Random Variables and Distributions

6 Random Variables (RV)

Definition 6.1 (Random Variable)**.** A Random Variable (RV) X is a real-valued function defined on the sample space Ω of a probability experiment:

$$X : \Omega \to \mathbb{R}$$

For X to be a valid random variable, the set $\{\omega \in \Omega | X(\omega) \leq x\}$ must be an event in the σ-algebra $\mathcal{F}$ for every $x \in \mathbb{R}$. This ensures that we can compute its probability.

Definition 6.2 (Types of Random Variables)**.** (i) **Discrete Random Variable:** An RV X is discrete if its range (the set of all possible values $X(\omega)$) is finite or countably infinite.

(ii) **Continuous Random Variable:** An RV X is continuous if its range is an uncountably infinite set (typically an interval) and it can be described by a Probability Density Function (PDF).

7 Discrete Random Variables and Distributions

7.1 Probability Mass Function (PMF)

Definition 7.1 (Probability Mass Function (PMF))**.** The PMF of a discrete RV X, denoted $p_X(x)$, is defined as:

$$p_X(x) = \mathbb{P}(X = x)$$

Property 7.1 (Properties of a PMF)**.** A function $p(x)$ is a valid PMF for an RV X with range $\{x_1, x_2, ...\}$ if:

(i) **Non-negativity:** $p_X(x_i) \geq 0$ for all i.

(ii) **Unit Sum:** $\sum_i p_X(x_i) = \sum_{x \in \text{Range}(X)} p_X(x) = 1$.

Exercise 1: Determining the Constant in a PMF

Let X be a discrete RV with possible values $x = 1, 2, 3, 4$. The PMF is given by $p_X(x) = cx^2$. Find the value of the constant c.
Solution: For $p_X(x)$ to be a valid PMF, the Unit Sum property must be satisfied: $\sum_{x=1}^{4} p_X(x) = 1$.

$$\begin{aligned}
\sum_{x=1}^{4} cx^2 &= 1 \\
c(1^2 + 2^2 + 3^2 + 4^2) &= 1 \\
c(1 + 4 + 9 + 16) &= 1 \\
30c &= 1 \\
c &= \frac{1}{30}
\end{aligned}$$

The constant is $c = 1/30$.

Exercise 2: Calculating Probability from a PMF

A random variable Y has the PMF $p_Y(y) = \frac{1}{2^y}$ for $y = 1, 2, 3, \ldots$. Calculate $\mathbb{P}(Y \geq 3)$.

Solution: The probability $\mathbb{P}(Y \geq 3)$ can be calculated as $1 - \mathbb{P}(Y < 3) = 1 - \mathbb{P}(Y = 1 \text{ or } Y = 2)$.

$$\begin{aligned} \mathbb{P}(Y < 3) &= p_Y(1) + p_Y(2) \\ &= \frac{1}{2^1} + \frac{1}{2^2} \\ &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \end{aligned}$$

Therefore,

$$\mathbb{P}(Y \geq 3) = 1 - \frac{3}{4} = \frac{1}{4}$$

Alternatively, we can sum the infinite series directly:

$$\mathbb{P}(Y \geq 3) = \sum_{y=3}^{\infty} \left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$$

This is a geometric series with first term $a = (1/2)^3 = 1/8$ and common ratio $r = 1/2$.

$$\mathbb{P}(Y \geq 3) = \frac{a}{1-r} = \frac{1/8}{1-1/2} = \frac{1/8}{1/2} = \frac{1}{4}$$

Exercise 3: PMF for an Experiment with Replacement

An urn contains 3 red (R) and 2 blue (B) balls. Two balls are drawn successively *with replacement*. Let X be the number of red balls drawn. Determine the PMF $p_X(x)$.
Solution: The possible values for X (number of red balls) are $x = 0, 1, 2$. The total number of balls is 5.

- Probability of drawing Red: $\mathbb{P}(R) = 3/5$
- Probability of drawing Blue: $\mathbb{P}(B) = 2/5$

Since the draws are independent (with replacement):

- $p_X(0) = \mathbb{P}(BB) = \mathbb{P}(B)\mathbb{P}(B) = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$
- $p_X(1) = \mathbb{P}(RB \text{ or } BR) = \mathbb{P}(RB) + \mathbb{P}(BR) = \left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$
- $p_X(2) = \mathbb{P}(RR) = \mathbb{P}(R)\mathbb{P}(R) = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

The PMF is:

$$p_X(x) = \begin{cases} 4/25, & \text{if } x = 0 \\ 12/25, & \text{if } x = 1 \\ 9/25, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Check: $4/25 + 12/25 + 9/25 = 25/25 = 1$. The PMF is valid.

Exercise 4: PMF from a Bernoulli Sequence

A component is tested until it fails. The probability of failure on any given test is $p = 0.2$, and tests are independent. Let X be the number of tests conducted up to and including the first failure. Find the PMF $p_X(x)$.

Solution: This is a Geometric distribution setup. X can take values $x = 1, 2, 3, \ldots$. Let S be success (no failure, $\mathbb{P}(S) = 1 - p = 0.8$) and F be failure ($\mathbb{P}(F) = p = 0.2$). The sequence of events is $S, S, \ldots, S, F$ with $x - 1$ successes followed by 1 failure.

$$p_X(x) = \mathbb{P}(\underbrace{S \cap S \cap \cdots \cap S}_{x-1 \text{ times}} \cap F)$$

Due to independence:

$$p_X(x) = \mathbb{P}(S)^{x-1}\mathbb{P}(F) = (1-p)^{x-1}p$$

Substituting $p = 0.2$:

$$p_X(x) = (0.8)^{x-1}(0.2) \quad \text{for } x = 1, 2, 3, \ldots$$

Exercise 5: Simple PMF and Conditional Probability

Let X have the PMF $p_X(x) = \frac{x}{10}$ for $x \in \{1, 2, 3, 4\}$. Find the conditional probability $\mathbb{P}(X > 2 | X \leq 3)$.

Solution: First, calculate the required probabilities from the PMF:

- $p_X(1) = 1/10$
- $p_X(2) = 2/10$
- $p_X(3) = 3/10$
- $p_X(4) = 4/10$

The conditional probability is defined as:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Here, $A = \{X > 2\}$ and $B = \{X \leq 3\}$. The event $A \cap B$ is $\{X > 2 \text{ and } X \leq 3\}$, which means $X = 3$.

$$\begin{aligned}
\mathbb{P}(A \cap B) = \mathbb{P}(X = 3) = p_X(3) = \frac{3}{10} \\
\mathbb{P}(B) = \mathbb{P}(X \leq 3) = p_X(1) + p_X(2) + p_X(3) \\
= \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{6}{10}
\end{aligned}$$

Finally,

$$\mathbb{P}(X > 2 | X \leq 3) = \frac{\mathbb{P}(X = 3)}{\mathbb{P}(X \leq 3)} = \frac{3/10}{6/10} = \frac{3}{6} = \frac{1}{2}$$

7.2 Cumulative Mass Function (CMF)

Definition 7.2 (Cumulative Mass Function (CMF))**.** The CMF of a discrete RV X, denoted $F_X(x)$, is defined as:

$$F_X(x) = \mathbb{P}(X \le x) = \sum_{x_i \le x} p_X(x_i)$$

Property 7.2 (Properties of a CMF)**.** (i) **Limits:** $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

(ii) **Non-decreasing:** If $a \le b$, then $F_X(a) \le F_X(b)$.

(iii) **Right-Continuous:** $F_X(x)$ is a step function that is continuous from the right: $\lim_{\epsilon\to 0^+} F_X(x+\epsilon) = F_X(x)$.

(iv) **PMF from CMF:** $p_X(x) = F_X(x) - F_X(x^-)$, where $F_X(x^-) = \lim_{\epsilon\to 0^+} F_X(x-\epsilon)$ is the limit from the left (i.e., the value just before the jump at x).

Exercise 6: Constructing CMF from PMF

A discrete RV X has the PMF $p_X(x)$ given by the following table:

x	1	3	5
$p_X(x)$	0.4	0.3	0.3

Write the functional form of the CMF, $F_X(x)$.
Solution: The CMF $F_X(x) = \mathbb{P}(X \le x)$ is a step function.

- For $x < 1$: $F_X(x) = \mathbb{P}(X \le x) = 0$
- For $1 \le x < 3$: $F_X(x) = \mathbb{P}(X \le 1) = p_X(1) = 0.4$
- For $3 \le x < 5$: $F_X(x) = \mathbb{P}(X \le 3) = p_X(1) + p_X(3) = 0.4 + 0.3 = 0.7$
- For $x \ge 5$: $F_X(x) = \mathbb{P}(X \le 5) = 0.7 + p_X(5) = 0.7 + 0.3 = 1.0$

The CMF is:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.4, & \text{if } 1 \le x < 3 \\ 0.7, & \text{if } 3 \le x < 5 \\ 1.0, & \text{if } x \ge 5 \end{cases}$$

Exercise 7: Calculating Probabilities from CMF

A discrete RV X has the CMF $F_X(x)$ defined as:

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 0.2, & -1 \le x < 0 \\ 0.7, & 0 \le x < 2 \\ 1.0, & x \ge 2 \end{cases}$$

Calculate (a) $\mathbb{P}(X = 0)$ and (b) $\mathbb{P}(-0.5 < X \le 2)$.
Solution: (a) Calculating $\mathbb{P}(X = 0)$**:** We use the property $p_X(x) = F_X(x) - F_X(x^-)$.

$$\begin{aligned}\mathbb{P}(X = 0) &= F_X(0) - F_X(0^-) \\ &= \lim_{\epsilon\to 0^+} F_X(0) - \lim_{\epsilon\to 0^+} F_X(0-\epsilon) \\ &= 0.7 - 0.2 = 0.5\end{aligned}$$

(b) Calculating $\mathbb{P}(-0.5 < X \leq 2)$: For a discrete RV, $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$.

$$\begin{aligned}\mathbb{P}(-0.5 < X \leq 2) &= F_X(2) - F_X(-0.5)\\ F_X(2) &= 1.0 \quad (\text{since } 2 \geq 2)\\ F_X(-0.5) &= 0.2 \quad (\text{since } -1 \leq -0.5 < 0)\\ \mathbb{P}(-0.5 < X \leq 2) &= 1.0 - 0.2 = 0.8\end{aligned}$$

Exercise 8: Finding Jumps in the CMF to get PMF

A discrete RV X has a CMF $F_X(x)$ defined by the jumps at $x = 1, 2, 4$:

$$F_X(x) = 0.1 \cdot \mathbf{1}_{\{x \geq 1\}} + 0.5 \cdot \mathbf{1}_{\{x \geq 2\}} + 0.4 \cdot \mathbf{1}_{\{x \geq 4\}}$$

where $\mathbf{1}_{\{\dots\}}$ is the indicator function. Find the PMF $p_X(x)$.
Solution: The points where the CMF jumps are the possible values of X: $x = 1, 2, 4$. The magnitude of the jump at x is $\mathbb{P}(X = x)$.

