State-Space System Representations

- State-Space Representation of CT Systems
  - [A,B,C,D] matrices
  - State-transition matrix
  - Controllability/Observability
  - Similarity Transformations and Canonical Forms
  - Comparison of System Representations
- State-Space Representation of DT Systems
- Computation of the State Transition Matrix

We lightly visit several of the key ideas with state-space models but don’t closely follow any of the optional references.

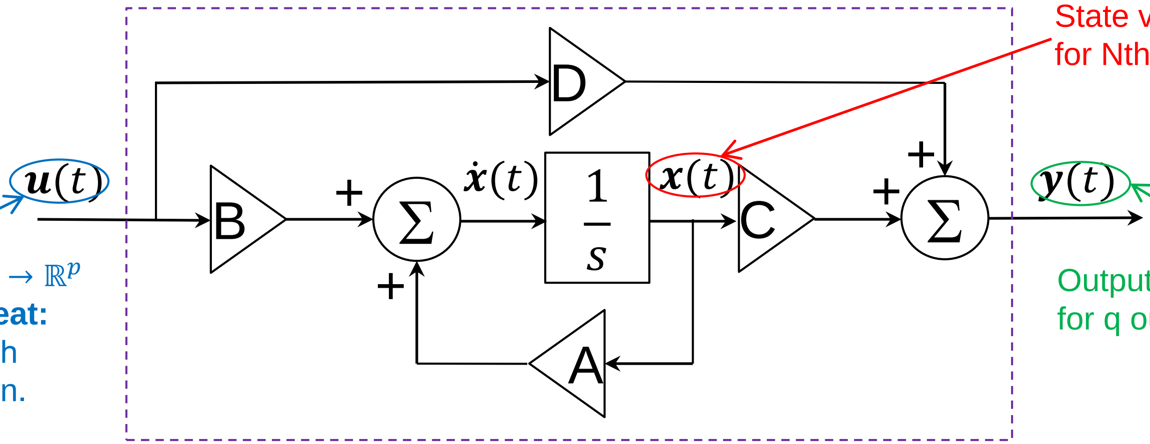
Concept of “State”

State: the status of the system such that, with the state known at time t_o for a continuous-time system (or step n_o for a discrete-time sequence), the state evolution can be completely determined from knowing the input $\mathbf{u}(t)$ for $[t_o,\infty)$ (or input sequence $\{\mathbf{u}[n_o]\ , \mathbf{u}[n_o+1], \mathbf{u}[n_o+2], \ldots\}$. <u>The state “summarizes” any effects of the previous history.</u>

Aside: This definition does apply to the Finite State Machines explored in CPEN 211. However, the states we explore in ELEC 221 are vector spaces (with an infinite continuum of possible states).

CT System State-Space LTI Model

State vector, $\boldsymbol{x}: \mathbb{R} \rightarrow \mathbb{R}^N$ for Nth order system.



Input vector, $\boldsymbol{u}: \mathbb{R} \rightarrow \mathbb{R}^p$ for p inputs. **Caveat:** Don't confuse with Heaviside function.

Output vector, $\boldsymbol{y}: \mathbb{R} \rightarrow \mathbb{R}^q$ for q outputs.

System Matrix $A \epsilon \mathbb{R}^{N \times N}$ describes internal dynamics

Input Matrix $B \epsilon \mathbb{R}^{N \times p}$ describes direct control of state

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \end{aligned}}$$

State Differential Equation

Output (or Response) Equation

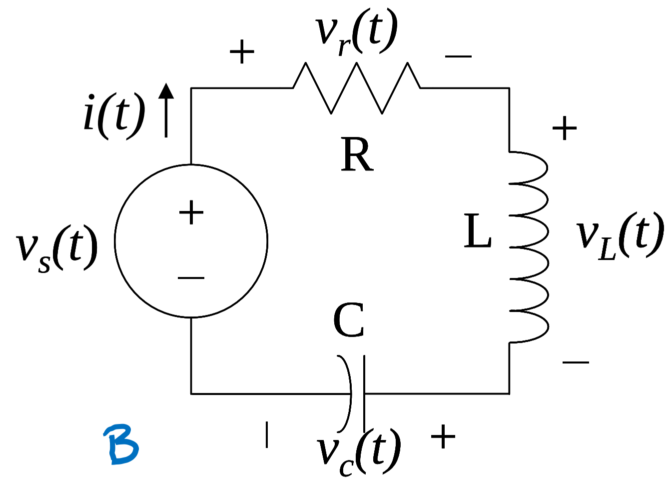
Output Matrix $C \epsilon \mathbb{R}^{q \times N}$ describes direct "observation" of state

Feedforward Matrix $D \epsilon \mathbb{R}^{q \times p}$ describes any direct I/O link

NB: All systems analysed so far are SISO (single-input, single-output) but state-space models nicely handle more general MIMO (multi-input, multi-output) systems. By convention, the signals are represented by $\boldsymbol{u}(t), \boldsymbol{x}(t) \,\&\, \boldsymbol{y}(t)$, as shown.

Eg: Series RLC Circuit

Find the system state-space representation if the state vector is taken as $\boldsymbol{x}(t) = \begin{bmatrix} v_c(t) \\ i(t) \end{bmatrix}$, the input is $v_s(t)$ and the output is $v_c(t)$.



