Analysis

Contents

1 Sets

- **Definition:** A set is a collection of objects, called the *elements* or *members* of the set. We write $x \in X$ if x is an element of the set X and $x \notin X$ if x is not an element of X.

  Two sets $X = Y$, if
  $$x \in X \iff x \in Y$$
  ("iff" or "⇔" both mean "if and only if").

  The empty set is denoted by $\varnothing$, that is, the set without any elements. X is *nonempty* if it has at least one element.

  We can define sets by listing their elements:
  $$X = \{a, b, c, d\}.$$

  We can also have infinite sets, for example:
  $$\mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \dots\},$$
  $$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}, \quad \mathbb{Q} = \left\{\tfrac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\right\},$$
  $$\mathbb{R} = \{\text{all numbers with decimal expansions}\}.$$

1.1 Subsets

A is a subset of a set X or A is included in X, written $A \subseteq X$, if every element of A belongs to X. A is a proper subset of X, written as $A \subset X$, when $A \subseteq X$, but $A \neq X$.

- **Def.:** The power set $\mathcal{P}(X)$ of a set X is the set of all subsets of X.
- **Ex.:** X ={1,2,3}, then $\mathcal{P}(X)$={∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}

The power set $\mathcal{P}(X)$ of a set X with $|X| = n$ elements has $|\mathcal{P}(X)| = 2^n$ elements because, in defining a subset, we have two independent choices for each element (does it belong to the subset or not?). Thus, the notation $2^X = \mathcal{P}(X)$ is also in use.

1.2 Set operations

The intersection $A \cap B$ of two sets A, B is the set of all elements that belong to both A and B. Two sets A, B are said to be disjoint if $A \cap B = \varnothing$; that is, if A and B have no elements in common.

The union $A \cup B$ is the set of all elements that belong to A or B. Note that we always use 'or' in an inclusive sense, so that $x \in A \cup B$ if x is an element of A or B, or both A and B. (Thus, $A \cap B \subseteq A \cup B$.)

The set-difference of two sets B and A is the set of elements of B that do not belong to A, that is $B \setminus A = \{x \in B : x \notin A\}$. If we consider sets that are subsets of a fixed set X (called the universe) that is understood from the context, then we write $A^c = \overline{A} = X \setminus A$ to denote the complement of $A \subseteq X$ in X. Note that $(A^c)^c = A$.

The Cartesian product $A \times B$ of sets A, B is the set whose members all possible ordered pairs (a, b) with $a \in A$, $b \in B$, thus $A \times B = \{(a, b) : a \in A, b \in B\}$ and $|A \times B| = |A||B|$.

1.3 Algebraic properties

Intersection is a commutative operation $A \cap B = B \cap A$; and an *associative* operation, that is:
$$(A \cap B) \cap C = A \cap (B \cap C), \text{ thus } = A \cap B \cap C$$
both are also true for the union $A \cup B$. Intersection distributes over union and union distributes over intersection:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

We have De Morgan's laws:

$$(a \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c$$

Arbitrary many unions and intersections: Let $\mathcal{C}$ be a collection of sets. Then

$$\bigcup \mathcal{C} = \bigcup_{A \in \mathcal{C}} A = \{x : x \in A, \text{ for some } A \in \mathcal{C}\}$$

$$\bigcap \mathcal{C} = \bigcap_{A \in \mathcal{C}} A = \{x : x \in A, \text{ for all } A \in \mathcal{C}\}$$

1.4 Relations

Any subset of the Cartesian product of two sets X, Y defines a (binary) relation $R \subseteq X \times Y$ between these two sets. Given $(x, y) \in R$ we may denote this inclusion simply as xRy. Notation: $\forall$ means 'for all', $\exists$ means 'exists'. A binary relation R is *univalent* if

$$\forall x \in X, \forall y \in Y, \forall z \in Y \text{ we have } ((x, y) \in R \text{ and } (x, z) \in R) \implies y = z$$

A binary relation R is *total* if

$$\forall x \in X, \exists y \in Y \text{ we have } (x, y) \in R$$

- **Def.:** A partially defined function is a univalent binary relation, and a function is a univalent and total binary relation. Thus a function $f : X \mapsto Y$ is defined by a univalent and total $xRy \iff y = f(x)$.

