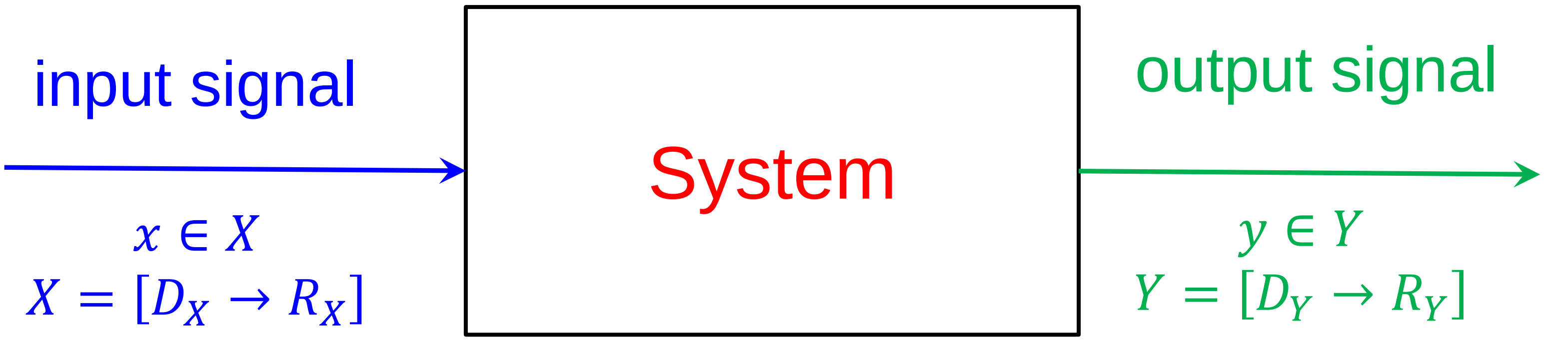
# Continuous-Time Systems

- System Definition
- System Classification
- Linear Time Invariant Systems
  - Linearity
  - Time Invariance
  - Modeling
  - Convolution
- Causality
- Stability

# System Mathematical Representation

A system is a *function* that maps a *domain signal* into a *range signal:*




$$S: [D_X \rightarrow R_X] \rightarrow [D_Y \rightarrow R_Y]$$
$$: X \rightarrow Y$$

$$y = S(x) \qquad \forall z \in D_Y, y(z) = S(x)(z) \in R_Y$$

# Some System Classification

Lumped parameter vs. Distributed parameter

Deterministic vs. Stochastic

Memoryless vs. Nonmemoryless

Continuous-Time vs. Hybrid vs. Discrete-Time

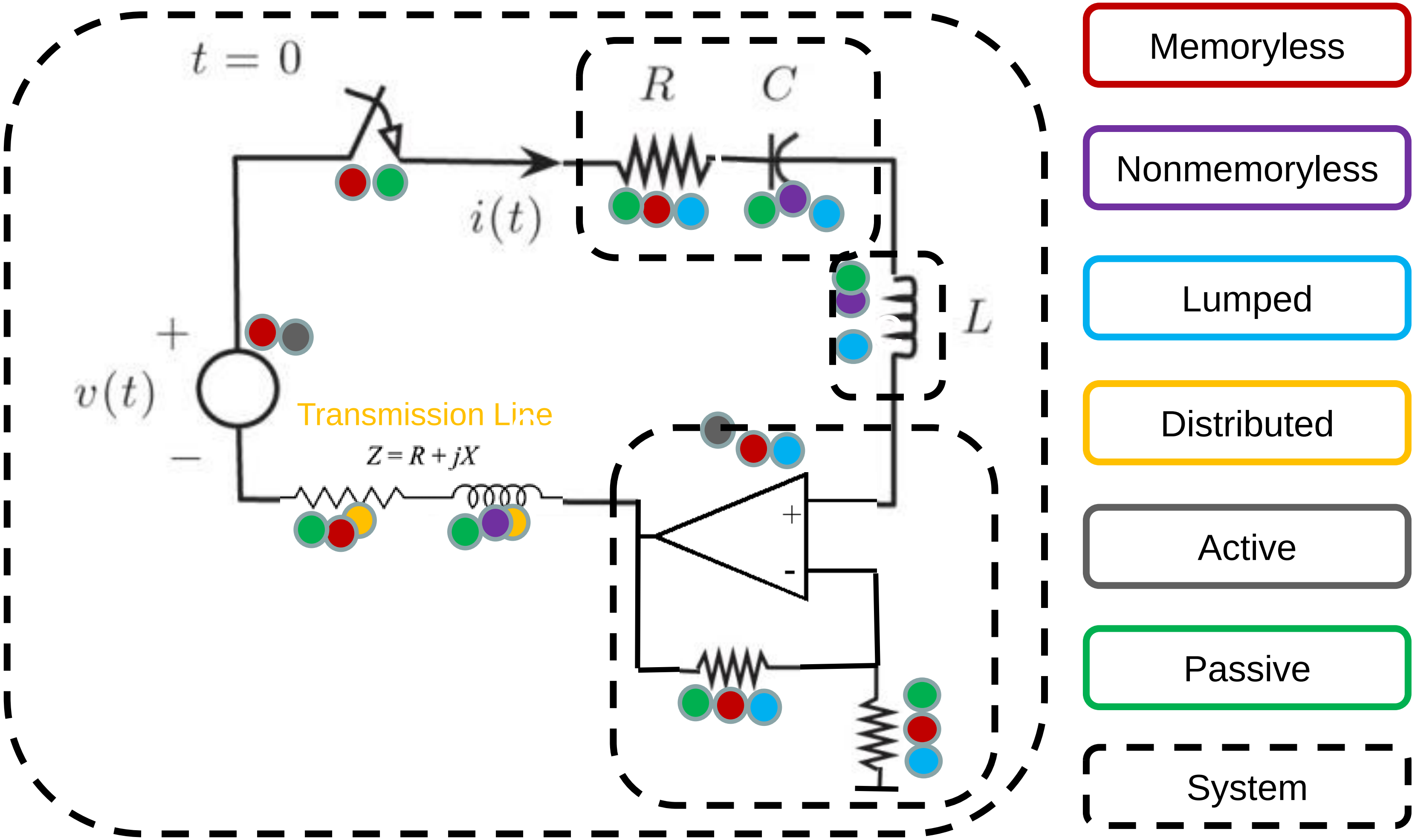
Linear vs. Nonlinear

Time invariant vs. Time varying

Causal vs. Acausal

Stable vs. Unstable

# Classification Example



# Memoryless Systems$\subset$Causal Systems

- A **Memoryless System** is one in which the instantaneous output at any given time is independent of the input at any other time: Let $y_i = S(x_i), i \in \{1,2\}$

