Multivariate and Vector Calculus Problems

Part I: Partial Derivatives, Vector-Valued Functions and Geometry

4.1 Advanced Partial Derivatives (25 Problems)

Instructions: Perform the required partial differentiation or derivation, often requiring implicit or chain rule application.

1. **(Implicit Differentiation)** Given $xe^y + ye^z + ze^x = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2. **(Mixed Partial)** Find the mixed second-order partial derivative $\frac{\partial^2 f}{\partial y \partial x}$ for $f(x, y) = \arctan\left(\frac{y}{x}\right)$.
3. **(Clairaut's Theorem)** For $f(x, y, z) = x\sin(yz)$, verify that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)$.
4. **(Chain Rule - Single Parameter)** Let $z = f(x, y)$, where $x = \cos t$ and $y = \sin t$. Derive the formula for $\frac{dz}{dt}$ in terms of t and the partials of f.
5. **(Chain Rule - Two Parameters)** Let $w = x^2y - z^2$, where $x = r\cos\theta$, $y = r\sin\theta$, and $z = r$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.
6. **(Homogeneous Function)** Prove that if $f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$ (Euler's Theorem for Homogeneous Functions).
7. **(Laplace's Equation)** Show that the function $u(x, y) = e^x \sin y$ satisfies the 2D **Laplace's equation**: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
8. **(Wave Equation)** Show that $u(x, t) = \sin(kx)\sin(akt)$ is a solution to the 1D **Wave Equation**: $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$.
9. **(Derivation)** If $z = f(x/y)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.
10. **(Chain Rule - General)** Let $z = f(x_1, x_2)$, where $x_1 = g_1(t_1, t_2)$ and $x_2 = g_2(t_1, t_2)$. Set up the matrix form of the Chain Rule to find the Jacobian $\begin{pmatrix} \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} \end{pmatrix}$.
11. **(Application)** Find the rate of change of the volume of a right circular cone $V = \frac{1}{3}\pi r^2 h$ with respect to the height h, assuming the radius r is constant.
12. **(Implicit)** Find $\frac{\partial y}{\partial z}$ for the equation $x^2 + y^2 + z^2 = \sin(yz)$.
13. **(Higher Order)** Find $\frac{\partial^3 f}{\partial z \partial y \partial x}$ for $f(x, y, z) = xy^2z^3$.
14. **(Conceptual)** If $f_{xy} = f_{yx}$ everywhere in a region, what does this imply about the continuity of the second partial derivatives of f?
15. **(Chain Rule - Logarithm)** Find $\frac{\partial w}{\partial x}$ if $w = \ln(r^2 + s^2)$ and $r = x + y, s = x - y$.
16. **(Tangent Plane/Normal Line)** Find the equation of the **normal line** to the surface $z = x^2 - 2y^2$ at the point $(2, 1, 2)$.
17. **(Directional Derivative/Geometric)** Find the **directional derivative** of $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$ in the direction of the vector $\mathbf{v} = \langle 1, 1\rangle$.

18. **(Gradient/Level Set)** Find the unit vector that is perpendicular to the level curve of $f(x, y) = x^3 - y^2$ at the point $(1, 1)$.

19. **(Schwarz's Theorem)** Explain the necessary condition for the equality of the mixed second partials $f_{xy} = f_{yx}$ (Schwarz's Theorem).

20. **(Application)** The temperature T at a point (x, y) is given by $T(x, y) = 40 - 2x^2 - y^2$. A bug is at $(1, 3)$. In what direction should the bug move to increase its temperature most rapidly?

21. **(Implicit/Derivation)** If $F(x, y, z) = 0$ defines x implicitly as a function of y and z, derive the formula for $\frac{\partial x}{\partial y}$.

22. **(Conceptual)** Under what conditions does the existence of all first-order partial derivatives guarantee the differentiability of a function $f(x, y)$?

23. **(Second Partials)** Calculate f_{xx} and f_{yy} for $f(x, y) = \cos(3x + 2y)$.

24. **(Product Rule)** Find $\frac{\partial f}{\partial x}$ for $f(x, y) = x^y \cos(x^2 y)$.

25. **(Jacobian)** Find the **Jacobian determinant** $\frac{\partial(u,v)}{\partial(x,y)}$ for the transformation $u = x/y, v = xy$.

4.2 Curvature, Motion, and Vector Fields

26. **(Curvature/Torsion)** A particle moves along the curve $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$. Find the **torsion** τ of the curve at $t = 1$.

