Discrete Math, Modern Algebra

Part I: Discrete Mathematics

2.1 Recurrence Relations (Using OGF)

Instructions: For problems 1-10, let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the Ordinary Generating Function (OGF) for the sequence a_n.

1. **(Conceptual)** If the sequence is $a_n = 1$ for all $n \geq 0$, what is the OGF $A(x)$? State the closed-form rational function.
2. **(Key Series)** Find the OGF $A(x)$ for the sequence $a_n = 5^n$ using the relevant Key Series.
3. **(Closed Form)** Express $A(x) = \sum_{n=0}^{\infty}(n+1)x^n$ as a closed-form rational function. What operation on the base series $\sum x^n$ generates this sequence?
4. **(Application)** A sequence a_n has $A(x) = \frac{1}{(1-3x)}$. Find the closed-form solution for a_n.
5. **(Application)** A sequence a_n is defined by the generating function $A(x) = \frac{1}{(1-x)^3}$. Determine the closed-form expression for a_n using the general coefficient formula $\binom{n+k-1}{k-1}$.
6. **(Recurrence Step)** Given $a_n = 3a_{n-1}$ for $n \geq 1$ with $a_0 = 1$. Write the exact expression for $\sum_{n=1}^{\infty} a_{n-1}x^n$ in terms of $A(x)$ and initial values, as required by the OGF algorithm.
7. **(OGF Method)** Solve the recurrence relation $a_n = a_{n-1} + n$, for $n \geq 1$, with $a_0 = 1$, using the OGF method. Express $A(x)$ algebraically before finding a_n.
8. **(Conceptual)** Let a_n be a sequence with OGF $A(x)$. Express the OGF for the sequence b_n where $b_n = a_{n-2}$ (with $b_0 = b_1 = 0$) in terms of $A(x)$ (assume a_0, a_1 are known).
9. **(OGF Method)** Apply the first four steps of the OGF algorithm to $a_n = 2a_{n-2} + 1$ for $n \geq 2$, with $a_0 = 0, a_1 = 1$. Solve for $A(x)$ algebraically.
10. **(Conceptual)** Explain how **Partial Fraction Decomposition** (Step 6) facilitates the conversion back to a power series (Step 7) to find the closed-form solution a_n.

2.2 Basics of Graph Theory

11. **(Handshaking Lemma)** An undirected graph $G = (V, E)$ has 11 edges. Use the **Handshaking Lemma** to determine the sum of the degrees of all vertices.
12. **(Directed Graphs)** In a directed graph, the sum of all in-degrees is k. What is the sum of all out-degrees, and how is this related to the total number of edges $|E|$?
13. **(Application)** A graph G has 7 vertices. If the degree of every vertex is 4, how many edges does the graph have?
14. **(Definitions)** Define a **Path** and a **Cycle** in terms of the sequence of vertices and edges.
15. **(Adjacency Matrix)** Let A be the adjacency matrix of a graph G. The (i, j) entry of A^k is 12. What specific object does the number 12 represent, and between which vertices?
16. **(Conceptual)** If G is a connected graph with n vertices, what is the minimum number of edges G must have?
17. **(Adjacency Matrix)** Prove or disprove: If A is the adjacency matrix, the trace of A^3 (sum of diagonal elements) counts twice the number of distinct cycles of length 3.

2.3 Eulerian and Hamiltonian Graphs

18. **(Eulerian Circuit)** State the **necessary and sufficient condition** for a connected graph to possess an **Eulerian Circuit**.

19. **(Eulerian Path)** A connected graph has exactly two vertices of odd degree. What kind of Eulerian trail is guaranteed to exist?

20. **(Contrast)** Differentiate between an **Eulerian Circuit** and a **Hamiltonian Cycle**.

21. **(Dirac's Theorem)** A graph G has $|V| = 14$. What is the minimum degree $\deg(v)$ required for *every* vertex to *guarantee* a Hamiltonian cycle by Dirac's Theorem?

22. **(Ore's Theorem)** A graph G has $|V| = 11$. If u and v are two non-adjacent vertices, what is the minimum sum of their degrees, $\deg(u) + \deg(v)$, that *guarantees* a Hamiltonian cycle by Ore's Theorem?

23. **(Conceptual)** Consider a connected bipartite graph G with $|V| \geq 3$. Can such a graph have a Hamiltonian cycle? Justify your answer.

24. **(Definition)** What is a **Hamiltonian Cycle**?

2.4 Graph Algorithms

25. **(Dijkstra's)** What is the primary **goal** of Dijkstra's Algorithm, and what critical **requirement** must the edge weights satisfy?

26. **(Dijkstra's)** In Dijkstra's Algorithm, what does the variable `alt` represent, and what is the purpose of the comparison `if alt < dist[v]`?

27. **(Floyd-Warshall)** What is the **goal** of the Floyd-Warshall Algorithm, and what is the key restriction on weights it can handle compared to Dijkstra's?