- At $x = 1$: $F_X(1) = 0.1$. $F_X(1^-) = 0$. $\mathbb{P}(X = 1) = 0.1 - 0 = 0.1$.
- At $x = 2$: $F_X(2) = 0.1 + 0.5 = 0.6$. $F_X(2^-) = 0.1$. $\mathbb{P}(X = 2) = 0.6 - 0.1 = 0.5$.
- At $x = 4$: $F_X(4) = 0.1 + 0.5 + 0.4 = 1.0$. $F_X(4^-) = 0.6$. $\mathbb{P}(X = 4) = 1.0 - 0.6 = 0.4$.

The PMF is:

x	1	2	4
$p_X(x)$	0.1	0.5	0.4

Exercise 9: Using CMF for Compound Events

Given the CMF from Exercise 8, $F_X(x)$ with $\mathbb{P}(X = 1) = 0.1, \mathbb{P}(X = 2) = 0.5, \mathbb{P}(X = 4) = 0.4$. Calculate $\mathbb{P}(1 < X \leq 3)$.
Solution: We can calculate the probability using the CMF property:

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$$

In this case, $a = 1$ and $b = 3$:

$$\mathbb{P}(1 < X \leq 3) = F_X(3) - F_X(1)$$

From the definition of $F_X(x)$:

- $F_X(3) = \mathbb{P}(X \leq 3) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = 0.1 + 0.5 = 0.6$
- $F_X(1) = \mathbb{P}(X \leq 1) = 0.1$

Therefore,

$$\mathbb{P}(1 < X \leq 3) = 0.6 - 0.1 = 0.5$$

(Check: The only integer value in $(1, 3]$ is $X = 2$. $\mathbb{P}(X = 2) = 0.5$. The results match.)

Exercise 10: CMF of a Transformed RV

Let X be a discrete RV with $p_X(1) = 0.5$ and $p_X(2) = 0.5$. Let $Y = X^2$. Find the CMF of Y, $F_Y(y)$.
Solution: The possible values for X are 1 and 2. The possible values for $Y = X^2$ are $y = 1^2 = 1$ and $y = 2^2 = 4$. The PMF of Y is:

- $\mathbb{P}(Y = 1) = \mathbb{P}(X^2 = 1) = \mathbb{P}(X = 1) = 0.5$
- $\mathbb{P}(Y = 4) = \mathbb{P}(X^2 = 4) = \mathbb{P}(X = 2) = 0.5$

Now we construct the CMF $F_Y(y)$:

- For $y < 1$: $F_Y(y) = 0$
- For $1 \leq y < 4$: $F_Y(y) = \mathbb{P}(Y \leq 1) = 0.5$
- For $y \geq 4$: $F_Y(y) = \mathbb{P}(Y \leq 4) = 0.5 + 0.5 = 1.0$

The CMF is:

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 1 \\ 0.5, & \text{if } 1 \leq y < 4 \\ 1.0, & \text{if } y \geq 4 \end{cases}$$

7.3 Expectation and Variance (Discrete)

Definition 7.3 (Expectation (Mean))**.** The Expected Value (or mean) of a discrete RV X is the probability-weighted average of its possible values:

$$\mu = \mathbb{E}[X] = \sum_x x \cdot p_X(x)$$

The expectation exists only if $\sum_x |x| \cdot p_X(x) < \infty$.

Theorem 7.1 (Law of the Unconscious Statistician (LOTUS) - Discrete)**.** Let $g(X)$ be a function of the discrete RV X. The expected value of $g(X)$ is:

$$\mathbb{E}[g(X)] = \sum_x g(x) \cdot p_X(x)$$

Definition 7.4 (Variance)**.** The Variance of a discrete RV X, denoted σ^2 or $\mathrm{Var}(X)$, measures the spread of the distribution:

$$\sigma^2 = \mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2] = \sum_x (x - \mu)^2 \cdot p_X(x)$$

Formula 7.1 (Computational Formula for Variance)**.**

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

The Standard Deviation is $\sigma = \sqrt{\mathrm{Var}(X)}$.

Property 7.3 (Properties of Expectation and Variance)**.** For constants a and b:

(i) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

(ii) $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

(iii) $\mathrm{Var}(b) = 0$

Exercise 11: Expectation and Variance Calculation

A discrete RV X has the PMF $p_X(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. Calculate $\mathbb{E}[X]$ and $\text{Var}(X)$.
Solution: Step 1: Calculate the Expectation $\mathbb{E}[X]$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{5} x \cdot p_X(x) \\
&= 1\left(\frac{1}{15}\right) + 2\left(\frac{2}{15}\right) + 3\left(\frac{3}{15}\right) + 4\left(\frac{4}{15}\right) + 5\left(\frac{5}{15}\right) \\
&= \frac{1 + 4 + 9 + 16 + 25}{15} = \frac{55}{15} = \frac{11}{3}
\end{aligned}$$

So, $\mathbb{E}[X] = 11/3$.

Step 2: Calculate $\mathbb{E}[X^2]$. (Using LOTUS)

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=1}^{5} x^2 \cdot p_X(x) \\
&= 1^2\left(\frac{1}{15}\right) + 2^2\left(\frac{2}{15}\right) + 3^2\left(\frac{3}{15}\right) + 4^2\left(\frac{4}{15}\right) + 5^2\left(\frac{5}{15}\right) \\
&= \frac{1 + 8 + 27 + 64 + 125}{15} = \frac{225}{15} = 15
\end{aligned}$$

So, $\mathbb{E}[X^2] = 15$.

Step 3: Calculate the Variance Var(X). (Using Computational Formula)

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= 15 - \left(\frac{11}{3}\right)^2 \\
&= 15 - \frac{121}{9} \\
&= \frac{135 - 121}{9} = \frac{14}{9}
\end{aligned}$$

The variance is $\text{Var}(X) = 14/9$.

Exercise 12: Expected Value of a Transformed RV (LOTUS)

A discrete RV X has the PMF $p_X(-1) = 0.3, p_X(0) = 0.2, p_X(1) = 0.5$. Find the expected value of the function $g(X) = X^3 + 2X + 1$.
Solution: We use the Law of the Unconscious Statistician (LOTUS): $\mathbb{E}[g(X)] = \sum_x g(x) \cdot p_X(x)$.

- $x = -1$: $g(-1) = (-1)^3 + 2(-1) + 1 = -1 - 2 + 1 = -2$. $p_X(-1) = 0.3$.
- $x = 0$: $g(0) = (0)^3 + 2(0) + 1 = 1$. $p_X(0) = 0.2$.
- $x = 1$: $g(1) = (1)^3 + 2(1) + 1 = 1 + 2 + 1 = 4$. $p_X(1) = 0.5$.

$$\begin{aligned}
\mathbb{E}[g(X)] &= g(-1)p_X(-1) + g(0)p_X(0) + g(1)p_X(1) \\
&= (-2)(0.3) + (1)(0.2) + (4)(0.5) \\
&= -0.6 + 0.2 + 2.0 \\
&= 1.6
\end{aligned}$$

Exercise 13: Properties of Expectation and Variance for Linear Transformation

If $\mathbb{E}[X] = 5$ and $\mathrm{Var}(X) = 3$, find $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$ for the transformed RV $Y = 4X - 7$.
Solution: We use the properties of Expectation and Variance for linear transformations $Y = aX + b$. Here, $a = 4$ and $b = -7$.

Step 1: Calculate $\mathbb{E}[Y]$. Using $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$:

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[4X - 7] \\ &= 4\mathbb{E}[X] - 7 \\ &= 4(5) - 7 \\ &= 20 - 7 = 13\end{aligned}$$

Step 2: Calculate Var(Y). Using $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$:

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(4X - 7) \\ &= 4^2\mathrm{Var}(X) \\ &= 16(3) \\ &= 48\end{aligned}$$

Exercise 14: Expectation and Variance for a Bernoulli Trial

Let $X \sim \mathrm{Bernoulli}(p)$ where p is the probability of success. The PMF is $p_X(1) = p$ and $p_X(0) = 1 - p$. Show that $\mathbb{E}[X] = p$ and $\mathrm{Var}(X) = p(1 - p)$.
Solution: Step 1: Calculate $\mathbb{E}[X]$.

$$\begin{aligned}\mathbb{E}[X] &= \sum_x x \cdot p_X(x) \\ &= 0 \cdot p_X(0) + 1 \cdot p_X(1) \\ &= 0(1 - p) + 1(p) \\ &= p\end{aligned}$$

Step 2: Calculate $\mathbb{E}[X^2]$. (Using LOTUS $g(X) = X^2$)

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_x x^2 \cdot p_X(x) \\ &= 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) \\ &= 0(1 - p) + 1(p) \\ &= p\end{aligned}$$

Step 3: Calculate Var(X). (Using Computational Formula)

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= p - (p)^2 \\ &= p(1 - p)\end{aligned}$$

Exercise 15: Finding PMF from Expectation Information

A discrete RV X takes values in $\{-1, 0, 1\}$. Its PMF is defined by $p_X(-1) = a, p_X(0) = b, p_X(1) = c$. We are given that $\mathbb{E}[X] = 0.5$ and $\mathbb{E}[X^2] = 0.7$. Find the values of a, b, c.

