Fourier Series

- Representations
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    - Parseval's Power Relation
    - Magnitude/Phase/Power Line Spectra
  - FS Convergence
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Fourier Series Representations

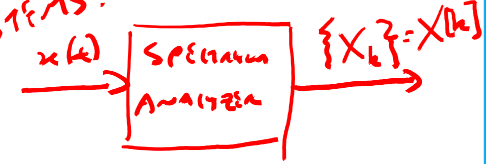
- Recall that a periodic signal $x(t)$ is one that:
  - is defined for $-\infty < t < \infty$
  - $\forall k \epsilon \mathbb{Z}, \forall t \epsilon \mathbb{R},$ then $x(t + kT_0) = x(t)$, where T_0 is the **fundamental period**; $\omega_0 = \frac{2\pi}{T_0}$ (sometimes Ω_0)
- Recall from MATH 255/256 that periodic signals can be represented by a Fourier Series. We'll see 3 common ways to write them:
  - Complex exponentials representation
  - Trigonometric representation with Sines & Cosines
  - Trigonometric representation with Cosines & Phase

Complex Exponentials Representation

The **Fourier Series representation** of a periodic signal $x(t)$, of fundamental period T_0, is given by an infinite sum of weighted complex exponentials (cosines and sines) with frequencies multiples of the **fundamental frequency** $\omega_0 = 2\pi/T_0$ (rad/sec) of the signal:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \tag{4.13}$$

IN SOME SENSE, THESE MAPPINGS / XFORMATIONS CAN BE CONSIDERED SYSTEMS.



where the Fourier coefficients $\{X_k\}$ are found according to

X_0 = DC VALUE = AVERAGE OF $x(t)$

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \underbrace{x(t)e^{-jk\omega_0 t}}_{= x(t)e^{-st}\big|_{s=jk\omega_0}} dt \tag{4.14}$$

for $k = 0, \pm 1, \pm 2, \ldots$ and any t_0. The form of the last equation indicates that the information needed for the Fourier series can be obtained from any period of $x(t)$.

Complex Exponentials Representation

- The function set $\{e^{jk\omega_0 t}\}, k\epsilon\mathbb{Z}$ provides an orthonormal basis for all periodic functions with fundamental period T_0:

$$\langle e^{jk\omega_0 t}, e^{jl\omega_0 t}\rangle = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t}\left[e^{jl\omega_0 t}\right]^* dt = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{cases} \quad k, l \in \mathbb{Z}$$

- The square magnitude of the Fourier coefficients ($\{|X_k|^2\} = \{X_k X_k^*\}$) provides the power spectrum (i.e., the power distribution showing each frequency contribution).
- The orthonormality of the basis functions results in superposition of power (**Parseval's power relation**):

The power P_x of a periodic signal $x(t)$, of fundamental period T_0, can be equivalently calculated in either the time or the frequency domain:

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2, \qquad \text{for any } t_0 \tag{4.17}$$

Power Line Spectrum & Symmetry

A periodic signal $x(t)$, of fundamental period T_0, is represented in the frequency by its

$$\text{Magnitude line spectrum} \qquad |X_k| \text{ vs } k\omega_0 \tag{4.18}$$

$$\text{Phase line spectrum} \qquad \angle X_k \text{ vs } k\omega_0 \tag{4.19}$$

The **power line spectrum**, $|X_k|^2$ vs. $k\omega_0$ of $x(t)$ displays the distribution of the power of the signal over frequency.

For a real-valued periodic signal $x(t)$, of fundamental period T_0, represented in the frequency domain by the Fourier coefficients $\{X_k = |X_k|e^{j\angle X_k}\}$ at harmonic frequencies $\{k\omega_0 = 2\pi k/T_0\}$, we have that

