MT1 Q1c (c.f., WW2, Q6c)

PERIODICITY OF $x(t)$ w/ $T_0 = 8$ RESULTS IN $y(t)$ HAVING SAME PERIOD

c) (9pts) Compute $y(t)$=$[x*h](t)$ for $h(t) = e^{-6t}u(t)$ and $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-8k)$.

$$\boxed{y(t) = \begin{cases} \dfrac{e^{-6t}}{1-e^{-48}} & 0 \le t < 8 \\ y(t-8k) & \text{OTHERWISE}, k \in \mathbb{Z} \end{cases}}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-6\tau}u(\tau) \sum_{k=-\infty}^{\infty} \underbrace{\delta(t-\tau-8k)}_{\text{NONZERO FOR } \tau = t-8k} d\tau$$

$$= \int_{0}^{\infty} e^{-6\tau} \sum_{k=-\infty}^{\infty} \delta(t-\tau-8k) d\tau$$

SOLVE FOR $0 \le t < 8$ ($\because$ $y(t)$ IS PERIODIC):

$\Rightarrow$ ONLY $k \le 0$ CONTRIBUTES

SIFTING PROPERTY $\Rightarrow$ $y(t) = \sum_{k=-\infty}^{0} e^{-6(t-8k)} = e^{-6t}(1+r+r^2+\cdots)$

WHERE $r = e^{-48}$

$$= e^{-6t} \times \frac{1}{1-r} = \frac{e^{-6t}}{1-e^{-48}}$$

RoC Intuition

For any function $f(t)$, $-\infty < t < \infty$, its one-sided Laplace transform $F(s)$ is defined as

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt, \quad \text{ROC} \tag{3.6}$$

or the two-sided Laplace transform of a causal or made-causal signal.

This ensures the signal being transformed is causal.

Infinitesimally before t=0, ensures no calculation ambiguity for impulse function.

Region of Convergence: area in ($\mathbb{C}$-valued) s-domain for which integral exists.

1-Sided LT Pair from Chaparro Table 3.2

(4)	$e^{-at}u(t), a > 0$ ← Unnecessary so Misleading	$\frac{1}{s+a}, \mathcal{R}e[s] > -a$

MT2 Q2 (S4 Causality)

2. Systems Classification

(17pts) For each system with the given input-output relationship, indicate in the table whether the classification is correct (☑), incorrect (☒) or there is insufficient information to determine ([?]). (Aside: Grading on the table is not linear since a single mistake in a row will result in a penalty of half of the associated marks; correct work outside the table justifying the choices may result in additional partial marks.)

i. $y_1(t) = S_1\big(x(t)\big) = x(t) \cdot x(t+1)$

ii. $y_2(t) = S_2\big(x(t)\big) = x(t) \cdot \cos\frac{\pi t}{4}$

iii. $y_3(t) = S_3\big(x(t)\big) = x(t) + 4$

iv. $y_4(t) = S_4\big(x(t)\big)$ so that $\frac{dy_4}{dt} - y_4(t) = x(t)$

	Linear	Time-Invariant	Causal	BIBO stable
S_1	✗	✓	✗	✓
S_2	✓	✗	✓	✓
S_3	✗	✓	✓	✓
S_4	✓	✓	?	?

$\mathcal{L} \Rightarrow (s-1)Y = X \Rightarrow H_4 = \frac{Y}{X} = \frac{1}{s-1}$

$\Rightarrow h_4(t) = \mathcal{L}^{-1}\{H_4(s)\} = \begin{cases} e^{t}u(t) & \text{IF CAUSAL [ROC: } Re(s)>1] \\ -e^{t}u(-t) & \text{IF ACAUSAL [ROC: } Re(s)<1] \end{cases}$

