Linear Algebra Practice Sheet

Reference : Howard Anton, Gil Strang

1. Linear Systems

1. Solve the following system using Gaussian elimination:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 8x_2 + 7x_3 &= 20 \\ 2x_1 + 7x_2 + 9x_3 &= 23 \end{aligned}$$

2. Solve the following homogeneous system:

$$\begin{aligned} 2x + 4y - 8z &= 0 \\ 3x + 6y - 12z &= 0 \\ -x - 2y + 4z &= 0 \end{aligned}$$

   Describe the solution set geometrically.

3. For what value(s) of k does the system

$$\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= 1 \end{aligned}$$

   have (a) no solution, (b) a unique solution, (c) infinitely many solutions?

4. Find the reduced row echelon form (RREF) of the matrix $A = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix}$.

5. Write the general solution of the system $Ax = b$ where $[A|b]$ has the RREF $\begin{bmatrix} 1 & -2 & 0 & 3 & | & 5 \\ 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$.

6. **True or False (Justify):** A system of 4 linear equations in 5 unknowns is always consistent.

7. **True or False (Justify):** If a homogeneous system $Ax = 0$ has a non-trivial solution, then $Ax = b$ must have infinitely many solutions for any b.

8. **Proof:** Prove that if x_p is a particular solution to $Ax = b$, then every solution to $Ax = b$ is of the form $x_p + x_h$, where x_h is a solution to the homogeneous system $Ax = 0$.

9. Suppose the RREF of an augmented matrix for a system is $\begin{bmatrix} 1 & 0 & -2 & | & 5 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & c \end{bmatrix}$. For what value of c is the system consistent?

10. **Conceptual:** Describe the three possible geometric interpretations of the solution set for a system of 3 linear equations in 3 unknowns.

11. Find a polynomial $p(t) = a + bt + ct^2$ that passes through the points $(1, 12), (2, 15), (3, 16)$. Set up the linear system and solve it.

12. **True or False (Justify):** If A is an $m \times n$ matrix with $m < n$, the system $Ax = 0$ must have a non-trivial solution.

13. **True or False (Justify):** If A is an $m \times n$ matrix with $m > n$, the system $Ax = b$ must be inconsistent for some $b \in \mathbb{R}^m$.

14. **Proof:** Prove that a system $Ax = b$ is consistent if and only if b is a linear combination of the columns of A.

15. Consider the system:

$$\begin{aligned} x_1 + 2x_2 &= a \\ x_1 + x_2 &= b \\ 2x_1 + 3x_2 &= c \end{aligned}$$

Find a condition on a, b, c that makes this system consistent.

16. **Conceptual:** What does the rank of the coefficient matrix A and the rank of the augmented matrix $[A|b]$ tell you about the solutions to $Ax = b$?

17. Find the RREF of $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{bmatrix}$.

18. Solve the system from the previous question with $b = (0, 0, 1)^\top$.

19. **True or False (Justify):** If A and B are row equivalent $m \times n$ matrices, then the systems $Ax = 0$ and $Bx = 0$ have the same solution set.

20. **True or False (Justify):** If $Ax = b$ and $Bx = b$ have the same solution set for a specific $b \neq 0$, then A and B are row equivalent.

2. Vectors and Linear Combinations

1. Let $u = (1, 2, 3)$ and $v = (-2, 0, 1)$. Compute $3u - 2v$.

2. Is the vector $b = (7, 8, 9)$ a linear combination of $v_1 = (1, 2, 3)$ and $v_2 = (1, 1, 1)$?

3. Describe geometrically the $\text{Span}\{v_1, v_2\}$ from the previous question.

4. Let $v_1 = (1, 0, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (1, 1, 0)$. Determine if these vectors are linearly independent.

5. **True or False (Justify):** Any set of 5 vectors in $\mathbb{R}^4$ must be linearly dependent.

6. **True or False (Justify):** If $\{v_1, v_2, v_3\}$ is a linearly dependent set, then v_1 must be a linear combination of v_2 and v_3.

7. **Proof:** Prove that a set of vectors $\{v_1, \ldots, v_k\}$ is linearly dependent if and only if at least one vector in the set is a linear combination of the others.

8. **Conceptual:** What is the geometric difference between a linearly independent set of two vectors in $\mathbb{R}^3$ and a linearly dependent set of two non-zero vectors in $\mathbb{R}^3$?

