Z-Transforms

- LT of Sampled Signals
- Two-Sided Z-Transform
- One-Sided Z-Transform
- Region of Convergence
- Transfer Functions
- Inverse Z-Transform

Z-transforms do for DT signals & systems what Laplace transforms do for CT signals & systems. Please compare and contrast accordingly.

Discrete-Time Domain ⇌ z-Domain

Discrete-Time domain $\qquad x[n] \qquad$ **$h[n]$** $\qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

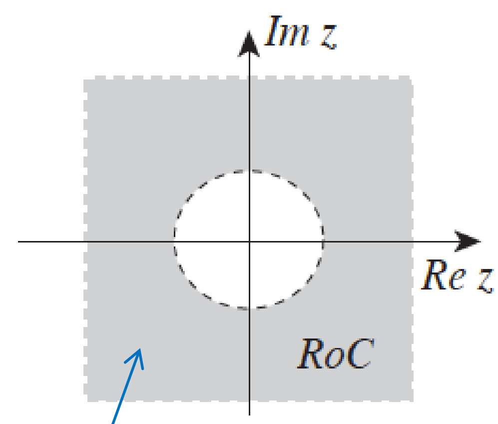
∞ ← n if $h[n]$ causal

$-\infty$ ← 0 if $x[n]$ causal

$\mathcal{Z}$ $\qquad$ System $\qquad$ $\mathcal{Z}^{-1}$

z-domain $\qquad X(z) \qquad$ **$H(z)$** $\qquad Y(z) = H(z)X(z)$

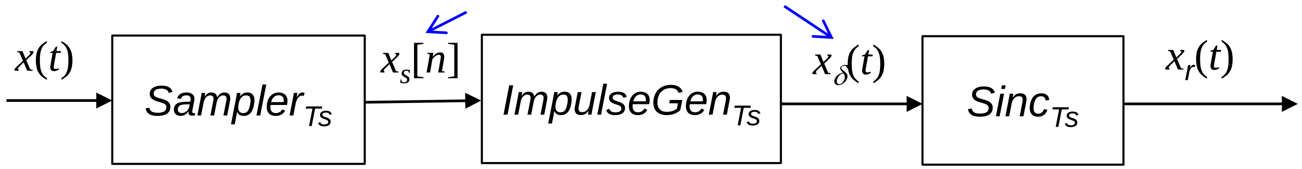
Z-transforms convert discrete-time domain difference equations into complex frequency domain algebraic equations. Time-domain convolution is more easily solved as frequency-domain multiplication.



Solution valid in the (shaded) Region of Convergence (RoC).

LT of Sampled Signals

Recall that these signals (one DT and the other CT) contain identical information and either one can be called the “sampled signal”.



The Laplace transform of a sampled signal

$$x_\delta(t) = \sum_n x(nT_s)\delta(t - nT_s) \tag{10.1}$$

is given by

$$X_\delta(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s)e^{-nsT_s} \tag{10.2}$$

By letting $z = e^{sT_s}$, we can rewrite (10.2) as

$$\mathcal{Z}[\ x_s[n]\ \ ] = \mathcal{L}[x_\delta(t)]|_{z=e^{sT_s}} = \sum_n x_s[n] \cdot z^{-n} \tag{10.3}$$

