Discrete-Time Signals & Systems

- DT Signals
  - Classification
  - Operations
  - Basic Signals
- DT Systems
  - Recursive (IIR) & Nonrecursive (FIR)
  - Representation by Difference Equations
  - Convolution Sum
  - Causality & Stability

“It's déjà vu all over again.” Quote by legendary baseball player/coach/manager “Yogi” Berra.

As we explore different topics, compare & contrast CT vs DT concepts.

Discrete-Time Signals

- A **Discrete-Time Signal** *x[n]* is a real-(or complex-) valued function of an integer sample index: $x: \mathbb{Z} \to \mathbb{R}\ (\mathbb{C})$.
- For E221, we focus on DT signals related to CT counterparts with uniform time intervals (e.g., through sampling).

Chaparro Ex 9.1: Consider $x(t) = 3\cos\left(2\pi t + \frac{\pi}{4}\right), \forall t \in \mathbb{R}$. Determine an appropriate sampling period T_s according to the Nyquist sampling rate condition, and obtain the discrete-time sampled signal $x_s[n]$ corresponding to the largest allowed sampling period.

A: The frequency of $x(t)$ is 1 Hz $\Rightarrow f_s = \frac{1}{T_s} > 2Hz \Rightarrow T_s < 0.5\ s$.

Caveat: Chaparro incorrectly says there is no loss of info if $T_s = 0.5\ s$ but in that case: $x_s[n] = 3\cos\left(2\pi n T_s + \frac{\pi}{4}\right) = 3\cos\left(\pi n + \frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}(-1)^n$.

If this DT signal is fed to the ideal interpolator, the reconstructed signal would be $x_r(t) = \frac{3}{\sqrt{2}}\cos(2\pi t) \neq x(t)$.

Classification I: Periodic vs Aperiodic

A discrete-time signal $x[n]$ is periodic if $\forall n \in \mathbb{Z}$

- it is defined for all possible values of n, $-\infty < n < \infty$, and
- there is a positive integer N, the fundamental period of $x[n]$, such that

Units are "samples" or "steps"

$$x[n + kN] = x[n] \qquad (9.2)$$

for any integer k.

An aperiodic signal does not satisfy one or both of the above conditions.

Periodic discrete-time sinusoids, of fundamental period N, are of the form

$$x[n] = A\cos\left(\frac{2\pi m}{N}n + \theta\right) \qquad -\infty < n < \infty \qquad (9.3)$$

where the discrete frequency is $\Omega_0 = 2\pi m/N$ (rad), for positive integers m and N which are not divisible by each other, and θ is the phase angle.

I prefer "rad/step"; alternatively $f_0 = \frac{m}{N}$ cycles/step

Recall for CT sinusoid, given $\omega = 2\pi f = 2\pi/T$ so $(\omega_1 \neq \omega_2) \Rightarrow (T_1 \neq T_2)$.

Consider $x_1[n] = \cos(\frac{2\pi}{5}n)$ and $x_2[n] = \cos\left(\frac{4\pi}{5}n\right)$. What are their discrete frequencies & periods? $f_1 = \frac{1}{5}\frac{cycles}{step}, f_2 = \frac{2}{5}\frac{cycles}{step}$ so $f_1 \neq f_2$ but $N_1 = N_2 = 5 steps$!

Chaparro Ex 9.3 & 9.4

For the following functions, determine if they are periodic and, if so, their corresponding fundamental periods:

$$x_1[n] = 2\cos(\pi n - \pi)\,, x_2[n] = 3\sin\left(3\pi n + \frac{\pi}{2}\right), x_3[n] = \cos\left(n + \frac{\pi}{4}\right)$$

$x_1[n]$: $2\pi \frac{m}{N} = \pi \Rightarrow \frac{m}{N} = \frac{1}{2} \Rightarrow$ IT IS PERIODIC W/ $N=2$

ASIDE: CAN SIMPLIFY $x_1[n] = -2\cos(\pi n) = 2(-1)^{n+1}$

$x_2[n]$: $2\pi \frac{m}{N} = 3\pi \Rightarrow \frac{m}{N} = \frac{3}{2} \Rightarrow$ IT IS PERIODIC W/ $N=2$

$x_3[n]$: $2\pi \frac{m}{N} = 1 \Rightarrow \frac{m}{N} = \frac{1}{2\pi}$ IRRATIONAL $\Rightarrow$ APERIODIC

$\Rightarrow$ UNIFORM SAMPLING OF A CT PERIODIC SIGNAL DOESN'T GUARANTEE THE DT SIGNAL IS PERIODIC!

Sampling an Analog Sinusoid

When sampling an analog sinusoid

$$x(t) = A\cos(\omega_0 t + \theta) \qquad -\infty < t < \infty \tag{9.4}$$

of fundamental period $T_0 = 2\pi/\omega_0, \omega_0 > 0$, we obtain a **periodic discrete sinusoid**

$$x[n] = A\cos(\omega_0 T_s n + \theta) = A\cos\left(\frac{2\pi T_s}{T_0} n + \theta\right) \tag{9.5}$$

provided that

$$\frac{T_s}{T_0} = \frac{m}{N} \tag{9.6}$$

I.e., Ratio of sampling interval and fundamental period must be rational.

for positive integers N and m which are not divisible by each other. To avoid frequency aliasing the sampling period should also satisfy the Nyquist sampling condition

$$T_s < \frac{\pi}{\omega_0} = \frac{T_0}{2} \tag{9.7}$$

In this sentence, the nonstrict inequality ("≤") is acceptable. Contrast with eqn (8.18) in slide 7.11.

Sum of Periodic DT Signals

The sum $z[n] = x[n] + y[n]$ of periodic signals $x[n]$ with fundamental period N_1, and $y[n]$ with fundamental period N_2 is periodic if the ratio of periods of the summands is rational, i.e.,

$$\frac{N_2}{N_1} = \frac{p}{q}$$

and p and q are integers not divisible by each other. If so, the fundamental period of $z[n]$ is $qN_2 = pN_1$.

Chaparro Ex 9.5: z[n] = v[n] + w[n] + y[n] is the sum of three periodic signals with fundamental periods N_1=2, N_2=3, and N_3=4, respectively. Determine if z[n] is periodic, and if so its fundamental period.

