Problem Set

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I. Limits and Continuity (Basic Evaluation)

1.1 (Standard) Evaluate the limit: $\lim_{x\to3}\frac{x^2-9}{x-3}$.

1.2 (Standard) Evaluate the limit: $\lim_{x\to-1}\frac{x^3+1}{x+1}$.

1.3 (Standard) Evaluate the limit: $\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}$.

1.4 (Standard) Evaluate the limit: $\lim_{h\to0}\frac{(5+h)^2-25}{h}$.

1.5 (Standard) Evaluate the limit: $\lim_{x\to\pi}\sin(x)\cos(x/2)$.

1.6 (Standard) Evaluate the limit: $\lim_{x\to0}\frac{\sin(4x)}{x}$.

1.7 (Standard) Evaluate the limit: $\lim_{x\to2^+}\frac{|x-2|}{x-2}$.

1.8 (Standard) Evaluate the limit: $\lim_{x\to0}\frac{e^x-e^{-x}}{2}$.

1.9 (Standard) Evaluate the limit: $\lim_{x\to\infty}\frac{3x^3-5x+1}{x^3+2x^2-4}$.

1.10 (Standard) Find the horizontal and vertical asymptotes of $f(x)=\frac{2x^2+x-1}{x^2-1}$.

1.11 (Intermediate) For the function $f(x)=\begin{cases}2x+a & \text{if } x\le1\\b-3x & \text{if } x>1\end{cases}$, find a and b such that $\lim_{x\to1}f(x)$ exists and $f(1)=5$.

1.12 (Intermediate) Evaluate the limit: $\lim_{x\to0}\frac{1-\cos(2x)}{x\sin(x)}$.

1.13 (Intermediate) Evaluate the limit: $\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{3x+5}$.

1.14 (Intermediate) Find the value of k that makes $f(x)=\begin{cases}\frac{\tan(kx)}{x} & \text{if } x\neq0\\5 & \text{if } x=0\end{cases}$ continuous at $x=0$.

1.15 (Intermediate) Evaluate the limit: $\lim_{x\to8}\frac{\sqrt[3]{x}-2}{x-8}$.

1.16 (Intermediate) Determine the type of discontinuity for $g(x)=\frac{x^2-4}{x^3-8}$ at $x=2$.

1.17 (Intermediate) Evaluate the limit: $\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)$.

1.18 (Complex) Prove that $f(x)=x^3-5x+1$ has a root in the interval $(0,1)$ using the Intermediate Value Theorem (IVT).

1.19 (Complex) Sketch the graph of a function f such that $\lim_{x\to3^-}f(x)=-\infty$, $\lim_{x\to3^+}f(x)=2$, and $f(3)=1$. Discuss its continuity.

1.20 (Complex) Find the number of points of discontinuity for $f(x)=\lfloor\sin(x)\rfloor$ on the interval $[0,2\pi]$.

II. Limits ($\epsilon - \delta$, Squeeze, L'Hôpital's Rule)

2.1 (Standard) Evaluate the limit: $\lim_{x \to 0} \frac{e^{5x} - 1}{x}$.

2.2 (Standard) Evaluate the limit: $\lim_{x \to 1} \frac{\ln(x)}{x-1}$.

2.3 (Standard) Evaluate the limit: $\lim_{x \to \infty} \frac{\ln(x)}{x^2}$.

2.4 (Standard) Evaluate the limit: $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$.

2.5 (Standard) Evaluate the limit: $\lim_{x \to 0^+} x \ln(x)$.

2.6 (Intermediate) Use the Squeeze Theorem to evaluate: $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$.

2.7 (Intermediate) Evaluate the limit: $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

2.8 (Intermediate) Evaluate the limit: $\lim_{x \to \infty} x^{1/x}$.

2.9 (Intermediate) Evaluate the limit: $\lim_{x \to 0^+} (\tan x)^{\sin x}$.

2.10 (Intermediate) Evaluate the limit: $\lim_{x \to 0} \frac{\arcsin(x)}{x}$.

2.11 (Intermediate) Evaluate the limit: $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

2.12 (Intermediate) Find $\lim_{x \to 0} f(x)$ if $2 - x^2 \le f(x) \le 2 + x^2$.

