Continuous-Time Signals

- Signal Definition and Function Notation
- Signal Classification
- Sampling, Quantization & Coding
- Signal Operations
- Basic Signals

Signals

- A Signal is a means to convey information or something that contains information.
- Mathematically, typically represent signal by a function which maps a domain (*D*, often time or space) into a range (*R*, often a physical measure).
- A Signal Space is a set of all eligible signals:

“such that”

$$\text{Notation: } X = [D \to R] = \{x \mid x: D \to R\}$$

Signal Space

Single Signal

Mapping Description

Signal Classification

There are numerous ways to classify & describe signals:

I: Deterministic vs Stochastic (Random)

- Can represented by formula or table of values.
- No uncertainty in value.

- Values have uncertainty, specified by probabilistic distribution.

II: Continuous-Time (CT) vs Discrete-Time (DT)

- Signal domain has continuous time interval (e.g. $D=\mathbb{R}$).

- Signal domain is a discrete set of times ($D\subseteq\mathbb{Z}$)

III: Continuous-Amplitude vs Discrete-Amplitude

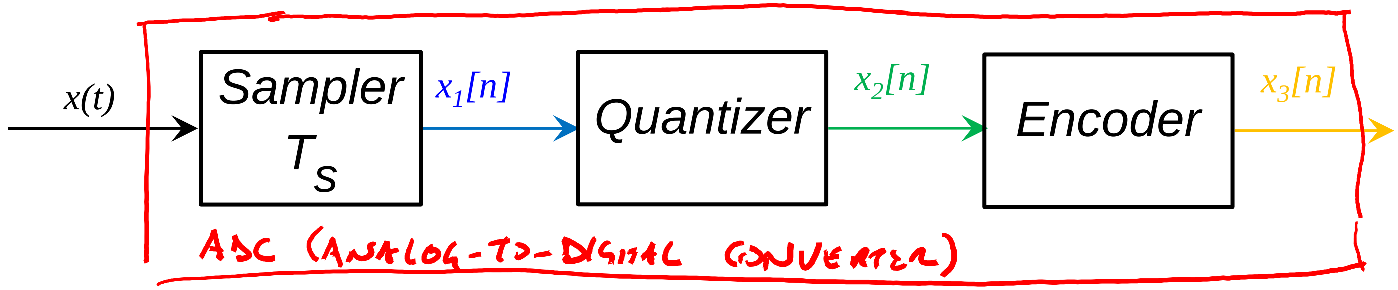
- Signal range is continuous (e.g., $R=\mathbb{R}$).

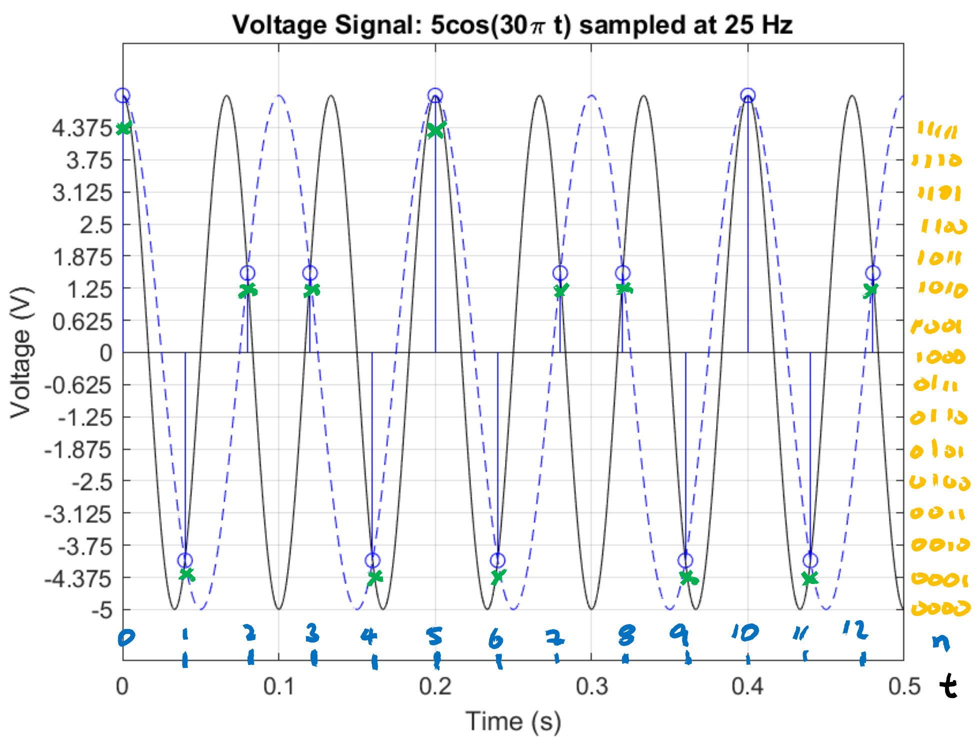
- Signal range is discrete (quantized) ($R\subseteq\mathbb{Z}$)

Signals that are both CT and Continuous-Amplitude are **"Analog Signals"**

Signals that are both DT and Discrete-Amplitude are **"Digital Signals"**

Eg: ADC: Sampling, Quantization, Coding





$x: \mathbb{R} \to \mathbb{R}$ (ANALOG)

$x_1: \mathbb{Z} \to \mathbb{R}$

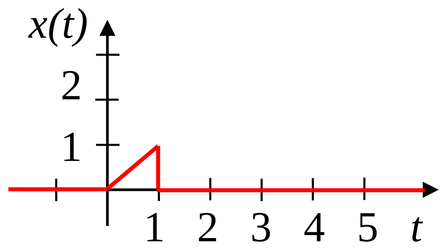
$x_2: \mathbb{Z} \to \{-5, -4.375, \ldots, 4.375\}$

$x_3: \mathbb{Z} \to \{0,1\}^4$ (DIGITAL, 4-BIT ENCODING)

Aside: The original 15 Hz signal sampled at 25 Hz is indistinguishable from a 10 Hz signal (i.e., it has been "aliased"). More later in Nyquist-Shannon Sampling Theorem.