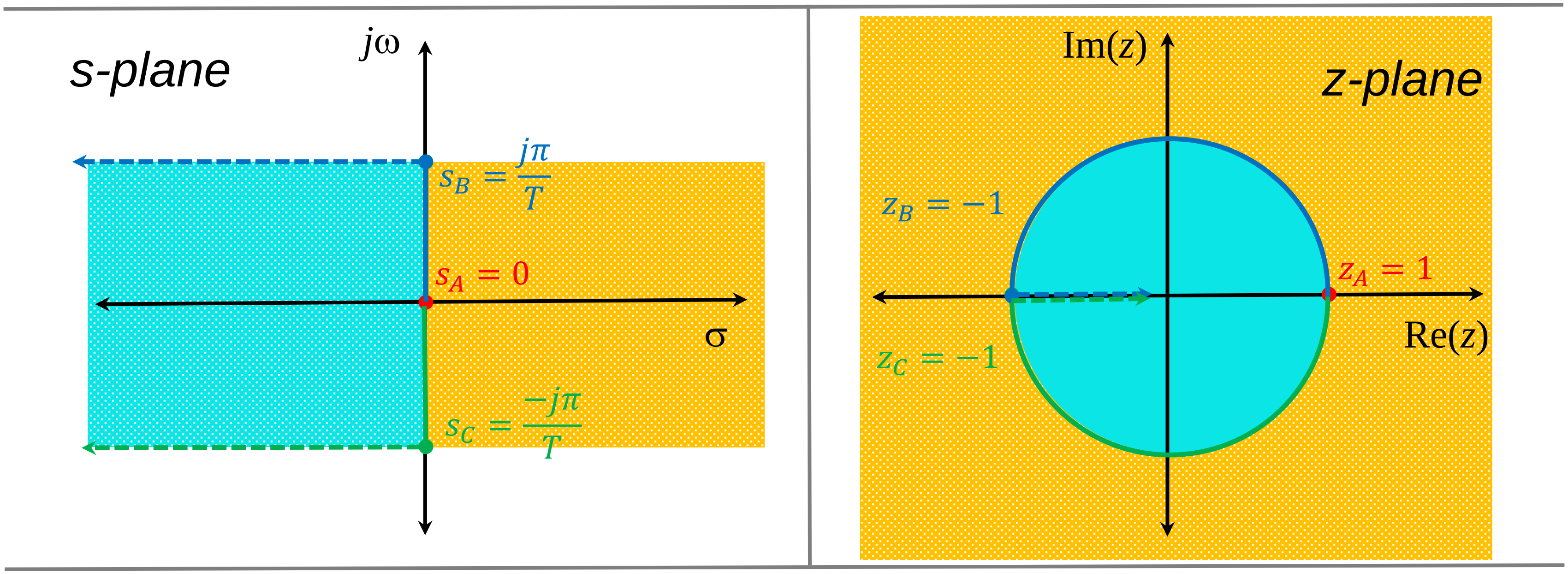
which is called the Z-transform of the sampled signal.

Mapping of s-plane to z-plane

Sampling Interval

$r = e^{\sigma T}$ $\Omega = \omega T$

$$z = e^{sT} = e^{(\sigma + j\omega)T} = re^{j\Omega}$$



$\frac{-\pi}{T} < \omega \le \frac{\pi}{T}$ in the s-plane maps to the entire z-plane. Frequencies outside this range are aliased to these lower frequencies after sampling (we get periodicity in frequency so strips of height $\frac{2\pi}{T}$ are repeatedly mapped to the same region).

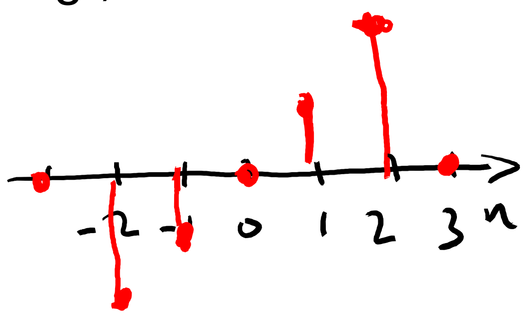
Two-Sided Z-Transform

Given a discrete-time signal $x[n]$, $-\infty < n < \infty$, its two-sided Z-transform is

$$\mathcal{Z}(x[n]) = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{10.4}$$

defined in a region of convergence (ROC) in the Z-plane.

E.g., Determine the Z-transform of $x[n] = \sum_{k=-2}^{2} k\delta[n-k]$.



$= -2\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\delta[n-2]$

$\Rightarrow X(z) = -2z^{2} - z + z^{-1} + 2z^{-2}$

$= \dfrac{-2z^{4} - z^{3} + z + 2}{z^{2}}$

DOUBLE POLE @ $z = 0$

$RoC(x[n]) = \{\mathbb{C} \setminus z = 0\}$

ie: ENTIRE $\mathbb{C}$-PLANE EXCEPT THE ORIGIN

One-Sided Z-Transform

The **one-sided Z-transform** is defined for causal signals, $x[n] = 0$ for $n < 0$, or for signals that are made causal by multiplying them with the unit-step signal $u[n]$:

$$X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n} \quad (10.5)$$

in a region of convergence $\mathcal{R}_c$

The two-sided Z-transform can be expressed in terms of the one-sided Z-transform as follows:

$$X(z) = \mathcal{Z}(x[n]u[n]) + \mathcal{Z}(x[-n]u[n])\,|_z - x[0] \quad (10.6)$$

i.e., replace z^{-1} by z
(or, replace z by z^{-1})