Solution: We have three unknown probabilities a, b, c, and three conditions:

Condition 1 (Unit Sum):

$$a + b + c = 1 \quad (1)$$

Condition 2 (Expectation $\mathbb{E}[X]$):

$$\begin{aligned} \mathbb{E}[X] &= \sum x p_X(x) \\ 0.5 &= (-1)a + (0)b + (1)c \\ 0.5 &= -a + c \quad (2) \end{aligned}$$

Condition 3 (Expectation $\mathbb{E}[X^2]$):

$$\begin{aligned} \mathbb{E}[X^2] &= \sum x^2 p_X(x) \\ 0.7 &= (-1)^2 a + (0)^2 b + (1)^2 c \\ 0.7 &= a + c \quad (3) \end{aligned}$$

Now we solve the system of linear equations (2) and (3) for a and c:

$$\begin{aligned} (3) + (2): \quad & (a + c) + (-a + c) = 0.7 + 0.5 \\ & 2c = 1.2 \implies c = 0.6 \\ (3) - (2): \quad & (a + c) - (-a + c) = 0.7 - 0.5 \\ & 2a = 0.2 \implies a = 0.1 \end{aligned}$$

Substitute a and c into Equation (1) to find b:

$$\begin{aligned} a + b + c &= 1 \\ 0.1 + b + 0.6 &= 1 \\ b + 0.7 &= 1 \implies b = 0.3 \end{aligned}$$

Thus, the PMF is $p_X(-1) = 0.1, p_X(0) = 0.3, p_X(1) = 0.6$.

8 Continuous Random Variables and Distributions

8.1 Probability Density Function (PDF)

Definition 8.1 (Probability Density Function (PDF))**.** An RV X is continuous if there exists a non-negative function $f_X(x)$, the PDF, such that for any set $B \subset \mathbb{R}$,

$$\mathbb{P}(X \in B) = \int_B f_X(x)dx$$

For an interval $[a, b]$, this means $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$.

Property 8.1 (Properties of a PDF)**.** A function $f(x)$ is a valid PDF if:

(i) **Non-negativity:** $f_X(x) \geq 0$ for all x.

(ii) **Unit Integral:** $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Note: For any continuous RV, $\mathbb{P}(X = a) = \int_a^a f_X(x)dx = 0$.

Exercise 16: Determining the Constant in a PDF

The continuous RV X has the PDF $f_X(x) = c(4x - 2x^2)$ for $0 \leq x \leq 2$, and $f_X(x) = 0$ otherwise. Find the value of the constant c.
Solution: For $f_X(x)$ to be a valid PDF, the Unit Integral property must be satisfied: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

$$\begin{aligned}
\int_0^2 c(4x - 2x^2)dx &= 1 \\
c\left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2 &= 1 \\
c\left[2x^2 - \frac{2}{3}x^3\right]_0^2 &= 1 \\
c\left[\left(2(2)^2 - \frac{2}{3}(2)^3\right) - \left(2(0)^2 - \frac{2}{3}(0)^3\right)\right] &= 1 \\
c\left[8 - \frac{16}{3}\right] &= 1 \\
c\left[\frac{24-16}{3}\right] &= 1 \\
c\left(\frac{8}{3}\right) &= 1 \\
c &= \frac{3}{8}
\end{aligned}$$

Exercise 17: Calculating Probability from a PDF

Using the PDF from Exercise 16, $f_X(x) = \frac{3}{8}(4x - 2x^2)$ for $0 \leq x \leq 2$, calculate the probability $\mathbb{P}(1 \leq X \leq 1.5)$.

Solution: The probability is calculated by integrating the PDF over the interval $[1, 1.5]$.

$$\begin{aligned}
\mathbb{P}(1 \le X \le 1.5) &= \int_1^{1.5} f_X(x)dx \\
&= \int_1^{1.5} \frac{3}{8}(4x - 2x^2)dx \\
&= \frac{3}{8}\left[2x^2 - \frac{2}{3}x^3\right]_1^{1.5} \\
&= \frac{3}{8}\left[\left(2(1.5)^2 - \frac{2}{3}(1.5)^3\right) - \left(2(1)^2 - \frac{2}{3}(1)^3\right)\right] \\
&= \frac{3}{8}\left[\left(2(2.25) - \frac{2}{3}(3.375)\right) - \left(2 - \frac{2}{3}\right)\right] \\
&= \frac{3}{8}\left[(4.5 - 2.25) - \left(\frac{4}{3}\right)\right] \\
&= \frac{3}{8}\left[2.25 - \frac{4}{3}\right] = \frac{3}{8}\left[\frac{9}{4} - \frac{4}{3}\right] \\
&= \frac{3}{8}\left[\frac{27 - 16}{12}\right] = \frac{3}{8}\left[\frac{11}{12}\right] \\
&= \frac{33}{96} = \frac{11}{32}
\end{aligned}$$

Exercise 18: PDF for an Exponential Distribution

An RV X represents the lifetime (in years) of a device. Its PDF is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$, and 0 otherwise. If the average lifetime is 1/5 year, find the value of λ and calculate the probability that the device lasts longer than 1 year.
Solution: This is an Exponential distribution, $\text{Exp}(\lambda)$, where $\mathbb{E}[X] = 1/\lambda$.

- Given average lifetime $\mathbb{E}[X] = 1/5$ year.
- $1/\lambda = 1/5 \implies \lambda = 5$.

The PDF is $f_X(x) = 5e^{-5x}$ for $x > 0$. We need to calculate $\mathbb{P}(X > 1)$:

$$\begin{aligned}
\mathbb{P}(X > 1) &= \int_1^{\infty} 5e^{-5x}dx \\
&= 5\left[\frac{e^{-5x}}{-5}\right]_1^{\infty} \\
&= \left[-e^{-5x}\right]_1^{\infty} \\
&= \left(\lim_{x\to\infty}(-e^{-5x})\right) - (-e^{-5(1)}) \\
&= 0 - (-e^{-5}) = e^{-5}
\end{aligned}$$

$\mathbb{P}(X > 1) = e^{-5} \approx 0.0067$.

Exercise 19: The PDF of a Transformed RV (Calculus)

Let X be a continuous RV uniformly distributed on $[0, 1]$, so $f_X(x) = 1$ for $0 \le x \le 1$. Let $Y = -\ln(X)$. Find the PDF $f_Y(y)$ for $y > 0$.

Solution: We use the method of transformations. First find the CDF of Y:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-\ln(X) \leq y)$$

Since $X \in [0, 1]$, $Y = -\ln(X)$ is non-negative, so $F_Y(y) = 0$ for $y \leq 0$. For $y > 0$:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(\ln(X) \geq -y) \\ &= \mathbb{P}(X \geq e^{-y}) \quad \text{(Since } e^x \text{ is an increasing function)} \end{aligned}$$

Since X is uniform on $[0, 1]$, $\mathbb{P}(X \geq a) = \int_a^1 f_X(x)dx = \int_a^1 1dx = 1 - a$ for $0 \leq a \leq 1$. Here, $a = e^{-y}$. Since $y > 0$, $0 < e^{-y} < 1$.

$$F_Y(y) = 1 - e^{-y}$$

To find the PDF $f_Y(y)$, we differentiate the CDF: $f_Y(y) = \frac{d}{dy}F_Y(y)$.

$$f_Y(y) = \frac{d}{dy}(1 - e^{-y}) = -(-e^{-y}) = e^{-y}$$

Thus, Y follows an Exponential distribution with $\lambda = 1$:

$$f_Y(y) = \begin{cases} e^{-y}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 20: PDF for a Piecewise Function

A continuous RV X has the PDF:

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 \leq x < 1 \\ c(2 - x), & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c.

Solution: The Unit Integral property must hold: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

$$\begin{aligned} \int_0^1 \frac{x}{2}dx + \int_1^2 c(2 - x)dx &= 1 \\ \left[\frac{x^2}{4}\right]_0^1 + c\left[2x - \frac{x^2}{2}\right]_1^2 &= 1 \\ \left(\frac{1}{4} - 0\right) + c\left[\left(2(2) - \frac{2^2}{2}\right) - \left(2(1) - \frac{1^2}{2}\right)\right] &= 1 \\ \frac{1}{4} + c\left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] &= 1 \\ \frac{1}{4} + c\left[2 - \frac{3}{2}\right] &= 1 \\ \frac{1}{4} + c\left[\frac{1}{2}\right] &= 1 \\ c\left(\frac{1}{2}\right) &= 1 - \frac{1}{4} = \frac{3}{4} \\ c &= \frac{3}{4} \cdot 2 = \frac{3}{2} \end{aligned}$$

8.2 Cumulative Distribution Function (CDF)

Definition 8.2 (Cumulative Distribution Function (CDF))**.** The CDF of a continuous RV X, denoted $F_X(x)$, is defined as:

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

Property 8.2 (Properties of a CDF)**.** (i) **Limits:** $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

(ii) **Non-decreasing:** If $a \leq b$, then $F_X(a) \leq F_X(b)$.

(iii) **Continuous:** $F_X(x)$ is a continuous function.

(iv) **Relationship to PDF:** $f_X(x) = \frac{d}{dx}F_X(x) = F'_X(x)$ at all points where F_X is differentiable.

(v) **Probability from CDF:** $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$. (Note: For continuous RVs, $\mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) = \ldots$ all are equal).

Exercise 21: Constructing CDF from PDF (Uniform)

Let $X \sim \text{Uniform}(0, 3)$. The PDF is $f_X(x) = 1/3$ for $0 \leq x \leq 3$, and 0 otherwise. Find the functional form of the CDF, $F_X(x)$.
Solution: The CDF $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$.

- **Case 1:** $x < 0$

$$F_X(x) = \int_{-\infty}^{x} 0dt = 0$$

- **Case 2:** $0 \leq x < 3$

$$F_X(x) = \int_{0}^{x} \frac{1}{3}dt = \frac{1}{3}[t]_0^x = \frac{x}{3}$$

- **Case 3:** $x \geq 3$

$$F_X(x) = \int_{0}^{3} \frac{1}{3}dt = 1$$

The CDF is:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x/3, & \text{if } 0 \leq x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}$$

Exercise 22: Finding PDF from CDF

A continuous RV X has the CDF $F_X(x) = 1 - e^{-(x-1)}$ for $x \geq 1$, and $F_X(x) = 0$ otherwise. Find the PDF $f_X(x)$.
Solution: The PDF is $f_X(x) = \frac{d}{dx}F_X(x)$.

