

ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

Jon Sterling

October 8, 2015

joint work with Darryl McAdams

INTRODUCTION

[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

($\Sigma p \in Woman$)

[[*A woman walked in.*]]

∇

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

SatDown(???)

[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(\pi_1(x))$

THE **require** ORACLE: STATICS

require : (0;1)

(operator)

require $x : A$ **in** $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

(notation)

require : (0;1)

(operator)

require $x : A$ **in** $N \stackrel{\text{def}}{=} \text{require}(A; x.N)$

(notation)

$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)

[[A woman walked in. She sat down]]

∇

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$ **require** $y : \text{Woman}$ **in** $\text{SatDown}(y)$

The meaning of a sentence is a logical proposition.

~~The meaning of a sentence is a logical proposition.~~

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

What we want:

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$ **require** $y : \text{Woman}$ **in** $\text{SatDown}(y)$

\sim

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

What we want:

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$ **require** $y : \text{Woman}$ **in** $\text{SatDown}(y)$

\sim

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where $M \sim N \stackrel{\text{def}}{=} (M \leq N) \ \& \ (N \leq M)$

$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

[[*The President ran a marathon*]]



[[*The President ran a marathon*]]

▽

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

\Downarrow

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

\Downarrow

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$

A NEGATIVE EXAMPLE

[[*The unicorn ran a marathon*]]



[[*The unicorn ran a marathon*]]

∇

require $x : \textit{Unicorn}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

[[*The unicorn ran a marathon*]]

∇

require $x : \textit{Unicorn}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

Yes, but we need two things:

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge
2. non-determinism must be eliminated

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

$$w \Vdash \mathcal{J}$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$w \Vdash \mathcal{I}$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$w \Vdash \mathcal{I}_\alpha$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$w \Vdash \mathcal{I}_\alpha$$

(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

All mathematics is a mental construction performed by an idealized subject, **subject to the following observations about knowledge:**

All mathematics is a mental construction performed by an idealized subject, **subject to the following observations about knowledge:**

1. experiences are never forgotten (**monotonicity, functoriality**)

All mathematics is a mental construction performed by an idealized subject, **subject to the following observations about knowledge**:

1. experiences are never forgotten (**monotonicity, functoriality**)
2. at a point in time, the subject knows whether or not it has experienced a judgment (**decidability**)

All mathematics is a mental construction performed by an idealized subject, **subject to the following observations about knowledge**:

1. experiences are never forgotten (**monotonicity, functoriality**)
2. at a point in time, the subject knows whether or not it has experienced a judgment (**decidability**)

Corollary

The meaning of a judgment \mathcal{J} must be explained in terms of its forcing condition, $w \Vdash \mathcal{J}$, for any stage/world w .

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathcal{J}. (w \Vdash \mathcal{J}) \vee \neg(w \Vdash \mathcal{J}) \quad (\text{Dummett's infelicity})$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

logical consequence \Rightarrow semantic consequence

Brouwer?, Martin-Löf, Sundholm \Rightarrow Brouwer?, Heyting, Allen, Zeilberger

logical consequence \Rightarrow semantic consequence

Brouwer?, Martin-Löf, Sundholm \Rightarrow Brouwer?, Heyting, Allen, Zeilberger

proof conditions \Rightarrow assertion conditions

Dummett, Martin-Löf, Sundholm \Rightarrow Brouwer, Heyting, Van Atten

logical consequence \Rightarrow **semantic consequence**

Brouwer?, Martin-Löf, Sundholm \Rightarrow *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions \Rightarrow **assertion conditions**

Dummett, Martin-Löf, Sundholm \Rightarrow *Brouwer, Heyting, Van Atten*

global meaning explanation \Rightarrow **local meaning explanation**

Husserl, Dummett, Martin-Löf \Rightarrow *Brouwer, Beth, Kripke,
Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)

assertion acts (judgments) are intensional (local)

$$|_x \mathcal{J}(x)$$

(general judgment)

assertion acts (judgments) are intensional (local)

$|_x \mathcal{I}(x)$

(general judgment)

