Differential Equations

Contents

1 Classification of Differential Equations **2**
- 1.1 Ordinary Differential Equations (ODE) vs. Partial Differential Equations (PDE) 2
- 1.2 Order and Degree of Differential Equations 2
- 1.3 Linear and Non-Linear Differential Equations 3

2 Deriving/Forming Ordinary Differential Equations **4**

3 Solving First-Order Ordinary Differential Equations **5**
- 3.1 Variable Separable Equations 6
- 3.2 Homogenous Equations 7
- 3.3 Equations Reducible to Homogenous Form 8
- 3.4 Exact Equations (and Integrating Factors) 11
- 3.5 First-Order Linear Equations (FOLDE) 13
- 3.6 Bernoulli's Equation 14

4 Higher-Order Linear Differential Equations **16**
- 4.1 Homogenous Linear Equations (HLE) 16
- 4.2 Non-Homogenous Linear Equations (NHLE) - Lagrange's Method (Variation of Parameters) 18

1 Classification of Differential Equations

1.1 Ordinary Differential Equations (ODE) vs. Partial Differential Equations (PDE)

Definition 1.1 (ODE)**.** An **Ordinary Differential Equation (ODE)** is an equation involving an unknown function of a **single independent variable** and its derivatives with respect to that variable.

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

Definition 1.2 (PDE)**.** A **Partial Differential Equation (PDE)** is an equation involving an unknown function of **two or more independent variables** and its partial derivatives with respect to those variables.

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \dots\right) = 0$$

Example #1: Classify: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = \sin x$

Solution: The equation involves derivatives of y with respect to a single independent variable, x. Therefore, it is an **ODE**.

Example #2: Classify: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution: The equation involves partial derivatives of the unknown function u with respect to two independent variables, t and x. Therefore, it is a **PDE** (specifically, the 1D Wave Equation).

Example #3: Classify: $\left(\frac{dy}{dt}\right)^3 + 5t^2 y = e^{3t}$

Solution: The equation involves derivatives of y with respect to a single independent variable, t. Therefore, it is an **ODE**.

1.2 Order and Degree of Differential Equations

Definition 1.3 (Order)**.** The **order** of a differential equation is the order of the **highest derivative** appearing in the equation.

Definition 1.4 (Degree)**.** The **degree** of a differential equation is the power of the **highest order derivative** after the equation has been rationalized (i.e., cleared of any fractional powers of the derivatives).

Example #1: Find the order and degree of: $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 - 5y = x$

Solution:

- Highest derivative is $\frac{d^3y}{dx^3}$. $\implies$ **Order = 3**.
- The power of the highest derivative ($\frac{d^3y}{dx^3}$) is 1. $\implies$ **Degree = 1**.

Example #2: Find the order and degree of: $\left(1+\frac{dy}{dx}\right)^{3/2} = \frac{d^2y}{dx^2}$

Solution: First, rationalize the equation by squaring both sides:

$$\left(1+\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

- Highest derivative is $\frac{d^2y}{dx^2}$. $\implies$ **Order = 2**.
- The power of the highest derivative ($\frac{d^2y}{dx^2}$) is 2. $\implies$ **Degree = 2**.

Example #3: Find the order and degree of: $\sqrt{\frac{d^2y}{dx^2}} = \frac{dy}{dx}+5$

Solution: First, rationalize the equation by squaring both sides:

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}+5\right)^2$$

- Highest derivative is $\frac{d^2y}{dx^2}$. $\implies$ **Order = 2**.
- The power of the highest derivative ($\frac{d^2y}{dx^2}$) is 1. $\implies$ **Degree = 1**.

1.3 Linear and Non-Linear Differential Equations

Definition 1.5 (Linear DE)**.** A differential equation is **linear** if it satisfies three conditions regarding the dependent variable (y) and its derivatives:

1. The dependent variable and its derivatives appear only to the **first power**.
2. The coefficients of the dependent variable and its derivatives are either **constants or functions of the independent variable only**.
3. There are **no products** of the dependent variable and/or its derivatives.

The general form of an n-th order linear ODE is:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Definition 1.6 (Non-Linear DE)**.** Any differential equation that violates any of the conditions for a linear DE is **non-linear**.

Example #1: Determine linearity: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0$

Solution:

- y and its derivatives are all to the first power. (✓)
- Coefficients ($x^2, x, 2$) are functions of x (the independent variable) or constants. (✓)
- No products of y or its derivatives. (✓)

The equation is **Linear**.

Example #2: Determine linearity: $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 - 3y = e^x$

Solution:

- The term $\left(\frac{dy}{dx}\right)^2$ has a power greater than 1.

The equation is **Non-Linear**.

Example #3: Determine linearity: $\frac{dy}{dx} + y\sin x = \cos y$

Solution:

- The term $\cos y$ is a non-linear function of the dependent variable y.

The equation is **Non-Linear**.

—

2 Deriving/Forming Ordinary Differential Equations

A differential equation is formed by eliminating the arbitrary constants from the general solution (or the primitive) of the equation.

Solution Algorithm 2.0.1. Procedure for Deriving a DE

Step 1 Count the number of arbitrary constants (n) in the given relation.

Step 2 The order of the resulting differential equation will be equal to n.

