Fourier Transform

- FT from the FS
- FT Properties & Pairs
- Dirichlet Conditions for Convergence
- FT from the LT
- Duality
- Spectral Representation

Fourier Transform

Aperiodic signals do not have a Fourier Series but might have a Fourier Transform.

An aperiodic, or non-periodic, signal $x(t)$ can be thought of as a periodic signal $\tilde{x}(t)$ with an infinite fundamental period. Using the Fourier series representation of this signal and a limiting process we obtain a Fourier transform pair

$$x(t) \quad \Leftrightarrow \quad X(\omega)$$

where the signal $x(t)$ is transformed into a function $X(\omega)$ in the frequency domain by the

$$\textbf{Fourier transform}: \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{5.1}$$

Whereas the FS involves a discrete sequence of frequencies (DC + fundamental + harmonics), the FT involves a continuum of frequencies.

while $X(\omega)$ is transformed into a signal $x(t)$ in the time-domain by the

$$\textbf{Inverse Fourier Transform}: \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\Omega t}\,d\omega \tag{5.2}$$

Fourier Transform from Fourier Series

Define a non periodic signal $x(t)$ in terms of a periodic signal $\tilde{x}(t)$ with infinite period $T_0 \to \infty$

Define $$x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$$

From F.S. definition $$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T_0}, \quad X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jn\omega_0 t} dt$$

issue of

Define Avoid ~~ratio X_n/T_0~~ going to zero $$\boxed{X(\omega_n)\Big|_{\omega_n = n\omega_0} = T_0 X_n}, \quad \Delta\omega = \omega_0 = \frac{2\pi}{T_0}$$

Derive $$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \frac{X(\omega_n)}{T_0} e^{jn\omega_n t} = \sum_{n=-\infty}^{\infty} X(\omega_n) e^{jn\omega_n t} \frac{\Delta\omega}{2\pi}; \quad X(\omega_n) = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-j\omega_n t} dt$$

Projection onto basis functions

As $T_0 \to \infty$

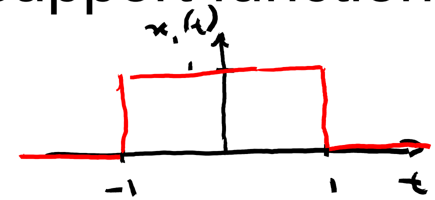
Fourier Transform Pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{X(\omega) e^{j\omega t}} \, d\omega \;\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}\; X(\omega) = \int_{-\infty}^{\infty} \boxed{x(t) e^{-j\omega t}} \, dt$$

Examples: Pulse and Windowed Ramp

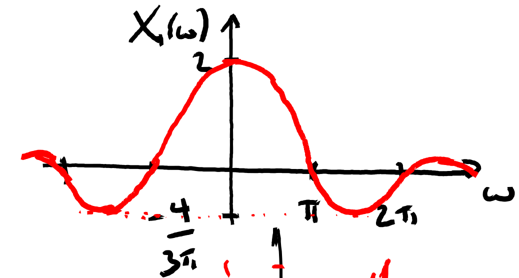
Find the FTs of these bounded and finite support functions.

$(a) x_1(t) = u(t+1) - u(t-1)$:



$$\int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt = \int_{-1}^{1} e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1}$$

$$= \frac{j}{\omega}\left(e^{-j\omega} - e^{j\omega}\right) = \frac{2\sin\omega}{\omega} = 2\,\mathrm{sinc}\,\frac{\omega}{\pi} = X_1(\omega)$$



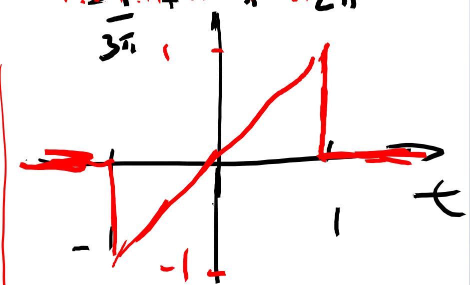
$(b) x_2(t) = t[u(t+1) - u(t-1)]$:

$$\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt = \int_{-1}^{1} t\, e^{-j\omega t}\, dt$$