$LCM(N_1, N_2, N_3) = 12 \Rightarrow$ z[n] IS PERIODIC w/ N=12 STEPS

$z[n+12] = v[n+6N_1] + w[n+4N_2] + y[n+3N_3] = v[n] + w[n] + y[n] = z[n]$

Classification II: Energy vs Power

For a discrete-time signal $x[n]$ we have the following definitions

$$\textbf{Energy}: \quad \varepsilon_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{9.9}$$

$$\textbf{Power}: \quad P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{9.10}$$

- $x[n]$ is said to have **finite energy** or to be **square summable** if $\varepsilon_x < \infty$.
- $x[n]$ is called **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{9.11}$$

- $x[n]$ is said to have **finite power** if $P_x < \infty$.

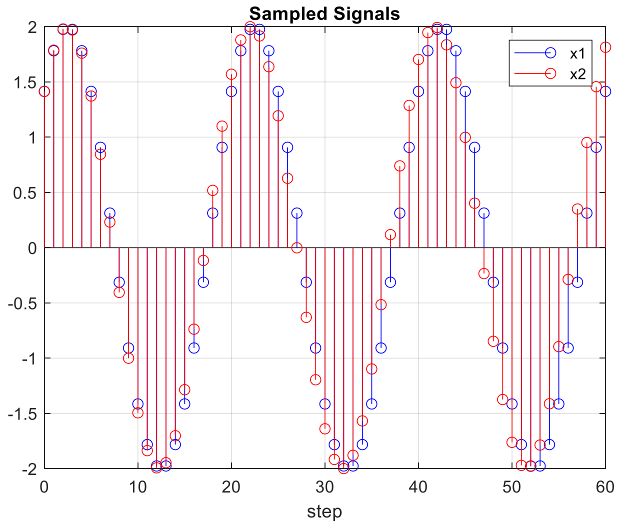
Chaparro Ex 9.7

Causal signal $x(t) = \begin{cases} 2\cos(\omega_0 t - \frac{\pi}{4}) & if\ t \geq 0 \\ 0 & otherwise \end{cases}$ is sampled with $T_s = 0.1\ s$

so $x_s[n] = \begin{cases} 2\cos(0.1\omega_0 n - \frac{\pi}{4}) & if\ n \geq 0 \\ 0 & otherwise \end{cases}$. Compute and compare the energy and power of the CT and DT signals for $\underbrace{\omega_0 = \pi}_{x_1[n]}$ and $\underbrace{\omega_0 = 3.2\ rad/s}_{x_2[n]}$

$x_1[n]$ PERIODIC $N_0 = 20$

$x_2[n]$ APERIODIC



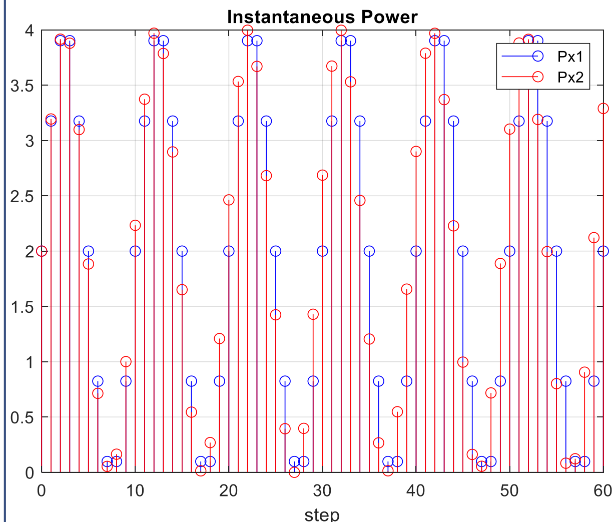
$x(t)$ HAS INFINITE ENERGY.

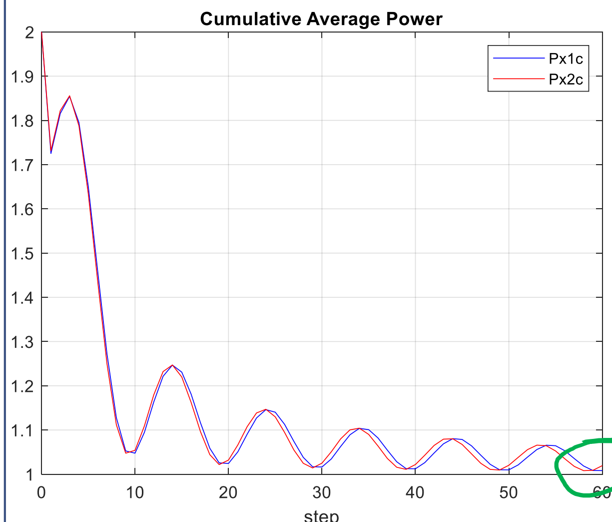
NOTICE $P_{x,AVG} = \begin{cases} 0 & \text{FOR } t<0 \\ \frac{2^2}{2}=2 & \text{FOR } t \geq 0 \end{cases} \Rightarrow P_x = \frac{2}{2} = 1$

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=0}^{\infty} 4\cos^2(0.1\Omega_0 n - \pi/4) \rightarrow \infty$$

$\Rightarrow$ BOTH $x_1[n]$ & $x_2[n]$ HAVE INFINITE ENERGY

Chaparro Ex 9.7 (cont.)





∵ SIGNAL IS CAUSAL

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{N}{2N+1} \underbrace{\left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right]}_{\text{power of period } n\geq 0}$$

VALID IF PERIODIC (i.e. OK FOR $x_1[n]$ BUT NOT $x_2[n]$)

$$= \lim_{N\to\infty} \frac{1}{2+1/N} \left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right] = \frac{1}{2N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 < \infty$$

i.e.: WHEN PERIODIC, REDUCES TO FINITE SUM OVER ONE PERIOD

BOTH $x_1[n]$ & $x_2[n]$ HAVE $P_x = 1$, THE VALUE ASYMPTOTICALLY APPROACHED BY THIS GRAPH

Classification III: Even vs Odd

Even and odd discrete-time signals are defined as

$$x[n] \text{ is } \textbf{even} \text{ if } x[n] = x[-n] \tag{9.12}$$

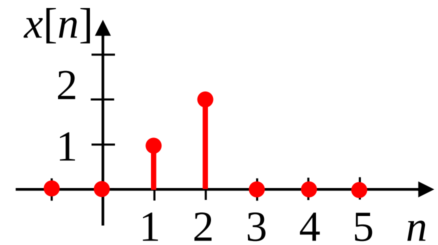
$$x[n] \text{ is } \textbf{odd} \text{ if } x[n] = -x[-n] \tag{9.13}$$

Any discrete-time signal $x[n]$ can be represented as the sum of an even and an odd component.

$$\begin{aligned} x[n] &= \underbrace{\frac{1}{2}\left(x[n] + x[-n]\right)}_{x_e[n]} + \underbrace{\frac{1}{2}\left(x[n] - x[-n]\right)}_{x_o[n]} \\ &= x_e[n] + x_o[n] \end{aligned} \tag{9.14}$$

The extension from CT systems here is straightforward.