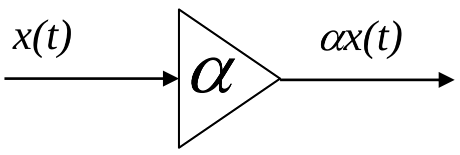
Basic Signal Operations

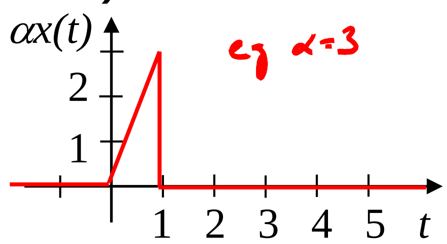
Consider the signal $x(t)$ shown.



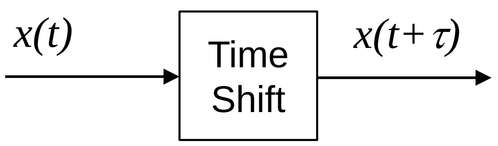
Unary Operators (only one signal input):

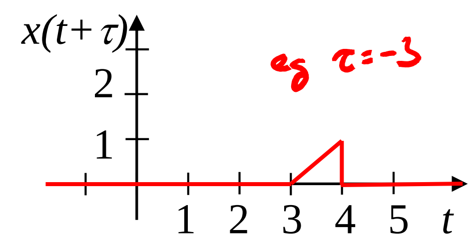
1a. (Amplitude) Scaling (constant multiplication).





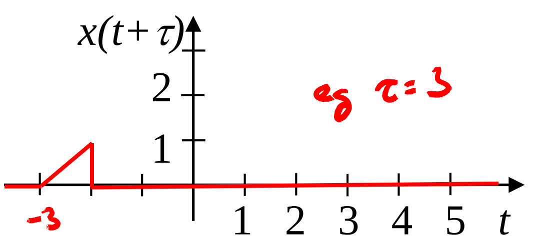
1b. Time shift:





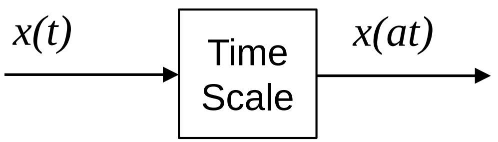
- Delay ("right shift") $\tau < 0$

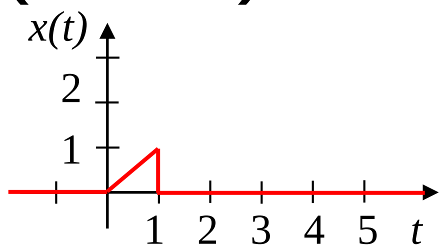
- Advance ("left shift") $\tau > 0$



Basic Signal Operations (cont.)

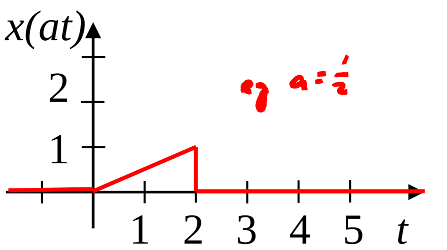
1c. Time Scaling:





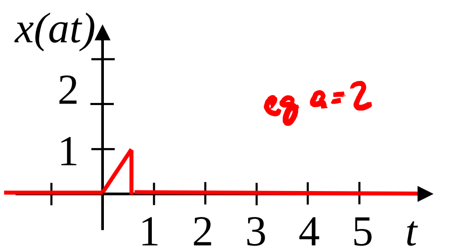
- Time stretch $|a| < 1$

$\Rightarrow$ LOWER FREQUENCIES

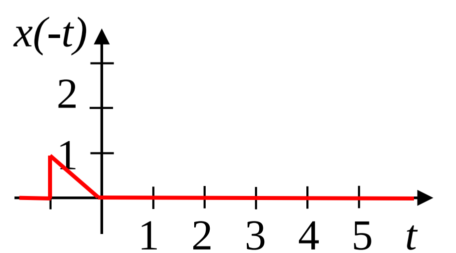


- Time contraction $|a| > 1$

$\Rightarrow$ HIGHER FREQUENCIES

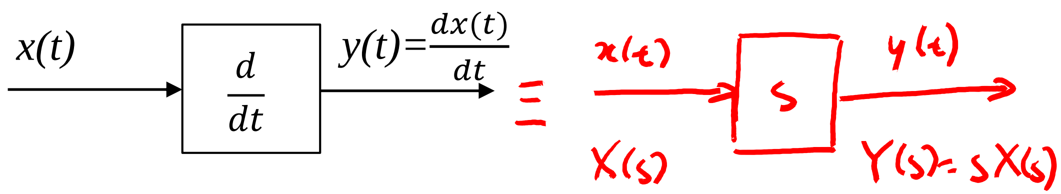


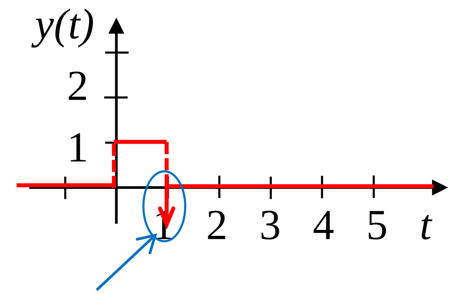
- Time reversal (reflection) $a = -1$



Basic Signal Operations (cont.)