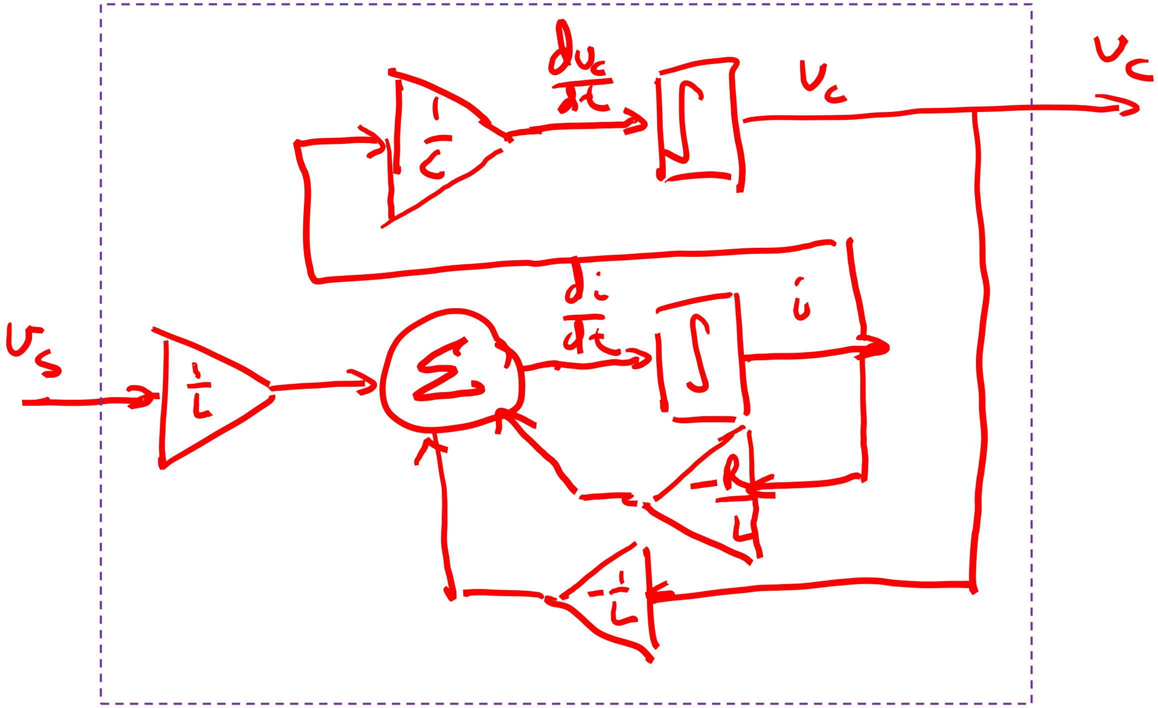
KVL: $v_s = v_L + v_r + v_c = L\frac{di}{dt} + Ri + v_c$

$i = C\dot{v}_c$

$$\Rightarrow \dot{x} = \begin{bmatrix} i/C \\ \frac{v_s - Ri - v_c}{L} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}^{B} v_s$$

$$y = v_c = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} v_s$$

Eg: Series RLC Circuit (cont.)



State Transition Matrix

For the CT LTI system, e^{At} is called the **state transition matrix**, and is used to compute the various responses.

$$\dot{x} = Ax + Bu \xrightarrow{\mathcal{L}} sX(s) - x_0 = AX(s) + BU(s)$$

$$(sI - A)X(s) = x_0 + BU(s)$$

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

$\mathcal{L}^{-1}$

State response: $$\boldsymbol{x}(t) = \underbrace{e^{At}\boldsymbol{x}_0}_{\text{ZERO-INPUT STATE RESPONSE}} + \underbrace{\int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau}_{\text{ZERO-STATE STATE RESPONSE}}$$

Output response: $$\boldsymbol{y}(t) = \underbrace{Ce^{At}\boldsymbol{x}_0}_{\text{ZIR}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau + D\boldsymbol{u}(\tau)}_{\text{ZSR}}$$

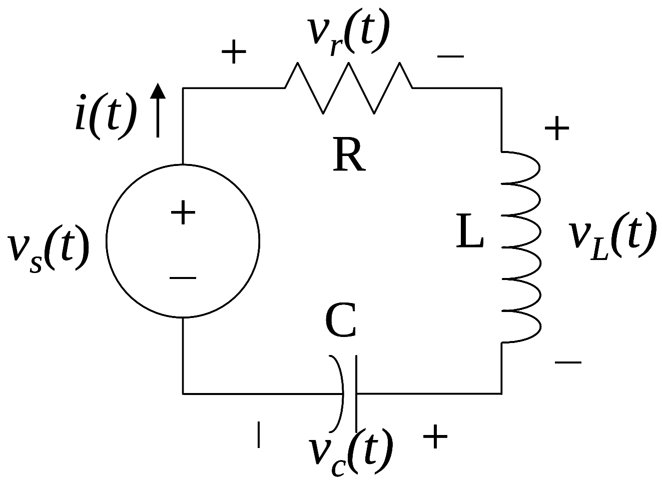
$\mathcal{L}^{-1}$

$$Y(s) = C(sI - A)^{-1}x_0 + [\underbrace{C(sI-A)^{-1}B + D}_{H(s)\ \leftarrow\ \text{T.F. MATRIX}}]U(s)$$

Eg: Revisit Series RLC Circuit

Recall from slide 11.4:

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = [0].$$



What is the transfer function?

$$H(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

$$\begin{bmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R}{L} \end{bmatrix}^{-1} = \frac{1}{s(s + \frac{R}{L}) + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{C} \\ -\frac{1}{L} & s \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{C} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \underline{\underline{\frac{1}{LCs^2 + RCs + 1}}}$$

THIS IS CONSISTENT W/ OUR VOLTAGE DIVIDER: $H(s) = \frac{V_c(s)}{V_s(s)} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{s^2LC + sCR + 1}$

Controllability

Controllability: Ability to drive systems from any initial state to any other final state in a finite time interval.

$$\boldsymbol{x}(t_f) = e^{At_f}\boldsymbol{x}_0 + \int_0^{t_f} e^{A(t_f-\tau)}\,B\boldsymbol{u}(\tau)d\tau$$

Can $\boldsymbol{u}(\cdot)$ be found to satisfy an arbitrary state $\boldsymbol{x}(t_f)$? Assume everything else in equation is specified. If the answer is "Yes", the system is said to be "(completely) controllable".