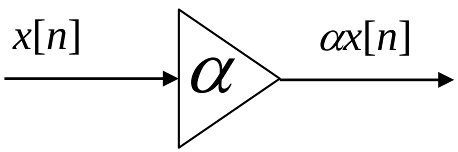
Basic Signal Operations

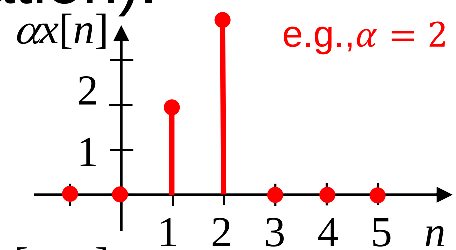


Consider the signal $x[n]$ shown.

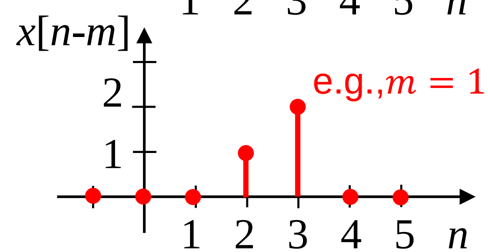
Unary Operators (only one signal input):

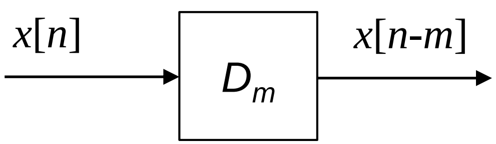
1a. (Amplitude) Scaling (constant multiplication).



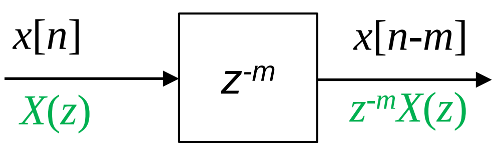


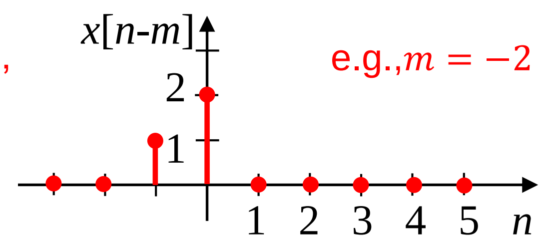
1b. Time shift (delay by m steps, $m \epsilon \mathbb{Z}$):





Aside: We'll see the DT unit delay (z^{-1}) is analogous to a CT integrator, as a "storage element" involving an I.C.,

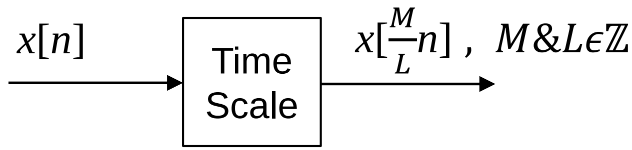


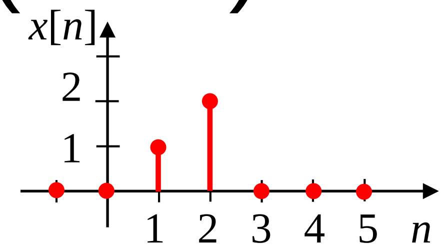


Z-Transforms will be explored in Slide Set 9

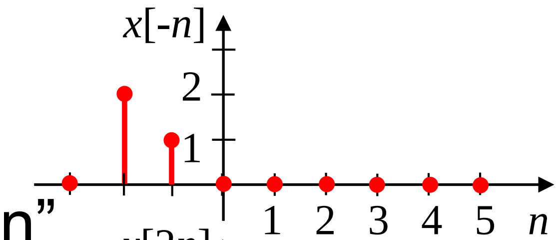
Basic Signal Operations (cont.)

1c. Time Scaling:

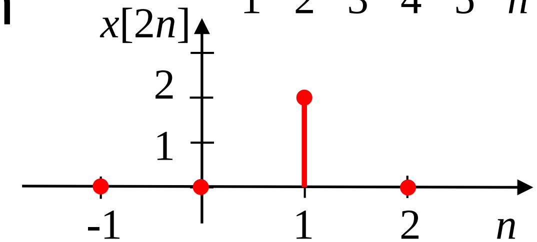




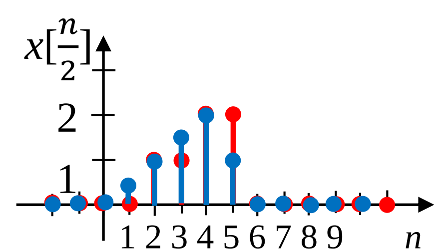
- Time reversal (reflection)



- Downsampling (M>L, aka “Decimation” or “Compression”): needed for multi-rate systems; discard samples if L=1, otherwise, different algorithms available.



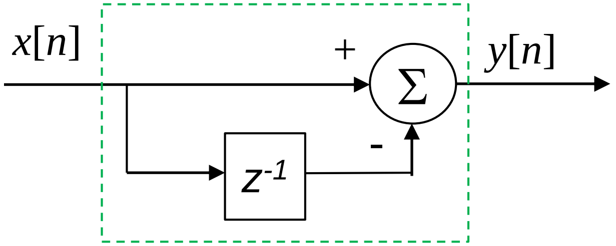
- Upsampling (L>M, aka “Interpolation” or “Expansion”): needed for multi-rate systems; different algorithms available, depending on application.



Basic Signal Operations (cont.)