INTEGRATE BY PARTS

$u = t \;;\; dV = e^{-j\omega t} dt$

$du = dt \;;\; V = \frac{e^{-j\omega t}}{-j\omega}$



$$= \left.\frac{t\, e^{-j\omega t}}{-j\omega}\right|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-j\omega t}}{-j\omega}\, dt$$

$$= \frac{e^{-j\omega} - (-1)\, e^{j\omega}}{-j\omega} - \left.\frac{1}{(j\omega)^2}\, e^{-j\omega t}\right|_{-1}^{1} = \frac{2j\cos\omega}{\omega} + \frac{1}{\omega^2}\left(2j\sin\omega\right) = \frac{2j}{\omega}\left(\cos\omega + \frac{\sin\omega}{\omega}\right) = X_2(\omega)$$

NB: As expected, $X_1(\omega)$ is real $\because x_1(t)$ is even and $X_2(\omega)$ is imaginary $\because x_2(t)$ is odd.

Table 5.1 Basic Properties of Fourier Transform

		Time Domain	Frequency Domain
	Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\omega), Y(\omega), Z(\omega)$
P1	Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
P2	Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\omega}{\alpha}\right)$
P3	Reflection	$x(-t)$	$X(-\omega)$
P4	Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
P5	Duality	$X(t)$	$2\pi x(-\omega)$
P6	Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1,$ integer	$(j\omega)^n X(\omega)$
P7	Frequency differentiation	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
P8	Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
P9	Time shifting	$x(t - \alpha)$	$e^{-j\alpha\omega} X(\omega)$
P10	Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
P11	Modulation	$x(t)\cos(\omega_c t)$	$0.5[X(\omega - \omega_c) + X(\omega + \omega_c)]$
P12	Periodic signals	$x(t) = \sum_k X_k e^{jk\omega_0 t}$	$X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$
P13	Symmetry	$x(t)$ real	$\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
			$\angle X(\omega) = -\angle X(-\omega)$
P14	Convolution in time	$z(t) = [x * y](t)$	$Z(\omega) = X(\omega)Y(\omega)$
P15	Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\omega)$
P16	Cosine transform	$x(t)$ even	$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt,$ real
P17	Sine transform	$x(t)$ odd	$X(\omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt,$ imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of ω
(1)	$\delta(t)$	1
(2)	$\delta(t - \tau)$	$e^{-j\omega\tau}$
(3)	$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
(4)	$u(-t)$	$\frac{-1}{j\omega} + \pi\delta(\omega)$
(5)	$\mathrm{sign}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\omega}$
(6)	$A, -\infty < t < \infty$	$2\pi A\delta(\omega)$
(7)	$Ae^{-at}u(t), a > 0$	$\frac{A}{j\omega+a}$
(8)	$Ate^{-at}u(t), a > 0$	$\frac{A}{(j\omega+a)^2}$
(9)	$e^{-a\|t\|}, a > 0$	$\frac{2a}{a^2+\omega^2}$
(10)	$\cos(\omega_0 t), -\infty < t < \infty$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
(11)	$\sin(\omega_0 t), -\infty < t < \infty$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
(12)	$p(t) = A[u(t + \tau) - u(t - \tau)], \tau > 0$	$2A\tau \frac{\sin(\omega\tau)}{\omega\tau}$
(13)	$\frac{\sin(\omega_0 t)}{\pi t}$	$P(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$
(14)	$x(t)\cos(\omega_0 t)$	$0.5[X(\omega - \omega_0) + X(\omega + \omega_0)]$

Dirichlet Conditions for Convergence

The Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

of a signal x(t) exists (i.e., we can calculate its Fourier transform via this integral) provided

- *x(t)* is absolutely integrable or the area under |*x(t)*| is finite,
- *x(t)* has only a finite number of discontinuities and a finite number of minima and maxima in any finite interval.

These are identical to those for the FS (c.f. slide 5.8).

Caveat: These are sufficient but not necessary conditions for the FT to exist. Table 5.2, signals 3, 4, 6, 10 & 11 are NOT absolutely integrable but have FTs (it's no coincidence that these FTs include the frequency-domain impulse).

FT from LT

If the RoC of $\hat{X}(s) = \mathscr{L}[x(t)]$ contains the $j\omega$-axis, then the FT of x(t) is given by : $X(\omega) = \mathscr{F}[x(t)] = \hat{X}(s)\big|_{s=j\omega}$

- This shows that the FT can be regarded as a special case of the LT for which $s=j\omega$.
- The notation used here is intended to show an explicit difference, that the LT is a function of a (2-D) complex parameter while the FT is a function of a (1-D) real-valued frequency.
- As with the Dirichlet conditions, this provides a sufficient, though not necessary condition for the FT to exist. The same functions that were problematic before do not have RoCs that include the $j\omega$-axis.

