Laplace Transforms

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One-Sided Laplace Transform

For causal signals and causal systems, the one-sided LT is used:

For any function $f(t)$, $-\infty < t < \infty$, its one-sided Laplace transform $F(s)$ is defined as

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt, \quad \text{ROC} \tag{3.6}$$

or the two-sided Laplace transform of a causal or made-causal signal.

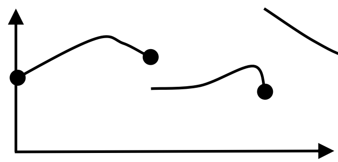
This ensures the signal being transformed is causal.

Infinitesimally before t=0, ensures no calculation ambiguity for impulse function.

Region of Convergence: area in (ℂ-valued) s-domain for which integral exists.

Aside: Chaparro starts the chapter with the 2-sided LT but I've opted to start with review of the 1-sided LT with which you are familiar through the pre-requisite MATH 255/256.

Time-Domain $\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftharpoons}}$ s-Domain



Focus here on I/O behavior so assume relaxed system (only showing ZSR)

Time-domain (t-domain)

$\mathcal{L}$

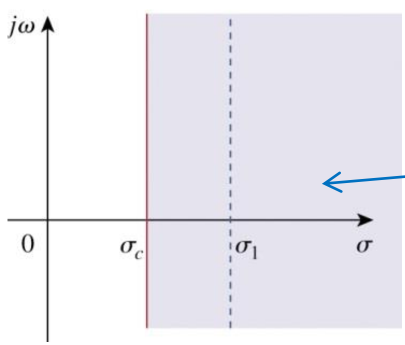
$x(t)$ → **System** $h(t)$ / $H(s)$ → $y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau = [x * h](t)$

$\mathcal{L}^{-1}$

Laplace/complex frequency domain (s-domain)

$X(s)$ → $H(s)$ → $Y(s) = H(s)X(s)$

OUTPUT CALCULATION IS MUCH EASIER IN THE S-DOMAIN



Solution only valid in the (shaded) Region of Convergence (RoC).

1-Sided LT Motivation

- Time-Domain convolution is equivalent to s-Domain multiplication
- Transform linear DEs into algebraic equations (easier to solve)
- Incorporate ICs in the solution automatically
- Provide the total response (natural+forced) in one operation

Additional Comments

- Not all functions have a LT (e.g., consider $f(t)=1/t$).
- $(f_1(t)=f_2(t)) \Rightarrow (F_1(s)=F_2(s))$. Is the converse true? Not quite.
  However, can say $(F_1(s)=F_2(s)) \Rightarrow (f_1(t)=f_2(t))$ except possibly at discontinuities. Exact values at discontinuities aren't important to engineers if they are finite.
- $f(t)$: $\mathbb{R} \rightarrow \mathbb{R}$. What about $F(s)$?
  $F(s)\colon \mathbb{C} \rightarrow \mathbb{C}$. This involves 4-D visualization (difficult to develop intuition).

1-Sided LT Properties

You must be familiar with the LT Properties and Pairs in the tables on slides 4.5 & 4.6. They will be provided on an exam but you must know how to employ them.

Table 3.1 Basic Properties of One-sided Laplace Transforms

	Causal functions and constants	$\alpha f(t), \beta g(t)$	$\alpha F(s), \beta G(s)$
P1	Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
P2	Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s} F(s)$
P3	Frequency shifting	$e^{\alpha t} f(t)$	$F(s-\alpha)$
P4	Multiplication by t	$tf(t)$	$-\frac{dF(s)}{ds}$
P5	Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
P6	Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0-) - f^{(1)}(0)$
P7	Integral	$\int_{0-}^{t} f(t')dt'$	$\frac{F(s)}{s}$
P8	Expansion/contraction	$f(\alpha t), \alpha \neq 0$	$\frac{1}{\|\alpha\|} F\left(\frac{s}{\alpha}\right)$
P9	Initial value	$f(0-) = \lim_{s \to \infty} sF(s)$	

1-Sided LT Pairs

What do you observe about f(t) when the ROC has $Re(s) > -a$?

Table 3.2 One-sided Laplace Transforms

	Function of time $f(t)$	**Function of** s**, ROC** $F(s)$
(1)	$\delta(t)$	1, whole $s-$plane
(2)	$u(t)$	$\frac{1}{s}, \mathcal{R}e[s] > 0$ — SIMPLE POLE AT ORIGIN
(3)	$r(t)$	$\frac{1}{s^2}, \mathcal{R}e[s] > 0$
(4)	$e^{-at}u(t), a > 0$	$\frac{1}{s+a}, \mathcal{R}e[s] > -a$ — SIMPLE POLE NOT AT ORIGIN
(5)	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}, \mathcal{R}e[s] > 0$
(6)	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\Omega_0^2}, \mathcal{R}e[s] > 0$
(7)	$e^{-at}\cos(\omega_0 t)u(t), a > 0$	$\frac{s+a}{(s+a)^2+\omega_0^2}, \mathcal{R}e[s] > -a$
(8)	$e^{-at}\sin(\omega_0 t)u(t), a > 0$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}, \mathcal{R}e[s] > -a$
(9)	$2Ae^{-at}\cos(\omega_0 t + \theta)u(t), a > 0$	$\frac{A\angle\theta}{s+a-j\omega_0} + \frac{A\angle-\theta}{s+a+j\omega_0}, \mathcal{R}e[s] > -a$
(10)	$\frac{1}{(N-1)!}t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$ — POLE OF DEGREE N
(11)	$\frac{1}{(N-1)!}t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
(12)	$\frac{2A}{(N-1)!}t^{N-1}e^{-at}\cos(\omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\omega_0)^N}, \mathcal{R}e[s] > -a$