9. Find a vector w in $\mathbb{R}^3$ that is not in the span of $u = (1, 1, 0)$ and $v = (0, 1, 1)$.

10. **Proof:** Prove that if A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if $Ax = 0$ has only the trivial solution.

11. Let $S = \{v_1, v_2, v_3, v_4\}$ be a set of vectors in $\mathbb{R}^3$. Provide a computational procedure to find a subset of S that is linearly independent and has the same span as S.

12. **True or False (Justify):** If v_4 is in $\text{Span}\{v_1, v_2, v_3\}$, then $\text{Span}\{v_1, v_2, v_3, v_4\} = \text{Span}\{v_1, v_2, v_3\}$.

13. **Proof:** Let $v_1, \ldots, v_k \in \mathbb{R}^n$. Prove that $\text{Span}\{v_1, \ldots, v_k\}$ is the smallest subspace of $\mathbb{R}^n$ that contains all v_i.

14. Let $v_1 = (1, a)$, $v_2 = (a, a+2)$. For what values of a are v_1, v_2 linearly independent?

15. **Conceptual:** If the columns of an $n \times n$ matrix A are linearly independent, what can you say about the RREF of A?

3. Matrices

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$. Compute AB and BA.

2. **Conceptual:** What does $AB \neq BA$ from the previous question imply about matrix multiplication?

3. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ using the Gauss-Jordan method $[A|I]$.

4. Solve $Ax = b$ using A^{-1} from the previous question, where $b = (1, 0, -1)^\intercal$.

5. **Proof:** Prove that if A is invertible, its inverse A^{-1} is unique.

6. **Proof:** Prove that $(AB)^\intercal = B^\intercal A^\intercal$.

7. **Proof:** Prove that if A and B are invertible $n \times n$ matrices, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

8. **True or False (Justify):** If $AB = 0$ for $n \times n$ matrices A and B, then $A = 0$ or $B = 0$. (If false, provide a counterexample).

9. **True or False (Justify):** If A is an $n \times n$ matrix and $A^2 = A$, and A is invertible, then $A = I$.

10. **True or False (Justify):** If A is an $n \times n$ matrix such that $A^k = 0$ for some integer $k \geq 1$ (a nilpotent matrix), then $I - A$ is invertible. (Hint: Consider $(I - A)(I + A + A^2 + \dots)$)

11. **Conceptual:** Define a symmetric matrix and a skew-symmetric matrix.

12. **Proof:** Prove that $A^\intercal A$ is always a symmetric matrix for any $m \times n$ matrix A.

13. **Proof:** Prove that any square matrix A can be uniquely written as $A = S + K$, where S is symmetric and K is skew-symmetric. (Hint: Find formulas for S and K in terms of A and $A^\intercal$).

14. A matrix A is idempotent if $A^2 = A$. Show that if A is idempotent, then $B = I - A$ is also idempotent and $AB = BA = 0$.

15. Find all 2x2 matrices A such that $A^2 = I$.

16. Find all 2x2 matrices A such that $A^2 = 0$.

17. **Conceptual:** What is an elementary matrix? Describe the three types.

18. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find an elementary matrix E such that EA performs the operation $R_2 \to R_2 - 3R_1$.

19. **Proof:** Prove that elementary matrices are invertible, and their inverses are also elementary matrices.

20. **Conceptual:** Explain the relationship between the statement "A is invertible" and its RREF.

21. Find A^5 if $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

22. Let A be a 4×3 matrix and B be a 3×4 matrix. Show that AB is not invertible. (Hint: what is the maximum possible rank of AB?)

23. Find the trace of $A = \begin{bmatrix} 1 & 7 & 8 \\ 0 & 5 & 2 \\ 1 & 1 & -3 \end{bmatrix}$.

24. **Proof:** Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any A $(m \times n)$ and B $(n \times m)$.

25. **True or False (Justify):** $\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$.

4. Determinants

1. Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

2. Compute the determinant of $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 4 & -1 \\ 3 & 0 & 0 & 2 \\ 1 & 0 & 1 & 5 \end{bmatrix}$ using cofactor expansion.

3. Compute the determinant of $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$ using row reduction.

4. **True or False (Justify):** $\det(A + B) = \det(A) + \det(B)$.

5. **True or False (Justify):** $\det(kA) = k^n \det(A)$ for an $n \times n$ matrix A.

6. **True or False (Justify):** $\det(A^\mathsf{T}) = \det(A)$.

7. **Proof:** Prove that if A is an $n \times n$ matrix, $\det(A) \neq 0$ if and only if A is invertible.

