Calculus Formulae Sheet

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Contents

1 Precalculus

1.1 Functions and Notation

$$f : X \to Y, \quad \mathrm{dom}(f),\ \mathrm{range}(f),\ f^{-1}(B) = \{x : f(x) \in B\}.$$

Types: polynomial, rational, exponential, logarithmic, trigonometric, inverse trig, hyperbolic.

1.2 Basic Identities

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1,\\
\sin(2x) &= 2\sin x\cos x,\\
\cos(2x) &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,\\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B},\\
\sinh^2 x + 1 &= \cosh^2 x.
\end{aligned}$$

1.3 Inverse functions and monotonicity

If f is strictly monotone on interval I then f^{-1} exists on $f(I)$ and $(f^{-1})'(y) = \frac{1}{f'(x)}$ with $y = f(x)$.

2 Limits

2.1 Intuitive notions

Limit describes approaching behavior. Useful heuristics: squeeze theorem, dominant terms for $x \to \infty$ or $x \to 0$.

2.2 Formal epsilon-delta definition

Definition 2.1. $\lim_{x\to a} f(x) = L$ iff $\forall\varepsilon > 0\ \exists\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$.

2.3 One-sided limits and infinite limits

$$\lim_{x\to a^+} f(x), \quad \lim_{x\to a^-} f(x), \quad \lim_{x\to a} f(x) = L \iff \text{both one-sided limits } = L.$$

$$\lim_{x\to a} f(x) = \infty \text{ means } \forall M > 0,\ \exists\delta > 0 : 0 < |x-a| < \delta \Rightarrow f(x) > M.$$

2.4 Standard limit techniques

- Algebraic simplification.
- Factor and cancel.
- Multiply by conjugate.
- L'Hôpital's rule: if $\lim f = \lim g = 0$ or $\pm\infty$, then $\lim \frac{f}{g} = \lim \frac{f'}{g'}$ when RHS exists.

- Series expansion.

2.5 Common limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}\frac{e^x-1}{x}=1,\quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

2.6 Continuity

$$f \text{ continuous at } x=a \iff \lim_{x\to a} f(x)=f(a),$$

Intermediate Value Theorem: f continuous on $[a,b] \Rightarrow \forall k\in[f(a),f(b)], \exists c\in[a,b] : f(c)=k$.

3 Sequences and Series (Single variable)

3.1 Sequences

$$a_n\to L \iff \forall\varepsilon>0,\ \exists N:\ n>N\Rightarrow |a_n-L|<\varepsilon.$$

Monotone convergence theorem: monotone bounded sequence converges.

3.2 Infinite series

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \text{partial sums } s_N=\sum_{n=0}^{N} a_n \text{ converge.}$$

3.3 Tests for convergence

- **Comparison test**: $0\le a_n\le b_n$ and $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.
- **Limit comparison**: $\lim \frac{a_n}{b_n}=L\in(0,\infty)\Rightarrow$ same behaviour.
- **Ratio test**: $L=\lim|a_{n+1}/a_n|$; $L<1$ conv, $L>1$ div, $L=1$ inconclusive.
- **Root test**: $L=\limsup\sqrt[n]{|a_n|}$ similar to ratio.
- **Alternating series (Leibniz)**: if $a_n\downarrow 0$ then $\sum(-1)^n a_n$ converges.
- **Absolute convergence**: $\sum|a_n|$ converges $\Rightarrow \sum a_n$ converges.

3.4 Power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,\quad R=\frac{1}{\limsup_{n\to\infty}\sqrt[n]{|c_n|}}.$$

Within radius of convergence series can be differentiated and integrated termwise.

3.5 Taylor and Maclaurin series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,\quad R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

4 Differential Calculus

4.1 Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$
$$(Df)(x) = \frac{df}{dx}, \quad \frac{d}{dx}(c) = 0, \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

4.2 Rules of Differentiation

$$\begin{aligned}
(uv)' &= u'v + uv', \\
\left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2}, \\
(f(g(x)))' &= f'(g(x))g'(x), \\
\frac{d}{dx}[a^x] &= a^x \ln a, \quad \frac{d}{dx}[\ln x] = \frac{1}{x}, \\
\frac{d}{dx}[\sin x] &= \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x, \\
\frac{d}{dx}[\tan x] &= \sec^2 x.
\end{aligned}$$

4.3 Higher derivatives and Taylor remainder

$$f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

Taylor remainder Lagrange form given above.