28. **(Floyd-Warshall)** Explain the significance of the outer loop variable k in the Floyd-Warshall recurrence relation $D_{ij}^{(k)} = min(D_{ij}^{(k-1)}, D_{ik}^{(k-1)} + D_{kj}^{(k-1)})$.

29. **(Kruskal's)** What is the **goal** of Kruskal's Algorithm, and what data structure is used to track connected components during the process?

30. **(Kruskal's)** Explain the role of the Disjoint Set operation `FIND-SET(u)` $\neq$ `FIND-SET(v)` in Kruskal's Algorithm in ensuring the resulting structure is a tree.

31. **(Contrast)** Explain why Kruskal's Algorithm uses the Disjoint Set structure, while Dijkstra's Algorithm uses a Priority Queue.

32. **(Kruskal's)** Kruskal's Algorithm relies on sorting the edges. Why must this sorting be done in **non-decreasing** order of weight?

2.5 Game Theory: Two-Player Zero-Sum Games

Instructions: For all problems, the given matrix A represents the payoff to Player A (Row Player).

34. **(Maximin/Minimax)** Consider the payoff matrix A_1:

$$A_1 = \begin{pmatrix} 4 & 2 & 5 \\ 1 & 3 & 0 \\ 5 & 0 & 4 \end{pmatrix}$$

Calculate Player A's Maximin value (v_1) and Player B's Minimax value (v_2).

35. **(Saddle Point)** For matrix A_1, does a saddle point exist? If so, state its location and the value of the game.

36. **(Dominance Reduction - Row)** Apply Row Dominance to A_1. Identify the row(s) Player A should never play and write down the resulting 2×3 matrix A_1'.

37. **(Dominance Reduction - Column)** Apply Column Dominance to A_1'. Identify the column(s) Player B should never play and write down the resulting 2×2 matrix A_1''.

38. **(Mixed Strategy Setup)** For the reduced 2×2 matrix A_1'', verify that a pure strategy solution (saddle point) does **not** exist, necessitating a mixed strategy.

39. **(Mixed Strategy for A)** Using the elements of $A_1'' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, calculate Player A's optimal probability p^* for playing their first remaining row.

40. **(Value of Game)** Calculate the Value of the Game (V) using the formula $V = \frac{ad-bc}{(a+d)-(b+c)}$ for A_1''.

41. **(Dominance Reduction)** Consider the payoff matrix A_2:

$$A_2 = \begin{pmatrix} 6 & 4 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}$$

Show that Row 3 is Row Dominated by another row. Write down the reduced 2×2 matrix A_2' after Player A applies Row Dominance.

42. **(Mixed Strategy for B)** Using the elements of A_2', calculate Player B's optimal probability q^* for playing their first column.

43. **(Strategy)** Explain, based on your result for q^* in Q42, which of the original two columns in A_2 Player B plays, and with what frequency.

44. **(Mixed Strategy Formula)** For a 2×2 game, derive the numerator for Player A's optimal probability p^* in terms of c and d (i.e., $d - c$).

45. **(Maximin/Minimax)** Consider the matrix $A_3 = \begin{pmatrix} -1 & 5 & 0 \\ 2 & 3 & 1 \end{pmatrix}$. Calculate Player A's maximin v_1 and Player B's minimax v_2.

46. **(Conceptual)** Why is the value of the game V always guaranteed to fall between Player A's maximin value v_1 and Player B's minimax value v_2?

47. **(Application)** Given a 2×2 matrix with $a = 6, b = 2, c = 1, d = 4$. Calculate the optimal probability q^* for Player B.

48. **(Definition)** Define **Maximin** (v_1) and **Minimax** (v_2) strategies.

49. **(Mixed Strategy)** If Player A's optimal strategy p^* is 0, what does this imply about the relationship between the rows c and d in the formula $p^* = \frac{d-c}{(a+d)-(b+c)}$?

50. **(Conceptual)** In a zero-sum game, if the game has no saddle point, why must the optimal strategy for both players be a mixed strategy?

Part II: Modern Algebra

3.1 Sets, Relations, and Groups

51. **(Equivalence Relation)** List the three defining properties that a relation $\sim$ on a set S must satisfy to be an **Equivalence Relation**.

52. **(Conceptual)** How does the set of all distinct equivalence classes (the **Quotient Set** $S/\sim$) partition the original set S?

53. **(Definition)** Define the property of **Commutativity** for a binary operation $*$.

54. **(Conceptual)** What must be true about an operation $*$ on a set S for it to be classified as a **Binary Operation** on S?