Step 3 Differentiate the given relation n times with respect to the independent variable.

Step 4 Eliminate the n arbitrary constants from the original relation and the n differentiated equations to obtain the desired differential equation.

Example #1: Form the DE for the primitive $y = Ax^2 + Bx$.

Solution: The arbitrary constants are A and B ($n = 2$).

1. Given: $y = Ax^2 + Bx \quad (1)$
2. Differentiate once: $\frac{dy}{dx} = 2Ax + B \quad (2)$
3. Differentiate twice: $\frac{d^2y}{dx^2} = 2A \quad (3)$
4. From (3), $A = \frac{1}{2}\frac{d^2y}{dx^2}$. Substitute A into (2):

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x + B \implies B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

5. Substitute A and B into (1):

$$y = \left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x^2 + \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x$$

$$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

$$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

6. Multiply by 2: $\mathbf{x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0}$.

Example #2: Form the DE for $y = a\cos(x) + b\sin(x)$.

Solution: The arbitrary constants are a and b ($n = 2$).

1. Given: $y = a\cos x + b\sin x \quad (1)$
2. Differentiate once: $\frac{dy}{dx} = -a\sin x + b\cos x \quad (2)$
3. Differentiate twice: $\frac{d^2y}{dx^2} = -a\cos x - b\sin x \quad (3)$
4. From (1) and (3), we can see:

$$\frac{d^2y}{dx^2} = -(a\cos x + b\sin x) = -y$$

5. Rearranging gives the DE: $\mathbf{\frac{d^2y}{dx^2} + y = 0}$.

Example #3: Form the DE for $y = ce^x + c^2$, where c is the arbitrary constant.

Solution: The arbitrary constant is c ($n = 1$).

1. Given: $y = ce^x + c^2 \quad (1)$
2. Differentiate once: $\frac{dy}{dx} = ce^x \quad (2)$
3. From (2), $c = e^{-x}\frac{dy}{dx}$.
4. Substitute c into (1):

$$y = \left(e^{-x}\frac{dy}{dx}\right)e^x + \left(e^{-x}\frac{dy}{dx}\right)^2$$

$$\mathbf{y = \frac{dy}{dx} + e^{-2x}\left(\frac{dy}{dx}\right)^2}$$

—

3 Solving First-Order Ordinary Differential Equations

The general form of a first-order ODE is $F\left(x, y, \frac{dy}{dx}\right) = 0$.

3.1 Variable Separable Equations

Definition 3.1 (Variable Separable)**.** A first-order ODE is **variable separable** if it can be written in the form:

$$f(x)dx + g(y)dy = 0$$

where $f(x)$ is a function of x only, and $g(y)$ is a function of y only.

Solution Algorithm 3.1.1. Solution Algorithm

Step 1 Rearrange the given ODE into the form $f(x)dx = g(y)dy$.

Step 2 Integrate both sides with respect to their respective variables: $\int f(x)dx = \int g(y)dy$.

Step 3 Add the arbitrary constant C (usually on the side of the independent variable) to get the general solution.

Example #1: Solve: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution:

1. Separate variables: $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$
2. Integrate both sides: $\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$
3. General solution: $\arctan(\mathbf{y}) = \arctan(\mathbf{x}) + \mathbf{C}$

Example #2: Solve: $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

Solution:

1. Separate variables: Divide by $\tan y(1 - e^x)$.

$$\frac{e^x}{1 - e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

$$\int \frac{e^x}{1 - e^x}dx + \int \frac{\sec^2 y}{\tan y}dy = C$$

2. Integrate: Let $u = 1 - e^x$, $du = -e^x dx$. Let $v = \tan y$, $dv = \sec^2 y dy$.

$$\int -\frac{du}{u} + \int \frac{dv}{v} = C$$

$$-\ln|1 - e^x| + \ln|\tan y| = \ln|C'| \quad \text{(using } \ln|C'| \text{ as the constant)}$$

3. Simplify: $\ln\left|\frac{\tan y}{1-e^x}\right| = \ln|C'| \implies \tan \mathbf{y} = \mathbf{C}(\mathbf{1} - \mathbf{e}^{\mathbf{x}})$

Example #3: Solve: $\frac{dy}{dx} = \sin(x)\cos^2(y)$

Solution:

1. Separate variables: $\frac{dy}{\cos^2(y)} = \sin(x)dx \implies \sec^2(y)dy = \sin(x)dx$
2. Integrate both sides: $\int \sec^2(y)dy = \int \sin(x)dx$
3. General solution: $\tan(\mathbf{y}) = -\cos(\mathbf{x}) + \mathbf{C}$

3.2 Homogenous Equations

Definition 3.2 (Homogenous)**.** A first-order ODE in the form $\frac{dy}{dx} = f(x, y)$ is **homogenous** if the function $f(x, y)$ is a homogenous function of degree zero. Alternatively, if the equation can be written as:

$$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$$

where $M(x, y)$ and $N(x, y)$ are homogenous functions of the same degree.

Theorem 3.1 (Substitution for Homogenous DE)**.** Any homogenous ODE $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ can be reduced to a variable separable form by the substitution:

$$y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Solution Algorithm 3.2.1. Solution Algorithm

Step 1 Check if the equation is homogenous.

