1. **"Short" Answers:**

a) (3pts) Compute the energy and the power for the signal $x(t) = e^{-t}u(t)$.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = \left.\frac{e^{-2t}}{-2}\right|_0^{\infty} = -\frac{1}{2}(0-1) = \underline{\underline{\frac{1}{2}}}$$

$$P_x = \underline{\underline{0}} \quad \because E_x < \infty$$

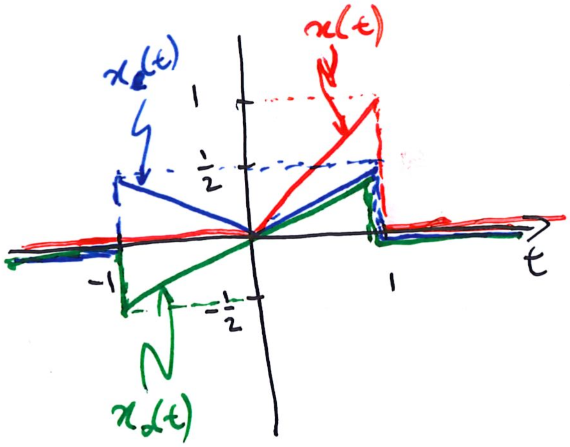
b) (7pts) Find the even and odd components of $x(t) = t[u(t) - u(t-1)] = x_e(t) + x_o(t)$. As well, sketch all 3 functions ($x(t)$, $x_e(t)$ and $x_o(t)$) on suitably-scaled graphs (if you can show it in a clear way, you can plot them all on the same graph).

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{t[u(t)-u(t-1)] + (-t)[u(-t)-u(-t-1)]}{2}$$

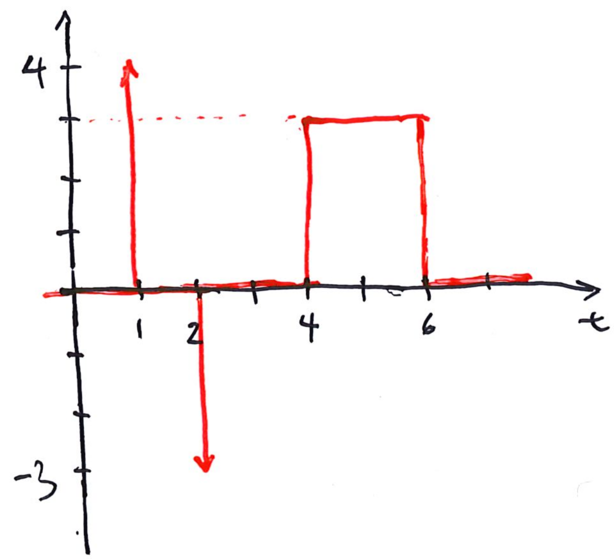
$$= \frac{t[u(-t-1) - u(-t) + u(t) - u(t-1)]}{2} = \underline{\underline{\frac{|t|[u(t+1)-u(t-1)]}{2}}}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \frac{t[u(t)-u(t-1)] - (-t)[u(-t)-u(-t-1)]}{2}$$

$$= \frac{t[-u(-t-1) + u(-t) + u(t) - u(t-1)]}{2} = \underline{\underline{\frac{t[u(t+1)-u(t-1)]}{2}}}$$



c) (9pts) For a system impulse response of $h(t) = 3\big(u(t-4) - u(t-6)\big) + 4\delta(t-1) - 3\delta(t-2)$, sketch this response and classify whether the system is linear, time-invariant, causal, memoryless or BIBO stable. For the classification, you may use your intuition except for time-invariance for which you must explicitly show how you arrived at your conclusion.



PREFACE: IN RETROSPECT, THE CLASSIFICATION ASPECT OF THIS QUESTION WAS NOT WELL-DEFINED, THERE TECHNICALLY IS NOT ENOUGH INFO TO CONCLUDE ANY OF THESE EXCEPT THE SYSTEM IS NOT MEMORYLESS.