1d. Difference Operator (cf. CT differentiator):

- Backward Difference: $y[n] = x[n] - x[n-1]$



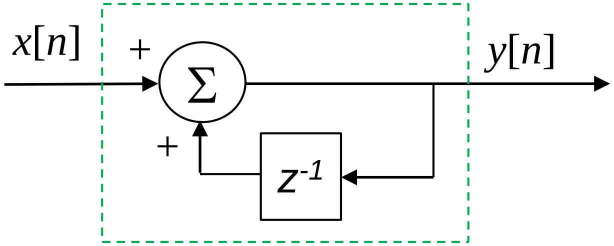
If $x[n] = v(nT_s)$ $(i.e, x[n]$ is a sampled version of the CT signal $v(t))$, then

$$\left.\frac{dv(t)}{dt}\right|_{t=nT_s} \approx \frac{y[n]}{T_s}$$

- Forward Difference: $y[n] = x[n+1] - x[n]$

1e. Accumulator (cf. CT integrator): $y[n] = \sum_{k=-\infty}^{n} x[k]$

$$= y[n-1] + x[n]$$



If $x[n] = v(nT_s)$, then

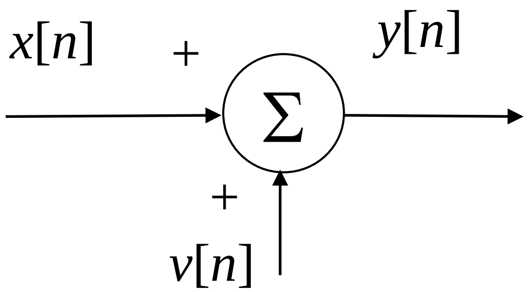
$$\int_{-\infty}^{nT_s} v(\tau)d\tau \approx y[n] \cdot T_s$$

Basic Signal Operations (cont.)

Binary Operators

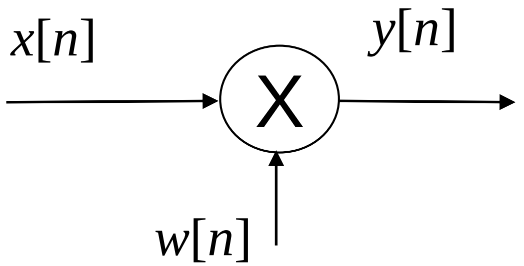
2a. Adder (aka “Summer”):

$$y[n] = x[n] + v[n]$$



2b. Time Windowing/Modulation:

$$y[n] = x[n] \cdot w[n]$$



Caveat: When operating on two sampled signals, be certain the effective sampling rate is the same. For multirate systems, this is one of the reasons that some signals may need interpolation/decimation.

Basic DT Signals: ℂ-Exponential

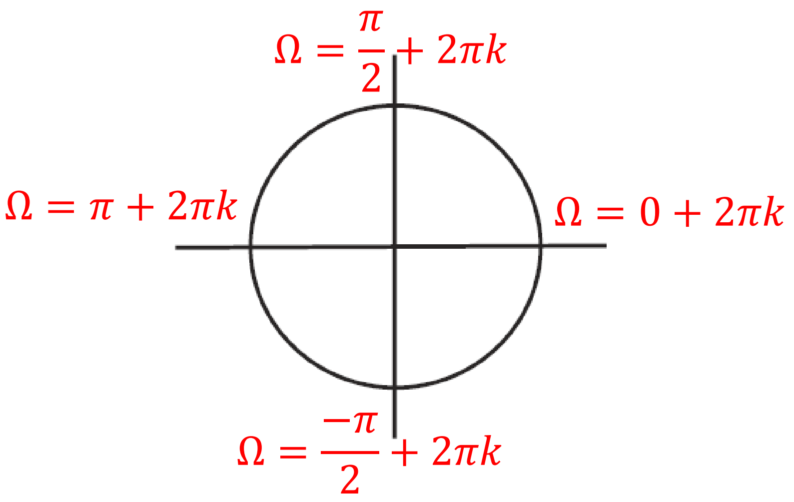
Given complex numbers $A = |A|e^{j\theta}$ and $\alpha = |\alpha|e^{j\Omega_0}$, a **discrete-time complex exponential** is a signal of the form

$$x[n] = A\alpha^n = |A||\alpha|^n e^{j(\Omega_0 n+\theta)}$$
$$= |A||\alpha|^n [\cos(\Omega_0 n + \theta) + j\sin(\Omega_0 n + \theta)] \quad (9.16)$$

where Ω_0 is a discrete frequency in radians/step.

Compare this to the CT ℂ-exponential $x(t) = Ae^{st}$ where $s = \sigma + j\omega$. Setting the sampling interval to T, we get $x(nT) = Ae^{snT} = A(e^{sT})^n$. Thus, the DT version can be determined by setting $\alpha = e^{sT}$.

DT frequencies (in rad/sample) are effectively periodic in the frequency domain with period 2π (i.e., $x[n] = Ae^{j(\Omega n+\theta)} = Ae^{j((\Omega+2\pi k)n+\theta)}\ \forall n, k \epsilon \mathbb{Z}$). Thus, we restrict frequencies to $-\pi < \Omega \leq \pi$.



"Correct" frequency

e.g.: $e^{j\left(\frac{19\pi}{6}n\right)} = e^{j\left(\frac{7\pi}{6}n\right)} = e^{j\left(\frac{-5\pi}{6}n\right)}\ \forall n \epsilon \mathbb{Z}$.