Step 2 Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation.

Step 3 The resulting equation in v and x will always be variable separable. Separate the variables:

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

Step 4 Integrate both sides.

Step 5 Substitute back $v = \frac{y}{x}$ to get the general solution in terms of x and y.

Example #1: Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Solution: The numerator and denominator are both homogenous of degree 2.

1. Substitute $y = vx$: $v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{2x(vx)} = \frac{x^2(1+v^2)}{2x^2v} = \frac{1+v^2}{2v}$
2. Separate variables:

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

3. Integrate: $\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$. Let $u = 1 - v^2$, $du = -2vdv$.

$$-\int \frac{du}{u} = \int \frac{dx}{x} \implies -\ln|1-v^2| = \ln|x| + \ln|C|$$

4. Simplify and substitute back $v = y/x$:

$$\ln\left|\frac{1}{1-v^2}\right| = \ln|Cx| \implies \frac{1}{1-(y/x)^2} = Cx$$

$$\frac{x^2}{x^2-y^2} = Cx \implies \mathbf{x = C(x^2 - y^2)}$$

Example #2: Solve: $(x^2 - y^2)dx - 2xy\,dy = 0$

Solution: Rearrange to $\frac{dy}{dx} = \frac{x^2-y^2}{2xy}$, which is homogenous of degree 2.

1. Substitute $y = vx$: $v + x\frac{dv}{dx} = \frac{x^2-(vx)^2}{2x(vx)} = \frac{1-v^2}{2v}$
2. Separate variables:
$$x\frac{dv}{dx} = \frac{1-v^2}{2v} - v = \frac{1-v^2-2v^2}{2v} = \frac{1-3v^2}{2v}$$
$$\frac{2v}{1-3v^2}dv = \frac{dx}{x}$$
3. Integrate: $\int \frac{2v}{1-3v^2}dv = \int \frac{dx}{x}$. Let $u = 1-3v^2$, $du = -6vdv$.
$$-\frac{1}{3}\int \frac{du}{u} = \int \frac{dx}{x} \implies -\frac{1}{3}\ln|1-3v^2| = \ln|x| + \ln|C'|$$
4. Simplify: $\ln|1-3v^2|^{-1/3} = \ln|C'x| \implies \frac{1}{\sqrt[3]{1-3v^2}} = Cx$
5. Substitute back $v = y/x$: $\frac{x}{\sqrt[3]{x^2-3y^2}} = Cx \implies \mathbf{x^2 - 3y^2 = C_1x^3}$

Example #3: Solve: $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

Solution: Rearrange to $\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}$, which is homogenous of degree 0.

1. Substitute $y = vx$: $v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$
2. Separate variables: $x\frac{dv}{dx} = \sqrt{1+v^2} \implies \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$
3. Integrate: $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$
$$\ln\left|v + \sqrt{1+v^2}\right| = \ln|x| + \ln|C|$$
4. Simplify and substitute back $v = y/x$:
$$\ln\left|\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right| = \ln|Cx|$$
$$\frac{y+\sqrt{x^2+y^2}}{x} = Cx \implies \mathbf{y + \sqrt{x^2+y^2} = Cx^2}$$

3.3 Equations Reducible to Homogenous Form

The equation of the form
$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$
is not homogenous due to the presence of constants c_1 and c_2.

Case 1: Intersecting Lines ($\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$)

Solution Algorithm 3.3.1. Solution Algorithm (Case 1)

Step 1 Solve the system of linear equations: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to find the intersection point (h, k).

Step 2 Apply the translation transformation: $x = X + h$ and $y = Y + k$. This implies $dx = dX$ and $dy = dY$, so $\frac{dy}{dx} = \frac{dY}{dX}$.

Step 3 The ODE transforms into a homogenous equation in X and Y: $\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$.

Step 4 Solve this homogenous equation using the substitution $Y = vX$.

Step 5 Substitute back $v = \frac{Y}{X}$ and then $X = x - h$, $Y = y - k$ to get the final solution.

Example #1: Solve: $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution: $\frac{a_1}{a_2} = 1/2$, $\frac{b_1}{b_2} = 2/1$. $1/2 \neq 2$. Case 1 applies.

1. Find intersection: $x + 2y - 3 = 0$ and $2x + y - 3 = 0$. Solving gives $x = 1, y = 1$. So $(h, k) = (1, 1)$.

2. Substitution: $x = X + 1, y = Y + 1$. $\frac{dY}{dX} = \frac{(X+1)+2(Y+1)-3}{2(X+1)+(Y+1)-3} = \frac{X+2Y}{2X+Y}$.

3. Let $Y = vX$: $v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v}$.

4. Separate variables:

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1+2v-v(2+v)}{2+v} = \frac{1-v^2}{2+v}$$

$$\frac{2+v}{1-v^2}dv = \frac{dX}{X} \implies \left(\frac{3/2}{1-v} - \frac{1/2}{1+v}\right)dv = \frac{dX}{X}$$

5. Integrate: $-\frac{3}{2}\ln|1-v| - \frac{1}{2}\ln|1+v| = \ln|X| + \ln|C'|$

$$\ln\left|\frac{1}{(1-v)^3(1+v)}\right|^{1/2} = \ln|C'X| \implies \frac{1}{(1-v)^3(1+v)} = CX^2$$

6. Substitute back $v = Y/X$, $X = x - 1$, $Y = y - 1$:

$$\frac{1}{\left(1 - \frac{y-1}{x-1}\right)^3\left(1 + \frac{y-1}{x-1}\right)} = C(x-1)^2$$

After simplification: $\mathbf{(x + y - 2) = C(x - y)^3}$.

