Discrete-Time Fourier Analysis

- Comparison of 4 Fourier Transforms*
- DTFT & DFT/DFS
- Frequency Response
- DTFT Tables & DFT Tables

*L&V Chap 10 provides a good comparison. Some examples/figures are from this resource.

4 Fourier Transforms: First Glance

Time Periodicity	Continuous-time (previously covered)	Discrete-time
Periodic	Fourier Series	Discrete Fourier series (DFS) or Discrete Fourier transform (DFT)*
Aperiodic	(Continuous-Time) Fourier transform (CTFT or FT)	Discrete-time Fourier transform (DTFT)

* Caveat: Some references consider the DFT and DFS as synonymous. Though they provide identical information, we'll follow most references which, for historical reasons, distinguish them by a scaling factor (slide 10.5).

Recap of CT Transforms

Time Domain	Conversion/Computation	Frequency Domain
$x(t)$ $x\colon \mathbb{R} \to \mathbb{R}$	Laplace Transform: $X_L(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$ $x(t) \rightarrow \boxed{\text{LT}} \rightarrow X_L(s)$	$X_L(s)$ $X_L\colon \mathbb{C} \to \mathbb{C}$
	Inverse Laplace Transform $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_L(s)\, e^{st} ds$ $x(t) \leftarrow \boxed{\text{ILT}} \leftarrow X_L(s)$	
	Fourier Transform: $X_F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ $x(t) \rightarrow \boxed{\text{FT}} \rightarrow X_F(\omega)$	$X_F(\omega)$ $X_F\colon \mathbb{R} \to \mathbb{C}$
	Inverse Fourier Transform: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_F(\omega)\, e^{j\omega t} d\omega$ $x(t) \leftarrow \boxed{\text{IFT}} \leftarrow X_F(\omega)$	
$x(t) = x(t+mT)$ $\forall m \epsilon \mathbb{Z}$ $\omega_0 = \frac{2\pi}{T}$	Fourier Series: $X_{FS}[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ $x(t) \rightarrow \boxed{\text{FS}_T} \rightarrow X_{FS}[k]$	$X_{FS}[k]$ $X_{FS}\colon \mathbb{Z} \to \mathbb{C}$
	Inverse Fourier Series: $x(t) = \sum_{k=-\infty}^{\infty} X_{FS}[k]\, e^{jk\omega_0 t}$ $x(t) \leftarrow \boxed{\text{IFS}_T} \leftarrow X_{FS}[k]$	

Extension for DT Transforms

Time Domain	Conversion/Computation	Freq Domain
$x[n]$ $x: \mathbb{Z} \to \mathbb{R}\ (or\ \mathbb{C})$ CCW closed contour within RoC that encircles origin	Z Transform: $X_Z(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ $x[n] \rightarrow \boxed{\text{ZT}} \rightarrow X_Z(z)$	$X_Z(z)$ $X_Z: \mathbb{C} \to \mathbb{C}$
	Inverse Z Transform $x[n] = \frac{1}{2\pi j} \oint_C X_Z(z) z^{n-1} dz$ $x[n] \leftarrow \boxed{\text{IZT}} \leftarrow X_Z(z)$	
	Discrete-Time Fourier Xform: $X_{DTFT}(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$ $x[n] \rightarrow \boxed{\text{DTFT}} \rightarrow X_{DTFT}(\Omega)$	$X_{DTFT}(\Omega) = X_{DTFT}(\Omega + 2\pi m)$ $\forall m \epsilon \mathbb{Z}$ $X_{DTFT}: \mathbb{R} \to \mathbb{C}$
	Inverse DTFT: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{DTFT}(\Omega)\, e^{j\Omega n} d\Omega$ $x[n] \leftarrow \boxed{\text{IDTFT}} \leftarrow X_{DTFT}(\Omega)$	
$x[n] = x[n+mN]$ $\forall m \epsilon \mathbb{Z}$ $\Omega_0 = \frac{2\pi}{N}$	Discrete Fourier Transform: $X_{DFT}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n}$ $x[n] \rightarrow \boxed{\text{DFT}} \rightarrow X_{DFT}[k]$	$X_{DFT}[k] = X_{DFT}[k + mN]$ $\forall m \epsilon \mathbb{Z}$ $X_{DFT}: \mathbb{Z} \to \mathbb{C}$
	Inverse DFT: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{DFT}[k]\, e^{jk\Omega_0 n}$ $x[n] \leftarrow \boxed{\text{IDFT}} \leftarrow X_{DFT}[k]$	