$$X_k = X_{-k}^* \tag{4.20}$$

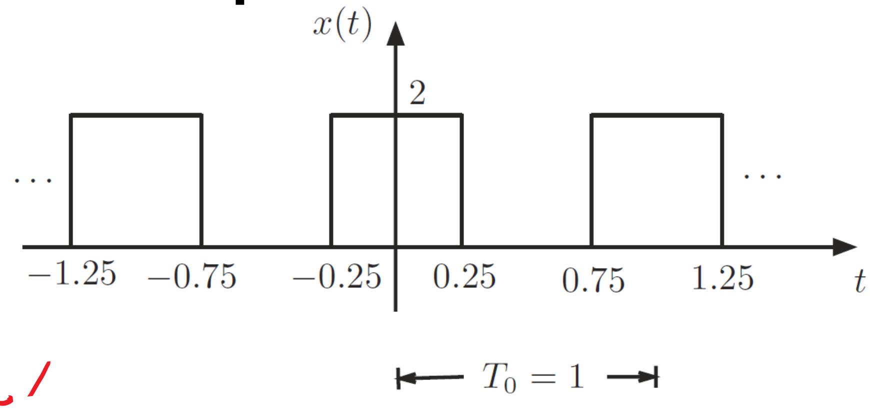
or equivalently that

$$\begin{aligned} &\text{(i)} \quad |X_k| = |X_{-k}|, \ \text{ i.e., magnitude } |X_k| \text{ is an even function of } k\omega_0 \\ &\text{(ii)} \quad \angle X_k = -\angle X_{-k}, \ \text{ i.e., phase } \angle X_k \text{ is an odd function of } k\omega_0 \end{aligned} \tag{4.21}$$

Thus, for real-valued signals we only need to display for $k \geq 0$ the magnitude line spectrum or a plot of $|X_k|$ vs $k\omega_0$, and the phase line spectrum or a plot of $\angle X_k$ *vs* $k\omega_0$ and to remember the even and odd symmetries of these spectra.

Chaparro Example 4.5

Find the Complex Exponential Fourier Series for the pulse train shown.



$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\Rightarrow \omega_0 = \frac{2\pi}{T_0} = 2\pi$$

$$X_0 = \frac{1}{1}\int_{-\frac{1}{4}}^{\frac{1}{4}} 2\, dt = 1 \checkmark$$

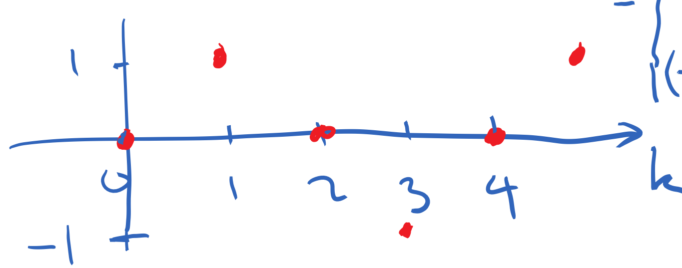
AGREES w/ INTUITION

$k \neq 0$:

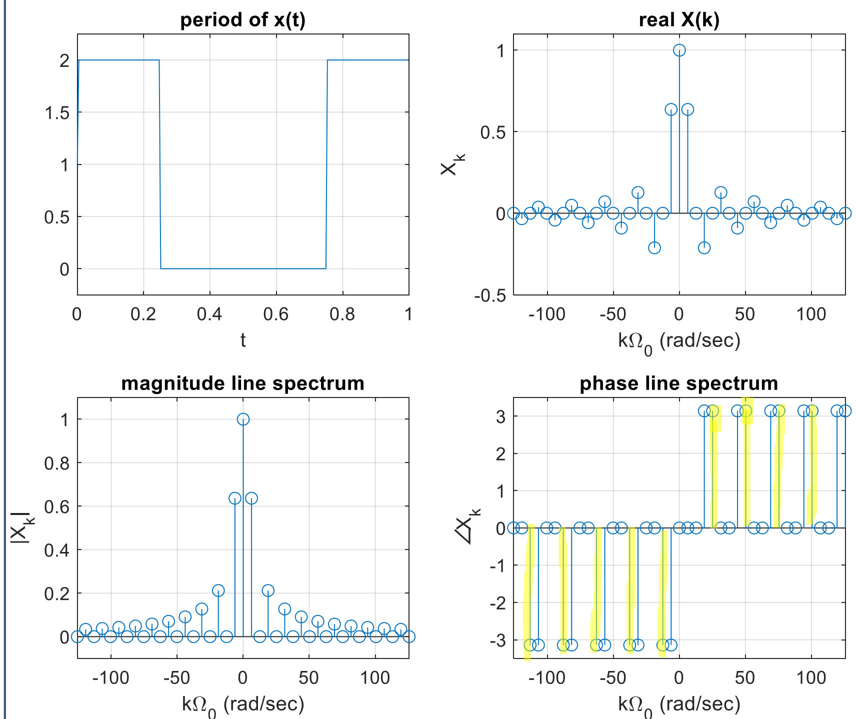
$$X_k = \int_{-\frac{1}{4}}^{\frac{1}{4}} 2e^{-jk\omega_0 t} = \frac{2e^{-jk\omega_0 t}}{-jk\omega_0}\Bigg|_{-\frac{1}{4}}^{\frac{1}{4}} = \frac{2\left[\cancel{\cos\left(\frac{k\omega_0}{4}\right)} - j\sin\left(\frac{k\omega_0}{4}\right) - \left[\cancel{\cos\left(\frac{k\omega_0}{4}\right)} + j\sin\left(\frac{k\omega_0}{4}\right)\right]\right]}{-jk\omega_0}$$