The set of all functions from X to Y is commonly denoted as

$$Y^X = \prod_{x \in X} Y$$

1.5 Orders and equivalences

- **Def.:** An order $\leq$ on a set X is a binary relation on X, s.t. for every $x, y, z \in X$:
  1. $x \leq x$ (reflexivity),
  2. If $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry),
  3. If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

An order is *linear* or *total* if $\forall x, y \in X$ either $x \leq y$ or $y \leq x$. If $\leq$ is an order, then we define a strict order by $x < y$ if $x \leq y$ and $x \neq y$.

If for a relation $\sim$ in 2. instead of antisymmetry we have *symmetry*: If $x \sim y$ then $y \sim x$ then $\sim$ is called an equivalence relation.

2 Functions

Per definition a function $f : X \mapsto Y$ is a univalent and total relation, that is for every $x \in X$ there is a unique $y = f(x) \in Y$. $Do(f) = X$ is called the *domain* of f, and $Ran(f) = \{y \in Y : \exists x \in X, y = f(x)\} \subseteq Y$ is called the range of f. Also $f(A) = \{y \in Y : \exists x \in A, y = f(x)\}$ for some $A \subseteq X$.

- **Ex.:** The identity function $id_X : X$ on a set X is the function that maps every element of X to itself, that is $id_X(x) = x$ for all $x \in X$.
- **Ex.:** the characteristic or indicator function $\chi_A : X \mapsto \{0, 1\}$ of $A \subseteq X$ is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The graph of a function $f : X \mapsto Y$ is defined as

$$G_f = \{(x, y) \in X \times Y : y = f(x)\}$$

2.1 Properties of functions

A function $f : X \mapsto Y$ is

1. injective (one-to-one) if it maps distinct elements to distinct elements, that is $x_1, x_2 \in X$ and $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$,

2. surjective (onto) if its range $Ran(f) =$ Y, that is for every y there exists an x, s.t. $y = f(x)$.

3. If a function is both injective and surjective then its bijective.

We define the composition $f \circ g(z) = f(g(z))$ of functions $f : Y \mapsto X$ and $g : Z \mapsto Y$. Note that we need the inclusion $Ran(g) \subseteq Do(f)$. $\circ$ is associative.

A bijective function $f : X \mapsto Y$ has an inverse $f^{-1} : Y \mapsto X$ defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

that is $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$.

If $f : x \mapsto Y$ is merely injective than still $f : X \mapsto Ran(f)$ is bijective, thus invertible on its range with inverse $f^{-1} : Ran(f) \mapsto X$.

2.2 Groups, monoids, fields

- **Def.:** Given a function $f : X \times X \mapsto X$ we may denote $f(x, y) = x * y$ and consider this as a binary operation on X. For example addition of integers is such an operation. The we say that * is/has

  1. Associative, if $x * (y * z) = (x * y) * z$,
  2. Commutative, if $x * y = y * x$,
  3. Neutral element, if there exists $e \in X$ (a neutral element), s.t. $x * e = e * x = x$,
  4. Inverse elements, if for all $x \in X$ there exists $x' \in X$ called an inverse of x, s.t. $x * x' = x' * x = e$ where e is a neutral element.

- **Def.:** $(X, *)$ is called a

  1. Semigroup, if * is associative,
  2. Monoid, if $(X, *)$ is a semigroup and has a neutral element,
  3. Group, if $(X, *)$ is a monoid and every element $x \in X$ has an inverse.

- **Theorem:** In a group $(X, *)$ the neutral element $e \in X$ and inverse x' for any fixed $x \in X$ are unique.

- **Proof.:** Indeed, if there would be two neutral elements e, e', then $e' = e * e' = e$. Also assuming $x * y = e = x * z$, implies $x' * (x * y) = x' * (x * z)$, that is $y = z = x'$.

  Let $(X, \cdot, +)$ be given with binary operations $\cdot$ and $+$.

- **Def.:** $(X, \cdot, +)$ is a field if

  1. $(X, \cdot, +)$ is a commutative group with neutral element 0,
  2. $(X \setminus \{0\}, \cdot)$ is a commutative group,
  3. $\cdot$ distributes over $+$ (distributivity), that is: $x \cdot (y + z) = x \cdot y + x \cdot z$. In this case $+$ is usually called addition and $\cdot$ multiplication.