Aside: I prefer
$$e^{\sigma t}u(t), \sigma < 0 \rightleftharpoons \frac{1}{s-\sigma}, Re(s) > \sigma$$
(more on slide 4.11)

Examples: Find the LTs

E.g.: $f(t)=\delta(t)+2u(t)-3e^{-2t}$ $\xrightarrow{\mathcal{L}} F(s) = 1 + \frac{2}{s} - \frac{3}{s+2} = \frac{s^2+s+4}{s(s+2)}$

use F1 F2 F4 P1

RoC: $Re(s) > 0$

E.g.: $f(t)=t^2\sin(2t)u(t)$ $\xrightarrow{\mathcal{L}} F(s) = -\frac{d}{ds}\left[-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right] = \frac{12s^2-16}{(s^2+4)^3}$

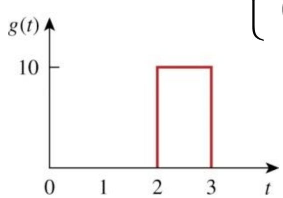
use F6, $(P4)^2$

RoC: $Re(s)>0$

E.g.: $g(t)=\begin{cases}10 & \text{for } 2\le t\le 3\\ 0 & \text{otherwise}\end{cases}$ $= 10\left[u(t-2)-u(t-3)\right]$

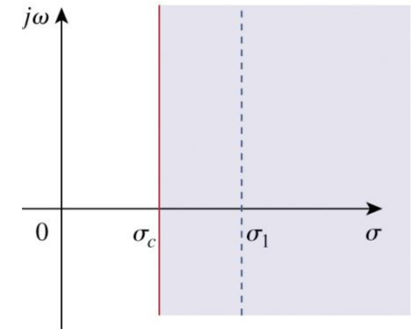
$\downarrow \mathcal{L}$

$G(s) = \frac{10}{s}\left[e^{-2s}-e^{-3s}\right], \quad Re(s)>0$



1-Sided Inverse LT

If the RoC for *F(s)* is Re(*s*)>σ_c, then the inverse Laplace transform is given by:



$$\boxed{\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds}$$

Fortunately, in E221, this computation isn't required but you'll need to generate a partial fraction expansion (PFE) and use look-up tables.

Algorithm to find inverse LT:

1. Find all poles of *F(s)*. ID them as *simple* vs. *repeated* vs. *complex*.
2. Find partial fraction expansion (PFE) in basic terms.
3. Look up inverse of each basic term in tables.

Consider *F(s)=N(s)/D(s)* where *N(s)* & *D(s)* are polynomials in s with degree (*N(s)*)<degree(*D(s)*)=n. "Poles" of *F(s)* are the roots p_i of $D(s)=0$ so we can write: $D(s)=(s-p_1)\,(s-p_2)\cdots(s-p_n)$

Examples: Find the Inverse LTs

Given $F(s) = 1 + \frac{4}{s+3} - \frac{5s}{s^2+16}$, $Re(s) > 0$, find $f(t)$.

After slide 4.23, we'll see that omitting this RoC allows for multiple solutions (3 in this example).

$$f(t) = \delta(t) + 4e^{-3t}u(t) - 5\cos 4t\, u(t) = \underline{\delta(t) + [4e^{-3t} - 5\cos 4t]u(t)}$$

Given $F(s) = \frac{6(s+2)}{(s+1)(s+3)(s+4)}$, $Re(s) > -1$, find $f(t)$.

$$= \frac{k_1}{s+1} + \frac{k_2}{s+3} + \frac{k_3}{s+4} \quad \text{(ie: FIND PFE)}$$

where $k_1 = \lim_{s \to -1} (s+1)F(s) = \frac{6(1)}{(2)(3)} = 1$; $k_2 = 3$; $k_3 = -4$

$$\Rightarrow F(s) = \frac{1}{s+1} + \frac{3}{s+3} - \frac{4}{s+4} \xrightarrow{\mathcal{L}^{-1}} \underline{f(t) = [e^{-t} + 3e^{-3t} - 4e^{-4t}]u(t)}$$

Poles of $F(s)$

There are 3 relatively distinct types of poles that $F(s)$ may have (for this slide, we'll assume the RoC is $Re(s) > Re(p_i)$):

Simple: p_i is real, negative (p_i <0) and occurs with degree 1.

IN PFE, APPEARS AS $F_i(s) = \frac{k_i}{s-p_i} \xrightarrow{\mathcal{L}^{-1}} f_i(t) = k_i e^{p_i t} u(t)$

Repeated: p_i <0 and occurs with degree m≥2.

