ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

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joint work with Darryl McAdams

INTRODUCTION

$[\![$*A woman walked in.*$]\!]$

$\bigtriangledown$

$$\llbracket \textit{A woman walked in.} \rrbracket$$

$$\triangledown$$

$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

INTRODUCING **require**

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$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\ \textbf{require}\ y : \mathit{Woman}\ \textbf{in}\ \mathit{SatDown}(y)$$

THE **require** ORACLE: STATICS

require — FORMAL RULES

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$$\mathbf{require} : (0; 1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require}\ x : A\ \textbf{in}\ N \in B} \qquad (\textit{require})$$

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What we want:

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$$\sim$$

$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\, \mathit{SatDown}(\pi_1(x))$$

require—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \qquad (??)$$

A POSITIVE EXAMPLE

$$[\![\textit{The President ran a marathon}]\!]$$

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$$\mathbf{require}\ x : \mathit{President}\ \mathbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

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$$(\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(\mathit{Obama}; y)$$

A NEGATIVE EXAMPLE

$$[\![\textit{The unicorn ran a marathon}]\!]$$
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$$\textbf{require } x : \mathit{Unicorn} \textbf{ in } (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

$$\llbracket\ \textit{The unicorn ran a marathon}\ \rrbracket$$

$$\triangledown$$

$$\textbf{require}\ x : \mathit{Unicorn}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\,\mathit{Ran}(x;y)$$

(not a proposition)

Is **require** computationally effective?

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(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

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Corollary

The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

REMARK ON DECIDABILITY

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathscr{J}. (w \Vdash \mathscr{J}) \lor \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

MEANING THEORY WITHOUT UNIFORMITY CONSTRAINTS

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Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions $\Rightarrow$ assertion conditions

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logical consequence ⇒ **semantic consequence**

Brouwer?, Martin-Löf, Sundholm ⇒ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions ⇒ **assertion conditions**

Dummett, Martin-Löf, Sundholm ⇒ *Brouwer, Heyting, Van Atten*

global meaning explanation ⇒ **local meaning explanation**

Husserl, Dummett, Martin-Löf ⇒ *Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal*

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BETH-KRIPKE SEMANTICS FOR ASSERTIONS

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$M = N \in A$	(membership)

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\begin{aligned}
w \Vdash |_x \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\
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where $\mathscr{D}$ is the (pre)sheaf of constructions that have been effected so far a statement is constructively valid iff it is forced at all nodes a statement is constructively valid iff it is forced at all nodes

Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects **(Fourman)**

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For a type A, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \ \mathit{true} \Longleftrightarrow \exists \mathfrak{B} \ \textbf{bars} \ w. \forall u \in \mathfrak{B}. \ u \Vdash A \ \mathit{verif} \qquad \text{(due to Dummett)}$$

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$$\bigwedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w(M', N') \end{array} \right\} \quad \rtimes \quad \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A\ \mathit{true} \\ 1 & \neg(i \Vdash A\ \mathit{true}) \end{cases} \tag{KS}$$

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \qquad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage w, in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

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3. a neighborhood may always be refined within the spread

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

THE **require** ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\ \rho(n) < n \ \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]_w^\alpha(M', N')}{w \Vdash M = N \in_\alpha A}$$

(double inference line)

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$$\frac{M \Downarrow_\alpha M' \quad N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha(M', N')}{M = N \in_\alpha A}$$

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

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require — DYNAMICS

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Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_\alpha M \quad [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our intuitive semantics:

$$\frac{A\ \mathit{true}_\alpha \quad x : A \vdash_\alpha N \in B}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

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$$

$$
\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M, M) \qquad \dfrac{}{M \in_\alpha A}}{}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred}M}} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha {\color{darkred}N'}}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha {\color{darkred}N'}} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}
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$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N', N')}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{\big|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\overline{[M/x]N \in_\alpha B}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{[M/x]N \in_\alpha B}}\ \mathscr{H}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\,\mathscr{D} \qquad \dfrac{\dfrac{}{\mid_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\,\mathscr{F}}{x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\,\mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N', N')}\,\mathscr{H}}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\,\mathit{require}$$

$$\frac{}{A\ \mathit{true}_\alpha}\ \mathscr{D}$$

$$\frac{\dfrac{}{\mid_{y,z} [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred}M}}}{M \in_\alpha A}}{[M/x]N \Downarrow_\alpha {\color{darkred}N'}}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N', N')}\ \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\frac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred}M}} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha {\color{darkred}N'}}\ \mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha {\color{darkred}N'}} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha(N', N')}\ \mathscr{H}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\overline{\Big|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ (M \in_\alpha A)}\ \mathscr{F}(M, M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}\ \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha (N', N')}\ \mathscr{H}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$\square$

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