Combinatorics and Probability

1 Set Theory and Generalized Inclusion-Exclusion

1.1 Fundamental Set Identities and Cardinality

Definition 1.1 (Key Set Operations)**.** Let A, B be sets in a universal set U.

(i) Difference: $A \setminus B = A \cap B^c$.

(ii) Symmetric Difference: $A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(iii) Cardinality of Symmetric Difference: $|A\Delta B| = |A| + |B| - 2|A \cap B|$

(iv) De Morgan's Laws (Generalized): $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n A_i^c$.

Theorem 1.1 (PIE for Two and Three Sets)**.** (i) $|A \cup B| = |A| + |B| - |A \cap B|$.

(ii) $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$

Solved Problem 1.1 (Difference and Symmetric Difference)**.** Given $U = \{1, 2, \ldots, 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$. Find $A \setminus B$, $A\Delta B$, and verify $|A\Delta B|$ using the cardinality formula.

Solution. Step 1: Calculate Intersection and Set Difference. $A \cap B = \{1, 3, 5\}$ $A \setminus B = A \cap B^c = \{7, 9\}$

Step 2: Calculate $B \setminus A$. $B \setminus A = B \cap A^c = \{2, 4\}$

Step 3: Calculate Symmetric Difference $A\Delta B$. $A\Delta B = (A \setminus B) \cup (B \setminus A) = \{7, 9\} \cup \{2, 4\} = \{2, 4, 7, 9\}$. Therefore, $|A\Delta B| = 4$.

Step 4: Verify with Cardinality Formula. $|A| = 5$, $|B| = 5$, $|A \cap B| = 3$. $|A\Delta B| = |A| + |B| - 2|A \cap B| = 5 + 5 - 2(3) = 10 - 6 = 4$. The formula is verified. ■

Solved Problem 1.2 (De Morgan's Law Verification)**.** Given $U = \{a, b, c, d, e\}$, $A = \{a, b, c\}$, $B = \{c, d\}$. Verify the generalized De Morgan's Law: $(\bigcup_{i=1}^2 A_i)^c = \bigcap_{i=1}^2 A_i^c$, where $A_1 = A, A_2 = B$.

Solution. Step 1: Calculate the LHS (Complement of the Union). $A \cup B = \{a, b, c\} \cup \{c, d\} = \{a, b, c, d\}$. $(\bigcup_{i=1}^2 A_i)^c = (A \cup B)^c = U \setminus (A \cup B) = \{a, b, c, d, e\} \setminus \{a, b, c, d\} = \{e\}$.

Step 2: Calculate the RHS (Intersection of the Complements). $A^c = U \setminus A = \{d, e\}$. $B^c = U \setminus B = \{a, b, e\}$. $\bigcap_{i=1}^2 A_i^c = A^c \cap B^c = \{d, e\} \cap \{a, b, e\} = \{e\}$.

Step 3: Comparison. Since LHS $= \{e\}$ and RHS $= \{e\}$, the identity $(\bigcup_{i=1}^2 A_i)^c = \bigcap_{i=1}^2 A_i^c$ is verified. ■

Solved Problem 1.3 (PIE for Two Sets - Application)**.** In a class of 50 students, 30 take Math (M) and 25 take Physics (P). If 10 students take both, how many students take at least one subject?

Solution. Step 1: Identify Given Cardinalities. $|M| = 30$ (Math students) $|P| = 25$ (Physics students) $|M \cap P| = 10$ (Students taking both)

Step 2: Apply the Principle of Inclusion-Exclusion (PIE) for two sets. The number of students taking at least one subject is $|M \cup P|$.

$$|M \cup P| = |M| + |P| - |M \cap P|$$

Step 3: Calculation.

$$|M \cup P| = 30 + 25 - 10 = 55 - 10 = 45$$

Thus, 45 students take at least one subject. ■

Solved Problem 1.4 (PIE for Three Sets - Application)**.** Out of 100 tourists, 50 speak English (E), 40 speak French (F), 30 speak German (G). 15 speak E and F, 10 speak E and G, 5 speak F and G. 3 speak all three. How many speak at least one language?

Solution. Step 1: List the Given Values. $|E| = 50, |F| = 40, |G| = 30$ $|E \cap F| = 15, |E \cap G| = 10, |F \cap G| = 5$ $|E \cap F \cap G| = 3$

Step 2: Apply the PIE for three sets. The number of tourists speaking at least one language is $|E \cup F \cup G|$.

$$|E \cup F \cup G| = (|E| + |F| + |G|) - (|E \cap F| + |E \cap G| + |F \cap G|) + |E \cap F \cap G|$$

Step 3: Calculation.

- Sum of singles (S_1): $50 + 40 + 30 = 120$
- Sum of doubles (S_2): $15 + 10 + 5 = 30$
- Sum of triples (S_3): 3

$$|E \cup F \cup G| = 120 - 30 + 3 = 90 + 3 = 93$$

Thus, 93 tourists speak at least one language. ■

Solved Problem 1.5 (Cardinality of Complement)**.** Given $|U| = 200$, $|A| = 120$, $|B| = 90$, and $|A \cup B| = 150$. Find the number of elements that are neither in A nor B, which is $|(A \cup B)^c|$.

Solution. Step 1: Define the target set. The number of elements neither in A nor B is the complement of the union, $|(A \cup B)^c|$.

Step 2: Use the Complement Rule.

$$|(A \cup B)^c| = |U| - |A \cup B|$$

Step 3: Calculation.

$$|(A \cup B)^c| = 200 - 150 = 50$$

Thus, 50 elements are neither in A nor B. (Note: We did not need to use the individual cardinalities $|A|$ and $|B|$ to solve this specific problem, but they would allow us to find $|A \cap B| = 120 + 90 - 150 = 60$). ■

1.2 Generalized Inclusion-Exclusion

Theorem 1.2 (Principle of Inclusion-Exclusion (PIE) General Form)**.** Let $A_1, A_2, \ldots, A_n$ be finite sets. The cardinality of their union is:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k-1} S_k$$

where $S_k = \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|$ (the sum of the cardinalities of all intersections of k distinct sets).

Theorem 1.3 (Generalized PIE: Exactly k Elements (E_k))**.** Let E_k be the number of elements belonging to exactly k of the n sets $A_1, \ldots, A_n$.

$$E_k = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} S_j$$

Solved Problem 1.6 (PIE for 4 Sets)**.** A coding system uses four properties A_1, A_2, A_3, A_4. In a set of 100 codes, assume all single intersections have size 20, all double intersections have size 10, all triple intersections have size 5, and the quadruple intersection has size 2. Find the size of the union $|\bigcup_{i=1}^{4} A_i|$.

Solution. **Step 1: Calculate the S_k terms.** The number of sets $n = 4$.

- S_1: Sum of all single intersections. There are $\binom{4}{1} = 4$ such intersections.

$$S_1 = 4 \times 20 = 80$$

- S_2: Sum of all double intersections. There are $\binom{4}{2} = 6$ such intersections.

$$S_2 = 6 \times 10 = 60$$

- S_3: Sum of all triple intersections. There are $\binom{4}{3} = 4$ such intersections.

$$S_3 = 4 \times 5 = 20$$

- S_4: Sum of all quadruple intersections. There is $\binom{4}{4} = 1$ such intersection.

$$S_4 = 1 \times 2 = 2$$

Step 2: Apply the General PIE Formula.

$$|\bigcup_{i=1}^{4} A_i| = S_1 - S_2 + S_3 - S_4$$

Step 3: Calculation.

$$|\bigcup_{i=1}^{4} A_i| = 80 - 60 + 20 - 2 = 20 + 20 - 2 = 38$$

Thus, 38 codes satisfy at least one of the four properties. ■

Solved Problem 1.7 (PIE for None)**.** Using the context of the previous example ($|U| = 100$, $|\bigcup_{i=1}^{4} A_i| = 38$), find the number of codes that satisfy *none* of the four properties.

Solution. **Step 1: Define the target quantity.** The number of codes satisfying none of the properties is the complement of the union: $|\bigcap_{i=1}^{4} A_i^c| = |U| - |\bigcup_{i=1}^{4} A_i|$.
Step 2: Apply the Complement Rule.

$$|\bigcap_{i=1}^{4} A_i^c| = |U| - |\bigcup_{i=1}^{4} A_i|$$

Step 3: Calculation.

$$|\bigcap_{i=1}^{4} A_i^c| = 100 - 38 = 62$$

Thus, 62 codes satisfy none of the four properties. ■

Solved Problem 1.8 (Exactly 2 Elements, E_2)**.** In a group of 80 people, $S_1 = 60$ people read magazine A, B, or C. $S_2 = 35$ people read the intersection of two magazines. $S_3 = 10$ people read all three. Find the number of people who read *exactly* two magazines (E_2).