- **Case 1:** $x < 1$

$$f_X(x) = \frac{d}{dx}(0) = 0$$

- **Case 2:** $x \geq 1$

$$f_X(x) = \frac{d}{dx}(1 - e^{-(x-1)}) = -(-1)e^{-(x-1)} = e^{-(x-1)}$$

The PDF is:

$$f_X(x) = \begin{cases} e^{-(x-1)}, & \text{if } x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 23: Calculating Probability from CDF

Given the CDF from Exercise 22, $F_X(x) = 1 - e^{-(x-1)}$ for $x \geq 1$. Calculate $\mathbb{P}(1 < X \leq 3)$.
Solution: We use the property $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$. Here $a = 1$ and $b = 3$.

$$\begin{aligned}
\mathbb{P}(1 < X \leq 3) &= F_X(3) - F_X(1) \\
F_X(3) &= 1 - e^{-(3-1)} = 1 - e^{-2} \\
F_X(1) &= 1 - e^{-(1-1)} = 1 - e^{0} = 1 - 1 = 0 \\
\mathbb{P}(1 < X \leq 3) &= (1 - e^{-2}) - 0 \\
&= 1 - e^{-2}
\end{aligned}$$

Exercise 24: Finding the Median using the CDF

Let X have the PDF $f_X(x) = 2x$ for $0 < x < 1$, and 0 otherwise. Find the median m of the distribution, which is defined such that $F_X(m) = 0.5$.
Solution: Step 1: Find the CDF $F_X(x)$. For $0 \leq x \leq 1$:

$$F_X(x) = \int_0^x 2t dt = \left[t^2\right]_0^x = x^2$$

Step 2: Find the median m. We solve for m such that $F_X(m) = 0.5$. Since $0 < 0.5 < 1$, the median must be in the range $(0, 1)$.

$$\begin{aligned}
F_X(m) &= m^2 \\
m^2 &= 0.5 \\
m &= \sqrt{0.5} = \frac{1}{\sqrt{2}} \approx 0.707
\end{aligned}$$

Since $0 < 1/\sqrt{2} < 1$, the median is $m = 1/\sqrt{2}$.

Exercise 25: CDF for a Piecewise PDF

Find the CDF $F_X(x)$ for the PDF from Exercise 20, $f_X(x)$:

$$f_X(x) = \begin{cases} x/2, & 0 \leq x < 1 \\ 3/2(2-x), & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution: Case 1: $x < 0 \implies F_X(x) = 0$.
Case 2: $0 \leq x < 1$

$$F_X(x) = \int_0^x \frac{t}{2} dt = \left[\frac{t^2}{4}\right]_0^x = \frac{x^2}{4}$$

Case 3: $1 \leq x < 2$

$$F_X(x) = \int_0^1 \frac{t}{2} dt + \int_1^x \frac{3}{2}(2-t) dt$$

The first integral is $F_X(1) = 1^2/4 = 1/4$. The second integral is:

$$\begin{aligned}
\int_1^x \frac{3}{2}(2-t) dt &= \frac{3}{2}\left[2t - \frac{t^2}{2}\right]_1^x \\
&= \frac{3}{2}\left[\left(2x - \frac{x^2}{2}\right) - \left(2(1) - \frac{1^2}{2}\right)\right] \\
&= \frac{3}{2}\left[2x - \frac{x^2}{2} - \frac{3}{2}\right]
\end{aligned}$$

$$F_X(x) = \frac{1}{4} + \frac{3}{2}\left(2x - \frac{x^2}{2} - \frac{3}{2}\right) = \frac{1}{4} + 3x - \frac{3x^2}{4} - \frac{9}{4} = 3x - \frac{3x^2}{4} - 2$$

Case 4: $x \geq 2$ $F_X(x) = F_X(2) = 3(2) - \frac{3(2)^2}{4} - 2 = 6 - 3 - 2 = 1$. The CDF is:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/4, & \text{if } 0 \leq x < 1 \\ 3x - 3x^2/4 - 2, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

8.3 Expectation and Variance (Continuous)

Definition 8.3 (Expectation (Mean))**.** The Expected Value of a continuous RV X is:

$$\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

The expectation exists only if $\int_{-\infty}^{\infty} |x| \cdot f_X(x)dx < \infty$.

Theorem 8.1 (Law of the Unconscious Statistician (LOTUS) Continuous)**.** Let $g(X)$ be a function of the continuous RV X. The expected value of $g(X)$ is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$$

Definition 8.4 (Variance)**.** The Variance of a continuous RV X is:

$$\sigma^2 = \mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)dx$$

Formula 8.1 (Computational Formula for Variance)**.**

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \left(\int_{-\infty}^{\infty} x^2 f_X(x)dx\right) - \left(\int_{-\infty}^{\infty} x f_X(x)dx\right)^2$$

Exercise 26: Expectation and Variance Calculation (Uniform)

Let $X \sim \mathrm{Uniform}(a, b)$. The PDF is $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and 0 otherwise. Show that $\mathbb{E}[X] = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.

Solution: Let $w = b - a$. $f_X(x) = 1/w$.

Step 1: Calculate $\mathbb{E}[X]$.

$$\begin{aligned} \mathbb{E}[X] &= \int_a^b x \cdot \frac{1}{w}dx = \frac{1}{w}\left[\frac{x^2}{2}\right]_a^b \\ &= \frac{1}{2w}(b^2 - a^2) = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

Step 2: Calculate $\mathbb{E}[X^2]$.

$$\begin{aligned} \mathbb{E}[X^2] &= \int_a^b x^2 \cdot \frac{1}{w}dx = \frac{1}{w}\left[\frac{x^3}{3}\right]_a^b \\ &= \frac{1}{3w}(b^3 - a^3) = \frac{b^3 - a^3}{3(b-a)} \end{aligned}$$

Note the identity: $b^3 - a^3 = (b-a)(b^2+ab+a^2)$.

$$\mathbb{E}[X^2] = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Step 3: Calculate Var(X).

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12} \\
&= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12} \\
&= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$

Exercise 27: Expected Value of a Transformed RV (LOTUS)

Let X have the PDF $f_X(x) = \frac{3}{8}(4x-2x^2)$ for $0 \le x \le 2$. Calculate $\mathbb{E}[e^X]$.
Solution: We use LOTUS: $\mathbb{E}[e^X] = \int_0^2 e^x f_X(x)dx = \int_0^2 e^x \frac{3}{8}(4x-2x^2)dx$. This requires integration by parts, but we can simplify using the linearity property of integration first.

$$\mathbb{E}[e^X] = \frac{3}{8}\int_0^2 (4xe^x - 2x^2e^x)dx = \frac{3}{8}\left(4\int_0^2 xe^x dx - 2\int_0^2 x^2e^x dx\right)$$

Using the integral formulas (via integration by parts):

- $\int xe^x dx = (x-1)e^x$
- $\int x^2e^x dx = (x^2-2x+2)e^x$

$$\begin{aligned}
\mathbb{E}[e^X] &= \frac{3}{8}\left[4\left[(x-1)e^x\right]_0^2 - 2\left[(x^2-2x+2)e^x\right]_0^2\right] \\
&= \frac{3}{8}\left[4\left(((2-1)e^2) - ((0-1)e^0)\right) - 2\left(((2^2-4+2)e^2) - ((0-0+2)e^0)\right)\right] \\
&= \frac{3}{8}\left[4(e^2-(-1)) - 2((2e^2)-(2))\right] \\
&= \frac{3}{8}\left[4e^2+4-4e^2+4\right] \\
&= \frac{3}{8}(8) = 3
\end{aligned}$$

Exercise 28: Expectation and Variance for an Exponential Distribution

Let $X \sim \mathrm{Exp}(\lambda)$. The PDF is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$. Use integration by parts to show that $\mathbb{E}[X] = 1/\lambda$ and $\mathbb{E}[X^2] = 2/\lambda^2$, and thus $\mathrm{Var}(X) = 1/\lambda^2$.

Solution: Step 1: Calculate $\mathbb{E}[X]$. Use integration by parts: $\int u dv = uv - \int v du$. Let $u = x$, $dv = \lambda e^{-\lambda x} dx$. Then $du = dx$, $v = -e^{-\lambda x}$.

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x\lambda e^{-\lambda x} dx \\
&= \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty (-e^{-\lambda x}) dx \\
&= (0-0) + \int_0^\infty e^{-\lambda x} dx \\
&= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty \\
&= (0) - \left(-\frac{1}{\lambda} e^0\right) = \frac{1}{\lambda}
\end{aligned}$$

Step 2: Calculate $\mathbb{E}[X^2]$. Use integration by parts again: $\int u dv = uv - \int v du$. Let $u = x^2$, $dv = \lambda e^{-\lambda x} dx$. Then $du = 2x dx$, $v = -e^{-\lambda x}$.

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\
&= \left[-x^2 e^{-\lambda x}\right]_0^\infty - \int_0^\infty (-e^{-\lambda x}) 2x dx \\
&= (0-0) + \frac{2}{\lambda} \int_0^\infty x\lambda e^{-\lambda x} dx
\end{aligned}$$

The remaining integral is exactly $\mathbb{E}[X]$, which is $1/\lambda$.

$$\mathbb{E}[X^2] = \frac{2}{\lambda} \cdot \mathbb{E}[X] = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

Step 3: Calculate Var(X).

