Problem Set

Muktadir Somio

Part I: Combinatorics, Set Theory, and PIE (40 Problems)

Q1: [S]A standard license plate has 3 letters followed by 4 digits. How many unique plates can be formed if repetition is allowed for both?

Q2: [S]How many ways can 7 distinct books be arranged on a shelf?

Q3: [S]From a set of 10 distinct points on a circle, how many different chords can be formed?

Q4: [S]In a class of 30 students, a president and a vice-president must be chosen. How many ways can this be done?

Q5: [S]How many distinct permutations are there of the letters in the word MISSISSIPPI?

Q6: [S]Given $|A| = 25$, $|B| = 35$, and $|A \cap B| = 10$. Find $|A \cup B|$.

Q7: [S]If $U = \{1, 2, \ldots, 100\}$, and A is the set of multiples of 5, find $|A'|$.

Q8: [S]How many 4-digit numbers can be formed using the digits $\{1, 2, 3, 4, 5\}$ if no digit is repeated?

Q9: [S]A committee of 4 people is to be selected from a group of 8 men and 6 women. How many committees are possible?

Q10: [S]Set $X = \{a, b\}$ and $Y = \{1, 2, 3\}$. List all elements of the Cartesian product $X \times Y$.

Q11: [S]How many subsets of the set $\{1, 2, 3, 4, 5, 6\}$ contain exactly 3 elements?

Q12: [S]A coin is tossed 5 times. What is the total number of outcomes in the sample space?

Q13: [S]Given A and B are disjoint sets with $|A| = 12$ and $|B| = 18$. Find $|A \cup B|$.

Q14: [S]Find the number of non-negative integer solutions to the equation $x_1 + x_2 = 10$. (Stars and Bars)

Q15: [S]How many ways can 5 people stand in a line?

Q16: [I]A team of 5 is selected from 8 men and 7 women. How many teams contain at least 3 men?

Q17: [I]How many ways can the letters of the word ALGEBRA be arranged such that the two A's are not consecutive?

Q18: [I]In a survey of 100 people, 60 read The Times (T), 45 read The Guardian (G), and 20 read neither. Find the number of people who read both. (PIE Application)

Q19: [I]Using the digits $\{0, 1, 2, 3, 4, 5\}$, how many distinct 5-digit odd numbers can be formed?

Q20: [I]Seven books are to be arranged on a shelf. Three specific books (A, B, C) must always be together. How many arrangements are possible?

Q21: [I]Three different prizes are to be distributed among 5 students. If a student can receive at most one prize, how many ways can this be done?

Q22: [I]Given $|A \setminus B| = 15$, $|B \setminus A| = 10$, and $|A \cup B| = 30$. Find $|A \cap B|$.

Q23: [I]Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 7$ where $x_3 \geq 2$.

Q24: [I]How many ways can 10 people be divided into two teams of 5 people each?

Q25: [I]Calculate the number of Derangements ($!n$) for $n = 5$. (Advanced Counting)

Q26: [I]In a set of 5 events $E_1, \ldots, E_5$, each with $P(E_i) = 0.1$. Use Boole's Inequality to find an upper bound for $P(E_1 \cup \cdots \cup E_5)$.

Q27: [I]From a standard deck of 52 cards, how many different 5-card hands contain exactly 2 Aces and 3 Kings?

Q28: [I]In a class of 50 students, 30 take Math, 25 take Physics. The number who take both is between 5 and 15. What is the maximum number of students who take exactly one subject?

Q29: [I]How many ways can 4 distinct red balls and 3 distinct blue balls be arranged in a line so that no two blue balls are consecutive?

Q30: [I]How many different words can be formed by rearranging the letters of the word ARRANGE?

Q31: [C]In a class of 100 students: 50 take Math (M), 40 take Physics (P), 35 take Chemistry (C). 15 take $M \cap P$, 10 take $P \cap C$, 20 take $M \cap C$. If 5 students take all three, how many students take none of the three subjects? (PIE for 3 Sets)

Q32: [C]How many 4-digit numbers greater than 5000 can be formed using the digits $\{1, 2, 3, 4, 5, 6, 7\}$ if repetition is not allowed?