Solution. **Step 1: Identify Given S_k values.** The number of sets $n = 3$. We are looking for E_2. S_1: Sum of single set sizes (not given directly, but S_1 is not needed for E_2 when $j \geq 2$). Wait, S_1 is the sum of single set sizes. The prompt incorrectly says S_1 is the union, which is $S_1 - S_2 + S_3$. Let's assume the question meant: $|A \cup B \cup C| = 60$ (The union) $S_2 = |A \cap B| + |A \cap C| + |B \cap C| = 35$ $S_3 = |A \cap B \cap C| = 10$ To find E_2, we need S_2 and S_3.

Step 2: Apply the E_k Formula for $k = 2$.

$$E_2 = \sum_{j=2}^{3} (-1)^{j-2} \binom{j}{2} S_j = (-1)^{2-2} \binom{2}{2} S_2 + (-1)^{3-2} \binom{3}{2} S_3$$

Step 3: Calculation.

$$E_2 = (+1) \times (1) \times S_2 + (-1) \times (3) \times S_3$$

$$E_2 = S_2 - 3S_3 = 35 - 3(10) = 35 - 30 = 5$$

Thus, 5 people read exactly two magazines. ■

Solved Problem 1.9 (Exactly 1 Element, E_1)**.** Using the context of the previous example, find the number of people who read *exactly* one magazine (E_1).

Solution. Step 1: Define E_1 using the generalized PIE. The total number of people in the union is $|A \cup B \cup C| = 60$. The total number of people is the sum of those reading exactly 1, exactly 2, and exactly 3 magazines:

$$|A \cup B \cup C| = E_1 + E_2 + E_3$$

Step 2: Find E_3 (Exactly 3 Elements). $E_3 = S_3 = 10$ (given).
Step 3: Use the relationship to find E_1.

$$E_1 = |A \cup B \cup C| - E_2 - E_3$$

From the previous problem, we found $E_2 = 5$.

$$E_1 = 60 - 5 - 10 = 45$$

Thus, 45 people read exactly one magazine. ■

Solved Problem 1.10 (Deriving $|A \cap B \cap C|$)**.** Given $|A| = 10, |B| = 12, |C| = 15, |A \cup B \cup C| = 25$. We know $|A \cap B| = 4, |A \cap C| = 5, |B \cap C| = 6$. Find $|A \cap B \cap C|$.

Solution. Step 1: State the PIE formula for three sets.

$$|A \cup B \cup C| = S_1 - S_2 + S_3$$

where $S_1 = |A| + |B| + |C|$, $S_2 = |A \cap B| + |A \cap C| + |B \cap C|$, and $S_3 = |A \cap B \cap C|$.
Step 2: Calculate S_1 and S_2.

$$S_1 = 10 + 12 + 15 = 37$$

$$S_2 = 4 + 5 + 6 = 15$$

Step 3: Substitute the known values and solve for S_3.

$$25 = 37 - 15 + S_3$$

$$25 = 22 + S_3$$

$$S_3 = 25 - 22 = 3$$

Thus, $|A \cap B \cap C| = 3$. ■

2 Advanced Combinatorics and Counting

2.1 Permutations and Combinations (12 Solved Problems)

Formula 2.1 (k-Permutations (Ordered, No Repetition))**.**

$$P(n,k) = \frac{n!}{(n-k)!}$$

Formula 2.2 (k-Combinations (Unordered, No Repetition))**.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Formula 2.3 (Combinations with Repetition (Multisets))**.** The number of ways to choose k items from n categories with repetition allowed:

$$\text{Multiset}\binom{n}{k} = \binom{n+k-1}{k}$$

Solved Problem 2.1 (k-Permutations: Club Officers)**.** How many ways can a president, vice-president, and treasurer be selected from a club of 15 members?

Solution. Step 1: Identify the parameters. The total number of items is $n = 15$. The number of items to choose is $k = 3$. Since the order matters (President $\neq$ Vice-President), this is a k-permutation.

Step 2: Apply the Permutation Formula $P(n,k)$.

$$P(15,3) = \frac{15!}{(15-3)!} = \frac{15!}{12!}$$

Step 3: Calculation.

$$P(15,3) = 15 \times 14 \times 13 = 2730$$

There are 2,730 ways to select the three officers. ■

Solved Problem 2.2 (k-Combinations: Pizza Toppings)**.** A restaurant offers 8 different toppings for a pizza. How many ways can a customer choose exactly 4 toppings?

Solution. Step 1: Identify the parameters. Total toppings $n = 8$. Toppings chosen $k = 4$. Since the order of toppings does not matter, this is a combination.

Step 2: Apply the Combination Formula $\binom{n}{k}$.

$$\binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8!}{4!4!}$$

Step 3: Calculation.

$$\binom{8}{4} = \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!} = \frac{8 \times 7 \times 6 \times 5}{24} = 7 \times 2 \times 5 = 70$$

There are 70 ways to choose 4 toppings. ■

Solved Problem 2.3 (Permutations with Repetition: Word Arrangement)**.** How many distinct arrangements of the letters in the word "MISSISSIPPI" are there?

Solution. **Step 1: Count total letters and repetitions.** Total letters $n = 11$. Repetitions: M (1), I (4), S (4), P (2). The formula for distinct permutations with repetition is $\frac{n!}{n_1!n_2!...n_k!}$.

Step 2: Apply the formula.

$$\text{Arrangements} = \frac{11!}{1! \times 4! \times 4! \times 2!}$$

Step 3: Calculation.

$$11! = 39,916,800$$

$$4! = 24$$

$$\text{Denominator} = 1 \times 24 \times 24 \times 2 = 1152$$

$$\text{Arrangements} = \frac{39,916,800}{1152} = 34,650$$

There are 34,650 distinct arrangements. ■

Solved Problem 2.4 (Stars and Bars: Non-negative Integer Solutions)**.** Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 7$.

Solution. **Step 1: Identify parameters for Stars and Bars.** Number of variables (categories) $n = 3$. Sum (stars) $k = 7$. We are looking for non-negative solutions.

Step 2: Apply the Stars and Bars formula. The number of solutions is $\binom{n+k-1}{k}$ (or $\binom{n+k-1}{n-1}$).

$$\text{Solutions} = \binom{3+7-1}{7} = \binom{9}{7}$$

Step 3: Calculation.

$$\binom{9}{7} = \binom{9}{9-7} = \binom{9}{2} = \frac{9 \times 8}{2 \times 1} = 36$$

There are 36 non-negative integer solutions. ■

Solved Problem 2.5 (Stars and Bars: Positive Integer Solutions)**.** Find the number of positive integer solutions (where $x_i \geq 1$) to $x_1 + x_2 + x_3 = 7$.

Solution. **Step 1: Transform the problem to non-negative solutions.** Since $x_i \geq 1$, we introduce new variables $y_i = x_i - 1$, where $y_i \geq 0$. The original equation becomes:

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) = 7$$

$$y_1 + y_2 + y_3 = 7 - 3 = 4$$

Step 2: Apply Stars and Bars for the transformed equation. Number of variables $n = 3$. Sum $k = 4$.

$$\text{Solutions} = \binom{n+k-1}{k} = \binom{3+4-1}{4} = \binom{6}{4}$$

Step 3: Calculation.

$$\binom{6}{4} = \binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15$$

There are 15 positive integer solutions. ■

Solved Problem 2.6 (Combinations with Repetition (Multisets))**.** A donut shop sells 6 different kinds of donuts. How many ways are there to choose a dozen (12) donuts?

Solution. **Step 1: Identify parameters.** Number of categories (kinds of donuts) $n = 6$. Number of items chosen (a dozen) $k = 12$. Repetition is allowed.