4.4 Mean Value Theorems

$$\begin{aligned}
\text{Rolle's Theorem: } & f(a) = f(b), f'(\xi) = 0 \text{ for some } \xi \in (a, b), \\
\text{Lagrange MVT: } & f'(\xi) = \frac{f(b) - f(a)}{b - a}, \\
\text{Cauchy MVT: } & \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\end{aligned}$$

4.5 Asymptotic notation

$$f(x) = o(g(x)) \iff \lim_{x \to a} \frac{f}{g} = 0, \quad f(x) \sim g(x) \iff \lim \frac{f}{g} = 1.$$

4.6 Convexity and second derivative test

If $f''(x) > 0$ on interval then f is convex. Local minima if $f'(x_0) = 0$ and $f''(x_0) > 0$.

5 Riemann Integration and Measure-theoretic remarks

5.1 Riemann sum definition

Partition $P = \{x_0 < \cdots < x_n\}$, sample points $\xi_i \in [x_{i-1}, x_i]$:

$$S(P, f) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \qquad \int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(P, f).$$

5.2 Riemann integrability criterion

Bounded f on $[a, b]$ is Riemann integrable iff set of discontinuities has measure zero.

5.3 Fundamental Theorems of Calculus

- FTC1: If $F(x) = \int_a^x f(t)\,dt$ and f is integrable then F is continuous and if f continuous at x then $F'(x) = f(x)$.
- FTC2: If F is any antiderivative of f on $[a, b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

5.4 Definite and Indefinite Integrals

$$\int f(x)\,dx = F(x) + C, \quad F'(x) = f(x),$$
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$
$$\text{Fundamental Theorem: } \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

5.5 Improper integrals

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx,$$

convergent if limit exists. Comparison and p-test: $\int_1^\infty \frac{1}{x^p}\,dx$ converges iff $p > 1$.

6 Integration Techniques

6.1 Basic Integration Rules

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,$$
$$\int e^x dx = e^x + C, \quad \int \frac{1}{x} dx = \ln|x| + C,$$
$$\int \sin x\,dx = -\cos x + C, \quad \int \cos x\,dx = \sin x + C.$$

6.2 Techniques of Integration

$$\text{Substitution:} \quad \int f(g(x))g'(x)\,dx = \int f(u)\,du,$$
$$\text{Integration by Parts:} \quad \int u\,dv = uv - \int v\,du,$$
$$\text{Partial Fractions:} \quad \frac{P(x)}{Q(x)} = \sum_i \frac{A_i}{(x-r_i)^k} + \cdots .$$

6.3 Trigonometric integrals

Use power reduction, substitution $t = \tan(x/2)$ for rational functions of $\sin, \cos$.

6.4 Reduction formulas

Examples:

$$I_n = \int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}.$$

6.5 Special definite integrals

$$\int_0^\infty e^{-ax} x^{\nu-1}\,dx = \frac{\Gamma(\nu)}{a^\nu}, \quad \mathrm{Re}(a) > 0,\ \mathrm{Re}(\nu) > 0.$$

6.6 Applications of Integration: Area, Volume, and Density

Area under a Curve (Single Variable)

$$A = \int_a^b f(x)\,dx, \quad \text{where } f(x) \geq 0,$$
$$A = \int_a^b |f(x)|\,dx \quad \text{if } f(x) \text{ changes sign on } [a,b].$$

Area between Two Curves

$$A = \int_a^b [f(x) - g(x)]\,dx, \quad \text{where } f(x) \geq g(x) \text{ on } [a,b].$$

Area using Vertical and Horizontal Slices

$$A_y = \int_{y_1}^{y_2} [x_{\text{right}}(y) - x_{\text{left}}(y)]\,dy.$$

Volume by Slicing

$$V = \int_a^b A(x)\,dx, \quad A(x) = \text{cross-sectional area at } x.$$

Volume of Revolution: Disk Method

$$V = \pi \int_a^b [f(x)]^2 \, dx, \quad \text{(rotation about x-axis)},$$
$$V = \pi \int_c^d [g(y)]^2 \, dy, \quad \text{(rotation about y-axis)}.$$