- **Ex.:** the rational numbers $\mathbb{Q}$ is a field, moreover an ordered field $(\mathbb{Q}, \cdot, +, \leq)$ equipped with the total order $x \leq y \iff 0 \leq y - x$, where 0 is the neutral element of $+$.

- **Ex.:** the set of real numbers $\mathbb{R}$ is also a totally ordered field $(\mathbb{R}, \cdot, +, \leq)$

- **Axiom:** the $\leq$ order of $\mathbb{R}$ satisfies

  I. $x \leq y$ implies $x + z \leq y + z$,

  II. $x < y$ and $z > 0$ implies $xz < yz$.

2.3 Supremum and infimum

- **Def.:** A set $A \subseteq \mathbb{R}$ is *bounded from above*, if $\exists M \in \mathbb{R}$ s.t. $x \leq M$ for all $x \in A$; and it is *bounded from below*, if $\exists m \in \mathbb{R}$ s.t. $x \geq m$ for all $x \in A$. If both holds for A, then it is bounded. ('s.t.' is short hand for 'such that')

- **Def.:** If $M \in \mathbb{R}$ is an upper bound of $A \subseteq \mathbb{R}$ s.t. for any other upper bound $M' \in \mathbb{R}$ of A we have $M \leq M'$, then M is called the least upper bound of A, denoted as

  $$M = \sup A$$

  Similarly, the greatest lower bound of $A \subseteq \mathbb{R}$, if exists, is denoted by

  $$m = \inf A$$

  meaning $m \geq m'$ for any lower bound m of A. If $A = \{x_i : i \in I\} \subseteq \mathbb{R}$ for an index set I, we also write:

  $$\sup A = \sup_{i \in I} x_i \text{ and } \inf A = \inf_{i \in I} x_i$$

  Fact: by the definition supremum and infimum of a set, if they exist, are both unique and $\sup A \geq \inf A$ for nonempty $A \subseteq \mathbb{R}$.

- **Def.:** if $\sup A \in A$, then we call it the maximum of A denoted by $\max A$, similarly if $\inf A \in A$, then we call it the minimum of A denoted by $\min A$.

  Ex.: Let $\mathbb{R} \supseteq A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\sup A = 1$ belongs to A, while $\inf A = 0$ does not belong to A.

- **Def.:** let us introduce the elements ∞,$-\infty$, so that $\infty > x > -\infty$ for any $x \in \mathbb{R}$ and define the extended real numbers as $\overline{\mathbb{R}} = \{\infty, -\infty\} \cup \mathbb{R}$. If a set $A \subseteq \mathbb{R}$ is not bounded from above then define $\sup A = \infty$, and if $A \subseteq \mathbb{R}$ is not bounded from below then define $\inf A = -\infty$. Also define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

2.4 (Order) Completeness

Consider $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$. This set is bounded from above but has no least upper bound in $\mathbb{Q}$.

- **Def(Completeness).:** a totally ordered field Z is complete, if all nonempty upper bounded subsets of Z have a least upper bound in Z. We call this the least upper bound property.

- **Theorem(Dedekind):** There exists a unique (up to $(\cdot, +, \leq)$-preserving transformation) ordered complete field satisfying the order axioms I., II. that contains $\mathbb{Q}$ and it is the field $\mathbb{R}$. Such a transformation $\phi : \mathbb{R} \mapsto \mathcal{M}$ satisfies $\phi(x + y) = \phi(x) + \phi(y), \phi(xy) = \phi(x)\phi(y), x \leq y \implies \phi(x) \leq \phi(y)$.

2.5 Archimedean property

- **Theorem(Archimedean property):** If $x \in \mathbb{R}$, then there exists $n \in \mathbb{Z}$ such that $x < n$.