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exercise 29: Expectation of a Function with Absolute Value

Let $X \sim \text{Uniform}(-1, 1)$, so $f_X(x) = 1/2$ for $-1 \le x \le 1$. Calculate $\mathbb{E}[|X|]$.
Solution: We use LOTUS with $g(X) = |X|$.

$$\mathbb{E}[|X|] = \int_{-1}^1 |x| f_X(x) dx = \int_{-1}^1 |x| \frac{1}{2} dx$$

Since $|x|$ is an even function and the interval $[-1, 1]$ is symmetric about 0:

$$\mathbb{E}[|X|] = 2 \int_0^1 x \frac{1}{2} dx = \int_0^1 x dx$$

$$\begin{aligned}
\int_0^1 x dx &= \left[\frac{x^2}{2}\right]_0^1 \\
&= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}
\end{aligned}$$

Thus, $\mathbb{E}[|X|] = 1/2$.

Exercise 30: Calculating Moments for a Truncated PDF

Let X have the PDF $f_X(x) = \frac{3}{x^4}$ for $x > 1$, and 0 otherwise. This is a Pareto distribution. Calculate the expected value $\mathbb{E}[X]$ and state why the variance does not exist.
Solution: Step 1: Calculate $\mathbb{E}[X]$.

$$\begin{aligned}
\mathbb{E}[X] &= \int_1^\infty x f_X(x)dx = \int_1^\infty x \cdot \frac{3}{x^4} dx \\
&= 3 \int_1^\infty x^{-3} dx \\
&= 3 \left[\frac{x^{-2}}{-2} \right]_1^\infty = -\frac{3}{2} \left[\frac{1}{x^2} \right]_1^\infty \\
&= -\frac{3}{2} \left(\lim_{x\to\infty} \frac{1}{x^2} - \frac{1}{1^2} \right) \\
&= -\frac{3}{2}(0-1) = \frac{3}{2}
\end{aligned}$$

Since $\mathbb{E}[X]$ is finite, the expectation exists. $\mathbb{E}[X] = 3/2$.

Step 2: Calculate $\mathbb{E}[X^2]$.

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_1^\infty x^2 f_X(x)dx = \int_1^\infty x^2 \cdot \frac{3}{x^4} dx \\
&= 3 \int_1^\infty x^{-2} dx \\
&= 3 \left[\frac{x^{-1}}{-1} \right]_1^\infty = -3 \left[\frac{1}{x} \right]_1^\infty \\
&= -3 \left(\lim_{x\to\infty} \frac{1}{x} - \frac{1}{1} \right) \\
&= -3(0-1) = 3
\end{aligned}$$

Since $\mathbb{E}[X^2]$ is finite, the variance $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ exists and is:

$$\text{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{12-9}{4} = \frac{3}{4}$$

(Self-Correction/Note to User: The prompt in the exercise stated the variance does not exist, which is true for Pareto distributions with $\alpha \leq 2$ (here $\alpha = 4$). In this case, both $\mathbb{E}[X]$ and Var(X) exist.)

9 Jointly Distributed Random Variables

9.1 Joint Distributions (Discrete and Continuous)

Definition 9.1 (Joint PMF (Discrete))**.** For two discrete RVs X and Y, the Joint PMF is:

$$p(x,y) = \mathbb{P}(X = x, Y = y)$$

Properties: (i) $p(x,y) \geq 0$, (ii) $\sum_x \sum_y p(x,y) = 1$

Definition 9.2 (Joint PDF (Continuous))**.** For two continuous RVs X and Y, the Joint PDF $f(x,y)$ satisfies:

$$\mathbb{P}((X,Y) \in A) = \iint_A f(x,y)dxdy$$

Properties: (i) $f(x,y) \geq 0$, (ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

Definition 9.3 (Joint CDF)**.**

$$F(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$

- **Discrete:** $F(x,y) = \sum_{s \leq x} \sum_{t \leq y} p(s,t)$.
- **Continuous:** $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dtds$.
- **PDF from CDF:** $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$.

Exercise 31: Determining the Constant in a Joint PMF

Let X and Y be discrete RVs with joint PMF $p(x,y) = c(x+y)$ for $x = 1,2$ and $y = 1,2$. Find the value of the constant c.
Solution: For $p(x,y)$ to be a valid joint PMF, the sum of all probabilities must be 1.

$$\sum_{x=1}^{2} \sum_{y=1}^{2} p(x,y) = 1$$

$$\begin{aligned} \sum_{x=1}^{2} \sum_{y=1}^{2} c(x+y) &= c\left[(1+1) + (1+2) + (2+1) + (2+2)\right] \\ &= c[2+3+3+4] \\ &= 12c \end{aligned}$$

Setting $12c = 1$, we find $c = 1/12$.

Exercise 32: Calculating Probability from a Joint PMF

Using the joint PMF from Exercise 31, $p(x,y) = \frac{1}{12}(x+y)$, calculate the probability $\mathbb{P}(X+Y \leq 3)$.
Solution: The possible pairs (x,y) such that $x+y \leq 3$ for $x \in \{1,2\}, y \in \{1,2\}$ are $(1,1), (1,2), (2,1)$.

- $\mathbb{P}(X = 1, Y = 1) = p(1,1) = \frac{1}{12}(1+1) = \frac{2}{12}$
- $\mathbb{P}(X = 1, Y = 2) = p(1,2) = \frac{1}{12}(1+2) = \frac{3}{12}$
- $\mathbb{P}(X = 2, Y = 1) = p(2,1) = \frac{1}{12}(2+1) = \frac{3}{12}$

$$\mathbb{P}(X+Y \leq 3) = \mathbb{P}(1,1) + \mathbb{P}(1,2) + \mathbb{P}(2,1) = \frac{2}{12} + \frac{3}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

Exercise 33: Determining the Constant in a Joint PDF

The joint PDF of continuous RVs X and Y is $f(x, y) = cxy$ for $0 < x < 1$ and $0 < y < 2$, and 0 otherwise. Find the value of the constant c.
Solution: For $f(x, y)$ to be a valid joint PDF, the Unit Integral property must be satisfied:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

$$\begin{aligned}
\int_0^2\int_0^1 cxydxdy &= 1\\
c\int_0^2 y\left(\int_0^1 xdx\right)dy &= 1\\
c\int_0^2 y\left[\frac{x^2}{2}\right]_0^1 dy &= 1\\
c\int_0^2 y\left(\frac{1}{2}\right)dy &= 1\\
\frac{c}{2}\int_0^2 ydy &= 1\\
\frac{c}{2}\left[\frac{y^2}{2}\right]_0^2 &= 1\\
\frac{c}{2}\left(\frac{4}{2}\right) &= 1\\
\frac{c}{2}(2) &= 1\\
c &= 1
\end{aligned}$$

Exercise 34: Calculating Probability from a Joint PDF over a Region

Using the joint PDF from Exercise 33, $f(x, y) = xy$ for $0 < x < 1, 0 < y < 2$, calculate $\mathbb{P}(Y > X)$.
Solution: We need to integrate $f(x, y)$ over the region $A = \{(x, y) : 0 < x < 1, 0 < y < 2, y > x\}$. The integration boundaries are: x from 0 to 1, and y from x to 2.

$$\mathbb{P}(Y > X) = \int_0^1\int_x^2 xydydx$$

$$\begin{aligned}
\int_0^1 x\left[\int_x^2 ydy\right]dx &= \int_0^1 x\left[\frac{y^2}{2}\right]_x^2 dx\\
&= \int_0^1 x\left(\frac{4}{2} - \frac{x^2}{2}\right)dx\\
&= \int_0^1 x\left(2 - \frac{x^2}{2}\right)dx\\
&= \int_0^1 \left(2x - \frac{x^3}{2}\right)dx\\
&= \left[x^2 - \frac{x^4}{8}\right]_0^1\\
&= \left(1 - \frac{1}{8}\right) - 0 = \frac{7}{8}
\end{aligned}$$

Exercise 35: Joint CDF and Finding Probability in a Rectangle

The joint CDF of X and Y is $F(x, y) = (1 - e^{-x^2})(1 - e^{-y^2})$ for $x > 0, y > 0$. Calculate $\mathbb{P}(1 < X \leq 2, 0 < Y \leq 1)$.
Solution: For a joint CDF, the probability over a rectangular region $[x_1, x_2] \times [y_1, y_2]$ is:

$$\mathbb{P}(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

Here, $x_1 = 1, x_2 = 2, y_1 = 0, y_2 = 1$. Since $F(x, y)$ is defined only for $x > 0, y > 0$, we need to check $F(x, 0)$ and $F(0, y)$.

$$F(x, 0) = (1 - e^{-x^2})(1 - e^{-0^2}) = (1 - e^{-x^2})(1 - 1) = 0$$

$$F(0, y) = (1 - e^{-0^2})(1 - e^{-y^2}) = 0(1 - e^{-y^2}) = 0$$

Since $F(x, y)$ can be factored into $F_X(x)F_Y(y)$, X and Y are independent (see IX.C). The probability can be calculated as a product of marginal probabilities:

$$\begin{aligned}\mathbb{P}(1 < X \leq 2, 0 < Y \leq 1) &= \mathbb{P}(1 < X \leq 2) \cdot \mathbb{P}(0 < Y \leq 1) \\ &= (F_X(2) - F_X(1)) \cdot (F_Y(1) - F_Y(0))\end{aligned}$$

Where $F_X(x) = 1 - e^{-x^2}$ and $F_Y(y) = 1 - e^{-y^2}$ for positive arguments.

- $F_X(2) - F_X(1) = (1 - e^{-4}) - (1 - e^{-1}) = e^{-1} - e^{-4}$
- $F_Y(1) - F_Y(0) = (1 - e^{-1}) - (1 - e^{-0}) = (1 - e^{-1}) - 0 = 1 - e^{-1}$

$$\mathbb{P}(1 < X \leq 2, 0 < Y \leq 1) = (e^{-1} - e^{-4})(1 - e^{-1})$$

9.2 Marginal Distributions

Definition 9.4 (Marginal Distributions)**.** The distribution of a single RV from a joint distribution.

- **Discrete (Marginal PMF):** $p_X(x) = \sum_y p(x, y) = \mathbb{P}(X = x)$.
- **Continuous (Marginal PDF):** $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$.

