Sampling & Reconstruction

- Sampling
- Reconstruction
- Nyquist Shannon Sampling Theorem

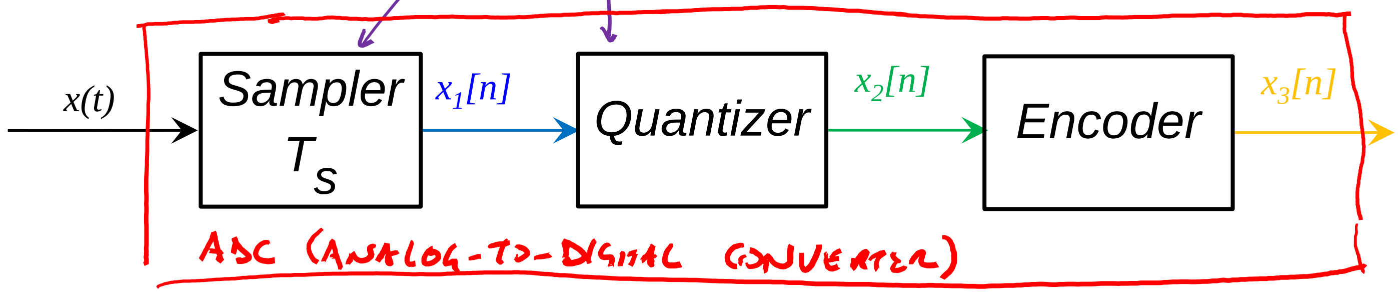
Aside: From Chaparro, we'll only work with Secs 8.1-2. Alternatively, Lee & Varaiya Chap 11 covers this topic well.

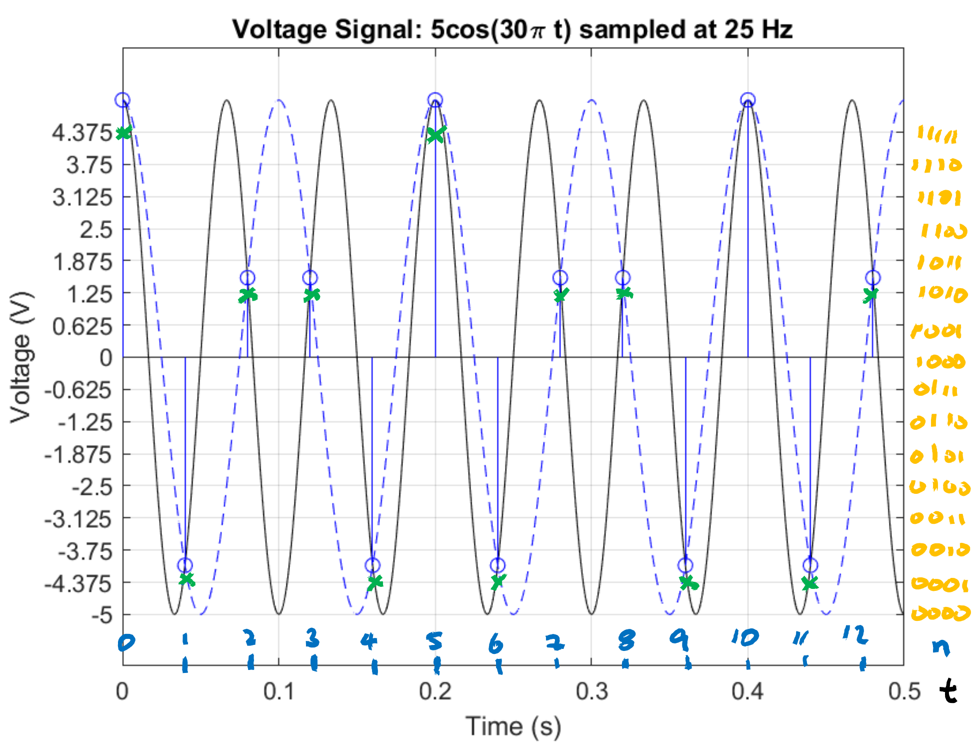
Bridging CT and DT Systems

- Many “signals” of interest to engineers represent physical attributes of the real world and are analog in nature (e.g., position, velocity, temperature, pressure, voltage, current, strain, electromagnetic field, etc.).
- Digital hardware (including computers) cannot directly use analog signals as it would require infinite memory (both in time and amplitude).
- The first step in an Analog-to-Digital Converter (ADC) is **sampling** to generate a discrete-time signal from a continuous-time one.

IN GENERAL, EXPECT LOSS OF INFO FROM THESE SYSTEMS

Recall the ADC from Slide 2.4





$x: \mathbb{R} \to \mathbb{R}$ (ANALOG)

$x_1: \mathbb{Z} \to \mathbb{R}$

$x_2: \mathbb{Z} \to \{-5, -4.375, \ldots, 4.375\}$

$x_3: \mathbb{Z} \to \{0,1\}^4$ (DIGITAL, 4-BIT ENCODING)

Aside: The original 15 Hz signal sampled at 25 Hz is indistinguishable from a 10 Hz signal (i.e., it has been “aliased”). More later in Nyquist-Shannon Sampling Theorem (slide 7.11).

(Uniform) Sampling

"bridge" converting CT signal to DT signal

$$\xrightarrow{x:\mathbb{R}\to\mathbb{C}} \boxed{Sampler_T} \xrightarrow{y:\mathbb{Z}\to\mathbb{C}}$$

$$y = Sampler_T(x) \colon \forall n \in \mathbb{Z}, y[n] = x(nT)$$

T — sampling interval

f_s=1/T — sampling frequency or sampling rate

Q1: Is y a digital signal?