$\mathcal{I}_2 (\mathcal{I}_1)$

(hypothetical judgment)

assertion acts (judgments) are intensional (local)

$ _x \mathcal{I}(x)$	(general judgment)
$\mathcal{I}_2 (\mathcal{I}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)

assertion acts (judgments) are intensional (local)

$ _x \mathcal{I}(x)$	(general judgment)
$\mathcal{I}_2 (\mathcal{I}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)
A type	(typehood)

assertion acts (judgments) are intensional (local)

$ _x \mathcal{I}(x)$	(general judgment)
$\mathcal{I}_2 (\mathcal{I}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)
$A \text{ type}$	(typehood)
$A \text{ verif}$	(verification)

assertion acts (judgments) are intensional (local)

$ _x \mathcal{I}(x)$	(general judgment)
$\mathcal{I}_2 (\mathcal{I}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)
$A \text{ type}$	(typehood)
$A \text{ verif}$	(verification)
$A \text{ true}$	(truth)

assertion acts (judgments) are intensional (local)

$ _x \mathcal{I}(x)$	(general judgment)
$\mathcal{I}_2 (\mathcal{I}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)
$A \text{ type}$	(typehood)
$A \text{ verif}$	(verification)
$A \text{ true}$	(truth)
$M = N \in A$	(membership)

assertion acts (judgments) are intensional (local)

$w \Vdash _x \mathcal{J}(x)$	(general judgment)
$w \Vdash \mathcal{J}_2 (\mathcal{J}_1)$	(hypothetical judgment)
$w \Vdash M \Downarrow N$	(evaluation)
$w \Vdash A \text{ type}$	(typehood)
$w \Vdash A \text{ verif}$	(verification)
$w \Vdash A \text{ true}$	(truth)
$w \Vdash M = N \in A$	(membership)

$w \Vdash |_x \mathcal{I}(x)$

(general judgment)

$w \Vdash \mathcal{I}_2 (\mathcal{I}_1)$

(hypothetical judgment)

$$w \Vdash |_x \mathcal{I}(x) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
\end{aligned}$$

where \mathcal{D}_w is the species of constructions that have been effected by stage w

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $w \Vdash P \text{ } \textit{verif}$ (and its synthesis).

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $w \Vdash P \text{ } \textit{verif}$ (and its synthesis).

For a type A , implicit in the explanation of $w \Vdash A \text{ } \textit{verif}$ is a \mathbb{W} -indexed family of PERs $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$ whose members **reflect** the computational content (extension) of verification acts.

Truth (**justification**) consists in recognizing the effectiveness of a procedure for **verification**.

Truth (**justification**) consists in recognizing the effectiveness of a procedure for **verification**.

In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$

The analytic judgments of type theory are reflections on mathematical activity.

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**

$$\mathcal{V}[[A]]_w(M, N) \bowtie w \Vdash A \text{ } \textit{verif}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**

$$w \Vdash M = N \in A \not\bowtie w \Vdash A \text{ true}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a bar**

$$w \Vdash M = N \in A \not\bowtie w \Vdash A \text{ true}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a bar**

$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathcal{V} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathcal{B} \text{ bars } w. \forall u \in \mathcal{B}. u \Vdash A \text{ } \textit{verif}$$

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$

Let $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{W}^{\text{op}}}$ be the presheaf of constructions of A *true* effected “so far” for each canonical proposition A .

Let $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{Wop}}$ be the presheaf of constructions of A *true* effected “so far” for each canonical proposition A .

We now can give a precise, but non-deterministic, dynamics to **require**:

Let $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{Wop}}$ be the presheaf of constructions of A *true* effected “so far” for each canonical proposition A .

We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathcal{K}_A(w)$. First, let κ_A be the choice sequence of lists given by enumerating $\mathcal{K}_A(w)$ at each stage w , in order of time.

We need a way to deterministically choose a representative of $\mathcal{K}_A(w)$. First, let κ_A be the choice sequence of lists given by enumerating $\mathcal{K}_A(w)$ at each stage w , in order of time.

Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

SPREADS: SETS OF CHOICE SEQUENCES

A spread direction \mathfrak{S} is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

SPREADS: SETS OF CHOICE SEQUENCES

A spread direction \mathfrak{S} is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle \rangle)$$

SPREADS: SETS OF CHOICE SEQUENCES

A spread direction \mathfrak{S} is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle \rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left(\mathfrak{S}(\vec{u} \smallfrown m) \right)$$

SPREADS: SETS OF CHOICE SEQUENCES

A spread direction \mathfrak{S} is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle \rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$\mid_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left(\mathfrak{S}(\vec{u} \smallfrown m) \right)$$

3. a neighborhood may always be refined within the spread

$$\mid_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \left(\mathfrak{S}(\vec{u}) \right)$$

THE require ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\langle \rangle)} \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\vec{u})} \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket_w^\alpha (M', N')}{w \Vdash M = N \in_\alpha A}$$

A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\vec{u})} \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{M \Downarrow_\alpha M' \quad N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket^\alpha (M', N')}{M = N \in_\alpha A}$$

Deterministic choice for \varkappa_A :

$$\frac{w \Vdash A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_{\alpha} M}$$

Deterministic choice for \varkappa_A :

$$\frac{A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(i) = M}{\varkappa_A \ni_{\alpha} M}$$

Deterministic choice for \varkappa_A :

$$\frac{A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(i) = M}{\varkappa_A \ni_{\alpha} M}$$

Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_{\alpha} M \quad [M/x]N \Downarrow_{\text{tl}(\alpha)} N'}{\mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_{\alpha} N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our semantics:

$$\frac{A \text{ true}_\alpha \quad x : A \vdash_\alpha N \in B}{\textbf{require } x : A \textbf{ in } N \in_\alpha B} \textit{require}$$

$$\frac{A \text{ true}_\alpha \quad x : A \vdash_\alpha N \in B}{\textbf{require } x : A \textbf{ in } N \in_\alpha B} \textit{require}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{}{\mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B} \text{require}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V}[[B]]^\alpha(N', N')}}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}$$

$$\overline{A \text{ true}_\alpha} \quad \mathcal{D} \qquad \overline{x : A \vdash_\alpha N \in B} \quad \mathcal{E}$$

$$\frac{\overline{\mathcal{K}_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')} \quad \text{require}$$

$$\frac{}{\text{require } x : A \text{ in } N \in_\alpha B}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D}}{\varkappa_A \ni_\alpha \textcolor{red}{M}} \quad \frac{}{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')} \text{require}$$

$$\frac{}{\text{require } x : A \text{ in } N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\overline{\not\prec_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')}}}{\overline{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}} \quad \mathcal{E}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \mathcal{F}(M,M)}{\overline{M \in_\alpha A}}$$

$$\frac{\overline{A \text{ true}_\alpha} \mathcal{D}}{\not\prec_A \ni_\alpha \textcolor{red}{M}} \quad \frac{\overline{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}}{\textbf{require } x : A \textbf{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{}{\mathcal{V}[[B]]^\alpha(N', N')} \text{require}$$

$$\frac{}{\textbf{require } x : A \textbf{ in } N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{\overline{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \quad \mathcal{F}}{x : A \vdash_\alpha N \in B} \quad \mathcal{E}}}$$

$$\frac{\overline{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M)} \quad \frac{\overline{\kappa_A \ni_\alpha M}}{M \in_\alpha A}}{\quad}$$

$$\frac{\overline{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D}} \quad \frac{\overline{\kappa_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{\overline{\mathcal{V}[[B]]^\alpha(N', N')}}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{}{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \mathcal{F}(M, M)}{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D}}{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}} \mathcal{E}} \frac{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}}{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}} \mathcal{E}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D}}{\frac{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}}{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}} \mathcal{E}} \frac{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}}{\frac{\checkmark}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}} \mathcal{E}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{}{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \mathcal{F}(M, M) \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\textcolor{red}{\lambda}_A \ni_\alpha \textcolor{red}{M}}{M \in_\alpha A}}{[M/x]N \in_\alpha B}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\textcolor{red}{\lambda}_A \ni_\alpha \textcolor{red}{M} \quad \frac{}{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')} \text{require}}{\text{require } x : A \text{ in } N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{\overline{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \quad \mathcal{F}}{x : A \vdash_\alpha N \in B} \quad \mathcal{E}}}$$