System controllability is determined solely by (A,B):

$$\boxed{(A,B)\text{ is controllable} \Leftrightarrow rank([B, AB, \ldots, A^{N-1}B]) = N}$$

"Controllability Matrix", M_c

Observability

Observability: Ability to infer initial state from observing input and output in a finite time interval

$$\boldsymbol{y}(t) = Ce^{At}\boldsymbol{x}_0 + C\int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau + D\boldsymbol{u}(\tau)$$

Can $\boldsymbol{x}_0$ be determined based on the observed $\boldsymbol{y}(t)$? Assume everything else in equation is known.
If the answer is "Yes", the system is said to be "(completely) observable".

System observability is determined solely by (A,C):

$$\boxed{(A, C) \text{ is observable} \Leftrightarrow rank\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}\right) = N}$$

"Observability Matrix" , M_o

Similarity Transformation

Consider an invertible matrix $T \epsilon \mathbb{R}^{N \times N}$ and let

$$\widetilde{\boldsymbol{x}}(t) = T\boldsymbol{x}(t) \Leftrightarrow T^{-1}\widetilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t)$$

$$\Rightarrow \dot{\widetilde{\boldsymbol{x}}}(t) = TA\boldsymbol{x}(t) + TB\boldsymbol{u}(t) = TAT^{-1}\widetilde{\boldsymbol{x}}(t) + TB\boldsymbol{u}(t)$$

$$= \tilde{A}\widetilde{\boldsymbol{x}}(t) + \tilde{B}\boldsymbol{u}(t)$$

$$\Rightarrow \boldsymbol{y}(t) = CT^{-1}\widetilde{\boldsymbol{x}}(t) + D\boldsymbol{u}(t)$$

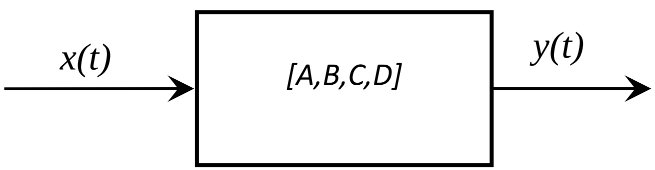
$$\boxed{[A, B, C, D] \rightarrow \left[\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}, \widetilde{D} = D\right]}$$

The state-space representation is not unique and any nonsingular NxN matrix allows for a valid transformation.

SISO Differential Equation

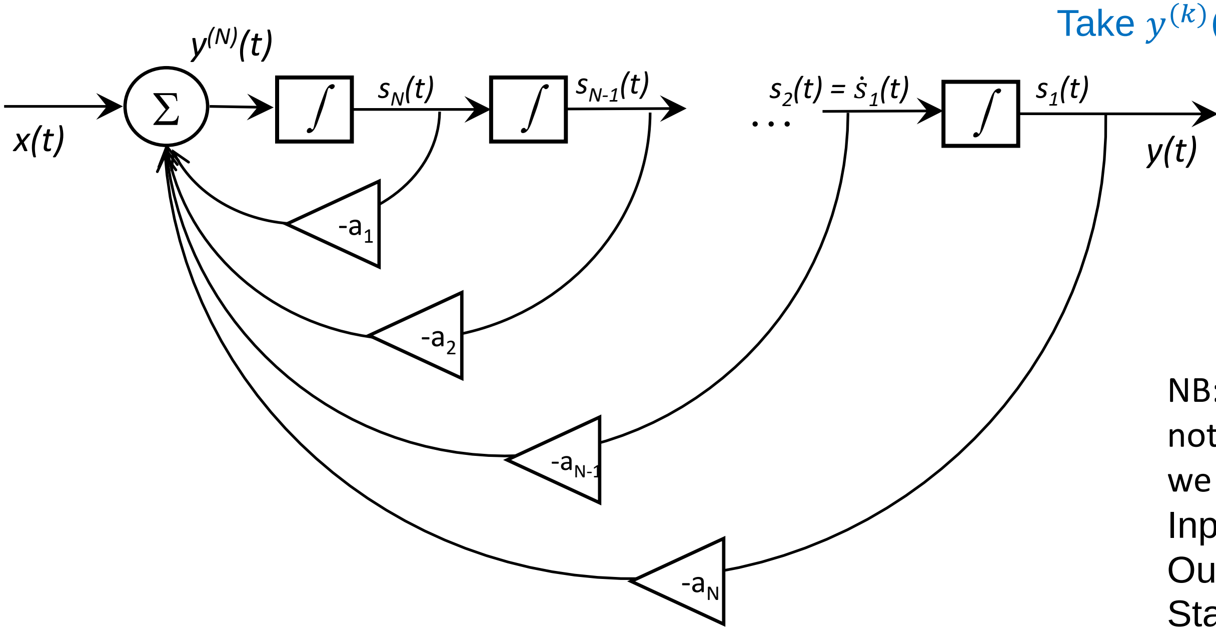
Consider the ODE relating input x(t) to output y(t):

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$



In the case where $b_i = \begin{cases} 1 & if \;\; i = N \\ 0 & if \;\; i \in [0, N-1] \end{cases}$, by choosing the state as $\boldsymbol{s}(t) = \begin{bmatrix} y^{(N-1)}(t) \\ \vdots \\ y(t) \end{bmatrix}$, the block diagram shows a valid realisation.