Step 2: Apply the Multiset Combination formula $\binom{n+k-1}{k}$.

$$\text{Ways} = \binom{6+12-1}{12} = \binom{17}{12}$$

Step 3: Calculation.

$$\binom{17}{12} = \binom{17}{17-12} = \binom{17}{5}$$

$$\binom{17}{5} = \frac{17 \times 16 \times 15 \times 14 \times 13}{5 \times 4 \times 3 \times 2 \times 1} = 17 \times 4 \times 7 \times 13 = 6188$$

There are 6,188 ways to choose a dozen donuts. ■

Solved Problem 2.7 (Circular Permutations)**.** In how many ways can 7 distinct guests be seated around a circular table?

Solution. **Step 1: Define Circular Permutation.** For n distinct objects arranged in a circle, the number of distinct arrangements is $(n-1)!$, as rotation is considered the same arrangement.

Step 2: Apply the formula. $n = 7$.

$$\text{Ways} = (7-1)! = 6!$$

Step 3: Calculation.

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

There are 720 ways to seat the 7 distinct guests. ■

Solved Problem 2.8 (Restricted Permutation: Adjacency)**.** How many ways can the letters A, B, C, D, E, F be arranged such that A and B are adjacent?

Solution. **Step 1: Group the adjacent elements.** Since A and B must be adjacent, treat the pair (AB) as a single block. The objects to arrange are now: (AB), C, D, E, F. This is a total of $n' = 5$ objects.

Step 2: Arrange the blocks. The number of ways to arrange $n' = 5$ objects is 5!.

$$5! = 120$$

Step 3: Arrange within the block. The letters A and B within the block (AB) can be arranged in $2! = 2$ ways: (AB) or (BA).

Step 4: Total Arrangements. By the Multiplication Principle, multiply the arrangements of the blocks by the internal arrangements of the block.

$$\text{Total Ways} = 5! \times 2! = 120 \times 2 = 240$$

There are 240 arrangements where A and B are adjacent. ■

Solved Problem 2.9 (Combination with Exclusion: At Least)**.** A committee of 5 people is to be formed from 6 men (M) and 4 women (W). How many committees can be formed if the committee must include at least 3 men?

Solution. **Step 1: Identify possible valid compositions.** The committee size is 5. "At least 3 men" means the committee can have:

(i) 3 Men and 2 Women ($\binom{6}{3}\binom{4}{2}$)

(ii) 4 Men and 1 Woman ($\binom{6}{4}\binom{4}{1}$)

(iii) 5 Men and 0 Women ($\binom{6}{5}\binom{4}{0}$)

Step 2: Calculate the combinations for each case.

(i) $\binom{6}{3}\binom{4}{2} = \left(\frac{6\times5\times4}{3\times2\times1}\right) \times \left(\frac{4\times3}{2\times1}\right) = 20 \times 6 = 120$

(ii) $\binom{6}{4}\binom{4}{1} = \left(\frac{6\times5}{2\times1}\right) \times 4 = 15 \times 4 = 60$

(iii) $\binom{6}{5}\binom{4}{0} = 6 \times 1 = 6$

Step 3: Sum the cases.

$$\text{Total Ways} = 120 + 60 + 6 = 186$$

There are 186 ways to form the committee with at least 3 men. ■

Solved Problem 2.10 (Grid Paths - Simple)**.** How many distinct paths are there from the origin $(0,0)$ to the point $(4,3)$ on a grid, only moving East (E) or North (N)?

Solution. Step 1: Define the path requirements. To reach $(4,3)$, a path must consist of 4 moves East (E) and 3 moves North (N). Total moves $n = 4 + 3 = 7$. The problem is equivalent to finding the number of distinct permutations of the sequence EEEENNN.

Step 2: Apply the Permutation with Repetition formula. The number of distinct paths is the number of ways to choose the 4 positions for the E moves (or 3 for the N moves) out of 7 total moves.

$$\text{Paths} = \binom{7}{4} = \binom{7}{3}$$

Step 3: Calculation.

$$\binom{7}{3} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

There are 35 distinct paths. ■

Solved Problem 2.11 (Grid Paths - Restricted)**.** How many distinct paths are there from $(0,0)$ to $(4,3)$ if the path *must* pass through the point $(2,2)$?

Solution. Step 1: Break the path into two stages. The total path is:

(i) Path from $(0,0)$ to $(2,2)$ (2E, 2N).

(ii) Path from $(2,2)$ to $(4,3)$ (2E, 1N).

Step 2: Calculate the number of paths for Stage (i).

$$P_1 = \text{Paths from } (0,0) \to (2,2) = \binom{2+2}{2} = \binom{4}{2} = \frac{4 \times 3}{2} = 6$$

Step 3: Calculate the number of paths for Stage (ii).

$$P_2 = \text{Paths from } (2,2) \to (4,3) = \binom{2+1}{2} = \binom{3}{2} = 3$$

Step 4: Total Paths. By the Multiplication Principle, multiply the number of paths for each stage.

$$\text{Total Paths} = P_1 \times P_2 = 6 \times 3 = 18$$

There are 18 distinct paths that pass through $(2,2)$. ■

Solved Problem 2.12 (Generating Subsets)**.** How many subsets does a set S with $n = 5$ elements have?

Solution. Step 1: State the general formula. The number of subsets of a set with n elements is 2^n. This can be seen by considering that for each element, there are two choices: either it is included in the subset or it is not.

Step 2: Apply the formula for $n = 5$.

$$\text{Number of Subsets} = 2^5$$

Step 3: Calculation.

$$2^5 = 32$$

A set with 5 elements has 32 subsets. ■

2.2 Counting Identities

Formula 2.4 (Derangements)**.** The number of permutations of n objects such that $\pi(i) \neq i$ for all i is the number of derangements, $!n$ (or D_n):

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$$

Formula 2.5 (Vandermonde's Identity)**.**

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}$$

Formula 2.6 (Hockey-Stick Identity)**.**

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

Solved Problem 2.13 (Derangements $!n$)**.** Four friends (A, B, C, D) exchange gifts, but no one is allowed to receive their own gift. How many ways can the gifts be distributed? (Calculate !4)

Solution. Step 1: Identify the problem type. The problem asks for the number of permutations of $n = 4$ objects such that no element remains in its original position, which is the derangement number !4.

Step 2: Apply the Derangement Formula.

$$!4 = 4! \sum_{k=0}^{4} \frac{(-1)^k}{k!}$$

$$!4 = 4! \left(\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} \right)$$

Step 3: Calculation.

$$!4 = 24 \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right)$$

$$!4 = 24 \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right)$$

$$!4 = 24 \left(\frac{12}{24} - \frac{4}{24} + \frac{1}{24} \right) = 24 \left(\frac{12 - 4 + 1}{24} \right) = 24 \left(\frac{9}{24} \right) = 9$$

There are 9 ways to distribute the gifts so no friend receives their own. ■

Solved Problem 2.14 (Vandermonde's Identity)**.** Verify Vandermonde's identity for the case $\binom{10}{5} = \sum_{k=0}^{5} \binom{4}{k} \binom{6}{5-k}$ by calculating both sides.

Solution. Step 1: Calculate the LHS $\binom{r+s}{n}$. Here $r = 4, s = 6, n = 5$.

$$\text{LHS} = \binom{4+6}{5} = \binom{10}{5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 2 \times 3 \times 2 \times 7 \times 3 = 252$$

Step 2: Calculate the RHS $\sum_{k=0}^{5} \binom{4}{k} \binom{6}{5-k}$. Note that $\binom{4}{k} = 0$ for $k > 4$, so the sum only runs from $k = 0$ to $k = 4$.

- $k = 0$: $\binom{4}{0}\binom{6}{5} = 1 \times 6 = 6$

- $k = 1$: $\binom{4}{1}\binom{6}{4} = 4 \times 15 = 60$
- $k = 2$: $\binom{4}{2}\binom{6}{3} = 6 \times 20 = 120$
- $k = 3$: $\binom{4}{3}\binom{6}{2} = 4 \times 15 = 60$
- $k = 4$: $\binom{4}{4}\binom{6}{1} = 1 \times 6 = 6$

Step 3: Sum the terms and compare.

$$\text{RHS} = 6 + 60 + 120 + 60 + 6 = 252$$

Since LHS = 252 and RHS = 252, Vandermonde's Identity is verified for this case. ■

Solved Problem 2.15 (Hockey-Stick Identity)**.** Calculate the sum $\sum_{i=2}^{5} \binom{i}{2}$ and verify it using the Hockey-Stick Identity $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.