- **Proof:** Suppose, for contradiction, that there exists $x \in \mathbb{R}$ s.t. $x > n$ for all $n \in \mathbb{Z}$. Then x is an upper bound of $\mathbb{Z} \subseteq \mathbb{R}$, so $M = \sup \mathbb{Z} \in \mathbb{R}$ exists. Since $n \leq M$ for all $n \in \mathbb{Z}$, we have $n - 1 \leq M - 1$ for all $n \in \mathbb{Z}$, which implies $n \leq M - 1$ for all $n \in \mathbb{Z}$. But then $M - 1$ is an upper bound of $\mathbb{Z}$ that is strictly less than $M = \sup \mathbb{Z}$, a contradiction to $M = \sup \mathbb{Z}$ being the least upper bound.

- **Corollary:** For every $0 < \varepsilon \in \mathbb{R}$, there exists an $n \in \mathbb{N}$, s.t. $0 < \frac{1}{n} < \varepsilon$.

- **Corollary(integer part):** If $x \in \mathbb{R}$, then there exists $[x] = n \in \mathbb{Z}$ called the integer part of x, such that $n \leq x < n + 1$.

2.6 Further properties

- **Def(dense set):** $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, if for any $0 < \varepsilon, x \in \mathbb{R}$ there exists $a \in A$, s.t. $x - \varepsilon < a < x + \varepsilon$.
- **Theorem(density of rationals):** $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$.
- **Proof:** Let $0 < \varepsilon, x \in \mathbb{R}$. Then for any $n \in \mathbb{N}$ we have

$$[nx] \leq nx < [nx] + 1$$

which gives

$$\frac{[nx]}{n} \leq x < \frac{[nx]}{n} + \frac{1}{n}$$

Pick $n \in \mathbb{N}$, s.t. $0 < \frac{1}{n} < \varepsilon$. Then we have

$$\frac{[nx]}{n} \leq x < \frac{[nx]}{n} + \frac{1}{n} < \frac{[nx]}{n} + \varepsilon$$

which implies $x - \varepsilon < \frac{[nx]}{n} < x + \varepsilon$ as wanted.

2.7 Properties of sup and inf

Theorem:

1. Given $A \in \mathbb{R}$, then $M = \sup A$ if and only if
   (a) M is an upper bound of A,
   (b) for every $M' < M$ there exists $x \in A$ s.t. $M' < x$.
2. If $A \subseteq B \subseteq \mathbb{R}$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$.
3. If $A \subseteq \mathbb{R}$, then $\inf A \leq \sup A$.
4. If $A \subseteq \mathbb{R}$, then $-\inf A = \sup(-A)$.
5. If $A \subseteq \mathbb{R} \ni \alpha \geq 0$, then $\sup(\alpha A) = \alpha \sup(A)$.
6. If $A, B \subseteq \mathbb{R}$, we have $\sup(A+B) \leq \sup A + \sup B$, $\inf(A+B) \geq \inf A + \inf B$ where $A + B = \{x \in \mathbb{R} : x = a + b, a \in A, b \in B\}$.
7. Let $\mathcal{C}$ be a family of sets in $\mathbb{R}$, then $\sup(\cup \mathcal{C}) = \sup\{\sup A : A \in \mathcal{C}\}$.

2.8 Intervals and topology of $\mathbb{R}$

Def.: Let $a, b \in \mathbb{R}$.

1. Closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$,
2. Open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$,
3. Half-open intervals $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}, [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
4. $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}, (a, \infty) = \{x \in \mathbb{R} : a < x\}, (-\infty, b] = \{x \inf \mathbb{R} : b \geq x\}, (-\infty, b) = \{x \in \mathbb{R} : b > x\}$

Def.: $A \subseteq \mathbb{R}$ is open if for every $x \in A$ there exists $0 < \epsilon \in \mathbb{R}$ s.t. $(x - \epsilon, x + \epsilon) \subseteq A$.
Def.: $B \subseteq \mathbb{R}$ is closed if $B^C = \{x \in \mathbb{R} : x \notin B\}$ is open.
Def.: $U \subseteq \mathbb{R}$ is a neighborhood of $z \in \mathbb{R}$, if there exists $0 < \epsilon \in \mathbb{R}$ s.t. $(z - \epsilon, z + \epsilon) \subseteq U$.
Theorem: Arbitrary union of open sets is open, and an intersection of finite number of open sets is open.
Def.: Let $A \subseteq \mathbb{R}$, then $x \in \mathbb{R}$ is