$$\forall x_1, x_2 \in X, \forall t_0 \in \mathbb{R}, \big(x_1(t_0) = x_2(t_0)\big) \Rightarrow \big(y_1(t_0) = y_2(t_0)\big)$$

- A **Causal System** is one in which the instantaneous output at any given time is independent of any future inputs (it only depends on the input up to that time): Let $y_i = S(x_i), i \in \{1,2\}$

$$\forall x_1, x_2 \in X, \forall t_0 \in \mathbb{R},$$
$$\left(x_1(t) = x_2(t) \forall t \le t_0\right) \Rightarrow \big(y_1(t_0) = y_2(t_0)\big)$$

- Every Memoryless System is Causal though most Causal Systems are not Memoryless.

# Linearity

**A system $S$ is linear**

2 DIFFERENT TESTS

$\Updownarrow$ $(iff)$

Homogeneity (aka "Scaling"):

$$\forall x \in [\mathbb{R} \to \mathbb{C}], \forall a \in \mathbb{C}, \quad S(ax) = aS(x)$$

(i.e.: I/O behaviours of red boxes are indistinguishable)

**and** Additivity:

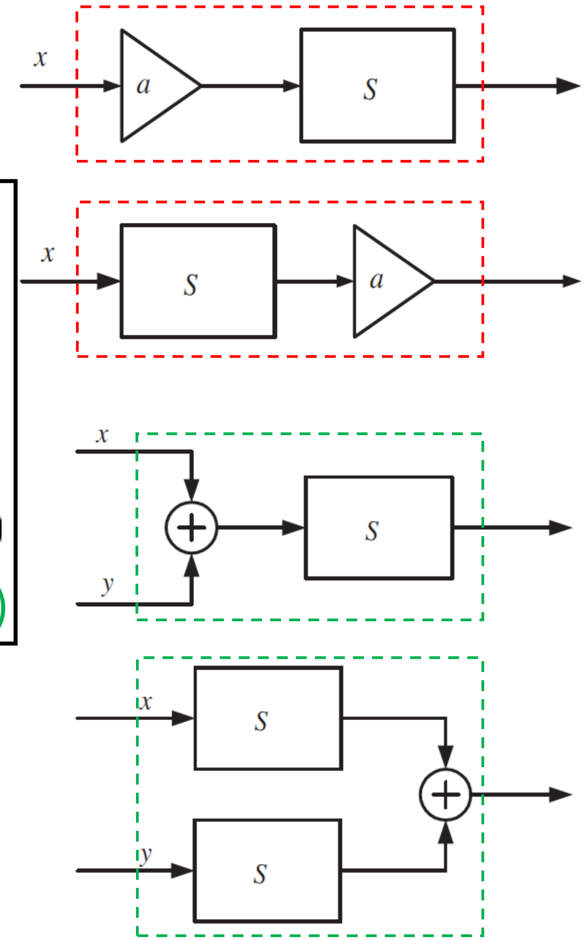
$$\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \quad S(x_1 + x_2) = S(x_1) + S(x_2)$$

(i.e.: I/O behaviours of green boxes are indistinguishable)

$\Updownarrow$ $(iff)$

Combined: $\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \forall a, b \in \mathbb{C}$,
$S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$

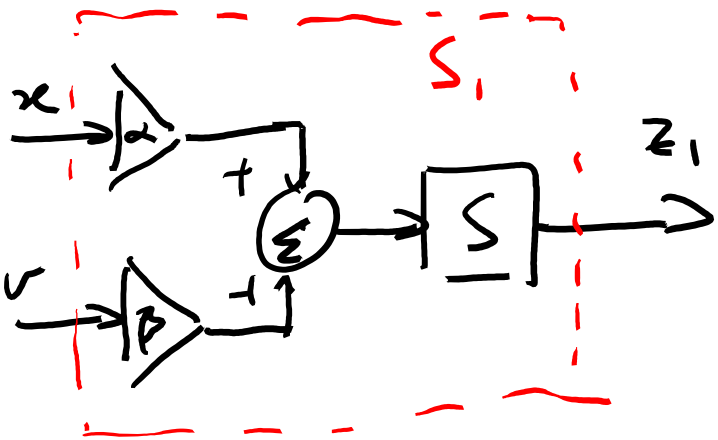
(1 TEST

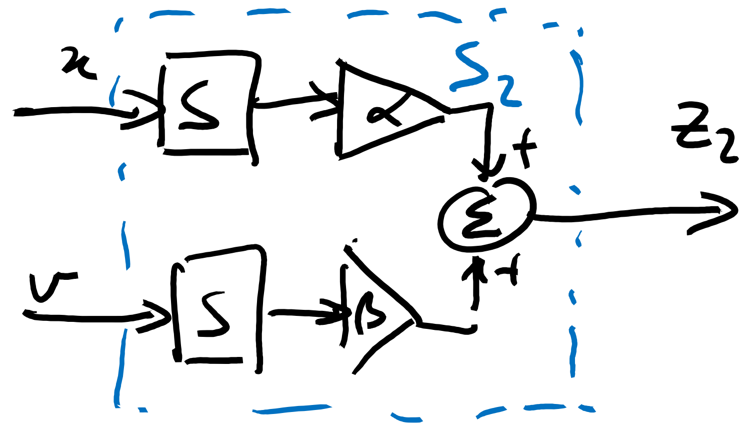


# Linearity

A system represented by $\mathcal{S}$ is said to be **linear** if for inputs *x(t)* and *v(t)*, and any constants $\alpha$ and $\beta$, **superposition** holds, that is

$$\boxed{\mathcal{S}[\alpha x(t) + \beta v(t)]} = \mathcal{S}[\alpha x(t)] + \mathcal{S}[\beta v(t)] \\ = \boxed{\alpha\mathcal{S}[x(t)] + \beta\mathcal{S}[v(t)]} \qquad (2.2)$$




