Analysis

Contents

1 Sets

Definition: A set is a collection of objects, called the elements or members of the set.

We write $x \in X$ if x is an element of the set X and $x \notin X$ if x is not an element of X.

Two sets $X = Y$ if $x \in X$ iff $x \in Y$ (where "iff" and $\iff$ means "if and only if").

The empty set is denoted by $\emptyset$, that is the set without any elements. X is nonempty if it has at least one element.

We can define sets by listing its elements: $X = \{a, b, c, d\}$.

We can have infinite sets, for example:

- The rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$

1.1 Subsets

A is a subset of a set X or A is included in X, written $A \subseteq X$, if every element of A belongs to X. A is a proper subset of X, written as $A \subset X$, when $A \subseteq X$, but $A \neq X$.

Definition: The power set $P(X)$ of a set X is the set of all subsets of X.

Example: $X = \{1, 2, 3\}$, then

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

The power set $P(X)$ of a set X with $|X| = n$ elements has $|P(X)| = 2^n$ elements because, in defining a subset, we have two independent choices for each element. Thus, the notation $2^X = P(X)$ is also in use.

The Cartesian product AxB of sets A, B is the set whose members all possible ordered pairs (a, b) with $a \in A$, $b \in B$, thus $A \times B = \{(a, b) : a \in A, b \in B\}$ and $|A \times B| = |A||B|$.

1.2 Relations

Any subset of the Cartesian product of two sets X, Y defines a (binary) relation $R \subseteq X \times Y$ between these two sets. Given $(x, y) \in R$ we may denote this inclusion simply as xRy.

Notation: $\forall$ means 'for all', $\exists$ means 'exists'.

- A binary relation R is univalent if $\forall x \in X, \forall y \in Y, \forall z \in Y$, we have $((x, y) \in R$ and $(x, z) \in R) \Rightarrow y = z$.
- A binary relation R is total if $\forall x \in X, \exists y \in Y$ we have $(x, y) \in R$.

Definition: A partially defined function is a univalent binary relation, and a function is a univalent and total binary relation. Thus a function $f : X \to Y$ is defined by a univalent and total $xRy \iff y = f(x)$.

The set of all functions from X to Y is commonly denoted as $Y^X = \prod_{x \in X} Y$.

1.3 Orders And Equivalences

Definition: An order $\leqq$ on a set X is a binary relation on X: s.t. for every $x, y, z \in X$:

- $x \leqq x$ (reflexivity)
- $If x \leqq y$ and $y \leqq x$ then $x = y$ (antisymmetry)
- $If x \leqq y$ and $y \leqq z$ then $x \leqq z$ (transitivity)

An order is linear or total if $\forall x, y \in X$ either $x \leqq y$ or $y \leqq x$. If $\leqq$ is an order, then we define a strict order by $x < y$ if $x \leqq y$ and $x \neq y$. If for a relation in 2. instead of antisymmetry we have symmetry: 2. If $x \sim y$ then $y \sim x$ then $\sim$ is called an equivalence relation.

1.4 Functions

Definition: Per definition a function $f : X \to Y$ is a univalent and total relation, that is for every $x \in X$ there is a unique $y = f(x) \in Y$. $Do(f) = X$ is called the domain of f, and $Ran(f) = y \in Y : \exists x \in X, y = f(x) \subseteq Y$ is called the range of f. Also $f(A) = y \in Y : \exists x \in A, y = f(x)$ for some $A \subseteq X$.

The identity function $id_X : X \to X$ on a set X is the function that maps every element of X to itself, that is $id_X(x) = x$ for all $x \in X$.

The characteristic or indicator function $\chi_A : X \to \{0, 1\}$ of $A \subseteq X$ is defined as $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.

The graph of a function $f : X \to Y$ is defined as $G_f = \{(x, y) \in X \times Y : y = f(x)\}$.

1.4.1 Properties of a Function

A function $f : X \to Y$ is

- injective (one-to-one) if it maps distinct elements to distinct elements, that is $x_1, x_2 \in X$ and $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.
- surjective (onto) if its range $Ran(f) = Y$, that is for every $y \in Y$ there exists an $x \in X$, s.t. $y = f(x)$.

If a function is both injective and surjective then its bijective.

We define the composition $f \circ g(z) = f(g(z))$ of functions $f : Y \to X$ and $g : Z \to Y$. Note that we need the inclusion $Ran(g) \subseteq Do(f)$. $\circ$ is associative. A bijective function $f : X \to Y$ has an inverse $f^{-1} : Y \to X$ defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ that is $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. If $f : X \to Y$ is merely injective than still $f : X \to Ran(f)$ is bijective, thus invertible on its range with inverse $f^{-1} : Ran(f) \to X$.

1.4.2 Groups, Monoids, Fields

Definition: Given a function $f : X \times X \to X$ we may denote $f(x, y) = x * y$ and consider this as a binary operation on X. For example addition of integers is such an operation. Then we say that $*$ is/has

- Associative, if $x * (y * z) = (x * y) * z$
- Commutative, if $x * y = y * x$
- Neutral element, if there exists $e \in X$ (a neutral element), s.t. $x * e = e * x = x$
- Inverse elements, if for all $x \in X$ there exists $x' \in X$ called an inverse of x, s.t. $x * x' = x' * x = e$ where e is a neutral element.

Definition: $(X, *)$ is called a

- Semigroup, if $*$ is associative,
- Monoid, if $(X, *)$ is a semigroup and has a neutral element,
- Group, if $(X, *)$ is a monoid and every element $x \in X$ has an inverse.

Theorem: In a group $(X, *)$ the neutral element $e \in X$ and inverse x' for any fixed $x \in X$ are unique.