A1: Not yet…values aren't quantized yet.

Q2: As the engineer, if you can specify the sampling rate, how should you do so? NB: Trade-off is that higher sampling rate generally means better approximation of CT signal (less information loss) but requiring more resources (in memory, processing power, cost, etc.)

A2: This depends on the highest frequencies expected. A remarkable fact is that certain sampling conditions allow for perfect reconstruction of the original signal (no loss of information)!

Sampling a Sinusoid & Aliasing

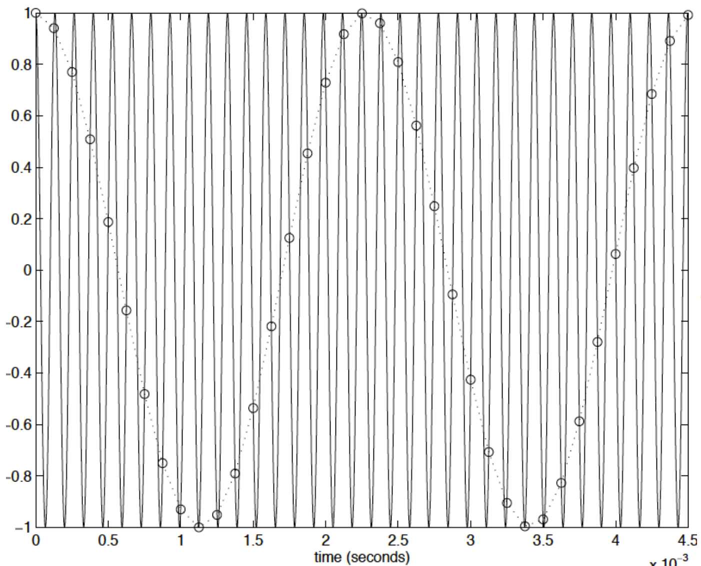
$\forall t \in \mathbb{R}, x(t) = \cos(2\pi f t)$ $\longrightarrow$ $\boxed{Sampler_T}$ $\longrightarrow$ $\forall n \in \mathbb{Z}, y[n] = \cos(2\pi f n T)$

How does output change if input is $v(t) = \cos(2\pi(f + N f_s)t)$ for $N \in \mathbb{Z}$?

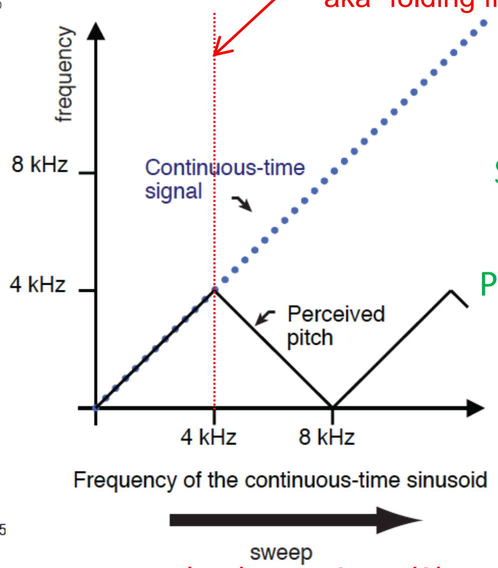
Observation: $v \neq x$, but $Sampler_T(v) = Sampler_T(x)$

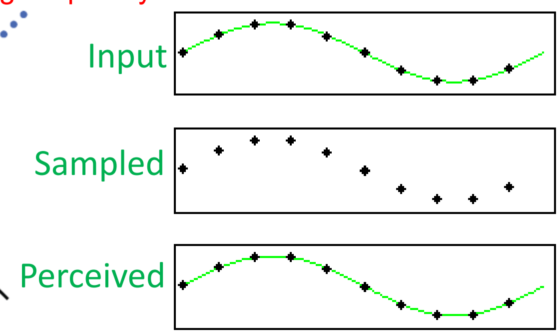
$\Rightarrow$ **Aliasing:** an effect that causes different signals to become indistinguishable (or *aliases* of one another) when sampled.

$\frac{f_s}{2}$ is the "Nyquist Frequency" of the Sampler, aka "folding frequency"



L&V Fig 11.3: 7.56kHz sinusoid sampled at 8kHz

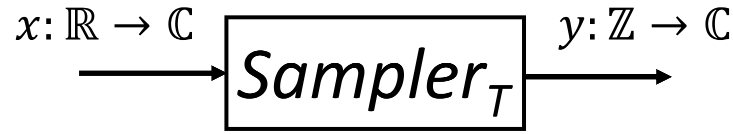




GIF image source: http://www.audiocheck.net/audiotests_aliasing.php

Perceived as 440 Hz (8kHz-7.56kHz)

Is Sampler$_T$ a LTI System?