The region of convergence of $X(z)$ is

$$\mathcal{R} = \mathcal{R}_c \cap \mathcal{R}_{ac}$$

where $\mathcal{R}_c$ is the region of convergence of $\mathcal{Z}(x[n]u[n])$ and $\mathcal{R}_{ac}$ the region of convergence of $\mathcal{Z}(x[-n]u[n])\,|_z$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]u[n]z^{-n} + \sum_{n=-\infty}^{0} x[n]u[-n]z^{-n} - x[0] \\ &= \mathcal{Z}(x[n]u[n]) + \sum_{m=0}^{\infty} x[-m]u[m]z^{m} - x[0] \end{aligned}$$

Region of Convergence (RoC)

The RoC is the area in the ($\mathbb{C}$-valued) z-domain for which the summation is finite.

$$|X(z)| = \left|\sum_n x[n]z^{-n}\right| \le \sum_n |x[n]||r^{-n}e^{j\omega n}| = \sum_n |x[n]|r^{-n} < \infty$$

$$RoC(x) = \{z = re^{j\Omega} \in \mathbb{C} \mid x[n]r^{-n} \text{ is absolutely summable}\}$$

Poles cannot belong to the RoC

The poles of a Z-transform $X(z)$ are complex values $\{p_k\}$ such that

$$X(p_k) \to \infty$$

while the zeros of $X(z)$ are the complex values $\{z_k\}$ that make

$$X(z_k) = 0$$

Possible Regions of Convergence

The region of convergence (ROC) of the Z-transform of a signal $x[n]$ of finite support $[N_0,\ N_1]$, where $-\infty < N_0 \le n \le N_1 < \infty$,

$$X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n} \qquad (10.7)$$

is the whole z-plane, excluding the origin $z = 0$ and/or $z = \pm\infty$ depending on N_0 and N_1.

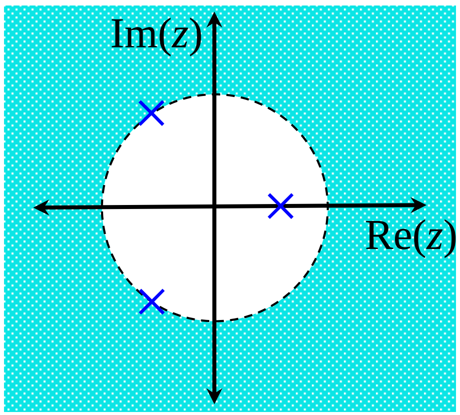
The Z-transform $X(z)$ of an infinite-support

1. **causal signal** $x[n]$ has a region of convergence $|z| > r_{min} = \max\{|p_i|\}$, where $\{p_i\}$ are the poles of $X(z)$, i.e., the region of convergence is the outside of a circle of radius r_{min}
2. **anti-causal signal** $x[n]$ has as region of convergence the inside of the circle defined by $|z| < r_{max} = \min\{|p_i|\}$.
3. **two – sided signal** $x[n]$ has as region of convergence $R_1 < |z| < R_2$, or the inside of a torus of inside radius R_1 and outside radius R_2 corresponding to the maximum and minimum radii of the poles of $X_c(z)$ *and* $X_a(z)$, or the Z-transforms of the causal and anti-causal components of $x[n]$.

Region of Convergence (RoC)

If ZT has poles (not at the origin or at ∞), RoC may appear as one of 3 forms, depending on signal causality.

Causal: $x[n] = 0 \; \forall n < 0$

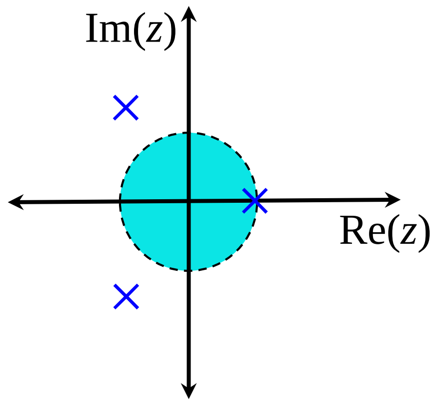


$R_c = \{z = re^{j\Omega}: r > \max\{|p_i|\}\}$

I.e., Outside circle of largest pole.

