^b$ may be required).

Q88: [C]Given $Mdx + Ndy = 0$. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = f(xy)(My - Nx)$, find the integrating factor $\mu(xy)$.

Q89: [C]Solve the DE: $(x^2 + y^2 + 2x)dx + 2ydy = 0$.

Q90: [C]Find the general solution to $\left(x^2y^3 + y\right)dx + \left(x^3y^2 - x\right)dy = 0$. (Requires a non-standard IF or inspection).

Part III: n-th Order Linear ODEs (60 Problems)

A. Homogeneous Equations with Constant Coefficients (Q91 - Q120)

Q91: [S]Find the general solution to $y'' - 5y' + 6y = 0$.

Q92: [S]Solve: $y'' + 4y = 0$.

Q93: [S]Find the general solution to $y'' + 6y' + 9y = 0$.

Q94: [S]Determine the characteristic equation for $y''' - 2y'' - y' + 2y = 0$.

Q95: [S]Find the general solution to $y'' - 2y' + 5y = 0$.

Q96: [S]Solve: $y^{(4)} - 16y = 0$.

Q97: [S]Find the general solution to the DE with characteristic roots $r_1 = 0, r_2 = 3, r_3 = 3$.

Q98: [S]Solve the IVP: $y'' + y = 0$, with $y(0) = 2, y'(0) = 0$.

Q99: [S]Find the DE with the characteristic equation $r(r^2 + 1) = 0$.

Q100: [S]If $y_1 = e^{2x}$ and $y_2 = e^x$ are solutions, find the general solution and the DE.

Q101: [S]Solve: $y''' + 2y'' = 0$.

Q102: [S]Find the solution to the Boundary Value Problem (BVP): $y'' - y = 0$, with $y(0) = 1, y(1) = e$.

Q103: [I]Find the general solution for $y''' - 6y'' + 12y' - 8y = 0$.

Q104: [I]Given that $y_1(x) = x$ and $y_2(x) = x\ln(x)$ are two solutions to a second-order linear homogeneous DE. Calculate their Wronskian $W(x)$.

Q105: [I]Find the general solution for $y^{(4)} + 2y'' + y = 0$. (Repeated complex roots).

Q106: [I]Solve the IVP: $y'' - 4y' + 3y = 0$, with $y(0) = 0, y'(0) = 2$.

Q107: [I]Find the DE whose characteristic roots are $r = 1 \pm 3i$ and $r = 2$.

Q108: [I]Determine if $y_1 = x$ and $y_2 = e^x$ are linearly independent solutions to a second-order DE on $(0, \infty)$ using the Wronskian.

Q109: [I]Find the solution to $y''' + 8y = 0$.

Q110: [I]Solve the BVP: $y'' + \lambda^2 y = 0$, where $y(0) = 0$ and $y(L) = 0$. Find the eigenvalues λ for which non-trivial solutions exist.

Q111: [I]A third-order homogeneous linear DE has $y = c_1 \cos(2x) + c_2 \sin(2x) + c_3 e^{-x}$ as its general solution. Formulate the DE.

Q112: [I]Find the general solution for $y^{(4)} - 4y''' + 4y'' = 0$.

Q113: [I]Prove that if $r = a + ib$ is a root of the characteristic polynomial with real coefficients, then $r = a - ib$ is also a root.

Q114: [C]Find the general solution to $y^{(5)} - 2y^{(4)} + y''' = 0$.

Q115: [C]Derive the general form of the solution $y(x)$ when the characteristic equation has a root r with multiplicity k.

Q116: [C]If a DE has characteristic roots $r = 0$ (multiplicity 3) and $r = i$ (multiplicity 2), write down the general solution.

Q117: [C]A DE models a critically damped system, $y'' + 6y' + 9y = 0$. If $y(0) = 1$ and $y(t) \to 0$ as $t \to \infty$, find $y(t)$ and the maximum displacement.

Q118: [C]Prove the Wronskian identity for the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$: $W(x) = Ce^{-\int P(x)dx}$. (Abel's Formula).

Q119: [C]Find the DE of lowest order that has $e^{2x}, xe^{2x}, \cos(x)$ as part of its basis of solutions.