2.13 (Complex) Use the $\epsilon - \delta$ definition to rigorously prove: $\lim_{x \to 4} (2x - 5) = 3$.

2.14 (Complex) Use the $\epsilon - \delta$ definition to prove: $\lim_{x \to -1} \frac{1}{x} = -1$. (Hint: Start by restricting δ to be less than a specific value).

2.15 (Complex) Evaluate $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x}$.

2.16 (Complex) Determine the value of a such that $\lim_{x \to 0} \frac{\sqrt{1 + ax} - 1}{x} = 4$.

2.17 (Complex) Prove that $f(x) = x \sin\left(\frac{1}{x}\right)$ is continuous but not differentiable at $x = 0$.

2.18 (Complex) Evaluate the limit: $\lim_{x \to \infty} \frac{\int_1^x \sqrt{1 + t^2}\, dt}{x^2}$. (Requires L'Hôpital's Rule and FTC).

2.19 (Complex) Use the Squeeze Theorem to prove that $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$. (Assumes prior knowledge of Taylor series for confirmation, but can be done with L'Hôpital's Rule three times).

2.20 (Complex) Prove or disprove: If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, then $\lim_{x \to a} [f(x)g(x)]$ cannot exist.

III. Derivative Definition and Basic Rules

3.1 (Standard) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 + x$.

3.2 (Standard) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = \frac{1}{x}$.

3.3 (Standard) Find $\frac{dy}{dx}$ for $y = 5x^{10} - 4\sqrt{x} + 2$.

3.4 (Standard) Find y' for $y = (x^3 + 1)(2x - 5)$. (Product Rule).

3.5 (Standard) Find $\frac{d}{dx}\left[\frac{\sin x}{x^2}\right]$. (Quotient Rule).

3.6 (Standard) Find the equation of the tangent line to $f(x) = x^3 - 4x$ at $x = 2$.

3.7 (Standard) If a particle's position is $s(t) = t^4 - 2t^3 + 1$, find its instantaneous velocity at $t = 1$.

3.8 (Standard) Find $\frac{d}{dx}[e^x \cdot \ln x]$.

3.9 (Standard) Find the points on the graph of $y = \frac{1}{3}x^3 - x$ where the tangent line is horizontal.

3.10 (Standard) Differentiate $f(x) = \tan(x) + \sec(x)$.

3.11 (Intermediate) Use the limit definition to find $f'(x)$ for $f(x) = \sqrt{x + 1}$.

3.12 (Intermediate) Find $\frac{dy}{dx}$ for $y = \frac{x \cos x}{x^2 + 1}$. (Combined Rules).

3.13 (Intermediate) Find the second derivative $f''(x)$ for $f(x) = x^2 e^x$.

3.14 (Intermediate) Prove the Quotient Rule using the Product Rule and the Chain Rule.

3.15 (Intermediate) Determine if $f(x) = x|x|$ is differentiable at $x = 0$.

3.16 (Intermediate) Find the rate of change of the area of a circle with respect to its radius r.

3.17 (Intermediate) Find the coordinates of the point where the tangent line to $y = x^2$ is parallel to the secant line passing through $(1, 1)$ and $(3, 9)$.

3.18 (Complex) Determine constants a and b so that $f(x) = \begin{cases} ax + b & \text{if } x < 1 \\ x^2 - 4x & \text{if } x \geq 1 \end{cases}$ is differentiable at $x = 1$.

3.19 (Complex) Prove the Product Rule using the limit definition of the derivative.

3.20 (Complex) Given that $f(x)$ is a differentiable function, evaluate $\lim_{h \to 0} \frac{f(x+3h) - f(x-3h)}{2h}$. (In terms of $f'(x)$).

IV. Chain Rule and Transcendental Functions

4.1 (Standard) Find y' for $y = \sin(x^3 - 2x)$.

4.2 (Standard) Find $\frac{dy}{dx}$ for $y = e^{\tan x}$.

4.3 (Standard) Find $f'(x)$ for $f(x) = \ln(\sec x)$.

4.4 (Standard) Find $\frac{d}{dx}\left[(x^2 + 5x)^{10}\right]$.

4.5 (Standard) Find the derivative of $y = \arctan(2x)$.