Acausal (or Noncausal): $\exists \; n < 0 \ni x[n] \neq 0$

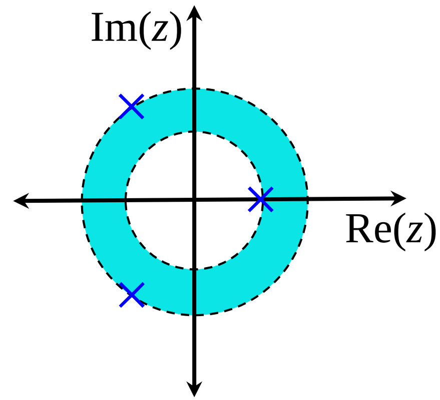
Anti-Causal: $x[n] = 0 \; \forall n > 0$



$R_{ac} = \{z = re^{j\Omega}: r < \min\{|p_i|\}\}$

I.e., Inside circle of smallest pole.

Two-Sided



$R_c \cap R_{ac}$

Plot all pole locations. The RoC is bordered by poles but may not contain any.

Examples: $u[n], -u[-n-1], 1$

Example: unit-step function *u[n]*

$$U(z) = \sum_{k=-\infty}^{\infty} u[k]z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad RoC(U(z)) = \{z \,|\, |z| > 1\}$$

NB: Same as for u[n] so they only differ in the RoC

Example: $v[n] = -u[-n-1] = u[n] - 1$

$$V(z) = \sum_{k=-\infty}^{\infty} -u[-k-1]z^{-k} = -\sum_{k=-\infty}^{-1} z^{-k} = -z\sum_{k=0}^{\infty} z^{k} = \frac{z}{z-1}$$

$$RoC(V(z)) = \{z \,|\, |z| < 1\}$$

Example: $\forall n \in \mathbb{Z}, x[n] = 1 = u[n] - v[n]$

$$X(z) = \sum_{k=-\infty}^{\infty} z^{-k}$$

$$RoC(X(z)) = RoC(U(z)) \cap RoC\big(V(z)\big) = \emptyset$$

Properties of the RoC

Property 1: The RoC is bordered by, but does not contain any poles.

Property 2: If *x[n]* is a finite sequence (has finite support), the RoC is the entire z-plane ($\mathbb{C}$-plane) except possibly *z=0* or *z* at infinity.

i.e., $\exists K \ni x[n] = 0 \forall n \leq K$

Property 3: If *x[n]* is a right-sided sequence (including causal signals), the RoC is of the form $|z| > r_{min} = \max_k |p_k|$ (p_k are the poles of *X(z)) (possibly excluding z at infinity)*.

i.e., $\exists L \ni x[n] = 0 \forall n \geq L$

Property 4: If *x[n]* is a left-sided sequence the RoC is of the form $|z| < r_{max} = \min_k |p_k|$(possibly excluding z=0).

Property 5: If *x[n]* is a two-sided sequence (infinite-duration on both sides) the RoC is of the form $r_1 < |z| < r_2$ where r_1 and r_2 are magnitudes of two poles of *X(z)*.

Z-Transform Pairs

Table 10.1 One-sided Z-transforms of Common Signals

One-sided Z-transforms		
Function of Time		**Function of z, ROC**
(1)	$\delta[n]$	1, Whole z-plane
(2)	$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
(3)	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
(4)	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
(5)	$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
(6)	$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
(7)	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(8)	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(9)	$\alpha^n\cos(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > 1$
(10)	$\alpha^n\sin(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > \lvert\alpha\rvert$

Z-Transform Properties

Table 10.2 Basic Properties of One-sided Z-transform

	Causal signals and constants	$\alpha x[n],\ \beta y[n]$	$\alpha X(z),\ \beta Y(z)$
P1	Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
P2	Convolution sum	$(x * y)[n] = \sum_k x[n]y[n-k]$	$X(z)Y(z)$
P3	Time shifting – causal	$x[n-N]$ N integer	$z^{-N}X(z)$
P4	Time shifting – non-causal	$x[n-N]$ $x[n]$ non-causal, N integer	$z^{-N}X(z) + x[-1]z^{-N+1}$ $+x[-2]z^{-N+2} + \cdots + x[-N]$
P5	Time reversal	$x[-n]$	$X(z^{-1})$
P6	Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$
P7	Multiplication by n^2	$n^2x[n]$	$z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz}$
P8	Finite difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z) - x[-1]$
P9	Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$
P10	Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
P11	Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Chaparro Ex. 10.22