Linearity Check:

$$y_1 = Sampler_T(x_1) \Rightarrow \forall n \in \mathbb{Z}, y_1[n] = x_1(nT)$$

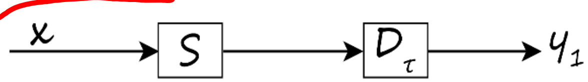
$$y_2 = Sampler_T(x_2) \Rightarrow \forall n \in \mathbb{Z}, y_2[n] = x_2(nT)$$

$$y = Sampler_T(ax_1 + bx_2) \Rightarrow \forall n \in \mathbb{Z}, y[n] = ax_1(nT) + bx_2(nT) = ay_1[n] + by_2[n]$$

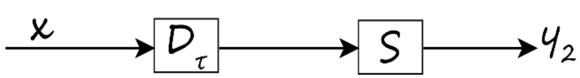
Linear ☑

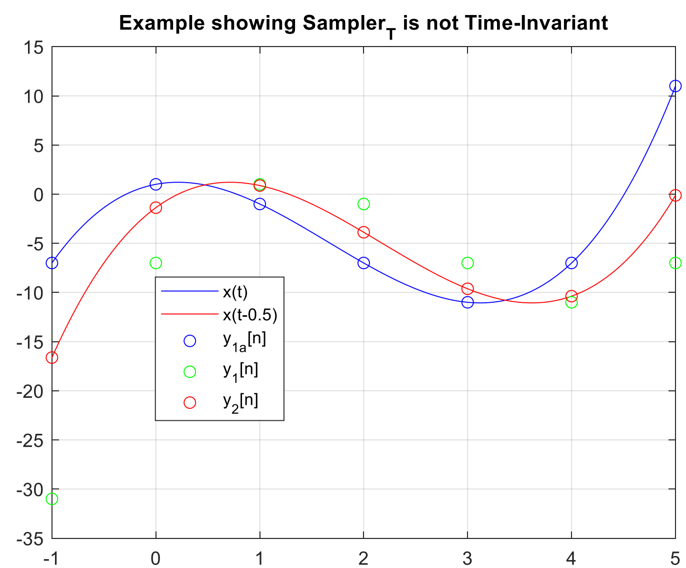
Time Invariance Check:

Recall: CT Systems, S, is Time-Invariant if $\forall \tau \epsilon \mathbb{R}, D_\tau \circ S = S \circ D_\tau$



Composition: $(D_\tau \circ S)(x) = D_\tau(S(x))$





NB: The delay to get y_1 is of a DT signal (more appropriately D_m where $m \in \mathbb{Z}$) while that to get y_2 is of a CT signal, quite different operators.

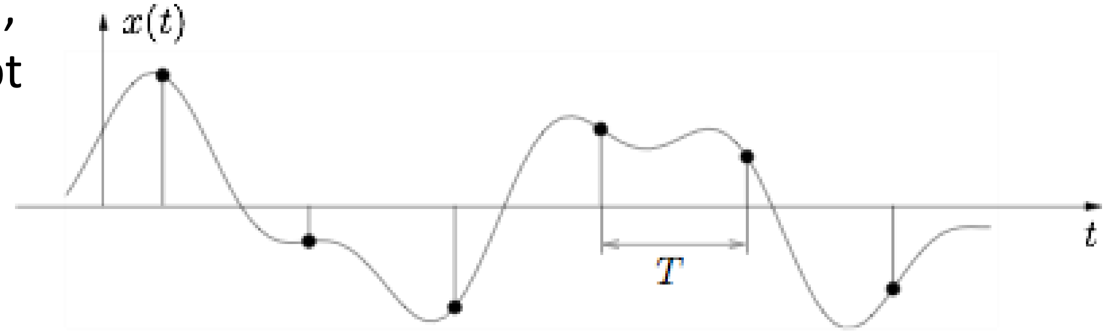
Time − Invariant☒

Reconstruction (from DT Signal)

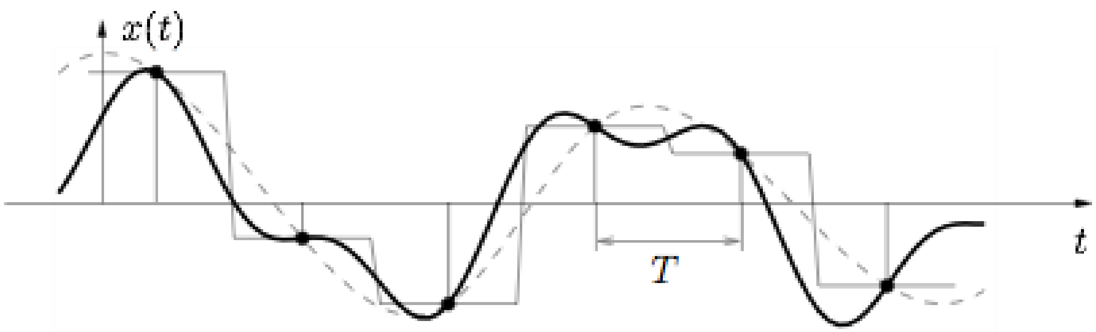
Other "bridge" converting DT signal to CT signal

$$y: \mathbb{Z} \to \mathbb{C} \longrightarrow \boxed{DiscToCont_T} \longrightarrow x: \mathbb{R} \to \mathbb{C}$$

Given a sampled signal, how should we attempt to reconstruct the original (unsampled) one?



NB: "Aliasing" results in distortion and loss of information, i.e., the original signal cannot confidently be reconstructed.