IN PFE, APPEARS AS $F_i(s) = \frac{k_1}{s-p_i} + \frac{k_2}{(s-p_i)^2} + \cdots + \frac{k_m}{(s-p_i)^m}$

$$\xRightarrow{\mathcal{L}^{-1}} f_i(t) = e^{p_i t}\left[\frac{k_1}{0!} + \frac{k_2 t}{1!} + \cdots + \frac{k_m t^{m-1}}{(m-1)!}\right]u(t)$$

Complex-Conjugate Pair: p_i=σ+jω with σ <0 and p_{i+1}=σ–jω= p_i^*

IN PFE, APPEARS AS $F_i(s) = \frac{k_1 s + k_2}{(s-p_i)(s-p_i^*)} = \frac{k_1 s + k_2}{(s-\sigma)^2 + \omega^2}$

$$\xRightarrow{\mathcal{L}^{-1}} f_i(t) = e^{\sigma t}\left[k_1 \cos\omega t + \frac{k_2 + k_1\sigma}{\omega} \sin\omega t\right]u(t)$$

$F(s)$ Partial Fraction Expansion

- Given $F(s)=N(s)/D(s)$ and the poles of $D(s)$, you need to find the coefficients in the PFE. I recommend the *Residue Method* for a pole's highest degree (i.e., $k_i = \lim_{s \to p_i} (s - p_i)F(s)$ if simple, $k_m = \lim_{s \to p_i} (s - p_i)^m F(s)$ if repeated). For the others (complex poles and lower degrees of a pole), I recommend a form of the *Algebraic Method* (examples on next two slides).
- Note subtle differences in my choice of notation compared to Chaparro text. Consider what reasons I might have for these differences.
  - Chaparro uses $(s+p_i)$ as a factor of $D(s)$ whereas I prefer $(s-p_i)$.

    I prefer to say that p_i is a pole (Chaparro says "$-p_i$" is the pole)

  - Chaparro uses $\{(s+\alpha)^2+\Omega^2\}$ as a factor but I prefer $\{(s-\sigma)^2+\omega^2\}$.

    It's more conventional to express the roots as $s = \sigma \pm j\omega$
    (instead of $s = -\alpha \pm j\Omega$)
- I specified the poles must be in the LHP. Why? Otherwise, signals in the time-domain grow exponentially (unstable) which is generally undesired. If simple poles allowed on the $j\omega$-axis, get steady-state periodic signals.

Example (Repeated Pole)

Given $G(s) = \frac{s^3+2s+6}{s(s+1)^2(s+3)}$, $Re(s) > 0$, find $g(t)$. $G(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} + \frac{k_4}{s+3}$

SIMPLE POLES: $k_1 = 2$; $k_4 = \frac{-27-6+6}{(-3)(-2)^2} = \frac{9}{4}$

REPEATED POLE: HIGHEST DEGREE: $k_3 = -\frac{3}{2}$

$G(1) = \frac{9}{16} = \frac{2}{1} + \frac{k_2}{2} + \frac{-3/2}{4} + \frac{9/4}{4} \Rightarrow k_2 = \frac{-13}{4}$) ALGEBRAIC METHOD

CHOOSE CONVENIENT VALUE BUT CANNOT BE A POLE

$\Rightarrow G(s) = \frac{2}{s} - \frac{13/4}{s+1} - \frac{3/2}{(s+1)^2} + \frac{9/4}{s+3}$

$\mathcal{L}^{-1} \rightarrow g(t) = \left[2 - \frac{13}{4}e^{-t} - \frac{3}{2}te^{-t} + \frac{9}{4}e^{-3t}\right]u(t)$

ASIDE: RESIDUE METHOD FOR k_2: $k_2 = \frac{1}{1!}\frac{d}{ds}\left[(s+1)^2 G(s)\right]\Big|_{s=-1}$

Example ($\mathbb{C}$-conjugate Pair of Poles)

Given $G(s) = \frac{10}{(s+1)(s^2+4s+13)}$, $Re(s) > -1$, find $g(t)$.

$$G(s) = \frac{k_1}{s+1} + \frac{k_2 s + k_3}{(s+2)^2 + 3^2} = \frac{N(s)}{D(s)} \qquad k_1 = 1$$

$$N(s) = (1)(s^2+4s+13) + (k_2 s + k_3)(s+1) = s^2[1+k_2] + s[4+k_2+k_3] + 13 + k_3 = 10$$

$$\left.\begin{aligned} 13 + k_3 = 10 &\Rightarrow k_3 = -3 \\ 1 + k_2 = 0 &\Rightarrow k_2 = -1 \end{aligned}\right\} \text{Check } 4 + k_2 + k_3 = 0 \checkmark$$

$$\Rightarrow G(s) = \frac{1}{s+1} - \frac{(s+3)}{(s+2)^2+3^2} = \frac{1}{s+1} - \frac{s+2}{(s+2)^2+3^2} - \left(\frac{1}{3}\right)\frac{3}{(s+2)^2+3^2}$$

$$\mathcal{L}^{-1} \Downarrow$$

$$g(t) = \left[e^{-t} - e^{-2t}\left[\cos 3t + \frac{1}{3}\sin 3t\right]\right]u(t)$$

Transfer Functions

A transfer function (TF) is a ratio of two *s*-domain signals, assuming the system is relaxed (i.e., all I.C.s are zero). The TF *H(s)* typically relates some output response *Y(s)* to the input excitation *X(s)* (i.e., *H(s)=Y(s)/X(s)*).

Impedance and admittance are 2 examples of familiar TFs where the input excitation and output response are measured "at the same locations".