Determine all possible impulse responses for a discrete-time filter with transfer function $H(z) = \frac{1+2z^{-1}+z^{-2}}{(1-0.5z^{-1})(1-z^{-1})}$. $= k_0 + \frac{k_1}{1-0.5z^{-1}} + \frac{k_2}{1-z^{-1}}$

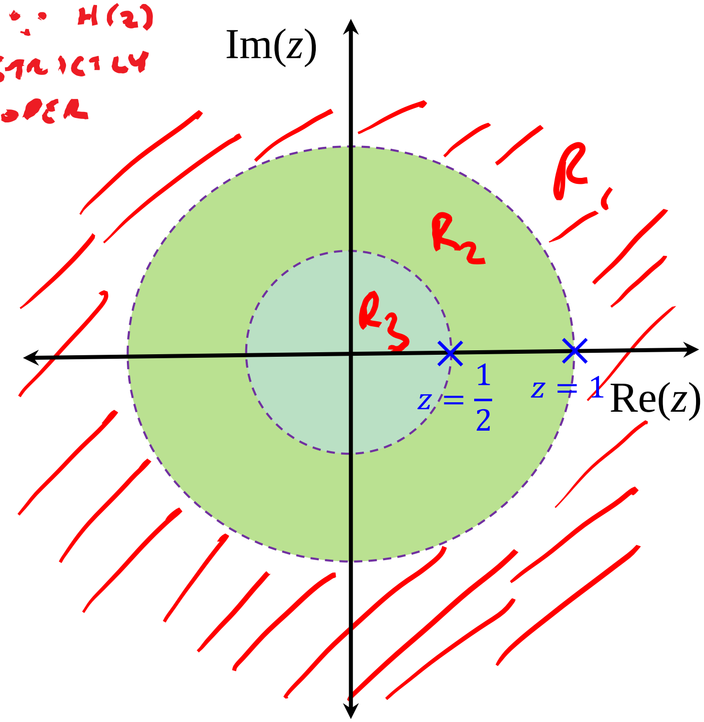
CAN USE PFE AS BEFORE:

NEEDED ∵ $H(z)$ NOT STRICTLY PROPER

$$k_0 = \lim_{z^{-1}\to\infty} H(z) = \frac{1}{(-\frac{1}{2})(-1)} = 2$$

$$k_1 = \lim_{z^{-1}\to 2} (1-0.5z^{-1})H(z) = \frac{1+4+4}{-1} = -9$$

$$k_2 = \lim_{z^{-1}\to 1} (1-z^{-1})H(z) = \frac{1+2+1}{1-\frac{1}{2}} = 8$$



ASIDE: $\mathcal{Z}^{-1}\{k_0\} = k_0\delta[n]$

WHICH IS BOTH CAUSAL AND ANTICAUSAL

Chaparro Ex. 10.22 (cont.)

$$\Rightarrow H(z) = 2 + \frac{-9}{1-0.5z^{-1}} + \frac{8}{1-z^{-1}}$$

$\Rightarrow$ 3 POSSIBLE ROCS:

R_1: $|z|>1$ (CAUSAL): $h_1[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$

R_2: $\frac{1}{2} < |z| < 1$ (TWO-SIDED): $h_2[n] = 2\delta[n] \underbrace{- 9\left(\frac{1}{2}\right)^n u[n]}_{\text{CAUSAL}} \underbrace{- 8u[-n-1]}_{\text{ANTI-CAUSAL}}$

R_3: $|z| < \frac{1}{2}$ (ANTI-CAUSAL): $h_3[n] = 2\delta[n] + \left[9\left(\frac{1}{2}\right)^n - 8\right]u[-n-1]$

ASIDE: NONE OF THESE ARE BIBO STABLE SINCE NONE CONTAIN THE UNIT CIRCLE

Convolution Sum & Transfer Function

The output $y[n]$ of a causal LTI system is calculated using the convolution sum

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k] \tag{10.18}$$

where $x[n]$ is a causal input and $h[n]$ the impulse response of the system. The Z-transform of $y[n]$ is the product

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z) \tag{10.19}$$

and the transfer function of the system is thus defined as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{output } y[n]]}{\mathcal{Z}[\text{input } x[n]]} \tag{10.20}$$

i.e., $H(z)$ transfers the input $X(z)$ into the output $Y(z)$.