A MORE WELL-DEFINED QUESTION WOULD BE TO, ASSUMING THE SYSTEM IS LTI, DETERMINE THE RELATIONSHIP $y(t) = S(x(t))$ AND CLASSIFY.

$$\Rightarrow y(t) = 4x(t-1) - 3x(t-2) + 3\int_{-\infty}^{t} x(\tau-4) - x(\tau-6)\, d\tau$$

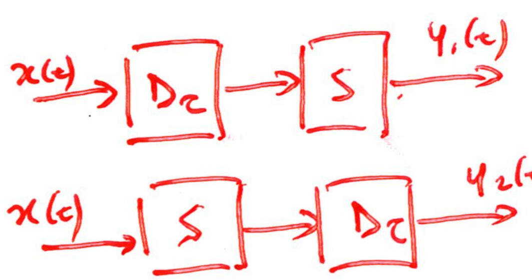
ASSUMING LTI:

$\Rightarrow$ CAUSAL ($\because h(t) = 0 \ \forall t < 0$)

$\Rightarrow$ BIBO STABLE ($\because \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$)

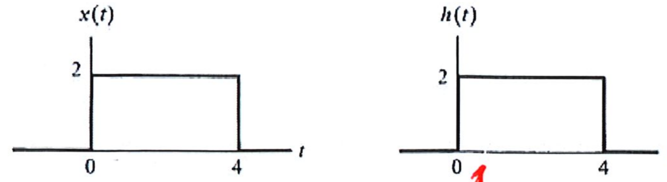
NOT MEMORYLESS

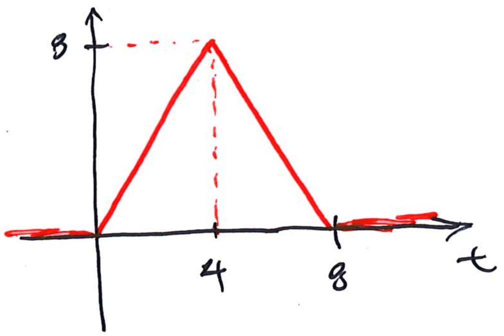
FOR TIME-INVARIANCE, REQUIRE



$y_1(t) \equiv y_2(t) \Leftrightarrow$ TIME-INVARIANT

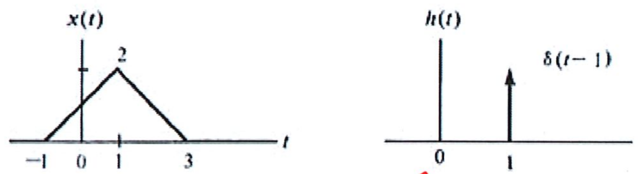
d) (3pts) Sketch $y(t)=[x*h](t)$ for $x(t)$ and $h(t)$ shown.

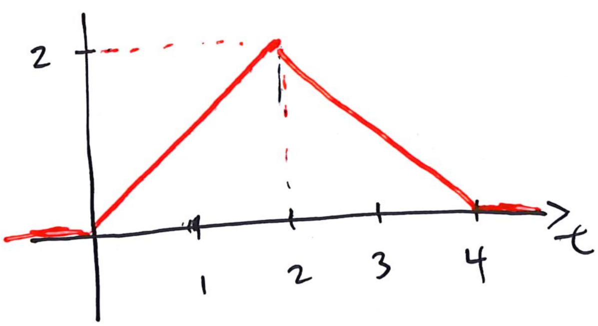




NB: RECALL $\mathcal{L}\{u(t)\} = \frac{1}{s}$ IS AN INTEGRATOR SO THIS WINDOW IS AN INTEGRATOR (x2) OF THE LAST 4 SECONDS

e) (2pts) Sketch $y(t)=[x*h](t)$ for $x(t)$ and $h(t)$ shown.