The **system function** *or* **transfer function** $H(s) = \mathcal{L}[h(t)]$, the Laplace transform of the impulse response $h(t)$ of a LTI system with input $x(t)$ and output $y(t)$, can be expressed as the ratio

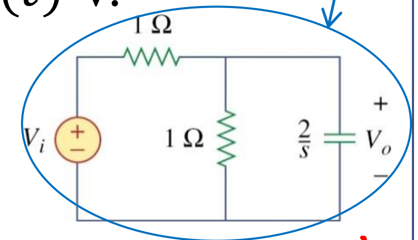
$$H(s) = \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]} == \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]} = \frac{Y(s)}{X(s)} \tag{3.24}$$

This function is called "transfer function" because it transfers the Laplace transform of the input to the output. Just as with the Laplace transform of signals, $H(s)$ characterizes *a LTI* system by means of its poles and zeros. Thus it becomes a very important tool in the analysis and synthesis of systems.

Example

All signals and systems (components) in this figure are represented in the s-domain.

Find (a) the TF $H(s) = V_0(s)/V_i(s)$, (b) the impulse response, (c) the unit step response, and (d) the response to , $v_i(t) = 8\cos(2t)\,u(t)$ V.



$\underbrace{}$ $1 \,//\, \frac{2}{s} = \frac{2/s}{1+2/s} = \frac{2}{s+2}$

a) $V_o = \dfrac{\frac{2}{s+2}}{\frac{2}{s+2}+1} V_i \Rightarrow \dfrac{V_o}{V_i} = \dfrac{2}{s+4} = H(s)$

b) $\mathcal{L}^{-1}\{H(s)\} = 2e^{-4t}u(t)\,V$

c) $Y_u(s) = \frac{1}{s} \times \frac{2}{s+4} = \frac{k_1}{s} + \frac{k_2}{s+4}$ where $k_1 = \frac{1}{2}$; $k_2 = -\frac{1}{2}$

$\Rightarrow y(t) = \frac{1}{2}(1 - e^{-4t})u(t)\,V$

d) $Y(s) = \left(\frac{8s}{s^2+4}\right)\left(\frac{2}{s+4}\right) = \frac{k_1}{s+4} + \frac{k_2 s + k_3}{s^2+4} \Rightarrow k_1 = -3.2$

$N(s) = -3.2(s^2+4) + k_2 s^2 + 4k_2 s + k_3 s + 4k_3 = 16s \Rightarrow \begin{cases} k_2 = 3.2 \\ k_3 = 3.2 \end{cases}$

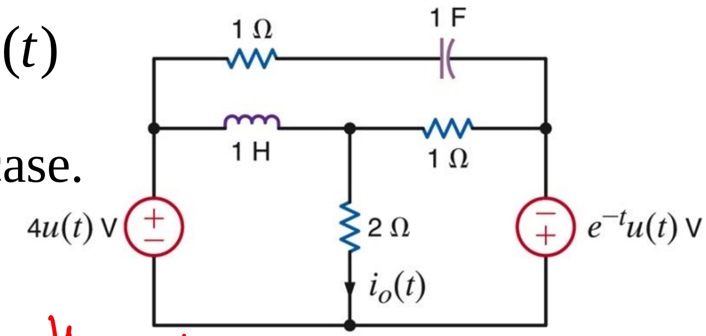
$\Rightarrow y(t) = 3.2\left[-e^{-4t} + \cos 2t + \frac{1}{2}\sin 2t\right]u(t)\,V$

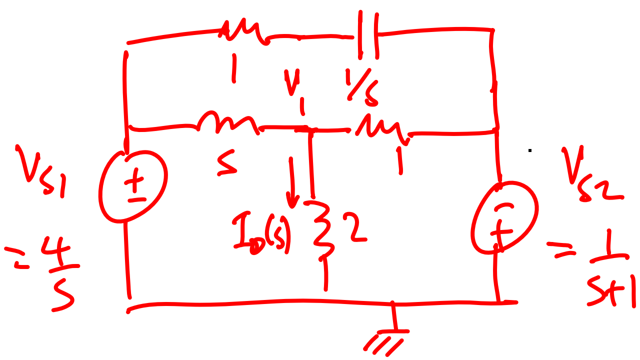
Example

ASIDE: THIS HAS 2 INPUTS SO IT COULD BE SOLVED W/ 2 TRANSFER FUNCTIONS

Find $I_o(s)$ and $i_o(t)$. Determine $i_o(0)$ and $\lim_{t\to\infty} i_o(t)$

Show that the IVT and FVT hold true in this case.





$$V_1\left(\frac{1}{s}+\frac{1}{2}+\frac{1}{1}\right) = \frac{V_{s1}}{s} - \frac{V_{s2}}{1}$$

$$\Rightarrow V_1(2+3s) = 2V_{s1} - 2sV_{s2} = \frac{8}{s} - \frac{2s}{s+1}$$

$$\Rightarrow V_1 = \frac{-2s^2+8s+8}{s(s+1)(3s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2/3}$$ WHERE $k_1 = 4$, $k_2 = -2$, $k_3 = -\frac{8}{3}$

$$\Rightarrow V_1(s) = \frac{4}{s} - \frac{2}{s+1} - \frac{8/3}{s+2/3} \Rightarrow I_o(s) = \frac{V_1(s)}{2} = \left(\frac{2}{s} - \frac{1}{s+1} - \frac{4}{3s+2}\right) A$$

$$\Rightarrow i_o(t) = \left[2 - e^{-t} - \frac{4}{3}e^{-2/3\,t}\right]u(t)\ A$$

$i_o(0) = 2 - 1 - \frac{4}{3} = -\frac{1}{3} A = \lim_{s\to\infty} sI_o(s) = 2 - 1 - \frac{4}{3}$ ✓ IVT; $i_o(\infty) = 2A = \lim_{s\to 0} sI_o(s) = 2A$ ✓ FVT