4.6 (Intermediate) Differentiate $f(x) = \sqrt{e^{3x} + \cos^2 x}$.

4.7 (Intermediate) Find y' for $y = \sin^4(\ln(x^2))$. (Nested Chain Rule).

4.8 (Intermediate) If $y = \log_2(x^3 + 1)$, find $\frac{dy}{dx}$. (Change of Base/Log Rule).

4.9 (Intermediate) Find $\frac{d}{dx}[x \cdot \arcsin(x)]$. (Product Rule and Inverse Trig).

4.10 (Intermediate) Given $f(x) = g(\cos x)$ and $g'(0) = 5$, find $f'(\pi/2)$.

4.11 (Intermediate) If $y = \sinh(x^2)$, find y''.

4.12 (Intermediate) Differentiate $h(t) = \frac{e^{-t}}{1+t^2}$ and evaluate $h'(0)$.

4.13 (Intermediate) Find $\frac{dy}{dx}$ for $y = \sqrt{x + \sqrt{x}}$.

4.14 (Intermediate) If $f(x)$ is an even differentiable function, prove that $f'(x)$ is an odd function.

4.15 (Intermediate) If $h(x) = f(x^2)g(x)$, find $h'(x)$ in terms of f, g, f', g'.

4.16 (Complex) If $y = f(\frac{2x+1}{x^2})$, and $f'(3) = 6$, find $\frac{dy}{dx}\big|_{x=1}$.

4.17 (Complex) Find the 100th derivative of $f(x) = e^{-x}$.

4.18 (Complex) Prove the derivative of $\arcsin(x)$ using implicit differentiation on $x = \sin y$.

4.19 (Complex) Find $f'(x)$ for $f(x) = x \cdot 2^x \cdot \sin(x)$.

4.20 (Complex) If $y = \cosh(x)\tanh(x)$, find $\frac{dy}{dx}$ and simplify using hyperbolic identities.

V. Implicit, Logarithmic Differentiation & Higher Order

5.1 (Standard) Use implicit differentiation to find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$.

5.2 (Standard) Find $\frac{dy}{dx}$ for $x^3 + xy - y^2 = 4$.

5.3 (Standard) Find $\frac{d^2y}{dx^2}$ for $y = x^5 - 3x^3 + 2x$.

5.4 (Standard) Find $\frac{dy}{dx}$ for $e^x \sin(y) = 1$.

5.5 (Standard) Find the slope of the tangent line to $y^2 = \frac{x-1}{x+1}$ at $(3, \frac{1}{2})$.

5.6 (Intermediate) Use logarithmic differentiation to find y' for $y = (x^2 + 1)^x$.

5.7 (Intermediate) Find $\frac{dy}{dx}$ for $y = \frac{(x^2+1)^3 \sqrt{x}}{\sin^2(x)}$. (Logarithmic Diff.).

5.8 (Intermediate) Find $\frac{dy}{dx}$ for $y = (\ln x)^{\cos x}$.

5.9 (Intermediate) Use implicit differentiation to find the equation of the tangent line to $x \cos y = y \sin x$ at $(\pi, \pi/2)$.

5.10 (Intermediate) Find $\frac{d^2y}{dx^2}$ implicitly for $xy + y^2 = 1$. Express your answer in terms of x and y.

5.11 (Intermediate) Find $\frac{d^2y}{dx^2}$ for the function defined parametrically by $x = t^2 + 1$, $y = t^3 - t$.

5.12 (Intermediate) If $f(x) = \frac{x^2}{x-1}$, find $f''(3)$.

5.13 (Intermediate) Determine the derivative $\frac{dy}{dx}$ for the folium of Descartes $x^3 + y^3 = 6xy$.

5.14 (Intermediate) Show that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$ intersect orthogonally (at right angles).

5.15 (Intermediate) If $y = (2x + 1)^n$, prove that $(2x + 1)y' = 2ny$.

5.16 (Complex) Find the 3rd derivative $\frac{d^3}{dx^3}(\ln(1 + x))$. Find the general formula for the n-th derivative.

5.17 (Complex) A function $f(x)$ is such that $f(x) + xf(x) = \sin x$. Find $f'(0)$ and $f''(0)$. (Implicit Diff. combined with Product Rule).