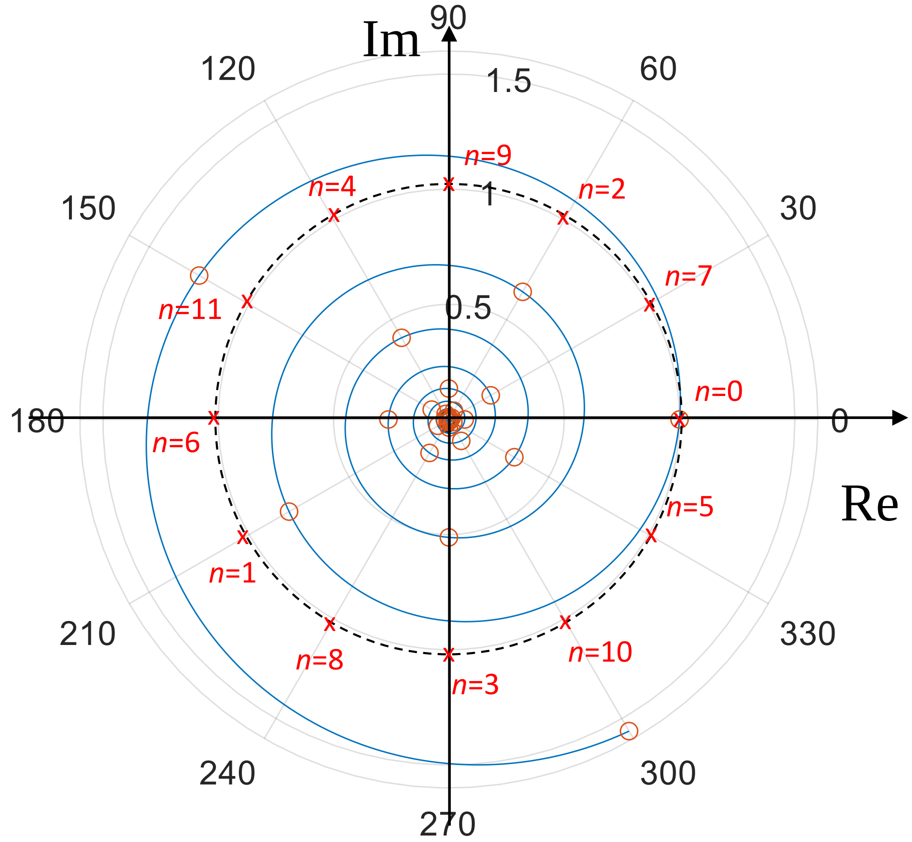
Eg. Plots of ℂ-Exponential in ℂ-Plane

$$x_1[n] = \mathrm{e}^{j\left(\frac{-5\pi}{6}n\right)}$$

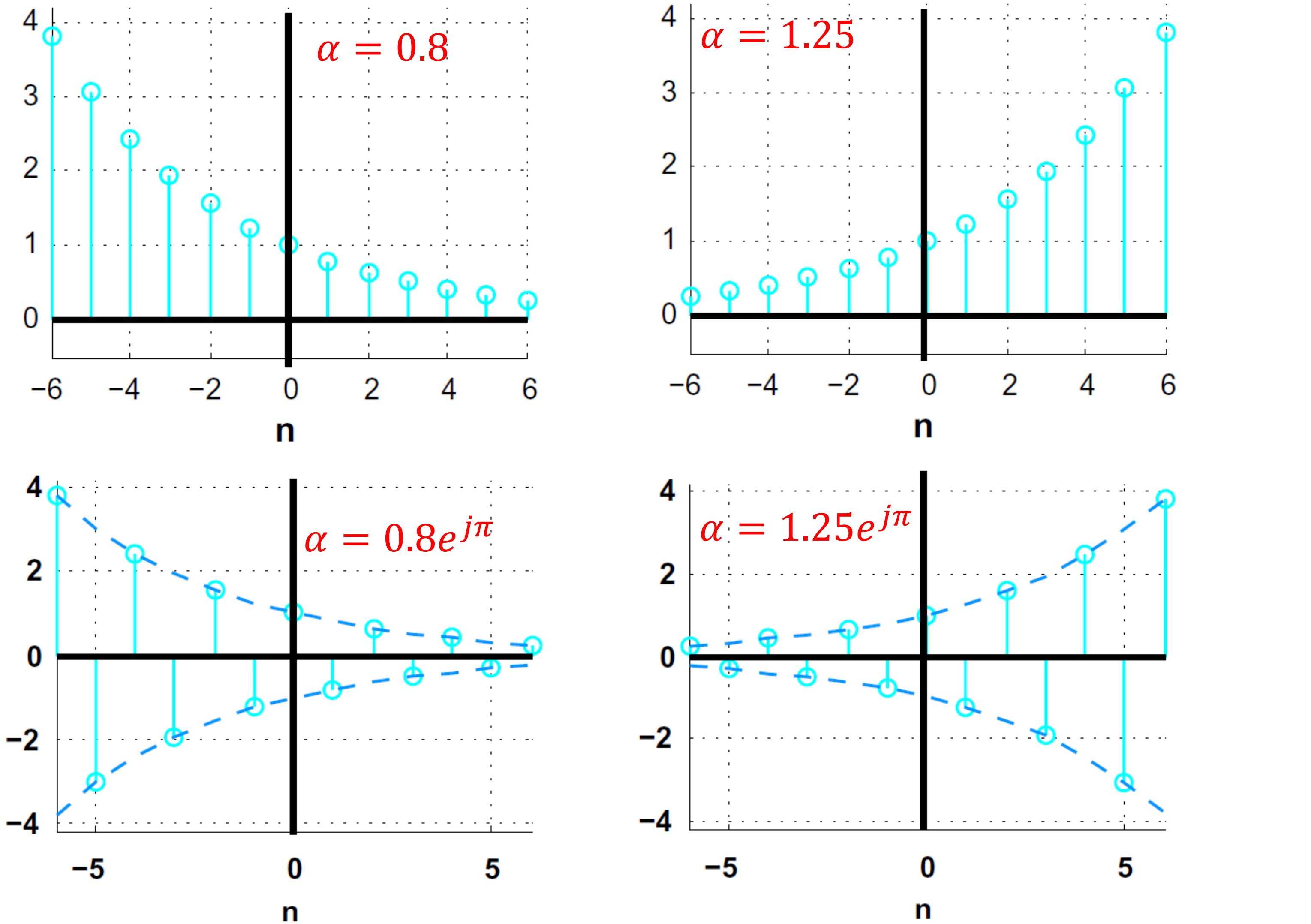
What is the period? 12

What is smallest n so $\sum_{k=0}^{n} x_1[k] = 0$? *n*=11

$$x_2[n] = \left(\frac{4}{5} e^{j\frac{-5\pi}{6}}\right)^n$$



Chaparro Ex 9.12 (Adapted): $x[n] = \alpha^n$



Basic DT Signals: Sinusoids

DT sinusoids can be written as the real part of a DT $\mathbb{C}$-exponential when $|\alpha|$=1 (so the magnitude doesn't change), $A \epsilon \mathbb{R}$:

$$x[n] = Re\left(Ae^{j(\Omega_0 n+\theta)}\right) = Acos(\Omega_0 n + \theta)$$

$$= A\frac{e^{j(\Omega_0 n+\theta)}+e^{-j(\Omega_0 n+\theta)}}{2} = Acos(-\Omega_0 n - \theta)$$

For DT $\mathbb{C}$-exponentials, we restricted $-\pi < \Omega \leq \pi$. In the case of DT sinusoids, we *can* further restrict frequencies to $0 \leq \Omega \leq \pi$. The fact that cosine is even allows us to reflect any negative frequencies into positive ones.

Chaparro Ex 9.14

For the following sinusoids, determine if they are periodic and, if so, their corresponding fundamental periods. Also comment as to whether they appear to be sampled sinusoids.