Case 2: Parallel Lines ($\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$)

Solution Algorithm 3.3.2. Solution Algorithm (Case 2)

Step 1 Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, let $a_1x + b_1y = k(a_2x + b_2y)$.

Step 2 Apply the substitution: $z = a_1x + b_1y$.

Step 3 Differentiate with respect to x: $\frac{dz}{dx} = a_1 + b_1\frac{dy}{dx}$, which allows $\frac{dy}{dx}$ to be expressed in terms of z.

Step 4 The ODE transforms into a variable separable equation in z and x.

Step 5 Solve the variable separable equation and substitute back z to get the final solution.

Example #2: Solve: $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

Solution: $\frac{a_1}{a_2} = 1/1$, $\frac{b_1}{b_2} = 1/1$. $1/1 = 1$. Case 2 applies.

1. Substitution: Let $z = x + y$. Then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{dz}{dx} - 1$.
2. Substitute into ODE:
$$\frac{dz}{dx} - 1 = \frac{z+1}{z-1} \implies \frac{dz}{dx} = 1 + \frac{z+1}{z-1} = \frac{z-1+z+1}{z-1} = \frac{2z}{z-1}$$
3. Separate variables: $\frac{z-1}{2z}dz = dx \implies \left(\frac{1}{2} - \frac{1}{2z}\right) dz = dx$
4. Integrate: $\int \left(\frac{1}{2} - \frac{1}{2z}\right) dz = \int dx$
$$\frac{1}{2}z - \frac{1}{2}\ln|z| = x + C'$$
5. Substitute back $z = x + y$:
$$\frac{1}{2}(x+y) - \frac{1}{2}\ln|x+y| = x + C'$$
6. Simplify (Multiply by 2 and let $2C' = -C$): $x+y-\ln|x+y| = 2x-C \implies \mathbf{y - x - \ln|x + y| = C}$

Example #3: Solve: $(x + y + 1)^2\frac{dy}{dx} = 1$

Solution: Rearrange to $\frac{dy}{dx} = \frac{1}{(x+y+1)^2}$. Here $a_1 = 0, b_1 = 0, c_1 = 1$ and $a_2 = 1, b_2 = 1, c_2 = 1$. This can be considered a special case where $\frac{dy}{dx} = f(ax + by)$.

1. Substitution: Let $z = x + y + 1$. Then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{dz}{dx} - 1$.
2. Substitute into ODE:
$$\frac{dz}{dx} - 1 = \frac{1}{z^2} \implies \frac{dz}{dx} = 1 + \frac{1}{z^2} = \frac{z^2+1}{z^2}$$
3. Separate variables: $\frac{z^2}{z^2+1}dz = dx \implies \left(1 - \frac{1}{z^2+1}\right) dz = dx$
4. Integrate: $\int \left(1 - \frac{1}{z^2+1}\right) dz = \int dx$
$$z - \arctan(z) = x + C$$
5. Substitute back $z = x + y + 1$:
$$(x + y + 1) - \arctan(x + y + 1) = x + C$$
6. Simplify: $\mathbf{y + 1} - \arctan(\mathbf{x + y + 1}) = \mathbf{C}$

3.4 Exact Equations (and Integrating Factors)

Definition 3.3 (Exact Equation)**.** A first-order ODE $M(x, y)dx + N(x, y)dy = 0$ is **exact** if there exists a function $f(x, y)$ such that $df(x, y) = M(x, y)dx + N(x, y)dy$.

Theorem 3.2 (Condition for Exactness)**.** The necessary and sufficient condition for the ODE $M(x, y)dx + N(x, y)dy = 0$ to be exact is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution Algorithm 3.4.1. Solution Algorithm (Exact Equations)

Step 1 Identify $M(x, y)$ and $N(x, y)$.

Step 2 Check for exactness: Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If they are equal, the equation is exact.

Step 3 The solution is given by $f(x, y) = C$, where $f(x, y) = \int M\, dx$+terms from N not containing x.

Step 4 Compute $f(x, y)$ via the following two methods which must yield the same result:

$$f(x, y) = \int M(x, y)dx + g(y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)$$

Step 5 Determine $g(y)$ by integrating $\frac{dg}{dy} = N - \frac{\partial}{\partial y}\left(\int M dx\right)$.

Step 6 The general solution is $\mathbf{f(x, y) = C}$.

Example #1: Solve: $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

Solution: $M = 3x^2 + 4xy$, $N = 2x^2 + 2y$.