8. **Proof:** Prove that $\det(AB) = \det(A)\det(B)$.

9. **Proof:** Using the previous result, prove that if A is invertible, $\det(A^{-1}) = 1/\det(A)$.

10. **Conceptual:** A matrix Q is orthogonal if $Q^\mathsf{T} Q = I$. What are the possible values for $\det(Q)$?

11. Find the determinant of the $n \times n$ "Vandermonde" matrix for $n = 3$: $V = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

12. Use Cramer's Rule to solve for x_2 in the system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ x_1 - x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_2 - x_3 &= 4 \end{aligned}$$

13. Find the adjugate (or adjoint) matrix $\mathrm{adj}(A)$ for $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

14. Verify the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ for the matrix A in the previous problem.

15. **Conceptual:** What is the geometric interpretation of $|\det(A)|$ for a 2x2 matrix A? For a 3x3 matrix A?

5. Euclidean and General Vector Spaces & Subspaces

1. **Conceptual:** List the 10 axioms that define a vector space V over a field F.

2. **Conceptual:** Prove from the axioms that for any vector $v \in V$, $0v = \mathbf{0}$ (where $0 \in F$ and $\mathbf{0} \in V$).

3. Determine if the set $V = \mathbb{P}_3$ (polynomials of degree ≤ 3) with standard polynomial addition and scalar multiplication is a vector space.

4. Determine if the set $V = \mathbb{R}^2$, with standard addition but non-standard scalar multiplication $k(x, y) = (kx, 0)$, is a vector space. If not, list all axioms that fail.

5. Determine if the set V of all 2×2 invertible matrices with standard matrix addition and scalar multiplication is a vector space.

6. **Conceptual:** Define a subspace W of a vector space V. What are the three conditions to check?

7. Determine if $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

8. Determine if $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$ is a subspace of $\mathbb{R}^3$.

9. Determine if $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ (the first quadrant) is a subspace of $\mathbb{R}^2$.

10. Determine if $W = \{p(t) \in \mathbb{P}_2 \mid p(1) = 0\}$ is a subspace of $\mathbb{P}_2$.

11. Determine if $W = \{A \in \mathbb{M}_{2\times 2} \mid A \text{ is symmetric}\}$ is a subspace of $\mathbb{M}_{2\times 2}$.

12. Determine if $W = \{A \in \mathbb{M}_{2\times 2} \mid \det(A) = 0\}$ is a subspace of $\mathbb{M}_{2\times 2}$.

13. **Proof:** Prove that the intersection of two subspaces, $W_1 \cap W_2$, is always a subspace.

14. **Conceptual:** Prove or disprove with a counterexample: The union of two subspaces, $W_1 \cup W_2$, is always a subspace.

15. **Conceptual:** Define the four fundamental subspaces of an $m \times n$ matrix A.

16. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$. Find a basis for the Null Space, $N(A)$.

17. For the matrix A above, find a basis for the Column Space, $C(A)$.

18. For the matrix A above, find a basis for the Row Space, $C(A^\mathsf{T})$.

19. For the matrix A above, find a basis for the Left Null Space, $N(A^\mathsf{T})$.

20. For the matrix A above, state the rank ($\operatorname{rank}(A)$) and nullity ($\operatorname{nullity}(A)$).

21. **Conceptual:** State the Rank-Nullity Theorem (or Dimension Theorem) for an $m \times n$ matrix A. Verify it for the matrix A above.

22. **Conceptual:** State the Fundamental Theorem of Linear Algebra, Part 1 (relating the dimensions of the four subspaces).

23. **Conceptual:** State the Fundamental Theorem of Linear Algebra, Part 2 (relating the orthogonality of the four subspaces).

24. **Proof:** Prove that $C(A) = C(AA^\mathsf{T})$ for any $m \times n$ matrix A. (Hint: Show $Ax = 0 \iff A^\mathsf{T}Ax = 0$).

25. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 1, 2, 3)$, $v_2 = (2, 2, 4, 6)$, $v_3 = (1, 0, 1, 1)$, $v_4 = (3, 2, 5, 7)$.

26. Let $W = \text{Span}\{v_1, v_2, v_3\}$ in $\mathbb{R}^3$. Describe a procedure to find a basis for W.

27. Let $S = \{t^2 + 1, t - 1, 2t\}$ be a set of vectors in $\mathbb{P}_2$. Determine if S is linearly independent.

28. Find a basis for $\mathbb{P}_2$. What is its dimension?

29. Find a basis for $\mathbb{M}_{2\times 2}$. What is its dimension?

30. Let $S = \{(1, 1, 0), (1, 0, 1)\}$. This is a linearly independent set in $\mathbb{R}^3$. Extend S to a basis for $\mathbb{R}^3$.