1. an interior point of a, if there exists $0 < \epsilon \in \mathbb{R}$ s.t. $(x - \epsilon, x + \epsilon) \subseteq A$;
2. an isolated point of A, if $x \in A$ and there exists $0 < \epsilon \in \mathbb{R}$ s.t. x is the only point of A that belongs to $(x - \epsilon, x + \epsilon)$;
3. a boundary point of A, if for every $0 < \epsilon \in \mathbb{R}$ the interval $(x - \epsilon, x + \epsilon)$ contains at least a point in A and at least a point not in A;
4. an accumulation point of A, if for every $0 < \epsilon \in \mathbb{R}$ the interval $(x - \epsilon, x + \epsilon)$ contains a point in A distinct from x.

3 The absolute value

- **Def.:** the absolute value of $x \in \mathbb{R}$ is defined by $\begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$.
- **Proposition**: $\forall x, y \in \mathbb{R}$ we have
  1. $|x| \geq 0$ and $|x| = 0 \iff x = 0$,
  2. $|-x| = |x|$,
  3. (triangle inequality) $|x + y| \leq |x| + |y|$,
  4. $|xy| = |x||y|$,
  5. $||x| - |y|| \leq |x - y|$.
- **Proof**: 1.,2. and 4. are trivial. To see 3. suppose without loss of generality that $x \geq 0$, $|x| \geq |y|$, in which case $x + y \geq 0$. If $y \geq 0$, then $|x + y| = x + y = |x| + |y|$. If $y < 0$, then $|x + y| = x + y = |x| - |y| \leq |x| + |y|$. To obtain 5. we use 3. to get $|x| = |x - y + y| \leq |x - y| + |y|$.

4 Sequences and limits

- **Def.:** a sequence x_n of real numbers is an ordered list of numbers $x_n \in \mathbb{R}$, called the terms of the sequence, indexed by the natural numbers $n \in \mathbb{N}$. It may be regarded as a function $f : \mathbb{N} \to \mathbb{R}$ with $x_n = f(n)$.
- **Def.:** A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$, written as $x = \lim\limits_{n \to \infty} x_n$, or $x_n \to x$ as $n \to \infty$, if $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $\forall n > N$ we have $|x_n - x| < \varepsilon$.

  A sequence (x_n) converges if it converges to a limit $x \in \mathbb{R}$, otherwise it diverges. Note that $x_n \to x$ and $|x_n - x| \to 0$ are equivalent statements.
- **Def.:** if (x_n) is a sequence, then $\lim\limits_{n \to \infty} x_n = \infty$, or $x_n \to \infty$ if $\forall M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ s.t. $\forall n > N$ we have $x_n > M$. Similarly, we define $\lim\limits_{n \to \infty} x_n = -\infty$, or $x_n \to -\infty$ if $\forall M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ s.t. $\forall n > N$ we have $x_n < M$.

4.1 Properties of limits

- **Proposition**: If a sequence converges, then its limit is unique.
- **Proof**: Suppose that (x_n) is a sequence such that $x_n \to x$ and $x_n \to x'$ as $n \to \infty$. Then $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon/2$ and $|x_n - x'| < \varepsilon/2$ for all $n > N$. Then $|x - x'| = |x - x_n + x_n - x'| \leq |x - x_n| + |x_n - x'| < \varepsilon$ for all $n > N$. Since $\varepsilon > 0$ was arbitrary, $|x - x'| < \varepsilon$ proves that $|x - x'| = 0$.
- **Ex.**: Let $x_n = \frac{1}{n} : n \in \mathbb{N}$. Then $x_n \to 0$. Indeed, let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Then $\forall n > N$ we have $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$, which proves that $\lim\limits_{n \to \infty} \dfrac{1}{n} = 0$.