1. Check exactness: $\frac{\partial M}{\partial y} = 4x$, $\frac{\partial N}{\partial x} = 4x$. Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is **exact**.
2. Integrate M w.r.t x:
$$\int (3x^2 + 4xy)dx = x^3 + 2x^2y + g(y)$$
3. Differentiate w.r.t y and set equal to N:
$$\frac{\partial}{\partial y}(x^3 + 2x^2y + g(y)) = 2x^2 + g'(y) = N = 2x^2 + 2y$$
$$g'(y) = 2y \implies g(y) = \int 2ydy = y^2$$
4. General solution: $\mathbf{x^3 + 2x^2y + y^2 = C}$.

Non-Exact Equations and Integrating Factors (IF)

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is non-exact. We multiply by an **Integrating Factor** (μ) such that $\mu M dx + \mu N dy = 0$ is exact.

Property 1 (Rules for Finding Integrating Factors). (i) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ (a function of x only), then $IF = e^{\int f(x)dx}$.

(ii) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ (a function of y only), then $IF = e^{\int g(y)dy}$.

(iii) If $Mdx + Ndy = 0$ is homogenous, then $IF = \frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$.

Example #2: Solve: $(x^2 + y^2 + x)dx + xydy = 0$

Solution: $M = x^2 + y^2 + x$, $N = xy$.

1. Check exactness: $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$. Since $2y \neq y$, it is **non-exact**.
2. Find Integrating Factor (IF) using rule (i):
$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{y}{xy} = \frac{1}{x} = f(x)$$
$$IF = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$
3. Multiply the ODE by $IF = x$: $(x^3 + xy^2 + x^2)dx + x^2ydy = 0$.
4. New $M' = x^3 + xy^2 + x^2$, $N' = x^2y$. Check exactness: $\frac{\partial M'}{\partial y} = 2xy$, $\frac{\partial N'}{\partial x} = 2xy$. It is **exact**.
5. Solution is $\int M'dx + g(y) = C$:
$$\int (x^3 + xy^2 + x^2)dx + g(y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + g(y)$$
6. $\frac{\partial}{\partial y} f = x^2y + g'(y) = N' = x^2y$. $g'(y) = 0 \implies g(y) = C_1$.
7. General solution: $\mathbf{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C_2}$ (or $3x^4 + 6x^2y^2 + 4x^3 = C_3$).

Example #3: Solve: $y(2xy + e^x)dx - e^x dy = 0$

Solution: $M = 2xy^2 + ye^x$, $N = -e^x$.

1. Check exactness: $\frac{\partial M}{\partial y} = 4xy + e^x$, $\frac{\partial N}{\partial x} = -e^x$. Non-exact.
2. Find Integrating Factor (IF) using rule (ii):
$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(2xy + e^x)}(-e^x - (4xy + e^x)) = \frac{-2e^x - 4xy}{y(2xy + e^x)} = -\frac{2(2xy + e^x)}{y(2xy + e^x)} = -\frac{2}{y} = g(y)$$
$$IF = e^{\int -\frac{2}{y}dy} = e^{-2\ln|y|} = y^{-2} = \frac{1}{y^2}$$
3. Multiply the ODE by $IF = 1/y^2$: $\left(2x + \frac{e^x}{y}\right)dx - \frac{e^x}{y^2}dy = 0$.

4. New $M' = 2x + \frac{e^x}{y}$, $N' = -\frac{e^x}{y^2}$. It is exact.

5. Solution is $\int M'dx + g(y) = C$:

$$\int \left(2x + \frac{e^x}{y}\right) dx = x^2 + \frac{e^x}{y} + g(y)$$

6. $\frac{\partial}{\partial y} f = -\frac{e^x}{y^2} + g'(y) = N' = -\frac{e^x}{y^2}$. $g'(y) = 0 \implies g(y) = C_1$.

7. General solution: $\mathbf{x^2 + \frac{e^x}{y} = C}$.

3.5 First-Order Linear Equations (FOLDE)

Definition 3.4 (First-Order Linear DE)**.** A first-order linear differential equation has the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only (or constants).

Theorem 3.3 (Integrating Factor for FOLDE)**.** The integrating factor (IF) for the FOLDE $\frac{dy}{dx} + P(x)y = Q(x)$ is:

$$\text{IF} = e^{\int P(x)dx}$$

Multiplying the ODE by the IF turns the left side into the derivative of a product:

$$\frac{d}{dx}\left(y \cdot \text{IF}\right) = Q(x) \cdot \text{IF}$$

Solution Algorithm 3.5.1. Solution Algorithm (FOLDE)

Step 1 Write the ODE in the standard form: $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 2 Compute the Integrating Factor: $IF = e^{\int P(x)dx}$.

Step 3 Multiply the entire standard form ODE by the IF.

Step 4 The solution is given by the formula:

$$y \cdot (IF) = \int Q(x) \cdot (IF)dx + C$$

Step 5 Solve the integral on the right-hand side and solve for y to get the general solution.

Example #1: Solve: $\frac{dy}{dx} + \frac{2}{x}y = x^2$

Solution: $P(x) = 2/x$, $Q(x) = x^2$.