27. **(Motion/Multiple Theorems)** A projectile is launched with $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \langle 10, 0, 30 \rangle$. If the acceleration is $\mathbf{a}(t) = \langle 0, 0, -9.8 \rangle$, find the minimum **speed** of the projectile and the time it occurs.

28. **(Tangent/Normal Plane)** Given $f(x, y, z) = x^2 + y^2 - z^2 = 9$. Find the equation of the **normal line** and the **tangent plane** to this surface at the point $(3, 1, 1)$.

29. **(Directional Derivative/Maximum Rate)** Find the unit vector in the direction of maximum increase of the scalar field $\phi(x, y, z) = xy^2z^3$ at the point $(1, 2, 1)$. What is this maximum rate?

30. **(Line Integral/Application)** Calculate the work done by the force field $\mathbf{F}(x, y) = \langle x^2y, xy^2 \rangle$ moving a particle along the line segment from $(0, 0)$ to $(1, 2)$.

31. **(Conservative Field/Potential)** Determine if the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 + 2z \rangle$ is **conservative**. If so, find the **potential function** $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

32. **(Chain Rule/Gradient)** Let $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Express the $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of f_x and f_y. Use this to show that $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$.

33. **(Divergence/Curl)** Calculate the **divergence** and **curl** of the vector field $\mathbf{F}(x, y, z) = \langle x^2, xy, yz \rangle$.

34. **(TNB Frame)** Find the **Unit Tangent Vector** $\mathbf{T}(t)$ and the **Unit Normal Vector** $\mathbf{N}(t)$ for $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ at $t = \pi/2$.

35. **(Kepler's Second Law)** Show that for a particle moving under a central force $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}$, the vector $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, implying that the motion is planar.

4.3 Multivariable Optimization and Taylor Series

37. **(Local Extrema/Hessian)** Find all **critical points** of $f(x, y) = x^3 - 3x + y^2$ and use the **Second Derivative Test** (Hessian) to classify them.

38. **(Lagrange Multipliers)** Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + 2y^2 = 1$.

39. **(Taylor Series/Approximation)** Find the **second-degree Taylor polynomial** $T_2(x, y)$ for $f(x, y) = e^{-x^2-y^2}$ centered at $(0, 0)$. Use it to approximate $f(0.1, -0.2)$.

40. **(Error Bound/Remainder Theorem)** Use the **Taylor's Remainder Theorem** to find an upper bound on the error $|R_2(x)|$ when $f(x) = \ln(1+x)$ is approximated by its second-degree Maclaurin polynomial $T_2(x)$ on the interval $[-0.1, 0.1]$.

41. **(Power Series/Convergence)** Find the **interval of convergence** for the power series $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^n}$.

42. **(Maclaurin Series/Application)** Find the Maclaurin series for $f(x) = \frac{x}{1-x^2}$ by manipulating the geometric series formula.

43. **(Hessian Matrix)** Calculate the **Hessian matrix** $H(x, y, z)$ for the function $f(x, y, z) = x^2 \sin(y) + z^3 y^2$.

44. **(Unconstrained Optimization)** For $f(x, y) = x^4 + y^4 - 4xy + 1$, find the critical points and determine if the Second Derivative Test is conclusive at the point $(1, 1)$.

45. **(Lagrange/Geometric Interpretation)** Use the method of Lagrange Multipliers to find the point on the plane $x + 2y - z = 4$ that is closest to the origin $(0, 0, 0)$.

46. **(Taylor Polynomial)** Find the **Taylor polynomial of degree 3** for $f(x) = \cos(2x)$ centered at $a = 0$.

4.4 Jacobian and Matrix Calculus

47. **(Jacobian/Change of Variables)** Find the **Jacobian** $\frac{\partial(x,y)}{\partial(u,v)}$ for the transformation $x = 2u + 3v$ and $y = u - v$. Use this to describe how area changes under this map.

48. **(Jacobian/Polar Coords)** Calculate the **Jacobian** for the transformation from rectangular to **polar coordinates**: $x = r\cos\theta, y = r\sin\theta$.

49. **(Hessian/Symmetry)** Prove that for a function $f : \mathbb{R}^n \to \mathbb{R}$ with continuous second partial derivatives, the **Hessian matrix** H is **symmetric**.

50. **(Jacobian/Chain Rule)** Let $\mathbf{G}(u, v) = \langle u^2 - v^2, 2uv \rangle$ and $f(x, y) = x + y$. Compute the Jacobian of the composition $\mathbf{F}(u, v) = f(\mathbf{G}(u, v))$ using the **multivariable Chain Rule** (Jacobian product).