- **Proposition**: A convergent sequence is bounded.
- **Proof**: After some index $N \in \mathbb{N}$ for all $n > N$ we have $|x_n - x| < 1$ which implies that $x-1 < x_n < x+1$, thus x_n is bounded.
- **Fact**: Convergence of x_n to x does not depend on any of the first finitely many elements of (x_n).
- **Theorem**: If (x_n), (y_n) are convergent sequences, then $x_n \le y_n$ for all $n > N \in \mathbb{N}$ implies $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.
- **Proof**: $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon/2$ and $|y_n - y| < \varepsilon/2$ for all $n > N$. Then $x = x_n + x - x_n < y_n + \frac{\varepsilon}{2} = y + y_n - y + \frac{\varepsilon}{2} < y + \varepsilon$, which implies $x \le y$.
- **Theorem(Squeeze, or Sandwich)**: If (x_n), (y_n) are convergent sequences with common limit L, then $x_n \le z_n \le y_n$ implies $\lim\limits_{n\to\infty} z_n = L$ as well.
- **Proof**: The assumption implies that $\forall \varepsilon > 0$ there exists an index $N \in \mathbb{N}$ s.t. for all $n > N$ we have $L - \varepsilon < x_n \le z_n \le y_n < L + \varepsilon$, which means that $z_n \to L$.
- **Theorem**: If (x_n), (y_n) are convergent sequences and $c \in \mathbb{R}$, then
  1. $\lim\limits_{n\to\infty} cx_n = c \lim\limits_{n\to\infty} x_n$;
  2. $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;
  3. $\lim\limits_{n\to\infty} (x_n y_n) = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$.
- **Proof**: If c = 0 then 1. is immediate. If $c \neq 0$, then $|x_n - x| < \varepsilon/|c|$ for all $n > N$ implies $|cx_n - cx| < \varepsilon$.

For the second statement we have $|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all large enough n, which proves that $x_n + y_n \to x + y$.

For the third statement $|x_n y_n - xy| = |(x_n - x)y_n + x(y_n - y)| \le |x_n - x||y_n| + |y_n - y||x| < \varepsilon$ for all large enough n, which proves that $x_n y_n \to xy$.

4.2 Monotone sequences

- **Definition:** A sequence (x_n) is monotone

  Increasing:
  if $x_n \le x_{n+1}$, *strictly* if $x_n < x_{n+1}$;

  Decreasing:
  if $x_n \ge x_{n+1}$, *strictly* if $x_n > x_{n+1}$.

- **Monotone Convergence Theorem:**

  If (x_n) is monotone increasing, then $\lim\limits_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

  If (x_n) is monotone decreasing, then $\lim\limits_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.

- **Proof:** We prove only the first statement, the second can be proved by choosing $y_n = -x_n$. The least upper bound of the set $\{x_n : n \in \mathbb{N}\}$ is $M = \sup\{x_n : n \in \mathbb{N}\}$, so $x_n \le M$. Suppose $M < \infty$. Then $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $M - \varepsilon < x_N \le x_n$ where the second inequality holds for all $n > N$ by monotonicity. This implies $M - \varepsilon < x_n < M + \varepsilon$ for all $n > N$, thus $\lim_{n\to\infty} x_n = M$. If $M = \infty$, then still $\forall K > 0$ there exists $N \in \mathbb{N}$ s.t. $K < x_N \le x_n$ for all $n > N$ by monotonicity. Thus, $\lim_{n\to\infty} x_n = \infty$.

4.3 $\lim sup x_n$ and $\lim inf x_n$

- **Def.:** for a sequence (x_n) we define

$$\lim_{n\to\infty} sup x_n = \lim_{n\to\infty} y_n \text{ where } y_n = \sup\{x_k : k \in \mathbb{N}, k \geq n\};$$
$$\lim_{n\to\infty} inf x_n = \lim_{n\to\infty} z_n \text{ where } z_n = \inf\{x_k : k \in \mathbb{N}, k \geq n\}.$$

- Note that the above limits exist, because $y_n \geq y_{n+1}$ and $z_n \leq z_{n+1}$.

- **Theorem:** We have $y = \lim_{n\to\infty} sup x_n \iff -\infty \leq y \leq \infty$ satisfies one of the following:

  1. $-\infty < y < \infty$ and for $\forall\varepsilon > 0$
     (a) there exists $N \in \mathbb{N}$ s.t. for all $n > N$ we have $x_n < y + \varepsilon$;
     (b) For every $N \in \mathbb{N}$ there exists $n > N$ s.t. $x_n > y - \varepsilon$.
  2. $y = \infty$ and for every $M \in \mathbb{R}$ there exists $n \in \mathbb{N}$ s.t. $x_n > M$.
  3. $y = -\infty$ and for every $m \in \mathbb{R}$ there exists $N \in \mathbb{N}$ s.t. $x_n < m$ for all $n > N$.