Reconstruction (from DT Signal)

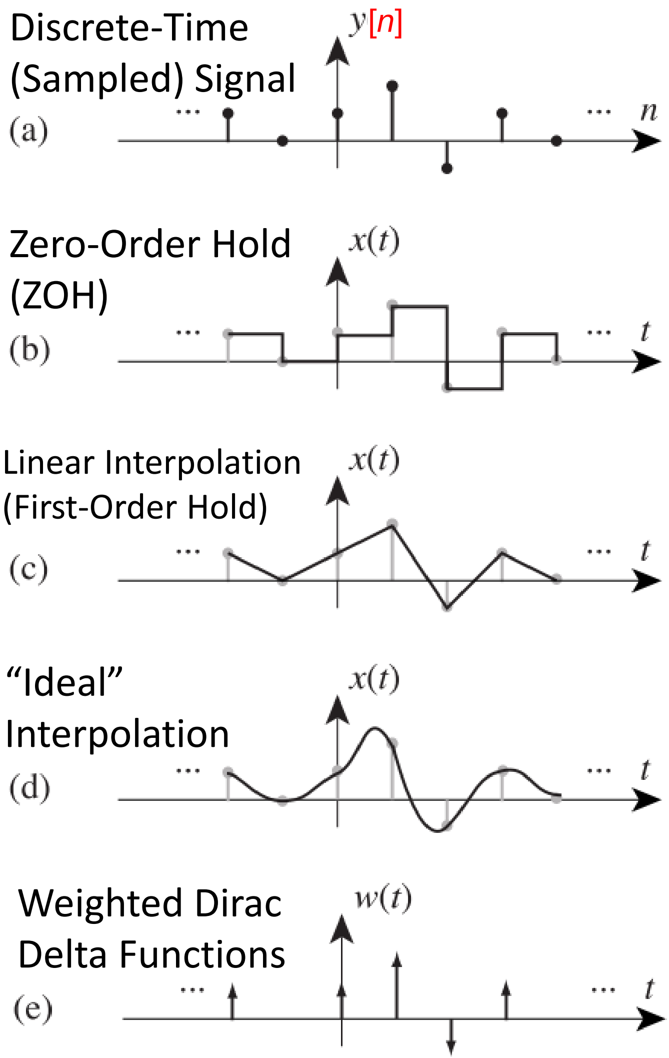
Discrete-Time (Sampled) Signal

Zero-Order Hold (ZOH)

Linear Interpolation (First-Order Hold)

"Ideal" Interpolation

Weighted Dirac Delta Functions



(b-d) show common reconstruction methods.

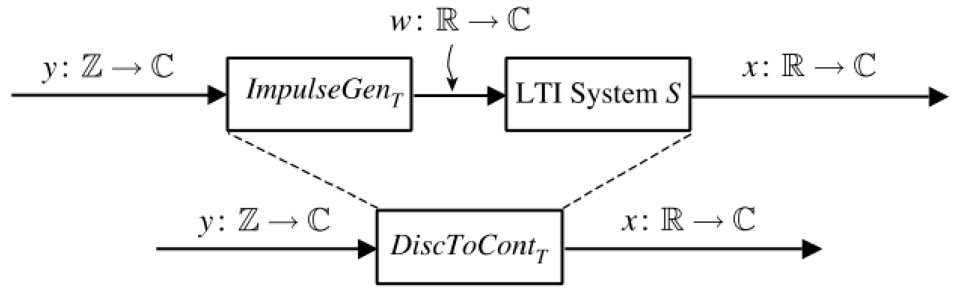


Figure 11.7: A model for reconstruction divides it into two stages.

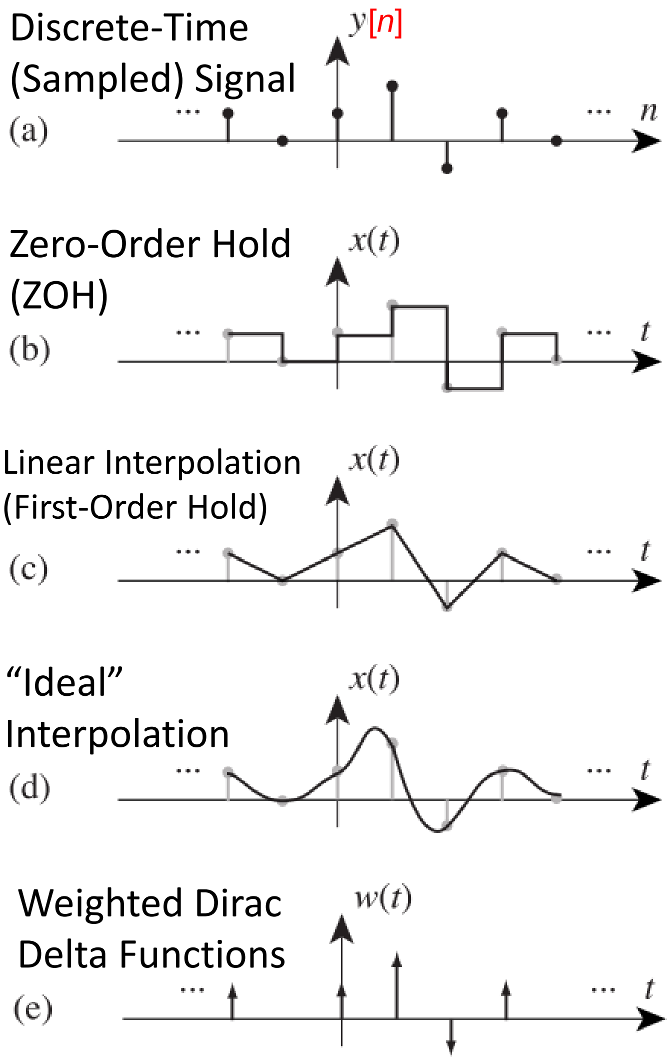
(e) shows the result from an ideal "Impulse Generator", a useful intermediate stage. Recall the "Impulse Train" (aka: Dirac Comb):

$$p: \mathbb{R} \to \mathbb{R}, \forall t \epsilon \mathbb{R}, \qquad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\Rightarrow w(t) = \sum_{k=-\infty}^{\infty} y[k]\delta(t - kT)$$