51. **(Inverse Function Theorem)** Let $\mathbf{F}(x, y) = \langle x + y^2, 2x^2 + y \rangle$. Calculate the **Jacobian** $J_{\mathbf{F}}(1, 1)$. If the Jacobian is non-singular, what does the **Inverse Function Theorem** guarantee about $\mathbf{F}$ near $(1, 1)$?

52. **(Vector Product Rule)** State and verify the vector product rule for the **curl** of a scalar function times a vector field: $\nabla \times (\phi\mathbf{F})$.

53. **(Gradient/Level Surface)** Show that the **gradient** ∇f at a point P is **orthogonal** to the level surface of f passing through P.

54. **(Implicit Differentiation)** If $F(x, y, z) = 0$ defines z implicitly as a function of x and y, use the chain rule for the gradient to derive the formula for $\frac{\partial z}{\partial x}$.

55. **(Conceptual)** In matrix calculus, what does the $\frac{\partial f}{\partial \mathbf{x}}$ (the gradient of a scalar field f with respect to a vector $\mathbf{x}$) represent?

56. **(Eigenvalues/Hessian)** If P is a critical point of $f(x, y)$ and the **eigenvalues** of the Hessian matrix $H(P)$ are $\lambda_1 = -2$ and $\lambda_2 = 5$, classify P.

Part II: Integration and Coordinate Systems

4.5 Double and Triple Integrals

58. **(Triple Integral/Application)** Calculate the **volume** of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - x^2 - y^2$. Use **cylindricalcoordinates**.

59. **(Double Integral/Change of Order)** Evaluate the double integral $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$ by **changing the order of integration**.

60. **(Double Integral/Area)** Find the **area** of the region bounded by the curves $y = x^2$ and $y = 2x$.

61. **(Triple Integral/Mass)** A solid is defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. If the density is $\delta(x, y, z) = x + y + z$, find the **total mass** of the solid.

62. **(Centroid)** Set up the iterated integral for the x-coordinate of the **centroid** of the triangular region with vertices $(0, 0), (1, 0), (0, 1)$, assuming uniform density.

63. **(Change of Variables)** Use the transformation $x = 2u$ and $y = 3v$ to evaluate the integral $\iint_R \sin(x^2/4 + y^2/9)\, dA$, where R is the elliptical region $x^2/4 + y^2/9 \le 1$.

64. **(Green's Theorem/Area)** Use **Green's Theorem** to find the **area** of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

65. **(Fundamental Theorem of Line Integrals)** Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y) = x^2y - y^3$ and C is any smooth curve from $(-1, 1)$ to $(1, 2)$.

66. **(Volume/Solid Geometry)** Set up and evaluate the triple integral for the volume of the region bounded by $z = 0$, $z = x + y + 5$, and the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

67. **(Green's Theorem/Work)** Calculate the work done by the force field $\mathbf{F}(x, y) = \langle y^2 - x, 2xy\rangle$ moving a particle once counterclockwise around the circle $x^2 + y^2 = 9$.

4.6 Polar, Cylindrical, and Spherical Coordinates

70. **(Polar Coords/Integration)** Evaluate the double integral $\iint_D e^{-(x^2+y^2)}\, dA$, where D is the entire $\mathbf{R}^2$ plane.

71. **(Cylindrical Coords/Application)** Find the **volume** of the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $z = 0$, and the plane $z = x + y + 10$.

72. **(Spherical Coords/Volume)** Set up the triple integral in **spherical coordinates** to find the **volume** of the region that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2+y^2+z^2 = 9$.

73. **(Jacobian/Spherical Coords)** State the **Jacobian** (Volume Element dV) for the transformation from rectangular to **spherical coordinates**.

74. **(Coordinate Conversion)** Convert the equation of the surface $x^2 + y^2 - z^2 = 1$ from rectangular to both **cylindrical** and **spherical** coordinates.

75. **(Spherical/Density)** A solid ball of radius a has a density function $\delta(\rho, \phi, \theta) = \rho^2$. Set up the triple integral in spherical coordinates to find the **total mass**.

76. **(Triple Integral/Application)** Use cylindrical coordinates to calculate $\iiint_E z^2\, dV$ where E is the solid that lies between the spheres $\rho = 1$ and $\rho = 3$ and above the cone $\phi = \pi/4$.

77. **(Surface Area/Parametric)** Find the **surface area** of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$. Use the standard surface area integral formula.