Convolution of Finite Sequences

- *The length of the convolution of two sequences of lengths $\widetilde{M}$ and $\widetilde{N}$ is $\widetilde{M} + \widetilde{N} - 1$.*
- *If one of the sequences is of infinite length, the length of the convolution is infinite. Thus, for infinite impulse response IIR or recursive filters the output is always of infinite length for any input signal, given that the impulse response of these filters is of infinite length.*

Why did I add the "~"? If these sequences are filter impulse responses, consider the relationship between sequence length and filter order.

E.g., Consider $h_1[n]$ & $h_2[n]$ to be causal with $h_1[n] \neq 0$ and $h_2[n] \neq 0$. Let $h_T[n] = (h_1 * h_2)[n]$.

	Length	"Order"
h_1	$\widetilde{M}$	$M = \widetilde{M} - 1$
h_2	$\widetilde{N}$	$N = \widetilde{N} - 1$
h_T	$\widetilde{M}+\widetilde{N} - 1 = \tilde{L}$	$L = \tilde{L} - 1 = M + N$

Efts: Consider what modifications are needed if we allow $h_1[n]$ & $h_2[n]$ to be noncausal and/or $h_1[n] = 0$ and/or $h_2[n] = 0$.

Non-Causal Right-Sided Sequences

Let $x_1[n]$ be the input to a non-causal LTI system, with an impulse response $h_1[n]$ such that $h_1[n] = 0$ for $n < -N_1 < 0$. Assume $x_1[n]$ is also non-causal, i.e., $x_1[n] = 0$ for $n < -N_0 < 0$. The output $y_1[n] = [x_1 * h_1][n]$ has a Z-transform

Implicit that $h_1[-N_1] \neq 0$ and $x_1[-N_0] \neq 0$.

$$Y_1(z) = X_1(z)H_1(z) = [z^{N_0}X(z)][z^{N_1}H(z)]$$

where $X(z)$ and $H(z)$ are the Z-transforms of a causal signal $x[n]$ and of a causal impulse response $h[n]$. If we let

$$y[n] = [x * h][n] = \mathcal{Z}^{-1}[X(z)H(z)]$$

then

$$y_1[n] = [x_1 * h_1][n] = y[n + N_0 + N_1].$$

$\Rightarrow$If $x_1[n]$ & $h_1[n]$ are non-causal right-sided sequences, delay them to get causal signals, convolve them, then advance them by the same total delay.

Inverse Z-Transform by Long Division

When a rational function $X(z) = B(z)/A(z)$, having as ROC the outside of a circle of radius R (i.e., $x[n]$ is causal), is expressed as

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

by dividing $B(z)$ *by* $A(z)$, then the inverse is the sequence $\{x[0],\ x[1],\ x[2], \ldots\}$ *or*

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

ie: USING LONG-DIVISION

eg $\dfrac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow$

$$\begin{array}{r|l} & 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \ldots \\ \hline 1-\frac{1}{2}z^{-1} & 1 \\ & -(1-\frac{1}{2}z^{-1}) \\ \hline & \frac{1}{2}z^{-1} \\ & -(\frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}) \\ \hline & \frac{1}{4}z^{-2} \end{array}$$

TF of FIR Systems

Non-recursive or FIR Systems. The impulse response $h[n]$ of an FIR or non-recursive system

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

has finite length and is given by

$$h[n] = b_0 \delta[n] + b_1 \delta[n-1] + \cdots + b_M \delta[n-M]$$

Its transfer function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\ &= \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^M} \end{aligned}$$

with all its poles at the origin $z = 0$ *(multiplicity M) and as such the system is BIBO stable.*

Every FIR system is automatically BIBO stable!

TF of IIR Systems

Recursive or IIR Systems. The impulse response $h[n]$ of an IIR or recursive system

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

has (possible) infinite length impulse response given by

$$h[n] = \mathcal{Z}^{-1}[H(z)] = \mathcal{Z}^{-1}\left[\frac{\sum_{m=0}^{M} b_m z^{-m}}{1+\sum_{k=1}^{N} a_k z^{-k}}\right]$$

$$= \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}\right] = \sum_{\ell=0}^{\infty} h[\ell]\delta[n-\ell]$$

where $H(z)$ *is the transfer function of the system. If the poles of* $H(z)$ *are inside the unit circle, or* $A(z) \neq 0$ *for* $|z| \geq 1$, *the system is BIBO stable.* (for a causal system)

BIBO stability of an IIR system depends on the TF pole locations.