$x_1[n] = \sin(0.1\pi n)$ $\Rightarrow 2\pi \frac{m_1}{N_1} = 0.1\pi \Rightarrow \frac{m_1}{N_1} = \frac{1}{20}$

$x_2[n] = \sin(0.2\pi n)$ $\Rightarrow 2\pi \frac{m_2}{N_2} = 0.2\pi \Rightarrow \frac{m_2}{N_2} = \frac{1}{10}$

$x_3[n] = \sin(0.6\pi n)$ $\Rightarrow 2\pi \frac{m_3}{N_3} = 0.6\pi \Rightarrow \frac{m_3}{N_3} = \frac{3}{10}$

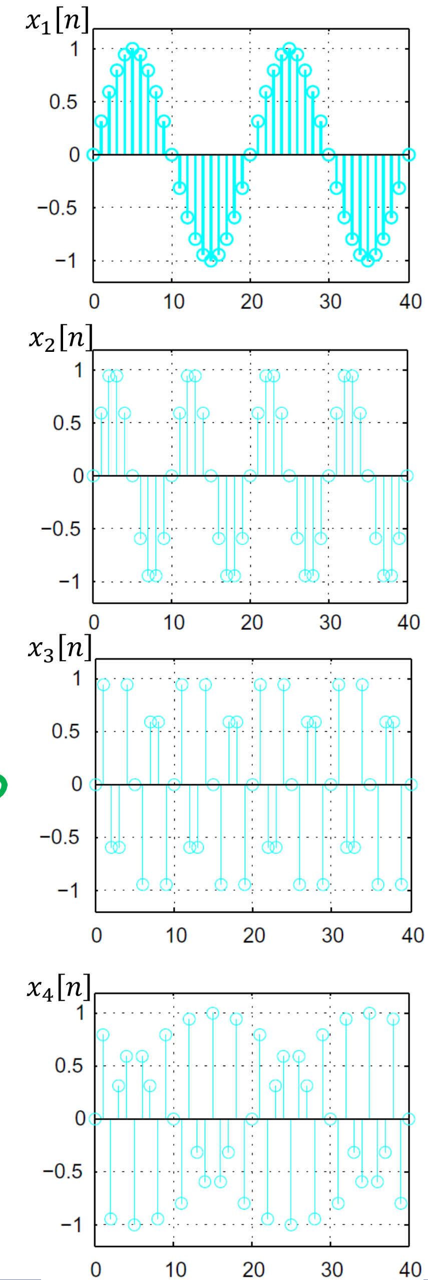
$x_4[n] = \sin(0.7\pi n)$ $\Rightarrow 2\pi \frac{m_4}{N_4} = 0.7\pi \Rightarrow \frac{m_4}{N_4} = \frac{7}{20}$

x_1 & x_4 HAVE PERIOD $N=20$

x_2 & x_3 HAVE PERIOD $N=10$

$x_1[n]$ & $x_2[n]$ appear as sinusoids.

$x_3[n]$ & $x_4[n]$ visually do not appear as sinusoids but if they are put through an ideal interpolator with $T_s = 0.1s$, a single frequency would be recovered in each case.



Basic DT Signals: Unit-Step & Impulse

The unit-step $u[n]$ and the unit-sample $\delta[n]$ discrete-time signals are defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{9.20}$$

a.k.a. (unit) impulse
or Kronecker Delta

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.21}$$

These two signals are related as follows

Change of variables:

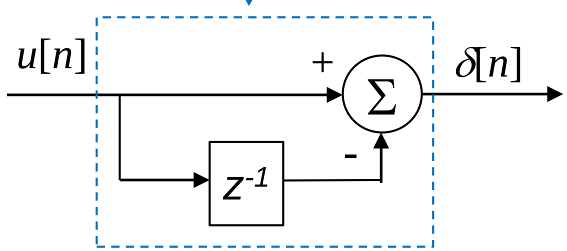
$$m = n - k \Rightarrow \begin{cases} k = 0 \Rightarrow m = n \\ k = \infty \Rightarrow m = -\infty \end{cases}$$

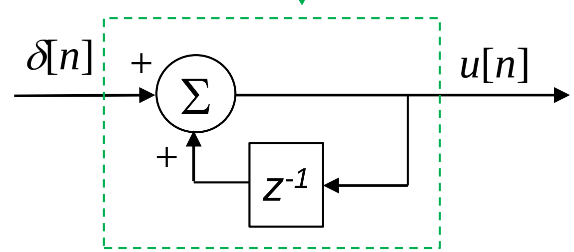
$$\delta[n] = u[n] - u[n-1] \tag{9.22}$$

Backward
Difference

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m] \tag{9.23}$$

Accumulator





Sifting Property for Generic DT Signal

Any discrete-time signal $x[n]$ is represented using unit-sample signals as

ANALOGY: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{9.24}$$

Chaparro Ex 9.15: Consider a discrete pulse $x[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0 \quad otherwise \end{cases}$.

Obtain representations of *x[n]* first using a linear combination of unit-sample signals and then a combination of unit-step signals.

$x[n] = \sum_{k=0}^{N-1} \delta[n-k]$ } USING UNIT-SAMPLE (IMPULSE) SIGNALS

$= u[n] - u[n-N]$ } USING UNIT STEP SIGNALS

Discrete-Time LTI Systems

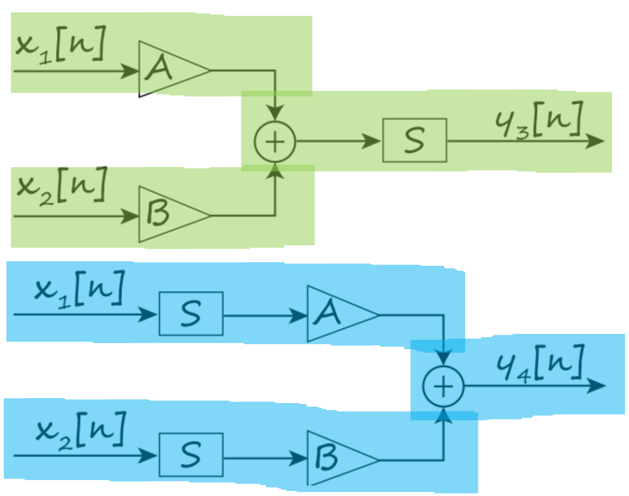
A discrete-time system $\mathcal{S}$ is said to be