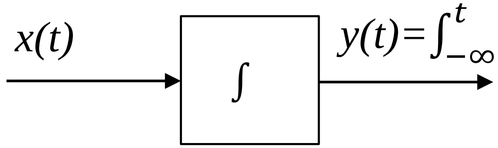
1d. Differentiation:



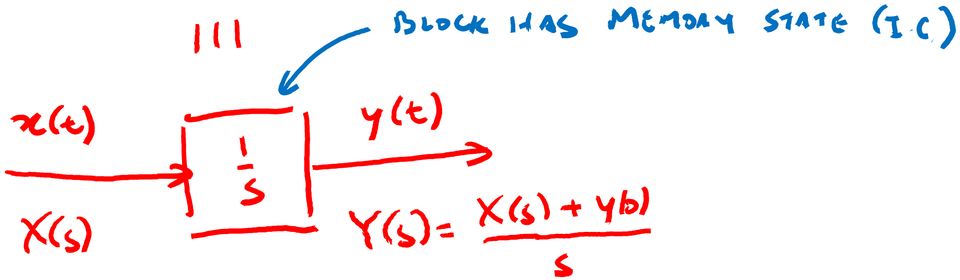


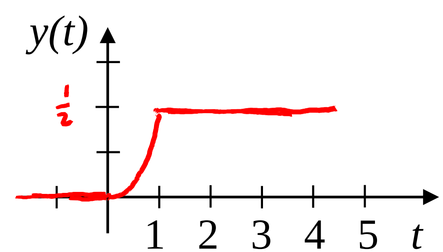
Arrow indicates -∞
(Impulse/Dirac function)

1e. Integration:



$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = y(0) + \int_{0}^{t} x(\tau)\,d\tau$$

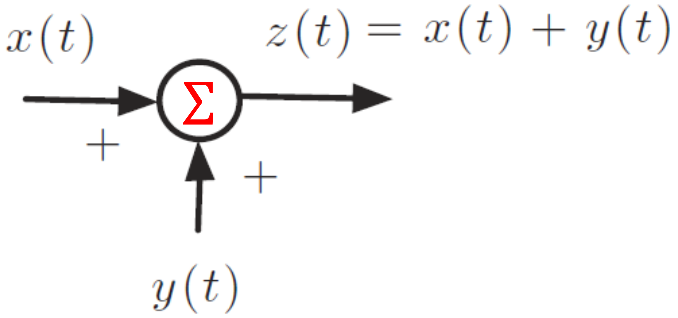




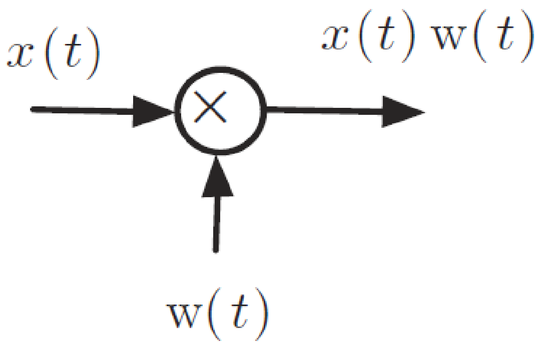
Basic Signal Operations (cont.)

Binary Operators

2a. Adder (aka "Summer"):



2b. Time Windowing:



Signal Classification (cont.)

IV: Even vs Odd vs Neither

Even and odd signals are defined as follows:

$$x(t) \quad \textbf{even} : x(t) = x(-t) \tag{1.4}$$

"MIRROR SYMMETRY"

$$x(t) \quad \textbf{odd} : x(t) = -x(-t) \tag{1.5}$$

"ROTATIONAL (OR POINT) SYMMETRY"

Even and odd decomposition: Any signal $y(t)$ is representable as a sum of an even component $y_e(t)$ and an odd component $y_o(t)$

$$y(t) = y_e(t) + y_o(t) \tag{1.6}$$

where

$$y_e(t) = 0.5\,[y(t) + y(-t)]$$
$$y_o(t) = 0.5\,[y(t) - y(-t)]$$

Signal Classification (cont.)

V: Periodic vs Aperiodic

- Signal repeats itself
- No consistent signal repetition

A continuous-time signal $x(t)$ is **periodic** *if*

- it is defined for all possible values of t, $-\infty < t < \infty$, and
- there is a positive real value T_0, the **fundamental period** of $x(t)$, such that

$$x(t + kT_0) = x(t) \qquad (1.7)$$

for any integer k.

The fundamental period of $x(t)$ is the smallest $T_0 > 0$ that makes the periodicity possible. Thus, although NT_0 for an integer $N > 1$ is a period of $x(t)$ it should not be considered the fundamental period.

Region in (time) Domain for which $x(t) \neq 0$.

Signal Classification (cont.)