Periodic DT Signals: DFS vs DFT

For a DT signal with period N: $x[n]=x[n+mN]\ \forall m\epsilon\mathbb{Z}, \Omega_0 = \frac{2\pi}{N}$.

The Discrete Fourier Series (DFS) is the representation:

NB: A big difference compared to the CT FS is a finite summation

$$x[n] = \sum_{k=0}^{N-1} X_{DFS}[k]\, e^{jk\Omega_0 n} \quad \text{where} \quad X_{DFS}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n}$$

Confusion can arise because these are all normally labeled simply as $X[k]$ or X_k (e.g. ,see Chaparro eqns 11.29, 11.30, 11.48 & 11.49). Subscripts on this slide are to show the explicit differences.

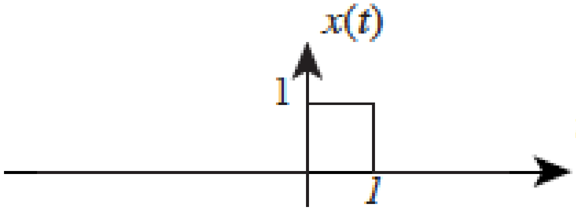
The Discrete Fourier Transform (DFT) is the representation:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_{DFT}[k]\, e^{jk\Omega_0 n} \quad \text{where} \quad X_{DFT}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n}$$

Both the DFS and DFT are commonly used in the literature. Efficient computing methods called FFTs (Fast-Fourier Transforms) are used for computing the DFT and IDFT.

Eg: Calculations for Square Pulse

Square Pulse: *x(t)=u(t)-u(t-1)*

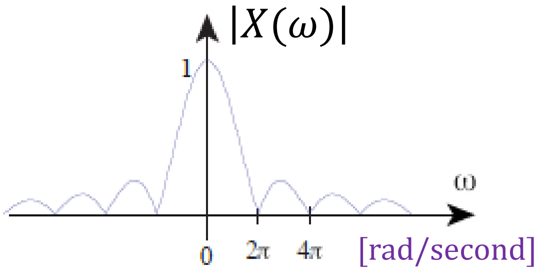


$$\forall\omega \in \mathbb{R}, X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{1} e^{-j\omega t} dt$$

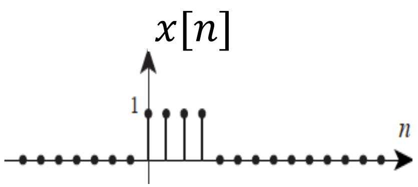
CTFT

$$= \frac{1}{-j\omega}\left(e^{-j\omega} - 1\right) = \frac{j\left(e^{-j\omega} - 1\right)}{\omega}$$

$$\Longrightarrow |X(\omega)| = \frac{|e^{-j\omega} - 1|}{\omega} = \frac{\sqrt{(\cos(\omega) - 1)^2 + \sin^2(\omega)}}{\omega} = \frac{\sqrt{2(1 - \cos(\omega))}}{\omega}$$



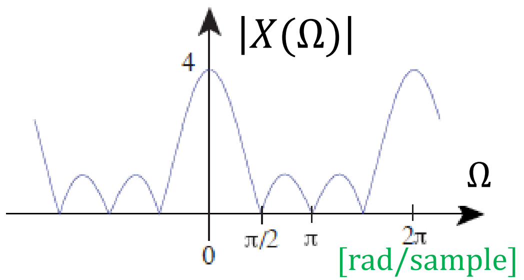
Discrete Square Pulse: *x[n]=u[n]-u[n-4]*



$$\forall\Omega \in \mathbb{R}, X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} = \sum_{n=0}^{3} e^{-j\Omega n} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}$$