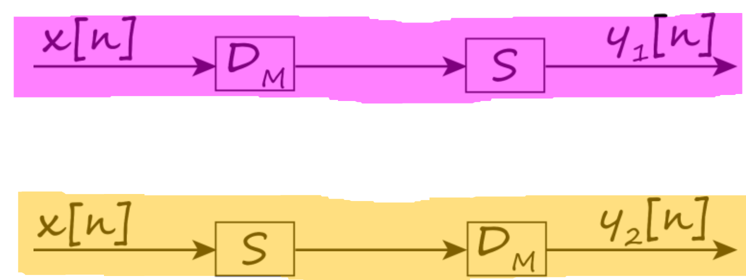
1. **Linear:** if for inputs $x[n]$ and $v[n]$, and constants a and b, it satisfies the following
   - **Scaling**: $\mathcal{S}\{ax[n]\} = a\mathcal{S}\{x[n]\}$
   - **Additivity**: $\mathcal{S}\{x[n] + v[n]\} = \mathcal{S}\{x[n]\} + \mathcal{S}\{v[n]\}$

   or equivalently if **superposition** applies, i.e.,

$$\mathcal{S}\{ax[n] + bv[n]\} = a\mathcal{S}\{x[n]\} + b\mathcal{S}\{v[n]\} \tag{9.26}$$

2. **Time-invariant:** if for an input $x[n]$ the corresponding output is $y[n] = \mathcal{S}\{x[n]\}$, the output corresponding to an advanced or a delayed version of $x[n]$, $x[n \pm M]$, for an integer M, is $y[n \pm M] = \mathcal{S}\{x[n \pm M]\}$, or the same as before but shifted as the input. In other words, the system is not changing with time. i.e. S and D_m commute





Recursive and Non-Recursive Systems

Depending on the relation between the input $x[n]$ and the output $y[n]$ two types of discrete-time systems of interest are:

- **Recursive system**

i.e: OUTPUT AT STEP n DEPENDS ON PREVIOUS OUTPUT VALUES

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m] \qquad n \geq 0 \tag{9.27}$$

initial conditions $y[-k], k = 1, \ldots, N$

This system is also called an **infinite impulse response (IIR)** system.

- **Non-recursive system**

i.e., if $x[n] = \delta[n]$ then $\nexists k\ s.t.\ y[n] = 0 \forall n \geq k$

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] \tag{9.28}$$

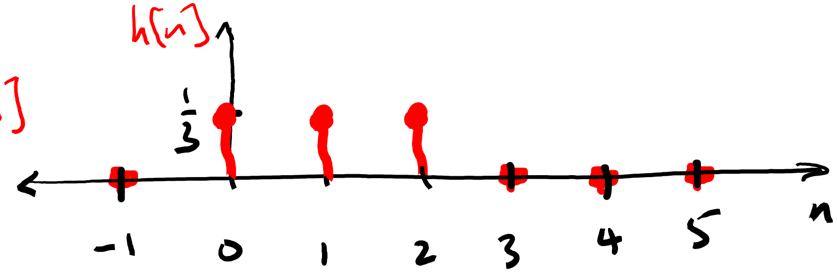
Observe that if $x[n] = \delta[n]$ then $y[n] = 0 \forall n > M$

This system is also called a **finite impulse response (FIR)** system.

Both of these can be shown to be LTI systems ((9.28) might be viewed as a special case of (9.27)).

Chaparro Ex. 9.19: Moving-Average

Consider the FIR system, *S*, relating input x[n] to output y[n] as follows: $y[n]=\frac{1}{3}(x[n]+x[n-1]+x[n-2])$. Plot the impulse response, h$[n]$, and check whether the system is LTI.



LINEARITY CHECK : LET $x[n]=\alpha x_1[n]+\beta x_2[n]$ AND $y_1=S(x_1)$, $y_2=S(x_2)$.

$\Rightarrow y[n]=\frac{1}{3}\{(\alpha x_1[n]+\beta x_2[n])+(\alpha x_1[n-1]+\beta x_2[n-1])+(\alpha x_1[n-2]+\beta x_2[n-2])\}$

$=\alpha y_1[n]+\beta y_2[n]$ $\Rightarrow$ SUPERPOSITION APPLIES

Linear☑

TIME-INVARIANCE CHECK: $D_M(S(x[n]))=\frac{1}{3}(x[n-M]+x[n-M-1]+x[n-M-2])$

$S(D_M(x[n]))=S(x[n-M])=\frac{1}{3}(x[n-M]+x[n-M-1]+x[n-M-2])$

Time − Invariant☑

Chaparro Ex. 9.20: Autoregression

Consider the recursive (**autoregressive**, or AR) discrete-time system represented by the following 1st-order difference equation:

$$y[n] = ay[n-1] + bx[n].$$

With an IC of $y[-1]$=0, determine the impulse response.

IF $x[n] = \delta[n]$

$\Rightarrow$

n	$x[n]$	$y[n]$
-1	0	0
0	1	b
1	0	ab
2	0	a^2b
3	0	a^3b
$\vdots$		
k	0	a^kb

$\Rightarrow \forall n \geq 0 \quad \underline{h[n] = a^n b}$ IS THE IMPULSE RESPONSE

(OR $h[n] = a^n b\, u[n]$).

ASIDE: FOR ARBITRARY CAUSAL INPUT $x[n]$

$$\Rightarrow y[n] = \sum_{k=0}^{n} ba^k x[n-k] = \underbrace{\sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} h[n-k]x[k]}_{\text{CONVOLUTION SUM}}$$

Representation by Difference Equations

A recursive discrete-time system is represented by a difference equation

System order=max(N,M)
=required # of memory elements

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m] \qquad n \geq 0 \qquad (9.30)$$

initial conditions $y[-k], k = 1, \ldots, N$

which naturally characterizes the dynamics of the system.

$\Rightarrow$Char eqn becomes $1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N} = 0 = z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N$

The complete response of a system represented by the difference equation can be shown to be composed of a **zero-input** and a **zero-state** responses, i.e., if $y[n]$ is the solution of the difference Equation (9.30) with initial conditions not necessarily equal to zero, then

$= y[n]\big|_{x[n]\equiv 0}$ $\qquad = y[n]\big|_{y[k]=0\,\forall k<0}$ (*i.e., ICs all* 0)