$$= \underbrace{\frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}}_{\text{OK TO LEAVE LIKE THIS}} \qquad \left(= \mathrm{sinc}\left(\frac{k}{2}\right)\right)$$

ASIDE: GRAPH OF NUMERATOR $\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & \text{IF EVEN} \\ (-1)^{\frac{k-1}{2}} & \text{IF ODD} \end{cases}$

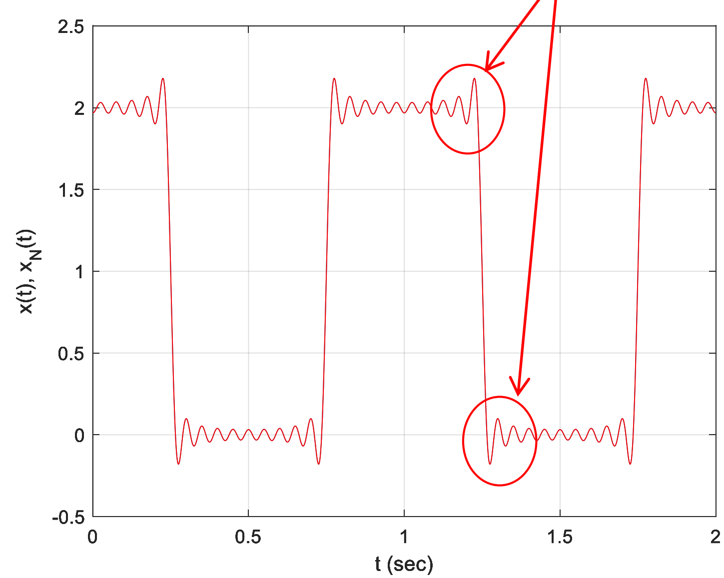


Chaparro Example 4.5 cont.



Magnitude and Phase spectra are as shown (NB: when $|X_k|=0$, the phase isn't important.

Gibb's Phenomenon shows "ringing" at discontinuity



This approximation shows $k \in \mathbb{Z}, -20 \leq k \leq 20.$

Convergence of the Fourier Series

The Fourier series of a piecewise smooth (continuous or discontinuous) periodic signal $x(t)$ converges for all values of t. The mathematician Dirichlet showed that for the Fourier series to converge to the periodic signal $x(t)$, the signal should satisfy the following sufficient (not necessary) conditions over a period:

1. be absolutely integrable,
2. have a finite number of maxima, minima, and discontinuities.

The infinite series equals $x(t)$ at every continuity point and equals the average

$$0.5[x(t+0_+) + x(t+0_-)]$$

of the right-hand limit $x(t+0_+)$ and the left-hand limit $x(t+0_-)$ at every discontinuity point. If $x(t)$ is continuous everywhere, then the series converges absolutely and uniformly.