Exercise 36: Finding Marginal PMFs from a Joint Table

The joint PMF $p(x, y)$ of X and Y is given by the table:

$p(x, y)$	$Y = 0$	$Y = 1$	$p_X(x)$
$X = 1$	0.1	0.2	0.3
$X = 2$	0.3	0.4	0.7
$p_Y(y)$	0.4	0.6	1.0

(The marginals are already provided in the table, but verify and state the marginal PMFs $p_X(x)$ and $p_Y(y)$ formally.)
Solution: Marginal PMF $p_X(x)$ (summing over y):

- $p_X(1) = \sum_y p(1, y) = p(1, 0) + p(1, 1) = 0.1 + 0.2 = 0.3$
- $p_X(2) = \sum_y p(2, y) = p(2, 0) + p(2, 1) = 0.3 + 0.4 = 0.7$

$$p_X(x) = \begin{cases} 0.3, & \text{if } x = 1 \\ 0.7, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Marginal PMF $p_Y(y)$ (summing over x):

- $p_Y(0) = \sum_x p(x, 0) = p(1, 0) + p(2, 0) = 0.1 + 0.3 = 0.4$
- $p_Y(1) = \sum_x p(x, 1) = p(1, 1) + p(2, 1) = 0.2 + 0.4 = 0.6$

$$p_Y(y) = \begin{cases} 0.4, & \text{if } y = 0 \\ 0.6, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 37: Finding Marginal PDFs (Continuous)

Let X and Y have the joint PDF $f(x, y) = x + y$ for $0 < x < 1$ and $0 < y < 1$, and 0 otherwise. Find the marginal PDF $f_X(x)$.

Solution: The marginal PDF $f_X(x)$ is found by integrating the joint PDF over the entire range of Y. For $0 < x < 1$:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 (x + y) dy \\ &= \left[xy + \frac{y^2}{2} \right]_0^1 \\ &= \left(x(1) + \frac{1^2}{2} \right) - \left(x(0) + \frac{0^2}{2} \right) \\ &= x + \frac{1}{2} \end{aligned}$$

The marginal PDF is:

$$f_X(x) = \begin{cases} x + 1/2, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(Check: $\int_0^1 (x + 1/2) dx = [x^2/2 + x/2]_0^1 = 1/2 + 1/2 = 1$).

Exercise 38: Finding Marginal PDFs for Non-Rectangular Support

The joint PDF is $f(x, y) = 2$ on the region $0 < x < 1, 0 < y < x$, and 0 otherwise. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution: Step 1: Find $f_X(x)$. For a fixed x, y ranges from 0 to x. Since x itself ranges from 0 to 1: For $0 < x < 1$:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^x 2 dy = [2y]_0^x = 2x$$

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Step 2: Find $f_Y(y)$. For a fixed y, x must be greater than y and less than 1. x ranges from y to 1. Since y itself ranges from 0 to 1: For $0 < y < 1$:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_y^1 2 dx = [2x]_y^1 = 2(1) - 2(y) = 2(1 - y)$$

$$f_Y(y) = \begin{cases} 2(1-y), & \text{if } 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 39: Expected Value using Marginal PMF

Using the marginal PMF $p_X(x)$ from Exercise 36 ($p_X(1) = 0.3, p_X(2) = 0.7$), calculate the expected value $\mathbb{E}[X]$.
Solution: The expected value of X only requires the marginal PMF $p_X(x)$:

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x)$$

$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot p_X(1) + 2 \cdot p_X(2) \\ &= 1(0.3) + 2(0.7) \\ &= 0.3 + 1.4 = 1.7 \end{aligned}$$

Exercise 40: Finding Expected Value using Marginal PDF

Using the marginal PDF $f_X(x)$ from Exercise 38 ($f_X(x) = 2x$ for $0 < x < 1$), calculate the expected value $\mathbb{E}[X]$.
Solution: The expected value of X is calculated using the marginal PDF $f_X(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\begin{aligned} \mathbb{E}[X] &= \int_0^1 x(2x) dx \\ &= \int_0^1 2x^2 dx \\ &= \left[\frac{2x^3}{3}\right]_0^1 \\ &= \frac{2(1)^3}{3} - 0 = \frac{2}{3} \end{aligned}$$

9.3 Independence and Covariance

Definition 9.5 (Statistical Independence of RVs)**.** Two RVs X and Y are independent if for all x, y:

- **General:** $F(x, y) = F_X(x)F_Y(y)$.
- **Discrete:** $p(x, y) = p_X(x)p_Y(y)$.
- **Continuous:** $f(x, y) = f_X(x)f_Y(y)$.

Theorem 9.1 (Expectation of a Product (Multiplication Theorem))**.** For any two RVs X, Y, and functions g, h:

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)] \quad \text{if } X, Y \text{ are independent.}$$

A special case is $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Definition 9.6 (Covariance)**.** The Covariance of X and Y measures their linear relationship:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

Formula 9.1 (Computational Formula for Covariance)**.**

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Property 9.1 (Covariance and Independence)**.** If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

Formula 9.2 (Variance of a Sum (General))**.**

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

If X, Y are independent, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Exercise 41: Testing for Independence (Discrete)

Use the joint PMF from Exercise 36 ($p_X(1) = 0.3, p_X(2) = 0.7$, $p_Y(0) = 0.4, p_Y(1) = 0.6$). Determine if X and Y are independent.
Solution: X and Y are independent if and only if $p(x, y) = p_X(x)p_Y(y)$ for all pairs (x, y). We check the condition for the pair $(1, 0)$:

- $p(1, 0) = 0.1$ (from the table)
- $p_X(1)p_Y(0) = (0.3)(0.4) = 0.12$

Since $p(1, 0) = 0.1 \neq 0.12 = p_X(1)p_Y(0)$, the condition for independence is violated. Therefore, X and Y are **not independent**.

Exercise 42: Testing for Independence (Continuous)

Let X and Y have the joint PDF $f(x, y) = x + y$ for $0 < x < 1, 0 < y < 1$. Determine if X and Y are independent.
Solution: From Exercise 37, the marginal PDFs are $f_X(x) = x + 1/2$ and $f_Y(y) = y + 1/2$ (due to symmetry). For independence, we must check if $f(x, y) = f_X(x)f_Y(y)$.

$$\begin{aligned} f_X(x)f_Y(y) &= (x + 1/2)(y + 1/2) \\ &= xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4} \end{aligned}$$

Since $f(x, y) = x + y$ and $x + y \neq xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4}$ for $x, y \in (0, 1)$, the condition for independence is violated. Therefore, X and Y are **not independent**.

Exercise 43: Calculating Covariance (Discrete)

Using the results from Exercise 36, calculate the covariance $\text{Cov}(X, Y)$.
Solution: We use the formula $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Step 1: Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. From Exercise 39, $\mathbb{E}[X] = 1.7$.

$$\mathbb{E}[Y] = \sum_y y p_Y(y) = 0 \cdot p_Y(0) + 1 \cdot p_Y(1) = 0(0.4) + 1(0.6) = 0.6$$

Step 2: Calculate $\mathbb{E}[XY]$.

$$\mathbb{E}[XY] = \sum_x \sum_y xyp(x, y)$$

$$\begin{aligned}\mathbb{E}[XY] &= (1)(0)p(1,0) + (1)(1)p(1,1) + (2)(0)p(2,0) + (2)(1)p(2,1) \\ &= 0 + 1(0.2) + 0 + 2(0.4) \\ &= 0.2 + 0.8 = 1.0\end{aligned}$$

Step 3: Calculate Cov(X, Y).

$$\begin{aligned}\text{Cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= 1.0 - (1.7)(0.6) \\ &= 1.0 - 1.02 \\ &= -0.02\end{aligned}$$

The negative covariance confirms a weak negative linear relationship and is consistent with the finding that X and Y are not independent.

Exercise 44: Variance of a Sum

Let X and Y be RVs such that $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$, and $\text{Cov}(X, Y) = 1$. Find $\text{Var}(2X - 3Y + 5)$.
Solution: We use the properties of variance and covariance for linear combinations. Let $W = 2X - 3Y + 5$. Using $\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$, where $a = 2, b = -3, c = 5$.

$$\begin{aligned}\text{Var}(W) &= \text{Var}(2X) + \text{Var}(-3Y) + 2\text{Cov}(2X, -3Y) + \text{Var}(5) \\ &= 2^2\text{Var}(X) + (-3)^2\text{Var}(Y) + 2(2)(-3)\text{Cov}(X,Y) + 0 \\ &= 4(4) + 9(9) + (-12)(1) \\ &= 16 + 81 - 12 \\ &= 97 - 12 = 85\end{aligned}$$

Exercise 45: Applying Expectation of a Product

X and Y are independent RVs. $\mathbb{E}[X] = 3$, $\mathbb{E}[Y] = 2$, $\text{Var}(X) = 5$, $\text{Var}(Y) = 6$. Find $\mathbb{E}[(X^2+1)(Y-3)]$.
Solution: Since X and Y are independent, any function of X, say $g(X) = X^2 + 1$, is independent of any function of Y, say $h(Y) = Y - 3$. By the Expectation of a Product Theorem:

$$\mathbb{E}[(X^2+1)(Y-3)] = \mathbb{E}[X^2+1] \cdot \mathbb{E}[Y-3]$$

Step 1: Calculate $\mathbb{E}[X^2 + 1]$. Using $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, we find $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = 5 + 3^2 = 5 + 9 = 14$$

By linearity of expectation:

$$\mathbb{E}[X^2+1] = \mathbb{E}[X^2] + 1 = 14 + 1 = 15$$

Step 2: Calculate $\mathbb{E}[Y - 3]$. By linearity of expectation:

$$\mathbb{E}[Y-3] = \mathbb{E}[Y] - 3 = 2 - 3 = -1$$

Step 3: Calculate the product.

$$\mathbb{E}[(X^2+1)(Y-3)] = 15 \cdot (-1) = -15$$

10 Conditional Distributions, Mean, and Variance

10.1 Conditional Distributions

Definition 10.1 (Conditional PMF (Discrete))**.** The conditional PMF of X given $Y = y$ is:

$$p_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{p(x, y)}{p_Y(y)} \quad \text{for } \mathbb{P}(Y = y) > 0$$

This is a valid PMF, so $\sum_x p_{X|Y}(x|y) = 1$ for any fixed y.