Take $y^{(k)}(t) = \frac{d^k y(t)}{dt^k}$



NB: For consistency with our notation prior to this slide set, we switch:

Input: $\boldsymbol{u}(t) \rightarrow x(t)$

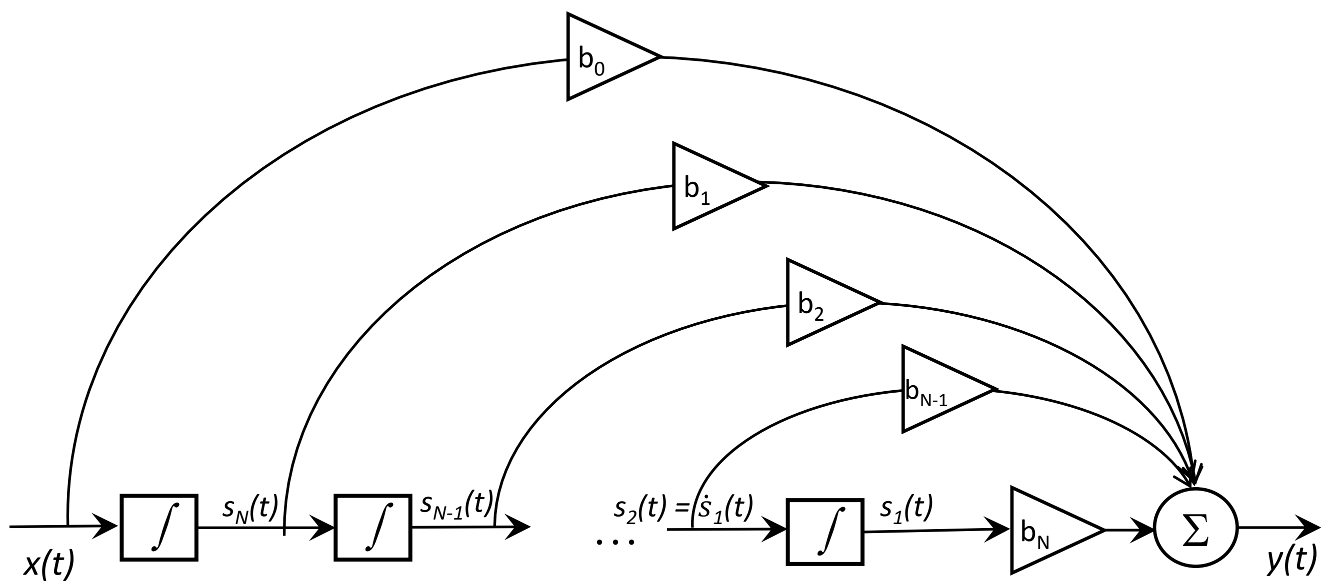
Output: $\boldsymbol{y}(t) \rightarrow y(t)$

State: $\boldsymbol{x}(t) \rightarrow \boldsymbol{s}(t)$

SISO Differential Equation

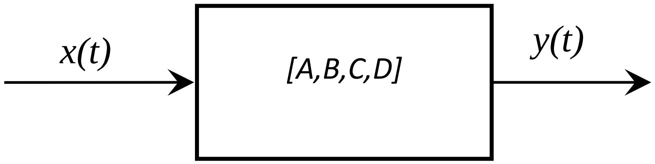
$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

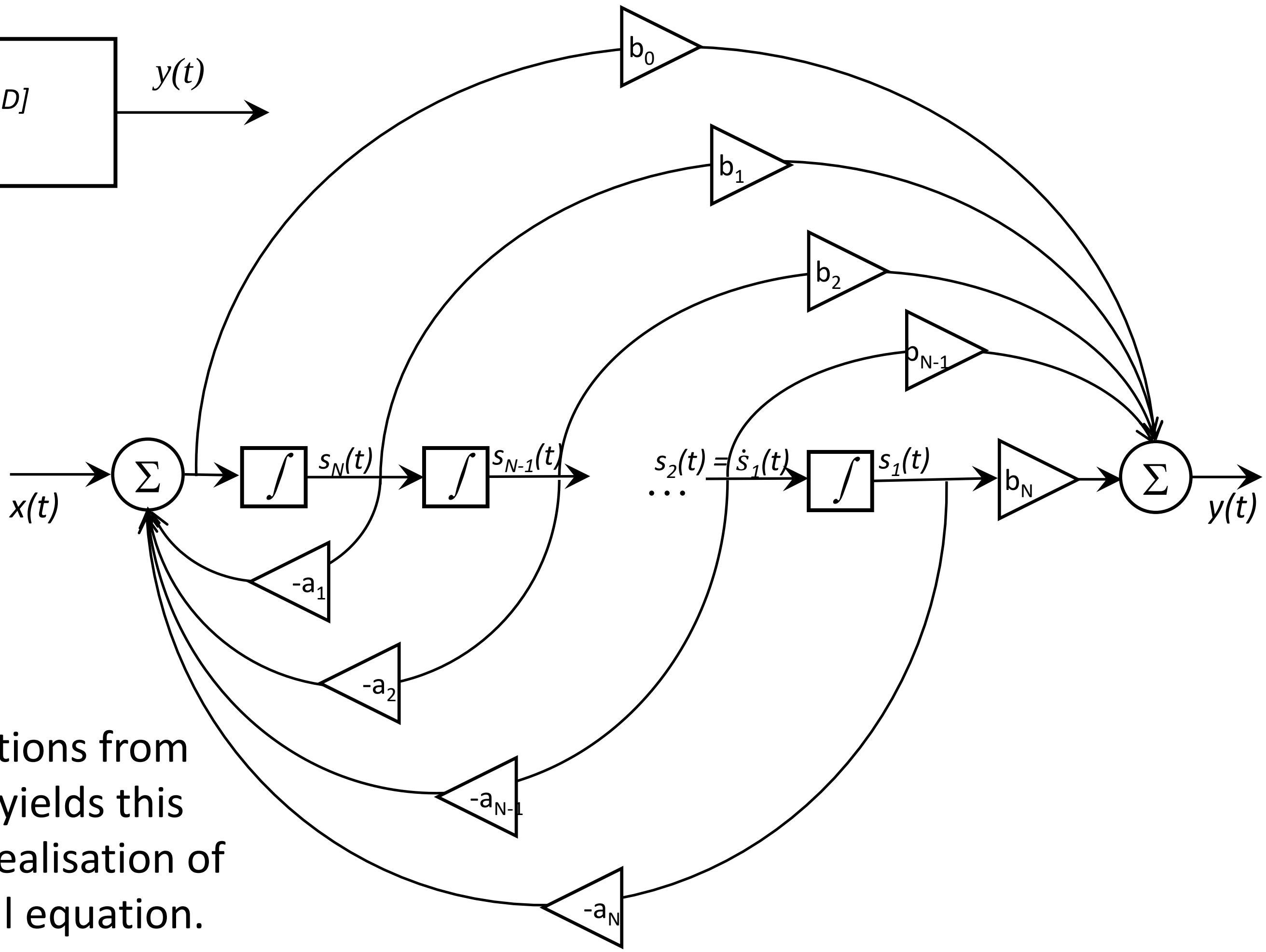
In the case where $a_i = 0 \; for \; i \in [1, N]$, by choosing the state as $\boldsymbol{s}(t) = \begin{bmatrix} x^{(N)}(t) \\ \vdots \\ x^{(1)}(t) \end{bmatrix}$, the block diagram shows a valid realisation.