$$\frac{\overline{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M)} \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\not\propto_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}{\overline{[M/x]N \in_\alpha B}}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\not\propto_A \ni_\alpha \mathbf{M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \mathcal{V}[[B]]^\alpha(N', N')} \quad \text{require} \quad \overline{\text{require } x : A \text{ in } N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M)}{\overline{\frac{\overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \mathcal{G} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')} \quad \mathcal{H}}{[M/x]N \in_\alpha B}} \quad \mathcal{D}} \quad \frac{\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\mathcal{K}_A \ni_\alpha \mathbf{M}}} \quad \mathcal{D}}{\overline{M \in_\alpha A}} \quad \mathcal{F}(M, M)$$

$$\frac{\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\mathcal{K}_A \ni_\alpha \mathbf{M}}} \quad \frac{\overline{[M/x]N \Downarrow_\alpha \mathbf{N}'}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'}} \quad \frac{\overline{\mathcal{V}[[B]]^\alpha(N', N')}}{\overline{\text{require } x : A \text{ in } N \in_\alpha B}} \quad \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D}}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M) \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D}}{\frac{\kappa_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}}{\frac{\frac{[M/x]N \Downarrow_\alpha \mathbf{N'}}{\mathcal{G}} \quad \mathcal{V}[[B]]^\alpha(N', N')}{[M/x]N \in_\alpha B} \quad \mathcal{H}}$$

$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D}}{\kappa_A \ni_\alpha \mathbf{M}} \quad \frac{\frac{\checkmark}{[M/x]N \Downarrow_\alpha \mathbf{N'}} \quad \mathcal{G}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N'}} \quad \mathcal{V}[[B]]^\alpha(N', N')}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\frac{\overline{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \quad \mathcal{F}}{x : A \vdash_\alpha N \in B} \quad \mathcal{E}}$$

$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D}}{\frac{\mathcal{K}_A \ni_\alpha M}{M \in_\alpha A}} \mathcal{F}(M, M)}{\frac{\frac{\checkmark}{[M/x]N \in_\alpha B \mid (M \in_\alpha A)} \mathcal{G} \quad \frac{\frac{[M/x]N \Downarrow_\alpha N'}{\mathcal{V}[\![B]\!]^\alpha(N', N')} \mathcal{H}}{[M/x]N \in_\alpha B}}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\checkmark}{[M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\checkmark}{\mathcal{V}[\![B]\!]^\alpha(N', N')} \mathcal{H}}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E} \quad \mathcal{F}$$

$$\frac{\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M) \quad \frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\checkmark}{\kappa_A \ni_\alpha M} \quad \mathcal{D}}{M \in_\alpha A}}{\frac{\overline{[M/x]N \Downarrow_\alpha N'} \quad \mathcal{G} \quad \overline{\mathcal{V}[B]^\alpha(N', N')} \quad \mathcal{H}}{[M/x]N \in_\alpha B} \quad \mathcal{F}(M, M)$$

$$\frac{\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\overline{\kappa_A \ni_\alpha M} \quad \mathcal{D} \quad \frac{\overline{[M/x]N \Downarrow_\alpha N'} \quad \mathcal{G}}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \mathcal{G}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \mathcal{G}} \quad \frac{\overline{\mathcal{V}[B]^\alpha(N', N')} \quad \mathcal{H}}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \mathcal{H}$$

□

QUESTIONS?



D. Bekki.

Representing anaphora with dependent types.

In N. Asher and S. Soloviev, editors, Logical Aspects of Computational Linguistics, volume 8535 of Lecture Notes in Computer Science, pages 14–29. Springer Berlin Heidelberg, 2014.



T. Coquand and G. Jaber.

A computational interpretation of forcing in type theory.

