ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

Jon Sterling

October 8, 2015

joint work with Darryl McAdams

INTRODUCTION

$$\llbracket \textit{A woman walked in.} \rrbracket$$
$$\nabla$$

$$\llbracket \textit{A woman walked in.} \rrbracket$$
$$\triangledown$$
$$(\Sigma p \in \textit{Woman})$$

$$\llbracket \textit{A woman walked in.} \rrbracket$$
$$\triangledown$$
$$(\Sigma p \in \textit{Woman})\, \textit{WalkedIn}(p)$$

$$⟦\textit{She sat down}⟧$$
$$\triangledown$$

⟦*She sat down*⟧
▽
SatDown(???)

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
$$\nabla$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in \textit{Woman}) \, \textit{WalkedIn}(p))$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in Woman) \, WalkedIn(p)) \, SatDown(\pi_1(x))$$

$[\![$*Every farmer who owns a donkey beats it.*$]\!]$
$$\triangledown$$
$(\Pi p \in (\Sigma x \in Farmer) \, (\Sigma y \in Donkey) \, Owns(x; y))$

⟦*Every farmer who owns a donkey beats it.*⟧

$\triangledown$

$(\Pi p \in (\Sigma x \in Farmer)\,(\Sigma y \in Donkey)\,Owns(x;y))\,Beats(???;???)$

$[\![$*Every farmer who owns a donkey beats it.*$]\!]$

$\triangledown$

$(\Pi p \in (\Sigma x \in \mathit{Farmer})\,(\Sigma y \in \mathit{Donkey})\,\mathit{Owns}(x; y))\,\mathit{Beats}(\pi_1(p); \pi_1(\pi_2(p)))$

- terms for presuppositions

- terms for presuppositions
- resolution of presuppositions

- terms for presuppositions
- resolution of presuppositions

- terms for presuppositions (this talk)
- resolution of presuppositions

THE **require** ORACLE: STATICS

$$\textbf{require} : (0; 1) \qquad \text{(operator)}$$

$$\textbf{require } x : A \textbf{ in } N \stackrel{\text{def}}{=} \textbf{require}(A; x.N) \qquad \text{(notation)}$$

require — FORMAL RULES

$$\textbf{require} : (0; 1) \qquad \text{(operator)}$$

$$\textbf{require } x : A \textbf{ in } N \stackrel{\text{def}}{=} \textbf{require}(A; x.N) \qquad \text{(notation)}$$

$$\frac{\Gamma \vdash A \ type \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require } x : A \textbf{ in } N \in B} \qquad (require)$$

⟦*A woman walked in. She sat down*⟧

▽

$(\Sigma x \in (\Sigma p \in Woman)\ WalkedIn(p))\ SatDown(???)$

$[\![A\ woman\ walked\ in.\ She\ sat\ down]\!]$
$$\bigtriangledown$$
$(\Sigma x \in (\Sigma p \in Woman)\ WalkedIn(p))\ \textbf{require}\ y : Woman\ \textbf{in}\ SatDown(y)$

The meaning of a sentence is a logical proposition.

~~The meaning of a sentence is a logical proposition.~~

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may <u>evaluate</u> to a canonical proposition.

The meaning of a sentence is a logical proposition.

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may <u>evaluate</u> to a canonical proposition.

What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \textbf{require } y : Woman \textbf{ in } SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \pi_1(x)$$

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may <u>evaluate</u> to a canonical proposition.

What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \textbf{require}\; y : Woman\; \textbf{in}\; SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \pi_1(x)$$

$$\boxed{\text{where}\; M \sim N \stackrel{\text{def}}{=} (M \leq N) \wedge (N \leq M)}$$

...but only some of them denote propositions (types)!

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \qquad (??)$$

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \tag{??}$$

Can the above be made precise?

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \qquad (??)$$

Can the above be made precise? There are two problems:

$$\frac{M \in A \quad [M/x]\, N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \qquad (??)$$

Can the above be made precise? There are two problems:

1. circularity

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \tag{??}$$

Can the above be made precise? There are two problems:

1. circularity
2. non-determinism

(HOLD THAT THOUGHT)

⟦ *The President ran a marathon* ⟧
▽

$\llbracket$ *The President ran a marathon* $\rrbracket$

$\triangledown$

require x : *President* **in** $(\Sigma y \in Marathon)\, Ran(x; y)$

require $x : President$ **in** $(\Sigma y \in Marathon) \, Ran(x; y)$

$$\textbf{require } x : \textit{President} \textbf{ in } (\Sigma y \in \textit{Marathon}) \, \textit{Ran}(x; y)$$
$$\Downarrow$$
$$(\Sigma y \in \textit{Marathon}) \, \textit{Ran}(\textit{Obama}; y)$$

⟦ *The unicorn ran a marathon* ⟧

∇

$⟦$ *The unicorn ran a marathon* $⟧$

$\triangledown$

require $x : Unicorn$ **in** $(\Sigma y \in Marathon)\, Ran(x; y)$

⟦ *The unicorn ran a marathon* ⟧

$\triangledown$

require $x : Unicorn$ **in** $(\Sigma y \in Marathon)\,Ran(x; y)$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

Yes, but we need two things:

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge
2. non-determinism must be eliminated

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

<u>Yes, but we need two things:</u>

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

$$w \Vdash \mathcal{J}$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$w \Vdash \mathcal{J}$$

<u>Yes, but we need two things:</u>

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$w \Vdash \mathscr{J}_\alpha$$

THE CREATING SUBJECT

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)
2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)
2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

...

