

# ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

## INTRODUCTION

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[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

( $\Sigma p \in Woman$ )

[[*A woman walked in.*]]

▽

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

*SatDown*(???)



[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

$\nabla$

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(\pi_1(x))$

## THE **require** ORACLE: STATICS

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**require** : (0;1)

(operator)

**require**  $x : A$  **in**  $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

(notation)

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)



[[A woman walked in. She sat down]]

$\nabla$

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(???)$

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▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$  **require**  $y : \text{Woman}$  **in**  $\text{SatDown}(y)$

The meaning of a sentence is a logical proposition.

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What we want:

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$  **require**  $y : \text{Woman}$  **in**  $\text{SatDown}(y)$

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where  $M \sim N \stackrel{\text{def}}{=} (M \leq N) \ \& \ (N \leq M)$



$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

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[[ *The President ran a marathon* ]]



[[ *The President ran a marathon* ]]

▽

**require**  $x : \textit{President}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

## A POSITIVE EXAMPLE

**require**  $x : \textit{President}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

**require**  $x : \textit{President}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

$\Downarrow$

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$



## A NEGATIVE EXAMPLE

[[ *The unicorn ran a marathon* ]]



[[ *The unicorn ran a marathon* ]]

∇

**require**  $x : \textit{Unicorn}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

[[ *The unicorn ran a marathon* ]]

$\nabla$

**require**  $x : \textit{Unicorn}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

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$$w \Vdash \mathcal{I}_\alpha$$

## THE CREATING SUBJECT

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## Corollary

*The meaning of a judgment  $\mathcal{J}$  must be explained in terms of its forcing condition,  $w \Vdash \mathcal{J}$ , for any stage/world  $w$ .*

...

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### Remark

*Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:*

$$\forall w. \forall \mathcal{J}. (w \Vdash \mathcal{J}) \vee \neg(w \Vdash \mathcal{J}) \quad (\text{Dummett's infelicity})$$

*The above is impossible in a Beth model.*



logical consequence  $\Rightarrow$  semantic consequence

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**proof conditions**  $\Rightarrow$  **assertion conditions**

*Dummett, Martin-Löf, Sundholm*  $\Rightarrow$  *Brouwer, Heyting, Van Atten*

**global meaning explanation**  $\Rightarrow$  **local meaning explanation**

*Husserl, Dummett, Martin-Löf*  $\Rightarrow$  *Brouwer, Beth, Kripke,  
Grothendieck, Lawvere, Joyal*

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$$|_x \mathcal{J}(x)$$

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$A \text{ true}$	(truth)
$M = N \in A$	(membership)

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$$w \Vdash |_x \mathcal{I}(x) \iff \dots$$

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$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
 w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
 w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
 \end{aligned}$$

where  $\mathcal{D}_w$  is the species of constructions that have been effected by stage  $w$



a statement is intuitionistically valid if it is forced at all nodes



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Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects (Fourman)

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for  $w \Vdash P \text{ } \textit{verif}$  (and its synthesis).

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For a type  $A$ , implicit in the explanation of  $w \Vdash A \text{ } \textit{verif}$  is a  $\mathbb{W}$ -indexed family of PERs  $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$  whose members **reflect** the computational content (extension) of verification acts.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$

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1. **Canonical membership** reflects **verification**
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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathcal{V} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathcal{B} \text{ bars } w. \forall u \in \mathcal{B}. u \Vdash A \text{ } \textit{verif}$$

## THE require ORACLE: DYNAMICS

---

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$

Let  $\mathcal{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\text{op}}}$  be the presheaf of constructions of  $A$  *true* effected “so far” for each canonical proposition  $A$ .



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We now can give a precise, but non-deterministic, dynamics to **require**:

Let  $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{Wop}}$  be the presheaf of constructions of  $A$  *true* effected “so far” for each canonical proposition  $A$ .

We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of  $\mathcal{K}_A(w)$ . First, let  $\kappa_A$  be the choice sequence of lists given by enumerating  $\mathcal{K}_A(w)$  at each stage  $w$ , in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

## SPREADS: SETS OF CHOICE SEQUENCES

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$$|_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left( \mathfrak{S}(\vec{u} \smallfrown m) \right)$$

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$$\mid_{\vec{u}, m} \mathfrak{S}(\vec{u}) \quad (\mathfrak{S}(\vec{u} \smallfrown m))$$

3. a neighborhood may always be refined within the spread

$$\mid_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \quad (\mathfrak{S}(\vec{u}))$$



A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\langle \rangle)} \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

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Reformulate type theory relative to an arbitrary  $\alpha \in \mathfrak{S}$ ! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket_w^\alpha (M', N')}{w \Vdash M = N \in_\alpha A}$$

Deterministic choice for  $\varkappa_A$ :

$$\frac{w \Vdash A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_{\alpha} M}$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash \varkappa_A \ni_{\alpha} M \quad w \Vdash [M/x]N \Downarrow_{\text{tl}(\alpha)} N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_{\alpha} N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