The system is stable if the TF RoC includes the unit circle.

Connections of LTI Systems

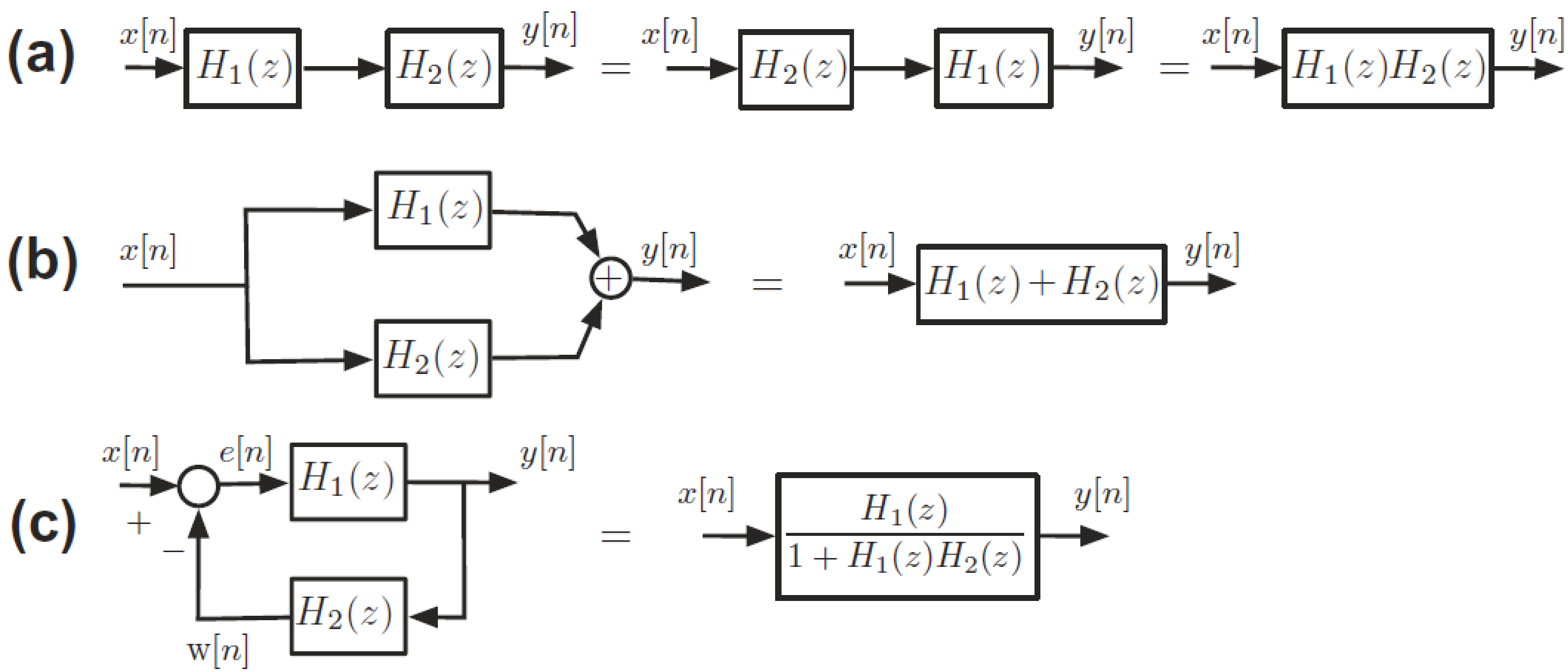


FIGURE 10.7

Connections of LTI systems: (a) cascade, (b) parallel, and (c) negative feedback.

DT connected systems work the same as CT connected systems.

Steady-State & Transient Responses

Just as with the Laplace transform, the steady-state response of a difference equation

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

is due to simple poles of $Y(z)$ *on the unit circle. Simple or multiple poles inside the unit circle give a transient, while multiple poles on the unit circle or poles outside the unit circle create an increasing response.*

Recall for CT systems, the steady-state response corresponded to simple poles on the $j\omega$-axis while transients corresponded to poles in the open LHP. Finally, poles in the RHP or repeated poles on the $j\omega$-axis result in instability. These results are consistent with the mappings from slide 9.4