Volume of Revolution: Washer Method

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx,$$
$$\text{where } R(x) = \text{outer radius}, \quad r(x) = \text{inner radius}.$$

Volume by Cylindrical Shells

$$V = 2\pi \int_a^b x f(x) \, dx \quad \text{(about y-axis)},$$
$$V = 2\pi \int_c^d y g(y) \, dy \quad \text{(about x-axis)}.$$

Density and Mass Distributions

$$\text{Mass: } M = \int_a^b \rho(x) \, dx,$$
$$\text{Center of Mass: } \bar{x} = \frac{1}{M} \int_a^b x \rho(x) \, dx,$$
$$\text{Moment of Inertia: } I = \int_a^b r^2 \rho(x) \, dx.$$

Double Integrals over Rectangular Regions

$$\iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx, \quad R = [a,b] \times [c,d].$$

Double Integrals over Non-Rectangular Regions

$$\iint_R f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx,$$
$$\text{or } \iint_R f(x,y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy.$$

Physical Applications of Double Integrals

$$\begin{aligned}
\text{Mass: } M &= \iint_R \rho(x,y)\,dA, \\
\text{Center of Mass: } \bar{x} &= \frac{1}{M}\iint_R x\rho(x,y)\,dA, \quad \bar{y} = \frac{1}{M}\iint_R y\rho(x,y)\,dA, \\
\text{Moment of Inertia: } I_x &= \iint_R y^2\rho(x,y)\,dA, \quad I_y = \iint_R x^2\rho(x,y)\,dA.
\end{aligned}$$

6.7 Integrals in Polar Coordinates

Polar Coordinate Relations

$$\begin{aligned}
x &= r\cos\theta, \quad y = r\sin\theta, \\
r^2 &= x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \\
dA &= r\,dr\,d\theta.
\end{aligned}$$

Conversion of Double Integral to Polar Form

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Area in Polar Coordinates

$$\begin{aligned}
A &= \int_{\theta_1}^{\theta_2} \frac{1}{2}[r(\theta)]^2\,d\theta, \\
\text{If bounded by } r = f(\theta): \quad A &= \frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2\,d\theta.
\end{aligned}$$

Area Between Two Polar Curves

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left([r_2(\theta)]^2 - [r_1(\theta)]^2\right)d\theta.$$

Arc Length in Polar Coordinates

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Surface Area of a Solid of Revolution (Polar Form)

$$\begin{aligned}
S &= 2\pi\int_{\theta_1}^{\theta_2} r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{(rotation about x-axis)}, \\
S &= 2\pi\int_{\theta_1}^{\theta_2} r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{(rotation about y-axis)}.
\end{aligned}$$

Mass and Center of Mass in Polar Coordinates

$$M = \iint_R \rho(r,\theta)\, r\, dr\, d\theta,$$
$$\bar{r} = \frac{1}{M} \iint_R r^2 \rho(r,\theta)\, dr\, d\theta,$$
$$\bar{\theta} = \frac{1}{M} \iint_R \theta\, r \rho(r,\theta)\, dr\, d\theta.$$

Moments and Inertia in Polar Coordinates

$$I_x = \iint_R (r\sin\theta)^2 \rho(r,\theta)\, r\, dr\, d\theta,$$
$$I_y = \iint_R (r\cos\theta)^2 \rho(r,\theta)\, r\, dr\, d\theta,$$
$$I_z = \iint_R r^2 \rho(r,\theta)\, r\, dr\, d\theta.$$

7 Special Functions

7.1 Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(n) = (n-1)!.$$

Reflection and duplication:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

7.2 Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

7.3 Stirling's approximation

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \quad (z \to \infty).$$

7.4 Error function and Gaussian integrals

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \qquad \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$

7.5 Bessel, Legendre, orthogonal polynomials (formulas)

Bessel ODE:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad J_\nu(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}.$$

8 Multivariable Calculus

8.1 Functions of several variables

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathrm{dom}(f) \subset \mathbb{R}^n.$$

8.2 Limits and continuity in $\mathbb{R}^n$

Definition uses $\|x - a\|$ and epsilon-delta. Path dependence matters for nonexistence.