5.18 (Complex) For $y = x^x$, find the value of $\frac{d^2y}{dx^2}$ at $x = 1$.

5.19 (Complex) Find the equation of the line perpendicular to the tangent line of the curve $x^2 - y^2 = 7$ at the point $(4, 3)$.

5.20 (Complex) Prove that the sum of the x-intercept and the y-intercept of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is the constant c.

VI. MVT, Rolle's Theorem & Linear Approximation

6.1 (Standard) State the three conditions for Rolle's Theorem to apply to a function $f(x)$ on $[a, b]$.

6.2 (Standard) Verify that $f(x) = x^2 - 4x + 3$ satisfies Rolle's Theorem on $[1, 3]$ and find all values of c.

6.3 (Standard) State the conclusion of the Mean Value Theorem (MVT) for a function $f(x)$ on $[a, b]$.

6.4 (Standard) Verify MVT for $f(x) = x^3 - 3x$ on $[0, 2]$ and find the value(s) of c.

6.5 (Standard) Use linear approximation to estimate $\sqrt{4.1}$.

6.6 (Standard) Use the differential dy to approximate the change in the volume of a sphere when its radius changes from $r = 5$ to $r = 5.02$.

6.7 (Standard) Find the linear approximation $L(x)$ of $f(x) = \cos x$ at $a = \pi/2$.

6.8 (Standard) Determine if Rolle's Theorem applies to $f(x) = |x|$ on $[-1, 1]$. Justify your answer.

6.9 (Standard) A car travels 100 miles in 2 hours. Use MVT to argue that the car's instantaneous speed must have been exactly 50 mph at least once.

6.10 (Standard) Use linear approximation to estimate $e^{0.01}$.

6.11 (Intermediate) A police helicopter clocks a vehicle at 60 mph and 10 minutes later clocks it at 70 mph. Use MVT to determine the minimum acceleration the vehicle must have experienced in that 10-minute interval.

6.12 (Intermediate) Show that $f(x) = x^3 + x - 1$ has exactly one real root. (Use IVT for existence and Rolle's Theorem for uniqueness).

6.13 (Intermediate) If $f(1) = 10$ and $f'(x) \geq 3$ for $1 \leq x \leq 4$, what is the smallest possible value for $f(4)$? (Use MVT).

6.14 (Intermediate) Find the differential dy and evaluate it for $y = \frac{x^2+1}{x}$ at $x = 1$ with $dx = 0.1$.

6.15 (Intermediate) Use linear approximation to estimate $\sin(3°)$. (Remember to convert to radians).

6.16 (Complex) Prove that if $f'(x) = 0$ for all x in an interval (a, b), then $f(x)$ is constant on (a, b). (Uses MVT).

6.17 (Complex) Show that $f(x) = 2x^2 + 3x + c$ can never satisfy Rolle's Theorem on any interval $[a, b]$ for which $a \neq b$.

6.18 (Complex) Prove that between any two real roots of $e^x \sin x = 1$, there exists at least one root of $e^x \cos x = 1$. (Consider $f(x) = e^{-x} - \sin x$).

6.19 (Complex) Let $f(x)$ be a differentiable function with two roots. Prove that $f'(x)$ must have at least one root.

6.20 (Complex) Use the MVT to prove the inequality $|\sin b - \sin a| \leq |b - a|$ for all real a and b.

VII. Related Rates (Geometry and Physics)

7.1 (Standard) A square's side length s is increasing at a rate of 2 cm/s. How fast is the area increasing when $s = 5$ cm?

7.2 (Standard) The radius of a circle is decreasing at a rate of 1 mm/s. How fast is the circumference changing?

7.3 (Standard) The position of a particle is given by $s(t) = 2t^3 - 15t^2 + 24t$. Find the time(s) when the acceleration is zero.

7.4 (Intermediate) A balloon is being inflated at a rate of 100 ft^3/min. How fast is the radius increasing when the diameter is 4 ft?

7.5 (Intermediate) Water is leaking from a conical tank (vertex down) at a rate of 2 ft^3/min. The tank has height 10 ft and radius 4 ft. How fast is the water level h dropping when $h = 5$ ft?