31. **Conceptual:** Let $T : V \to W$ be a linear transformation. Define the kernel (or null space) $\ker(T)$ and range (or image) $R(T)$.

32. **Proof:** Prove that $\ker(T)$ is a subspace of V.

33. **Proof:** Prove that $R(T)$ is a subspace of W.

34. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x, y) = (x + y, x - y, 2x)$. Find the standard matrix for T.

35. Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be the differentiation operator $T(p(t)) = p'(t)$. (a) Show T is a linear transformation. (b) Find $\ker(T)$. (c) Find $R(T)$.

36. **Conceptual:** Let $T : V \to W$ be a linear transformation. What is the relationship between $\dim(V)$, $\dim(\ker(T))$, and $\dim(R(T))$?

37. Let $B = \{(1, 1), (1, -1)\}$ be a basis for $\mathbb{R}^2$. Let $v = (3, 1)$. Find the coordinate vector $[v]_B$.

38. Let $C = \{(1, 0), (0, 1)\}$ be the standard basis for $\mathbb{R}^2$. Find the change of basis matrix $P_{B\to C}$ (from basis B in the previous problem to C).

39. Find the change of basis matrix $P_{C\to B}$.

40. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 90 degrees counter-clockwise. (a) Find the standard matrix A for T. (b) Find the matrix $[T]_B$ for T relative to the basis $B = \{(1, 1), (1, -1)\}$.

6. Inner Product Spaces and Gram-Schmidt

1. Let $u = (1, 2, -1, 3)$ and $v = (4, 0, 1, -2)$ in $\mathbb{R}^4$. (a) Find the dot product $u \cdot v$. (b) Find the norms $||u||$ and $||v||$. (c) Find the angle θ between u and v.

2. **Conceptual:** State the Cauchy-Schwarz Inequality in $\mathbb{R}^n$.

3. **Conceptual:** State the Triangle Inequality in $\mathbb{R}^n$.

4. **Conceptual:** Define an inner product $\langle u, v\rangle$ on a general real vector space V. List the 4 axioms.

5. Let $V = C[0, 1]$ (continuous functions on $[0, 1]$). Show that $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ is a valid inner product.

6. Using the inner product from the previous question, find $\langle x, e^x\rangle$.

7. Using the inner product from $C[0,1]$, find the "norm" $||x^2||$ and the "angle" between $f(x) = 1$ and $g(x) = x$.

8. Let $W = \text{Span}\{(1,1,1)\}$ in $\mathbb{R}^3$. Find a basis for the orthogonal complement, $W^\perp$.

9. **Proof:** Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is also a subspace of $\mathbb{R}^n$.

10. **Proof:** Let W be a subspace of $\mathbb{R}^n$. Prove that $(W^\perp)^\perp = W$.

11. **Conceptual:** Let A be $m \times n$. State the orthogonal relationship between $C(A)$ and $N(A^\mathsf{T})$.

12. **Conceptual:** Define an orthogonal set of vectors and an orthonormal set.

13. **Proof:** Prove that if $S = \{v_1, \dots, v_k\}$ is an orthogonal set of non-zero vectors, then S is linearly independent.

14. Let $B = \{q_1, q_2, q_3\}$ be an orthonormal basis for $\mathbb{R}^3$. Let $x \in \mathbb{R}^3$. Find a simple formula for the coordinates of x in this basis, i.e., c_1, c_2, c_3 such that $x = c_1q_1 + c_2q_2 + c_3q_3$.

15. Let $y = (7,2,3)$ and $u = (1,2,2)$. Find the orthogonal projection of y onto the line spanned by u.

16. Let $W = \text{Span}\{v_1, v_2\}$ where $v_1 = (1,1,0)$ and $v_2 = (0,1,1)$. Find the orthogonal projection of $y = (2,3,4)$ onto the subspace W.

17. For y and W above, find the vector $z \in W^\perp$ such that $y = \hat{y} + z$ (where $\hat{y} = \text{proj}_W\, y$). Find the shortest distance from y to W.

18. **Conceptual:** What is the least-squares problem for an inconsistent system $Ax = b$?