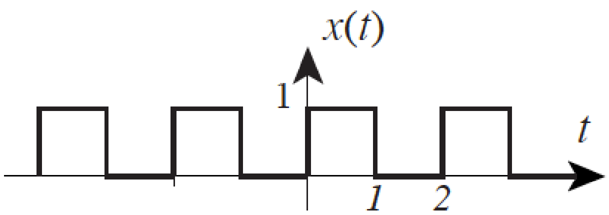
DTFT

$$\Longrightarrow |X(\Omega)| = \frac{|1 - e^{-j4\Omega}|}{|1 - e^{-j\Omega}|} = \sqrt{\frac{1 - \cos(4\Omega)}{1 - \cos(\Omega)}}$$



Eg: Calculations for Square Wave

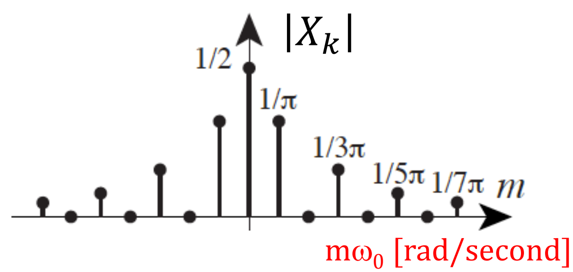
Square Wave: $x(t) = \begin{cases} u(t) - u(t-1), 0 \le t < 2 \\ x(t-2m) \forall m \epsilon \mathbb{Z} otherwise \end{cases}$



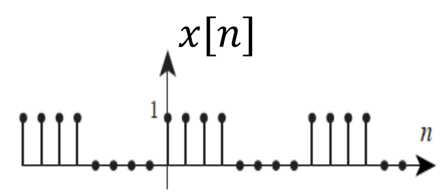
$$\forall \mathrm{k} \in \mathbb{Z}, X[k] = \frac{1}{2}\int_0^2 x(t) e^{-jk\pi t} dt = \frac{1}{2}\int_0^1 e^{-jk\pi t} dt$$

FS

$$= \frac{1}{-2jk\pi}\left(e^{-jk\pi} - e^0\right) = \begin{cases} 1/2 & \text{, if } k = 0 \\ -j/(k\pi) & \text{, if } k \text{ odd} \\ 0 & \text{, otherwise} \end{cases}$$



Discrete Square Wave: $x[n] = \begin{cases} u[n] - u[n-4], 0 \le n < 8 \\ x[n-8m] \forall m \epsilon \mathbb{Z} otherwise \end{cases}$

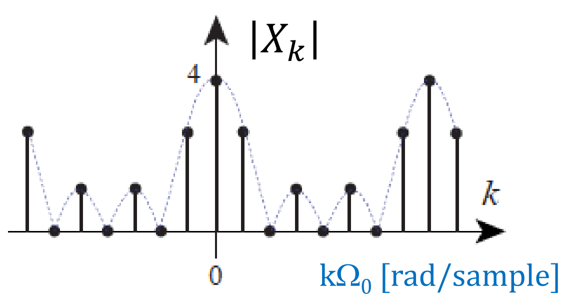


$$\Omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\forall \mathrm{k} \in \mathbb{Z}, X[k] = \sum_{n=0}^{7} x[n]\, e^{-jk\Omega_0 n} = \sum_{n=0}^{3} e^{-\frac{jkn\pi}{4}} = \frac{1 - e^{-jk\pi}}{1 - e^{-\frac{jk\pi}{4}}}$$