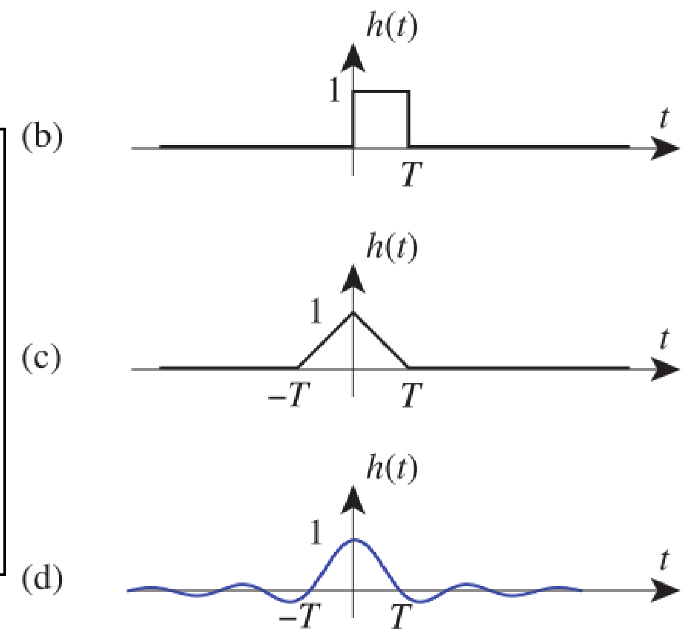
$$= \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) = x(t)p(t)$$

Reconstruction (from DT Signal)



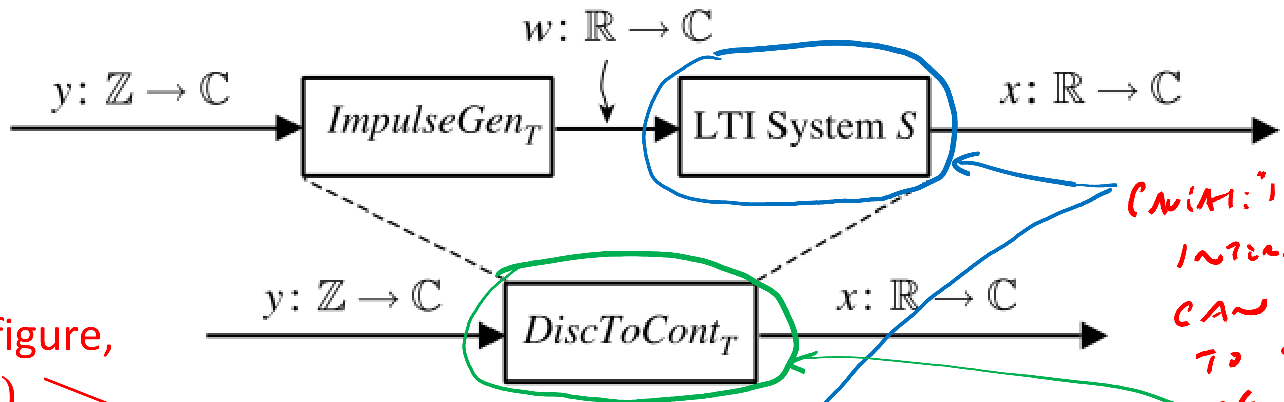
Convolve (e) with impulse response to yield reconstruction

Below are required impulse responses of S for each reconstruction method.



Ideal Interpolator: $h(t) = \dfrac{sin(\pi t/_{T})}{\pi t/_{T}} = sinc(t/_{T})$

Ideal Interpolation



I.e., in this figure,
$x = Sinc_T(w)$

CAUTION: "IDEAL INTERPOLATION" CAN REFER TO EITHER OF THESE PAIRS

The Ideal Interpolator (aka $Sinc_T$) is the LTI System, S, with impulse response:

$$\forall t \in \mathbb{R}, h(t) = \frac{sin\left({}^{\pi t}/_{T}\right)}{{}^{\pi t}/_{T}} = sinc\left({}^{t}/_{T}\right)$$

FT

$$H(\omega) = \begin{cases} T, & -\pi/T < \omega < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

i.e., Ideal Low-Pass Filter

$$\Rightarrow \quad IdealInterpolator_T = Sinc_T \circ ImpulseGen_T$$

NB: This impulse response is practically unachievable because $\nexists \tau \epsilon \mathbb{R}$ such that $\forall t < \tau, h(t) = 0$ (i.e., the noncausal portion never vanishes).