78. **(Fundamental Theorem)** Explain how the **Fundamental Theorem of Calculus for Line Integrals** is used to simplify calculations when the vector field is conservative.

4.7 Advanced Theorems (Stokes', Gauss's, etc.)

80. **(Stokes' Theorem/Application)** Use **Stokes' Theorem** to evaluate $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle xz, xy, 3yz \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 0$, oriented upward.

81. **(Gauss's Theorem/Divergence)** Use the **Divergence Theorem (Gauss's)** to evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ and S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

82. **(Stokes' Theorem/Line Integral)** Use **Stokes' Theorem** to calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x+y, y+z, z+x \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, oriented counterclockwise when viewed from above.

83. **(Gauss's Theorem/Volume)** Let E be a solid region with boundary S. Prove that the **volume** of E can be calculated by the formula $\text{Volume}(E) = \frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n}\, d\mathbf{S}$, where $\mathbf{r} = \langle x, y, z \rangle$.

84. **(Surface Integral/Flux)** Set up the surface integral for the **flux** of $\mathbf{F}(x, y, z) = \langle x, y, z^2 \rangle$ across the surface S, which is the part of the cylinder $x^2 + z^2 = 4$ that lies between $y = 0$ and $y = 3$, oriented outward.

85. **(Conceptual/Divergence vs. Curl)** Explain the fundamental geometric difference between the physical interpretations of the **Divergence** and the **Curl** of a vector field $\mathbf{F}$.

86. **(Stokes' Theorem/Conceptual)** If $\text{curl}\mathbf{F} = \mathbf{0}$ everywhere in a region D, use **Stokes' Theorem** to explain why the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ must be zero for any closed curve C in D.

87. **(Conservative/Double Application)** Given $\mathbf{F} = \nabla f$ is a conservative field, show that $\text{curl}\mathbf{F} = \mathbf{0}$ and $\text{div}(\text{curl}\mathbf{F}) = 0$.

88. **(Flux/Cylindrical Coords)** Calculate the **outward flux** of the field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ across the closed surface of the cylinder $x^2 + y^2 = R^2$ bounded by $z = 0$ and $z = H$.

89. **(Triple Integral/Volume)** Set up and evaluate the triple integral for the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the planes $y = x$ and $y = \sqrt{3}x$ in the first octant.

4.8 Series, Remainder, and Approximation

91. **(Maclaurin Series/Integration)** Use the Maclaurin series for $f(x) = \frac{1}{1+x}$ to find the Maclaurin series for $g(x) = \ln(1 + x)$.

92. **(Remainder Theorem/Application)** Given $f(x) = e^{-x}$, find the maximum value of the **Lagrange error bound** $|R_4(x)|$ when approximating $f(x)$ on the interval $[-1, 1]$ using the Maclaurin polynomial $T_4(x)$.

93. **(Taylor Series)** Find the **Taylor series** for $f(x) = \frac{1}{x}$ centered at $a = 1$.

94. **(Taylor/Multivariable)** Write the general formula for the **Taylor Series expansion** of $f(x, y)$ centered at (a, b) up to the second degree.

95. **(Series Convergence)** State the **Ratio Test** and use it to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

96. **(Series/Differentiation)** Starting with the geometric series for $\frac{1}{1-x}$, find the power series representation for $\frac{1}{(1-x)^2}$.

97. **(Approximation)** Use a third-degree Taylor polynomial for $f(x) = \sin(x)$ centered at $a = 0$ to approximate $\sin(0.1)$.

98. **(Vector Series)** The path of a particle is given by $\mathbf{r}(t) = \langle e^t, \cos t \rangle$. Find the **second-degree Taylor polynomial** for $\mathbf{r}(t)$ centered at $t = 0$.

99. **(Remainder Theorem)** Explain why the **alternating series estimation theorem** provides a simpler error bound than the general Taylor's Remainder Theorem for certain series.

100. **(Series Integration)** Express $\int_0^{0.5} \frac{1}{1+x^4}\, dx$ as an infinite series.

4.9 Advanced Multivariable Techniques (Application)

102. **(Line Integral/Green's)** Use Green's Theorem to show that $\oint_C x^2\,dy$ gives the area enclosed by the simple closed curve C.

103. **(Jacobian/Volume Scaling)** The transformation $T(u,v,w) = \langle au, bv, cw\rangle$ maps the unit cube $0 \le u, v, w \le 1$ to a rectangular solid R. Use the **Jacobian** to find the volume of R.