Definition 10.2 (Conditional PDF (Continuous))**.** The conditional PDF of X given $Y = y$ is:

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad \text{where } f_Y(y) > 0$$

This is a valid PDF, so $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1$ for any fixed y.

Property 10.1 (Independence and Conditionals)**.** X and Y are independent if and only if $p_{X|Y}(x|y) = p_X(x)$ (or $f_{X|Y}(x|y) = f_X(x)$) for all x, y.

Exercise 46: Finding Conditional PMF

Using the joint PMF from Exercise 36, find the conditional PMF $p_{X|Y}(x|y = 1)$.

$p(x,y)$	$Y = 0$	$Y = 1$	$p_X(x)$
$X = 1$	0.1	0.2	0.3
$X = 2$	0.3	0.4	0.7
$p_Y(y)$	0.4	0.6	1.0

Solution: We condition on $Y = 1$. First, we need the marginal probability $p_Y(1) = 0.6$. The conditional PMF is $p_{X|Y}(x|1) = \frac{p(x,1)}{p_Y(1)}$.

- For $x = 1$:
$$p_{X|Y}(1|1) = \frac{p(1, 1)}{p_Y(1)} = \frac{0.2}{0.6} = \frac{1}{3}$$
- For $x = 2$:
$$p_{X|Y}(2|1) = \frac{p(2, 1)}{p_Y(1)} = \frac{0.4}{0.6} = \frac{2}{3}$$

Check: $p_{X|Y}(1|1) + p_{X|Y}(2|1) = 1/3 + 2/3 = 1$. The conditional PMF is valid.

Exercise 47: Finding Conditional PDF (Continuous)

Let X and Y have the joint PDF $f(x, y) = x + y$ for $0 < x < 1, 0 < y < 1$. Find the conditional PDF $f_{X|Y}(x|y)$.

Solution: The conditional PDF is $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. From Exercise 37 (by symmetry), the marginal PDF is $f_Y(y) = y + 1/2$ for $0 < y < 1$. Thus, for $0 < x < 1$ and a fixed $y \in (0, 1)$:

$$f_{X|Y}(x|y) = \frac{x + y}{y + 1/2} = \frac{x + y}{\frac{2y+1}{2}} = \frac{2(x + y)}{2y + 1}$$

This is the conditional PDF for X given $Y = y$. For other x values, $f_{X|Y}(x|y) = 0$.

Exercise 48: Calculating Probability using Conditional PDF

Using the conditional PDF $f_{X|Y}(x|y)$ from Exercise 47, calculate $\mathbb{P}(X < 0.5|Y = 0.5)$.
Solution: Step 1: Determine the conditional PDF $f_{X|Y}(x|Y = 0.5)$. Substitute $y = 0.5$ into the conditional PDF from Exercise 47:

$$f_{X|Y}(x|0.5) = \frac{2(x+0.5)}{2(0.5)+1} = \frac{2x+1}{1+1} = \frac{2x+1}{2} = x + \frac{1}{2} \quad \text{for } 0 < x < 1$$

Step 2: Calculate the probability.

$$\mathbb{P}(X < 0.5|Y = 0.5) = \int_0^{0.5} f_{X|Y}(x|0.5)dx$$

$$\begin{aligned}\mathbb{P}(X < 0.5|Y = 0.5) &= \int_0^{0.5} (x + 1/2)dx \\ &= \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^{0.5} \\ &= \left(\frac{(0.5)^2}{2} + \frac{0.5}{2}\right) - 0 \\ &= \frac{0.25}{2} + \frac{0.5}{2} = \frac{0.125 + 0.25}{1} = 0.375 \text{ or } \frac{3}{8}\end{aligned}$$

Exercise 49: Reversing the Conditional PDF

Let X and Y have the joint PDF $f(x, y) = 2$ on the region $0 < x < 1, 0 < y < x$. Find the conditional PDF $f_{Y|X}(y|x)$.
Solution: The conditional PDF is $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$. From Exercise 38, the marginal PDF is $f_X(x) = 2x$ for $0 < x < 1$. For a fixed $x \in (0, 1)$, the support of Y is $0 < y < x$.

$$f_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x} \quad \text{for } 0 < y < x$$

For other y values, $f_{Y|X}(y|x) = 0$. This means that given $X = x$, Y is uniformly distributed over the interval $(0, x)$.

Exercise 50: Independence Check using Conditional Distribution Property

The joint PDF is $f(x, y) = 4xy$ for $0 < x < 1, 0 < y < 1$. Find $f_{X|Y}(x|y)$ and use the result to determine if X and Y are independent.
Solution: Step 1: Find the marginal PDF $f_Y(y)$. For $0 < y < 1$:

$$f_Y(y) = \int_0^1 4xydx = 4y\left[\frac{x^2}{2}\right]_0^1 = 4y\left(\frac{1}{2}\right) = 2y$$

Step 2: Find the conditional PDF $f_{X|Y}(x|y)$. For $0 < x < 1$ and $0 < y < 1$:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{4xy}{2y} = 2x$$

Step 3: Check for Independence. The marginal PDF of X is:

$$f_X(x) = \int_0^1 4xydy = 4x\left[\frac{y^2}{2}\right]_0^1 = 4x\left(\frac{1}{2}\right) = 2x$$

Since $f_{X|Y}(x|y) = 2x$ and $f_X(x) = 2x$, we have $f_{X|Y}(x|y) = f_X(x)$. Therefore, X and Y **are independent**.

10.2 Conditional Expectation

Definition 10.3 (Conditional Expectation)**.** The Conditional Expectation of X given $Y = y$ is the mean of the conditional distribution:

- **Discrete:** $\mathbb{E}[X|Y = y] = \sum_x x \cdot p_{X|Y}(x|y)$
- **Continuous:** $\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)dx$

Note: $\mathbb{E}[X|Y = y]$ is a number. The function $g(y) = \mathbb{E}[X|Y = y]$ defines a new random variable $\mathbb{E}[X|Y] = g(Y)$.

Theorem 10.1 (Law of Total Expectation (Adam's Law / Tower Property))**.** The expected value of X is the expected value of its conditional expectation given Y.

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$$

Exercise 51: Conditional Expectation (Discrete)

Using the results from Exercise 46, calculate the conditional expectation $\mathbb{E}[X|Y = 1]$.
Solution: From Exercise 46, the conditional PMF is $p_{X|Y}(1|1) = 1/3$ and $p_{X|Y}(2|1) = 2/3$.

$$\mathbb{E}[X|Y = 1] = \sum_x x \cdot p_{X|Y}(x|1)$$

$$\begin{aligned}
\mathbb{E}[X|Y = 1] &= 1 \cdot p_{X|Y}(1|1) + 2 \cdot p_{X|Y}(2|1) \\
&= 1\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) \\
&= \frac{1}{3} + \frac{4}{3} = \frac{5}{3}
\end{aligned}$$

Exercise 52: Conditional Expectation Function (Continuous)

Let X and Y have the joint PDF $f(x, y) = x + y$ for $0 < x < 1, 0 < y < 1$. Find the conditional expectation function $\mathbb{E}[X|Y]$.
Solution: Step 1: Use the conditional PDF $f_{X|Y}(x|y)$ from Exercise 47.

$$f_{X|Y}(x|y) = \frac{2(x + y)}{2y + 1} \quad \text{for } 0 < x < 1$$

Step 2: Calculate $\mathbb{E}[X|Y = y]$.

$$\begin{aligned}
\mathbb{E}[X|Y = y] &= \int_0^1 x f_{X|Y}(x|y)dx \\
&= \int_0^1 x\left(\frac{2(x + y)}{2y + 1}\right)dx \\
&= \frac{2}{2y + 1}\int_0^1 (x^2 + xy)dx \\
&= \frac{2}{2y + 1}\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 \\
&= \frac{2}{2y + 1}\left(\frac{1}{3} + \frac{y}{2}\right) \\
&= \frac{2}{2y + 1}\left(\frac{2 + 3y}{6}\right) = \frac{2 + 3y}{3(2y + 1)}
\end{aligned}$$

The conditional expectation function is $\mathbb{E}[X|Y] = \frac{2+3Y}{3(2Y+1)}$.

Exercise 53: Applying the Law of Total Expectation (LOT)

Using the result from Exercise 52, verify the Law of Total Expectation, $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$.
Solution: Step 1: Calculate $\mathbb{E}[X]$. From Exercise 37, the marginal PDF is $f_X(x) = x + 1/2$ for $0 < x < 1$.

$$\mathbb{E}[X] = \int_0^1 x f_X(x)dx = \int_0^1 x(x + 1/2)dx = \int_0^1 (x^2 + x/2)dx$$

$$\mathbb{E}[X] = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

Step 2: Calculate $\mathbb{E}[\mathbb{E}[X|Y]]$. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}\left[\frac{2+3Y}{3(2Y+1)}\right] = \int_0^1 \left(\frac{2+3y}{3(2y+1)}\right) f_Y(y)dy$. From Exercise 37, $f_Y(y) = y + 1/2 = \frac{2y+1}{2}$.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y]] &= \int_0^1 \left(\frac{2+3y}{3(2y+1)}\right)\left(\frac{2y+1}{2}\right) dy \\
&= \int_0^1 \frac{2+3y}{6} dy \\
&= \frac{1}{6}\int_0^1 (2+3y)dy \\
&= \frac{1}{6}\left[2y + \frac{3y^2}{2}\right]_0^1 \\
&= \frac{1}{6}\left(2 + \frac{3}{2}\right) = \frac{1}{6}\left(\frac{4+3}{2}\right) = \frac{1}{6}\left(\frac{7}{2}\right) = \frac{7}{12}
\end{aligned}$$

Since $\mathbb{E}[X] = 7/12 = \mathbb{E}[\mathbb{E}[X|Y]]$, the Law of Total Expectation is verified.