Q33: [C]Show that the number of ways to arrange n objects in a circle, where arrangements are considered the same if one can be obtained from the other by rotation, is $\frac{(n-1)!}{2}$ if the arrangement can be flipped (e.g., beads on a necklace).

Q34: [C]Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 \leq 8$. (Hint: Introduce a slack variable).

Q35: [C]Derive a general formula for the number of arrangements of n distinct objects where exactly k objects are in their original position (not a derangement).

Q36: [C]A shelf holds 12 books: 5 Math, 4 Science, and 3 History. In how many ways can the books be arranged such that all books of the same subject are grouped together?

Q37: [C]Prove Bonferroni's Inequality for two events A and B: $P(A \cap B) \geq P(A) + P(B) - 1$.

Q38: [C]Calculate the minimum number of people needed in a room such that the probability that at least two people share the same birthday is greater than 0.5. (Conceptual setup only, assume 365 days).

Q39: [C]Let A be the set of integers divisible by 2, B by 3, and C by 5, in the set $\{1, 2, \ldots, 300\}$. Find the number of integers divisible by none of 2, 3, or 5.

Q40: [C]A class of 10 students has 5 pairs of twins. If a group of 4 students is selected randomly, how many groups can be formed that contain exactly one pair of twins?

Part II: Fundamental, Joint, and Conditional Probability (40 Problems)

Q41: [S]A fair six-sided die is rolled. Find the probability of rolling a number greater than 4.

Q42: [S]Two events A and B are mutually exclusive. $P(A) = 0.3$ and $P(B) = 0.5$. Find $P(A \cup B)$.

Q43: [S]A card is drawn from a standard 52-card deck. Find the probability it is a red face card (King, Queen, or Jack).

Q44: [S]Given $P(A) = 0.7$, $P(B) = 0.4$, and $P(A \cap B) = 0.2$. Find $P(A \cup B)$.

Q45: [S]A jar contains 5 red and 5 blue marbles. Two marbles are drawn with replacement. Find the probability that both are blue.

Q46: [S]If $P(A) = 0.6$ and $P(A \cap B) = 0.3$. Find the conditional probability $P(B|A)$.

Q47: [S]Two fair coins are tossed. Find the probability of getting at least one head.

Q48: [S]For two independent events E and F, $P(E) = 0.2$ and $P(F) = 0.5$. Find $P(E \cap F)$.

Q49: [S]What is the probability of drawing a King or a Spade from a standard deck?

Q50: [S]A system has two independent components, A and B. $P(A \text{ fails}) = 0.1$, $P(B \text{ fails}) = 0.2$. Find the probability that the system fails (i.e., at least one component fails).

Q51: [S]Given $P(A|B) = 0.8$ and $P(B) = 0.5$. Find $P(A \cap B)$.

Q52: [S]In a class, 70% of students pass Math. If 3 students are chosen randomly, find the probability that all 3 passed Math, assuming independence.

Q53: [S]A spinner has 4 equal sectors labeled 1, 2, 3, 4. It is spun twice. Find the probability that the sum of the results is 6.

Q54: [S]If $P(A \cup B) = 0.9$ and $P(A \cap B) = 0.1$, and $P(A) = 0.4$. Find $P(B)$.

Q55: [S]What is the probability of drawing two cards of the same rank (a pair) when drawing two cards without replacement from a deck?

Q56: [I]A bag contains 4 red and 6 blue balls. Two balls are drawn without replacement. Find the probability that the second ball is red.

Q57: [I]Let A and B be two events. Given $P(A') = 0.4$, $P(B') = 0.3$, and $P(A \cup B)' = 0.1$. Find $P(A|B)$.

Q58: [I]Two cards are drawn without replacement. Find the probability that the first card is a King and the second card is a Queen.