NB: THIS SYSTEM IS A PURE DELAY BY 1 SECOND

$\omega_1 = \frac{6\pi}{7}$ $\omega_2 = \frac{3\pi}{5}$

f) (6pts) For $x(t) = 4\cos\left(\frac{6\pi}{7}t\right) + \cos\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)$, determine the fundamental frequency ω_0 and find the complex exponential Fourier coefficients $\{X_k\}$.

$$\omega_0 = GCF(\omega_1, \omega_2) = \underline{\underline{\frac{3\pi}{35}}} = \frac{2\pi}{T_0} \Rightarrow T_0 = \frac{70}{3}$$

$$\cos\left(\omega t - \frac{\pi}{2}\right) = \sin(\omega t)$$

$$\Rightarrow x(t) = 4\cos\left(\frac{6\pi}{7}t\right) + \sin\left(\frac{3\pi}{5}t\right)$$

$$= \frac{4}{2}\left[e^{j\frac{6\pi}{7}t} + e^{-j\frac{6\pi}{7}t}\right] + \frac{1}{2j}\left[e^{j\frac{3\pi}{5}t} - e^{-j\frac{3\pi}{5}t}\right]$$

$\omega_1 = 10\omega_0$ $\omega_2 = 7\omega_0$

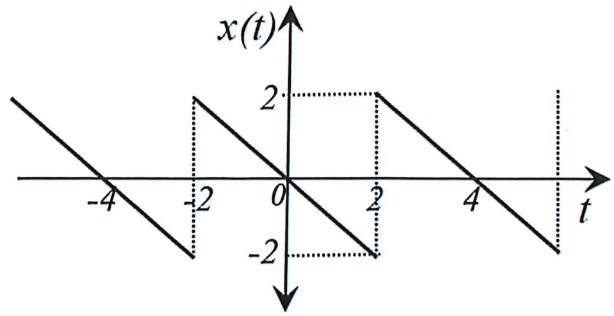
$$\Rightarrow X_k = \begin{cases} \pm\frac{1}{2j} = \mp\frac{j}{2} & \text{FOR } k = \pm 7 \\ 2 & \text{FOR } k = \pm 10 \\ 0 & \text{OTHERWISE} \end{cases}$$

2. **Fourier Series of a Sawtooth:**

a) (10pts) For the signal shown, compute the complex exponential Fourier Series representation (i.e., find $\{X_k\}$).

b) (4pts) Determine the coefficients $\{c_k\}$ & $\{d_k\}$ for the trigonometric representation.

c) (2pts) If the signal is input to an ideal low pass filter with cutoff frequency ω_c=10rad/s, determine the highest harmonic (value of k) that appears in the Fourier Series representation of the output.



$T_0 = 4 \Rightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$

a) $X_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (-t)e^{-jk\omega_0 t}\,dt$

$X_0 = 0$ BY INSPECTION

FOR $k \neq 0$:

$X_k = -\frac{1}{4}\int_{-2}^{2} t\, e^{-jk\omega_0 t}\,dt$

INTEGRATE BY PARTS:

$u = t \quad dv = e^{-jk\frac{\pi}{2}t}dt$

$du = dt \quad v = \frac{e^{-jk\frac{\pi}{2}t}}{-jk\pi/2}$

$= -\frac{1}{4}\left[\frac{j2t\, e^{-jk\frac{\pi}{2}t}}{k\pi}\Big|_{-2}^{2} - \cancel{\int_{-2}^{2}\frac{e^{-jk\frac{\pi}{2}t}}{-jk\pi/2}\,dt}^{\,0}\right]$

$\therefore$ AVERAGE VALUE OF $e^{-jk\omega_0 t}$ OVER A PERIOD IS 0 (THE SAME REASON DC TERMS DON'T CONTRIBUTE WHEN $k \neq 0$)

$= \frac{-j}{2k\pi}\left[2e^{-jk\pi} - (-2)e^{jk\pi}\right]$

$= \frac{-j2\cos k\pi}{k\pi} = \frac{-j2(-1)^k}{k\pi}$

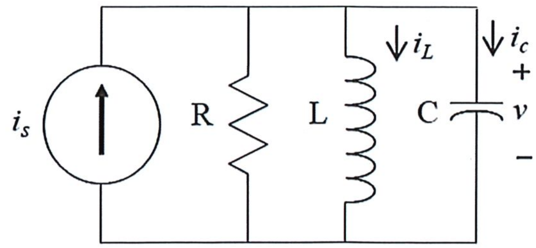
b) $c_k = Re\{X_k\} = 0 \quad k = 0, 1, 2, \ldots$