Chaparro Example 5.1

Discuss whether it is possible to use the LT to obtain the FT of the following functions:

$(a) x_1(t) = u(t)$: $\hat{X}_1(s) = \frac{1}{s}$

ROC: Re(s) > 0 doesn't include $j\omega$-axis

$\Rightarrow$ cannot use LT to find F.T

$(b) x_2(t) = e^{-2t}u(t)$: $\hat{X}_2(s) = \frac{1}{s+2}$

ROC: Re(s) > -2 does include $j\omega$-axis

$\Rightarrow X_2(\omega) = \mathcal{F}\{x_2(t)\} = \frac{1}{j\omega+2}$

$(c) x_3(t) = e^{-|t|}$: $\hat{X}_3(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{\cancel{1}\ 2}{1-s^2}$

$= e^{-t}u(t) + e^{t}u(-t)$

ROC₁: Re(s) > -1

ROC₂: Re(s) < 1

ROC = ROC₁ ∩ ROC₂

-1 < Re(s) < 1

does include $j\omega$-axis

$\Rightarrow X_3(\omega) = \frac{2}{1+\omega^2}$

"Rules of Thumb" for FT Calculation

Depending on the signal $x(t)$ there are several techniques for finding the Fourier transform:

1. Bounded signals with a finite time support
   $\Rightarrow$ Use FT integral directly or LT (e.g., slide 6.4)
2. Infinite time signals that include $j\omega$-axis in RoC
   $\Rightarrow$ Use LT, substituting $s = j\omega$ (e.g., slide 6.9)
3. Periodic signals
   $\Rightarrow$ Use $\delta(\omega\text{-}k\omega_0)$ scaled by FS coeffs (e.g., slide 6.11)
4. Signals not classified by 1 – 3
   $\Rightarrow$ Apply FT Properties, FT Pairs, Duality, etc. (e.g., slides 6.15 & 6.16)

Fourier Transform from Fourier Series

Representing a periodic signal x(t), of period T_0, by its Fourier series we have the following Fourier pair:

$$x(t) = \sum_k X_k e^{jk\omega_0 t} \Leftrightarrow X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0) \qquad (5.17)$$

Find the FT for the following:

$(a) x_1(t) = A$: THIS IS A DC (CONSTANT) TERM SO $X_0 = A$

$\Rightarrow X_1(\omega) = 2\pi A \delta(\omega)$ (PAIR (6) IN TABLE 5.2)

$$(b) x_2(t) = x(t) = \underbrace{4\cos\left(\frac{6\pi}{7}t\right)}_{2\left(e^{j\frac{6\pi}{7}t} + e^{-j\frac{6\pi}{7}t}\right)} + \underbrace{\cos\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)}_{\frac{1}{2}\left(e^{j\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)} + e^{-j\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)}\right)};$$

$\omega_0 = \frac{3\pi}{35} \Rightarrow \omega_1 = \frac{6\pi}{7} = 10\omega_0$, $\omega_2 = \frac{3\pi}{5} = 7\omega_0$

$X_{10} = 2 = X_{-10}$

$\Rightarrow X_7 = \frac{1}{2} e^{-j\frac{\pi}{2}} = \frac{-j}{2}$; $X_{-7} = \frac{j}{2}$

$$\Rightarrow X_2(\omega) = 2\pi\left[X_{-10}\delta(\omega + 10\omega_0) + X_{-7}\delta(\omega + 7\omega_0) + X_7\delta(\omega - 7\omega_0) + X_{10}\delta(\omega - 10\omega_0)\right]$$

$$= 2\pi\left[\frac{j}{2}\left(\delta\left(\omega + \frac{3\pi}{5}\right) - \delta\left(\omega - \frac{3\pi}{5}\right)\right) + 2\left(\delta\left(\omega + \frac{6\pi}{7}\right) + \delta\left(\omega - \frac{6\pi}{7}\right)\right)\right]$$

Duality

Notice the similarity between the FT and the IFT integrals.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Aside: The similarity is even more striking when the frequency parameter is in Hz instead of rad/s: $f = {}^{\omega}/_{2\pi}$

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \qquad\qquad x(t) = \int_{-\infty}^{\infty} \hat{X}(f) e^{j2\pi f t}\, df$$

The FT and IFT are duals of each other, allowing **“mirroring” of concepts and pairs of equations**. This means that **problems solved in one domain** (time or frequency) often **allow for ready solutions in the complementary domain** (frequency or time).