19. **Conceptual:** Derive the normal equations $A^\mathsf{T} A\hat{x} = A^\mathsf{T} b$ for the least-squares solution $\hat{x}$.

20. Find the least-squares solution to the system:

$$\begin{aligned} x + y &= 4 \\ 2x + y &= 5 \\ x - y &= 0 \end{aligned}$$

This corresponds to finding the "best fit" line $y = c_0 + c_1x$ for points $(1,4), (2,5), (1,0)$.

21. **Iterative Process:** Let $v_1 = (1,1,0)$, $v_2 = (1,2,0)$, $v_3 = (0,1,2)$. Apply the Gram-Schmidt process to find an orthonormal basis $\{q_1, q_2, q_3\}$ for $\text{Span}\{v_1, v_2, v_3\}$.

22. **Conceptual:** Let A be an $m \times n$ matrix with linearly independent columns. What is the Gram-Schmidt process in matrix form? (This leads to $A = QR$).

23. Let $V = \mathbb{P}_2$ with inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$. The set $B = \{1, x, x^2\}$ is a basis. Apply Gram-Schmidt to the first two vectors $(1, x)$ to find an orthogonal basis.

24. **Proof:** Let Q be an $n \times n$ matrix. Prove that Q is orthogonal ($Q^\mathsf{T} Q = I$) if and only if its columns form an orthonormal basis for $\mathbb{R}^n$.

25. **Proof:** Let Q be an orthogonal matrix. Prove that Q preserves lengths ($||Qx|| = ||x||$) and angles ($\langle Qx, Qy\rangle = \langle x, y\rangle$).

7. Eigenvalues and Eigenvectors

1. **Conceptual:** Define eigenvalue and eigenvector for a matrix A.
2. Find the characteristic polynomial for $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.
3. Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.
4. Find the eigenvalues and a basis for each eigenspace for $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
5. **True or False (Justify):** The eigenvalues of a triangular matrix are its diagonal entries.
6. **Proof:** Prove that $\lambda = 0$ is an eigenvalue of A if and only if A is singular.
7. **Proof:** Prove that A and $A^\top$ have the same eigenvalues.
8. Find a 2x2 matrix A such that A and $A^\top$ do not share the same eigenvectors.
9. **Proof:** Let λ be an eigenvalue of an invertible matrix A with eigenvector v. Prove that $1/\lambda$ is an eigenvalue of A^{-1} with the same eigenvector v.
10. **Proof:** Let λ be an eigenvalue of A. Prove that λ^k is an eigenvalue of A^k for any $k \geq 1$.
11. **Inter-connected:** A matrix A has eigenvalues $1, 2, 4$. (a) What are the eigenvalues of A^3? (b) What are the eigenvalues of A^{-1}? (c) What are the eigenvalues of $A - 3I$? (d) What is $\det(A)$? (e) What is $\operatorname{tr}(A)$?
12. **Conceptual:** What is the relationship between the trace of A and its eigenvalues?
13. **Conceptual:** What is the relationship between the determinant of A and its eigenvalues?
14. **Conceptual:** Define "diagonalizable." What is the test for diagonalizability?
15. Determine if $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ is diagonalizable.
16. Determine if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.
17. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
18. **Iterative Process:** Using the previous result, compute A^{10}.
19. **Conceptual:** Define algebraic multiplicity and geometric multiplicity of an eigenvalue. How do they relate to diagonalizability?
20. **Proof:** Prove that eigenvectors corresponding to distinct eigenvalues are linearly independent.
21. **Iterative Process:** Solve the system of differential equations $\mathbf{u}'(t) = A\mathbf{u}(t)$ with $\mathbf{u}(0) = (1, 5)$, where $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

22. Find the eigenvalues of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

23. Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a rotation matrix). What does this tell you about real vs. complex eigenvalues?

24. **Conceptual:** What is the Spectral Theorem for real symmetric matrices? (State its main consequences).

25. **Proof:** Prove that if A is a real symmetric matrix, then eigenvectors from different eigenspaces are orthogonal.

26. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^\mathsf{T}$.

27. **Conceptual:** Define a positive definite matrix S.

28. **Conceptual:** List 4 (or 5) equivalent conditions for a symmetric matrix S to be positive definite.

29. Determine if $S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are positive definite. Use two different methods (e.g., eigenvalues, principal minors).

30. **Conceptual:** A matrix A is similar to B if $A = PBP^{-1}$. Prove that similar matrices have the same eigenvalues.