LTI Systems Represented by ODEs

The **complete response** $y(t)$ of a system represented by an Nth-order linear ordinary differential equation with constant coefficients,

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = \quad y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M \tag{3.38}$$

where $x(t)$ is the input and $y(t)$ the output of the system, and the initial conditions are

$$\{y^{(k)}(0), \quad 0 \leq k \leq N-1\} \tag{3.39}$$

is obtained by inverting the Laplace transform

$$Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s) \tag{3.40}$$

where $Y(s) = \mathcal{L}[y(t)]$, $X(s) = \mathcal{L}[x(t)]$ and

$a_N = 1$

System Transfer Function

System Characteristic Polynomial (provides system poles)

$$A(s) = \sum_{k=0}^{N} a_k s^k,$$

$$B(s) = \sum_{\ell=0}^{M} b_\ell s^\ell$$

$$I(s) = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right), \qquad a_N = 1$$

i.e., $I(s)$ depends on the initial conditions.

Complete Response=ZSR+ZIR

Letting

$$H(s) = \frac{B(s)}{A(s)} \quad \text{and} \quad H_1(s) = \frac{1}{A(s)}$$

the **complete response** $y(t) = \mathcal{L}^{-1}[Y(s)]$ of the system is obtained by the inverse Laplace transform of

$$Y(s) = H(s)X(s) + H_1(s)I(s) \tag{3.42}$$

which gives

$$y(t) = y_{zs}(t) + y_{zi}(t) \tag{3.43}$$

where

$y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$ is the system's zero-state response

$y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$ is the system's zero-input response

(Compare to slide 3.20.)

Complete Response=SS+Transient

The other common decomposition which also yields much insight separates the complete response into the steady-state response (permanent) and the transient response (nonpermanent).

In summary, when solving ordinary differential equations—with or without initial conditions—using Laplace we have:

(i) The steady-state component of the complete solution is given by the inverse Laplace transforms of the partial fraction expansion terms of $Y(s)$ that have simple poles (real or complex conjugate pairs) in the $j\omega$-axis.

(ii) The transient response is given by the inverse Laplace transform of the partial fraction expansion terms with poles in the left-hand s-plane, independent of whether the poles are simple or multiple, real or complex.

(iii) Multiple poles in the $j\omega$-axis and poles in the right-hand s-plane give terms that will increase as t increases making the complete response unbounded.

Q: Does (i) include a simple pole at the origin?

A: Yes, it's the Steady-State DC Value.

2-Sided Laplace Transform

The 1-sided LT is used for causal signals. For more general signals that include nonzero values for t<0, it usually is more appropriate to use the more general 2-sided LT.

The two-sided Laplace transform of a continuous-time function $f(t)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \quad s \in \text{ROC} \tag{3.3}$$

Caveat: Not to be confused with "Damping Ratio": $\zeta = -\sigma/\omega$

where the variable $s = \sigma + j\omega$, with σ a damping factor and ω frequency in rad/sec. ROC stands for the region of convergence of $F(s)$, i.e., where the infinite integral exists. The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds \quad \sigma \in \text{ROC} \tag{3.4}$$

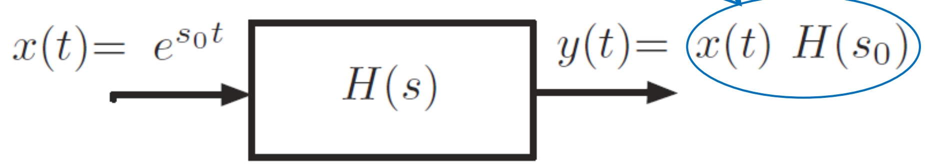
LTI System Eigenfunctions

Consider an LTI System with an input of the form $x(t) = e^{s_0 t}, s_0 \in \mathbb{C}$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau$$

$$= e^{s_0 t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-\tau s_0}d\tau}_{H(s_0)} = x(t)H(s_0)$$

I.e., Output is a scaled version of the input.



An input $x(t) = e^{s_0 t}$, $s_0 = \sigma_0 + j\omega_0$, is called an **eigenfunction** of a LTI system with impulse response $h(t)$ if the corresponding output of the system is

$$y(t) = x(t)\int_{-\infty}^{\infty} h(t)e^{-s_0 t}\, dt = x(t)H(s_0)$$

where $H(s_0)$ is the Laplace transform of $h(t)$ computed at $s = s_0$. This property is only valid for LTI systems, it is not satisfied by time-varying or non-linear systems.

The Inverse LT computes *f(t)* as an "infinite sum" of these eigenfunctions.