Nyquist-Shannon Sampling Theorem

If a low-pass continuous-time signal *x(t)* is band-limited (i.e., it has a spectrum $X(\omega)$ such that $X(\omega) = 0$ for $|\omega| > \omega_{max}$, where ω_{max} is the maximum frequency in x(t)) we then have:

- The information in x(t) is preserved by a sampled signal $x_s[n]$, with samples $x_s[n]=x(nT_s) = x(t)|_{t=nTs}, n = 0, \pm 1, \pm 2, \ldots,$ provided that the sampling frequency $\omega_s = 2\pi/T_s$ (rad/sec) is such that

$$\omega_s > 2\omega_{max} \qquad \underline{\textit{Nyquist sampling rate condition}} \qquad (8.17)$$

or equivalently if the sampling rate f_s (samples/sec) or the sampling period T_s (sec/sample) are

Often mistakenly includes equality. Equality doesn't cause aliasing but likely results in lost info.

$$f_s = \frac{1}{T_s} > \frac{\omega_{max}}{\pi} \qquad (8.18)$$

- When the Nyquist sampling rate condition is satisfied, the original signal *x(t)* can be reconstructed by passing the sampled signal $x_s[n]$ through an ideal low-pass filter with frequency response:

ideal Impulse Generator cascaded to an

$$H(j\omega) = \begin{cases} T_s & -\omega_s/2 < \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases} = T_s\left[u\left(\omega + \frac{\omega_s}{2}\right) - u\left(\omega - \frac{\omega_s}{2}\right)\right]$$

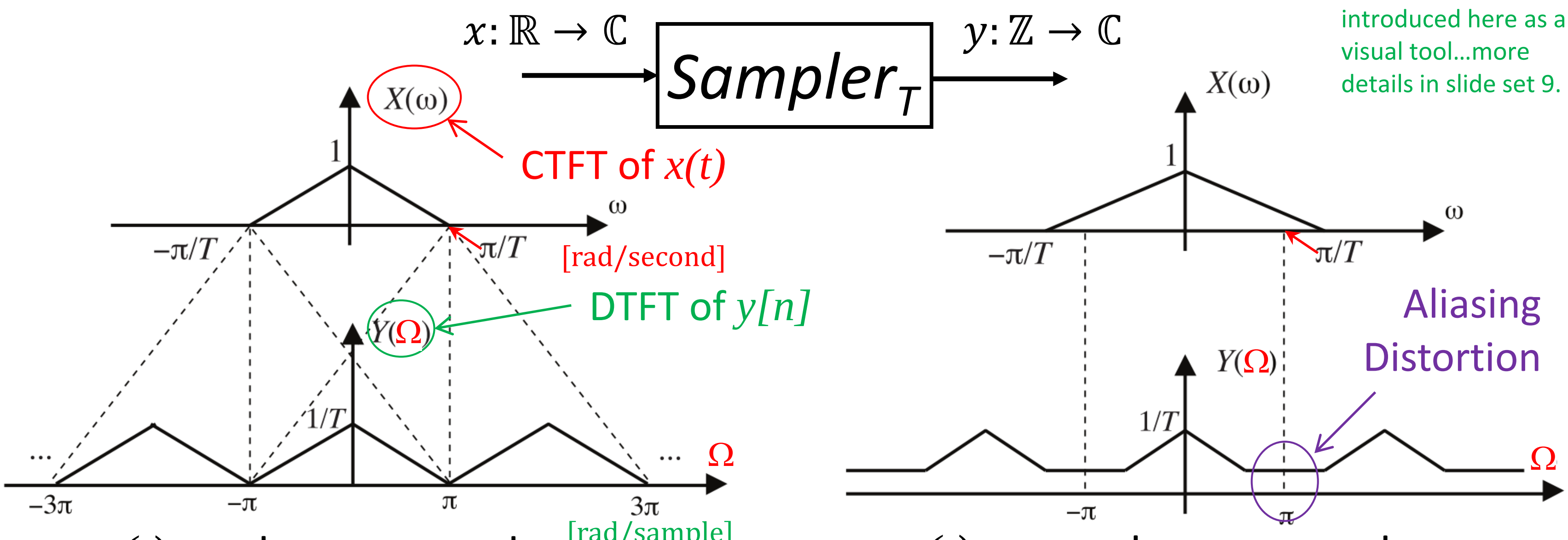
The reconstructed signal is given by the following sinc interpolation from the samples

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s} \qquad (8.19)$$

Sampling Theorem & FTs

The Nyquist-Shannon Sampling Theorem is understood better in the frequency domain so we resort to Fourier Transforms (for the original analog signal, the FT is also commonly called the CTFT (Continuous-Time FT) while the sampled signal has a DTFT (Discrete-Time FT) that similarly describes the signal's frequency content. NB: ω of CTFT & Ω of DTFT related by $\omega T=\Omega$.

Aside: The DTFT is introduced here as a visual tool…more details in slide set 9.