Q59: [I]A company has two machines, M1 and M2, which operate independently. $P(\text{M1 fails}) = 0.1$ and $P(\text{M2 fails}) = 0.05$. Find the probability that exactly one machine fails.

Q60: [I]Prove that if A and B are independent, then A and B' are also independent.

Q61: [I]In a box, 60% of the parts are good (G) and 40% are defective (D). If two parts are drawn without replacement, find $P(\text{second part is G — first part is D})$.

Q62: [I]Three fair dice are rolled. Find the probability that the sum of the three faces is 5.

Q63: [I]A discrete random variable X has $P(X = x) = c(x + 1)$ for $x = 0, 1, 2$. Find the value of the constant c and calculate $P(X > 1)$.

Q64: [I]Given $P(A) = 0.5$, $P(B) = 0.6$, and $P(A|B) = 0.7$. Find $P(B|A)$.

Q65: [I]Show that if A and B are mutually exclusive, they cannot be independent, unless $P(A) = 0$ or $P(B) = 0$.

Q66: [I]A poker hand of 5 cards is dealt from a standard deck. Find the probability of getting a "Flush" (5 cards of the same suit).

Q67: [I]What is the probability of having exactly 1 heart when drawing 3 cards from a deck without replacement?

Q68: [I]If A, B, and C are independent events with $P(A) = 0.3$, $P(B) = 0.4$, and $P(C) = 0.5$. Find $P(A' \cap B' \cap C')$.

Q69: [I]Two teams A and B play a series of games. The first team to win 2 games wins the series. Team A has a probability of 0.6 of winning any single game. Find the probability that Team A wins the series.

Q70: [I]A box contains 3 fair coins and 1 biased coin ($P(\text{Heads}) = 0.8$). A coin is selected at random and tossed. Find the probability of getting a Head.

Q71: [C]Events $A_1, A_2, \ldots, A_n$ are independent. Show that $P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \prod_{i=1}^{n}(1 - P(A_i))$.

Q72: [C]Given $P(A|B) = 2P(A)$, and $P(B|A) = 3P(B)$. Find $P(A)$ and $P(B)$, assuming $P(A) > 0$ and $P(B) > 0$.

Q73: [C]Derive the inclusion-exclusion principle formula for the probability of the union of four events $P(A \cup B \cup C \cup D)$.

Q74: [C]Three drawers contain two coins each: Drawer 1 (2 Gold), Drawer 2 (2 Silver), Drawer 3 (1 Gold, 1 Silver). A drawer is chosen randomly, and one coin is drawn. It is Gold. What is the probability that the remaining coin in that drawer is also Gold? (Conceptual Setup for Bayes)

Q75: [C]A sequence of independent Bernoulli trials with success probability p is performed. Find the probability that the first success occurs at an even numbered trial.

Q76: [C]A bag contains N balls, K of which are red. n balls are drawn without replacement. Let X be the number of red balls drawn. Find the probability $P(X = k)$. (Hypergeometric PMF derivation)

Q77: [C]Prove the identity $P(A|B) + P(A'|B) = 1$.

Q78: [C]Let $A_1, A_2, \ldots, A_n$ be an exhaustive set of events (partitions of Ω). Show that for any event B, $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$ (Law of Total Probability).

Q79: [C]Two events A and B are such that $P(A \cup B) = 1$ and $P(A \cap B) = 0.2$. If A and B are equally likely, find $P(A|B)$.

Q80: [C]Find the probability of drawing exactly k aces when drawing n cards without replacement from a standard deck.

Part III: Law of Total Probability and Bayes' Theorem (40 Problems)

Q81: [S]A drug test is 98% accurate for users (true positive) and 99% accurate for non-users (true negative). If 0.5% of the population use the drug, find the probability that a random person tests positive. (LTP)

Q82: [S]A manufacturing plant has three lines L1, L2, L3, producing 50%, 30%, and 20% of products. Defect rates are 1%, 2%, and 3% respectively. Find the overall probability of a defective product.

Q83: [S]Given $P(A_1) = 0.6$, $P(A_2) = 0.4$ (a partition), $P(B|A_1) = 0.1$, $P(B|A_2) = 0.2$. Find $P(B)$.