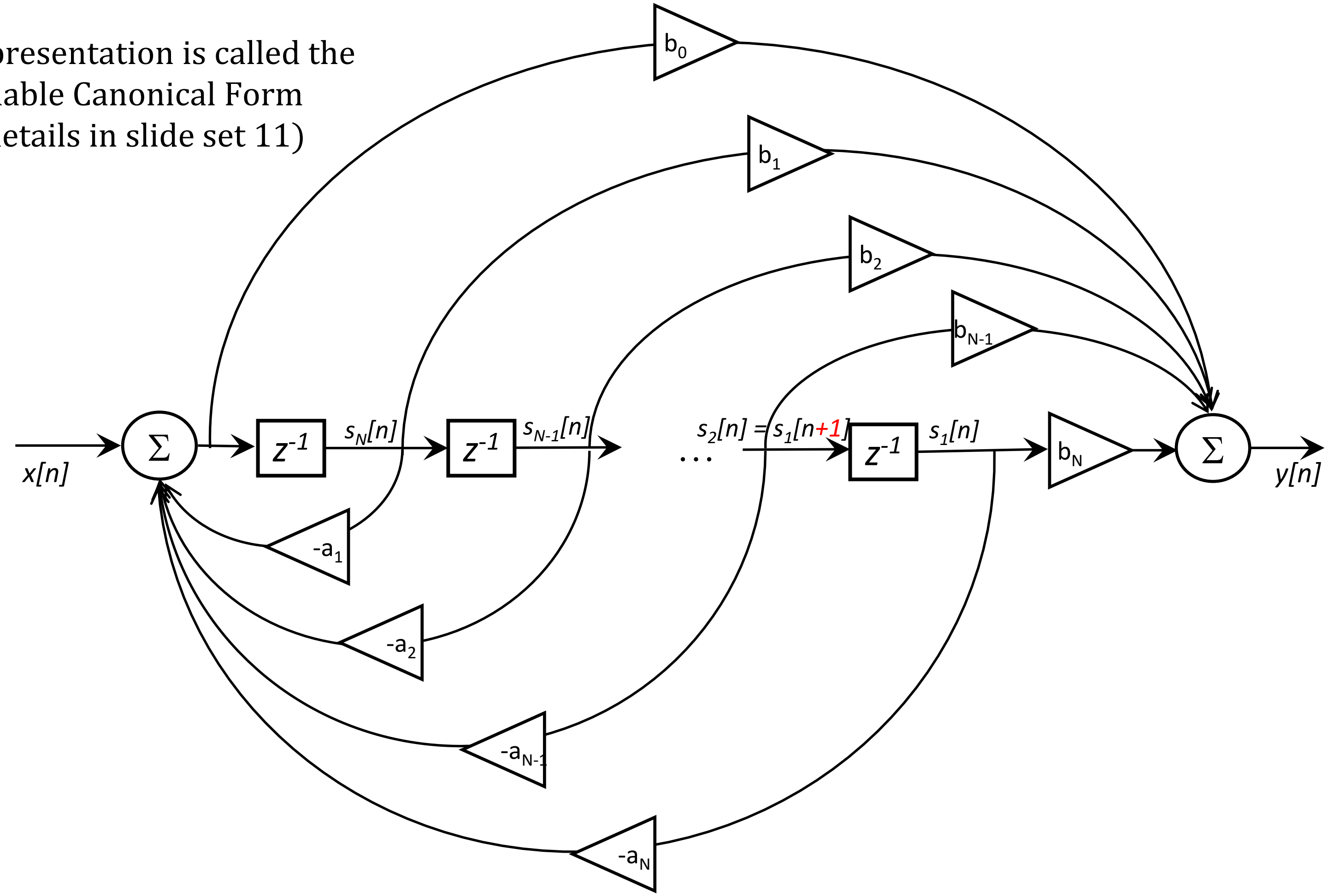
$$y[n] = y_{zi}[n] + y_{zs}[n]. \qquad (9.31)$$

As we might expect, we might also find a decomposition into **steady-state** and **transient** responses.

Nth Order Difference Equation CCF

A block diagram for Difference Equation: $y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + \cdots + b_N x[n-N]$

This representation is called the Controllable Canonical Form (more details in slide set 11)



Convolution Sum

Let $h[n]$ be the **impulse response** of a linear time-invariant (LTI) discrete-time system, or the output of the system corresponding to an impulse $\delta[n]$ as input, and initial conditions (if needed) equal to zero.

Using the **generic representation** of the input $x[n]$ of the LTI system

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{9.32}$$

the output of the LTI system is given by either of the following equivalent forms of the **convolution sum**:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] \tag{9.33}$$

Causal Systems

A discrete-time system $\mathcal{S}$ is **causal** if:

- whenever the input $x[n] = 0$, and there are no initial conditions, the output is $y[n] = 0$,
- the present output $y[n]$ does not depend on future inputs.

- An LTI discrete-time system is **causal** if the impulse response of the system is such that

$$h[n] = 0 \qquad n < 0 \tag{9.35}$$

- A signal $x[n]$ is said to be **causal** if

$$x[n] = 0 \qquad n < 0 \tag{9.36}$$

- For a causal LTI discrete-time system with a causal input $x[n]$ its output y[n] is given by

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] \qquad n \geq 0 \tag{9.37}$$

where the lower limit of the sum depends on the input causality, $x[k] = 0$ for $k < 0$, and the upper limit on the causality of the system, $h[n-k] = 0$ for $n - k < 0$ or $k > n$.

DT BIBO Stability

An LTI discrete-time system is said to be BIBO stable if its impulse response $h[n]$ is absolutely summable

$$\sum_{k} |h[k]| < \infty \qquad (9.38)$$

DT Internal Stability

Internal stability: *zero input, arbitrary ICs, output is z[n]*

	Marginally stable	**Asymptotically stable**
Definition	$\forall n \epsilon \mathbb{N} = \mathbb{Z}_+, \|z[n]\| \leq M < \infty$ i.e., with no input, output never exceeds a bound	$\lim_{n \to \infty} \|z[n]\| = 0$
LTI Conditions	$\|\lambda_i\|<1$, for repeated eigenvalues $\|\lambda_i\| \leq 1$, for nonrepeated eigenvalues	$\|\lambda_i\|<1$ i.e., poles of TF must be in unit circle

cf CT BIBO Stability

A LTI system is **bounded-input bounded-output (BIBO) stable** provided that the system impulse response *h(t)* is **absolutely integrable,** i.e.,

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \tag{2.23}$$

cf CT Internal Stability

Internal stability: *zero input, arbitrary ICs, output is z(t)*

	Marginally stable	**Asymptotically stable**
Definition	$\forall t \epsilon \mathbb{R}_+, \|z(t)\| \leq M < \infty$	$\lim_{t\to\infty} \|z(t)\| = 0$
LTI Conditions	Re(λ_i)<0, for repeated eigenvalues Re(λ_i)≤0, for nonrepeated eigenvalues	Re(λ_i)<0 i.e., poles of TF must be open LHP