$\Rightarrow x(t)$ can be recovered

$\Rightarrow x(t)$ cannot be recovered

Nyquist Frequency vs Nyquist Rate

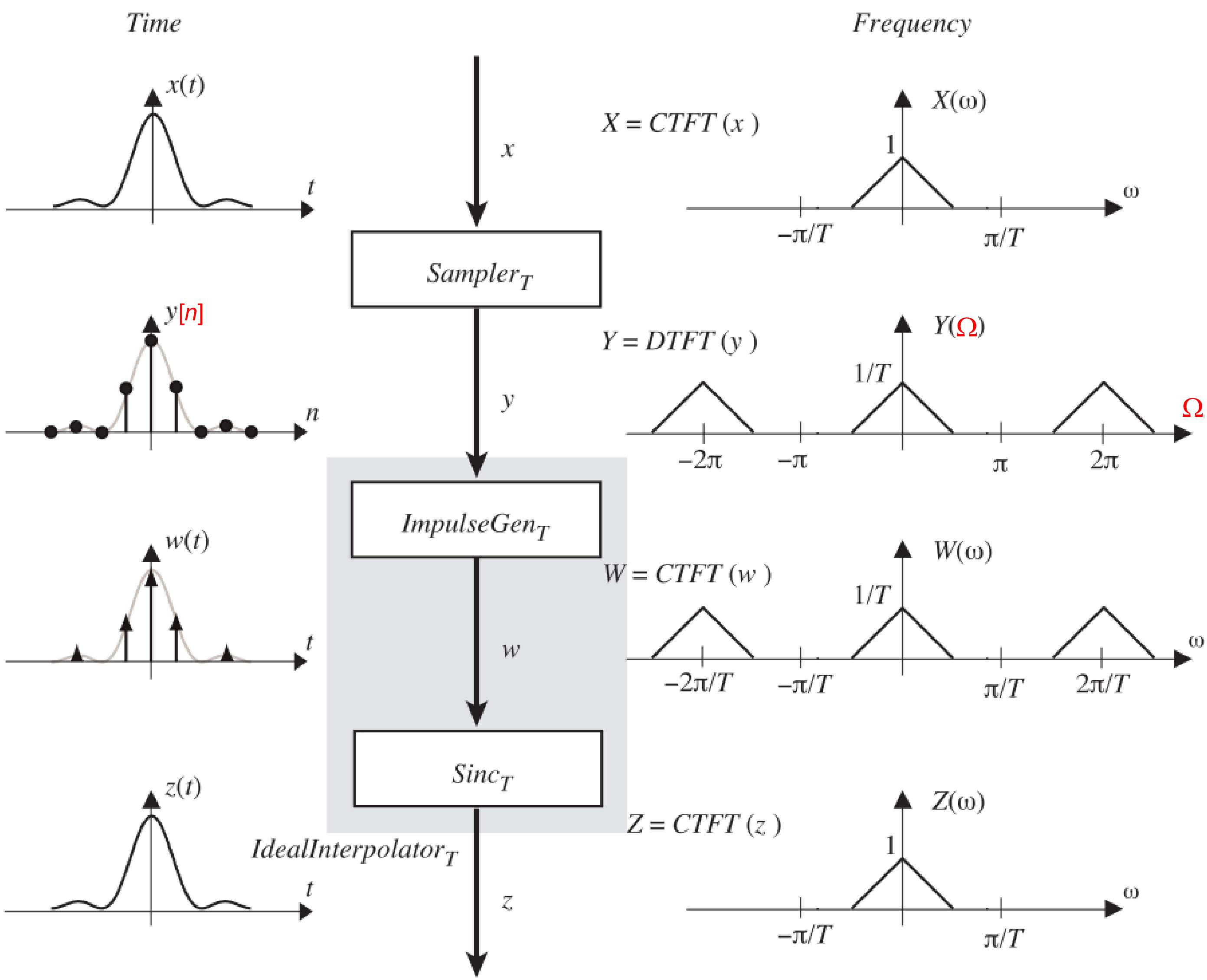
A subtle but important distinction arises in the use of these terms which address two different problems (often confused in some resources).

Nyquist Frequency: Given a system with sampling rate f_s, the Nyquist Frequency is the upper bound on an input signal's frequency so that it can be completely recovered through ideal interpolation (it's a property of the system): $f_{Nyquist} = {}^{f_s}/_2$. $(X(f) = 0 \text{ for } f \geq f_{Nyquist}) \Rightarrow x(t) \text{ recoverable}$

Nyquist (Sampling) Rate: Given a bandlimited signal with maximum frequency f_{max}, the Nyquist Rate is the minimum sampling rate required for complete recovery of the signal through ideal interpolation (it's a property of the signal or a set of signals): $f_{Sampling} > 2f_{max}$

Efts: For the edge-case $x(t) = Acos(2\pi f_{Nyquist} + \phi)$, observe that we don't get aliasing but explore why $x(t)$ still isn't generally recoverable (try plotting the sampled signal).

Sampling & Reconstruction

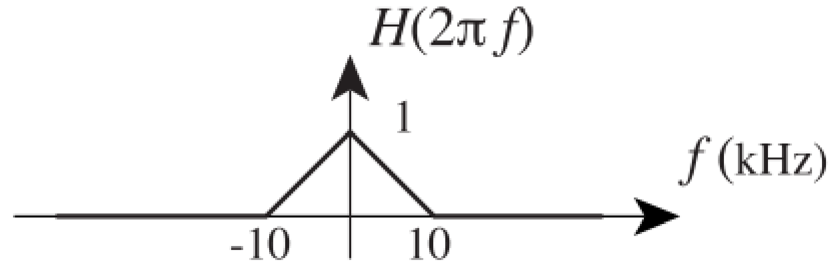


Perfect reconstruction possible when the Nyquist-Shannon Sampling condition is satisfied! If it is not satisfied, an anti-aliasing filter placed BEFORE *Sampler*$_T$ may be applied to reduce distortion.