LT Existence & Function Poles/Zeros

For the Laplace transform $F(s)$ of $f(t)$ to exist we need that

$$\left| \int_{-\infty}^{\infty} f(t) e^{-st}\, dt \right| = \left| \int_{-\infty}^{\infty} f(t) e^{-\sigma t} e^{-j\omega t}\, dt \right|$$
$$\leq \int_{-\infty}^{\infty} | f(t) e^{-\sigma t} |\, dt < \infty$$

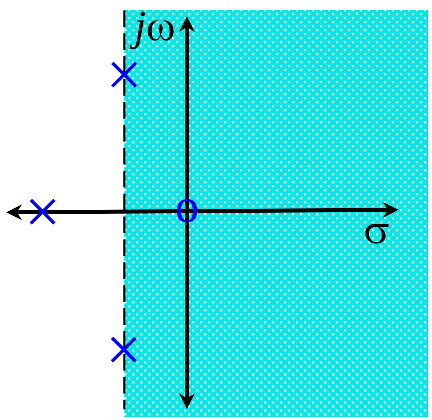
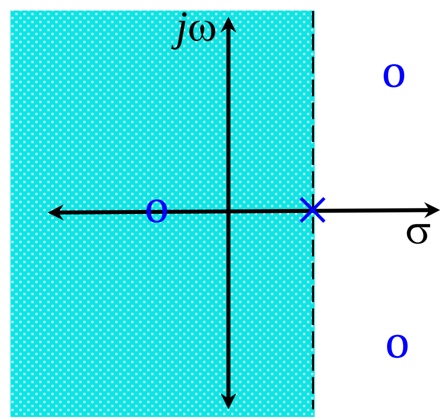
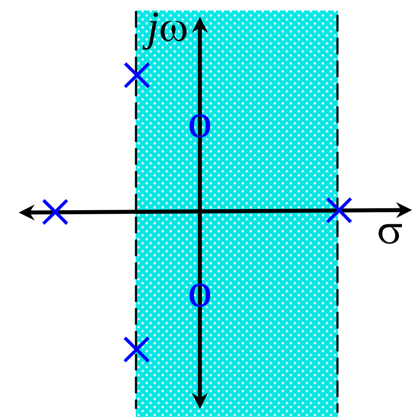
or that $f(t)e^{-\sigma t}$ be absolutely integrable. This may be possible by choosing an appropriate σ even in the case when $f(t)$ is not absolutely integrable. The value chosen for σ determines the ROC of $F(s)$. The frequency ω does not affect the ROC.

For a rational function $F(s) = \mathcal{L}[f(t)] = N(s)/D(s)$, its **zeros** are the values of s that make the function $F(s) = 0$, and its **poles** are the values of s that make the function $F(s) \to \infty$. Although only finite zeros and poles are considered, infinite zeros and poles are also possible.

Region of Convergence (RoC)

If the LT has poles, the RoC may appear as one of 3 forms, depending on the signal's causality.

Acausal (or Noncausal): $\exists t<0 \ni x(t)\neq 0$

Causal: $x(t)=0,\ t<0$	Anti-Causal: $x(t)=0,\ t>0$	Two-Sided
		
$R_c = \{(\sigma,\omega): \sigma > \max\{\sigma_i\}, \infty < \omega < \infty\}$	$R_{ac} = \{(\sigma,\omega): \sigma < min\{\sigma_i\}, \infty < \omega < \infty\}$	$R_c \cap R_{ac}$
I.e., Right of right-most pole.	I.e., Left of left-most pole.	

Plot all pole locations. The RoC is bordered by poles but may not contain any.

Using 1-Sided LT to Solve 2-Sided LT

The Laplace transform of a

- Finite support function $f(t)$, i.e., $f(t) = 0$ for $t < t_1$ and $t > t_2$, $t_1 < t_2$,

$$F(s) = \mathcal{L}[f(t)[u(t-t_1) - u(t-t_2)]] \quad \text{ROC : whole } s\text{-plane} \tag{3.7}$$

- Causal function $g(t)$, i.e., $g(t) = 0$ for $t < 0$, is

$$G(s) = \mathcal{L}[g(t)u(t)] \quad \mathcal{R}_c = \{(\sigma, \omega) : \sigma > \max\{\sigma_i\}, -\infty < \omega < \infty\} \tag{3.8}$$

where $\{\sigma_i\}$ are the real parts of the poles of $G(s)$.

- Anti-causal function $h(t)$, i.e., $h(t) = 0$ for $t > 0$, is

This signal is causal so can use 1-Sided LT

$$H(s) = \mathcal{L}[h(-t)u(t)]_{(-s)} \quad \mathcal{R}_{ac} = \{(\sigma, \omega) : \sigma < \min\{\sigma_i\}, -\infty < \omega < \infty\} \tag{3.9}$$

I.e., substitute "s" by "-s" in the LT.

where $\{\sigma_i\}$ are the real parts of the poles of $H(s)$.

- Two-sided function $p(t)$, i.e., $p(t) = p_{ac}(t) + p_c(t) = p(t)u(-t) + p(t)u(t)$, is

$$P(s) = \mathcal{L}[p(t)] = \mathcal{L}[p_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[p_c(t)u(t)] \quad \mathcal{R}_c \bigcap \mathcal{R}_{ac} \tag{3.10}$$

LT of Anti-causal Signals

Find $\mathscr{L}(u(-t))$

Method A (from Definition):

$$\mathcal{L}\{u(-t)\} = \int_{-\infty}^{\infty} u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{-\infty}^{0} = \frac{1}{-s} \quad \text{ROC}: \text{Re}(s) < 0$$

Method B (from 1-sided LT; see Chaparro eqn (3.9) from slide 4.24):

$$\mathcal{L}\{u(-t)\} = \left.\mathcal{L}\{u(t)\}\right|_{-s} = \frac{1}{-s}$$

POLE @ $s=0$ AND ANTICAUSAL

$\Rightarrow$ ROC : $\text{Re}(s) < 0$

Find $\mathscr{L}(e^{-at}u(-t))$

Method B:

$$\mathcal{L}\{e^{-at}u(-t)\} = \left.\mathcal{L}\{e^{at}u(t)\}\right|_{-s} = \frac{1}{-s-a} = \frac{-1}{s+a}$$

POLE @ $s=-a$ AND ANTICAUSAL

$\Rightarrow$ ROC : $\text{Re}(s) < -a$

Chaparro Example 3.6

Consider a non-causal LTI system with impulse response $h(t) = e^{-t}u(t) + e^{2t}u(-t) = h_c(t) + h_{ac}(t)$. Find the system function $H(s)$, its ROC, and indicate whether we could compute $H(j\Omega)$ from it.