  Analogous results hold for the lim inf as well.

- **Proof:** First suppose that $-\infty < y < \infty$. Then (x_n) is bounded from above and $y_n = \sup\{x_k : k \in \mathbb{N}, k \geq n\}$ is a monotone decreasing sequence with limit y. Therefore $\forall\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $y_N < y + \varepsilon$. Since $x_n \leq y_N$ for all $n > N$ we have $x_n < y + \varepsilon$ proving 1.a.

  To prove 1.b. let $\varepsilon > 0$ and suppose that $N \in \mathbb{N}$ is arbitrary. $y_N \geq y$ is the sup of $\{x_k : k \in \mathbb{N}, k \geq N\}$, there exists $n \geq N$ s.t. $x_n > y_N - \varepsilon \geq y - \varepsilon$ which proves 1.b.

  Conversely, suppose that $-\infty < y < \infty$ satisfies 1. Then given any $\varepsilon > 0$, 1.a. implies that there exists $N \in \mathbb{N}$ s.t. $y_n = \sup\{x_k : k \in \mathbb{N}, k \geq n\} < y + \varepsilon$ for all $n > N$, and 1.b. implies that $y_n > y - \varepsilon$ for all $n \in \mathbb{N}$. Thus, $|y_n - y| < \varepsilon$ for all $n > N$, so $y_n \to y$.

- **Theorem:** for a sequence (x_n) we have
  $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x \iff \lim_{n\to\infty} x_n = x.$.

- **Proof:** First suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$. Then $y_n = \sup\{x_k : k \in \mathbb{N}, k \geq n\}$ is monotone decreasing to x, while $z_n = inf x_k : k \in \mathbb{N}, k \geq n$ is monotone increasing to x and also $z_n \leq x_n \leq y_n$, so the squeeze theorem proves $\lim_{n\to\infty} x_n = x$. The reverse implication follows from $x - \varepsilon < x_n < x + \varepsilon$ implying also $x - \varepsilon < z_n \leq y_n < x + \varepsilon$ for all $n > N$ where N is chosen accordingly to given $\varepsilon > 0$. Thus, $z_n, y_n \to x$.

  In the remaining cases a sequence (x_n) diverges to $\infty \iff \liminf_{n\to\infty} x_n = \infty$, and then $\limsup_{n\to\infty} x_n = \infty$, since $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$. Similarly, (x_n) diverges to $-\infty \iff \limsup_{n\to\infty} x_n = -\infty$ and then $\liminf_{n\to\infty} x_n = -\infty$ as well.

- **Corollary:** $\lim_{n\to\infty} x_n = x \iff \limsup_{n\to\infty} |x_n - x| = 0$.

4.4 Cauchy sequences

- **Def(Cauchy sequence):** a sequence (x_n) is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m > N$.

- **Theorem:** A sequence (x_n) converges $\iff$ it is Cauchy.

- **Proof:** First suppose that (x_n) converges to a limit $x \in \mathbb{R}$. Then $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \frac{\varepsilon}{2}$ for all $n > N$. It follows that if $n, m > N$, then $|x_n - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon$ proving that (x_n) is Cauchy.

  Conversely suppose that (x_n) is Cauchy. Then there exists $N_1 \in \mathbb{N}$ s.t. for all $n, m > N_1$, we have that $|x_n - x_m| < 1$. Then for $n > N_1$ we have $|x_n| \leq |x_n - x_{N_1+1}| + |x_{N_1+1}| \leq 1 + |x_{N_1+1}|$.

  Thus, the sequence is bounded with

  $|x_n| \leq \max\{|x_1|, ..., |x_{N_1}|, 1 + |x_{N_1+1}|\}$.