Solution. Step 1: Calculate the Sum Directly. Here $r = 2$ and $n = 5$. The sum is:

$$\sum_{i=2}^{5} \binom{i}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}$$
$$= 1 + 3 + 6 + 10 = 20$$

Step 2: Apply the Hockey-Stick Identity.

$$\sum_{i=2}^{5} \binom{i}{2} = \binom{5+1}{2+1} = \binom{6}{3}$$

Step 3: Calculation of the Identity Result.

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

Since the direct sum (20) equals the identity result (20), the Hockey-Stick Identity is verified. ■

Solved Problem 2.16 (Derangements - Application for At Least One)**.** In a class of 5 students, a teacher collects and shuffles their exams. The teacher then randomly hands the exams back. What is the number of ways that *at least one* student receives their correct exam?

Solution. Step 1: Identify Total and Forbidden Permutations. Total number of ways to hand back $n = 5$ exams is $5! = 120$. The number of ways that *none* of the students receive their correct exam is the derangement $!5$.

Step 2: Calculate the derangement $!5$.

$$!5 = 5!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$!5 = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$
$$!5 = 60 - 20 + 5 - 1 = 44$$

(Alternatively, using $!n = (n) \times !(n-1) + (-1)^n$, $!5 = 5 \times !4 - 1 = 5 \times 9 - 1 = 44$).

Step 3: Calculate the target number (At least one). The number of ways for at least one student to receive their correct exam is:

$$\text{Total Permutations} - \text{Derangements } (!5)$$

$$\text{Ways (At least one)} = 5! - !5 = 120 - 44 = 76$$

There are 76 ways that at least one student receives their correct exam. ■

Solved Problem 2.17 (Vandermonde's Identity - Combination Interpretation)**.** A committee of 8 people is chosen from a group of 5 mathematicians ($r = 5$) and 7 physicists ($s = 7$). Use Vandermonde's Identity to calculate the total number of ways to form the committee ($n = 8$).

Solution. Step 1: State the total number of ways (LHS). The total number of people is $r + s = 5 + 7 = 12$. The committee size is $n = 8$.

$$\text{Total Ways} = \binom{12}{8} = \binom{12}{4} = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495$$

Step 2: State the breakdown using Vandermonde's Identity (RHS). The committee can be formed by choosing k mathematicians (from 5) and $n - k = 8 - k$ physicists (from 7). Since we can't choose more than 5 mathematicians or more than 7 physicists, k must run from $\max(0, 8 - 7) = 1$ to $\min(8, 5) = 5$.

$$\text{Total Ways} = \sum_{k=1}^{5} \binom{5}{k}\binom{7}{8-k}$$

Step 3: Calculate the sum.

- $k = 1$: $\binom{5}{1}\binom{7}{7} = 5 \times 1 = 5$
- $k = 2$: $\binom{5}{2}\binom{7}{6} = 10 \times 7 = 70$
- $k = 3$: $\binom{5}{3}\binom{7}{5} = 10 \times 21 = 210$
- $k = 4$: $\binom{5}{4}\binom{7}{4} = 5 \times 35 = 175$
- $k = 5$: $\binom{5}{5}\binom{7}{3} = 1 \times 35 = 35$

Step 4: Comparison.

$$\text{RHS} = 5 + 70 + 210 + 175 + 35 = 495$$

Since $495 = 495$, the identity is verified and the number of ways is 495. ∎

2.3 Stirling Numbers

Definition 2.1 (Stirling Numbers of the Second Kind, $\{ {n \atop k} \}$)**.** The number of ways to partition a set of n distinct objects into k non-empty, indistinguishable subsets.

Formula 2.7 (Stirling Numbers of the Second Kind (Recurrence))**.**

$$\{{n \atop k}\} = \{{n-1 \atop k-1}\} + k\{{n-1 \atop k}\}$$

Formula 2.8 (Stirling Numbers of the First Kind (Recurrence))**.**

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

Solved Problem 2.18 (Stirling Numbers of the Second Kind $\{ {n \atop k} \}$ - Definition)**.** Find $\{ {4 \atop 2} \}$, the number of ways to partition a set of 4 distinct elements, $S = \{1, 2, 3, 4\}$, into 2 non-empty, indistinguishable subsets. List the partitions.

Solution. Step 1: Identify partition types. To partition 4 elements into 2 non-empty subsets, the sizes of the subsets must be either $(3, 1)$ or $(2, 2)$.

Step 2: Count partitions of type (3, 1). Choose 3 elements for the first set ($\binom{4}{3}$) and 1 element for the second ($\binom{1}{1}$).

$$\binom{4}{3}\binom{1}{1} = 4 \times 1 = 4$$

Partitions: $\{\{1, 2, 3\}, \{4\}\}$, $\{\{1, 2, 4\}, \{3\}\}$, $\{\{1, 3, 4\}, \{2\}\}$, $\{\{2, 3, 4\}, \{1\}\}$.

Step 3: Count partitions of type (2, 2). Choose 2 elements for the first set ($\binom{4}{2}$) and 2 for the second ($\binom{2}{2}$). Since the two subsets are indistinguishable, we must divide by 2!.

$$\frac{1}{2!}\binom{4}{2}\binom{2}{2} = \frac{1}{2} \times 6 \times 1 = 3$$

Partitions: $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$.

Step 4: Total ways.

$$\{{4 \atop 2}\} = 4 + 3 = 7$$

■

Solved Problem 2.19 (Stirling Numbers of the Second Kind - Explicit Formula)**.** Use the explicit formula $\{ {n \atop k} \} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$ to compute $\{ {5 \atop 3} \}$.

Solution. Step 1: Apply the formula for $n = 5, k = 3$.

$$\{{5 \atop 3}\} = \frac{1}{3!} \sum_{j=0}^{3} (-1)^{3-j} \binom{3}{j} j^5$$

Step 2: Expand the sum.

$$3! = 6$$

$$6\{{5 \atop 3}\} = (-1)^{3-0}\binom{3}{0}0^5 + (-1)^{3-1}\binom{3}{1}1^5 + (-1)^{3-2}\binom{3}{2}2^5 + (-1)^{3-3}\binom{3}{3}3^5$$

$$6\{{5 \atop 3}\} = (-1)(1)(0) + (1)(3)(1) + (-1)(3)(32) + (1)(1)(243)$$

Step 3: Calculation.

$$6\{{5 \atop 3}\} = 0 + 3 - 96 + 243 = 150$$

$$\{{5 \atop 3}\} = \frac{150}{6} = 25$$

$\blacksquare$

Solved Problem 2.20 (Stirling Numbers of the Second Kind - Recurrence)**.** Given $\{{5 \atop 2}\} = 15$ and $\{{5 \atop 3}\} = 25$, use the recurrence relation to compute $\{{6 \atop 3}\}$.

Solution. Step 1: State the Recurrence Relation.

$$\{{n \atop k}\} = \{{n-1 \atop k-1}\} + k\{{n-1 \atop k}\}$$

Step 2: Substitute $n = 6, k = 3$.

$$\{{6 \atop 3}\} = \{{5 \atop 2}\} + 3\{{5 \atop 3}\}$$

Step 3: Calculation with given values.

$$\{{6 \atop 3}\} = 15 + 3(25) = 15 + 75 = 90$$

$\blacksquare$

Solution. Step 1: Analyze the cycle structure. A permutation of 4 elements has 3 disjoint cycles only if the structure is (length 2 cycle) $\times$ (length 1 cycle) $\times$ (length 1 cycle), which is a $2+1+1$ partition of 4. This means one pair is swapped, and the other two elements are fixed points.

Step 2: Count the ways to choose the cycle elements.

- Choose 2 elements for the cycle of length 2: $\binom{4}{2} = 6$.
- The remaining 2 elements form the two cycles of length 1: $\binom{2}{1}\binom{1}{1} = 2$.

The number of permutations is the number of ways to choose the elements that form the 2-cycle.

Step 3: Calculation.

$$[{4 \atop 3}] = \binom{4}{2} = 6$$

Permutations (in cycle notation): (12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2). $\blacksquare$

Solved Problem 2.22 (Stirling Numbers of the First Kind - Recurrence)**.** Given $[{5 \atop 2}] = 50$ and $[{5 \atop 3}] = 35$, use the recurrence relation to compute $[{6 \atop 3}]$.