## Theorem

*The following rule is valid in our semantics:*

$$\frac{w \Vdash A \text{ true}_\alpha \quad w \Vdash x : A \vdash_\alpha N \in B}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B} \text{ require}$$

$$\frac{w \Vdash A \text{ true}_{\alpha} \quad w \Vdash x : A \vdash_{\alpha} N \in B}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_{\alpha} B} \text{ require}$$

$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

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$$\frac{\frac{}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \text{require}$$



$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

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$$\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{}{w \Vdash \neg_A \ni_\alpha M} \quad \frac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')} \text{require}$$

$$\frac{}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B}$$

$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{w \Vdash \not\prec_A \ni_\alpha M \quad [M/x]N \Downarrow_\alpha N'}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')} \text{require}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha \textcolor{red}{M} \quad \frac{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha \textcolor{red}{N'}}} \frac{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \text{require}
\end{array}$$

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\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \quad \mathcal{F}(w, M, M)}{w \Vdash M \in_\alpha A} \checkmark$$

$$\begin{array}{c}
\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{w \Vdash \mathcal{K}_A \ni_\alpha \textcolor{red}{M} \quad [M/x]N \Downarrow_\alpha \textcolor{red}{N'}} \checkmark \\
\frac{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'} \quad \overline{\mathcal{V}[\![B]\!]_w^\alpha(N', N')}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \text{require}
\end{array}$$

$$\frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{\frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \mathcal{F}(w, M, M)}{\overline{w \Vdash M \in_\alpha A} \quad \frac{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}{w \Vdash M \in_\alpha A}}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}} \quad \overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \overline{\mathcal{V}[[B]_w^\alpha(N', N')}}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}' \quad \text{require}}$$

$$\begin{array}{c}
\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{\mathcal{F}} \\
\frac{w \Vdash A \text{ true}_\alpha \quad \mathcal{D} \quad \frac{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)}{\mathcal{E}}}{w \Vdash x : A \vdash_\alpha N \in B}
\end{array}$$

$$\begin{array}{c}
\frac{\overline{\checkmark}}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{\checkmark}}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}} \mathcal{D} \\
\frac{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B \quad \mathcal{F}(w, M, M) \quad \frac{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}{w \Vdash M \in_\alpha A}}{\mathcal{E}}
\end{array}$$

$$\begin{array}{c}
\frac{\overline{\checkmark}}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \\
\frac{w \Vdash \varkappa_A \ni_\alpha \mathbf{M} \quad \overline{[M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \\
\frac{\quad \overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \text{ require}
\end{array}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B} \quad \mathcal{F}}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)} \quad \mathcal{E}}}{w \Vdash x : A \vdash_\alpha N \in B}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \quad \mathcal{F}(w, M, M) \quad \frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \frac{\overline{w \Vdash M \in_\alpha A}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D} \quad \overline{w \Vdash \varkappa_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]_w^\alpha(N', N')}}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha N'} \quad \text{require}$$



$$\begin{array}{c}
\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D} \\
\hline
\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B} \quad \mathcal{F} \\
\hline
\overline{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)} \quad \mathcal{E} \\
\hline
w \Vdash x : A \vdash_\alpha N \in B
\end{array}$$

$$\begin{array}{c}
\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \quad \mathcal{F}(w, M, M) \\
\hline
\overline{w \Vdash [M/x]N \in_\alpha B} \\
\hline
\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D} \\
\hline
\overline{w \Vdash \not\prec_A \ni_\alpha \textcolor{red}{M}} \\
\hline
\overline{w \Vdash M \in_\alpha A}
\end{array}$$

$$\begin{array}{c}
\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D} \\
\hline
\overline{w \Vdash \not\prec_A \ni_\alpha \textcolor{red}{M} \quad [M/x]N \Downarrow_\alpha \textcolor{red}{N'}} \\
\hline
\overline{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \mathcal{V}[\llbracket B \rrbracket_w^\alpha(N', N')] \\
\hline
w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B \quad \textit{require}
\end{array}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}
\end{array}$$

$$\frac{\frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \text{require}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash \textbf{require } x : A \text{ in } N \Downarrow_\alpha N'}}{w \Vdash \textbf{require } x : A \text{ in } N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \textbf{require } x : A \text{ in } N \in_\alpha B} \mathcal{H}}
\end{array}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \mathcal{F}(w, M, M) \quad \frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A}}{\frac{\frac{}{w \Vdash [M/x]N \Downarrow_\alpha \mathbf{N}'} \mathcal{G} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket_w^\alpha (\mathbf{N}', \mathbf{N}')} \mathcal{H}}{w \Vdash [M/x]N \in_\alpha B}}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{G} \quad \frac{\frac{}{\mathcal{V} \llbracket B \rrbracket_w^\alpha (\mathbf{N}', \mathbf{N}')} \mathcal{H}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \mathcal{H}
\end{array}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \mathcal{F}(w, M, M) \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A} \mathcal{D}}{\frac{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{E} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{\mathcal{H}}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{F} \quad \frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{E}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{H} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{\mathcal{H}} \text{ require}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \mathcal{H}
\end{array}$$

□

QUESTIONS?