S IS LINEAR $\Leftrightarrow z_1(t) = S_1(x(t), v(t)) = S_2(x(t), v(t)) = z_2(t)$

$\forall t \in \mathbb{R}, \forall x, v \in X, \forall \alpha, \beta \in \mathbb{C}$

# Linearity Check Examples

**Eg1:** $y(t)=S(x(t))=|x(t)|$

$X = [\mathbb{R} \to \mathbb{C}]$

Homogeneity check: $\forall x \in [\mathbb{R} \to \mathbb{C}], \forall a \in \mathbb{C}, \quad S(ax) \overset{?}{=} aS(x)$

$S(ax(t)) = |ax(t)| = |a||x(t)| \neq a|x(t)| = aS(x)$ if $angle(a) \neq 0$

Homogeneity ☒

Additivity check: $\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \quad S(x_1 + x_2) \overset{?}{=} S(x_1) + S(x_2)$

$S(x_1 + x_2) = |x_1 + x_2| \neq |x_1| + |x_2| = S(x_1) + S(x_2)$

Additivity ☒

⇒NONLINEAR (disqualified for Linearity on both checks though either one was sufficient for disqualification)!

Q: Can we have additivity without homogeneity (or the converse)?

A: Yes…that’s why they both are needed for proof!

# Linearity Check Examples

**Eg2:** $y=S(x)=x^*=Re(x)-jIm(x)$ $= |x| e^{-j\theta_x}$ if $x = |x| e^{j\theta_x}$

Homogeneity check: $\forall x \in [\mathbb{R} \to \mathbb{C}], \forall a \in \mathbb{C}, \quad S(ax) \stackrel{?}{=} aS(x)$ $\quad a = |a| e^{j\theta_a}$

$S(ax) = S\left(|a||x| e^{j(\theta_a+\theta_x)}\right)$

$= |a||x| e^{-j(\theta_a+\theta_x)}$

$aS(x) = |a||x| e^{j(\theta_a-\theta_x)}$

$\neq$ HOMOGENEITY ✗

⇒ NONLINEAR

Additivity check: $\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \quad S(x_1+x_2) \stackrel{?}{=} S(x_1) + S(x_2)$

$S(x_1+x_2) = (x_1+x_2)^* = x_1^* + x_2^*$

$S(x_1) + S(x_2) = x_1^* + x_2^*$

$=$ ⇒ ADDITIVITY ✓

# Linearity Check Examples

**Eg3:** $y(t)=S(x(t))=x(2t)$

Superposition check:

$$\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \forall a, b \in \mathbb{C}, \quad \underbrace{S(ax_1 + bx_2)}_{z_1} \overset{?}{=} \underbrace{aS(x_1) + bS(x_2)}_{z_2}$$

$$z_1(t) = S\left(ax_1(t) + bx_2(t)\right) = a x_1(2t) + b x_2(2t)$$

$$z_2(t) = a x_1(2t) + b x_2(2t) = z_1(t)$$

$\Rightarrow$ SUPERPOSITION ✓ $\Rightarrow$ LINEAR

# Linearity Check Examples

$x \rightarrow \boxed{S} \rightarrow y$

Eg4: $y(t) + a_1 \frac{dy(t)}{dt} = b_0 x(t)$

Check: $\forall x_1, x_2 \in [\mathbb{R} \to \mathbb{C}], \forall \alpha, \beta \in \mathbb{C}, S(\alpha x_1 + \beta x_2) \stackrel{?}{=} \alpha S(x_1) + \beta S(x_2)$

Let $z_1 = S_1(x_1, x_2) = LHS$

$$\Rightarrow z_1(t) + a_1 \frac{dz_1(t)}{dt} = b_0(\alpha x_1(t) + \beta x_2(t))$$

Let $z_2 = S_2(x_1, x_2) = RHS$

$\Rightarrow z_2\ (t) = \alpha y_1(t) + \beta y_2(t)$ where $\begin{cases} y_1(t) + a_1 \frac{dy_1(t)}{dt} = b_0 x_1(t) \\ y_2(t) + a_1 \frac{dy_2(t)}{dt} = b_0 x_2(t) \end{cases}$

$$= \alpha \left( b_0 x_1(t) - a_1 \frac{dy_1(t)}{dt} \right) + \beta \left( b_0 x_2(t) - a_1 \frac{dy_2(t)}{dt} \right)$$

$$= b_0(\alpha x_1(t) + \beta x_2(t)) - a_1 \left( \alpha \frac{dy_1(t)}{dt} + \beta \frac{dy_2(t)}{dt} \right)$$

$$= b_0(\alpha x_1(t) + \beta x_2(t)) - a_1 \left( \frac{dz_2(t)}{dt} \right)$$

$\Rightarrow S_1\ \ and\ S_2$ exhibit the same I/O characteristics and are indistinguishable!