Trigonometric Representations

The **trigonometric Fourier Series** of a real-valued, periodic signal $x(t)$, of fundamental period T_0, is an equivalent representation that uses sinusoids rather than complex exponentials as the basis functions. It is given by

$(X_k = |X_k| e^{j\theta_k})$

$$\begin{aligned} x(t) &= X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \theta_k) \\ &= c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)] \qquad \omega_0 = \frac{2\pi}{T_0} \end{aligned} \tag{4.23}$$

where $X_0 = c_0$ is called the **dc-component**, and $\{2|X_k|\cos(k\omega_0 t + \theta_k)\}$ are the kth **harmonics** for $k = 1, 2, \ldots$. The coefficients $\{c_k, d_k\}$ are obtained from $x(t)$ as follows

$= Re(X_k)$. These account for even component of *x(t)*.

$= -Im(X_k)$. These account for odd component of *x(t)*.

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\,dt \qquad k = 0, 1, \ldots \\ d_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\,dt \qquad k = 1, 2, \ldots \end{aligned} \tag{4.24}$$

The coefficients $X_k = |X_k|e^{j\theta_k}$ are connected with the coefficients c_k *and* d_k *by*

$$|X_k| = \sqrt{c_k^2 + d_k^2}$$

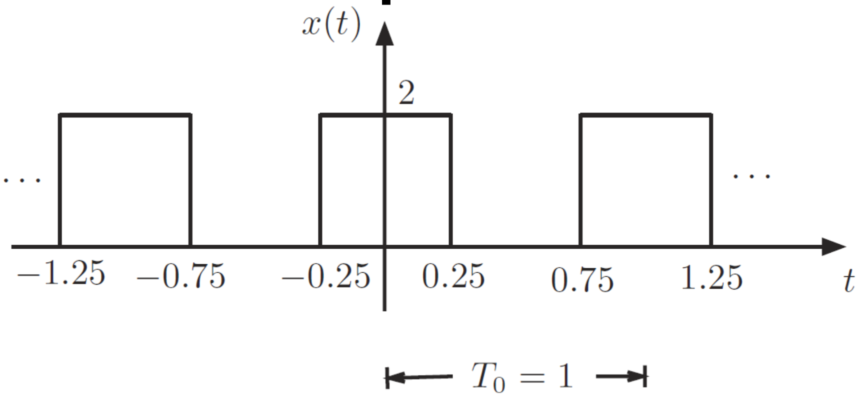
$$\theta_k = -\tan^{-1}\left[\frac{d_k}{c_k}\right]$$

$k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$
NATURAL NUMBERS
INCLUDING 0

The sinusoidal basis functions $\{\sqrt{2}\cos(k\omega_0 t), \sqrt{2}\sin(k\omega_0 t)\}, k = 0, 1, \ldots$, are orthonormal in $[0, T_0]$.

Revisit Chaparro Example 4.5

Find the Trigonometric Fourier Series for the pulse train shown.



$$c_n = \int_{-\frac{1}{4}}^{\frac{1}{4}} 2\cos(n\omega_0 t)\,dt = \left.\frac{2\sin(n\omega_0 t)}{n\omega_0}\right|_{-\frac{1}{4}}^{\frac{1}{4}}$$

$$= \frac{4\sin\left(n\frac{2\pi}{4}\right)}{n(2\pi)} = \frac{\sin\left(n\frac{\pi}{2}\right)}{(n\pi/2)} \quad (= X_n)$$

$$d_n = \int_{-\frac{1}{4}}^{\frac{1}{4}} 2\sin(n\omega_0 t)\,dt = \left.\frac{-2\cos(n\omega_0 t)}{n\omega_0}\right|_{-\frac{1}{4}}^{\frac{1}{4}} = 0$$

AGREES W/ INTUITION SINCE $x(t)$ IS EVEN FUNCTION

F.S. Coeffs from L.T.