Eg: Adapted from L&V Exercise 11.6

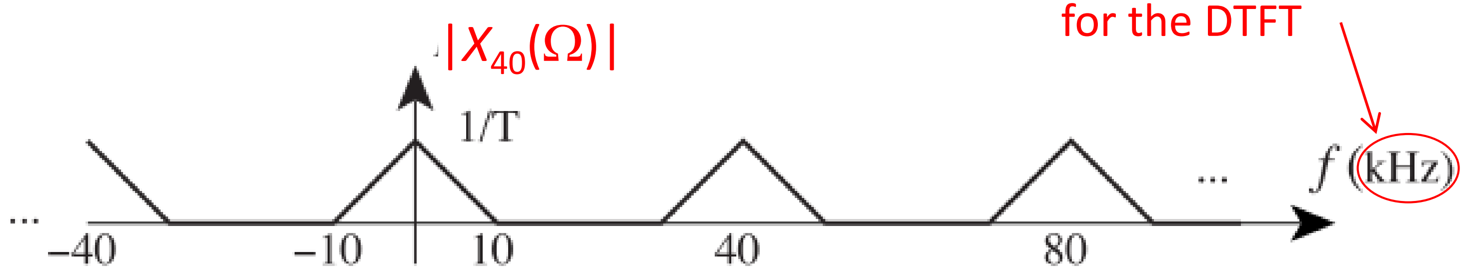
Consider a CT audio signal x with CTFT shown below, bandlimited to 10 kHz. Suppose that it is sampled at 40 kHz (resulting in a new DT signal x_{40}). Let X_{40} be the DTFT of x_{40}.

(a) Sketch $|X_{40}(\Omega)|$, carefully marking magnitudes and frequencies.



Technically, these aren't valid units for the DTFT

Ans:



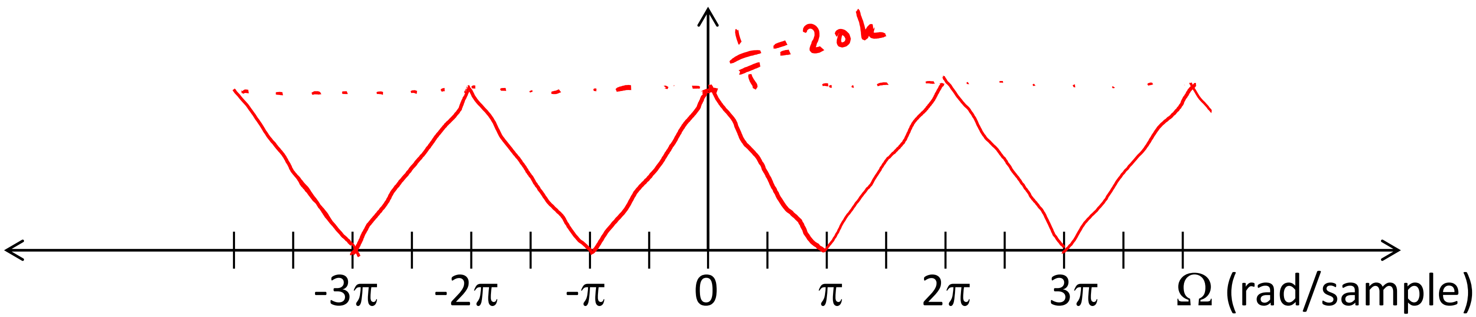
Frequency axis remedied using f_s=40 ksamples/sec

-1	-1/4	1/4	1	2	f (cycles/sample)
-2π	$-\pi/2$	$\pi/2$	2π	4π	Ω (rad/sample)

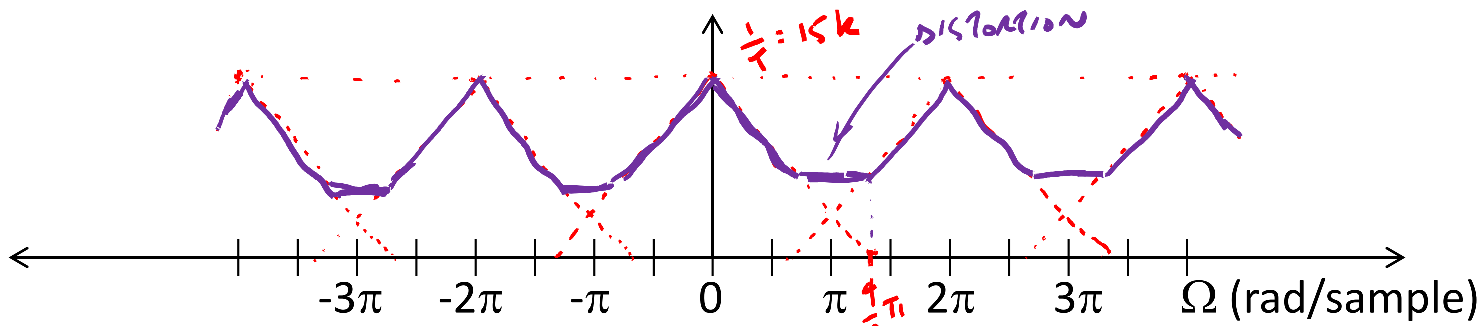
The height of each peak is 1/T=40,000 here.

Eg: L&V Exercise 11.6 (cont.)

(b) Sketch $|X_{20}(\Omega)|$ where X_{20} is the DTFT of x_{20}, sampling at 20 kHz.



(c) Sketch $|X_{15}(\Omega)|$ where X_{15} is the DTFT of x_{15}, sampling at 15 kHz. What ideal anti-aliasing filter should be applied before sampling to reduce distortion?



FOR SAMPLING @ 15 kHz, APPLY A LPF w/ CUT-OFF @ THE NYQUIST FREQ OF 7.5 kHz. THIS MAINTAINS FIDELITY OF ALL SIGNALS UP TO 7.5 kHz (THOUGH THE FREQS FROM 7.5 – 10 kHz ARE LOST). W/O THIS FILTER, GET DISTORTION ABOVE 5 kHz.