SISO Controllable Canonical Form

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$





Combining observations from the two prior slides yields this CCF block diagram realisation of the given differential equation.

SISO Controllable Canonical Form

The block diagram can be used to generate the "controller/controllable canonical form" realization

$$\dot{\boldsymbol{s}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & & -a_2 & -a_1 \end{bmatrix} \boldsymbol{s}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [b_N - a_N b_0 \quad b_{N-1} - a_{N-1} b_0 \quad \ldots \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0]\boldsymbol{s}(t) + b_0 x(t)$$

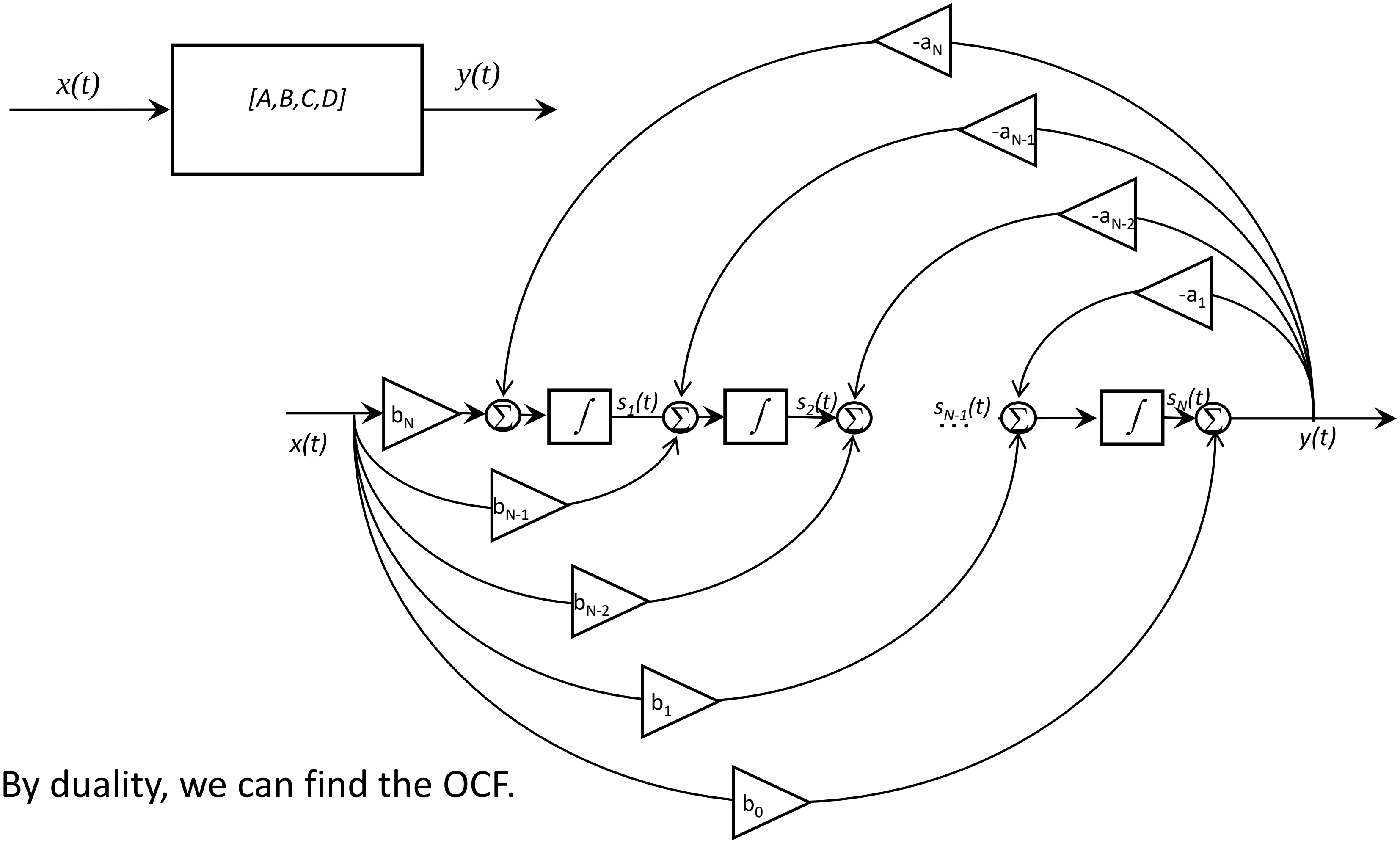
$* $ - INDICATES A VALUE THAT DOESN'T MATTER

$$M_c = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & * \\ \vdots & \vdots & \vdots & \iddots & * & \vdots \\ 0 & 1 & * & & \vdots & \vdots \\ 1 & * & * & \cdots & * & * \end{bmatrix} \Rightarrow |M_c| \neq 0 \Rightarrow \text{CONTROLLABLE}$$

Caveat: When determined from the ODE, canonical forms may not be accurate representations of the "internal dynamics"; then the original schematic/block diagram should be used instead.

SISO Observable Canonical Form

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$



By duality, we can find the OCF.