$d_k = -Im\{X_k\} = \frac{2(-1)^k}{k\pi} \quad k = 1, 2, 3, \ldots$

c) $6\left(\frac{\pi}{2}\right) < \omega_c < 7\left(\frac{\pi}{2}\right)$

$\hookrightarrow$ HIGHEST k IS $\underline{\underline{6}}$ (TIMES THE FUNDAMENTAL FREQ)

3. RLC Parallel Circuit



a) (7pts) For the circuit shown, determine the ODE relating the voltage $v(t)$ to the current source $i_s(t)$ as well as the corresponding transfer function $Z(s) = \frac{V(s)}{I_s(s)}$. [Hint: It might be easier to find the latter before the former.]

b) (12pts) For R=0.8Ω, C=0.25F and L=1H, determine the (i) impulse response, (ii) the unit step response and (iii) the response to a pure sinusoid $i_s(t) = \cos(\pi t)$ *Amperes* (note that this last function is ∀*t*, not just t>0). [Hint: The system eigenvalues for these parameters should be integers.]

a) $Z(s) = \left(\frac{1}{R} + \frac{1}{sL} + sC\right)^{-1} = \underline{\underline{\frac{sLR}{s^2LCR + sL + R}}}$

ASIDE: THIS IS THE NET IMPEDANCE SEEN BY THE CURRENT SOURCE

$\Rightarrow (s^2LCR + sL + R)V = sLRI_s$

$\mathcal{L}^{-1}$ $\Rightarrow \underline{\underline{LCR\ddot{v} + L\dot{v} + Rv = LR\frac{di_s}{dt}}}$ ODE (ALSO CAN BE DERIVED USING KCL)

OR $\underline{\frac{di_s}{dt} = C\ddot{v} + \frac{\dot{v}}{R} + \frac{v}{L}}$

b) (i) FOR $i_s(t) = \delta(t)$

$V(s) = \mathcal{L}\{\delta(t)\} \cdot Z(s) = \frac{s}{\frac{s^2}{4} + \frac{5s}{4} + 1} = \frac{4s}{(s+1)(s+4)} = \frac{k_1}{s+1} + \frac{k_2}{s+4}$

$k_1 = -\frac{4}{3}$; $k_2 = \frac{-16}{-3} = \frac{16}{3} \Rightarrow \underline{\underline{v(t) = \left[-\frac{4}{3}e^{-t} + \frac{16}{3}e^{-4t}\right]u(t)}}$ Volts

(ii) FOR $i_s(t) = u(t)$

$V(s) = \frac{Z(s)}{s} = \frac{4}{(s+1)(s+4)} = \frac{k_1}{s+1} + \frac{k_2}{s+4} \Rightarrow k_1 = \frac{4}{3}, k_2 = -\frac{4}{3}$

$\Rightarrow \underline{\underline{v(t) = \frac{4}{3}(e^{-t} - e^{-4t})u(t)}}$ Volts

(iii) FOR $i_s(t) = \cos \pi t \Rightarrow \omega = \pi \Rightarrow Z(j\pi) = \frac{4j\pi}{-\pi^2 + 5j\pi + 4} = \frac{4\pi}{5\pi + j(\pi^2 - 4)}$

$\Rightarrow \underline{\underline{v(t) = \frac{4\pi}{\sqrt{(5\pi)^2 + (\pi^2-4)^2}} \cos\left(\pi t - \tan^{-1}\left(\frac{\pi^2-4}{5\pi}\right)\right)}}$

4. **LTI System Transfer Function** $H(s) = \frac{s^3+9s^2+9s+60\cancel{s}}{(s^2+4s)(s^2+2s+5)}$

a) (18pts) Determine all the possible RoCs and for each one, determine the corresponding impulse response. [Hint: If done correctly, your coefficients in your PFE will all be integers.]