7.6 (Intermediate) A 13 ft ladder is leaning against a wall. The bottom of the ladder slides away from the wall at 0.5 ft/s. How fast is the top of the ladder sliding down when the bottom is 5 ft from the wall?

7.7 (Intermediate) Car A travels North at 50 mph and car B travels East at 60 mph. Both leave the intersection at the same time. At what rate is the distance between them increasing 2 hours later?

7.8 (Intermediate) A spotlight on the ground shines on a wall 12 m away. A person 2 m tall walks from the spotlight toward the wall at 1.6 m/s. How fast is the length of the shadow on the wall decreasing when the person is 4 m from the wall?

7.9 (Intermediate) The altitude of a triangle is increasing at 1 cm/min while the area of the triangle is increasing at 2 cm^2/min. At what rate is the base changing when the altitude is 10 cm and the area is 100 cm^2?

7.10 (Intermediate) A spherical snowball melts such that its surface area decreases at a rate of 1 cm^2/min. Find the rate at which the radius decreases when the volume is 36π cm^3.

7.11 (Complex) The angle of elevation of a rocket is measured by an observer 500 m from the launch site. The rocket is rising vertically. How fast is the rocket's velocity increasing when the angle of elevation is $\pi/4$ and the angle is increasing at a rate of 0.1 rad/s?

7.12 (Complex) A trough is 10 ft long and its ends are isosceles triangles that are 3 ft across at the top and 1 ft deep. If water is poured into the trough at a rate of 12 ft^3/min, how fast is the water level rising when the water is 0.5 ft deep?

7.13 (Complex) Two ships leave port at noon. Ship A sails North at 10 knots. Ship B sails East at 15 knots for 1 hour, then turns North. How fast is the distance between them changing at 2:00 PM?

7.14 (Complex) Gas is escaping from a spherical balloon at a rate proportional to its surface area. Show that the radius is decreasing at a constant rate.

7.15 (Complex) The current I (in Amperes) in a circuit with a battery and a resistor is given by $I = \frac{V}{R}$. If V is constant at 12 Volts, but R is increasing at 0.5 Ohms/s, how fast is the power dissipated $P = I^2R$ changing when $R = 10$ Ohms?

7.16 (Complex) A particle moves along the curve $y = \sqrt{x^2 + 1}$. When $x = 2$, the x-coordinate is increasing at 3 units/s. How fast is the distance from the particle to the origin changing at that instant?

7.17 (Complex) Find the constant velocity v_x of a particle moving along $y = x^2$ such that the rate of change of the angle θ of its position vector with the positive x-axis is a constant k. (i.e., $\frac{d\theta}{dt} = k$).

7.18 (Complex) A man walks at 5 ft/s toward a street light that is 15 ft high. His shadow shortens at a rate of 2.5 ft/s. How tall is the man? (Set up the relation and solve for the constant height).

7.19 (Complex) The base of an isosceles triangle is 20 cm. The two equal sides are decreasing at a rate of 1 cm/s. How fast is the area decreasing when the triangle is equilateral?

7.20 (Complex) A reservoir is shaped like a hemisphere of radius $R = 5$ m. Water is pumped out at a rate of 0.1 m^3/min. The volume of a spherical cap with height h is $V = \pi h^2(R - h/3)$. Find the rate at which the water level h drops when $h = 3$ m.

VIII. Curve Sketching (Monotonicity, Concavity, Asymptotes)

8.1 (Standard) Find the critical numbers of $f(x) = x^3 - 6x^2 + 5$.

8.2 (Standard) Find the interval(s) where $f(x) = x^4 - 4x^3$ is increasing.

8.3 (Standard) Find the local extrema of $f(x) = x + \frac{4}{x}$ on $(0, \infty)$.

8.4 (Standard) Find the interval(s) where $f(x) = \frac{1}{1+x^2}$ is concave up.

8.5 (Standard) Find the points of inflection for $f(x) = x^4 - 2x^2$.

8.6 (Standard) Find all asymptotes (Vertical, Horizontal, or Slant) for $f(x) = \frac{3x^2+1}{x-1}$.

8.7 (Standard) Determine the absolute maximum and minimum values of $f(x) = x^2 - 2x + 1$ on the interval $[0, 3]$.

8.8 (Intermediate) Analyze the concavity and inflection points for $f(x) = xe^{-x}$.