NOT BIBO STABLE

BIBO STABLE

Time-Invariance Proof

TIME-INVARIANT $\Leftrightarrow z_3(t) = z_4(t)$ WHERE

$$x \rightarrow \boxed{S} \xrightarrow{v} \boxed{D_\tau} \xrightarrow{z_3} \qquad x \rightarrow \boxed{D_\tau} \xrightarrow{w} \boxed{S} \xrightarrow{z_4}$$

$$y_1(t) = S_1\big(x(t)\big) = x(t) \cdot x(t+1)$$

$$v(t) = x(t)\,x(t+1) \Rightarrow z_3(t) = v(t-\tau) = x(t-\tau)\,x(t-\tau+1)$$

$$w(t) = x(t-\tau) \quad \Rightarrow z_4(t) = w(t)\,w(t+1) = x(t-\tau)\,x(t+1-\tau) = z_3(t) \Rightarrow TI$$

$$y_2(t) = S_2\big(x(t)\big) = x(t) \cdot \cos\frac{\pi t}{4}$$

$$v(t) = x(t)\cos\left(\frac{\pi t}{4}\right) \Rightarrow z_2(t) = v(t-\tau) = x(t-\tau)\cos\left(\frac{\pi(t-\tau)}{4}\right)$$

$$w(t) = x(t-\tau) \Rightarrow z_4(t) = x(t-\tau)\cos\left(\frac{\pi t}{4}\right) \neq z_3(t) \Rightarrow \text{TIME-VARYING}$$

Table 5.1 Basic Properties of Fourier Transform

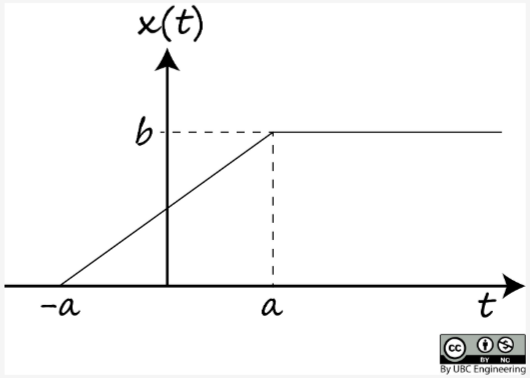
		Time Domain	Frequency Domain
	Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\omega), Y(\omega), Z(\omega)$
P1	Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
P2	Expansion/contraction in time	$x(\alpha t),\ \alpha \neq 0$	$\frac{1}{\|\alpha\|} X\left(\frac{\omega}{\alpha}\right)$
P3	Reflection	$x(-t)$	$X(-\omega)$
P4	Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \|x(t)\|^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|X(\omega)\|^2 d\omega$
P5	Duality	$X(t)$	$2\pi x(-\omega)$
P6	Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1,$ integer	$(j\omega)^n X(\omega)$
P7	Frequency differentiation	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
P8	Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
P9	Time shifting	$x(t - \alpha)$	$e^{-j\alpha\omega} X(\omega)$
P10	Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
P11	Modulation	$x(t)\cos(\omega_c t)$	$0.5[X(\omega - \omega_c) + X(\omega + \omega_c)]$
P12	Periodic signals	$x(t) = \sum_k X_k e^{jk\omega_0 t}$	$X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$
P13	Symmetry	$x(t)$ real	$\|X(\omega)\| = \|X(-\omega)\|$
			$\angle X(\omega) = -\angle X(-\omega)$
P14	Convolution in time	$z(t) = [x * y](t)$	$Z(\omega) = X(\omega)Y(\omega)$
P15	Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\omega)$
P16	Cosine transform	$x(t)$ even	$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt,$ real
P17	Sine transform	$x(t)$ odd	$X(\omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt,$ imaginary

"Dummy Variable"

Convolution in Frequency

WW5, Q1

Consider the signal $x(t)$ given in the figure below. Assume $a = \frac{1}{3}$ and $b = 3.33333333333333$.



a) Find a closed form expression for the Fourier transform $X(\omega)$ of the signal $x(t)$.

Hint: Use the integration and differentiation properties, as well as the Fourier transform pair of a rectangular pulse.

$X(\omega) =$

b) Find the Fourier transform $G(\omega)$ of the signal $g(t) = x(t) - \frac{5}{3}$.