8.3 Vector-Valued Functions

Definition and Representation

$$\begin{aligned} \vec{r}(t) &= \langle x(t), y(t), z(t) \rangle, \quad t \in I \subset \mathbb{R}, \\ \text{Domain: } &\{t \in \mathbb{R} : x(t), y(t), z(t) \text{ are defined}\}, \\ \text{Range: } &\{\vec{r}(t) : t \in I\} \subset \mathbb{R}^3. \end{aligned}$$

Limits and Continuity

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle,$$
$$\vec{r}(t) \text{ continuous at } t = a \iff x, y, z \text{ are continuous at } t = a.$$

Differentiation

$$\begin{aligned} \vec{r}'(t) &= \frac{d\vec{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle, \\ \text{Tangent vector: } \vec{T}(t) &= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \\ \text{Derivative rules: } (c\vec{r})' &= c\vec{r}', \quad (\vec{r}_1 + \vec{r}_2)' = \vec{r}_1' + \vec{r}_2', \\ \frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) &= \vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2', \\ \frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) &= \vec{r}_1' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2'. \end{aligned}$$

Velocity, Speed, and Acceleration

$$\begin{aligned} \vec{v}(t) &= \vec{r}'(t), \quad \text{velocity vector}, \\ v &= \|\vec{v}(t)\|, \quad \text{speed}, \\ \vec{a}(t) &= \vec{v}'(t) = \vec{r}''(t), \quad \text{acceleration vector}. \end{aligned}$$

Arc Length and Reparameterization

$$\begin{aligned} s(t) &= \int_{t_0}^{t} \|\vec{r}'(u)\| \, du, \\ \frac{ds}{dt} &= \|\vec{r}'(t)\|, \quad \frac{d\vec{r}}{ds} = \vec{T}(s). \end{aligned}$$

Curvature and Torsion

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3},$$
$$\text{Principal normal: } \vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|},$$
$$\text{Binormal: } \vec{B} = \vec{T} \times \vec{N},$$
$$\tau = -\frac{d\vec{B}/ds \cdot \vec{N}}{1} = \frac{(\vec{r}'(t), \vec{r}''(t), \vec{r}'''(t))}{\|\vec{r}'(t) \times \vec{r}''(t)\|^2}.$$

Tangential and Normal Components of Acceleration

$$a_T = \frac{d}{dt}\|\vec{v}\| = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|},$$
$$a_N = \sqrt{\|\vec{a}\|^2 - a_T^2} = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}.$$

Projectile and Circular Motion

$$\text{Projectile: } \vec{r}(t) = (v_0 \cos\alpha)t\,\hat{\imath} + (v_0 \sin\alpha)t - \frac{1}{2}gt^2\,\hat{\jmath},$$
$$\text{Circular: } \vec{r}(t) = \langle r\cos(\omega t), r\sin(\omega t)\rangle,$$
$$\vec{v}(t) = r\omega\langle -\sin(\omega t), \cos(\omega t)\rangle, \quad \|\vec{v}\| = r\omega,$$
$$\vec{a}(t) = -r\omega^2\langle \cos(\omega t), \sin(\omega t)\rangle.$$

8.4 Partial derivatives and differentiability

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots) - f(x)}{h}.$$

Differentiable at a iff

$$f(a + h) = f(a) + Df(a)h + o(\|h\|)$$

where $Df(a)$ is the gradient (row) or Jacobian.

8.5 Gradient, directional derivative, Hessian

$$\nabla f = \begin{pmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix}, \quad D_{\boldsymbol{u}} f(a) = \nabla f(a) \cdot \boldsymbol{u}.$$

Hessian $H_{ij} = \partial^2_{x_i x_j} f$. Second derivative test uses eigenvalues of Hessian.

8.6 Chain rule (multivariable)

If $z = f(x, y)$ with $x = x(t), y = y(t)$ then

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

General matrix form available via Jacobians.