Q84: [S]A disease affects 1% of the population. A test is 90% accurate for the disease (True Positive). The false positive rate is 5%. Find $P(\text{Disease}|\text{Positive Test})$.

Q85: [S]A university student is taking two courses, Math (M) and English (E). 70% pass M, and 80% pass E. If they fail M, the probability of failing E is 0.4. What is the probability of failing both?

Q86: [S]A worker commutes via Bus (B) 60% of the time and Car (C) 40% of the time. $P(\text{Late}|B) = 0.3$, $P(\text{Late}|C) = 0.1$. Find the probability that the worker is on time.

Q87: [S]Using the data from Q82, if a product is found to be defective, what is the probability it came from line L1? (Bayes)

Q88: [S]In a city, 60% of people own a smartphone (S). 40% own a laptop (L). 80% of smartphone owners own a laptop. Find the probability that a person owns a smartphone given they own a laptop.

Q89: [S]Three friends A, B, C shoot at a target with probabilities of hitting $P(A) = 0.5$, $P(B) = 0.6$, $P(C) = 0.7$. Find the probability that the target is hit at least once.

Q90: [S]Given $P(A) = 0.3$, $P(B) = 0.5$, $P(A \cap B) = 0.15$. Are A and B independent? Justify.

Q91: [S]In a box, 70% of items are of type X, and 30% are of type Y. The probability that a type X item is faulty is 0.05, and for type Y is 0.01. Find the probability that a randomly chosen faulty item is of type X.

Q92: [S]Let $P(A_1) = p$ and $P(A_2) = 1 - p$ partition the sample space. If $P(B|A_1) = a$ and $P(B|A_2) = b$. Write $P(A_1|B)$ in terms of p, a, b.

Q93: [S]A bag has 5 red and 5 blue marbles. Two marbles are drawn without replacement. Find $P(\text{Second is Blue}|\text{First is Red})$.

Q94: [S]$P(A) = 0.8$. If $P(B) = 0.2$, and A and B are disjoint, find $P(A \cap B')$.

Q95: [S]A factory runs two shifts: Day (D, 60% of production) and Night (N, 40%). The error rate on the Day shift is 2% and on the Night shift is 5%. What is the probability that an error occurred?

Q96: [I]A company uses two screening methods M1 and M2 sequentially. The probability of an item being faulty is 0.1. M1 misses 5% of faulty items. M2 misses 10% of the faulty items that M1 passed. Find the probability that a faulty item is passed by both screens.

Q97: [I]Three drawers contain 3 coins each. Drawer D_i has i gold coins ($i = 1, 2, 3$). A drawer is chosen at random and a coin is drawn. If the coin is gold, what is the probability that it came from the drawer with the most gold coins (D_3)?

Q98: [I]$P(A|B) = 0.7$, $P(A|B') = 0.3$, $P(B) = 0.4$. Calculate $P(A)$.

Q99: [I]In a community, 30% of residents are smokers. The probability that a smoker develops lung disease is 0.6, and the probability for a non-smoker is 0.1. If a resident has lung disease, what is the probability they are a smoker?

Q100: [I]Prove that if A and B are independent, then $P(A|B) = P(A|B')$.

Q101: [I]A signal can be high (H, $P(H) = 0.6$) or low (L, $P(L) = 0.4$). The channel adds noise such that $P(\text{receive L}|\text{send H}) = 0.2$ and $P(\text{receive H}|\text{send L}) = 0.1$. If L is received, what is the probability that L was sent?

Q102: [I]A card is drawn from a deck. Let A be the event of drawing an Ace, B be the event of drawing a Heart. Are A and B independent? Justify formally.

Q103: [I]Two marbles are drawn without replacement from a bag containing 3 red and 7 blue marbles. Find the probability that the two marbles are of different colors.

Q104: [I]Show that for any events A and B, $P(A) = P(A|B)P(B) + P(A|B')P(B')$.