8. Decompositions (CR, LU, QR, Eigen, SVD)

1. **CR Decomposition:** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. (a) Find the RREF of A. (b) Find the rank r of A. (c) Find the $A = CR$ decomposition, where C contains the r pivot columns of A, and R contains the r non-zero rows of the RREF.

2. **Conceptual:** If $A = CR$ is a rank r decomposition, what are the dimensions of C and R?

3. **CR Decomposition:** A rank-1 matrix can always be written as $A = uv^\mathsf{T}$. Find this decomposition for $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$.

4. **Conceptual:** How does the $A = CR$ decomposition express every column of A as a linear combination of the basis columns in C?

5. **LU Decomposition:** Find the LU decomposition (where L is unit lower triangular) for $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 2 \\ 6 & 1 & 1 \end{bmatrix}$.

6. **Iterative Process:** Solve $Ax = b$ using your LU decomposition, where $b = (2, 4, 8)^\mathsf{T}$. (Solve $Ly = b$, then $Ux = y$).

7. **Conceptual:** What is the condition under which an $n \times n$ matrix A has an $A = LU$ decomposition (without pivoting)?

8. **LU Decomposition:** Find the $A = PLU$ decomposition (with partial pivoting) for $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$.

9. **Conceptual:** Find the $A = LDU$ decomposition for $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

10. **QR Decomposition:** Find the QR decomposition for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ using the Gram-Schmidt process.

11. **Conceptual:** If $A = QR$, show how the least-squares problem $A^{\mathsf{T}} A\hat{x} = A^{\mathsf{T}} b$ simplifies to $R\hat{x} = Q^{\mathsf{T}} b$.

12. **Iterative Process:** Use the QR decomposition from problem 175 to find the least-squares solution to $Ax = (1, 3, 1)^{\mathsf{T}}$.

13. **Eigen (Spectral) Decomposition:** For the symmetric matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, find its spectral decomposition $A = \lambda_1 q_1 q_1^{\mathsf{T}} + \lambda_2 q_2 q_2^{\mathsf{T}}$.

14. **Conceptual:** What is the geometric action of A in the previous problem, expressed in terms of its spectral decomposition? (Stretching along which axes?)

15. **Conceptual:** Define the Singular Value Decomposition (SVD) for an $m \times n$ matrix A. Define all three matrices U, Σ, and V.

16. **Conceptual:** How are the singular values σ_i of A related to the eigenvalues of $A^{\mathsf{T}} A$?

17. **Conceptual:** How are the columns of U (left singular vectors) and V (right singular vectors) found?

18. Find the singular values $\sigma_1 \geq \sigma_2$ for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

19. **SVD Process:** Find the full SVD ($A = U\Sigma V^{\mathsf{T}}$) for $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

20. **SVD Process:** Find the full SVD for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

21. **SVD Process:** Find the full SVD for $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

22. **SVD Process:** Find the SVD for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

23. **Conceptual:** How does the SVD provide orthonormal bases for all four fundamental subspaces of A?

24. **Conceptual:** State the Eckart-Young Theorem. What is the best rank-k approximation A_k to a matrix A?

25. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find its singular values and use them to find $||A||_2$ (the spectral norm) and the Frobenius norm $||A||_F$.

9. Principal Component Analysis (PCA)

1. **Conceptual:** What is the primary goal of Principal Component Analysis (PCA)?
2. **Conceptual:** What does the "first principal component" represent in a data set?
3. **Conceptual:** Outline the (mathematical) steps to perform PCA on a data matrix X, which is $n \times p$ (n samples, p features).
4. Let the data matrix (already centered) be $X = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -2 \end{bmatrix}$. (a) Find the covariance matrix $S = \frac{1}{n-1} X^\top X$ (or just $X^\top X$ for simplicity). (b) Find the eigenvalues and eigenvectors of S.
5. From the previous problem: (a) What is the first principal component (the direction vector v_1)? (b) What is the total variance in the data (sum of eigenvalues)? (c) What percentage of the total variance is explained by the first principal component?
6. **Conceptual:** What is a "scree plot" and how is it used?
7. **Conceptual:** How is PCA related to the SVD of the (centered) data matrix X?
8. **Iterative Process:** Using the data from problem 194, project the original data points onto the first principal component. What is the new, 1-dimensional representation of the data?
9. **Conceptual:** Why is it generally important to standardize (scale) the features before applying PCA?
10. **Conceptual:** What is the "curse of dimensionality," and how does PCA serve as a "dimensionality reduction" technique?