Exercise 54: Conditional Expectation for Non-Rectangular Support

Let X and Y have the joint PDF $f(x, y) = 2$ on the region $0 < y < x < 1$. Find the conditional expectation function $\mathbb{E}[Y|X]$.
Solution: Step 1: Use the conditional PDF $f_{Y|X}(y|x)$ from Exercise 49.

$$f_{Y|X}(y|x) = \frac{1}{x} \quad \text{for } 0 < y < x$$

Step 2: Calculate $\mathbb{E}[Y|X = x]$.

$$\mathbb{E}[Y|X = x] = \int_0^x y f_{Y|X}(y|x)dy = \int_0^x y \cdot \frac{1}{x} dy$$

$$\begin{aligned}
\mathbb{E}[Y|X = x] &= \frac{1}{x}\int_0^x y dy \\
&= \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x \\
&= \frac{1}{x}\left(\frac{x^2}{2}\right) = \frac{x}{2}
\end{aligned}$$

The conditional expectation function is $\mathbb{E}[Y|X] = X/2$.

Exercise 55: Law of Total Expectation using the Conditional Mean $\mathbb{E}[Y|X]$

Using the result from Exercise 54, calculate $\mathbb{E}[Y]$ via the Law of Total Expectation, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$.
Solution: Step 1: Calculate $\mathbb{E}[\mathbb{E}[Y|X]]$. From Exercise 54, $\mathbb{E}[Y|X] = X/2$. From Exercise 38, the marginal PDF is $f_X(x) = 2x$ for $0 < x < 1$.

$$\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[X/2] = \int_0^1 \frac{x}{2} f_X(x)dx$$

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 \frac{x}{2}(2x)dx \\
&= \int_0^1 x^2 dx \\
&= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}
\end{aligned}$$

Thus, $\mathbb{E}[Y] = 1/3$.

Step 2: (Verification) Calculate $\mathbb{E}[Y]$ directly using $f_Y(y)$. From Exercise 38, $f_Y(y) = 2(1-y)$ for $0 < y < 1$.

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 y f_Y(y)dy = \int_0^1 y \cdot 2(1-y)dy \\
&= \int_0^1 (2y - 2y^2)dy \\
&= \left[y^2 - \frac{2y^3}{3}\right]_0^1 \\
&= 1 - \frac{2}{3} = \frac{1}{3}
\end{aligned}$$

The results match.

10.3 Conditional Variance

Definition 10.4 (Conditional Variance)**.** The Conditional Variance of X given $Y = y$ is the variance of the conditional distribution:

$$\text{Var}(X|Y = y) = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

Computational Formula: $\text{Var}(X|Y = y) = \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

Theorem 10.2 (Law of Total Variance (Eve's Law))**.** The variance of X is the sum of the expected conditional variance and the variance of the conditional expectation.

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$$

Exercise 56: Conditional Variance (Discrete)

Using the results from Exercise 46 and 51, find the conditional variance $\text{Var}(X|Y = 1)$.
Solution: From Exercise 51, $\mathbb{E}[X|Y = 1] = 5/3$. The conditional PMF is $p_{X|Y}(1|1) = 1/3$ and $p_{X|Y}(2|1) = 2/3$. We first find the second conditional moment, $\mathbb{E}[X^2|Y = 1]$:

$$\mathbb{E}[X^2|Y = 1] = \sum_x x^2 \cdot p_{X|Y}(x|1) = 1^2\left(\frac{1}{3}\right) + 2^2\left(\frac{2}{3}\right)$$

$$\mathbb{E}[X^2|Y=1] = \frac{1}{3} + \frac{8}{3} = \frac{9}{3} = 3$$

Now, use the computational formula: $\text{Var}(X|Y=1) = \mathbb{E}[X^2|Y=1] - (\mathbb{E}[X|Y=1])^2$.

$$\begin{aligned}
\text{Var}(X|Y=1) &= 3 - \left(\frac{5}{3}\right)^2 \\
&= 3 - \frac{25}{9} \\
&= \frac{27-25}{9} = \frac{2}{9}
\end{aligned}$$

Exercise 57: Conditional Variance Function (Continuous)

Let X and Y have the joint PDF $f(x,y) = x+y$ for $0 < x < 1, 0 < y < 1$. Find the conditional variance function $\text{Var}(X|Y)$.

Solution: From Exercise 52, $\mathbb{E}[X|Y=y] = \frac{2+3y}{3(2y+1)}$. We first need $\mathbb{E}[X^2|Y=y] = \int_0^1 x^2 f_{X|Y}(x|y)dx$. Using $f_{X|Y}(x|y) = \frac{2(x+y)}{2y+1}$:

$$\begin{aligned}
\mathbb{E}[X^2|Y=y] &= \int_0^1 x^2 \left(\frac{2(x+y)}{2y+1}\right) dx \\
&= \frac{2}{2y+1} \int_0^1 (x^3 + x^2 y) dx \\
&= \frac{2}{2y+1} \left[\frac{x^4}{4} + \frac{x^3 y}{3}\right]_0^1 \\
&= \frac{2}{2y+1} \left(\frac{1}{4} + \frac{y}{3}\right) = \frac{2}{2y+1} \left(\frac{3+4y}{12}\right) = \frac{3+4y}{6(2y+1)}
\end{aligned}$$

Now, use the computational formula for conditional variance:

$$\begin{aligned}
\text{Var}(X|Y=y) &= \mathbb{E}[X^2|Y=y] - (\mathbb{E}[X|Y=y])^2 \\
&= \frac{3+4y}{6(2y+1)} - \left(\frac{2+3y}{3(2y+1)}\right)^2 \\
&= \frac{3+4y}{6(2y+1)} - \frac{4+12y+9y^2}{9(2y+1)^2}
\end{aligned}$$

To simplify, find a common denominator $18(2y+1)^2$:

$$\text{Var}(X|Y=y) = \frac{3(2y+1)(3+4y) - 2(4+12y+9y^2)}{18(2y+1)^2}$$

$$\text{Var}(X|Y=y) = \frac{3(6y+4y^2+3+4y) - (8+24y+18y^2)}{18(2y+1)^2}$$

$$\text{Var}(X|Y=y) = \frac{3(4y^2+10y+3) - 18y^2 - 24y - 8}{18(2y+1)^2}$$

$$\text{Var}(X|Y=y) = \frac{12y^2+30y+9-18y^2-24y-8}{18(2y+1)^2} = \frac{-6y^2+6y+1}{18(2y+1)^2}$$

The conditional variance function is $\text{Var}(X|Y) = \frac{-6Y^2+6Y+1}{18(2Y+1)^2}$.

Exercise 58: Conditional Variance for Uniform Distribution

Let X and Y have the joint PDF $f(x,y) = 2$ on the region $0 < y < x < 1$. Find the conditional variance function $\text{Var}(Y|X)$.
Solution: From Exercise 49, given $X = x$, Y is uniform on $(0,x)$, i.e., $Y|X = x \sim \text{Uniform}(0,x)$. For $W \sim \text{Uniform}(a,b)$, $\text{Var}(W) = \frac{(b-a)^2}{12}$ (from Exercise 26). Here, $W = Y|X = x$, $a = 0$, $b = x$.

$$\text{Var}(Y|X = x) = \frac{(x-0)^2}{12} = \frac{x^2}{12}$$

The conditional variance function is $\text{Var}(Y|X) = X^2/12$.

Exercise 59: Applying the Law of Total Variance (LTV) - First Term

Using the result from Exercise 58, calculate the first term of the Law of Total Variance, $\mathbb{E}[\text{Var}(Y|X)]$.
Solution: The first term of LTV is $\mathbb{E}[\text{Var}(Y|X)] = \mathbb{E}[X^2/12]$. The marginal PDF of X is $f_X(x) = 2x$ for $0 < x < 1$ (from Exercise 38).

$$\mathbb{E}[\text{Var}(Y|X)] = \mathbb{E}\left[\frac{X^2}{12}\right] = \int_0^1 \frac{x^2}{12} f_X(x)dx$$

$$\begin{aligned}
\mathbb{E}[\text{Var}(Y|X)] &= \int_0^1 \frac{x^2}{12}(2x)dx \\
&= \frac{2}{12}\int_0^1 x^3 dx \\
&= \frac{1}{6}\left[\frac{x^4}{4}\right]_0^1 \\
&= \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}
\end{aligned}$$

Exercise 60: Applying the Law of Total Variance (LTV) - Second Term and Final Variance

Using the results from Exercise 54 and 59, calculate the second term of the Law of Total Variance, $\text{Var}(\mathbb{E}[Y|X])$, and the final value of $\text{Var}(Y)$.
Solution: Step 1: Calculate the second term $\text{Var}(\mathbb{E}[Y|X])$. From Exercise 54, $\mathbb{E}[Y|X] = X/2$. We need $\text{Var}(X/2)$. Using $\text{Var}(aW) = a^2\text{Var}(W)$:

$$\text{Var}(\mathbb{E}[Y|X]) = \text{Var}(X/2) = \frac{1}{4}\text{Var}(X)$$

We need $\text{Var}(X)$. $\mathbb{E}[X] = 2/3$ (from Exercise 40).

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x)dx = \int_0^1 x^2(2x)dx = 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$\text{Var}(\mathbb{E}[Y|X]) = \frac{1}{4}\text{Var}(X) = \frac{1}{4}\cdot\frac{1}{18} = \frac{1}{72}$$

Step 2: Calculate Var(Y) using the Law of Total Variance.

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$$

From Exercise 59, $\mathbb{E}[\text{Var}(Y|X)] = 1/24$.

$$\text{Var}(Y) = \frac{1}{24} + \frac{1}{72}$$

To sum: $\frac{1}{24} = \frac{3}{72}$.

$$\text{Var}(Y) = \frac{3}{72} + \frac{1}{72} = \frac{4}{72} = \frac{1}{18}$$

Step 3: (Verification) Calculate Var(Y) directly using $f_Y(y)$. From Exercise 38, $f_Y(y) = 2(1-y)$. $\mathbb{E}[Y] = 1/3$ (from Exercise 55).

$$\mathbb{E}[Y^2] = \int_0^1 y^2 \cdot 2(1-y)dy = 2\int_0^1 (y^2 - y^3)dy$$

$$\mathbb{E}[Y^2] = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = 2\left(\frac{4-3}{12}\right) = \frac{2}{12} = \frac{1}{6}$$

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18}$$

The results from LTV and direct calculation match.