$\Rightarrow LHS = RHS$

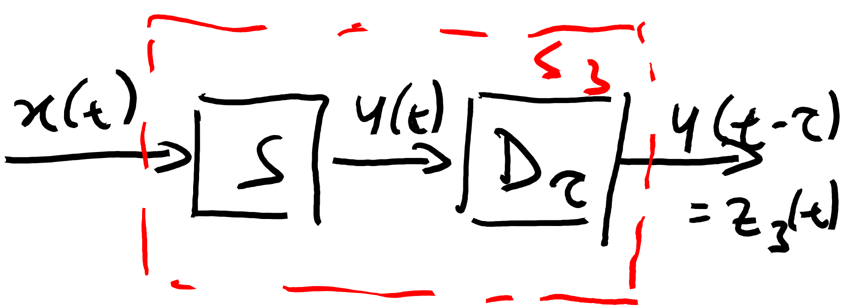
Linearity☑

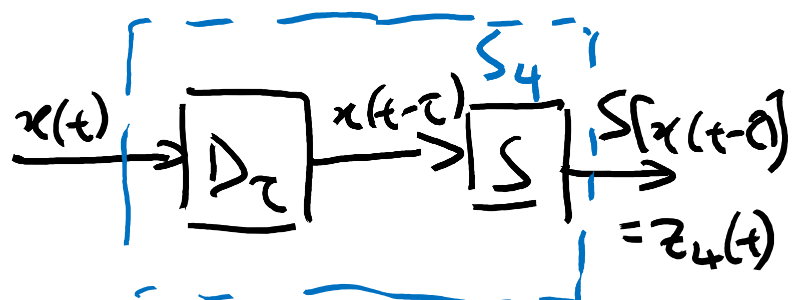
# Time-Invariance

A continuous-time system $\mathcal{S}$ is **time-invariant** if whenever for an input $x(t)$, with a corresponding output $y(t) = \mathcal{S}[x(t)]$, the output corresponding to a shifted input $x(t \mp \tau)$ (delayed or advanced) is the original output equally shifted in time, $y(t \mp \tau) = \mathcal{S}[x(t \mp \tau)]$ (delayed or advanced). Thus

$$
\begin{aligned}
x(t) &\Rightarrow y(t) = \mathcal{S}[x(t)] \\
x(t \mp \tau) &\Rightarrow \boxed{y(t \mp \tau)} = \boxed{\mathcal{S}[x(t \mp \tau)]}
\end{aligned}
\tag{2.6}
$$

That is, the system does not age—its parameters are constant.




$z_3(t) = S_3[x(t)]$ $\qquad$ $z_4(t) = S_4[x(t)]$

S IS TIME-INVARIANT $\Leftrightarrow z_3(t) = z_4(t) \quad \forall t, \forall x \in X$

# TI Check Examples

**Eg1:** $y(t)=S(x(t))=|x(t)|$

$z_3(t) = y(t-\tau) = |x(t-\tau)|$

$z_4(t) = S(x(t-\tau)) = |x(t-\tau)|$

$\Rightarrow$ TIME-INVARIANT ✓

**Eg2:** $y=S(x)=x^*=Re(x)-jIm(x)$ $\Rightarrow y(t) = x^*(t)$

$z_3(t) = y(t-\tau) = x^*(t-\tau)$

$z_4(t) = S(x(t-\tau)) = x^*(t-\tau)$

$\Rightarrow$ TIME-INVARIANT ✓

**Eg3:** $y(t)=S(x(t))=x(2t)$

$z_3(t) = y(t-\tau) = x(2(t-\tau)) = x(2t-2\tau)$

$z_4(t) = S(x(t-\tau)) = x(2t-\tau)$ $\neq$ if $\tau \neq 0$

$\Rightarrow$ TIME-VARYING

# TI Check Examples

Eg4: $y(t) + a_1 \frac{dy(t)}{dt} = b_0 x(t)$

$z_3(t) = y(t-\tau)$

$\Rightarrow y(t-\tau) + a_1 \frac{dy(t-\tau)}{dt} = b_0 x(t-\tau)$

$z_3(t) + a_1 \frac{dz_3(t)}{dt} = b_0 x(t-\tau)$

$z_4(t) + a_1 \frac{dz_4(t)}{dt} = b_0 x(t-\tau)$

$\Rightarrow$ $S_3$ & $S_4$ ARE INDISTINGUISHABLE

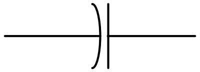
$\Rightarrow$ TIME-INVARIANT ✓

# Modeling Comments

- Most systems in ELEC 221 are modeled as LTI (Linear Time-Invariant) systems so the analysis is simplified.
- In reality, all systems are NL (Nonlinear, due to limitations such as saturation, breakdown voltage, ultimate strength, etc.) and TV (Time-Varying, due to creation, aging, expiration, environment changes, etc.).
- LTI models are suitable and can be quite accurate if the system is kept within an appropriate operating region. The greater accuracy of NLTV models seldom justifies the extra complexity and resources involved in applying them. NL devices sometimes are usefully modeled by several distinct regions (e.g., BJTs have cut-off, active and saturation regions).

# Ideal Device Models in ELEC 221

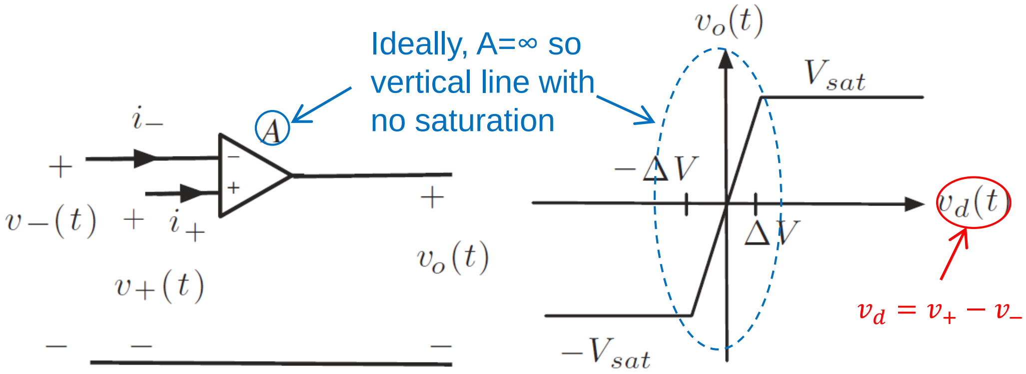
## Passive components:

| | Dissipative | Storage | |
|---|---|---|---|
| | $R$  | $C$  | $L$  |
| Time Domain | $v(t) = i(t)R$ (Ohm's Law) | $i(t) = C\dfrac{dv(t)}{dt}$ (Capacitance Law) | $v(t) = L\dfrac{di(t)}{dt}$ (Inductance Law) |
| Phasor Analysis | $\boldsymbol{V} = R\boldsymbol{I}$ | $\boldsymbol{I} = j\omega C\boldsymbol{V}$ | $\boldsymbol{V} = j\omega L\boldsymbol{I}$ |
| $s-$Domain | $V(s) = I(s)R$ | $I(s) = sCV(s) - Cv(0)$ | $V(s) = sLI(s) - Li(0)$ |
| Impedance | $R$ | $\dfrac{1}{sC}$ | $sL$ |
| Power/Energy | $p(t) = \dfrac{v^2(t)}{R}$ (power dissipated) | $w(t) = \dfrac{Cv^2(t)}{2}$ (energy | $w(t) = \dfrac{Li^2(t)}{2}$ stored) |

Instantaneous Power

NB: We may regard phasor analysis as a special case of L.T. analysis where we set $s=j\omega$ (i.e., only one frequency) and I.C.s are ignored.

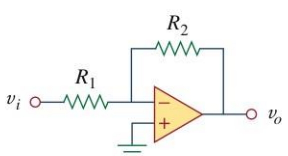
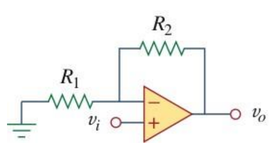
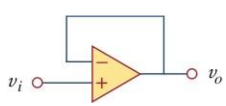
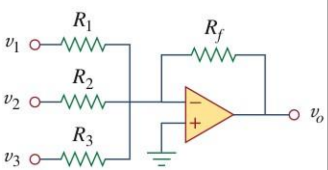
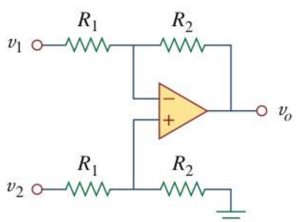
# Ideal Device Models in ELEC 221




Ideal Op-Amp
"Golden Rules":

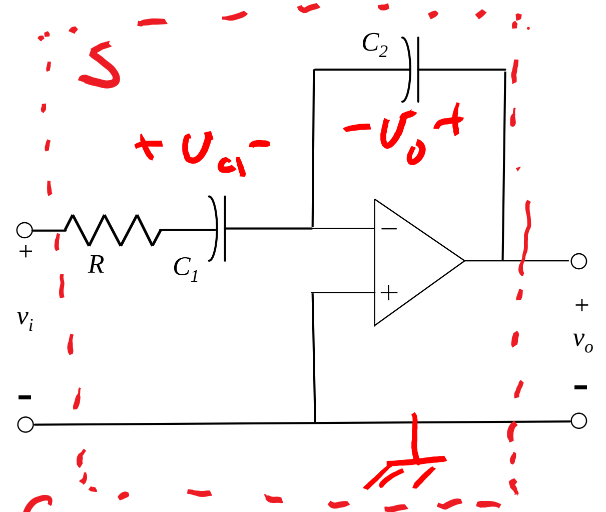
$$\boxed{\begin{aligned} i_+ &= i_- = 0 \\ v_+ &= v_- \end{aligned}}$$

## Typical Configurations

| Inverting Amplifier | Noninverting Amplifier | Voltage Follower | Summer | Difference Amplifier |
|---|---|---|---|---|
|  |  |  |  |  |
| $v_o = -\frac{R_2}{R_1} v_i$ | $v_o = \left(1 + \frac{R_2}{R_1}\right) v_i$ | $v_o = v_i$ | $v_o = -R_f \left(\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3}\right)$ | $v_o = \frac{R_2}{R_1}(v_2 - v_1)$ |

# Example

Find the differential equation relating input $v_i(t)$ to output $v_o(t)$. Assume that if/when $v_o(t)$=0 that both capacitors are discharged.




METHOD A: TIME-DOMAIN

KCL: $\frac{v_i - v_{c1}}{R} = C_1 \dot{v}_{c1} = -C_2 \dot{v}_o \Rightarrow v_{c1} = -\frac{C_2}{C_1} v_o$

$\Rightarrow \frac{v_i}{R} + \frac{C_2}{RC_1} v_o = -C_2 \dot{v}_o \Rightarrow v_o + RC_1 \dot{v}_o = -\frac{C_1}{C_2} v_i$ $\left(\text{cf. } y + a_1 \frac{dy}{dt} = bx\right)$

METHOD B: S-DOMAIN $\left(V_i(s) = \mathcal{L}\{v_i(t)\},\ V_o(s) = \mathcal{L}\{v_o(t)\}\right)$

INVERTING AMPLIFIER CONFIGURATION $\Rightarrow \frac{V_o}{V_i} = \frac{-Z_2}{Z_1} = \frac{-1/sC_2}{R + 1/sC_1} = \frac{-C_1/C_2}{1 + sC_1R}$

$\Rightarrow (1 + sC_1R)V_o = -\frac{C_1}{C_2} V_i \xRightarrow[\mathcal{L}^{-1}]{} v_o + RC_1 \dot{v}_o = \frac{C_1}{C_2} v_i$

# ODE System Representation

The requirement of being “relaxed” (unenergized) to label as LTI is unnecessary. IC’s shouldn’t affect classification and can be remedied with ZIR (next slide)

Most CT dynamic systems with lumped parameters can be represented by linear ODEs with constant coefficients.

> A system represented by a linear ordinary differential equation, of any order N, having constant coefficients, and with input *x(t)* and output *y(t)*:
>
> $$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \quad t \geq 0 \qquad (2.8)$$
>
> is linear time-invariant ~~if the system is **not initially energized**~~ (i.e., the initial conditions are zero, and the input *x(t)* is zero for t < 0).