VI: Finite-Energy vs Finite-Power vs Infinite-Power

The **energy** and the **power** of a continuous-time signal $x(t)$ are defined for either finite or infinite support signals as:

IMPLICITLY "AVERAGE POWER"

INSTANTANEOUS POWER

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{1.8}$$

A signal $x(t)$ is then said to be **finite-energy**, or **square integrable**, whenever

$$E_x < \infty \tag{1.9}$$

⇒ "ENERGY SIGNAL"

A signal $x(t)$ is said to be **finite-power** if

$$P_x < \infty \tag{1.10}$$

⇒ "POWER SIGNAL"

($P_x \neq 0$ ⇒ INFINITE ENERGY)

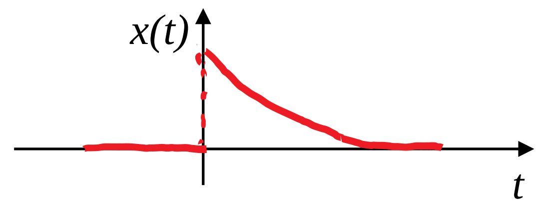
Caveat: What are the units of this energy & power? Joules & Watts?

A: Without more info about signal and application, these are "normalized" unitless versions that allow general comparisons of information (even if useless) content. To relate more concretely to metric units, consider $x(t)$ to be voltage across (or current through) a 1Ω resistor; then E_x and P_x are corresponding total dissipated energy and power, respectively.

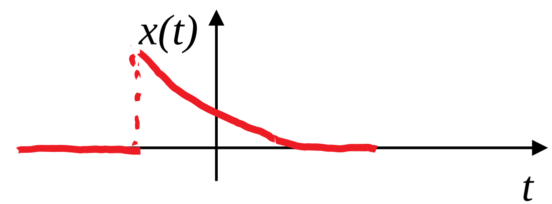
Signal Classification (cont.)

VII: Causal vs Acausal vs Anti-Causal

Causal: $x(t) = 0 \quad \forall t < 0$

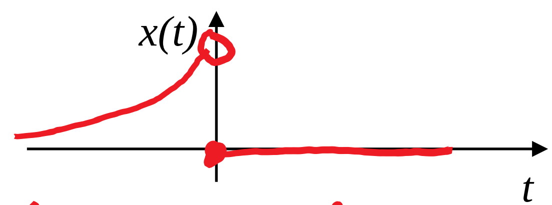


Acausal: $\exists t < 0 \text{ s.t. } x(t) \neq 0$



Anti-Causal: $x(t) = 0 \quad \forall t \geq 0$

AND $\exists t < 0 \text{ s.t. } x(t) \neq 0$

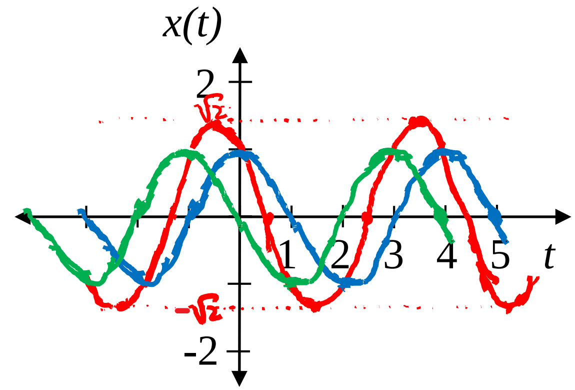


$\mathcal{X}$ = SET OF ALL SIGNALS = CAUSAL SIGNALS $\cup$ ACAUSAL SIGNALS ; ANTI-CAUSAL SIGNALS $\subset$ ACAUSAL SIGNALS

NB: In effect, a “causal” signal is one which is potentially a valid impulse response for a causal system.

Chaparro Example 1.1

Characterize $x(t) = \sqrt{2}\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right), -\infty < t < \infty$



I : DETERMINISTIC

II : C.T. } ANALOG

III : CTS AMPLITUDE

IV : NEITHER ODD NOR EVEN

V : PERIODIC ($\omega = \frac{\pi}{2} = 2\pi f \Rightarrow f = \frac{1}{4}\,\text{Hz} \Rightarrow T = \frac{1}{f} = 4\,\text{s}$)

$$x_e(t) = \frac{1}{2}\left[\sqrt{2}\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) + \sqrt{2}\cos\left(-\frac{\pi t}{2} + \frac{\pi}{4}\right)\right] = \cos\left(\frac{\pi t}{2}\right)$$

$$x_o(t) = \frac{1}{2}\left[\sqrt{2}\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) - \sqrt{2}\cos\left(-\frac{\pi t}{2} + \frac{\pi}{4}\right)\right] = -\sin\left(\frac{\pi t}{2}\right)$$

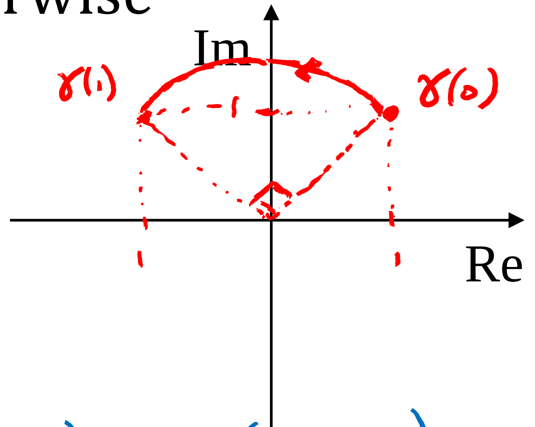
VI : POWER SIGNAL : $P_x = 1$ (INFINITE ENERGY)

VII : ACAUSAL

Chaparro Example 1.2

$\sqrt{2}e^{\frac{j\pi}{4}}$

Characterize $\gamma(t) = \begin{cases}(1+j)e^{\frac{j\pi t}{2}}, & \text{if } 0 \le t \le 10 \\ 0, & \text{otherwise}\end{cases}$ and express it in terms of $x(t)$ from slide 2.13.