Duality: Temporal vs Freq Scaling

Table 5.1 includes the **Temporal Scaling** Property (P2) (expansion/contraction in time):

$$\boxed{\mathcal{F}[x(\alpha t)] = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)}$$

Simply replacing $\beta = {}^{1}/_{\alpha}$ and multiplying by |α| allows us to see the effect of **Frequency Scaling**:

$$|\alpha|\mathcal{F}[x(\alpha t)] = \frac{1}{|\beta|}\mathcal{F}\left[x\left(\frac{t}{\beta}\right)\right] = X(\beta\omega)$$

$$\boxed{\mathcal{F}^{-1}[X(\beta\omega)] = \frac{1}{|\beta|} x\left(\frac{t}{\beta}\right)}$$

Furthermore, notice that Property P3 (**Time Reversal** or **Reflection**) is simply a special case of this with α=-1.

Duality: Other Examples

Temporal Shifting (P9) is dual to **Frequency Shifting** (P10)

$$\boxed{\mathcal{F}[x(t-\alpha)] = e^{-j\alpha\omega}X(\omega)}$$

$$\boxed{\mathcal{F}^{-1}[X(\omega-\alpha)] = e^{j\alpha t}x(t)}$$

Temporal Differentiation (P6) is dual to **Frequency Differentiation** (P7)

$$\boxed{\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right] = (j\omega)^n X(\omega)}$$

$$\boxed{\mathcal{F}^{-1}\left[\frac{d^n X(\omega)}{d\omega^n}\right] = (-jt)^n x(t)}$$

P5 (**Duality**) allows quick determination of new FT Pairs:

$$\boxed{\mathcal{F}[x(t)] = X(\omega) \Rightarrow \mathcal{F}[X(t)] = 2\pi x(-\omega)}$$

Example: Heaviside

Recall that slide 6.10 Rules of Thumb 1-3 don’t apply well to the unit step and one might be tempted to conclude that $\mathcal{F}[u(t)]$ doesn’t exist but it does.

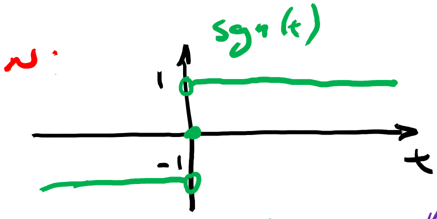
Suppose don’t have pair (2) from Table 5.2 but have (1) $\mathcal{F}[\delta(t)]=1=D(\omega)$.

METHOD A: USE P3 WITH $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

$$\Rightarrow \mathcal{F}\{u(t)\} = \frac{D(\omega)}{j\omega} + \pi D(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

METHOD B: EXPRESS $u(t)$ AS SUM OF EVEN AND ODD FUNCTIONS:

$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$ WHERE $\mathrm{sgn}(t) = \begin{cases} 1 & \text{if } t>0 \\ -1 & \text{if } t<0 \\ 0 & \text{if } t=0 \end{cases}$



$\mathcal{F}$

$$U(\omega) = \frac{1}{2}\left(2\pi\delta(\omega)\right) + \frac{1}{j\omega}$$
$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\frac{d}{dt}\mathrm{sgn}(t) = 2\delta(\omega) \Rightarrow \mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2\mathcal{F}\{\delta(\omega)\}}{j\omega} = \frac{2}{j\omega}$$

DC AVERAGE ZERO SO NO IMPULSE

ASIDE: TRY $\mathcal{F}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-j\omega t}\,dt$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_0^\infty = \frac{\lim_{t\to\infty} e^{-j\omega t} - 1}{-j\omega}$$

FAILS ∵ DOESN'T CONVERGE

TRY $\mathcal{L}\{u(t)\} = \frac{1}{s} \Rightarrow \mathcal{F}\{u(t)\} = \frac{1}{j\omega}$ FAILS ∴ IGNORES DC AVERAGE

PROBLEM IS THAT ROC DOES NOT INCLUDE $j\omega$-AXIS

Example: Sinc

Let's also apply duality to find $\mathcal{F}[sinc(t)]$.

