1. **"Short" Answers:**

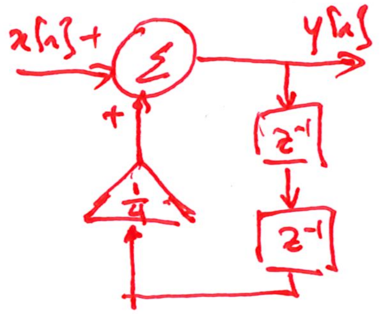
a) (10pts) Consider a system represented by the difference equation $y[n]=0.25y[n-2]+x[n]$ where $x[n]$ is the input and $y[n]$ is the output.

i. Sketch a block diagram of this system (use only adders, scaling blocks and delay blocks) and determine the impulse response.

ii. For the zero-input case, find the initial conditions $y[-1]$ and $y[-2]$ so that $y[n]=0.5^n u[n]$.

iii. For zero initial conditions, find the input $x[n]$ so that $y[n]=0.5^n u[n]$.

(i)




| n | x | y |
|---|---|---|
| -1 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 2 | 0 | 1/4 |

$$y[n] = \frac{1}{2}^n\left[\frac{1+(-1)^n}{2}\right]u[n] = h[n]$$

(ii)
$$y[0] = \tfrac{1}{4}y[-2] + 0 = \left(\tfrac{1}{2}\right)^0(1) \Rightarrow \boxed{y[-2]=4}$$
$$y[1] = \tfrac{1}{4}y[-1] + 0 = \left(\tfrac{1}{2}\right)^1(1) \Rightarrow \boxed{y[-1]=2}$$

(iii)
$$y[0] = 0 + x[0] = \left(\tfrac{1}{2}\right)^0(1) \Rightarrow x[0]=1$$
$$y[1] = 0 + x[1] = \left(\tfrac{1}{2}\right)^1(1) \Rightarrow x[1]=\tfrac{1}{2}$$
$$y[2] = \tfrac{1}{4}(1) + x[2] = \left(\tfrac{1}{2}\right)^2(1) \Rightarrow x[2]=0$$
$$y[3] = \tfrac{1}{4}\left(\tfrac{1}{2}\right) + x[3] = \left(\tfrac{1}{2}\right)^3(1) \Rightarrow x[3]=0$$

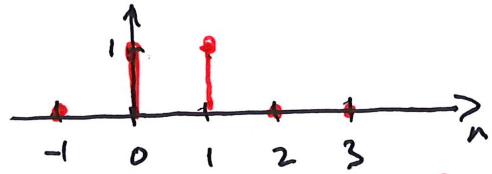
$$x[n] = \delta[n] + \tfrac{1}{2}\delta[n-1]$$

b) (9pts) Consider a causal LTI system with impulse response $h[n] = u[n] - u[n-2]$.

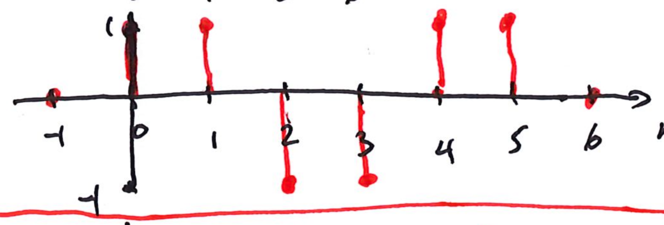
i. Sketch $y[n]$ for input $x[n] = x_1[n] - x_1[n-2] + x_1[n-4]$ where $x_1[n] = u[n] - u[n-2]$.

ii. When cascaded with another causal LTI system with impulse response $g[n]$, the overall impulse response is $h_T[n] = \begin{cases} 1, & n = 0,1,2,3 \\ 0, & otherwise \end{cases}$. Determine $g[n]$,.