SISO Observable Canonical Form

The block diagram can be used to generate the "observer/observable canonical form" realization

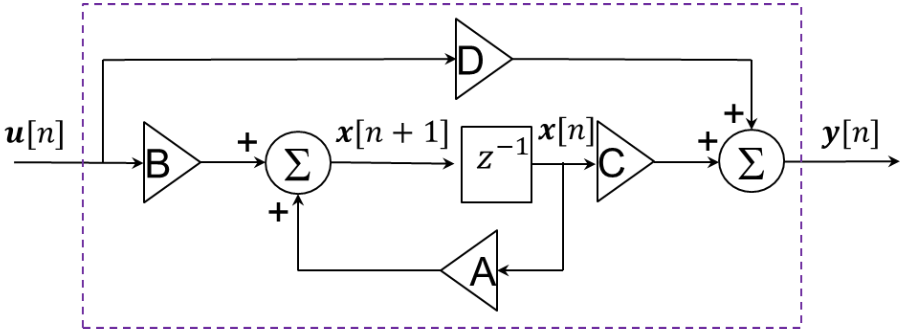
$$\dot{\boldsymbol{s}}(t) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -a_N \\ 1 & 0 & & 0 & 0 & -a_{N-1} \\ 0 & 1 & & 0 & 0 & -a_{N-2} \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{bmatrix} \boldsymbol{s}(t) + \begin{bmatrix} b_N - a_N b_0 \\ b_{N-1} - a_{N-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \boldsymbol{s}(t) + b_0 x(t)$$

$$M_o = \begin{bmatrix} 0 & \cdots & & 0 & 1 \\ 0 & \cdots & 0 & 1 & * \\ \vdots & & & & \vdots \\ 0 & 1 & & & * \\ 1 & * & \cdots & & * \end{bmatrix} \Rightarrow |M_o| \neq 0 \Rightarrow \text{OBSERVABLE}$$

Aside: The Jordan Canonical Form (outside the scope of this course) is another interesting common system representation, allowing decoupling of different "modes".

DT System State-Space LTI Model



State Difference Equation

$$\boxed{\begin{aligned} \boldsymbol{x}[n+1] &= A\boldsymbol{x}[n] + B\boldsymbol{u}[n] \\ \boldsymbol{y}[n] &= C\boldsymbol{x}[n] + D\boldsymbol{u}[n] \end{aligned}}$$

Output (or Response) Equation

State transition matrix, A^n, is used to find the response:

$$\boldsymbol{x}[n] = \underbrace{A^n \boldsymbol{x}_0}_{\text{ZISR}} + \underbrace{\sum_{k=0}^{n-1} A^{n-1-k} B\boldsymbol{u}[k]}_{\text{ZSSR}}$$

$$\boldsymbol{y}[n] = \underbrace{CA^n \boldsymbol{x}_0}_{\text{ZIR}} + \underbrace{\sum_{k=0}^{n-1} CA^{n-1-k} B\boldsymbol{u}[k] + D\boldsymbol{u}[n]}_{\text{ZSR}}$$

$$Y(z) = C(zI-A)^{-1}x_0 + \underbrace{[C(zI-A)^{-1}B + D]}_{H(z)}U(z)$$

DT System Controllability

Controllability: For a DT system, a system is controllable if it can be driven between any initial and final states in N steps (or less).

$$\boldsymbol{x}\,[N] = A^N \boldsymbol{x}_0 + [B \quad AB \quad \cdots \quad A^{N-1}B] \begin{bmatrix} \boldsymbol{u}[N-1] \\ \vdots \\ \boldsymbol{u}[0] \end{bmatrix} = A^N \boldsymbol{x}_0 + M_c \begin{bmatrix} \boldsymbol{u}[N-1] \\ \vdots \\ \boldsymbol{u}[0] \end{bmatrix}$$

The condition for controllability is the same as for CT but the rationale is now more obvious: $\boxed{(A, B) \text{ is controllable} \Leftrightarrow rank(M_c) = N}$

In the case of a single input system that is controllable, the required control can be easily computed:

$$\begin{bmatrix} u[N-1] \\ \vdots \\ u[0] \end{bmatrix} = M_c^{-1}\left(\boldsymbol{x}_f - A^N \boldsymbol{x}_0\right)$$

Aside: When the number of steps to reach the final state isn't constrained, a more interesting problem (beyond the scope of this course) is to optimize the control based on both shortest time and least control effort (LQR or Linear Quadratic Regulator).

DT System Observability

Observability: For a DT system, a system is observable if the initial state can be computed from observing N output values (or less).

Define $\Delta\boldsymbol{y}[n] = \boldsymbol{y}[n] - \sum_{k=0}^{n-1} CA^{n-1-k}\, B\boldsymbol{u}[k] - D\boldsymbol{u}[n] = CA^n\boldsymbol{x}_0$

$$\text{From N output values} \Rightarrow \begin{bmatrix} \Delta\boldsymbol{y}[0] \\ \Delta\boldsymbol{y}[1] \\ \vdots \\ \Delta\boldsymbol{y}[N-1] \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} \boldsymbol{x}_0 = M_o\boldsymbol{x}_0$$