For a periodic signal $x(t)$, of fundamental period T_0, if we know or can easily compute the Laplace transform of a period of $x(t)$

$$x_1(t) = x(t)[u(t - t_0) - u(t - t_0 - T_0)] \qquad \text{for any } t_0$$

then the Fourier coefficients of $x(t)$ are given by

$$X_k = \frac{1}{T_0}\mathcal{L}[x_1(t)]_{s=jk\Omega_0} \quad \omega_0 = \frac{2\pi}{T_0} \text{ (fundamental frequency)}, \quad k = 0, \pm 1, \cdots \qquad (4.25)$$

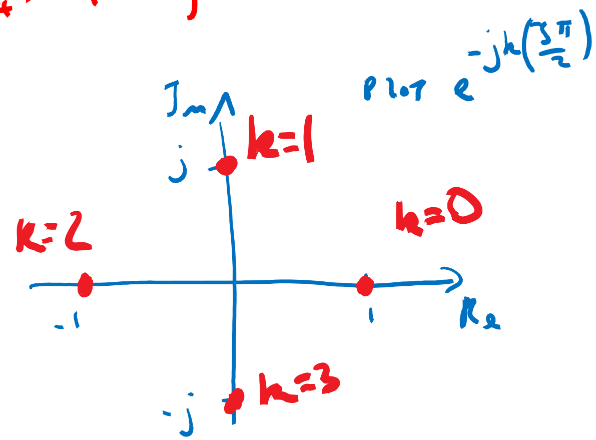
Let's verify this for the pulse train from slide 5.6:

For $0 < t < T_0$: $x_1(t) = 2\left[u(t) - u(t-\tfrac{1}{4}) + u(t-\tfrac{3}{4}) - u(t-1)\right]$

$$X_1(s) = \frac{2}{s}\left[1 - e^{-\frac{1}{4}s} + e^{-\frac{3}{4}s} - e^{-s}\right]$$

$$X_k = X_1(jk \times 2\pi) = \frac{2}{jk2\pi} \times \left[\cancel{1} - e^{-jk\frac{\pi}{2}} + \underbrace{e^{-jk\left(\frac{3\pi}{2}\right)}}_{= e^{jk\frac{\pi}{2}}} - \cancel{1}\right]$$

$$= \frac{2\sin\left(k\frac{\pi}{2}\right)}{k\pi} \quad \checkmark \text{ AGREES}$$



A Pure Sinusoid from Eigenfunctions

For a stable LTI system, any function of the form $e^{j\omega t}$ is an eigenfunction. Consider representations of a pure sinusoid in both the *t*-domain and *s*-domain.

$$x(t) = A\cos(\omega t + \theta) = \frac{A}{2}e^{j\theta}e^{j\omega t} + \frac{A}{2}e^{-j\theta}e^{-j\omega t}$$

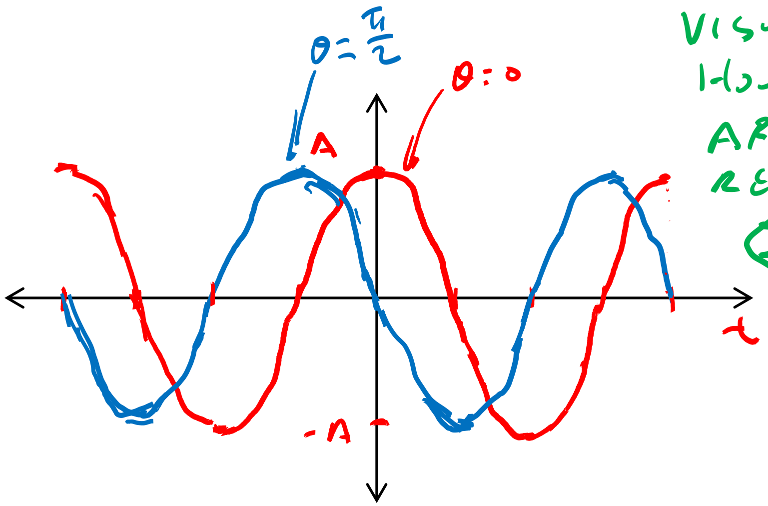
$$= A\,\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}$$