This form allows easy determination of the **characteristic polynomial**:

$$a_0 + a_1 s + \cdots + s^N = \prod_{k=1}^{N} (s - p_k)$$

The polynomial roots ($p_k$) are the system **natural frequencies** or **eigenvalues**, providing great insight into the intrinsic dynamics.

E.g., Damping classification, stability, time constants, etc.

# ODE System Superposition

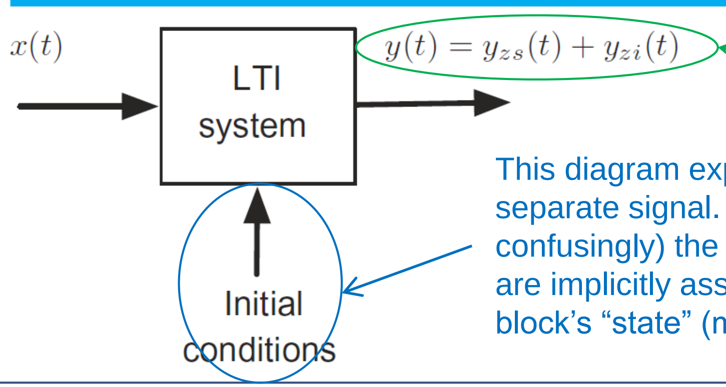
Given a dynamic system represented by a linear ordinary differential equation with constant coefficients,

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \quad t \geq 0 \qquad (2.9)$$

with $N$ initial conditions: $y(0)$, $d^k y(t)/dt^k|_{t=0}$, for $k = 1, \cdots, N-1$ and input $x(t) = 0$ for $t < 0$, the system's **complete response** $y(t)$, $t \geq 0$, has two components:

- *the* ***zero-state response***, $y_{zs}(t)$, *due exclusively to the input as the initial conditions are zero, and*
  ZSR (aka “Forced Response”)
- *the* ***zero-input response***, $y_{zi}(t)$, *due exclusively to the initial conditions as the input is zero.* Thus,
  ZIR (aka “Natural Response”)

$$y(t) = y_{zs}(t) + y_{zi}(t). \qquad (2.10)$$




“Response” refers to the output

This diagram explicitly illustrates the ICs as a separate signal. More typically (and less confusingly) the arrow is omitted and the ICs are implicitly assumed as part of the system block’s “state” (memory).

# Convolution Integral

If $\mathcal{S}$ is the transformation corresponding to a LTI system, so that the response of the system is

$$y(t) = \mathcal{S}[x(t)] \text{ to an input } x(t)$$

Intuition: If new input is a linear combination of time-shifted versions of the original, then the new output is also a corresponding linear combination.

then we have that by superposition and time-invariance

$$\mathcal{S}\left[\sum_k A_k x(t-\tau_k)\right] = \sum_k A_k \mathcal{S}[x(t-\tau_k)] = \sum_k A_k y(t-\tau_k)$$

$$\mathcal{S}\left[\int g(\tau)x(t-\tau)d\tau\right] = \int g(\tau)\mathcal{S}[x(t-\tau)]d\tau = \int g(\tau)y(t-\tau)d\tau$$

This extension allows for a more general new input made of a continuum of differentials of the original.

# Impulse Response

(as an input)

The **impulse response** of a continuous-time LTI system, $h(t)$, is the output of the system corresponding to an impulse $\delta(t)$ and initial conditions equal to zero.

**Key Idea**: An LTI system's impulse response *gives us everything* needed to compute the response to any signal. In fact, *h(t)* is a valid representation of the system.

# Convolution with Impulse Response

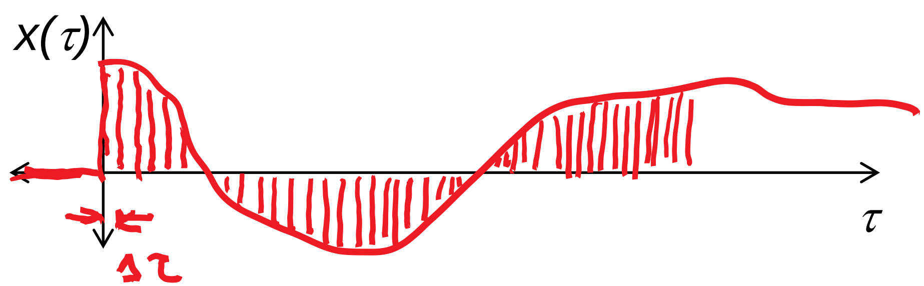
The response of a LTI system $\mathcal{S}$, represented by its impulse response *h(t)*, to any signal *x(t)* is the **convolution integral**

NB: If $x(t) = \delta(t)$, then $y(t) = h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \qquad (2.17)$$
$$= [x*h](t) = [h*x](t)$$

where the symbol * stands for the convolution integral of the input signal *x(t)* and the impulse response *h(t)* of the system.

Intuition for this (first consider *x(t)* and *h(t)* being causal):




$$y(t) \approx [x(0)\Delta\tau]h(t) + [x(\Delta\tau)\Delta\tau]h(t-\Delta\tau) + \ldots + [x((n-1)\Delta\tau)\Delta\tau]h(\Delta\tau) \quad \text{WHERE} \quad \Delta\tau = \frac{t}{n}$$

$$\approx \sum_{i=0}^{n-1} x(i\Delta\tau)h(t-i\Delta\tau)\Delta\tau = \lim_{\Delta\tau \to 0} \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