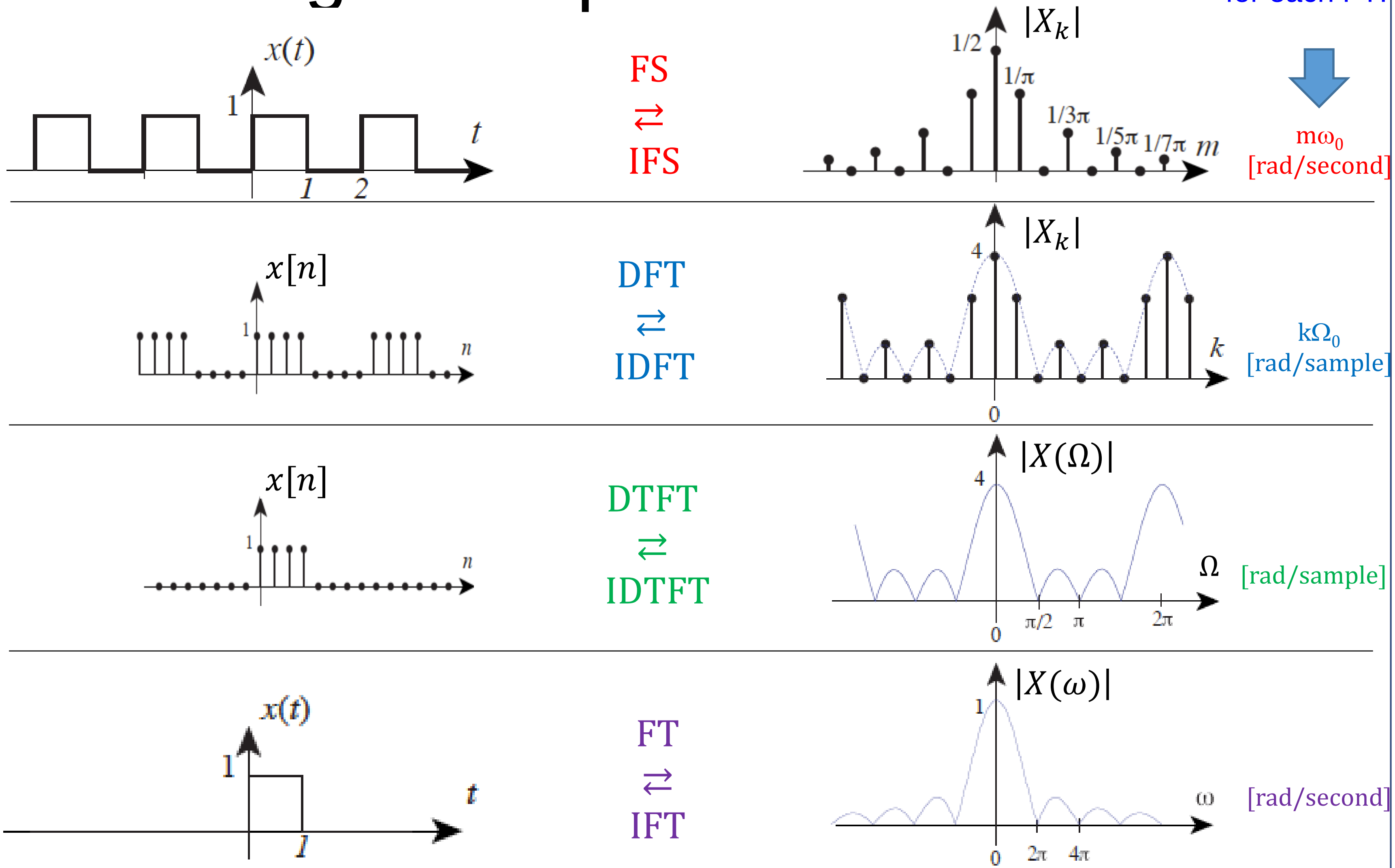
DFT

$$\Longrightarrow |X(\Omega)| = \frac{\left|1 - e^{-jk\pi}\right|}{\left|1 - e^{-\frac{jk\pi}{4}}\right|} = \sqrt{\frac{1 - \cos(\mathrm{k}\pi)}{1 - \cos\left(\frac{\mathrm{k}\pi}{4}\right)}}$$



Eg. Juxtaposition of 4 FTs

NB: Careful about freq units for each FT!