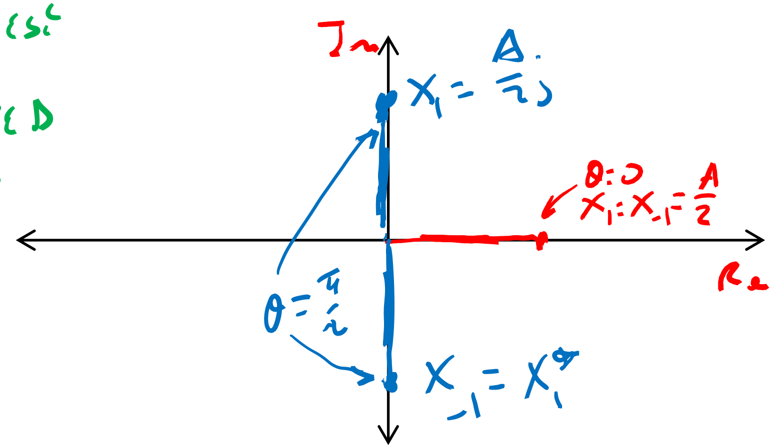
$$\underbrace{\tfrac{A}{2}e^{j\theta}}_{X_1} \qquad \underbrace{\tfrac{A}{2}e^{-j\theta}}_{X_{-1} = X_1^*}$$

FOURIER COEFFS

EIGENFUNCTIONS

VISUALIZE HOW THESE ARE RELATED





LTI System Frequency Response

Eigenfunction Property: In steady state, the response to a complex exponential (or a sinusoid) of a certain frequency is the same complex exponential (or sinusoid), but its amplitude and phase are affected by the frequency response of the system at that frequency.

If the input $x(t)$ of a causal and stable LTI system, with impulse response $h(t)$, is periodic of fundamental period T_0 and has the Fourier series

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \angle X_k) \qquad \omega_0 = \frac{2\pi}{T_0} \tag{4.43}$$

the steady-state response of the system is

$$y(t) = X_0 |H(j0)| + 2\sum_{k=1}^{\infty} |X_k| |H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0)) \tag{4.44}$$

where

$$H(jk\omega_0) = |H(jk\omega_0)| e^{j\angle H(jk\omega_0)} = \int_0^{\infty} h(\tau) e^{-jk\omega_0 \tau}\, d\tau = H(s)|_{s=jk\omega_0} \tag{4.45}$$

$\rightarrow H(s) = \mathcal{L}\{h(t)\}$ = system T.F.

Gives us freq response

is the **frequency response of the system at** $k\omega_0$.

Freq Response from Eigenfunctions

$x_1(t)=e^{j\omega t}$ → **System** ($h(t)$, $H(s)$) → $y_1(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t+\angle H(j\omega))}$

$x_2(t) = e^{-j\omega t}$ → $y_2(t) = H(-j\omega)e^{-j\omega t} = H^*(j\omega)e^{-j\omega t}$

$= y_1^*(t) = |H(j\omega)|e^{-j(\omega t+\angle H(j\omega))}$

IF $x(t) = \dfrac{x_1(t)+x_2(t)}{2} = \cos\omega t$ IS THE INPUT, THE OUTPUT WOULD BE

$$y(t) = \frac{y_1(t)+y_2(t)}{2} = \underbrace{|H(j\omega)|}\cos(\omega t + \underbrace{\angle H(j\omega)})$$

$H(j\omega)$ GIVES FREQ RESPONSE

CONCLUSION: FOR LTI SYSTEM W/ T.F $H(s)$, IF INPUT $x(t)$ IS SINUSOID OF FREQ ω, OUTPUT $y(t)$ IS ALSO SINUSOID OF FREQ ω BUT IS SCALED BY $|H(j\omega)|$ AND PHASE SHIFTED BY $\angle H(j\omega)$.

Reflection & Even/Odd Decomposition

Reflection: If the Fourier coefficients of a periodic signal $x(t)$ are $\{X_k\}$ then those of $x(-t)$, the time-reversed signal with the same period as $x(t)$, are $\{X_{-k}\}$.