$$\boxed{(A, C) \text{ is observable} \Leftrightarrow rank(M_o) = N}$$

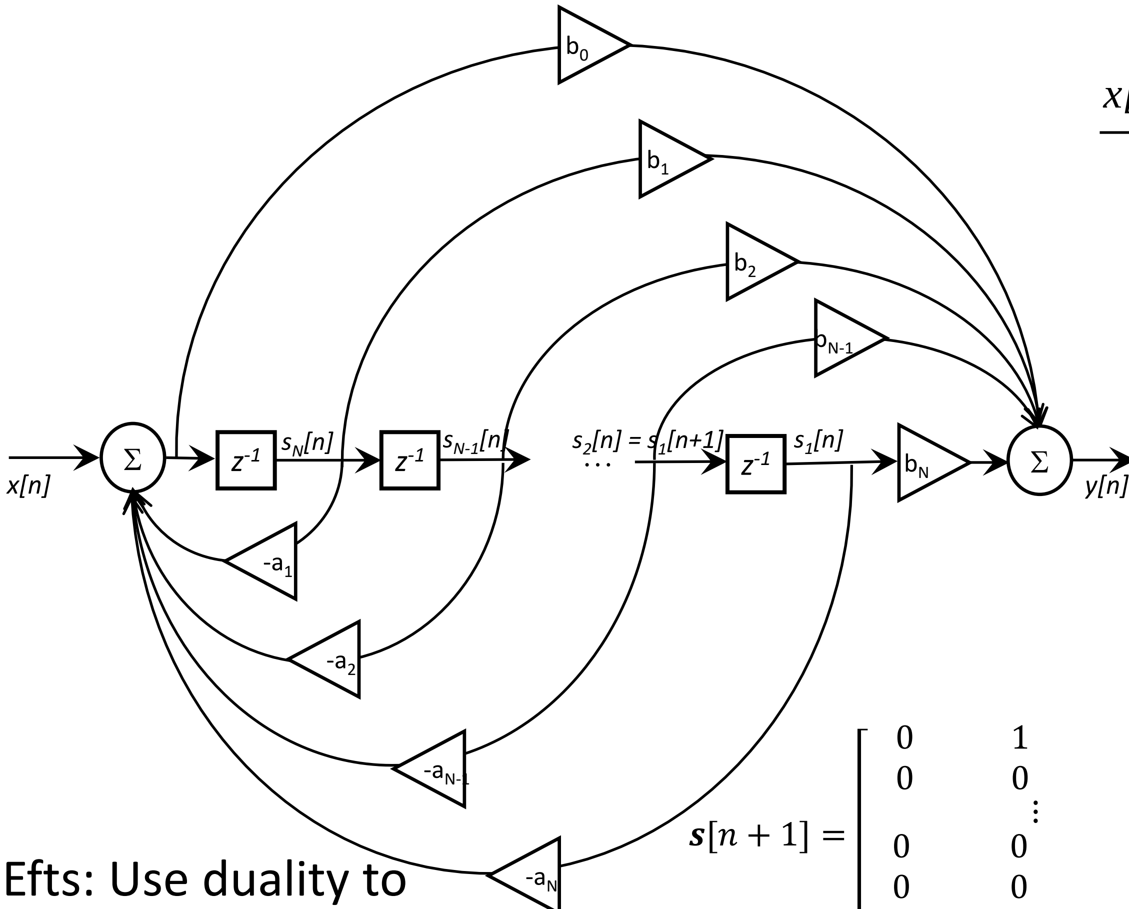
In the case of a single output system that is observable, the initial state can be easily computed:

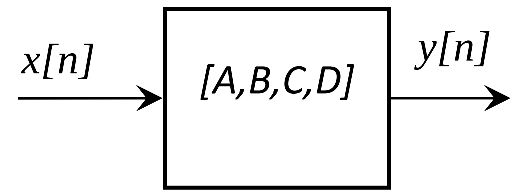
$$\boldsymbol{x}_0 = M_o^{-1} \begin{bmatrix} \Delta y[0] \\ \Delta y[1] \\ \vdots \\ \Delta y[N-1] \end{bmatrix}$$

Aside: As more output values are observed, the problem becomes overconstrained. If Gaussian noise is assumed in these measurements (so the signal is “stochastic”), a more interesting problem (beyond the scope of this course) is to find the least-squares-error solution (Kalman filter).

DT SISO CCF

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$





Extension of the CT CCF to the DT CCF from the difference equation is straightforward.

$$\boldsymbol{s}[n+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & & -a_2 & -a_1 \end{bmatrix} \boldsymbol{s}[n] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = [b_N - a_N b_0 \quad b_{N-1} - a_{N-1} b_0 \quad \ldots \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0]\boldsymbol{s}[n] + b_0 x[n]$$

Efts: Use duality to extend these ideas to the OCF

Computation of State Transition Matrix

Method 1: Eigendecomposition of A

$A \epsilon \mathbb{R}^{N\times N}$ is diagonalizable $\Leftrightarrow$ A has N linearly independent eigenvectors (otherwise, more generally need "Jordan form" when repeated eigenvalues have geometric multiplicity <algebraic multiplicity).

Find the eigenvalues & eigenvectors of A: $|A-\lambda I|=0, A\boldsymbol{v_i}=\lambda_i \boldsymbol{v_i}$.

$$\Rightarrow A=[\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_N}]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N\end{bmatrix}[\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_N}]^{-1}=TDT^{-1}$$

$\Rightarrow$State transition matrix

CT

$$e^{At}=Te^{Dt}T^{-1}=T\begin{bmatrix}e^{\lambda_1 t} & 0 & \cdots & 0\\ 0 & e^{\lambda_2 t} & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_N t}\end{bmatrix}T^{-1}$$

DT

$$A^n=TD^nT^{-1}=T\begin{bmatrix}\lambda_1^n & 0 & \cdots & 0\\ 0 & \lambda_2^n & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N^n\end{bmatrix}T^{-1}$$

Computation of State Transition Matrix

Method 2: Analytic Functions of Square Matrices

Cayley-Hamilton Theorem: Every square matrix satisfies its own characteristic equation.

$\Rightarrow$ Any analytic function of A can be written as $f(A) = \sum_{k=0}^{N-1} c_k A^k$.
Coefficients $\{c_k\}$ found from system of equations generated by every λ_i.