Suppose we don't have pair (13) from Table 5.2

RECALL FROM SLIDE 6.4 THM

$$x_1(t) = u(t+1) - u(t-1) \xrightarrow{\mathcal{F}} X_1(\omega) = 2\,sinc\left(\frac{\omega}{\pi}\right)$$

(USING P1 & P2 FROM TABLE 5.1

$$\text{LET } x(t) = \frac{x_1\left(\frac{t}{\pi}\right)}{2\pi} = \frac{u\left(\frac{t}{\pi}+1\right) - u\left(\frac{t}{\pi}-1\right)}{2\pi} \xrightarrow{\mathcal{F}} X(\omega) = \frac{\pi X_1(\pi\omega)}{2\pi} = sinc(\omega)$$

USE P5

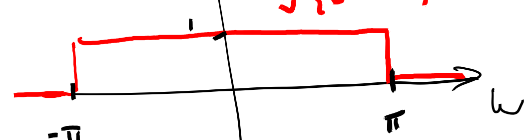
$$X(t) = sinc(t) \xrightarrow{\mathcal{F}} 2\pi x(-\omega) = \cancel{2\pi}\left[\frac{u\left(-\frac{\omega}{\pi}+1\right) - u\left(-\frac{\omega}{\pi}-1\right)}{\cancel{2\pi}}\right]$$

$$= u(\omega+\pi) - u(\omega-\pi)$$

$$u\left(-\frac{\omega}{\pi}+1\right) = \begin{cases} 1 & \text{if } \omega < \pi \\ 0 & \text{OTHERWISE} \end{cases}$$

$$u\left(-\frac{\omega}{\pi}-1\right) = \begin{cases} 1 & \text{if } \omega < -\pi \\ 0 & \text{OTHERWISE} \end{cases}$$

$\mathcal{F}\{sinc(t)\}$



Spectral Representation

Parseval's Energy Relation for Energy Signals:

For an aperiodic, finite-energy signal x(t), with Fourier transform $X(\omega)$, its energy E_x is conserved by the transformation:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{5.18}$$

Thus $|X(\omega)|^2$ is an energy density—indicating the amount of energy at each of the frequencies ω. The plot $|X(\omega)|^2$ vs ω is called the energy spectrum of x(t), and displays how the energy of the signal is distributed over frequency.

$|x(t)|^2$ GIVES TEMPORAL ENERGY DENSITY (ENERGY PER UNIT TIME)

$|X(\omega)|^2$ GIVES FREQUENCY ENERGY DENSITY (ENERGY PER UNIT Hz; $\Delta f = \frac{\Delta\omega}{2\pi}$)

EFTS: Compare and contrast Parseval's Power Relation for the FS (slide 5.4) and Parseval's Energy Relation for the FT.

Energy Spectrum

If $X(\omega)$ is the Fourier transform of a real-valued signal x(t), periodic or aperiodic, the magnitude $|X(\omega)|$ and the real part $\mathcal{R}e[X(\omega)]$ are even functions of ω:

$$|X(\omega)| = |X(-\omega)| \tag{5.19}$$

$$\mathcal{R}e[X(\omega)] = \mathcal{R}e[X(-\omega)]$$

and the phase $\angle X(\omega)$ and the imaginary part $\mathcal{I}m[X(\omega)]$ are odd functions of ω:

$$\angle X(\omega) = -\angle X(-\omega) \tag{5.20}$$

$$\mathcal{I}m[X(\omega)] = -\mathcal{I}m[X(-\omega)]$$

We then call the plots

$|X(\omega)|$ *vs* ω **Magnitude Spectrum**

$\angle X(\omega)$ *vs* ω **Phase Spectrum**

$|X(\omega)|^2$ *vs* ω **Energy/Power Spectrum**

ie: $x(t)$ IS REAL-VALUED $\Rightarrow X(-\omega) = X^*(\omega)$

Frequency Response Applied to FT

If the input x(t) (periodic or aperiodic) of a stable LTI system has Fourier transform $X(\omega)$, and the system has a frequency response $H(j\omega) = \mathcal{F}[h(t)]$, where *h(t)* is the impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform

$$Y(\omega) = X(\omega)H(j\omega) \tag{5.21}$$

In particular, if the input signal *x(t)* is periodic the output is also periodic of the same fundamental period, and with Fourier transform

$$Y(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\omega_0)\delta(\omega - k\omega_0) \tag{5.22}$$

where $\{X_k\}$ are the Fourier series coefficients of *x(t)* and ω_0 its fundamental frequency.