Q105: [I]A company's stock price rises (R) 65% of the time. If it rises today, the probability of it rising tomorrow is 70%. If it doesn't rise today, the probability of it rising tomorrow is 40%. What is the probability it will rise tomorrow?

Q106: [I]A continuous uniform random variable X is distributed over $[0, 5]$. Find $P(X \geq 3|X \geq 2)$.

Q107: [I]In a class, 50% of students who are on the Dean's list (D) are also honor students (H). 20% of students are on the Dean's list. 60% of students are Honor students. Find the probability that an honor student is on the Dean's list.

Q108: [I]Two boxes. Box A: 3 red, 2 blue. Box B: 2 red, 4 blue. A coin is flipped. If Heads ($P(H) = 0.6$), Box A is chosen. Otherwise, Box B is chosen. Find the probability of drawing a red ball.

Q109: [I]Using the setup in Q113, if a red ball is drawn, what is the probability that Box B was chosen?

Q110: [I]Let A, B, C be three events. Given $P(A \cap B \cap C') = 0.1$, $P(A \cap B' \cap C) = 0.15$, $P(A \cap B \cap C) = 0.05$. Find $P(A \cap (B \cup C))$.

Q111: [C]A test for a rare genetic condition (P=0.001) has a sensitivity (True Positive) of 0.99 and a specificity (True Negative) of 0.95. A person tests positive. Find the posterior probability of having the condition. Discuss why the result is counter-intuitive.

Q112: [C]A, B, C are three events such that A and B are independent, B and C are independent, and A and C are disjoint. Given $P(A) = 0.5$, $P(B) = 0.4$, $P(C) = 0.3$. Find $P(A \cup B \cup C)$.

Q113: [C]An urn contains n balls, k of which are red. A ball is drawn, its color is noted, and it is returned along with c additional balls of the same color (Polya's Urn). If the first ball drawn is red, find the probability that the second ball drawn is red.

Q114: [C]Prove that if $A_1, A_2, \ldots, A_n$ are independent events, then $P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$.

Q115: [C]A survey shows that 80% of students use a specific study app (A). 60% of students who use the app score an A grade (G). 10% of students who do not use the app also score an A grade. Find the probability that a student who scored an A grade *did not* use the app.

Q116: [C]Generalize Bayes' Theorem for three nested events $E_1 \subset E_2 \subset E_3$: $\frac{P(E_3|E_2)P(E_2|E_1)}{P(E_3|E_1)}$. Simplify the expression.

Q117: [C]In a sequence of coin tosses, show that the probability of seeing two consecutive Heads (HH) before seeing a Head followed by a Tail (HT) is 1/3, assuming the coin is fair.

Q118: [C]Let $P(A) = a$, $P(B) = b$, and $P(A \cap B) = c$. Express $P(A^c \cup B^c)$ in terms of a, b, c.

Q119: [C]Derive a formula for $P(A \cap B \cap C')$ purely in terms of the probabilities of the 3-set union and pairwise intersections.

Q120: [C]A multiple-choice test has 5 options per question. A student knows the answer with probability p and guesses randomly with probability $1-p$. If the student answers correctly, find the probability that they actually knew the answer.

Part IV: Random Variables, PMF, PDF, Expectation, and Variance (40 Problems)

Q121: [S]Classify the following as a Discrete or Continuous Random Variable: The weight of a newborn baby.

Q122: [S]A discrete RV X has the following PMF: $P(X = 1) = 0.2, P(X = 2) = 0.3, P(X = 3) = 0.5$. Calculate $\mathbb{E}[X]$.

Q123: [S]For the RV in Q122, calculate $\mathbb{E}[X^2]$.

Q124: [S]A continuous RV X has a PDF $f(x) = 0.5$ for $0 \le x \le 2$. Find $P(X \le 1)$.

Q125: [S]Find the constant c such that $f(x) = cx^2$ for $0 \le x \le 1$ is a valid PDF.

Q126: [S]A random variable X has $\mathbb{E}[X] = 4$ and $\mathrm{Var}(X) = 9$. Find $\mathbb{E}[2X + 5]$.