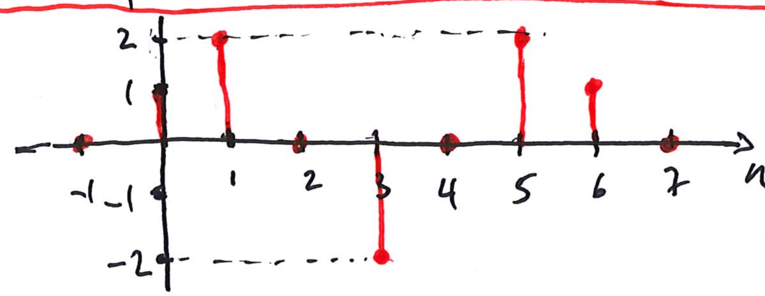
(i) $h[n] = x_1[n]$



$x[n]$



$y[n]$



(ii) $\underbrace{h_T[n]}_{\text{SUPPORT}=4} = \underbrace{h[n]}_{\text{SUPPORT}=2} * g[n] \Rightarrow \text{SUPPORT} = 3$

$\Rightarrow g[n] = g[0]\delta[n] + g[1]\delta[n-1] + g[2]\delta[n-2]$

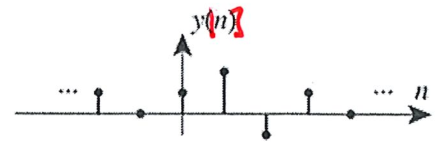
$h_T[0] = h[0]g[0] = 1 \Rightarrow g[0] = 1$

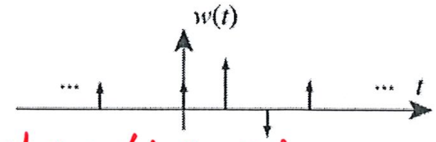
$h_T[1] = h[0]g[1] + h[1]g[0] = 1 \Rightarrow g[1] = 0$

$h_T[2] = h[0]g[2] + h[1]g[1] = 1 \Rightarrow g[2] = 1$

$g[n] = \delta[n] + \delta[n-2]$

c) (5pts) For the "sampled signals" $y$ and $w$ used during sampling and reconstruction, as shown, explain how they differ mathematically (at least two points) and write the equation relating them to each other.




$y \in [\mathbb{Z} \to \mathbb{R}]$

$w \in [\mathbb{R} \to \underbrace{\mathbb{R}_{**}}_{=\mathbb{R} \cup \{-\infty, \infty\}}]$

(i) $y$ IS DISCRETE-TIME
$w$ IS CONTINUOUS-TIME

(ii) $\forall n \in \mathbb{Z}, |y(n)| < \infty$ (i.e. FINITE).
THERE MAY EXIST $t \in \mathbb{R}$ s.t. $|w(t)| = \infty$

$$w(t) = \sum_{k=-\infty}^{\infty} y(k)\,\delta(t - kT)$$ WHERE $T$ = SAMPLING INTERVAL

d) (7pts) For signals $x(t) = \begin{cases} \cos(t), & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $y(t) = x\left(\frac{t}{2}\right)$, find their Fourier Transforms $X(\omega)$ and $Y(\omega)$.

$$x(t) = \cos(t)[u(t) - u(t-1)] = p(t)\cos(t)$$

$$\Rightarrow X(\omega) = \frac{1}{2}[P(\omega - 1) + P(\omega + 1)]$$

WHERE $P(\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)(1 - e^{-j\omega}) = \left(\frac{1}{j\omega}\right)(1 - e^{-j\omega})$

$$= \frac{e^{-j\omega/2}}{j\omega}\left(e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) = 2e^{-j\frac{\omega}{2}}\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}$$

[ASIDE: IT'S TEDIOUS TO SUB INTO $X(\omega)$ SO WE DON'T DO IT HERE]

$$Y(\omega) = 2X(2\omega) = P(2\omega - 1) + P(2\omega + 1)$$

e) (9pts) Consider an ideal low pass filter with the following frequency response:

$$|H(j\omega)| = \begin{cases} 1, & -4 \le \omega \le 4 \\ 0, & \text{otherwise} \end{cases} \qquad \angle H(j\omega) = \begin{cases} -\pi/2, & \omega \ge 0 \\ \pi/2, & \omega < 0 \end{cases}$$

Determine the impulse response $h(t)$ and the output for $x(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}\cos(3kt/2)$.