1. Find IF: $IF = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$.

2. General Solution Formula: $y \cdot x^2 = \int x^2 \cdot x^2 dx + C$

$$yx^2 = \int x^4 dx + C$$

$$yx^2 = \frac{x^5}{5} + C$$

3. Solve for y: $\mathbf{y = \frac{x^3}{5} + Cx^{-2}}$

Example #2: Solve: $\cos x \frac{dy}{dx} + y \sin x = 1$

Solution:

1. Standard form (Divide by $\cos x$): $\frac{dy}{dx} + (\tan x)y = \sec x$. $P(x) = \tan x$, $Q(x) = \sec x$.
2. Find IF: $IF = e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$.
3. General Solution Formula: $y \cdot \sec x = \int \sec x \cdot \sec x dx + C$
$$y \sec x = \int \sec^2 x dx + C$$
$$y \sec x = \tan x + C$$
4. Solve for y: $\mathbf{y = \tan x \cos x + C \cos x} \implies \mathbf{y = \sin x + C \cos x}$

Example #3: Solve: $x \ln x \frac{dy}{dx} + y = 2 \ln x$

Solution:

1. Standard form (Divide by $x \ln x$): $\frac{dy}{dx} + \frac{1}{x \ln x} y = \frac{2}{x}$. $P(x) = \frac{1}{x \ln x}$, $Q(x) = \frac{2}{x}$.
2. Find IF: $IF = e^{\int \frac{1}{x \ln x} dx}$. Let $u = \ln x$, $du = 1/x dx$. $\int \frac{du}{u} = \ln u = \ln(\ln x)$.
$$IF = e^{\ln(\ln x)} = \ln x$$
3. General Solution Formula: $y \cdot \ln x = \int \frac{2}{x} \cdot \ln x dx + C$
$$y \ln x = 2 \int \ln x \left(\frac{1}{x} dx \right) + C \quad \text{(Let } u = \ln x)$$
$$y \ln x = 2 \int u du + C = u^2 + C = (\ln x)^2 + C$$
4. Solve for y: $\mathbf{y = \frac{(\ln x)^2}{\ln x} + \frac{C}{\ln x}} \implies \mathbf{y = \ln x + \frac{C}{\ln x}}$

3.6 Bernoulli's Equation

Definition 3.5 (Bernoulli's Equation)**.** Bernoulli's equation is a non-linear first-order ODE of the form:
$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$
where n is any real number and $n \neq 0, 1$.

Solution Algorithm 3.6.1. Solution Algorithm (Bernoulli's Equation)

Step 1 Divide the entire equation by y^n: $y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$.

Step 2 Substitute $v = y^{1-n}$.

Step 3 Differentiate the substitution w.r.t x: $\frac{dv}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx} = (1-n)y^{-n}\frac{dy}{dx}$.

$$\implies y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

Step 4 Substitute this back into the modified ODE to get a FOLDE in terms of v:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Step 5 Solve this FOLDE for v using the standard integrating factor method.

Step 6 Substitute back $v = y^{1-n}$ to get the general solution in terms of y.

Example #1: Solve: $\frac{dy}{dx} + \frac{1}{x}y = xy^2$

Solution: Bernoulli's equation with $n = 2$.

1. Divide by y^2: $y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x$.
2. Substitution: Let $v = y^{1-2} = y^{-1}$. $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \implies y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$.
3. FOLDE in v: $-\frac{dv}{dx} + \frac{1}{x}v = x \implies \frac{dv}{dx} - \frac{1}{x}v = -x$.
4. Solve for v: $P(x) = -1/x$. $IF = e^{\int -1/x dx} = e^{-\ln|x|} = 1/x$.

$$v \cdot \frac{1}{x} = \int (-x)\cdot\frac{1}{x}dx + C = \int -1 dx + C = -x + C$$

$$v = Cx - x^2$$

5. Substitute back $v = 1/y$: $\mathbf{\frac{1}{y} = Cx - x^2 \implies y = \frac{1}{Cx - x^2}}$.

Example #2: Solve: $\frac{dy}{dx} + y = xy^3$

Solution: Bernoulli's equation with $n = 3$.

1. Divide by y^3: $y^{-3}\frac{dy}{dx} + y^{-2} = x$.
2. Substitution: Let $v = y^{1-3} = y^{-2}$. $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \implies y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$.
3. FOLDE in v: $-\frac{1}{2}\frac{dv}{dx} + v = x \implies \frac{dv}{dx} - 2v = -2x$.
4. Solve for v: $P(x) = -2$. $IF = e^{\int -2dx} = e^{-2x}$.

$$ve^{-2x} = \int (-2x)e^{-2x}dx + C$$

5. Integration by parts ($\int udv = uv - \int vdu$): Let $u = -2x, dv = e^{-2x}dx$.

$$ve^{-2x} = -2x\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right)(-2dx) + C$$

$$ve^{-2x} = xe^{-2x} - \int e^{-2x}dx + C = xe^{-2x} - \left(-\frac{1}{2}e^{-2x}\right) + C$$

$$v = x + \frac{1}{2} + Ce^{2x}$$

6. Substitute back $v = 1/y^2$: $\mathbf{\frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}}$.

Example #3: Solve: $2\frac{dy}{dx} - \frac{y}{x} = y^3 \cos x$

Solution: Divide by 2: $\frac{dy}{dx} - \frac{1}{2x}y = \frac{1}{2}y^3 \cos x$. $n = 3$.