8.7 Taylor series in several variables

Second order expansion:

$$f(\boldsymbol{x}+\boldsymbol{h}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{h}^\top H f(\boldsymbol{x})\boldsymbol{h} + o(\|\boldsymbol{h}\|^2).$$

8.8 Multiple integrals

$$\iint_D f(x,y)\,dA = \int_{y_0}^{y_1}\int_{x_0(y)}^{x_1(y)} f(x,y)\,dx\,dy.$$

Fubini's theorem for integrable functions allows iterated integrals.

8.9 Change of variables and Jacobian

If $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$ is a C^1 bijection,

$$\iint_{D_x} f(\boldsymbol{x})\,d\boldsymbol{x} = \iint_{D_u} f(\boldsymbol{x}(\boldsymbol{u}))|\det J_{\boldsymbol{x}}(\boldsymbol{u})|\,d\boldsymbol{u}.$$

8.10 Common coordinate systems

$$\text{Polar: } (r,\theta),\ dA = r\,dr\,d\theta.$$

$$\text{Cylindrical: } (r,\theta,z),\ dV = r\,dr\,d\theta\,dz.$$

$$\text{Spherical: } (\rho,\phi,\theta),\ dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

8.11 Line integrals

For curve $\boldsymbol{r}(t)$, $a \le t \le b$:

$$\int_C f\,ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt.$$

For vector field $\boldsymbol{F}$:

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\,dt.$$

8.12 Conservative fields and potentials

$\boldsymbol{F} = \nabla\phi$ implies $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \phi(b)-\phi(a)$. Condition in simply connected domain: $\nabla\times\boldsymbol{F} = 0$.

8.13 Surface integrals

Parameterize S by $\boldsymbol{r}(u,v)$:

$$\iint_S f\,dS = \iint_D f(\boldsymbol{r}(u,v))\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,du\,dv.$$

Flux:

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}(\boldsymbol{r})\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv.$$

8.14 Gradient, Divergence, Curl

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\
\nabla \cdot \vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \\
\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.
\end{aligned}
$$

8.15 Integral Theorems

$$
\begin{aligned}
\text{Green's Theorem: } & \oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA, \\
\text{Stokes' Theorem: } & \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}, \\
\text{Divergence Theorem: } & \iiint_V (\nabla \cdot \vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}.
\end{aligned}
$$

8.16 Optimization with constraints

Lagrange multipliers: optimize $f(\boldsymbol{x})$ subject to $g(\boldsymbol{x}) = 0$ via

$$\nabla f = \lambda \nabla g.$$

Multiple constraints use vector of multipliers.

9 Ordinary Differential Equations

9.1 Classification and existence

General form: $F(x, y, y', \dots) = 0$. First order explicit: $y' = f(x, y)$. Picard-Lindelöf theorem: if f is Lipschitz in y then local uniqueness and existence.

9.2 Linear first-order ODEs

$$y' + p(x)y = q(x).$$

Integrating factor $\mu(x) = e^{\int p(x)\, dx}$ gives

$$y = \frac{1}{\mu}\left(\int \mu q\, dx + C\right).$$

9.3 Separable equations

$$\frac{dy}{dx} = g(x)h(y) \Rightarrow \int \frac{dy}{h(y)} = \int g(x)\, dx.$$

9.4 Second-order linear ODEs with constant coefficients

$$y'' + ay' + by = f(x).$$

Homogeneous solution from characteristic equation $r^2 + ar + b = 0$. Particular solution via undetermined coefficients or variation of parameters.

9.5 Method of undetermined coefficients

Assume particular solution form matching RHS type (polynomial, exponential, sine/cosine) multiplied by x^k if resonance occurs.

9.6 Variation of parameters (2nd order)

If y_1, y_2 fundamental solutions of homogeneous then particular solution:

$$y_p = -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx, \qquad W = y_1 y_2' - y_1' y_2.$$

9.7 Linear systems and matrix exponentials

System $\boldsymbol{x}' = A\boldsymbol{x}$. Solution:

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0), \quad e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

Compute via diagonalization $A = S\Lambda S^{-1}$ or Jordan form.