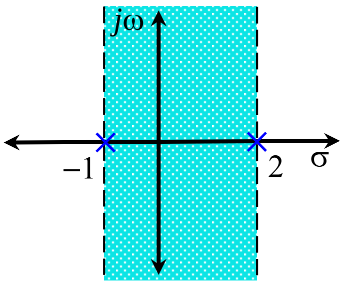
Solution:

Causal Portion: $H_c(s) = \dfrac{1}{s+1}$ $RoC: \sigma > -1$

Anti-causal Portion: $\mathcal{L}[h_{ac}(t)] = \mathcal{L}[h_{ac}(-t)u(t)]_{(-s)} = \dfrac{1}{-s+2}$ $RoC: \sigma < 2$

Total: $H(s) = \dfrac{1}{s+1} + \dfrac{1}{-s+2} = \dfrac{-3}{(s+1)(s-2)}$

RoC: $\{(\sigma, \Omega) : -1 < \sigma < 2, -\infty < \Omega < \infty\}$



RoC includes $j\omega$-axis so $H(j\omega)$ can be computed (we'll see later on that this means the signal Fourier Transform also exists).

RoC for $x(t) = u(t) - u(t-1)$

According to slide 4.24, Chaparro eqn (3.7), the RoC should be the entire s-plane since it has finite support (and is absolutely integrable).

$$\mathcal{L}\{x(t)\} = X(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

Since there is a pole at s=0, does this contradict the statement that the RoC cannot contain any poles?

CONSIDER TAYLOR SERIES EXPANSION OF e^{-s}:

$$e^{-s} = 1 - s + \frac{s^2}{2!} - \cdots = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \Rightarrow 1 - e^{-s} = -\sum_{k=1}^{\infty} \frac{(-s)^k}{k!}$$

$$\Rightarrow \frac{1-e^{-s}}{s} = \sum_{k=1}^{\infty} \frac{(-s)^{k-1}}{k!} = 1 - \frac{s}{2!} + \frac{s^2}{3!} - \cdots$$ SO THERE ARE NO ACTUAL POLES!

ALTERNATIVELY SEE $1-e^{-s}$ HAS A ZERO SO GET "POLE-ZERO CANCELLATION"

Revisit Slide 4.12

Given $G(s) = \frac{s^3+2s+6}{s(s+1)^2(s+3)}$ is the 2-sided LT, determine all possible RoCs and the associated time signals *g(t)*.

4 possibles

Solution:

Slide 4.12: RoC with Re(s)>0: $g(t) = \left[2 - \frac{13}{4}e^{-t} - \frac{3}{2}te^{-t} + \frac{9}{4}e^{-3t}\right]u(t)$

RoC: $-1 < Re(s) < 0$: $g(t) = -2u(-t) + \left[-\frac{13}{4}e^{-t} - \frac{3}{2}te^{-t} + \frac{9}{4}e^{-3t}\right]u(t)$

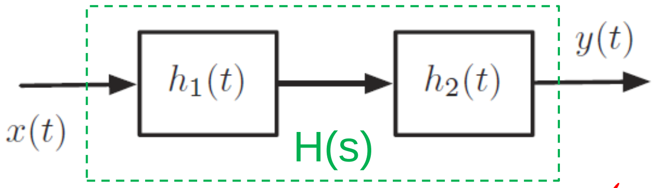
RoC: $-3 < Re(s) < -1$: $g(t) = \left[-2 + \left(\frac{13}{4} + \frac{3}{2}t\right)e^{-t}\right]u(-t) + \frac{9}{4}e^{-3t}u(t)$

RoC: $Re(s) < -3$: $g(t) = \left[-2 + \left(\frac{13}{4} + \frac{3}{2}t\right)e^{-t} - \frac{9}{4}e^{-3t}\right]u(-t)$

Revisit System Interconnections

Determine the overall TF given $H_i(s)=\mathscr{L}(h_i(t))$

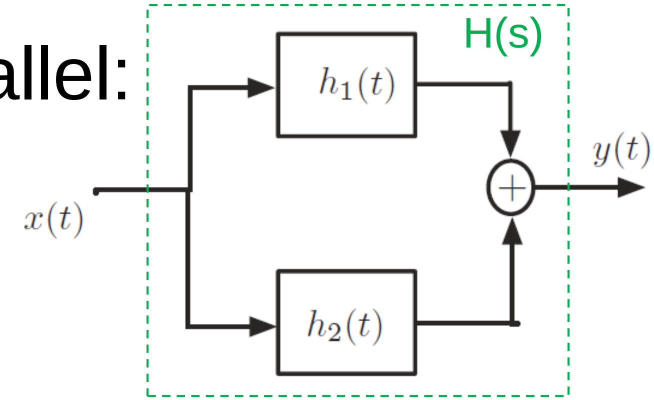
Cascade (Series):