I : DETERMINISTIC

II : C.T.

III : CTS AMPLITUDE

IV : NEITHER ODD NOR EVEN

V : APERIODIC

VI $E_\gamma = \int_0^{10} |\sqrt{2}|^2 dt = 20, \quad P_\gamma = 0$

VII CAUSAL

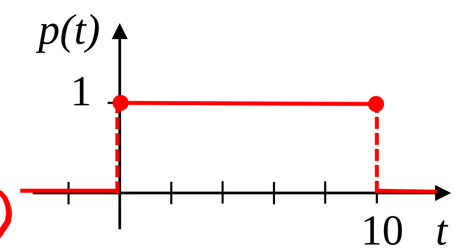
FOR $0 \le t \le 10$:

$$\gamma(t) = \underbrace{\sqrt{2}\cos\left(\frac{\pi t}{2}+\frac{\pi}{4}\right)}_{x(t)} + j\underbrace{\sqrt{2}\sin\left(\frac{\pi t}{2}+\frac{\pi}{4}\right)}_{x(t)\text{ DELAYED BY }90^\circ}$$

$$\Rightarrow \gamma(t) = \begin{cases} x(t) + jx(t-1) & 0 \le t \le 10 \\ 0 & \text{OTHERWISE}\end{cases}$$

Chaparro Example 1.3

Characterize pulse $p(t) = \begin{cases} 1\,, \text{if } 0 \le t \le 10 \\ 0, \quad \text{otherwise} \end{cases}$ and use it along with $x(t)$ from slide 2.13 to represent $y(t)$ from slide 2.14.



I : DETERMINISTIC

II : C.T.

III : CTS AMPLITUDE (ACCEPTABLE TO LABEL DISCRETE-AMPLITUDE)

IV : NEITHER ODD NOR EVEN

V : APERIODIC

VI $E_p = 10$, $P_p = 0$

VII CAUSAL

$$y(t) = [x(t) + j\,x(t-1)]\,p(t)$$

USE OF THIS "SINGULARITY FUNCTION" ALLOWS FOR COMPACT EQN REPRESENTATION (VALID $\forall t$ INSTEAD OF SEPARATE INTERVALS)

Basic Signals

A complex exponential *is a signal of the form*

REAL PART TELLS US ABOUT STABILITY

IMAGINARY PART TELLS US FREQUENCY

$$x(t) = Ae^{at} \quad \leftarrow \text{Conventionally, use } Ae^{st} \text{ where } s = \sigma + j\omega$$

$$= |A|e^{rt}\left[\cos(\Omega_0 t + \theta) + j\sin(\Omega_0 t + \theta)\right] \quad -\infty < t < \infty \qquad (1.15)$$

where $A = |A|e^{j\theta}$ *and* $a = r + j\Omega_0$ *are complex numbers.*

A sinusoid is of the general form

ie: COSINE LEADS SINE BY 90°

$$A\cos(\Omega_0 t + \theta) = A\sin(\Omega_0 t + \theta + \pi/2) \quad -\infty < t < \infty \qquad (1.16)$$

where A is the amplitude of the sinusoid, $\Omega_0 = 2\pi f_0$ (rad/sec) is its analog frequency, and θ its phase shift. The fundamental period T_0 of the above sinusoid is inversely related to the frequency:

$$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

Aside: Chapparo uses: $\omega \rightarrow$ discrete frequency variable in radians/sample

$\Omega \rightarrow$ continuous frequency variable in radians/sec

By convention (most textbooks): $\omega \rightarrow$ continuous frequency variable in radians/sec

$\Omega \rightarrow$ discrete frequency variable in radians/sample

Basic Signals (cont.)

The **unit-impulse signal** $\delta(t)$ (aka: "DIRAC DELTA")

- is zero everywhere except at the origin where its value is not well defined, i.e., $\delta(t) = 0, t \neq 0$, undefined at $t = 0$,
- has an integral

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{1.24}$$

so that the area under the impulse is unity.

AND INSTANTANEOUS POWER @ $t=0$ ALLOWS SIGNAL TO IMPART NONZERO FINITE ENERGY

The **unit-step** signal is (aka: "HEAVISIDE")

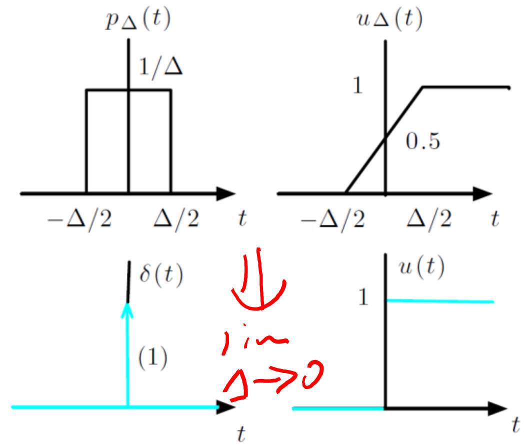
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

($0, \frac{1}{2}$ OR 1 @ $t=0$ DEPENDING ON WHO YOU ASK

The $\delta(t)$ and $u(t)$ are related as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{1.25}$$

$$\delta(t) = \frac{du(t)}{dt} \tag{1.26}$$



$\delta(t)$ & $u(t)$ ARE MOST IMPORTANT "SINGULARITY" OR "SWITCHING" FUNCTIONS IN ELEC 221

By the sifting property of the impulse function $\delta(t)$ any signal $x(t)$ can be represented by the following **generic representation**:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau. \tag{1.32}$$

Basic Signals (cont.)