Q127: [S]For the RV in Q126, find $\mathrm{Var}(2X + 5)$.

Q128: [S]State the two defining properties of a Probability Mass Function (PMF).

Q129: [S]The CDF of a DRV is $F(x) = 0$ for $x < 0$, 0.3 for $0 \le x < 1$, and 1 for $x \ge 1$. Find $P(X = 0)$.

Q130: [S]If $X \sim \mathrm{Uniform}(a, b)$, what is the PDF $f(x)$?

Q131: [S]A fair coin is tossed until the first Head. Let X be the number of tosses. List the range of X.

Q132: [S]Let X be a DRV with $\mathbb{E}[X] = 5$. Find $\mathbb{E}[X - \mathbb{E}[X]]$.

Q133: [S]If a continuous RV X has PDF $f(x)$. Write the formula for its CDF, $F(x)$.

Q134: [S]Given a PMF $P(X = x) = 1/4$ for $x \in \{1, 2, 3, 4\}$. Calculate the standard deviation $\mathrm{SD}(X)$.

Q135: [S]A bus arrives between 8:00 AM and 8:15 AM with equal probability. Let X be the arrival time in minutes past 8:00 AM. Find $\mathrm{Var}(X)$.

Q136: [I]A bag contains 2 red and 3 blue balls. Let X be the number of red balls drawn when 2 balls are drawn without replacement. Form the full PMF of X.

Q137: [I]A DRV X has PMF $P(X = x) = k(x^2 + 1)$ for $x = 0, 1, 2$. Find the constant k and calculate $\mathbb{E}[X]$.

Q138: [I]A CRV X has PDF $f(x) = k(3-x)$ for $0 \le x \le 3$. Find the constant k and calculate $P(X > 1)$.

Q139: [I]The CDF of a DRV is $F(x) = 0$ for $x < 0$, 0.2 for $0 \le x < 1$, 0.7 for $1 \le x < 2$, and 1 for $x \ge 2$. Find $\mathrm{Var}(X)$.

Q140: [I]Derive the expectation formula for a $\mathrm{Uniform}(a, b)$ distribution.

Q141: [I]Let X be the number of successes in $n = 3$ independent Bernoulli trials with $p = 0.5$. Calculate $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ using the properties $\mathbb{E}[X_1+X_2+X_3]$ and $\mathrm{Var}(X_1+X_2+X_3)$.

Q142: [I]A continuous RV has PDF $f(x) = 2x$ for $0 \le x \le 1$. Find the median value m such that $P(X \le m) = 0.5$.

Q143: [I]Show that $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Q144: [I]A DRV X has $P(X = 1) = 1/2$, $P(X = 2) = 1/3$, $P(X = 3) = 1/6$. Find $\mathbb{E}[1/X]$.

Q145: [I]If X is a CRV with PDF $f(x)$. Write the formula for the variance $\mathrm{Var}(X)$ using integration.

Q146: [I]For a function $g(X)$, state the general formula for $\mathbb{E}[g(X)]$ for a discrete random variable X.

Q147: [I]The height H of a student is a CRV with $\mathbb{E}[H] = 170$ cm and $\mathrm{SD}(H) = 10$ cm. Find the expected value and variance of the height converted to meters ($M = H/100$).

Q148: [I]Given that $F(x)$ is the CDF of a CRV, how is the PDF $f(x)$ related to $F(x)$?

Q149: [I]Let X be the number of distinct people you meet until you find one with your same birthday. Assume 365 days. Find $\mathbb{E}[X]$.

Q150: [I]Calculate the mode of the CRV with PDF $f(x) = 6x(1-x)$ for $0 \le x \le 1$.

Q151: [C]A random variable X is Exponential($\lambda = 0.5$) with PDF $f(x) = 0.5e^{-0.5x}$ for $x \ge 0$. Derive the expected value $\mathbb{E}[X]$ using integration by parts.