104. **(Surface Integral/Mass)** The surface S is the cone $z = \sqrt{x^2+y^2}$ for $0 \le z \le 1$. If the mass density is $\delta(x,y,z) = z$, set up the integral for the **total mass** of S.

105. **(Gauss's Theorem/Non-spherical)** Use the **Divergence Theorem** to find the outward flux of $\mathbf{F}(x,y,z) = \langle e^z, \sin(x), y^2\rangle$ across the surface of the cube $0 \le x, y, z \le 1$.

106. **(Change of Order/Spherical)** A region E is defined by $\rho \le 1$ and $\pi/4 \le \phi \le \pi/2$. Set up the triple integral of $f(\rho,\phi,\theta)$ in the order $d\rho\, d\phi\, d\theta$.

107. **(Hessian/Optimization)** Given $f(x,y)$, if the Hessian matrix $H(a,b)$ is positive definite, what does the **Second Derivative Test** conclude about the critical point (a,b)?

108. **(Conservative Field/Path Independence)** Prove that if $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every simple closed curve C in an open region D, then $\mathbf{F}$ is a **conservative vector field** in D.

109. **(Stokes' Theorem/Surface Choice)** Let C be the boundary of the surface S. Explain why, when using **Stokes' Theorem**, you can replace S with any other simple, oriented surface S' that shares the same boundary C.

110. **(Jacobian/Non-linear)** Find the Jacobian $\frac{\partial(x,y)}{\partial(r,\phi)}$ for the transformation $x = r\cosh\phi, y = r\sinh\phi$.

111. **(Line Integral/Circulation)** Calculate the **circulation** of the vector field $\mathbf{F}(x,y) = \langle -y, x\rangle$ around the unit circle C, and use this to state the physical interpretation of circulation.

4.10 Geometrical Synthesis and Advanced Concepts

112. **(Geometric Measure)** Find the volume of the region that lies inside the sphere $x^2+y^2+z^2 = 4$ and outside the cylinder $x^2+y^2 = 1$. Use the most appropriate coordinate system.

113. **(Curvature/Radius of Curvature)** Calculate the **radius of curvature** ρ for the parabola $y = x^2$ at the origin $(0,0)$.

114. **(Arc Length/Vector)** Find the arc length of the curve $\mathbf{r}(t) = \langle t^2, 4t^{3/2}, 2t\rangle$ from $t = 0$ to $t = 1$.

115. **(3D Area)** Find the area of the part of the plane $z = x+2y$ that lies inside the cylinder $x^2+y^2 = 4$.

116. **(Polar/Area)** Find the area of the region outside the cardioid $r = 1+\cos\theta$ and inside the circle $r = 3$.

117. **(Level Curves/Gradient)** Sketch the level curves of $f(x,y) = x^2 - y^2$ and the gradient vector ∇f at two distinct points, illustrating the orthogonality property.

118. **(Second Derivative Test)** For $f(x,y)$, if $D(a,b) = f_{xx}f_{yy} - (f_{xy})^2 = 0$ at a critical point (a,b), explain what the **Second Derivative Test** concludes.

119. **(Series Manipulation)** Show that the series $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ by differentiating the geometric series.

120. **(Coordinate Systems)** Describe a solid E (using inequalities) that would make the integral $\iiint_E f\,dV$ simplest to evaluate using **cylindrical** coordinates but complex in **spherical** coordinates.

121. **(Jacobian/Conceptual)** If a non-singular transformation T has a constant Jacobian magnitude of 5, how does the volume of any region R change when mapped to $T(R)$?

122. **(Tangent Plane/Implicit)** Find the tangent plane to the surface defined implicitly by $\ln(xyz) = x+y+z-3$ at the point $(1,1,1)$.

123. **(Flux/Geometric)** Calculate the **outward flux** of the vector field $\mathbf{F}(x, y, z) = \langle x, 2y, 3z \rangle$ across the surface of a cube defined by $0 \leq x, y, z \leq 2$. Use the Divergence Theorem.

124. **(Lagrange/Proof)** Explain the geometric motivation behind the **Lagrange Multiplier** condition $\nabla f = \lambda \nabla g$ when optimizing f subject to $g = c$.

125. **(Series/Convergence)** Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is **absolutely convergent**, **conditionally convergent**, or **divergent**.

126. **(Vector/Acceleration)** A particle has position $\mathbf{r}(t)$. Decompose the acceleration $\mathbf{a}(t)$ into its **tangential** (a_T) and **normal** (a_N) components using dot and cross products.