The **ramp** signal is defined as

$$r(t) = tu(t) \tag{1.27}$$

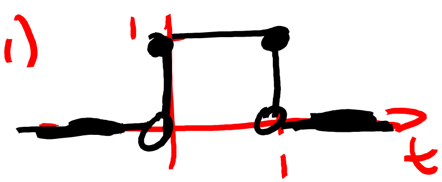
The relation between the ramp, the unit-step and the unit-impulse signals is given by

$$\frac{dr(t)}{dt} = u(t) \tag{1.28}$$

$$\frac{d^2r(t)}{dt^2} = \delta(t) \tag{1.29}$$

The **unit pulse**: (aka: “rectangle function” or “gate function”):

$$\text{rect}(t) = \sqcap(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} = u(t) - u(t-1)$$

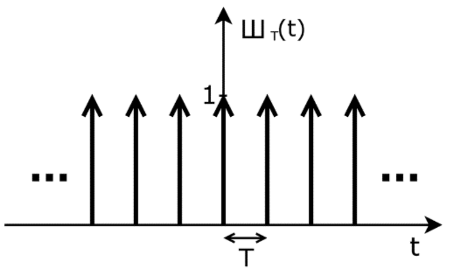


eg slide 2.15 $p(t) = \text{rect}\left(\frac{t}{T_0}\right)$

Other Special Signals

Impulse Train: (aka: "Dirac Comb" or "Sampling Function"):

$$\boxed{\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)}$$

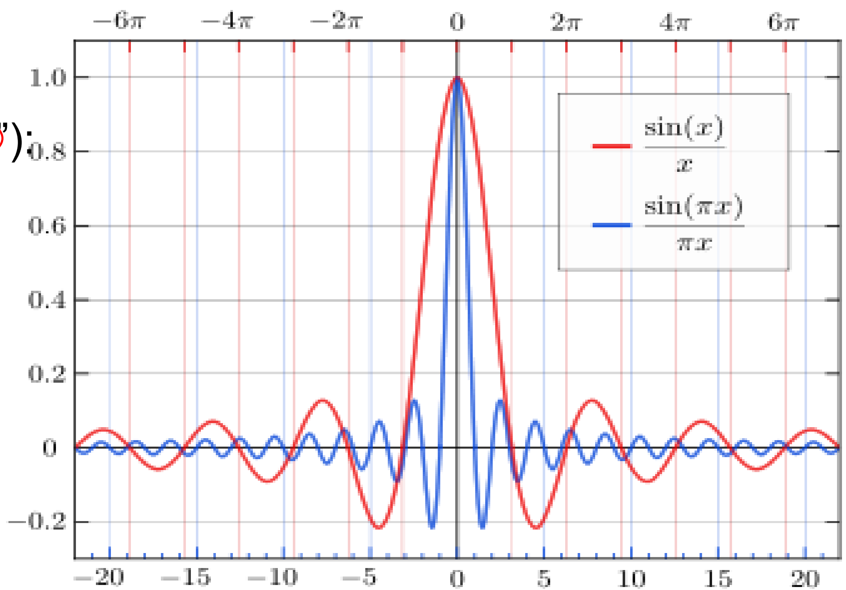


[Images Source: Wikipedia]

Argh! Ambiguous so I avoid this term.

Sinc Function: (aka: "Cardinal Sine", "Ideal Interpolator" or "Sampling Function"):