> 2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

…

> 2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

Remark

Contra Dummett, I <u>by no means</u> take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathscr{J}. \ [\![w \Vdash \mathscr{J}]\!] \lor \neg [\![w \Vdash \mathscr{J}]\!] \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

proof conditions ⇒ assertion conditions

Martin-Löf, Sundholm ⇒ Brouwer, Heyting, Van Atten

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

proof conditions ⇒ assertion conditions

Martin-Löf, Sundholm ⇒ Brouwer, Heyting, Van Atten

global meaning explanation ⇒ local meaning explanation

Husserl, Dummett, Martin-Löf ⇒   *Brouwer?, Beth, Kripke, Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$\mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$\mathscr{J}_2 (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$
$$M \Downarrow N \qquad \text{(evaluation)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$\mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$
$$M \Downarrow N \qquad \text{(evaluation)}$$
$$A \ type \qquad \text{(typehood)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$

$$\mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

$$M \Downarrow N \qquad \text{(evaluation)}$$

$$A \ type \qquad \text{(typehood)}$$

$$A \ verif \qquad \text{(verification)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x)$$ (general judgment)

$$\mathscr{J}_2\ (\mathscr{J}_1)$$ (hypothetical judgment)

$$M \Downarrow N$$ (evaluation)

$$A\ type$$ (typehood)

$$A\ verif$$ (verification)

$$A\ true$$ (truth)

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$\mathscr{J}_2\ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$
$$M \Downarrow N \qquad \text{(evaluation)}$$
$$A\ type \qquad \text{(typehood)}$$
$$A\ verif \qquad \text{(verification)}$$
$$A\ true \qquad \text{(truth)}$$
$$M = N \in A \qquad \text{(membership)}$$

assertion acts (judgments) are intensional (local)

$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$
$$w \Vdash M \Downarrow N \qquad \text{(evaluation)}$$
$$w \Vdash A \ type \qquad \text{(typehood)}$$
$$w \Vdash A \ verif \qquad \text{(verification)}$$
$$w \Vdash A \ true \qquad \text{(truth)}$$
$$w \Vdash M = N \in A \qquad \text{(membership)}$$

$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

$$w \Vdash |_x \mathscr{J}(x) \quad \iff \quad \cdots$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \quad \iff \quad \cdots$$

$$w \Vdash |_x \mathscr{J}(x) \quad \Longleftrightarrow \quad \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x)$$
$$w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) \quad \Longleftrightarrow \quad \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2$$

$$w \Vdash |_x \mathscr{J}(x) \quad \Longleftrightarrow \quad \forall u \geq w. \forall x \in \mathscr{D}_u. \ u \Vdash \mathscr{J}(x)$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \quad \Longleftrightarrow \quad \forall u \geq w. \ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2$$

where $\mathscr{D}_w$ is the species of constructions that have been effected by stage w

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $\boxed{w \Vdash P \; verif}$ (and its synthesis).

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $\boxed{w \Vdash P\ \textit{verif}}$ (and its synthesis).

For a type A, implicit in the explanation of $w \Vdash A\ \textit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

In the model, this corresponds to the inevitability of verification (i.e. a <u>bar</u>, in which verification occurs at all nodes):

$w \Vdash A\ true \iff \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ verif$ (due to Dummett)

The analytic judgments of type theory are reflections on mathematical activity.

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**

$$\mathscr{V}[\![A]\!]_w (M, N) \;\;\bowtie\;\; w \Vdash A \; \mathit{verif}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**

$$w \Vdash M = N \in A \quad \bowtie \quad w \Vdash A \; true$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a <u>bar</u>**

$$w \Vdash M = N \in A \quad \bowtie \quad w \Vdash A \; true$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a <u>bar</u>**

$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ w \Vdash \mathscr{V}[\![A]\!]_w (M', N') \end{array} \right\} \quad \bowtie \quad \exists \mathfrak{B} \ \mathbf{bars} \ w. \forall u \in \mathfrak{B}. \ u \Vdash A \ \mathit{verif}$$

THE require ORACLE: DYNAMICS

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \; true \\ 1 & \neg(i \Vdash A \; true) \end{cases} \qquad \text{(KS)}$$

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of A *true* effected prior to w for each canonical proposition A.

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of A *true* effected prior to w for each canonical proposition A.

We now can give a precise, but non-deterministic, dynamics to **require**:

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of A *true* effected prior to w for each canonical proposition A.

We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow N'} \qquad (\star)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each world w, in order of construction.

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each world w, in order of construction.

Idea: reformulate Type Theory relative to a choice sequence of "choosers".

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

SPREADS: SETS OF CHOICE SEQUENCES

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

SPREADS: SETS OF CHOICE SEQUENCES

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$

SPREADS: SETS OF CHOICE SEQUENCES

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u},m}\, \mathfrak{S}(\vec{u})\, \big(\mathfrak{S}(\vec{u} \frown m)\big)$$

3. inclusion of neighborhoods is closed under refinement/extension

$$|_{\vec{u}}\, \mathfrak{S}(\vec{u} \frown m)\, \big(\mathfrak{S}(\vec{u})\big)$$

A spread for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n \, \rho(n) < n \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

require — DYNAMICS

A spread for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n \, \rho(n) < n \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

Deterministic dynamics for **require**:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = j \quad w \Vdash \big[\varkappa_{A'}(j)/x\big] N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'}$$

$$(\text{for } \alpha \in \mathfrak{S})$$

QUESTIONS?