Caveat: To carry out the inverse (←), phase info (∠X) needed in addition to |X|

Frequency Response

Eigenfunctions are $\mathbb{C}$-exponentials

$$x[n]=e^{j\Omega n} \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n]=H(\Omega)e^{j\Omega n}$$

The symmetry property, like other properties, also applies to systems. If h[n] is the impulse response of an LTI discrete-time system, and it is real-valued, its DTFT is

$$H(\Omega) = H_z(e^{j\Omega}) = \mathcal{Z}(h[n])\,|_{z=e^{j\Omega}} = H_z(z)|_{z=e^{j\Omega}}$$

if the region of convergence of $H_z(z)$ includes the unit circle. As with the DTFT of a signal, the frequency response of the system, $H(\Omega)$, has a magnitude that is an even function of Ω, and a phase that is an odd function of ω. Thus, the **magnitude response** of the system is such that

$$|H(\Omega)| = |H(-\Omega)| \tag{11.25}$$

and the **phase response** is such that

$$\angle H(\Omega) = -\angle H(-\Omega) \tag{11.26}$$

$$H(\Omega) = H^*(-\Omega)$$

According to these symmetries and that the frequency response is periodic, it is only necessary to give these responses in $[0, \pi]$ rather than in $(-\pi, \pi]$.

Example: 2-Sample Moving Average

Consider the system modeled by $y[n] = \frac{1}{2}(x[n] + x[n-1])$.

Determine the frequency response and sketch both the magnitude and phase responses.

METHOD A: Assume $x[n] = e^{j\Omega n} \Rightarrow y[n] = H(\Omega)e^{j\Omega n} = \frac{1}{2}\left(e^{j\Omega n} + e^{j\Omega(n-1)}\right)$

$\Rightarrow H(\Omega) = \frac{1}{2}\left[1 + e^{-j\Omega}\right]$

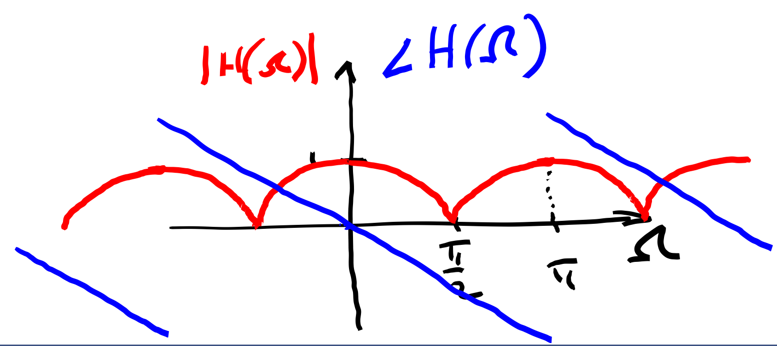
METHOD B: $Y(z) = \frac{1}{2}\left[X(z) + z^{-1}X(z)\right] \Rightarrow H_z(z) = \frac{Y(z)}{X(z)} = \frac{1}{2}\left[1 + z^{-1}\right]$

$\Rightarrow H(\Omega) = H_z(e^{j\Omega}) = \frac{1}{2}\left[1 + e^{-j\Omega}\right]$

METHOD C: $h[n] = \frac{1}{2}\left[\delta[n] + \delta[n-1]\right] \xrightarrow{DTFT} H(\Omega) = \frac{1}{2}\left[1 + e^{-j\Omega}\right] = \frac{1 + \cos\Omega - j\sin\Omega}{2}$

$|H(\Omega)| = \sqrt{\frac{(1+\cos\Omega)^2 + \sin^2\Omega}{2}} = \sqrt{\frac{1+\cos\Omega}{2}}$