8.9 (Intermediate) Find the interval(s) where $g(x) = \frac{\ln x}{x}$ is decreasing.

8.10 (Intermediate) Sketch the graph of $f(x) = \frac{x^2}{x^2-4}$, identifying intercepts, symmetry, asymptotes, and extrema.

8.11 (Intermediate) Use the Second Derivative Test to classify the critical points of $f(x) = x^3 - 12x$.

8.12 (Intermediate) For $f(x) = 2x + \cot x$ on $(0, \pi)$, find the intervals of monotonicity.

8.13 (Intermediate) Find the value of k such that the function $f(x) = x^3 + kx^2 + x - 1$ has a point of inflection at $x = 1$.

8.14 (Intermediate) Sketch a continuous function $f(x)$ where $f'(x) > 0$ for all x, and $f''(x) > 0$ for $x < 2$, and $f''(x) < 0$ for $x > 2$.

8.15 (Intermediate) Find the slant asymptote for $y = \frac{x^3+2x^2-1}{x^2+1}$.

8.16 (Intermediate) Determine the behavior of the derivative $f'(x)$ given that $f(x)$ is always concave down and increasing for all x.

8.17 (Intermediate) Find the local extrema of $f(x) = (x-2)^{2/3} + 1$.

8.18 (Complex) The profit function for a company is $P(q) = R(q) - C(q)$, where $R(q) = 5q$ and $C(q) = 0.01q^2 + 2q + 100$. Find the quantity q that maximizes profit, and determine the maximum profit. (Marginal analysis).

8.19 (Complex) Find the number of distinct real roots for the equation $x^4 - 4x^3 - 20x^2 + 8 = 0$. (Uses extrema and IVT).

8.20 (Complex) Sketch the graph of $f(x) = 2x - 3x^{2/3}$, paying special attention to the cusp at $x = 0$.

IX. Optimization (Applied Max/Min)

9.1 (Standard) Find two positive numbers whose sum is 20 and whose product is a maximum.

9.2 (Standard) A rectangular field is to be fenced off next to a straight river. No fence is needed along the river. If 400 m of fencing is available, find the maximum area of the field.

9.3 (Standard) Find the point on the line $y = 2x + 3$ that is closest to the origin $(0,0)$.

9.4 (Standard) An open-top box is to be made from a 10 inch by 16 inch piece of cardboard by cutting out squares of equal size from the four corners. Find the size of the square that should be cut out to maximize the volume.

9.5 (Standard) Find the dimensions of a rectangle with perimeter P that has the largest area. (Answer in terms of P).

9.6 (Intermediate) Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

9.7 (Intermediate) A cylindrical can must hold V cubic units of liquid. Find the ratio of the height h to the radius r that minimizes the surface area of the can.

9.8 (Intermediate) A farmer wants to construct a fence separating two adjacent rectangular pens. The two pens must total 100 m^2 in area. What are the dimensions that minimize the amount of fencing used?

9.9 (Intermediate) Find the maximum area of a rectangle inscribed under the parabola $y = 12 - x^2$ and above the x-axis.

9.10 (Intermediate) What is the largest possible product of two non-negative numbers, where one number is at most 5 and the sum of the two numbers is 12?

9.11 (Intermediate) The speed of a boat in a river is $v = c \cdot \frac{L^2}{h^2}(h - L)$, where L is the depth of the river, h is the height of the boat's hull, and c is a constant. For a fixed depth L, find the height h that maximizes the boat's speed.

9.12 (Intermediate) Find the shortest distance from the point $(0, c)$ to the parabola $y = x^2$ for $c > \frac{1}{2}$.

9.13 (Complex) A gutter is made by folding up the sides of a 30-inch wide rectangular sheet of metal. The cross-section of the gutter is an isosceles trapezoid with a 10-inch base and 10-inch sides. Find the angle θ (the angle the side makes with the base) that maximizes the cross-sectional area.

9.14 (Complex) A wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a maximum?

9.15 (Complex) An airline offers a flight with 200 seats at a ticket price of $100. For every $10 increase in price, 5 fewer seats are sold. Find the ticket price that maximizes the revenue.