Q152: [C]Let X be the DRV of the number of coin tosses needed to get the first head. $P(X = k) = (1-p)^{k-1}p$. Show that $\sum_{k=1}^{\infty} P(X = k) = 1$. (Geometric Series Proof)

Q153: [C]Prove the property $\text{Var}(aX+b) = a^2\text{Var}(X)$ starting from the definition of variance.

Q154: [C]A CRV X has PDF $f(x)$. Derive the CDF of $Y = X^2$ in terms of $F_X(x)$, assuming X is symmetric around 0.

Q155: [C]Find the median of the Exponential(λ) distribution.

Q156: [C]A stick of length L is broken at a random point $X \sim \text{Uniform}(0, L)$. Find the expected length of the longer piece.

Q157: [C]The moment generating function is defined as $M_X(t) = \mathbb{E}[e^{tX}]$. Show how $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ can be derived from $M_X(t)$.

Q158: [C]A DRV X has $P(X = x) = \frac{1}{2^x}$ for $x = 1, 2, 3, \ldots$. Calculate $\mathbb{E}[X]$. (Hint: Use derivative of geometric series).

Q159: [C]Let X be a continuous RV. Show that $\mathbb{E}[|X - c|]$ is minimized when c is the median of X.

Q160: [C]Derive the general formula for the k-th moment, $\mathbb{E}[X^k]$, for a continuous uniform distribution on $[a, b]$.

Part V: Bivariate Distributions, Covariance, and Correlation (40 Problems)

Q161: [S]State the range of the correlation coefficient, $\rho(X, Y)$.

Q162: [S]Given $\mathbb{E}[X] = 3$, $\mathbb{E}[Y] = 5$, and $\mathbb{E}[XY] = 18$. Calculate $\text{Cov}(X, Y)$.

Q163: [S]If X and Y are independent random variables, what is the value of $\text{Cov}(X, Y)$?

Q164: [S]Given $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$, and $\text{Cov}(X, Y) = 6$. Calculate $\rho(X, Y)$.

Q165: [S]If $\rho(X, Y) = 1$, what does this imply about the relationship between X and Y?

Q166: [S]Write the formula for the $\text{Cov}(X, Y)$ in terms of $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[XY]$.

Q167: [S]Given X and Y are independent, $\text{Var}(X) = 2$, $\text{Var}(Y) = 3$. Find $\text{Var}(X + Y)$.

Q168: [S]For the variables in Q167, find $\text{Var}(2X - 3Y)$.

Q169: [S]For a discrete joint distribution $P(x, y)$, state the formula for the marginal PMF of X, $P_X(x)$.

Q170: [S]A joint PMF is $P(X = 1, Y = 1) = 0.2$, $P(X = 1, Y = 2) = 0.3$, $P(X = 2, Y = 1) = 0.1$, $P(X = 2, Y = 2) = 0.4$. Find $P(X = 1)$.

Q171: [S]Using the joint PMF from Q170, find the conditional probability $P(Y = 2|X = 1)$.

Q172: [S]If X and Y are independent, $f(x, y) = f_X(x)f_Y(y)$. State the analogous condition for conditional PDFs.

Q173: [S]Let X and Y be two RVs. State the general formula for $\mathbb{E}[aX + bY]$.

Q174: [S]Given X and Y are independent, $\mathbb{E}[X] = 5, \mathbb{E}[Y] = 10$. Find $\mathbb{E}[XY]$.

Q175: [S]A bivariate continuous RV has joint PDF $f(x, y)$. Write the formula for the conditional PDF $f_{Y|X}(y|x)$.

Q176: [I]Two independent RVs X and Y have $\mathbb{E}[X] = 1, \mathrm{Var}(X) = 4, \mathbb{E}[Y] = 2, \mathrm{Var}(Y) = 9$. Find $\mathrm{SD}(2X - 5Y + 1)$.

Q177: [I]A DRV pair (X, Y) has the following joint PMF: $P(1, 1) = 0.3, P(1, 2) = 0.2, P(2, 1) = 0.2, P(2, 2) = 0.3$. Determine if X and Y are independent.