$$\boxed{sinc(t) = \frac{\sin(\pi t)}{\pi t}, -\infty < t < \infty}$$

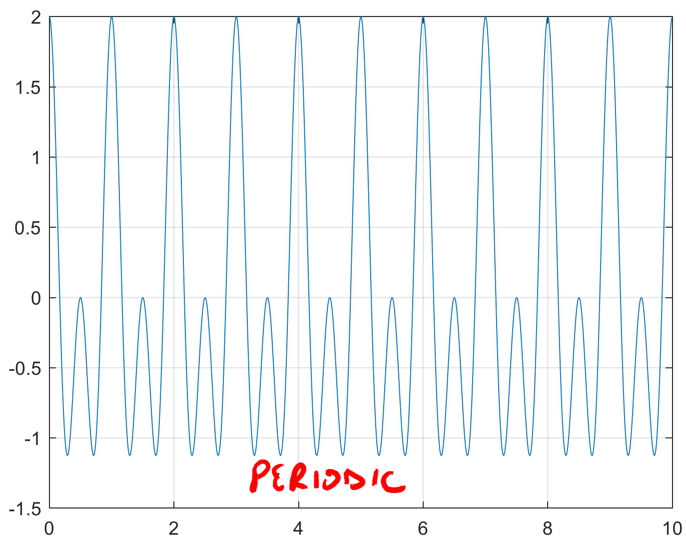


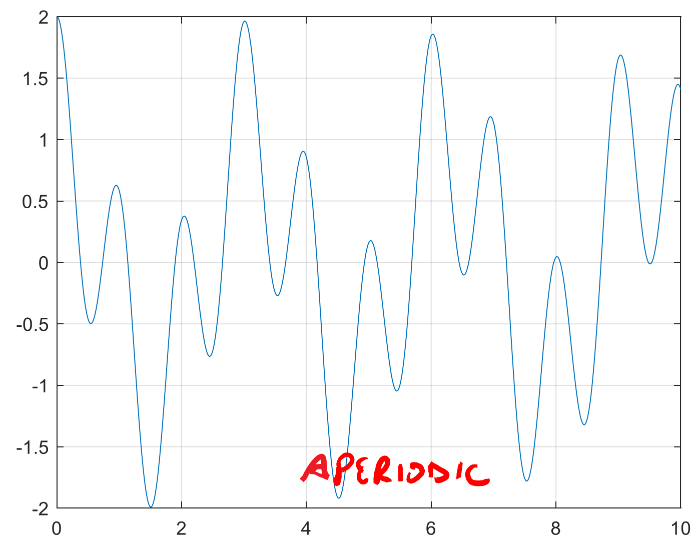
Caveat: The above sinc function is the convention used in Signal Processing and Information Theory. Mathematicians use $sinc(t) = \frac{\sin(t)}{t}, -\infty < t < \infty$

Chaparro Example 1.14

$T = 1s$ $\quad$ $T = \frac{1}{2} s$ $\quad$ $T = 1s$ $\quad$ $T = \pi \, s$

Consider the signals $x(t) = \cos(2\pi t) + \cos(4\pi t)$ and $y(t) = \cos(2\pi t) + \cos(2t)$, $-\infty < t < \infty$. Determine if these signals are periodic and if so find their fundamental periods. Compute the power of these signals.





$P_x = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) = 1$

$P_y = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) = 1$

Chaparro Example 1.16

Write a Matlab script to generate and plot the signal:

$$y(t) = 3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3)$$

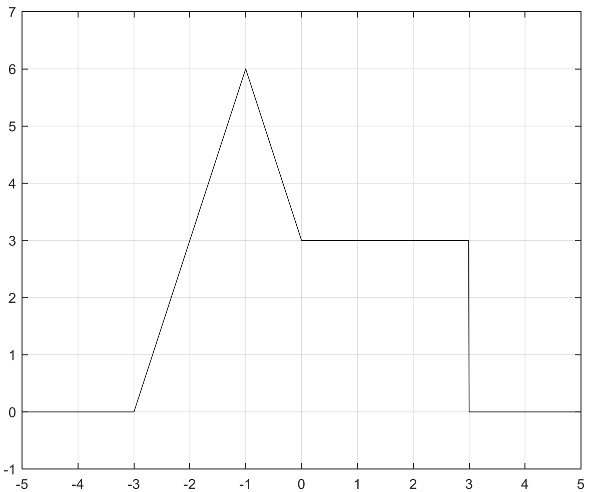
```
function y = ramp(t,m,ad)
% ramp generation
% t: time support
% m: slope of ramp
% ad : advance (positive), delay (negative) factor
% USE: y = ramp(t,m,ad)
N = length(t);y = zeros(1,N);
for i = 1:N,
          if t(i)>= -ad,
          y(i) = m*(t(i) + ad);
          end
end
```

I would exclude this in the function (use as function coefficient instead).

Unnecessary (redundant to function declaration).

```
function y = ustep(t,ad)
% generation of unit step
% t: time
% ad : advance (positive), delay (negative)
% USE y = ustep(t,ad)
N = length(t);y = zeros(1,N);
for i = 1:N,
          if t(i)>= -ad,
          y(i) = 1;
          end
end
```

```
% Example 1.16 -- signal generation
clear; clf
Ts=0.01; t=-5:Ts:5; % support of signal
% ramps with support [-5, 5]
y1=ramp(t,3,3); % slope of 3 and advanced by 3
y2=ramp(t,-6,1); % slope of -6 and advanced by 1
y3=ramp(t,3,0); % slope of 3
% unit-step signal with support [-5,5]
y4=-3*ustep(t,-3); % amplitude -3 and delayed by 3
y=y1+y2+y3+y4;
plot(t,y,'k'); axis([-5 5 -1 7]); grid
```



Chaparro Example 1.17

What is the function y, plotted by this Matlab script?