9.16 (Complex) Light travels from point A in a medium with speed v_1 to point B in a medium with speed v_2. Use the principle that the path taken minimizes the total time (Fermat's Principle) to derive Snell's Law of Refraction: $\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$.

9.17 (Complex) A storage container is to be constructed in the shape of a right circular cylinder with a flat top and a hemispherical bottom. The total volume must be V. Find the ratio of the height of the cylindrical part to the radius r that minimizes the cost of material (surface area).

9.18 (Complex) Find the maximum possible volume of a right circular cone inscribed in a sphere of radius R. Express the answer in terms of R.

9.19 (Complex) A right triangle with legs of length a and b is given. Find the dimensions of the rectangle of maximum area that can be inscribed in the triangle, with one side on the hypotenuse. (Requires combining geometry and optimization, complex setup).

9.20 (Complex) A window is in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is P, find the dimensions that admit the most light (maximize the area).

X. Conceptual and Combined Problems

10.1 (Standard) True or False: If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.

10.2 (Standard) True or False: If a limit is of the indeterminate form $\frac{0}{0}$, its value must be 1.

10.3 (Standard) If $F(x) = \int_0^x \tan(t^2)dt$, find $F'(x)$ using the Fundamental Theorem of Calculus (FTC).

10.4 (Standard) If $\frac{dy}{dt} = ky$, solve the differential equation for $y(t)$.

10.5 (Standard) True or False: A function can have an infinite number of critical points.

10.6 (Standard) State the conditions under which L'Hôpital's Rule can be applied to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$.

10.7 (Standard) Let $f(x)$ be a differentiable function. Write the equation of the line tangent to $g(x) = f(x^2)$ at $x = a$.

10.8 (Standard) If $f(x)$ is a polynomial of degree n, what is the maximum number of inflection points $f(x)$ can have?

10.9 (Standard) Give an example of a function that is continuous everywhere but not differentiable at $x = 0$.

10.10 (Standard) If $\lim_{x \to 2} f'(x) = 5$, must $\lim_{x \to 2} f(x)$ exist?

10.11 (Complex) Prove that if $f(x)$ is differentiable on $\mathbb{R}$ and $f'(x) \to L$ as $x \to \infty$, then $\lim_{x \to \infty} \frac{f(x)}{x} = L$. (Hint: Use L'Hôpital's Rule).

10.12 (Complex) A cost function is $C(q) = 100 + 4q + 0.02q^2$. Show that the quantity q that minimizes the average cost $\bar{C}(q) = \frac{C(q)}{q}$ is the quantity where the marginal cost $C'(q)$ equals the average cost $\bar{C}(q)$.

10.13 (Complex) Find all values of a and b such that the function $f(x) = \begin{cases} \arctan(x) & \text{if } x < 1 \\ ax^2 + bx & \text{if } x \geq 1 \end{cases}$ is differentiable at $x = 1$.

10.14 (Complex) Consider the logistic differential equation $\frac{dP}{dt} = kP(M - P)$, where k and M are constants. Find the population P at which the growth rate $\frac{dP}{dt}$ is maximized. (Requires maximizing the right-hand side with respect to P).

10.15 (Complex) Let f and g be two functions whose graphs are tangent at $x = a$. If $h(x) = \min(f(x), g(x))$, determine if $h(x)$ is differentiable at $x = a$.

10.16 (Complex) If $f(x)$ and $g(x)$ are both concave up on an interval I, is $h(x) = f(x) + g(x)$ necessarily concave up on I? Prove or give a counterexample.

10.17 (Complex) Determine the value of k for which the function $f(x) = \frac{x^2 - 1}{x^2 + kx - 2}$ has exactly one vertical asymptote.

10.18 (Complex) Let $f(x)$ be a function such that $f''(x) = 0$ for all x. What must the graph of $f(x)$ look like? Justify your answer using integration (or repeated application of MVT).

10.19 (Complex) A particle's velocity is $v(t) = 4 - t^2$ for $0 \leq t \leq 3$. Find the total distance traveled by the particle over the interval $[0, 3]$. (Requires integrating the absolute value of the velocity).

10.20 (Complex) Use the fact that $\frac{d}{dx}(\ln(\sec x + \tan x)) = \sec x$ to find the derivative of $y = \int_0^{\arctan x} \sec(t)dt$.