b) (2pts) Explain which, if any of the systems from (a) are BIBO stable.

a) $H(s) = \underbrace{\frac{k_1}{s} + \frac{k_2}{s+4} + \frac{k_3 s + k_4}{s^2+2s+5}}$ $\Rightarrow k_1 = \frac{60}{(4)(5)} = 3;\ k_2 = \frac{-64+(9)(-4)(-3)+60}{(-4)(16-8+5)} = -2$

POLES @ $s = 0, -4, -1 \pm j2$

$$s^3+9s^2+9s+60 = k_1(s+4)(s^2+2s+5) + k_2 s(s^2+2s+5) + (k_3 s + k_4)(s^2+4s)$$
$$= s^3(k_1+k_2+k_3) + s^2(6k_1+2k_2+4k_3+k_4) + s(13k_1+5k_2+4k_4) + 20k_1$$

$\Rightarrow 1 = k_1+k_2+k_3 \Rightarrow k_3 = 1-3-(-2) = 0$

$\Rightarrow 9 = 6k_1+2k_2+4k_3+k_4 \Rightarrow k_4 = 9-(6)(3)-(2)(-2) = 5$

$\Rightarrow H(s) = \frac{3}{s} - \frac{2}{s+4} - \frac{5}{2}\underbrace{\frac{2}{(s+1)^2+2^2}}$

WRITTEN THIS WAY SO TABLE 3.2 PAIR (8) USED

4 POSSIBLE RoCs:

(I) (CAUSAL) $Re(s) = \sigma > 0 \Rightarrow h(t) = \left[3 - 2e^{-4t} - \frac{5}{2}e^{-t}\sin 2t\right]u(t)$

(II) $-1 < \sigma < 0 \rightarrow h(t) = -3u(-t) - \left[2e^{-4t} + \frac{5}{2}e^{-t}\sin 2t\right]u(t)$

(III) $-4 < \sigma < -1 \rightarrow h(t) = \left[-3 + \frac{5}{2}e^{-t}\sin 2t\right]u(-t) - 2e^{-4t}u(t)$

(II) and (III): TWO SIDED

(IV) (ANTICAUSAL) $\sigma < -4 \Rightarrow h(t) = \left[-3 + \frac{5}{2}e^{-t}\sin 2t + 2e^{-4t}\right]u(-t)$

b) BIBO STABILITY REQUIRES RoC CONTAINS $j\omega$-AXIS

$\Rightarrow$ NONE OF THESE ARE BIBO STABLE

ASIDE: (I) & (II) ARE MARGINALLY STABLE

5. Transfer Function from System Characteristics

(15pts) Determine the transfer function $H(s)$ for an LTI system with the following characteristics:

A • Its impulse response $h(t)$ is real-valued $\forall t \in \mathbb{R}$.

B • $H(s)$ has exactly two zeros with one occurring at $s=1+j$.

C • The signal $\frac{d^2}{dt^2}h(t) + 3\frac{d}{dt}h(t) + 2h(t)$ consists of an impulse with unknown energy, the first derivative of an impulse with unknown energy and a unit step.

D • $\lim_{t\to\infty} h(t) = 3$.

$$C \Rightarrow \frac{d^2h}{dt^2} + 3\frac{dh}{dt} + 2h = a\delta(t) + b\frac{d}{dt}\delta(t) + u(t)$$

$$\mathcal{L}\{ \quad s^2H + 3sH + 2H = a + bs + \frac{1}{s} \Rightarrow H(s) = \frac{a + bs + \frac{1}{s}}{s^2+3s+2} = \frac{bs^2+as+1}{s(s+1)(s+2)}$$

$$A \,\&\, B \Rightarrow \quad bs^2+as+1 = K[s-(1+j)][s-(1-j)] = K(s^2-2s+2)$$

$$D \Rightarrow \lim_{t\to\infty} h(t) = \lim_{s\to 0} sH(s) = K = 3$$

$$\Rightarrow H(s) = \underline{\underline{\frac{3(s^2-2s+2)}{s(s+1)(s+2)}}}$$