Q178: [I]Calculate $\mathbb{E}[X + Y]$ for the joint PMF in Q177.

Q179: [I]Calculate $\mathrm{Cov}(X, Y)$ for the joint PMF in Q177.

Q180: [I]A joint PDF is given by $f(x, y) = c(x + y)$ for $0 \le x \le 1, 0 \le y \le 1$. Find the constant c.

Q181: [I]Using the PDF from Q180, find the marginal PDF $f_X(x)$.

Q182: [I]Let X and Y be the number of Heads and Tails in 3 coin tosses. Find $\mathrm{Cov}(X, Y)$.

Q183: [I]If $\mathbb{E}[X|Y = y] = y$ and $Y \sim \mathrm{Uniform}(0, 1)$. Find $\mathbb{E}[X]$. (Law of Total Expectation)

Q184: [I]The correlation between a stock (X) and a bond (Y) is $\rho(X, Y) = -0.5$. $\mathrm{SD}(X) = 10, \mathrm{SD}(Y) = 5$. A portfolio is $P = 0.5X + 0.5Y$. Find $\mathrm{Var}(P)$.

Q185: [I]Prove the identity $\mathrm{Cov}(aX + b, cY + d) = ac\mathrm{Cov}(X, Y)$ starting from the definition of covariance.

Q186: [I]Find the marginal distribution $P_X(x)$ for a Multinomial distribution where $n = 2$ and $k = 2$ categories (effectively a Binomial distribution).

Q187: [I]Given a joint PDF $f(x, y)$. Write the formula for the expected value of a function $g(X, Y)$, $\mathbb{E}[g(X, Y)]$.

Q188: [I]For two RVs X and Y, show that $\mathrm{Cov}(X, Y) = 0$ does not necessarily imply independence. (Conceptual only, state a counter-example condition).

Q189: [I]Given $\text{Var}(X+Y) = 10$ and $\text{Var}(X-Y) = 6$. If $\text{Var}(X) = 4$, find $\text{Cov}(X,Y)$.

Q190: [I]Two students arrive independently at a restaurant, one between 12:00 and 1:00 PM, the other between 12:00 and 1:30 PM. Let X and Y be their arrival times (in minutes past 12:00). Find $P(X < Y)$.

Q191: [C]Derive the formula for the correlation coefficient $\rho(X,Y)$ from the definition of covariance and standard deviation.

Q192: [C]Given the joint PDF $f(x,y) = 1$ for $0 \leq x \leq 1$ and $x \leq y \leq x+1$. Are X and Y independent? Justify your answer by checking the boundaries of the support.

Q193: [C]Prove the Cauchy-Schwarz Inequality for random variables: $|\mathbb{E}[XY]|^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2]$.

Q194: [C]Consider a sequence of n independent trials. Let X be the number of successes in the first k trials, and Y be the number of successes in the last $n-k$ trials. Show that X and Y are independent. (Conceptual proof setup).

Q195: [C]Given $f_{Y|X}(y|x) = xe^{-xy}$ for $y \geq 0$, and $f_X(x) = e^{-x}$ for $x \geq 0$. Find the joint PDF $f(x,y)$.

Q196: [C]Using the joint PDF from Q195, find the marginal PDF $f_Y(y)$.

Q197: [C]For a portfolio $P = \sum_{i=1}^{n} w_i X_i$, where w_i are weights and X_i are asset returns. Derive the general formula for $\text{Var}(P)$ in terms of variances and covariances.

Q198: [C]Let $X_1, X_2, \ldots, X_n$ be a set of independent and identically distributed (i.i.d.) random variables. Let $S = \sum X_i$ and $\bar{X} = S/n$. Show that $\text{Cov}(X_i - \bar{X}, \bar{X}) = 0$. (Independence of sample mean and residual).

Q199: [C]Prove the Law of Total Variance: $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$.

Q200: [C]Let X and Y be non-negative RVs. Prove that if X and Y are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$ for any non-negative measurable functions g and h.