Q120: [C]Solve the third-order BVP: $y''' - y' = 0$ with $y(0) = 0, y'(0) = 1, y(1) = 1$.

B. Non-Homogeneous Equations (Undetermined Coefficients) (Q121 - Q150)

Q121: [S]Find the form of the particular solution y_p for $y'' - 3y' + 2y = 4x^2$ using Undetermined Coefficients (UC).

Q122: [S]Find the general solution to $y'' - y = 2e^{3x}$.

Q123: [S]Solve: $y'' + 4y = 8\sin(2x)$. (Case of overlap with y_c).

Q124: [S]Find the particular solution y_p for $y'' + y' - 6y = 5e^{-3x}$.

Q125: [S]Solve: $y'' + 2y' + y = e^{-x}$.

Q126: [S]Find the general solution to $y'' - 4y' + 4y = 2$.

Q127: [S]Find the form of y_p for $y'' - y' = x$.

Q128: [S]Solve the IVP: $y'' - 2y' = 4x$, with $y(0) = 0, y'(0) = 1$.

Q129: [S]Find the general solution to $y'' + 9y = e^{-x} + x^2$.

Q130: [S]Determine the form of y_p for $y'' - 5y' + 6y = xe^x \sin(x)$.

Q131: [S]Solve: $y''' + y'' = 6$.

Q132: [S]Find the general solution to $y'' - 4y' = 2\cos(4x)$.

Q133: [I]Find the particular solution for $y'' - 4y' + 5y = e^{2x} \sin(x)$. (Complex overlap case).

Q134: [I]Solve: $y'' + 2y' + 5y = 4e^{-x} \cos(2x)$.

Q135: [I]A mass-spring system is modeled by $y'' + 4y' + 3y = 2\sin(t)$. Find the transient and steady-state components of the solution.

Q136: [I]Solve the IVP: $y'' + 2y' + 10y = 25x$, with $y(0) = 0, y'(0) = 0$.

Q137: [I]Find the form of the particular solution y_p for $y^{(4)} - 2y''' + y'' = x^2 e^x + \cos(x)$.

Q138: [I]Solve the third-order non-homogeneous DE: $y''' - y = e^x$.

Q139: [I]Use the method of Undetermined Coefficients to find the general solution of $y'' + 4y = \sin(2x) + x\cos(2x)$.

Q140: [I]Explain the reasoning behind the "multiplication rule" (multiplying the guess by x^k) in the method of Undetermined Coefficients.

Q141: [I]Solve: $y'' - 6y' + 9y = 9xe^{3x}$.

Q142: [I]Find the DE whose general solution is $y = c_1 e^x + c_2 e^{2x} - 3\sin(x)$.

Q143: [I]If the Method of Undetermined Coefficients fails, which other general method can be used to find the particular solution y_p? (Conceptual).

Q144: [C]Find the general solution to $x^2 y'' + xy' - y = x$ using the substitution $x = e^t$ (Cauchy-Euler, then UC).

Q145: [C]Use the method of Variation of Parameters to find the particular solution y_p for $y'' + y = \sec(x)$.

Q146: [C]A forced harmonic oscillator is described by $y'' + 2\gamma y' + \omega_0^2 y = F_0 \cos(\omega t)$. Derive the conditions for resonance in the undamped case ($\gamma = 0$) and find the form of the solution at resonance.

Q147: [C]Find the DE of the lowest order that has $y_1 = x^2$ and $y_2 = x^3$ as solutions to the homogeneous part, and $\frac{1}{2}x^4$ as a particular solution.

Q148: [C]The general solution to $y'' + P(x)y' + Q(x)y = f(x)$ is $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$. Prove the superposition principle for the non-homogeneous solution, i.e., if $f(x) = f_1(x) + f_2(x)$, then $y_p = y_{p1} + y_{p2}$.

Q149: [C]Find the particular solution to $y''' - 3y'' + 3y' - y = e^x$.

Q150: [C]Solve the fourth-order non-homogeneous IVP: $y^{(4)} - 5y'' + 4y = 40\cosh(x)$, with $y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0$.