9.8 Stability of equilibria

For autonomous $\boldsymbol{x}' = f(\boldsymbol{x})$, linearize at equilibrium $\boldsymbol{x}_0$ using Jacobian $J = f'(\boldsymbol{x}_0)$. Eigenvalues of J determine local stability.

9.9 Phase plane methods

Nullclines, critical points classification (node, saddle, spiral, center) by eigenvalues.

9.10 Boundary value problems and Sturm-Liouville

Form $-(p(x)y')' + q(x)y = \lambda w(x)y$ with BCs. Eigenfunctions form orthogonal basis for weighted L^2.

10 Applications

10.1 Optimization

$$\text{Critical points: } f'(x) = 0, \quad f''(x) > 0 \Rightarrow \text{local min}, \; f''(x) < 0 \Rightarrow \text{local max}.$$

11 Partial Differential Equations: Classical Examples and Methods

11.1 Canonical PDEs

- **Heat equation**: $u_t = \kappa \Delta u$.
- **Wave equation**: $u_{tt} = c^2 \Delta u$.
- **Laplace's equation**: $\Delta u = 0$.

11.2 Separation of variables

Assume $u(x,t) = X(x)T(t)$ leading to ODEs and eigenvalue problems.

11.3 Fourier series and transforms

Fourier series on $[-L, L]$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

coefficients given by orthogonality. Fourier transform:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

11.4 Green's functions

For linear operator $\mathcal{L}$, solution to $\mathcal{L}u = f$ can be represented via $u = \mathcal{G}f$ where $\mathcal{G}$ involves Green's function kernel solving $\mathcal{L}G(x,s) = \delta(x-s)$ with BCs.

12 Transforms and Operational Methods

12.1 Laplace transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

Properties: linearity, shift, differentiation in time $\mathcal{L}\{f'\} = sF(s) - f(0)$, convolution theorem $\mathcal{L}\{f * g\} = F(s)G(s)$.

12.2 Z-transform (discrete)

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

12.3 Fourier transform properties

Linearity, modulation, differentiation, convolution.

13 Numerical Methods

13.1 Root finding

- Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
- Secant method and bisection (guaranteed convergence for continuous sign change).

13.2 Numerical integration

$$\text{Trapezoidal rule: } \int_a^b f(x)\,dx \approx \frac{b-a}{2}(f(a)+f(b))$$

Simpson's rule and composite versions. Error estimates involve higher derivatives.

13.3 Numerical ODEs

- Euler forward: $y_{n+1} = y_n + hf(t_n, y_n)$.
- Improved Euler/Heun and classical RK4:

$$k_1 = f(t_n, y_n),\ k_2 = f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1),\ k_3 = \dots,\ k_4 = f(t_n + h, y_n + hk_3),$$

$$y_{n+1} = y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

14 Miscellaneous Topics

14.1 Uniform convergence

Sequence of functions f_n converges uniformly to f if $\sup_x |f_n(x) - f(x)| \to 0$. Uniform convergence permits termwise integration and differentiation under conditions.

14.2 Dominated convergence theorem (measure-theory remark)

If $f_n \to f$ a.e. and $|f_n| \le g$ with $g \in L^1$ then $\lim \int f_n = \int f$.

14.3 Asymptotic expansions and stationary phase

Common tools for approximating integrals and series for large parameters.

14.4 Calculus of variations (principles)

Euler-Lagrange equation: stationary action for functional $J[y] = \int_a^b L(x, y, y')\,dx$ yields

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

15 Cheat Sheet: Common Identities and Integrals

15.1 Derivatives

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

15.2 Integrals

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C.$$

15.3 Useful limits and series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x.$$

16 Reference Theorems (Statements)

Theorem 16.1 (Taylor's theorem). *If $f^{(n+1)}$ exists on interval containing a and x then*

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Theorem 16.2 (Uniform convergence interchange). *If $f_n \to f$ uniformly on $[a,b]$ and each f_n integrable then $\int_a^b f_n \to \int_a^b f$.*

Theorem 16.3 (Green-Stokes-Gauss collection). *The respective integral theorems relating derivatives inside a domain to boundary integrals hold under stated regularity.*