Solution. Step 1: State the Recurrence Relation.

$$[{n \atop k}] = [{n-1 \atop k-1}] + (n-1)[{n-1 \atop k}]$$

Step 2: Substitute $n = 6, k = 3$.

$$[{6 \atop 3}] = [{5 \atop 2}] + (6-1)[{5 \atop 3}]$$

$$[{6 \atop 3}] = [{5 \atop 2}] + 5[{5 \atop 3}]$$

Step 3: Calculation with given values.

$$[{6 \atop 3}] = 50 + 5(35) = 50 + 175 = 225$$

$\blacksquare$

3 Axiomatic Probability and Total Probability

3.1 Axiomatic Foundations

Axiom 3.1 (Kolmogorov's Axioms)**.** A probability measure $\mathbb{P}$ on a sample space Ω with σ-algebra $\mathcal{F}$ satisfies:

(i) Non-negativity: $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$

(ii) Unit Measure: $\mathbb{P}(\Omega) = 1$.

(iii) Countable Additivity: If $A_1, A_2, \ldots$ are pairwise disjoint events, then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Property 3.1 (Derived Properties)**.** (i) $\mathbb{P}(\emptyset) = 0$

(ii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(iii) Monotonicity: If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Solved Problem 3.1 (Unit Measure and Non-negativity)**.** A sample space is $\Omega = \{1, 2, 3, 4\}$. If $\mathbb{P}(\{1\}) = 0.1$, $\mathbb{P}(\{2\}) = 0.3$, and $\mathbb{P}(\{3\}) = 0.4$, find $\mathbb{P}(\{4\})$ and show that all axioms are satisfied.

Solution. Step 1: Find $\mathbb{P}(\{4\})$ using Countable Additivity (Axiom iii) and Unit Measure (Axiom ii). Since the outcomes $\{1\}, \{2\}, \{3\}, \{4\}$ are disjoint and their union is Ω, we must have:

$$\mathbb{P}(\Omega) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{4\}) = 1$$

$$1 = 0.1 + 0.3 + 0.4 + \mathbb{P}(\{4\})$$

$$1 = 0.8 + \mathbb{P}(\{4\}) \implies \mathbb{P}(\{4\}) = 0.2$$

Step 2: Check Axioms.

(i) Non-negativity: $0.1, 0.3, 0.4, 0.2$ are all ≥ 0. Satisfied.

(ii) Unit Measure: $\mathbb{P}(\Omega) = 1$. Satisfied by construction.

(iii) Countable Additivity: The sum of probabilities of disjoint elementary events equals the probability of their union. For any event $A \in \mathcal{F}$, $\mathbb{P}(A)$ is the sum of its elementary probabilities, which is ≤ 1. Satisfied.

■

Solved Problem 3.2 (Complement Rule)**.** The probability that a new phone model is a success (S) is $\mathbb{P}(S) = 0.75$. What is the probability that it is a failure ($F = S^c$)?

Solution. Step 1: Define the events. F is the complement of S, $F = S^c$.
Step 2: Apply the Derived Property (ii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

$$\mathbb{P}(F) = \mathbb{P}(S^c) = 1 - \mathbb{P}(S)$$

Step 3: Calculation.

$$\mathbb{P}(F) = 1 - 0.75 = 0.25$$

The probability of failure is 0.25. ■

Solved Problem 3.3 (Countable Additivity - Finite Case)**.** Two disjoint events are A (rolling a 1 or 2 on a fair die) and B (rolling a 5 or 6). Find $\mathbb{P}(A \cup B)$.

Solution. **Step 1: Calculate individual probabilities.** The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$, so $\mathbb{P}(\{i\}) = 1/6$ for all i. $A = \{1, 2\}$, so $\mathbb{P}(A) = 1/6 + 1/6 = 2/6 = 1/3$. $B = \{5, 6\}$, so $\mathbb{P}(B) = 1/6 + 1/6 = 2/6 = 1/3$.

Step 2: Verify disjointness. $A \cap B = \emptyset$, so A and B are pairwise disjoint.

Step 3: Apply Axiom (iii) Countable Additivity (finite form).

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Step 4: Calculation.

$$\mathbb{P}(A \cup B) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

■

Solved Problem 3.4 (Monotonicity)**.** A fair six-sided die is rolled. Let A be the event of rolling an even number, and B be the event of rolling a 2 or a 4. Verify Monotonicity.

Solution. **Step 1: Define the events and their probabilities.** $A = \{2, 4, 6\}$, so $\mathbb{P}(A) = 3/6 = 1/2$. $B = \{2, 4\}$, so $\mathbb{P}(B) = 2/6 = 1/3$.

Step 2: Check the subset condition. Every element in B is also in A. Thus, $B \subseteq A$.

Step 3: Verify Monotonicity (Property iii). Since $B \subseteq A$, we must have $\mathbb{P}(B) \leq \mathbb{P}(A)$.

$$\frac{1}{3} \leq \frac{1}{2}$$

Since $0.333\cdots \leq 0.5$, the property of Monotonicity is verified. ■

Solved Problem 3.5 (Countable Additivity - Infinite Case)**.** Let $\Omega = \{1, 2, 3, \dots\}$ and $\mathbb{P}(\{i\}) = \frac{1}{2^i}$. Find $\mathbb{P}(\Omega)$ to verify the Unit Measure axiom.

Solution. **Step 1: Apply Countable Additivity (Axiom iii).** Since the elementary events $\{i\}$ are disjoint and their union is Ω:

$$\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} \mathbb{P}(\{i\}) = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

Step 2: Recognize the geometric series. This is a geometric series with first term $a = 1/2$ (when $i = 1$) and common ratio $r = 1/2$. The sum of an infinite geometric series is $\frac{a}{1-r}$.

Step 3: Calculation.

$$\mathbb{P}(\Omega) = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1$$

Since $\mathbb{P}(\Omega) = 1$, the Unit Measure axiom is satisfied. ■

3.2 Conditional Probability and Independence

Definition 3.1 (Conditional Probability)**.** For events A and B where $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition 3.2 (Independence)**.** Events A and B are independent if and only if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Formula 3.1 (Multiplication Rule (Chain Rule))**.**

$$\mathbb{P}(\bigcap_{i=1}^{n} A_i) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)\dots\mathbb{P}(A_n|\bigcap_{i=1}^{n-1} A_i)$$

Theorem 3.1 (Law of Total Probability (LTP) - Generalized)**.** If $B_1, B_2, \dots, B_n$ form a partition of the sample space Ω (disjoint, union is Ω, and $\mathbb{P}(B_i) > 0$), then for any event A:

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

Solved Problem 3.6 (Conditional Probability - Dependent Events)**.** Two cards are drawn without replacement from a standard 52-card deck. What is the probability that the second card is an Ace (A_2) given the first card was an Ace (A_1)?

Solution. Step 1: Identify probabilities before the draw. Total cards = 52. Total Aces = 4. $\mathbb{P}(A_1) = 4/52$.

Step 2: Determine the sample space after the first event (A_1). Given A_1 occurred, the remaining deck has 51 cards, and there are 3 Aces left.

Step 3: Apply the Conditional Probability Definition (in context).

$$\mathbb{P}(A_2|A_1) = \frac{\text{Number of ways to get } A_2 \text{ after } A_1}{\text{Total remaining outcomes}} = \frac{3}{51}$$

Step 4: Simplification.

$$\mathbb{P}(A_2|A_1) = \frac{3}{51} = \frac{1}{17}$$

■

Solved Problem 3.7 (Independence Verification)**.** A fair coin is flipped twice. Let A be the event of getting a Head on the first flip, and B be the event of getting a Head on the second flip. Show that A and B are independent.

Solution. Step 1: Define the sample space Ω and events. $\Omega = \{HH, HT, TH, TT\}$. $|\Omega| = 4$. $\mathbb{P}(A) = \mathbb{P}(\{HH, HT\}) = 2/4 = 1/2$. $\mathbb{P}(B) = \mathbb{P}(\{HH, TH\}) = 2/4 = 1/2$. $\mathbb{P}(A \cap B) = \mathbb{P}(\{HH\}) = 1/4$.

Step 2: Check the Independence Condition $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

$$\mathbb{P}(A)\mathbb{P}(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Step 3: Comparison. Since $\mathbb{P}(A \cap B) = 1/4$ and $\mathbb{P}(A)\mathbb{P}(B) = 1/4$, the condition holds. Events A and B are independent. ■

Solved Problem 3.8 (Multiplication Rule)**.** A box contains 3 red balls (R) and 7 blue balls (B). Two balls are drawn without replacement. What is the probability that both balls are red ($R_1 \cap R_2$)?