1. Divide by y^3: $y^{-3}\frac{dy}{dx} - \frac{1}{2x}y^{-2} = \frac{1}{2}\cos x$.
2. Substitution: Let $v = y^{-2}$. $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \implies y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$.
3. FOLDE in v: $-\frac{1}{2}\frac{dv}{dx} - \frac{1}{2x}v = \frac{1}{2}\cos x \implies \frac{dv}{dx} + \frac{1}{x}v = -\cos x$.
4. Solve for v: $P(x) = 1/x$. $IF = e^{\int 1/x dx} = x$.

$$v \cdot x = \int (-\cos x) \cdot x dx + C$$

$$vx = -\int x \cos x dx + C$$

5. Integration by parts ($\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x$):

$$vx = -(x \sin x + \cos x) + C$$

$$v = -\sin x - \frac{\cos x}{x} + \frac{C}{x}$$

6. Substitute back $v = 1/y^2$: $\mathbf{\frac{1}{y^2} = \frac{C - x\sin x - \cos x}{x}}$.

—

4 Higher-Order Linear Differential Equations

We focus on the n-th order linear ODE with constant coefficients:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

4.1 Homogenous Linear Equations (HLE)

The general form of an n-th order HLE with constant coefficients is:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Theorem 4.1 (Principle of Superposition)**.** If $y_1, y_2, \ldots, y_n$ are n solutions to a homogenous linear ODE, then any linear combination $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also a solution.

Definition 4.1 (Linear Dependence/Independence)**.** A set of n functions $\{y_1, y_2, \ldots, y_n\}$ is **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$ for all $x \in I$. Otherwise, the set is **linearly independent**.

The Wronskian Method

The Wronskian is a determinant used to test the linear independence of solutions.

Definition 4.2 (Wronskian)**.** The **Wronskian** of n functions $\{y_1, y_2, \ldots, y_n\}$ is the determinant:

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem 4.2 (Wronskian Test)**.** Let $y_1, y_2, \ldots, y_n$ be n solutions to an n-th order HLE on an interval I.

- The solutions are **linearly independent** if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for at least one point in I.
- The general solution of the ODE is $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ if and only if $W \neq 0$.

Solution Algorithm 4.1.1. Procedure for Checking Linear Independence (Wronskian)

Step 1 Form the Wronskian matrix using the functions y_i and their derivatives up to $(n-1)$.

Step 2 Compute the determinant $W(x)$.

Step 3 If $W(x)$ is non-zero on the interval, the functions are linearly independent. If $W(x)$ is zero everywhere, the functions are linearly dependent.

Example #1: Determine if $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent.

Solution: $n = 2$. $y_1' = e^x$, $y_2' = -e^{-x}$.

1. Compute Wronskian:
$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$
2. Determinant: $W(x) = e^x(-e^{-x}) - e^{-x}(e^x) = -1 - 1 = -2$.
3. Conclusion: Since $W(x) = -2 \neq 0$ for all x, y_1 and y_2 are **linearly independent**.

Example #2: Determine if $y_1 = x^2$ and $y_2 = 3x^2$ are linearly independent.

Solution: $n = 2$. $y_1' = 2x$, $y_2' = 6x$.

1. Compute Wronskian:
$$W(x^2, 3x^2) = \begin{vmatrix} x^2 & 3x^2 \\ 2x & 6x \end{vmatrix}$$
2. Determinant: $W(x) = x^2(6x) - 3x^2(2x) = 6x^3 - 6x^3 = 0$.
3. Conclusion: Since $W(x) = 0$ for all x, y_1 and y_2 are **linearly dependent**.

Example #3: Determine if $y_1 = x, y_2 = x^2, y_3 = x^3$ are linearly independent.

Solution: $n = 3$. $y_1' = 1, y_1'' = 0$; $y_2' = 2x, y_2'' = 2$; $y_3' = 3x^2, y_3'' = 6x$.

1. Compute Wronskian:
$$W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$
2. Determinant (cofactor expansion along the first column):
$$W(x) = x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - 1 \begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix} + 0$$
$$W(x) = x(12x^2 - 6x^2) - (6x^3 - 2x^3) = x(6x^2) - 4x^3 = 6x^3 - 4x^3 = 2x^3$$
3. Conclusion: Since $W(x) = 2x^3 \neq 0$ for $x \neq 0$, the functions are **linearly independent**.

4.2 Non-Homogenous Linear Equations (NHLE) - Lagrange's Method (Variation of Parameters)

The general solution of a non-homogenous equation is the sum of the complementary function y_c (solution to the corresponding HLE) and a particular solution y_p:
$$y(x) = y_c(x) + y_p(x)$$

Theorem 4.3 (Existence and Uniqueness)**.** For a linear ODE, the general solution y_c is always composed of n linearly independent solutions, forming the basis.

The **Method of Variation of Parameters** (also known as Lagrange's Method) finds the particular solution y_p for an n-th order NHLE $y'' + P(x)y' + Q(x)y = f(x)$ when the complementary solution $y_c = c_1y_1 + c_2y_2$ is known.

Solution Algorithm 4.2.1. Lagrange's Method (for 2nd Order NHLE)

Step 1 Find the complementary solution $y_c = c_1y_1 + c_2y_2$ by solving the corresponding HLE.