$y(t) = (x * h)(t)$ where $h(t) = (h_1 * h_2)(t)$

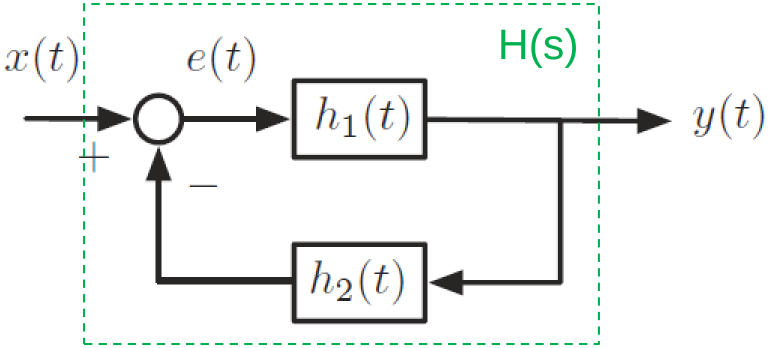
$Y(s) = H(s)X(s)$ where $H(s) = H_1(s)H_2(s)$

Parallel:



$Y(s) = H(s)X(s)$ where $H(s) = H_1(s) + H_2(s)$

Feedback:



$Y(s) = H_1(s)\left[X(s) - H_2(s)Y(s)\right]$

$\Rightarrow \left(1 + H_1(s)H_2(s)\right)Y(s) = H_1(s)X(s)$

$\Rightarrow Y(s) = H(s)X(s)$ where $H(s) = \dfrac{H_1(s)}{1 + H_1(s)H_2(s)}$

BIBO Stability from H(s) or h(t)

Two very important generalizations of the results in the above example are:

1. A LTI with a transfer function $H(s)$ and region of convergence $\mathcal{R}$ is BIBO stable if the $j\omega$-axis is contained in the region of convergence.
2. A causal LTI system with impulse response $h(t)$ or transfer function $H(s) = \mathcal{L}[h(t)]$ is BIBO stable if the following equivalent conditions are satisfied

(i) $H(s) = \mathcal{L}[h(t)] = \dfrac{N(s)}{D(s)}$, $j\omega$-axis in ROC of $H(s)$

(ii) $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt < \infty$, $h(t)$ is absolutely integrable

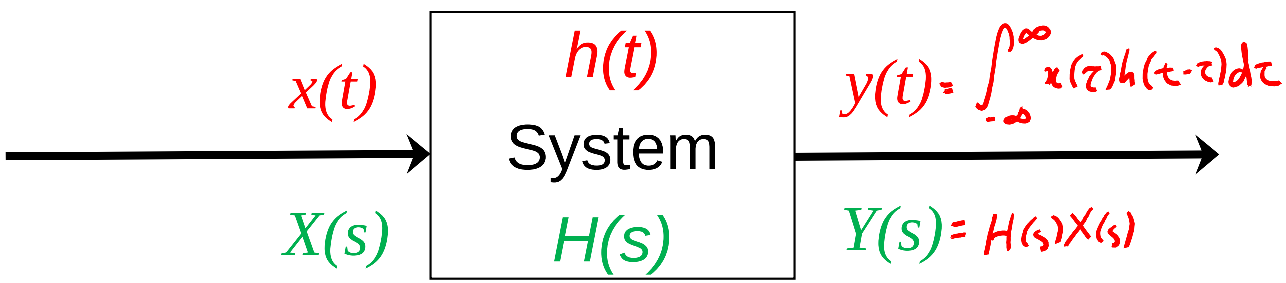
(iii) Poles of $H(s)$ are in the open left-hand s-plane (not including the $j\omega$-axis).

LT of Signal or System?

Q: Does the Laplace Transform apply to a signal or a system?

A: A signal. Do NOT ever talk about the LT of a system. You CAN talk about the LT of the system impulse response or equivalently, the system TRANSFER FUNCTION: $H(s)=\mathscr{L}(h(t))=Y(s)/X(s)$

but remember that this is the ratio of two LTs.



Impulse Response of Ideal LPF

$x(t)=\delta(t)$ → **Ideal Low-Pass Filter ($\omega_c=\pi$ rad/s)** → $y(t)=?$

CUTOFF FREQ

ONLY FREQS BELOW CUT-OFF ARE ALLOWED THRU

$$H(s) = H(\sigma+j\omega) = \begin{cases} 1 & \text{IF } |\omega| \le \omega_c \\ 0 & \text{OTHERWISE} \end{cases}$$

$$X(s) = \mathcal{L}\{x(t)\} = 1 \qquad Y(s) = H(s)X(s) = H(s)$$

A RARE TIME I'LL USE THE INVERSE L.T. EQN (SLIDE 4.8):

$$y(t) = \frac{1}{2\pi}\int_{-j\infty}^{j\infty} Y(s)e^{st}\,ds = \frac{1}{2\pi}\int_{-j\pi}^{j\pi} e^{st}\,ds = \frac{1}{2\pi}\left.\frac{e^{st}}{t}\right|_{-j\pi}^{j\pi}$$

FOR CONVENIENCE, CHOOSE $\sigma_i = 0$ IN LIMITS OF INTEGRATION

$$= \boxed{\frac{\sin \pi t}{\pi t} = \text{sinc}(t)}$$

WOW! IDEAL LPF IS IDEAL INTERPOLATOR (BUT IT'S NOT CAUSAL!)