FOR OUTPUT, ONLY FREQS FOR $k=1$ & $2$ ARE BELOW CUTOFF

$$\Rightarrow y(t) = 2\cos\left(\frac{3t}{2} - \frac{\pi}{2}\right) + \frac{2}{4}\cos\left(\frac{3(2)t}{2} - \frac{\pi}{2}\right)$$

$$= 2\sin\left(\frac{3t}{2}\right) + \frac{1}{2}\sin(3t)$$

IMPULSE RESPONSE:

METHOD A: $H(j\omega) = \begin{cases} j & \text{IF } -4<\omega<0 \\ -j & \text{IF } 0<\omega<4 \\ 0 & \text{OTHERWISE} \end{cases} = j\left[\frac{\text{sgn}(\omega-4)+\text{sgn}(\omega+4)}{2} - \text{sgn}(\omega)\right]$

RECALL $\mathcal{F}\{\text{sgn}(t)\} = \frac{2}{j\omega}$

DUALITY $\Rightarrow \mathcal{F}\left\{\frac{2}{jt}\right\} = 2\pi\,\text{sgn}(-\omega) = -2\pi\,\text{sgn}(\omega) \Rightarrow \mathcal{F}^{-1}\{\text{sgn}(\omega)\} = \frac{-1}{\pi j t}$

$$\Rightarrow h(t) = j\left[\frac{e^{j4t}+e^{-j4t}}{2}\left(-\frac{1}{\pi j t}\right) - \left(-\frac{1}{\pi j t}\right)\right] = \frac{1-\cos 4t}{\pi t}$$

METHOD B:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\int_0^4 e^{j(\omega t - \frac{\pi}{2})}\,d\omega + \int_{-4}^{0} e^{j(\omega t + \frac{\pi}{2})}\,d\omega\right]$$

$$= \frac{-j}{2\pi}\left[\frac{e^{j\omega t}}{jt}\Big|_0^4 - \frac{e^{j\omega t}}{jt}\Big|_{-4}^{0}\right] = \frac{1}{2\pi t}\left[-(e^{j4t}-1)+(1-e^{-j4t})\right]$$

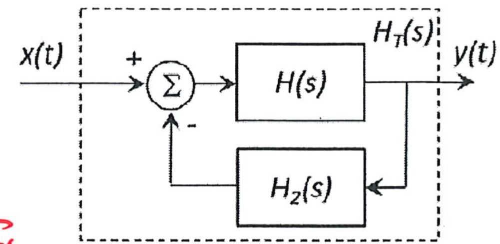
$$= \frac{1-\cos 4t}{\pi t}$$

**2. Frequency Response:** A continuous time LTI system is represented by the ordinary differential equation $\frac{dy(t)}{dt} = -y(t) + x(t)$ where $x(t)$ is the input and $y(t)$ the output.

a. (7pts) Determine the frequency response $H(j\omega)$ and the impulse response $h(t)$ of this system.

b. (7pts) For the input $x(t) = \sin(t)/(\pi t)$, determine the Fourier transform of the output $Y(\omega)$ and sketch a plot of $|Y(\omega)|$ vs $\omega$ (use a linear scale as this is not a Bode plot).

c. (6pts) If a feedback system with transfer function $H_2(s) = \frac{s}{s+2}$ is implemented as shown, determine the net transfer function $H_T(s)$.




a) $sY = -Y + X \Rightarrow Y(s+1) = X$

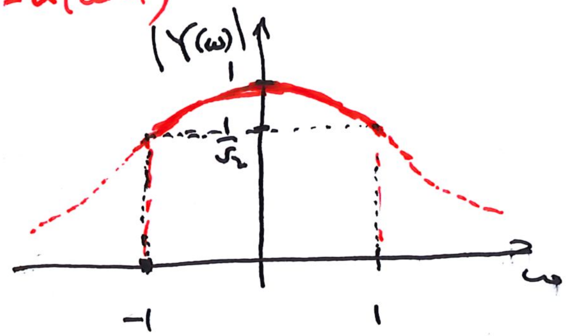
$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1} \Rightarrow H(j\omega) = \frac{1}{j\omega+1}$