In P. Dybjer, S. Lindström, E. Palmgren, and G. Sundholm, editors, Epistemology versus Ontology, volume 27 of Logic, Epistemology, and the Unity of Science, pages 203–213. Springer Netherlands, 2012.



M. Dummett.

Elements of intuitionism, volume 39 of Oxford Logic Guides.

The Clarendon Press Oxford University Press, New York, second edition, 2000.



M. P. Fourman.

Notions of choice sequence.

In D. van Dalen and A. Troelstra, editors, L.E.J. Brouwer Centenary Symposium, pages 91–105. North-Holland, 1982.



M. P. Fourman.

Continuous truth I: non-constructive objects.

In G. Lolli, G. Longo, and A. Marcja, editors, Proc. Logic Colloquium, Florence 1982, Stud. Logic Found. Math. 112, pages 161–180. Elsevier Science Publishers B.V. (North-Holland), 1984.
Invited Paper.

REFERENCES III



R. Harper.

Constructing type systems over an operational semantics.

Journal of Symbolic Computation, 14(1):71 – 84, 1992.



D. Howe.

On computational open-endedness in Martin-Löf's type theory.

In Logic in Computer Science, 1991. LICS '91., Proceedings of Sixth Annual IEEE Symposium on, pages 162–172, July 1991.



S. Mac Lane and I. Moerdijk.

Sheaves in geometry and logic : a first introduction to topos theory.

Universitext. Springer, New York, 1992.



P. Martin-Löf.

Constructive mathematics and computer programming.

In L. J. Cohen, J. Łoś, H. Pfeiffer, and K.-P. Podewski, editors, Logic, Methodology and Philosophy of Science VI, Proceedings of the Sixth International Congress of Logic, Methodology and Philosophy of Science, Hannover 1979, volume 104 of Studies in Logic and the Foundations of Mathematics, pages 153–175. North-Holland, 1982.



P. Martin-Löf.

Analytic and synthetic judgements in type theory.

In P. Parrini, editor, Kant and Contemporary Epistemology, volume 54 of The University of Western Ontario Series in Philosophy of Science, pages 87–99. Springer Netherlands, 1994.



P. Martin-Löf.

On the meanings of the logical constants and the justifications of the logical laws.

Nordic Journal of Philosophical Logic, 1(1):11–60, 1996.



D. McAdams and J. Sterling.

Dependent types for pragmatics.

In O. Pombo, A. Nepumuceno, and J. Redmond, editors,
Epistemology, Knowledge and the Impact of Interaction, Logic,
Epistemology, and the Unity of Science. Springer, 2016.



S. Muller, U. Acar, W. Duff, and R. Harper.

Interactive computation in an open world.

POPL, 2016.

In Review.



V. Rahli and M. Bickford.

A nominal exploration of intuitionism.

Unpublished, 2015.



A. Ranta.

Type-theoretical Grammar.

Oxford University Press, Oxford, UK, 1994.



J. Sterling.

Remark on hypothetical judgment.

<http://arxiv.org/abs/1508.01600>, 2015.



J. Sterling.

Type theory and its meaning explanations.

<http://www.jonmsterling.com/pdfs/meaning-explanations.pdf>, 2015.



G. Sundholm.

Proof theory and meaning.

In D. Gabbay and F. Guenther, editors, Handbook of Philosophical Logic, volume 166 of Synthese Library, pages 471–506. Springer Netherlands, 1986.



G. Sundholm.

Constructive recursive functions, Church's Thesis, and Brouwer's theory of the creating subject: Afterthoughts on a parisian joint session.

In J. Dubucs and M. Bourdeau, editors, Constructivity and Computability in Historical and Philosophical Perspective, volume 34 of Logic, Epistemology, and the Unity of Science, pages 1–35. Springer Netherlands, 2014.



M. van Atten.

On Brouwer.

Wadsworth Philosophers Series. Thompson/Wadsworth, Toronto, Canada, 2004.



M. van Atten.

Brouwer Meets Husserl: On the Phenomenology of Choice Sequences.

Springer, 2007.