  Given that (x_n) is bounded, its lim sup and lim inf is bounded. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. by the assumption $x_n - \varepsilon < x_m < x_n + \varepsilon$ for all $m \geq n > N$. Then we have

  $x_n - \varepsilon \leq \inf\{x_m : m \in \mathbb{N}, m \geq n\}$ and $\sup\{x_m : m \in \mathbb{N}, m \geq n\} \leq x_n + \varepsilon$,

  which implies

  $\sup\{x_m : m \in \mathbb{N}, m \geq n\} - \varepsilon \leq \inf\{x_m : m \in \mathbb{N}, m \geq n\} + \varepsilon$.

  Taking the limit $n \to \infty$, we get that

  $\limsup\limits_{n\to\infty} x_n - \varepsilon \leq \liminf\limits_{n\to\infty} x_n + \varepsilon$

  and since $\varepsilon > 0$ was arbitrary, this yields $\limsup\limits_{n\to\infty} x_n \leq \liminf\limits_{n\to\infty} x_n$, thus

  $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n$ which implies convergence.

4.5 Subsequences

- **Def(subsequence):** a subsequence of a sequence (x_n) is a sequence of the form (x_{n_k}) with index $k \in \mathbb{N}$ where $n_1 < n_2 < ... < n_k < ....$ In other words, n_k is a strictly increasing integer valued function with domain $\mathbb{N}$.

- **Proposition:** Every subsequence of a convergent sequence converges to the limit of the sequence.

- **Proof:** Suppose (x_n) is a convergent sequence with $\lim\limits_{n\to\infty} x_n = x$ and (x_{n_k}) is a subsequence. Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n > N$. Since $n_k \to \infty$ as $k \to \infty$, there exists $K \in \mathbb{N}$ s.t. $n_k > N$ if $k > K$. Then $k > K$ implies $|x_{n_k} - x| < \varepsilon$.

- **Corollary:** If a sequence has subsequences that converge to different limits, then the sequence diverges.

4.6 Bolzano-Weierstrass theorem

- **Theorem(Bolzano-Weierstrass):** Every bounded sequence of real numbers has a convergent subsequence.

- **Proof:** The statement will follow from the Monotone Convergence theorem of monotone bounded sequences using the following

  Claim: every sequence (x_n) has a monotone subsequence.

  Proof: Let us call a positive integer-valued index n of a sequence a "peak" of the sequence when $x_m \leq x_n$ for all $m > n$. Suppose first that the sequence has infinitely many peaks, which means there is a subsequence (x_{n_k}) consisting of these peaks for which we have $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq ... \geq x_{n_k} \geq ...$, so (x_{n_k}) is a monotone decreasing subsequence. Now suppose that on the contrary there are only finitely many peaks of (x_n). Let the index N be the index of the final peak in (x_n) and let $n_1 = N + 1$. Since n_1 comes after the final peak, it is not a peak, thus it implies the existence of $n_2 > n_1$ s.t. $x_{n_2} \geq x_{n_1}$. Again n_2 comes after the final peak, it is not a peak, so again it implies the existence $n_3 > n_2$ s.t. $x_{n_3} \geq x_{n_2}$. Repeating this we obtain $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq ... \leq x_{n_k} \leq ...$ a monotone increasing subsequence (x_{n_k}).

5 Binomial theorem

- Let $\binom{n}{k} = B(n,k) = \frac{n!}{k!(n-k)!}$ for $n \geq k \geq 0$ integers. It is called the binomial coefficient.

- **Proposition:** $\binom{n}{k}$ gives the number of ways, disregarding order, that k objects can be chosen from among n objects; more formally, the number of k-element subsets (or k-combinations) of an n-element set. Also, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

- **Theorem(Binomial theorem):** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

- **Proof:** $(x+y)^n = \underbrace{(x+y)}_{1} \cdots \underbrace{(x+y)}_{n}.$

  and you count the number of times the term $x^k y^{n-k}$ occurs in the expansion. It is exactly how many ways k objects can be chosen from among n objects.

- **Corollary:** For $1 < x = 1+\delta$ the binomial theorem implies that $x^n > 1+n\delta$, thus $x^n \to \infty$. Similarly, when $1 > x > 0$ we have that $\frac{1}{x} > 1$, so $\frac{1}{x^n} \to \infty$ implies that $x^n \to 0$. When $-1 < x < 1$ these imply $x^n \to 0$.