$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t}u(t)$

b) $x(t) = \frac{\sin(t)}{\pi t} \xrightarrow{\mathcal{F}} X(\omega) = u(\omega+1) - u(\omega-1)$

$\Rightarrow Y(\omega) = \frac{u(\omega+1) - u(\omega-1)}{j\omega+1}$

$|Y(\omega)| = \frac{1}{\sqrt{\omega^2+1}}\left(u(\omega+1) - u(\omega-1)\right)$




c) $Y = H(X - H_2 Y)$

$(1 + HH_2)Y = HX \Rightarrow H_T = \frac{Y}{X} = \frac{H}{1+HH_2} = \frac{1/(s+1)}{1 + \frac{1}{s+1} \times \frac{s}{s+2}}$

$\Rightarrow H_T = \frac{s+2}{(s+1)(s+2)+s} = \frac{s+2}{s^2+4s+2}$

3. **Sampling & Reconstruction:** Consider $x(t) = \cos(50\pi t + \frac{\pi}{11})$, $y(t) = \cos(300\pi t - \frac{\pi}{11})$, and $z(t) = 3\sin(200\pi t - \frac{\pi}{11})$.

a) (6pts) If sampled at $f_s = 400\ Hz$, state whether the following signals are periodic or aperiodic, and if so, determine their period:
- $p[n] = y(nT_s) + z(nT_s)$
- $q[n] = y(nT_s)x(nT_s)$

b) (3pts) Compute the Nyquist sampling rate of the signal $u(t) = y(t)\cos(50\pi t)$.

c) (6pts) For the continuous time signal $v(t) = x(t) - y(t) + z(t)$, determine the discrete time signal $v_s[n] = v(nT_s)$ using a sampling rate of $f_s = 180\ Hz$.

d) (5pts) From the discrete time signal $v_s[n]$ found in Part (c), determine the reconstructed signal using an ideal interpolator and a sampling interval of $T_s = \frac{1}{180}$ seconds. Indicate which, if any, frequencies have been aliased.

a) $p[n] = \cos\left(\underbrace{\frac{300\pi}{400}}n - \frac{\pi}{11}\right) + 3\sin\left(\underbrace{\frac{200\pi}{400}}n - \frac{\pi}{11}\right)$

$\frac{300\pi}{400} = 2\pi\frac{m_1}{N_1} \Rightarrow \frac{m_1}{N_1} = \frac{3}{8}$ $\quad \frac{200\pi}{400} = 2\pi\frac{m_2}{N_2} \Rightarrow \frac{m_2}{N_2} = \frac{1}{4} \Rightarrow LCM(N_1, N_2) = 8$

$\Rightarrow$ p[n] IS PERIODIC W/ PERIOD 8 SAMPLES

$q[n] = \frac{1}{2}\left[\cos\left(\frac{350\pi}{400}n\right) + \cos\left(\frac{250\pi}{400}n - \frac{2\pi}{11}\right)\right]$

$\frac{350\pi}{400} = 2\pi\frac{m_1}{N_1} \Rightarrow \frac{m_1}{N_1} = \frac{7}{16}$ $\quad \frac{250\pi}{400} = 2\pi\frac{m_2}{N_2} \Rightarrow \frac{m_2}{N_2} = \frac{5}{16} \Rightarrow LCM(N_1, N_2) = 16$

$\Rightarrow$ q[n] IS PERIODIC W/ PERIOD 16 SAMPLES

b) $u(t) = \frac{1}{2}\left[\cos\left(350\pi t - \frac{\pi}{11}\right) + \cos\left(250\pi t - \frac{\pi}{11}\right)\right]$

HIGHEST FREQ IS $350\pi$ rad/s OR 175 Hz

$\Rightarrow$ NYQUIST SAMPLING RATE IS 350 Hz

c) $v_s[n] = \cos\left(\frac{50\pi n}{180} + \frac{\pi}{11}\right) - \cos\left(\frac{300\pi n}{180} - \frac{\pi}{11}\right) + 3\sin\left(\frac{200\pi n}{180} - \frac{\pi}{11}\right)$