$t$ → $\infty$ IF $h(t)$ ACAUSAL

$0$ → $-\infty$ IF $x(t)$ ACAUSAL

# Responses to Impulse, Step & Ramp

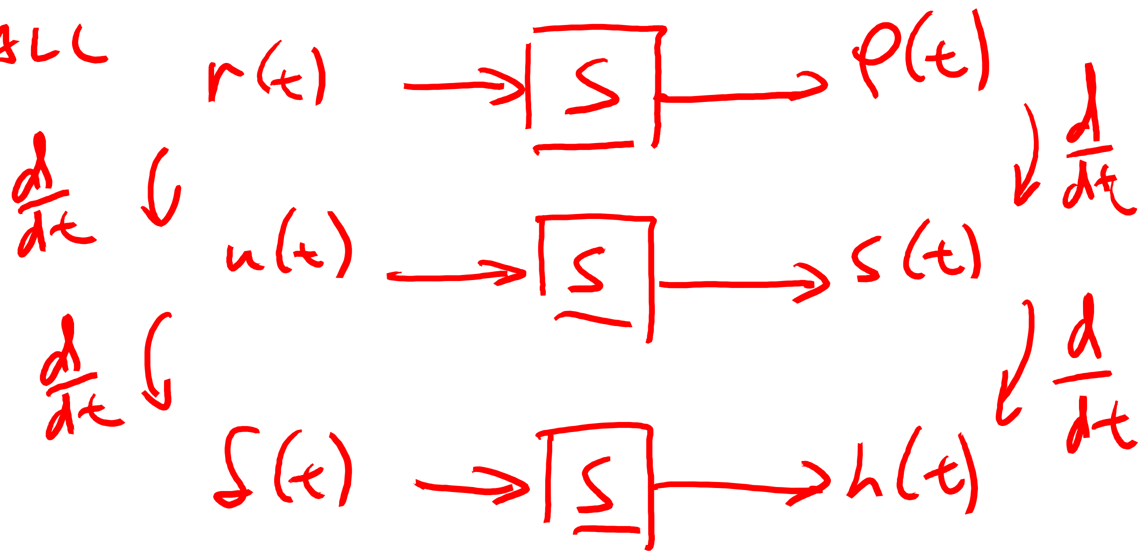
The impulse response $h(t)$, the unit-step response $s(t)$ and the ramp response $\rho(t)$ are related by

The step response is used more often in practice to characterize a system.

Potential ambiguity with Laplace variable (s) so I generally won't use s(t).

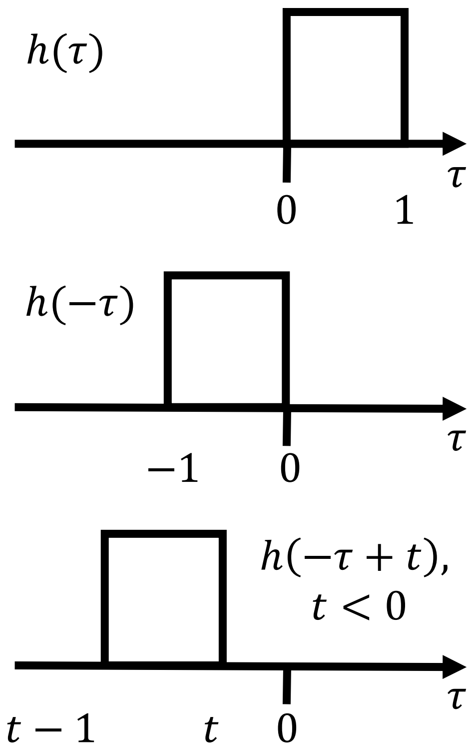
$$h(t) = \begin{cases} ds(t)/dt \\ d^2\rho(t)/dt^2 \end{cases} \tag{2.18}$$

RECALL

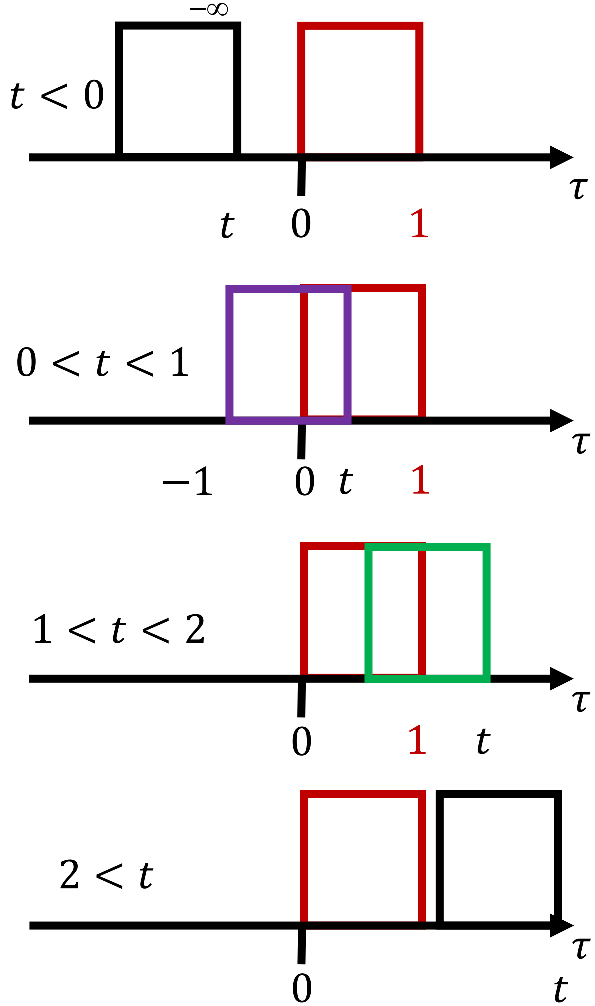


# Convolution Graphical Example

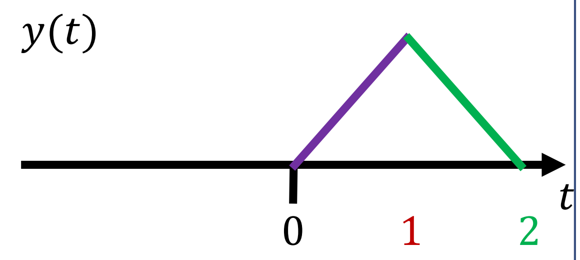
Impulse Response




$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$




System Output




$$y(t)=\begin{cases}0, & t<0\\ t, & 0\le t<1\\ (2-t), & 1\le t<2\\ 0, & 2\le t\end{cases}$$