Step 2 Assume the particular solution has the form: $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where u_1 and u_2 are functions to be determined.

Step 3 Calculate the Wronskian $W(y_1, y_2)$.

Step 4 Compute $u_1'(x)$ and $u_2'(x)$ using the formulas:
$$u_1'(x) = -\frac{y_2(x)f(x)}{W(y_1, y_2)} \quad \text{and} \quad u_2'(x) = \frac{y_1(x)f(x)}{W(y_1, y_2)}$$
where $f(x)$ is the non-homogenous term (assuming the leading coefficient of y'' is 1).

Step 5 Integrate to find $u_1(x) = \int u_1'(x)dx$ and $u_2(x) = \int u_2'(x)dx$. (Constants of integration are omitted for y_p).

Step 6 The particular solution is $y_p = u_1y_1 + u_2y_2$.

Step 7 The general solution is $\mathbf{y} = \mathbf{y_c} + \mathbf{y_p}$.

Example #1: Solve $y'' + y = \sec x$ using Lagrange's method.

Solution: HLE is $y'' + y = 0$. Characteristic equation $m^2 + 1 = 0 \implies m = \pm i$.

1. Complementary solution: $y_c = c_1 \cos x + c_2 \sin x$. $y_1 = \cos x, y_2 = \sin x$. $f(x) = \sec x$.

2. Wronskian: $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$.

3. Find u_1' and u_2':
$$u_1' = -\frac{\sin x \sec x}{1} = -\frac{\sin x}{\cos x} = -\tan x$$
$$u_2' = \frac{\cos x \sec x}{1} = \frac{\cos x}{\cos x} = 1$$

4. Integrate:
$$u_1 = \int -\tan x dx = \ln|\cos x|$$
$$u_2 = \int 1 dx = x$$

5. Particular solution: $y_p = u_1 y_1 + u_2 y_2 = \ln|\cos x| \cdot \cos x + x \cdot \sin x$.

6. General solution: $\mathbf{y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x}$.

Example #2: Solve $y'' - 4y' + 4y = \frac{e^{2x}}{x}$ using Lagrange's method.

Solution: HLE is $y'' - 4y' + 4y = 0$. Characteristic equation $m^2 - 4m + 4 = 0 \implies (m-2)^2 = 0$. $m = 2, 2$.

1. Complementary solution: $y_c = c_1 e^{2x} + c_2 x e^{2x}$. $y_1 = e^{2x}, y_2 = xe^{2x}$. $f(x) = 1/xe^{2x}$.

2. Wronskian: $y_1' = 2e^{2x}$, $y_2' = (2x+1)e^{2x}$.
$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix} = e^{4x}(2x+1) - e^{4x}(2x) = e^{4x}$$

3. Find u_1' and u_2':
$$u_1' = -\frac{xe^{2x}(e^{2x}/x)}{e^{4x}} = -\frac{e^{4x}}{e^{4x}} = -1$$
$$u_2' = \frac{e^{2x}(e^{2x}/x)}{e^{4x}} = \frac{1}{x}$$

4. Integrate: $u_1 = \int -1 dx = -x$. $u_2 = \int \frac{1}{x} dx = \ln|x|$.

5. Particular solution: $y_p = u_1 y_1 + u_2 y_2 = -xe^{2x} + \ln|x|(xe^{2x})$.

6. General solution: $y = c_1 e^{2x} + c_2 x e^{2x} - xe^{2x} + xe^{2x} \ln|x|$.
$$\mathbf{y = c_1 e^{2x} + c_3 x e^{2x} + x e^{2x} \ln|x|} \quad (\text{where } c_3 = c_2 - 1)$$

Example #3: Solve $y'' - 2y' + y = \frac{e^x}{x^2}$.

Solution: HLE is $y'' - 2y' + y = 0$. Characteristic equation $m^2 - 2m + 1 = 0 \implies (m-1)^2 = 0$. $m = 1, 1$.

1. Complementary solution: $y_c = c_1 e^x + c_2 x e^x$. $y_1 = e^x, y_2 = xe^x$. $f(x) = e^x/x^2$.

2. Wronskian: $y_1' = e^x$, $y_2' = (x+1)e^x$.

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}(x+1) - xe^{2x} = e^{2x}$$

3. Find u_1' and u_2':

$$u_1' = -\frac{xe^x(e^x/x^2)}{e^{2x}} = -\frac{1}{x}$$
$$u_2' = \frac{e^x(e^x/x^2)}{e^{2x}} = \frac{1}{x^2}$$

4. Integrate: $u_1 = \int -\frac{1}{x}dx = -\ln|x|$. $u_2 = \int x^{-2}dx = -x^{-1} = -1/x$.

5. Particular solution: $y_p = u_1y_1 + u_2y_2 = (-\ln|x|)e^x + (-1/x)xe^x = -e^x\ln|x| - e^x$.

6. General solution: $y = c_1e^x + c_2xe^x - e^x\ln|x| - e^x$.

$$\mathbf{y} = \mathbf{Ce^x} + \mathbf{c_2xe^x} - \mathbf{e^x}\ln|\mathbf{x}| \quad (\text{where } C = c_1 - 1)$$