Solution. Step 1: State the Multiplication Rule for two events.

$$\mathbb{P}(R_1 \cap R_2) = \mathbb{P}(R_1)\mathbb{P}(R_2|R_1)$$

Step 2: Calculate individual probabilities.

$$\mathbb{P}(R_1) = \frac{\text{Number of Red Balls}}{\text{Total Balls}} = \frac{3}{10}$$

$$\mathbb{P}(R_2|R_1) = \frac{\text{Remaining Red Balls}}{\text{Remaining Total Balls}} = \frac{3-1}{10-1} = \frac{2}{9}$$

Step 3: Calculation.

$$\mathbb{P}(R_1 \cap R_2) = \frac{3}{10} \times \frac{2}{9} = \frac{6}{90} = \frac{1}{15}$$

■

Solved Problem 3.9 (Law of Total Probability (LTP))**.** In a factory, Machine 1 (M_1) produces 60% of items and Machine 2 (M_2) produces 40%. Machine 1 produces 5% defective items ($D|M_1$), and Machine 2 produces 3% defective items ($D|M_2$). What is the probability that a randomly chosen item is defective (D)?

Solution. Step 1: Define the partition and conditional probabilities. The events $\{M_1, M_2\}$ form a partition: $\mathbb{P}(M_1) = 0.6, \mathbb{P}(M_2) = 0.4$. Conditional probabilities: $\mathbb{P}(D|M_1) = 0.05, \mathbb{P}(D|M_2) = 0.03$.

Step 2: Apply the Law of Total Probability.

$$\mathbb{P}(D) = \mathbb{P}(D|M_1)\mathbb{P}(M_1) + \mathbb{P}(D|M_2)\mathbb{P}(M_2)$$

Step 3: Calculation.

$$\mathbb{P}(D) = (0.05)(0.6) + (0.03)(0.4)$$

$$\mathbb{P}(D) = 0.030 + 0.012 = 0.042$$

The probability that a randomly chosen item is defective is 4.2%. ■

Solved Problem 3.10 (Conditional Probability with Three Events)**.** Three events A, B, C are such that $\mathbb{P}(A \cap B \cap C) = 0.1$ and $\mathbb{P}(A \cap B) = 0.4$. Find $\mathbb{P}(C|A \cap B)$.

Solution. Step 1: State the Definition of Conditional Probability.

$$\mathbb{P}(C|A \cap B) = \frac{\mathbb{P}(C \cap (A \cap B))}{\mathbb{P}(A \cap B)}$$

Step 2: Simplify the numerator. The intersection $C \cap (A \cap B)$ is the same as $A \cap B \cap C$.

$$\mathbb{P}(C|A \cap B) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)}$$

Step 3: Calculation with given values.

$$\mathbb{P}(C|A \cap B) = \frac{0.1}{0.4} = \frac{1}{4} = 0.25$$

■

4 Bayes' Theorem and Generalized Formulae

Theorem 4.1 (Bayes' Theorem)**.** For events A and B (with $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$):

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Theorem 4.2 (Generalized Bayes' Theorem for N Events (Partition))**.** If $B_1, B_2, \ldots, B_n$ form a partition of the sample space Ω, then for any event A (with $\mathbb{P}(A) > 0$) and any $j \in \{1, \ldots, n\}$:

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$$

Solved Problem 4.1 (Basic Bayes' Theorem - Medical Test)**.** A rare disease (D) affects $\mathbb{P}(D) = 1/1000 = 0.001$. A test ($T$) is 99% accurate for those who have the disease ($\mathbb{P}(T|D) = 0.99$) and 98% accurate for those who do not ($\mathbb{P}(T^c|D^c) = 0.98$). If a person tests positive (T), what is the probability that they actually have the disease ($\mathbb{P}(D|T)$)?

Solution. Step 1: Define Prior and Conditional Probabilities.

- Prior: $\mathbb{P}(D) = 0.001$, $\mathbb{P}(D^c) = 1 - 0.001 = 0.999$.
- Likelihoods: $\mathbb{P}(T|D) = 0.99$ (True Positive), $\mathbb{P}(T^c|D^c) = 0.98$ (True Negative).
- Derived Likelihood: $\mathbb{P}(T|D^c) = 1 - \mathbb{P}(T^c|D^c) = 1 - 0.98 = 0.02$ (False Positive).

Step 2: Calculate $\mathbb{P}(T)$ using the Law of Total Probability.

$$\mathbb{P}(T) = \mathbb{P}(T|D)\mathbb{P}(D) + \mathbb{P}(T|D^c)\mathbb{P}(D^c)$$

$$\mathbb{P}(T) = (0.99)(0.001) + (0.02)(0.999)$$

$$\mathbb{P}(T) = 0.00099 + 0.01998 = 0.02097$$

Step 3: Apply Bayes' Theorem to find $\mathbb{P}(D|T)$.

$$\mathbb{P}(D|T) = \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T)}$$

$$\mathbb{P}(D|T) = \frac{0.00099}{0.02097} \approx 0.0472$$

The probability of actually having the disease given a positive test is approximately 4.72%. ∎

Solved Problem 4.2 (Bayes with Law of Total Probability - Generalized)**.** (Using Problem 41 data) In the factory, if a randomly chosen item is found to be defective (D), what is the probability it was produced by Machine 1 (M_1)?

Solution. Step 1: List the probabilities. $\mathbb{P}(M_1) = 0.6, \mathbb{P}(M_2) = 0.4$. $\mathbb{P}(D|M_1) = 0.05, \mathbb{P}(D|M_2) = 0.03$. From LTP (Example 41), $\mathbb{P}(D) = 0.042$.

Step 2: Apply the Generalized Bayes' Theorem for $\mathbb{P}(M_1|D)$.

$$\mathbb{P}(M_1|D) = \frac{\mathbb{P}(D|M_1)\mathbb{P}(M_1)}{\mathbb{P}(D|M_1)\mathbb{P}(M_1) + \mathbb{P}(D|M_2)\mathbb{P}(M_2)}$$

Step 3: Calculation.

$$\mathbb{P}(M_1|D) = \frac{(0.05)(0.6)}{0.042} = \frac{0.030}{0.042}$$

$$\mathbb{P}(M_1|D) = \frac{30}{42} = \frac{5}{7} \approx 0.7143$$

The probability that the defective item came from Machine 1 is approximately 71.43%. ∎

Solved Problem 4.3 (Bayes for 3 Events - Urns)**.** Three urns U_1, U_2, U_3 are chosen with $\mathbb{P}(U_1) = 0.5, \mathbb{P}(U_2) = 0.3, \mathbb{P}(U_3) = 0.2$.

- U_1 has 2 Red (R), 1 Black (B). $\mathbb{P}(R|U_1) = 2/3$.
- U_2 has 1R, 2B. $\mathbb{P}(R|U_2) = 1/3$.
- U_3 has 3R, 3B. $\mathbb{P}(R|U_3) = 3/6 = 1/2$.

A ball is drawn and is found to be red (R). What is the probability it came from U_3 ($\mathbb{P}(U_3|R)$)?

Solution. Step 1: Calculate $\mathbb{P}(R)$ using the Law of Total Probability.

$$\mathbb{P}(R) = \mathbb{P}(R|U_1)\mathbb{P}(U_1) + \mathbb{P}(R|U_2)\mathbb{P}(U_2) + \mathbb{P}(R|U_3)\mathbb{P}(U_3)$$

$$\mathbb{P}(R) = \left(\frac{2}{3}\right)(0.5) + \left(\frac{1}{3}\right)(0.3) + \left(\frac{1}{2}\right)(0.2)$$

$$\mathbb{P}(R) = \frac{1}{3} + 0.1 + 0.1 = \frac{1}{3} + \frac{2}{10} = \frac{1}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15}$$

Step 2: Apply Generalized Bayes' Theorem for $\mathbb{P}(U_3|R)$.

$$\mathbb{P}(U_3|R) = \frac{\mathbb{P}(R|U_3)\mathbb{P}(U_3)}{\mathbb{P}(R)}$$

Step 3: Calculation.

$$\mathbb{P}(U_3|R) = \frac{(1/2)(0.2)}{8/15} = \frac{0.1}{8/15} = \frac{1/10}{8/15}$$

$$\mathbb{P}(U_3|R) = \frac{1}{10} \times \frac{15}{8} = \frac{15}{80} = \frac{3}{16}$$

■

Solved Problem 4.4 (Prior and Posterior Probability)**.** A stock analyst believes the probability of a market boom (B) is $\mathbb{P}(B) = 0.6$. If the stock index I goes up, the probability $\mathbb{P}(I|B) = 0.8$ and $\mathbb{P}(I|B^c) = 0.3$. If I goes up, what is the updated (posterior) probability of a boom ($\mathbb{P}(B|I)$)?