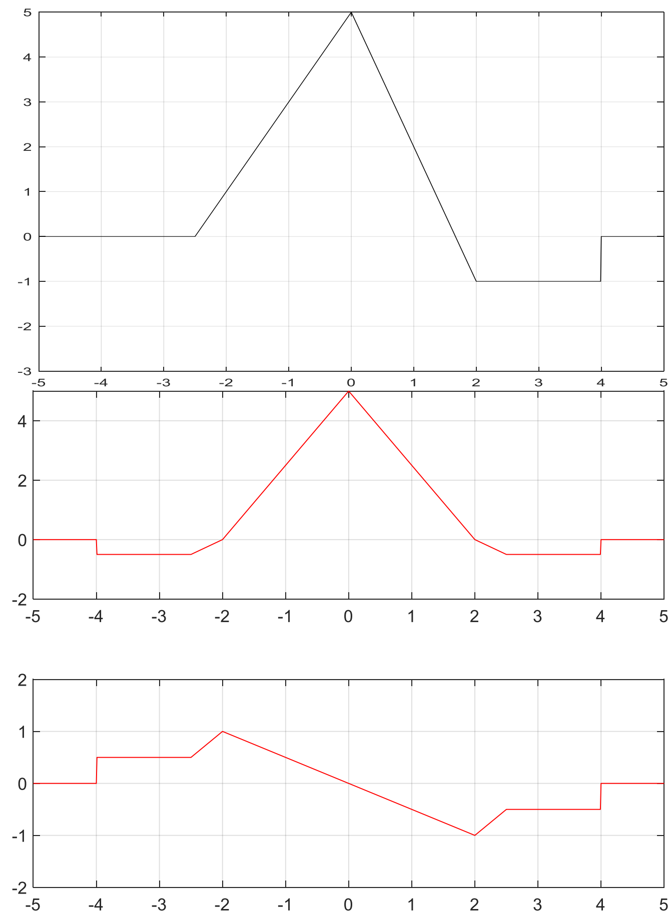
2r(t+2.5)-5r(t)+3r(t-2)+u(t-4)

```
% Example 1.17 -- signal generation
clear all; clf
t = -5:0.01:5;
y1 = ramp(t,2,2.5);
y2 = ramp(t,-5,0);
y3 = ramp(t,3,-2);
y4 = ustep(t,-4);
y = y1 + y2 + y3 + y4;
plot(t,y,'k'); axis([-5 5 -3 5]); grid

[ye, yo] = evenodd(t,y);
subplot(211);plot(t,ye,'r');grid
axis([min(t) max(t) -2 5])
subplot(212);plot(t,yo,'r');grid
axis([min(t) max(t) -2 2])
```

```
function [ye,yo] = evenodd(t,y)
% even/odd decomposition
% t: time
% y: analog signal
% ye, yo: even and odd components
% USE [ye,yo] = evenodd(t,y)
yr = fliplr(y);
ye = 0.5*(y + yr);
yo = 0.5*(y-yr);
```

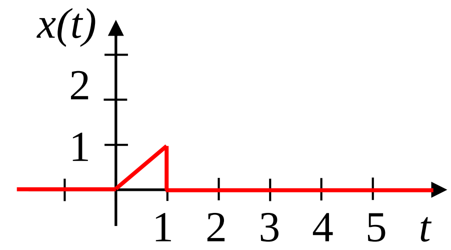
Should mention support must be symmetric wrt t=0.

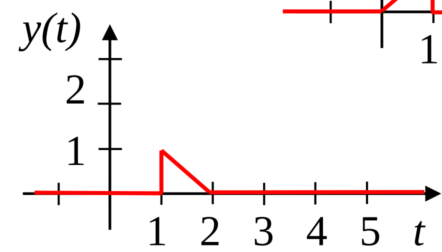


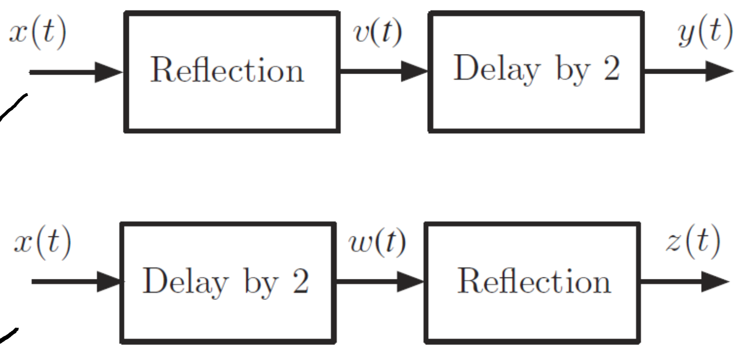
Chaparro Prob 1.4

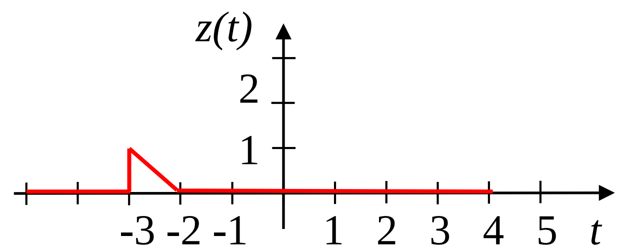
Do reflection and time-shifting commute?

E.g., for *x(t)* shown, does *y(t)*=*z(t)*?









MORE GENERALLY

$v(t)=x(-t)$; $y(t)=v(t-2)=x(-(t-2))=x(-t+2)$

$\neq$

$w(t)=x(t-2)$; $z(t)=w(-t)=x(-t-2)$

DO NOT COMMUTE

Example

Given that $x(t)$ is even and that $x(t+1)$ is also even, show that $x(t)$ is periodic.

$$y(t) = x(t+1) \Rightarrow y(t) = y(-t)$$

$$= x(-t+1)$$

(A)

$$x(t) = x(-t)$$

(B)

(C)

$$x(t+1) = x(-t+1) = x(t-1) \quad \forall t$$

PERIOD IS 2

(D) CHANGE OF DUMMY VARIABLES $\tau = t-1 \Leftrightarrow \tau+1 = t$

$$\Rightarrow x(\tau+2) = x(\tau) \quad \forall \tau \Rightarrow$$ PERIOD IS 2

ASIDE: THIS DOES NOT PRECLUDE THE POSSIBILITY OF A SHORTER FUNDAMENTAL PERIOD