1^{st} Order Linear ODEs

ODE with constant coefficient, a, and forcing function (input), $f(t)$. If $f(t) \equiv 0$, the system is said to be "source-free" and (*) is the *homogenous equation*.

Problem Definition:

$$\text{Given } \begin{cases} \boxed{\dfrac{dy(t)}{dt} + ay(t) = f(t)} & (*) \\ \boxed{y(t_0) = y_0} & (**) \end{cases}, \text{ find } y(t).$$

Initial Condition (IC)

A Fundamental Theorem of Linear DEs:

Suppose that $y_p(t)$ is any particular integral solution to (*) and $y_c(t)$ is any complementary solution to the homogenous equation. I.e.,

Soln: $y_c(t) = Ke^{-at}$

$$\frac{dy_p(t)}{dt} + ay_p(t) = f(t) \text{ and } \boxed{\frac{dy_c(t)}{dt} + ay_c(t) = 0}$$

$$\boxed{\Rightarrow y(t) = y_p(t) + y_c(t) \text{ is also a solution to } (*).}$$

NB: This is therefore also a "particular solution". There are an infinite number of complementary solutions and of particular solutions. It is the I.C. that uniquely identifies the correct solution.

Complete Response

Particular integral solution ($y_p(t)$)

- Sometimes, use forced response, a.k.a. zero-state response (ZSR), the unique particular solution satisfying zero IC.
- Sometimes more convenient to use steady-state (permanent) response.

Complementary solution ($y_c(t)$)

- Sometimes, use natural response, a.k.a. zero-input response (ZIR), the unique complementary solution satisfying the given IC.
- Sometimes more convenient to use transient response (vanishes as $t \to \infty$).

$\Rightarrow$ Complete response = $\boxed{\text{Forced response}}$ + $\boxed{\text{Natural response}}$

Due to $f(t)$; compute assuming $y_p(0) = 0$

Due to IC $y_c(0) = y_0$; take $f(t) \equiv 0$

$\Rightarrow$ Complete response = $\boxed{\text{Steady-state}}$ + $\boxed{\text{Transient}}$

Aside: This form can't be used for unstable systems.

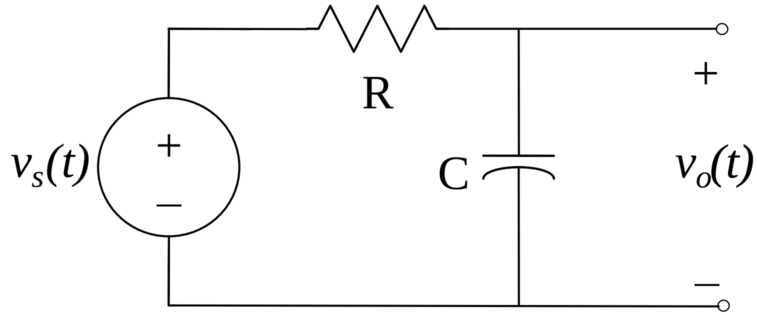
Periodic (including DC)

- Constant if DC input
- If Sinusoidal input then sinusoid of same frequency.

Usually combination of decaying exponentials

Example: RC Circuit

Given $v_s(t) = V_s$ for $t \geq 0$ and $v_0(0) = V_0$, determine the steady-state response, the transient response, the ZIR, the ZSR and the complete response (for $t \geq 0$).



KCL: $\frac{v_s - v_o}{R} = C\dot{v}_o \Leftrightarrow \underbrace{\dot{v}_o + \frac{1}{RC}v_o = \frac{1}{RC}v_s}$

$\Rightarrow v_{o,c}(t) = Ke^{-t/RC}$

COMPLETE SOLUTION: $v_o(t) = V_s + (V_o - V_s)e^{-t/RC}$

STEADY-STATE RESPONSE: $v_{o,ss}(t) = V_s$

TRANSIENT RESPONSE: $v_{o,trans}(t) = (V_o - V_s)e^{-t/RC}$

ZIR : $v_{o,ZIR}(t) = V_o e^{-t/RC}$

ZSR : $v_{o,ZSR}(t) = V_s(1 - e^{-t/RC})$

2^{nd} Order Linear ODEs

Problem Definition:

Given $\left\{\begin{array}{l} \dfrac{d^2y(t)}{dt^2} + a_1\dfrac{dy(t)}{dt} + a_0 y(t) = f(t) \ \ (*) \\ y(t_0) = y_0 \ and \ \left.\dfrac{dy(t)}{dt}\right|_{t=t_0} = y_0' \ \ (**) \end{array}\right\}$, find $y(t)$.

As before: $\Rightarrow y(t) = y_p(t) + y_c(t)$ is also a solution to (*).

We focus on the homogenous equation and rewrite DE coeffs:

$$\frac{d^2y_c(t)}{dt^2} + 2\zeta\omega_0\frac{dy_c(t)}{dt} + \omega_0^2 y_c(t) = 0 \ \ (***)$$

where ζ is damping ratio and ω_0 is natural (undamped) frequency.

Assume $y_c(t) = Ke^{st}$. $(***) \Rightarrow s^2(Ke^{st}) + 2\zeta\omega_0 s(Ke^{st}) + \omega_0^2(Ke^{st}) = 0$

$\Rightarrow$ Characteristic Equation: $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$

Characteristic Equation

$$\boxed{s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0}$$

Roots provide useful information about system qualitative (especially transient) characteristics.

Quadratic equation yields 2 roots (eigenvalues): $s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$

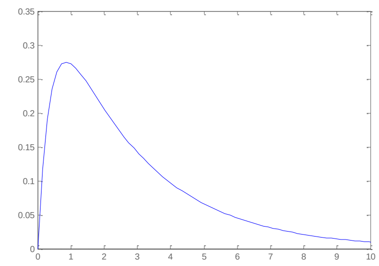
For stable systems, investigate 3 cases of ζ.

Case 1: $\zeta > 1$ (overdamped)

$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

(two negative real and unequal roots)

Solution has form:
$y_c(t) = Ae^{s_1 t} + Be^{s_2 t}$

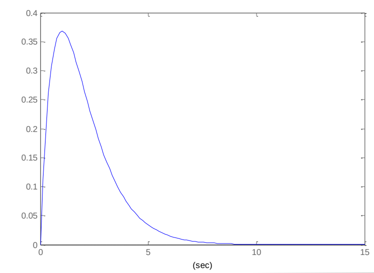


Case 2: $\zeta = 1$ (critically damped)

$$s_{1,2} = -\omega_0$$

(single repeated negative root)

Solution has form:
$y_c(t) = e^{-\omega_0 t}[A + Bt]$



Case 3: $1 > \zeta > 0$ (underdamped)

$$s_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$$
$$= \sigma \pm j\omega_d$$

Damped frequency: $\omega_d = \omega_0\sqrt{1 - \zeta^2}$

(complex-conjugate pair of roots)

Solution has form:
$y_c(t) = e^{\sigma t} C\cos(\omega_d t + \phi)$
$= e^{\sigma t}[A\cos(\omega_d t) + B\sin(\omega_d t)]$