Even periodic signal x(t): its Fourier coefficients X_k are real, and its trigonometric Fourier series is

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} X_k \cos(k\omega_0 t) \tag{4.29}$$

Odd periodic signal x(t): its Fourier coefficients X_k are imaginary, and its trigonometric Fourier series is

$$x(t) = 2\sum_{k=1}^{\infty} jX_k \sin(k\omega_0 t) \tag{4.30}$$

Any periodic signal $x(t)$ can be written $x(t) = x_e(t) + x_o(t)$, where $x_e(t)$ and $x_o(t)$ are the even and the odd components of $x(t)$ then

$$X_k = X_{ek} + X_{ok} \tag{4.31}$$

where $\{X_{ek}\}$ are the Fourier coefficients of $x_e(t)$ and $\{X_{ok}\}$ are the Fourier coefficients of $x_o(t)$ or

$$X_{ek} = 0.5[X_k + X_{-k}] \tag{4.32}$$

$$X_{ok} = 0.5[X_k - X_{-k}]$$

Addition of Periodic Signals

Same fundamental frequency: If $x(t)$ and $y(t)$ are periodic signals with the same fundamental frequency ω_0, then the Fourier series coefficients of $z(t) = \alpha x(t) + \beta y(t)$ for constants α and β are

$$Z_k = \alpha X_k + \beta Y_k \tag{4.47}$$

where X_k *and* Y_k *are the Fourier coefficients of* $x(t)$ *and* $y(t)$.

Different fundamental frequencies: If $x(t)$ is periodic of fundamental period T_1, and $y(t)$ is periodic of fundamental period T_2 such that $T_2/T_1 = N/M$, for non-divisible integers N and M, then $z(t) = \alpha x(t) + \beta y(t)$ is periodic of fundamental period $T_0 = MT_2 = NT_1$, and its Fourier coefficients are

$$Z_k = \alpha X_{k/N} + \beta Y_{k/M}, \tag{4.48}$$

$$\text{for } k = 0, \pm 1, \ldots \textit{ such that } k/N,\, k/M \textit{ are integers}$$

where X_k and Y_k are the Fourier coefficients of $x(t)$ and $y(t)$.

E.g. $x(t) = \cos 2\pi t \Rightarrow \omega_1 = 2\pi,\ T_1 = 1,\ X_k = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$

$y(t) = \sin 3\pi t \Rightarrow \omega_2 = 3\pi,\ T_2 = \frac{2}{3},\ Y_k = \begin{cases} \mp \frac{1}{2}j, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$

$\frac{T_2}{T_1} = \frac{2}{3} = \frac{N}{M};\ T_0 = 2;\ \omega_0 = \pi$

$z(t) = 4\cos 2\pi t + 5 \sin 3\pi t \Rightarrow Z_0 = 0,\ Z_{\pm 1} = 0,\ Z_{\pm 2} = 4(\frac{1}{2}) = 2,\ Z_{\pm 3} = 5(\mp \frac{1}{2}j) = \mp \frac{5}{2}j$

Product of Periodic Signals

If $x(t)$ and $y(t)$ are periodic signals of the same fundamental period T_0, then their product

$$z(t) = x(t)y(t) \tag{4.49}$$

is also periodic of fundamental period T_0 and with Fourier coefficients which are the convolution sum of the Fourier coefficients of $x(t)$ and $y(t)$:

$$Z_k = \sum_m X_m Y_{k-m}. \tag{4.50}$$

Efts: Explore what happens if you multiply periodic signals of different fundamental periods.

FS Basic Properties (c.f. Slide 4.5)

LT Properties from Slide 4.5

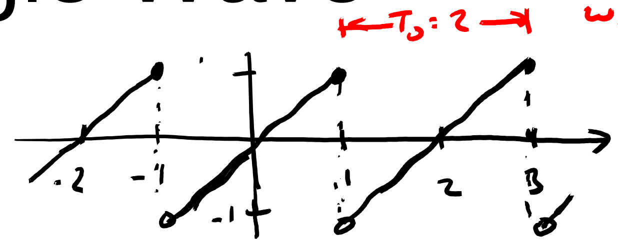
Table 4.1 Basic Properties of Fourier Series