Solution. Step 1: List the probabilities. $\mathbb{P}(B) = 0.6, \mathbb{P}(B^c) = 0.4$. $\mathbb{P}(I|B) = 0.8, \mathbb{P}(I|B^c) = 0.3$.

Step 2: Calculate $\mathbb{P}(I)$ (Evidence).

$$\mathbb{P}(I) = \mathbb{P}(I|B)\mathbb{P}(B) + \mathbb{P}(I|B^c)\mathbb{P}(B^c)$$

$$\mathbb{P}(I) = (0.8)(0.6) + (0.3)(0.4) = 0.48 + 0.12 = 0.60$$

Step 3: Apply Bayes' Theorem $\mathbb{P}(B|I)$.

$$\mathbb{P}(B|I) = \frac{\mathbb{P}(I|B)\mathbb{P}(B)}{\mathbb{P}(I)}$$

$$\mathbb{P}(B|I) = \frac{0.48}{0.60} = \frac{48}{60} = \frac{4}{5} = 0.8$$

The posterior probability of a boom, given the index went up, is 0.8 (an increase from the prior of 0.6). ■

Solved Problem 4.5 (Sequential Bayes)**.** An initial opinion is $\mathbb{P}(A) = 0.5$. A first piece of evidence E_1 arrives, where $\mathbb{P}(E_1|A) = 0.9$ and $\mathbb{P}(E_1|A^c) = 0.2$. Find the updated $\mathbb{P}(A|E_1)$.

Solution. Step 1: Calculate $\mathbb{P}(E_1)$ (Evidence).

$$\mathbb{P}(E_1) = \mathbb{P}(E_1|A)\mathbb{P}(A) + \mathbb{P}(E_1|A^c)\mathbb{P}(A^c)$$

Since $\mathbb{P}(A) = 0.5$, $\mathbb{P}(A^c) = 0.5$.

$$\mathbb{P}(E_1) = (0.9)(0.5) + (0.2)(0.5) = 0.45 + 0.10 = 0.55$$

Step 2: Apply Bayes' Theorem $\mathbb{P}(A|E_1)$.

$$\mathbb{P}(A|E_1) = \frac{\mathbb{P}(E_1|A)\mathbb{P}(A)}{\mathbb{P}(E_1)}$$

$$\mathbb{P}(A|E_1) = \frac{0.45}{0.55} = \frac{45}{55} = \frac{9}{11} \approx 0.8182$$

The updated probability of A given the evidence E_1 is approximately 81.82%. ■

5 Probability Inequalities

5.1 Bounds on Probability

Inequality 5.1 (Boole's Inequality (Union Bound))**.** For any sequence of events $A_1, A_2, \ldots$:

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Inequality 5.2 (Bonferroni Inequalities)**.** Let $S_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$ be the k-th term sum.

(i) Lower Bound (m odd): $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \geq \sum_{k=1}^{m} (-1)^{k-1} S_k$

(ii) Upper Bound (m even): $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{k=1}^{m} (-1)^{k-1} S_k$

Solved Problem 5.1 (Boole's Inequality - Finite Union)**.** A fair die is rolled. Let A_1 be the event of rolling an even number ($\{2, 4, 6\}$), A_2 be the event of rolling a number greater than 4 ($\{5, 6\}$), and A_3 be the event of rolling a 1 ($\{1\}$). Calculate $\mathbb{P}(A_1 \cup A_2 \cup A_3)$ directly and compare it to the Boole's inequality bound.

Solution. Step 1: Calculate individual probabilities.

$$\mathbb{P}(A_1) = 3/6 = 1/2$$

$$\mathbb{P}(A_2) = 2/6 = 1/3$$

$$\mathbb{P}(A_3) = 1/6$$

Step 2: Calculate the Union Directly. The union $A_1 \cup A_2 \cup A_3 = \{1, 2, 4, 5, 6\}$.

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = 5/6$$

Step 3: Calculate the Boole's Bound.

$$\mathbb{P}(\bigcup_{i=1}^{3} A_i) \leq \sum_{i=1}^{3} \mathbb{P}(A_i)$$

$$\sum_{i=1}^{3} \mathbb{P}(A_i) = \frac{3}{6} + \frac{2}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

Step 4: Comparison.

$$5/6 \leq 1$$

Since $5/6 \approx 0.833$ and 1.0 is the bound, the inequality holds. ∎

Solved Problem 5.2 (Boole's Inequality - Infinite Case)**.** Let $\mathbb{P}(A_i) = \frac{1}{i^2}$ for $i = 1, 2, 3, \ldots$ Calculate the Boole's upper bound for $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i)$ using the known result $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.

Solution. Step 1: Apply Boole's Inequality.

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Step 2: Calculate the summation.

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Step 3: State the bound.

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \frac{\pi^2}{6}$$

Since $\pi^2/6 \approx 9.87/6 \approx 1.645$, and probability cannot exceed 1, the Boole's bound provides an upper limit of 1.645. In the context of probability, the bound is $\min(1, \sum \mathbb{P}(A_i)) = 1$. However, the mathematical statement of the bound is $\pi^2/6$. ■

Solved Problem 5.3 (Bonferroni - Lower Bound)**.** Given $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.6$, and $\mathbb{P}(A \cap B) = 0.2$. Calculate the union $\mathbb{P}(A \cup B)$ exactly and verify the Bonferroni lower bound using $m = 1$.

Solution. Step 1: Calculate the exact union (PIE for $n = 2$).

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\mathbb{P}(A \cup B) = 0.5 + 0.6 - 0.2 = 0.9$$

Step 2: Apply the Bonferroni Lower Bound ($m = 1$). The bound is $\mathbb{P}(\bigcup A_i) \geq \sum_{k=1}^{1}(-1)^{k-1}S_k = S_1$. Here $S_1 = \mathbb{P}(A) + \mathbb{P}(B) = 0.5 + 0.6 = 1.1$.

Step 3: Verification.

$$0.9 \geq 1.1 \quad \text{(This is False)}$$

Correction: The provided Bonferroni theorem in the reference notes a sign change: (b) If m is odd ($m + 1$ is even), $R_m \geq 0$ so $\mathbb{P}(\bigcup A_i) \geq \sum_{k=1}^{m}(-1)^{k-1}S_k$. The error $R_m = \mathbb{P}(\bigcup A_i) - \sum_{k=1}^{m}(-1)^{k-1}S_k$. For $m = 1$: $\mathbb{P}(A \cup B) - S_1 = 0.9 - 1.1 = -0.2$. This should be $R_1 \leq 0$. The bound should be $\mathbb{P}(\bigcup A_i) \leq S_1$. (This is Boole's inequality). Let's stick to the stated theorem: Lower Bound for m odd.

$$\mathbb{P}(A \cup B) \geq S_1 = 1.1$$

This is clearly false, highlighting that the $m = 1$ bound in the Bonferroni sequence is **Boole's Upper Bound**, and the Bonferroni Lower Bound starts at $m = 2$: $m = 1$ (Upper): $\mathbb{P}(\bigcup A_i) \leq S_1 = 1.1$. (True: $0.9 \leq 1.1$). $m = 2$ (Lower): $\mathbb{P}(\bigcup A_i) \geq S_1 - S_2$.

$$S_1 - S_2 = 1.1 - 0.2 = 0.9.$$

$$\mathbb{P}(A \cup B) \geq 0.9.$$

(True: $0.9 \geq 0.9$). **We verify the $m = 1$ Upper Bound from the general sequence, which is S_1.**

$$\mathbb{P}(A \cup B) \leq S_1$$

$$0.9 \leq 1.1$$

(Verified) ■

Solved Problem 5.4 (Bonferroni - Upper Bound)**.** Using the data from Problem 50 ($\mathbb{P}(A \cup B) = 0.9, S_1 = 1.1, S_2 = 0.2$), verify the Bonferroni upper bound using $m = 2$.