$$= r(t)-2r(t-1)+r(t-2)$$

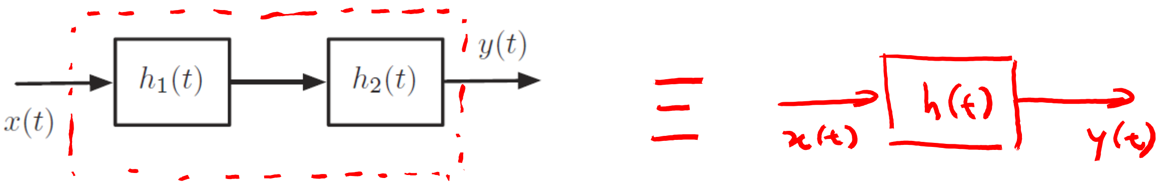
# Block Diagram Interconnections

Two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ connected in **cascade** have as an overall impulse response

…or "in series"…

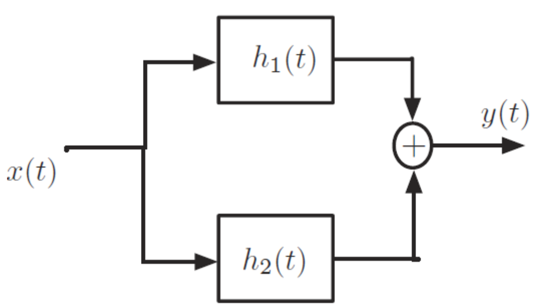
$$h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$$

where $h_1(t)$ and $h_2(t)$ commute (i.e., they can be interchanged).




If we connect in parallel two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$, the impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t)$$



# Block Diagram of Feedback Control

Given two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$, a negative feedback connection (Figure 2.11(c)) is such that the output is

$$y(t) = [h_1 * e](t)$$

where the error signal is

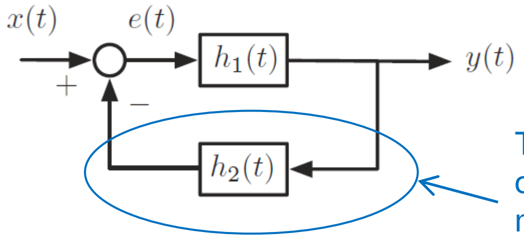
$$e(t) = x(t) - [y * h_2](t)$$

The overall impulse response $h(t)$, or the impulse response of the **closed-loop** system, is given by the following implicit expression

Not very useful. Main takeaway: Solving in the time-domain is hard but transfer functions (and Laplace Transforms) are much easier.

$$h(t) = [h_1 - h * h_1 * h_2](t)$$

If $h_2(t) = 0$, i.e., there is no feedback, the system is called **open-loop** and $h(t) = h_1(t)$.




This "feedback" path closes the loop and is an important feature of Control Systems. When the "open-loop" performance doesn't meet requirements, the Controls Engineer might design and implement $h_2(t)$ (e.g., PID controller) as a remedy.

# Causal Systems

A continuous-time system $S$ is called **causal** if

- *whenever its input $x(t) = 0$, and there are no initial conditions, the output is $y(t) = 0$,*
- *the output $y(t)$ does not depend on future inputs.*

A LTI system represented by its impulse response $h(t)$ is **causal** if

$$h(t) = 0 \qquad \text{for } t < 0 \tag{2.20}$$

The output of a causal LTI system with a causal input $x(t)$, i.e., $x(t) = 0$ for $t < 0$, is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau \tag{2.21}$$

# BIBO Stability

In general, practical systems are designed to be stable. There are several important stability concepts.

**Bounded-input bounded-output (BIBO) stability** establishes that for a bounded (that is what is meant by well-behaved) input $x(t)$ the output of a BIBO stable system $y(t)$ is also bounded. This means that if there is a finite bound $M < \infty$ such that $|x(t)| \leq M$ (you can think of it as an envelope $[-M, M]$ inside which the input is in) the output is also bounded. That is, there is a bound $L < \infty$ such that $|y(t)| \leq L < \infty$.

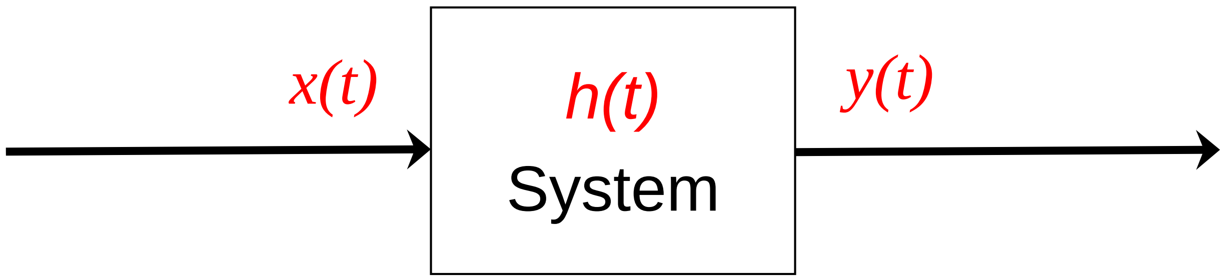
A LTI system is **bounded-input bounded-output (BIBO) stable** provided that the system impulse response $h(t)$ is **absolutely integrable,** i.e.,

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \qquad (2.23)$$

# Impulse Response

Q: Is the "Impulse Response" a signal or a system?

A: Yes…the term is used for both! Context allows you to determine which usage is more relevant.




As a signal: $h(t)=y(t)$ if $x(t)=\delta(t)$ (i.e., it's the response if the input is the impulse function)

As a system: $h(t)$ completely characterizes the system I/O behaviour, allowing determination of the ZSR for any input.