Basic Properties of Fourier Series			
		Time Domain	Frequency Domain
	Signals and constants	$x(t)$, $y(t)$ periodic with period T_0, α, β	X_k, Y_k
$P1$	Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
	Parseval's power relation	$P_x = \frac{1}{T_0} \int_{T_0} \|x(t)\|^2 dt$	$P_x = \sum_k \|X_k\|^2$
$P5$	Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 X_k$
$P7$	Integration	$\int_{-\infty}^{t} x(t')dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\omega_0}$ $k \neq 0$
$P2$	Time shifting	$x(t - \alpha)$	$e^{-j\alpha\omega_0 k} X_k$
$P3$	Frequency shifting	$e^{jM\omega_0 t} x(t)$	X_{k-M} $(M \epsilon \mathbb{Z})$
	Symmetry	$x(t)$ real	$\|X_k\| = \|X_{-k}\|$ even function of k $\angle X_k = -\angle X_{-k}$ odd function of k
		I.e.: $X_k = X_{-k}^*$	
	Convolution in time	$z(t) = [x * y](t)$	$Z_k = X_k Y_k$

$EFTS$: Explore what happened to $P4, P6, P8$ & $P9$

F.S. of Triangle Wave

Consider $x(t) = \begin{cases} t & \text{for } -1 < t \le 1 \\ x(t-2) & \text{otherwise} \end{cases}$



$\omega_0 = \frac{2\pi}{T_0} = \pi$

Find the ℂ–exponential FS Representation.

Method A (see eqn (4.14) on slide 5.3):

$X_k = \frac{1}{T_0}\int_{-1}^{1} t\, e^{-jk\omega_0 t}\, dt$ $\qquad$ $\underline{X_0 = 0}$ BY INSPECTION (DC VALUE)

$k \neq 0$: $X_k = \frac{1}{2}\int_{-1}^{1} t\, e^{-jk\pi t}\, dt$ $\qquad$ LET $u = t$, $dv = e^{-jk\pi t}dt$, $du = dt$, $v = \frac{e^{-jk\pi t}}{-jk\pi}$ $\qquad$ INTEGRATE BY PARTS.

$$= \frac{1}{2}\left[\frac{t e^{-jk\pi t}}{-jk\pi}\Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-jk\pi t}}{-jk\pi}\, dt \right]$$

$$\frac{t e^{-jk\pi t}}{-jk\pi}\Bigg|_{-1}^{1} = \frac{j}{k\pi}\left(e^{-jk\pi} - (-1)e^{jk\pi}\right) = \frac{2j\cos k\pi}{k\pi}$$

$$\int_{-1}^{1} \frac{e^{-jk\pi t}}{-jk\pi}\, dt = \frac{-j}{k\pi}\,\frac{e^{-jk\pi t}}{-jk\pi}\Bigg|_{-1}^{1} = \frac{1}{(k\pi)^2}\left(e^{-jk\pi} - e^{jk\pi}\right) = \cancel{\frac{-2j\sin k\pi}{(k\pi)^2}}\ 0 \quad \because k \in \mathbb{Z}$$

$\Rightarrow$ For $k \neq 0$: $\underline{X_k = \frac{j(-1)^k}{k\pi}}$

NB: AGREES W/ REQUIREMENT THAT X_k IS IMAGINARY $\because$ $x(t)$ IS ODD FUNCTION

Method B (see eqn (4.25) on slide 5.11):

$$x_1(t) = t\,[\underbrace{u(t+1)}_{\frac{e^{s}}{s}} - \underbrace{u(t-1)}_{\frac{e^{-s}}{s}}]$$

$$\mathcal{L}\{x_1(t)\} = -\frac{d}{ds}\left[\frac{e^{s}}{s}\right] + \frac{d}{ds}\left[\frac{e^{-s}}{s}\right] = -\left[\frac{se^{s}-e^{s}}{s^2}\right] + \left[\frac{-se^{-s}-e^{-s}}{s^2}\right]$$

$$= \frac{1}{s^2}\left[e^{s}(1-s) - e^{-s}(s+1)\right]$$

$$X_k = \frac{\mathcal{L}\{x_1(t)\}}{T_0}\Big|_{s=jk\omega_0} = \frac{1}{-2(k\pi)^2}\left[e^{jk\pi}(\cancel{1}-jk\pi) - e^{-jk\pi}(jk\pi+\cancel{1})\right]$$

$$= \frac{(-1)^k}{\cancel{-2}(k\pi)^2}(\cancel{-2}jk\pi) = \frac{(-1)^k j}{k\pi}$$