Solution. Step 1: Define the $m = 2$ term sum. The $m = 2$ bound uses the first two terms: $S_1 - S_2$.

$$S_1 - S_2 = 1.1 - 0.2 = 0.9$$

Step 2: Apply the Bonferroni Upper Bound (m even). The bound is $\mathbb{P}(\bigcup A_i) \leq \sum_{k=1}^{2}(-1)^{k-1}S_k = S_1 - S_2$.

$$\mathbb{P}(A \cup B) \leq 0.9$$

Step 3: Verification.

$$0.9 \leq 0.9$$

(Verified) The bound is exact in this case because $n = 2$ and the $m = 2$ partial sum equals the full PIE sum. ■

Solved Problem 5.5 (Bonferroni - Three Events)**.** For three events A, B, C, let $S_1 = 1.8$, $S_2 = 1.0$, $S_3 = 0.2$. Calculate the Bonferroni lower bound for $\mathbb{P}(A \cup B \cup C)$ using $m = 2$ and the upper bound using $m = 1$.

Solution. Step 1: Calculate the exact Union (for comparison).

$$\mathbb{P}(\bigcup A_i) = S_1 - S_2 + S_3 = 1.8 - 1.0 + 0.2 = 1.0$$

Step 2: Calculate the Upper Bound ($m = 1$, odd term count).

$$\mathbb{P}(\bigcup A_i) \leq S_1$$

$$\mathbb{P}(\bigcup A_i) \leq 1.8$$

(Verified: $1.0 \leq 1.8$)

Step 3: Calculate the Lower Bound ($m = 2$, even term count).

$$\mathbb{P}(\bigcup A_i) \geq S_1 - S_2$$

$$\mathbb{P}(\bigcup A_i) \geq 1.8 - 1.0 = 0.8$$

(Verified: $1.0 \geq 0.8$) The exact probability is bounded by $0.8 \leq \mathbb{P}(\bigcup A_i) \leq 1.8$. Since probability cannot exceed 1, the tightest bound is $0.8 \leq \mathbb{P}(\bigcup A_i) \leq 1.0$. ■

5.2 Moment Inequalities

Inequality 5.3 (Markov's Inequality)**.** If X is a non-negative random variable ($X \geq 0$) and $a > 0$ then:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Inequality 5.4 (Chebyshev's Inequality)**.** For any random variable X with finite mean $\mu = \mathbb{E}[X]$ and finite variance $\sigma^2 = \text{Var}(X)$, and for any $\epsilon > 0$:

$$\mathbb{P}(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Solved Problem 5.6 (Markov's Inequality)**.** A non-negative random variable X (e.g., waiting time) has an expected value $\mathbb{E}[X] = 10$ minutes. Use Markov's Inequality to find an upper bound on the probability that the waiting time is at least 30 minutes.

Solution. Step 1: Identify parameters. Random variable X is non-negative. Expected value $\mathbb{E}[X] = 10$. Threshold $a = 30$.

Step 2: Apply Markov's Inequality.

$$\mathbb{P}(X \geq 30) \leq \frac{\mathbb{E}[X]}{30} = \frac{10}{30}$$

Step 3: Calculation.

$$\mathbb{P}(X \geq 30) \leq \frac{1}{3} \approx 0.3333$$

The probability that the waiting time is 30 minutes or more is at most 1/3, or 33.33%. ■

Solved Problem 5.7 (Chebyshev's Inequality - Basic)**.** A random variable X has mean $\mu = 5$ and variance $\sigma^2 = 4$. Use Chebyshev's Inequality to bound the probability that X is outside the interval $[1, 9]$.

Solution. Step 1: Determine ϵ. The interval $[1, 9]$ is centered at $\mu = 5$. The distance from the mean to the endpoints is $\epsilon = 9-5 = 4$ (or $5-1 = 4$). We are looking for $\mathbb{P}(|X-\mu| \geq \epsilon) = \mathbb{P}(|X-5| \geq 4)$.

Step 2: Apply Chebyshev's Inequality.

$$\mathbb{P}(|X - 5| \geq 4) \leq \frac{\sigma^2}{\epsilon^2} = \frac{4}{4^2}$$

Step 3: Calculation.

$$\mathbb{P}(|X - 5| \geq 4) \leq \frac{4}{16} = \frac{1}{4} = 0.25$$

The probability that X falls outside the interval $[1, 9]$ is at most 0.25. ■

Solved Problem 5.8 (Chebyshev's Inequality - Percentage)**.** A machine produces items with an average weight of $\mu = 100$ grams and a standard deviation $\sigma = 5$ grams. Use Chebyshev's Inequality to find the minimum percentage of items that fall within the range 90 to 110 grams.

Solution. Step 1: Determine ϵ and σ^2. $\mu = 100$. Standard deviation $\sigma = 5$, so variance $\sigma^2 = 25$. The range is 100 ± 10, so $\epsilon = 10$.

Step 2: Find the bound for the complement event $\mathbb{P}(|X - \mu| \geq \epsilon)$.

$$\mathbb{P}(|X - 100| \geq 10) \leq \frac{\sigma^2}{\epsilon^2} = \frac{25}{10^2} = \frac{25}{100} = 0.25$$

Step 3: Find the minimum probability within the range. The probability of X being within the range is the complement:

$$\mathbb{P}(|X - \mu| < \epsilon) = 1 - \mathbb{P}(|X - \mu| \geq \epsilon)$$

Since $\mathbb{P}(|X-\mu| \geq \epsilon) \leq 0.25$, we have:

$$\mathbb{P}(|X-100| < 10) \geq 1 - 0.25 = 0.75$$

The minimum percentage of items falling within the range is 75%. ■

Solved Problem 5.9 (Markov vs. Chebyshev Comparison)**.** Let X be a non-negative random variable with $\mathbb{E}[X] = 5$. (a) Find the Markov bound for $\mathbb{P}(X \geq 10)$. (b) If $\mathrm{Var}(X) = 1$, find the Chebyshev bound for $\mathbb{P}(|X-5| \geq 5)$. (c) Compare the two bounds.

Solution. (a) Markov Bound ($a = 10$).

$$\mathbb{P}(X \geq 10) \leq \frac{\mathbb{E}[X]}{10} = \frac{5}{10} = 0.5$$

(b) Chebyshev Bound ($\mu = 5, \sigma^2 = 1, \epsilon = 5$).

$$\mathbb{P}(|X-5| \geq 5) \leq \frac{\sigma^2}{\epsilon^2} = \frac{1}{5^2} = \frac{1}{25} = 0.04$$

Note that since X is non-negative and $\mu = 5$, the event $|X-5| \geq 5$ is equivalent to $X \geq 10$.

(c) Comparison. Markov Bound ≤ 0.5. Chebyshev Bound ≤ 0.04. The Chebyshev bound is much tighter (lower) because it incorporates the variance (measure of spread) of the distribution, while Markov's inequality only uses the mean. ■

Solved Problem 5.10 (General Bound on Variance)**.** For a random variable X with $\mathbb{E}[X] = 0.5$ and $\sigma^2 = 0.04$, use Chebyshev's inequality to find an interval around the mean such that X falls within this interval with at least 96% probability. (Find ϵ such that $\mathbb{P}(|X-\mu| < \epsilon) \geq 0.96$).

Solution. Step 1: Set up the complementary bound. We want $\mathbb{P}(|X-\mu| < \epsilon) \geq 0.96$. This means the probability of being outside the interval must be at most $1 - 0.96 = 0.04$.

$$\mathbb{P}(|X-\mu| \geq \epsilon) \leq 0.04$$

Step 2: Apply Chebyshev's Inequality and solve for ϵ.

$$\mathbb{P}(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

We set the bound equal to the maximum allowable probability:

$$\frac{0.04}{\epsilon^2} = 0.04$$

$$\frac{1}{\epsilon^2} = 1 \implies \epsilon^2 = 1 \implies \epsilon = 1$$

Step 3: State the final interval. The interval around the mean $\mu = 0.5$ is $\mu \pm \epsilon$:

$$[0.5 - 1, \quad 0.5 + 1] = [-0.5, \quad 1.5]$$

Thus, X falls within the interval $[-0.5, 1.5]$ with a probability of at least 96%. ■