$= \cos\left(\frac{5\pi n}{18} + \frac{\pi}{11}\right) - \cos\left(\frac{5\pi n}{3} - \frac{\pi}{11}\right) + 3\sin\left(\frac{10\pi n}{9} - \frac{\pi}{11}\right)$

$= \cos\left(\frac{5\pi n}{18} + \frac{\pi}{11}\right) - \cos\left(\frac{\pi n}{3} + \frac{\pi}{11}\right) - 3\sin\left(\frac{8\pi n}{9} + \frac{\pi}{11}\right)$

d) $v_r(t) = \cos\left(50\pi t + \frac{\pi}{11}\right) - \cos\left(-60\pi t - \frac{\pi}{11}\right) + 3\sin\left(-160\pi t - \frac{\pi}{11}\right)$

$= \cos\left(50\pi t + \frac{\pi}{11}\right) - \cos\left(60\pi t + \frac{\pi}{11}\right) - 3\sin\left(160\pi t + \frac{\pi}{11}\right)$

25 Hz NOT ALIASED | 150 Hz ALIASED TO 30 Hz | 100 Hz ALIASED TO 80 Hz

**4. Discrete-Time Systems Classification**

a) (14pts) For the system represented by $y_1[n] = x_1[n] \cdot x_1[n+2]$, **show** whether or not the system is linear, time-invariant, causal and/or BIBO stable.

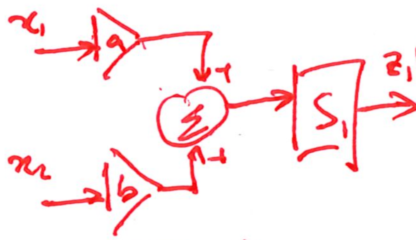
b) (6pts) For each of the systems represented by difference equations, classify the system according to its linearity, time-invariance, causality and BIBO stability (it isn't necessary to explain how you concluded these):

i. $y_2[n] = x_2[n] \cdot \cos\frac{n\pi}{4}$

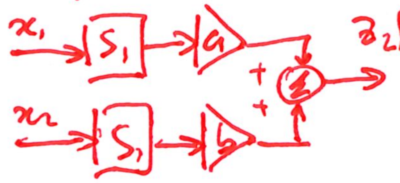
ii. $y_3[n] = x_3[n] + 4$

iii. $y_4[n] = y_4[n-2] + x_4[n] + x_4[n+2]$

a) LINEARITY:

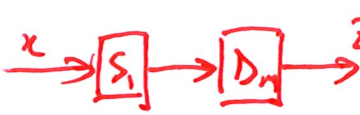



$z_1[n] = \{a x_1[n] + b x_2[n]\}\{a x_1[n+2] + b x_2[n+2]\}$

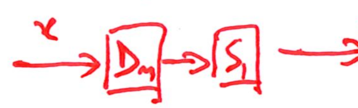



$z_2[n] = a x_1[n]\, x_1[n+2] + b\, x_2[n]\, x_2[n+2] \neq z_1[n]$

$\Rightarrow$ NONLINEAR




$z_3[n] = x[n-M]\, x[n+2-M]$




$z_4[n] = x[n-M]\, x[n-M+2] = z_3[n] \Rightarrow$ TIME-INVARIANT

$y_1[n]$ DEPENDS ON $x_1[n+2] \Rightarrow$ NOT CAUSAL

IF $x[n] \leq M \ \forall n \in \mathbb{Z} \Rightarrow y[n] \leq M^2 \ \forall n \in \mathbb{Z} \Rightarrow$ BIBO STABLE

b)

$S_2$ IS LINEAR, TIME-VARYING, CAUSAL AND BIBO STABLE

$S_3$ IS NONLINEAR, TIME-INVARIANT, CAUSAL AND BIBO STABLE

$S_4$ IS LINEAR, TIME-INVARIANT, NOT CAUSAL, NOT BIBO STABLE