$G(\omega) =$

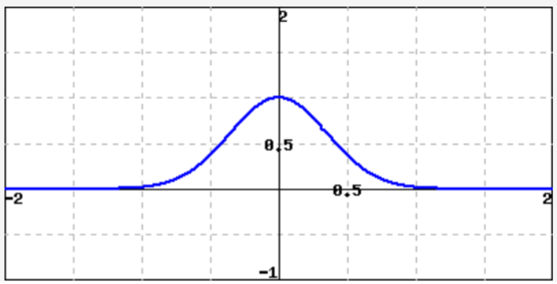
Correct Answer

$$\frac{10\sin\left(\frac{w}{3}\right)}{jw^2} + \frac{10\pi D(w)}{3}$$

$$\frac{10\sin\left(\frac{w}{3}\right)}{jw^2}$$

WW5, Q4a: FT of $x(t) = e^{-at^2}$

Hint: For (A), one approach to find the FT is to apply the properties of Duality and of differentiation (both in time and frequency).

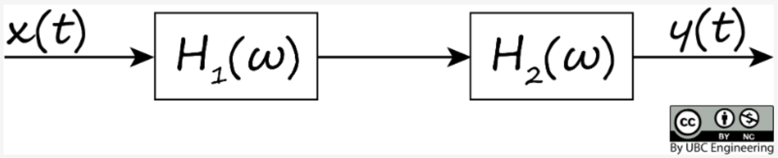
	Signal, $x(t)$	FT of signal, $X(\omega)$	Figure Number
A	e^{-4t^2} 		?

Correct Answer

$$\sqrt{\frac{\pi}{4}}e^{\frac{-w^2}{16}}$$

WW5, Q6

Two filters with frequency responses $H_1(\omega) = jw$ and $H_2(\omega) = e^{-4jw}$ for $-\infty < \omega < \infty$, are cascaded together so that the output of the first filter is fed as the input to the second, as shown in the figure below.



Suppose that the input to this cascaded system is the signal $x(t) = cos\left(\frac{\pi t}{5}\right)[u(t+5) - u(t-5)]$.

a) Find the output, $y(t)$, of this cascaded system.

$y(t) =$

Hint: think about what each filter does and do your calculations in the time domain.

b) We now reverse the cascading order of the two filters. Find the output, $y'(t)$, of this new cascaded system.

$y'(t) =$

c) Does the output depend on the order of cascading? ?

Correct Answer
$\frac{-\pi}{5}\sin\left(\frac{\pi}{5}(t-4)\right)(u(t+5-4) - u(t-5-4))$
$\frac{-\pi}{5}\sin\left(\frac{\pi}{5}(t-4)\right)(u(t+5-4) - u(t-5-4))$
No

WW5, Q8

NB: In this Webwork problem, take $sinc(t) = sin(t)/t$ (in contrast, in Signal Processing literature, $sinc(t) = sin(\pi t)/(\pi t)$.

Find the Fourier transform $X_1(\omega)$, $X_2(\omega)$, and $X_3(\omega)$ of the signals $x_1(t)$, $x_2(t)$, and $x_3(t)$, using the Fourier transform pair $x(t) = u(t+1) - u(t-1) \longleftrightarrow X(\omega) = 2sinc(\omega)$. Then select the Fourier transform property you used for each signal, from the corresponding drop-down menu.

In your answers, enter "w" for omega.

a) $x_1(t) = -4u(t+2) + 9u(t) - 5u(t-2)$

$X_1(\omega) =$ [ ] [? ▾]

b) $x_2(t) = cos(10\pi t)[u(t+1) - u(t-1)]$

$X_2(\omega) =$ [ ] [? ▾]

c) $x_3(t) = 3[u(t+\frac{1}{3}) - u(t-\frac{1}{3})]$

$X_3(\omega) =$ [ ] [? ▾]

Correct Answer
$2\,\mathrm{sinc}(w)\left(-4e^{iw} + 5e^{-iw}\right)$
Time-shift
$\mathrm{sinc}(w + \pi \cdot 10) + \mathrm{sinc}(w - \pi \cdot 10)$
Modulation
$2\,\mathrm{sinc}\left(\frac{w}{3}\right)$
Compression in time