$\angle H(\Omega) = \tan^{-1}\left(\frac{-\sin\Omega}{1+\cos\Omega}\right) = -\frac{\Omega}{2}$



DTFT & DFT Tables

Table 11.1 DTFT of Common Signals and DTFT Properties

Discrete-time Fourier Transforms (DTFT)

	Discrete-time signal	DTFT $X(e^{j\omega})$, periodic of period 2π
(1)	$\delta[n]$	$1, -\pi \le \omega < \pi$
(2)	A	$2\pi A\delta(\omega), -\pi \le \omega < \pi$
(3)	$e^{j\omega_0 n}$	$2\pi\delta(W - \omega_0), -\pi \le \omega < \pi$
(4)	$\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha e^{-j\omega}}, -\pi \le \omega < \pi$
(5)	$n\,\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}, -\pi \le \omega < \pi$
(6)	$\cos(\omega_0 n)u[n]$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)], -\pi \le \omega < \pi$
(7)	$\sin(\omega_0 n)u[n]$	$-j\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)], -\pi \le \omega < \pi$
(8)	$\alpha^{\lvert n\rvert}, \lvert\alpha\rvert < 1$	$\frac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}, -\pi \le \omega < \pi$
(9)	$p[n] = u[n+N/2] - u[n-N/2]$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}, -\pi \le \omega < \pi$
(10)	$\alpha^n\cos(\omega_0 n)u[n]$	$\frac{1-\alpha\cos(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \le \omega < \pi$
(11)	$\alpha^n\sin(\omega_0 n)u[n]$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \le \omega < \pi$

Properties of the DTFT

Z-transform:	$x[n], X(z), \lvert z\rvert = 1 \in ROC$	$X(e^{j\omega}) = X(z)\rvert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)})$, k *integer*
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N}X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω
		$\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=\infty}^{\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$	

Table 11.2 Properties of Discrete Fourier Series and Discrete Fourier Transform (DFT).

Fourier Series of Discrete-time Periodic signals

	$x[n]$ periodic signal of period N	$X[k]$ periodic FS coefficients of period N
Z-transform	$x_1[n] = x[n](u[n] - u[n-N])$	$X[k] = \frac{1}{N}\mathcal{Z}(x_1[n])\rvert_{z=e^{j2\pi k/N}}$
DTFT	$x[n] = \sum_k X[k]e^{j2\pi nk/N}$	$X(e^{j\omega}) = \sum_k 2\pi X[k]\delta(\omega - 2\pi k/N)$
LTI response	input $x[n] = \sum_k X[k]e^{j2\pi nk/N}$	output: $y[n] = \sum_k X[k]H(e^{jk\omega_0})e^{j2\pi nk/N}$ $H(e^{j\omega})$ (frequency response of system)
Time-shift (circular shift)	$x[n-M]$	$X[k]e^{-j2\pi kM/N}$
Modulation	$x[n]e^{j2\pi Mn/N}$	$X[k-M]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1}X[m]Y[k-m]$ periodic convolution
Periodic convolution	$\sum_{m=0}^{N-1}x[m]y[n-m]$	$NX[k]Y[n]$

Discrete Fourier Transform (DFT)

	$x[n]$ finite-length N aperiodic signal	$\tilde{x}[n]$ periodic extension of period $L \ge N$
	$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{L-1}\tilde{X}[k]e^{j2\pi nk/L}$	$\tilde{X}[k] = \sum_{n=0}^{L-1}\tilde{x}[n]e^{-j2\pi nk/L}$
IDFT/DFT	$x[n] = \tilde{x}[n]W[n], W[n] = u[n] - u[n-N]$	$X[k] = \tilde{X}[k]W[n], W[k] = u[k] - u[k-N]$
Circular convolution	$(x \otimes_L y)[n]$	$X[k]Y[k]$
Circular and linear convolution	$(x \otimes_L y)[n] = (x * y)[n], L \ge M + K - 1$ $M =$ length of $x[n]$, $K =$ length of $y[n]$	