55. **(Group Axioms)** List the four axioms that a set G with a binary operation $*$ must satisfy to be a **Group**.
56. **(Definition)** What distinguishes an **Abelian group** from a general group?
57. **(Order of Element)** Define the **Order of an Element** $\text{ord}(a)$ in a group G.
58. **(Subgroup Test)** State the necessary and sufficient condition for a non-empty subset $H \subseteq G$ to be a subgroup, using the **One-Step Subgroup Test**.
59. **(Definition)** Define the **Right Coset** of a subgroup H in a group G, generated by an element $a \in G$.
60. **(Normal Subgroup)** What is the condition on the left and right cosets that defines a **normal subgroup** N of G?
61. **(Lagrange's Theorem)** State the conclusion of **Lagrange's Theorem** for a finite group G and its subgroup H.
62. **(Index)** If $|G| = 60$ and H is a subgroup of G with $|H| = 12$, what is the **Index** $[G : H]$?
63. **(Normal Subgroup Test)** State the equivalent test for a subgroup N to be a normal subgroup using the element-wise conjugation property: gng^{-1}.
64. **(Quotient Group Operation)** Given a normal subgroup N of G, define the binary operation in the **Quotient (Factor) Group** G/N.
65. **(Cyclic Group)** Define the **Cyclic Group** $\langle a \rangle$ generated by an element a using set notation.

3.4 Homomorphisms and Isomorphisms (Groups)

66. **(Homomorphism)** State the defining property of a group **homomorphism** $\phi : G \to G'$.
67. **(Kernel)** Define the **Kernel** $\text{Ker}(\phi)$ of a group homomorphism ϕ using set notation, assuming e' is the identity of G'.
68. **(Kernel Property)** What important group-theoretic property does the $\text{Ker}(\phi)$ always possess within the domain group G?
69. **(Image)** Define the **Image** $\text{Im}(\phi)$ of a group homomorphism ϕ using set notation.
70. **(Image Property)** What group-theoretic property does the $\text{Im}(\phi)$ always possess within the codomain group G'?
71. **(Isomorphism)** What must be true about the kernel and the image of a homomorphism ϕ for it to be an **Isomorphism**?
72. **(First Isomorphism Theorem)** State the conclusion of the **First Isomorphism Theorem for Groups**, using the symbol $\cong$.
73. **(Application)** If $|G| = 24$ and $|\text{Im}(\phi)| = 6$ for a homomorphism $\phi : G \to G'$, use the Isomorphism Theorem to find the order of $\text{Ker}(\phi)$.
74. **(Proof)** If $G \cong H$ and G is Abelian, prove that H must also be Abelian (proof based on the definition of isomorphism).
75. **(Conceptual)** If G is a cyclic group, must every group isomorphic to G also be cyclic? Justify your answer.

3.5 Rings and Basic Properties

76. **(Ring Axioms)** List the three main structural requirements that a set R with two binary operations $(R, +, \cdot)$ must satisfy to be a **Ring**.
77. **(Conceptual)** Why must the set R under the operation of addition, $(R, +)$, necessarily be an Abelian group?
78. **(Definition)** What two properties define a **Commutative Ring with Unity**?
79. **(Definition)** Define a **Unit** u in a ring R.
80. **(Definition)** Define a **Zero Divisor** a in a ring R.
81. **(Integral Domain)** List the three defining properties of an **Integral Domain**.
82. **(Definition)** Define a **Field**.
83. **(Conceptual)** Every field is an integral domain. Provide the counterexample (or condition) that shows not every integral domain is a field.
84. **(Proof)** Prove or disprove: In an Integral Domain, if $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.
85. **(Conceptual)** If a finite ring R is an Integral Domain, what additional structure is it guaranteed to have, according to the provided properties?
86. **(Application)** In the ring $\mathbb{Z}_6$, list all the **zero divisors** and all the **units**.

3.6 Ideals and Factor Rings

88. **(Ideal)** What is the essential "absorption" property that distinguishes an **Ideal** I of a ring R from a general subring?
89. **(Factor Ring)** Given an ideal I of R, state the elements of the **Factor Ring** R/I (the set of cosets).
90. **(Factor Ring Operation)** Define the addition operation and the multiplication operation in the Factor Ring R/I.
91. **(Ring Homomorphism)** State the two properties that a ring homomorphism $\phi : R \to S$ must satisfy.
92. **(Kernel of Ring Homo.)** Define the **Kernel** $\text{Ker}(\phi)$ of a ring homomorphism ϕ and state what type of substructure it always is in R.
93. **(Application)** Let R be a commutative ring with unity. If an ideal I contains the multiplicative identity 1_R, what must be true about the ideal I? Justify.
94. **(First Isomorphism Theorem for Rings)** State the conclusion of the **First Isomorphism Theorem for Rings**.
95. **(Homomorphism Property)** If $\phi : R \to S$ is a ring homomorphism, prove that $\phi(0_R) = 0_S$.
96. **(Contrast)** Explain the difference in the required structure (group vs. ring) between the object formed by cosets in a **Quotient Group** and a **Factor Ring**.
97. **(Integral Domain/Ideal)** In a commutative ring R with unity, the factor ring R/I is an integral domain if and only if I has a certain property. What is that property (in relation to zero divisors in R/I)?
98. **(Definition)** Define a **subring** S of R.
99. **(Factor Ring Unit)** If R/I is a factor ring and $(r + I)$ is a unit in R/I, what must be true about the element r in the original ring R (in relation to I)?
100. **(Conceptual)** If $\phi : \mathbb{Z} \to \mathbb{Z}$ is a ring homomorphism, what is the most general form of $\phi(n)$? (Hint: Consider $\phi(1)$).