$\Rightarrow$ State transition matrix

CT: $e^{At} = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \epsilon [1, N], e^{\lambda_i t} = \sum_{k=0}^{N-1} c_k \lambda_i^k$

DT: $A^n = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \epsilon [1, N],\ \lambda_i^n = \sum_{k=0}^{N-1} c_k \lambda_i^k$

Ex: Computing A^{-1}, A^n and e^{At}

For $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, compute A^{-1}, A^n and e^{At} (write each as a single 2x2 matrix).

$$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) \Rightarrow \begin{cases} \lambda_1 = -1 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = -2 \Rightarrow v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{cases}$$

$$\Rightarrow A = TDT^{-1} \text{ WHERE } T = [v_1 \ \ v_2] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}; D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

EIV DECOMPOSITION APPROACH:

$$A^{-1} = TD^{-1}T^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1.5 & -0.5 \\ 1 & 0 \end{bmatrix}$$

$$A^n = TD^nT^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (-1)^n & -(-2)^n \\ -(-1)^n & 2(-2)^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)^n - (-2)^n & (-1)^n - (-2)^n \\ 2(-2)^n - 2(-1)^n & 2(-2)^n - (-1)^n \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2[e^{-2t} - e^{-t}] & 2e^{-2t} - e^{-t} \end{bmatrix}$$

CHT APPROACH: $A^2+3A+2I=0 \Rightarrow A^{-1}=\dfrac{-(A+3I)}{2}=\underline{\begin{bmatrix}-1.5 & -0.5\\ 1 & 0\end{bmatrix}}$

$A^n=\alpha_0 I+\alpha_1 A$

$$\left.\begin{matrix}\lambda_1^n=\alpha_0+\alpha_1\lambda_1\\ \lambda_2^n=\alpha_0+\alpha_1\lambda_2\end{matrix}\right\}\Rightarrow\begin{bmatrix}\lambda_1^n\\ \lambda_2^n\end{bmatrix}=\begin{bmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}\Rightarrow\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}=\frac{1}{\lambda_2-\lambda_1}\begin{bmatrix}\lambda_2 & -\lambda_1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^n\\ \lambda_2^n\end{bmatrix}=\frac{1}{(-1)}\begin{bmatrix}2(-1)^{n+1}+(-2)^n\\ (-2)^n-(-1)^n\end{bmatrix}$$

$$\Rightarrow A^n=\left[2(-1)^n-(-2)^n\right]I+\left[(-1)^n-(-2)^n\right]A=\underline{\begin{bmatrix}2(-1)^n-(-2)^n & (-1)^n-(-2)^n\\ 2\left[(-2)^n-(-1)^n\right] & 2(-2)^n-(-1)^n\end{bmatrix}}$$

$$e^{At}=\underline{\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ 2\left[e^{-2t}-e^{-t}\right] & 2e^{-2t}-e^{-t}\end{bmatrix}}$$

Ex: From 2018 MT2

For your convenience, you are given that matrix $A = \begin{bmatrix} -1 & 1 & -4 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ can be diagonalised using $T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for which $T^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$)

a) Determine A^n and e^{At} (write each as a single 3x3 matrix).

$$|\lambda I - A| = (\lambda+1)(\lambda+2)(\lambda+3) = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$$

$$A = TDT^{-1} \quad \text{where } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow A^n = TD^nT^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & & 0 \\ & (-2)^n & \\ 0 & & (-3)^n \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} (-1)^n & -(-2)^n & 2(-3)^n \\ 0 & (-2)^n & 0 \\ 0 & 0 & (-3)^n \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} (-1)^n & (-1)^n - (-2)^n & 2\left[(-3)^n - (-1)^n\right] \\ 0 & (-2)^n & 0 \\ 0 & 0 & (-3)^n \end{bmatrix}}$$

Ex: From 2018 MT2 (cont.)

$$e^{At} = T e^{\Lambda t} T^{-1} = T \begin{bmatrix} e^{-t} & & 0 \\ & e^{-2t} & \\ 0 & & e^{-3t} \end{bmatrix} T^{-1}$$

$$= \underbrace{\begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} & 2(e^{-3t} - e^{-t}) \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}}$$

FORM READILY APPARENT BASED ON SIMILARITY TO A^n RESULT

ASIDE: THE APPROACH USING CHT IS POSSIBLE BUT SOMEWHAT MORE TEDIOUS.

Recall $\lambda_1 = -1$
$\lambda_2 = -2$
$\lambda_3 = -3$

Ex: From 2018 MT2 (cont.)

b) If A is part of the state-space realization for a discrete-time system, explain if the internal dynamics are stable. How about for a continuous-time system?

DT: $|\lambda_2| > 1$ AND $|\lambda_3| > 1$ SO THIS IS UNSTABLE. THIS IS ALSO EVIDENT FROM THE GROWTH OF A^n AS n INCREASES.

CT: $Re\{\lambda_i\} < 0 \ \forall i \in \{1,2,3\} \Rightarrow$ ASYMPTOTICALLY STABLE. THIS IS ALSO EVIDENT BECAUSE $\lim_{t \to \infty} e^{At} = 0$.

Recall $A = \begin{bmatrix} -1 & 1 & -4 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Ex: From 2018 MT2 (cont.)

c) Given matrices $B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (one input) and $B_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$ (two inputs), generate the respective controllability matrices M_{c1} and M_{c2} and use them to determine the controllability of the two different state-space realizations that include (A,B_1) and (A,B_2).

$$M_{c1} = [B_1 \mid AB_1 \mid A^2B_1] = \begin{bmatrix} 0 & -3 & 13 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \Rightarrow |M_{c1}| = 3(9-4) + 13(-3+2) = 2 \neq 0$$

$\Rightarrow (A, B_1)$ IS CONTROLLABLE

$$M_{c2} = [B_2 \mid AB_2 \mid A^2B_2] = \begin{bmatrix} 1 & 1 & -1 & -2 & 1 & 4 \\ 0 & -1 & 0 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rk(M_{c2}) = 2 < N = 3$$

$\Rightarrow (A, B_2)$ IS NOT CONTROLLABLE

ASIDE: NOTE THE COLUMN VECTORS OF B_2 ARE EIGENVECTORS OF A AND THIS IS A BIG REASON FOR THE LOST CONTROLLABILITY.