ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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1. Model Construction

We will now ground the intuitive semantics for the extended theory in a concrete model construction, using Kripke logical relations. Let $\mathcal{B}_\Psi$ be the set of abstract binding trees (terms) generated by the signature Ψ. Then, we define a world w as a triple $\langle \mathbf{\Psi}_w, \mathbf{T}_w, \mathbf{K}_w \rangle$, where $\mathbf{T}_w$ is the finite list of terms which have been recognized as canonical types so far, and $\mathbf{K}_w\,(A)$ is the finite list of witnesses of the truth of A that have been constructed so far, for each A in $\mathbf{T}_w$. In other words, the set of worlds $\mathcal{W}$ is defined as follows, where for any set S, the operator $[-] : S\star \to \mathbf{Set}$ takes a list to its extension as a set.

$$\mathcal{W} \triangleq \coprod_{\Psi \in \mathsf{sig}} \coprod_{T \in \mathcal{B}_\Psi \star} [T] \to \mathcal{B}_\Psi \star$$

Let $\mathbf{D}_w$ be the domain of discourse $\mathcal{B}_{\Psi w}$. The accessibility relation $\preceq$ is defined as follows:

$$\frac{\mathbf{\Psi}_u \preceq \mathbf{\Psi}_v \quad [\mathbf{T}_u] \subseteq [\mathbf{T}_v] \quad \forall A \in \mathbf{T}_u.\ \mathbf{K}_u\,(A) \subseteq \mathbf{K}_v\,(A)}{u \preceq v}$$

Theorem 1.1. *$\langle \mathcal{W}, \preceq \rangle$ is a Kripke frame.*

Proof. It suffices to show that $\preceq$ is reflexive and transitive, which is immediate from the corresponding properties of signature subsumption and subsethood. □

1.1. **Operational Semantics.** We begin with an intensional (i.e. world-indexed) operational semantics for the fragment of type theory which is characterized by the signature **TT**:

$$\begin{gathered}
\mathsf{void} : () \\
\mathsf{unit} : (), \mathsf{Ax} : () \\
\Pi : (0,1), \lambda : (1), \mathsf{ap} : (0,0) \\
\Sigma : (0,1), \langle -,- \rangle : (0,0), \mathsf{spread} : (0,2) \\
\mathsf{require} : (0,1)
\end{gathered}$$

Let the judgment $M \Downarrow_\alpha M'$ be pronounced "The value of M is M' under choice sequence α." Then, we will explain its meaning with respect to any world $w \in \mathcal{W}$ such that $\mathbf{\Psi}_w \succeq \mathbf{TT}$.

The canonical forms all evaluate to themselves; the non-canonical operators evaluate in the following way:

$$\frac{w \Vdash M \Downarrow_\alpha (\lambda x)E \quad w \Vdash [N/x]E \Downarrow_\alpha E'}{w \Vdash \mathsf{ap}(M; N) \Downarrow_\alpha E'}$$

$$\frac{w \Vdash M \Downarrow_\alpha \langle M_1, M_2 \rangle \quad w \Vdash [M_1/u, M_2/v]E \Downarrow_\alpha E'}{w \Vdash \mathsf{spread}(M; u, v.E) \Downarrow_\alpha E'}$$

$$\frac{A \Downarrow_\alpha A' \quad \mathbf{K}_w A' \ni_{\mathsf{hd}(\alpha)} M \quad [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \mathsf{require}(A; x.N) \Downarrow_\alpha N'}$$

The type system is defined by mutual recursion with the world-wellformedness judgment, $w \Vdash u\ \mathit{world}_\alpha$ (presupposing $u \succeq w$):

$$\frac{\mathbf{T}_u \setminus \mathbf{T}_w \equiv \{\}}{w \Vdash u\ \mathit{world}_\alpha}$$

$$\frac{\mathbf{T}_u \setminus \mathbf{T}_w \equiv \ldots, B \quad \forall M \in [\mathbf{K}_u B].\ w \Vdash \mathcal{E}_\alpha [\![B]\!] (M, M) \quad w \Vdash \langle \mathbf{\Psi}_u, \mathbf{T}_u \setminus \{B\}, \mathbf{K}_u \rangle\ \mathit{world}_\alpha}{w \Vdash u\ \mathit{world}_\alpha}$$

$$\frac{}{w \Vdash \mathcal{V}_\alpha [\![\mathsf{unit}]\!] (\mathsf{Ax}, \mathsf{Ax})}$$

$$\frac{\begin{array}{l} \forall u \succeq w.\ \forall M, N \in \mathbf{D}_u. \\ \quad w \Vdash u\ \mathit{world}_\alpha \\ \quad u \Vdash \mathcal{E}^\star_\alpha [\![A]\!] (M, N) \Rightarrow u \Vdash \mathcal{E}^\star_\alpha [\![[M/x]B]\!] ([M/x]E, [N/y]F) \end{array}}{w \Vdash \mathcal{V}_\alpha [\![(\Pi x \in A)B]\!] ((\lambda x)E, (\lambda y)F)}$$

$$\frac{w \Vdash \mathcal{E}^\star_\alpha [\![A]\!] (M, M') \quad w \Vdash \mathcal{E}^\star_\alpha [\![[M/x]B]\!] (N, N')}{w \Vdash \mathcal{V}_\alpha [\![(\Sigma x \in A)B]\!] (\langle M, N \rangle, \langle M', N' \rangle)}$$

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad w \Vdash \mathcal{V}_\alpha [\![A]\!] (M', N')}{w \Vdash \mathcal{E}_\alpha [\![A]\!] (M, N)}$$

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad w \Vdash \mathcal{E}_\alpha [\![A']\!] (M, N)}{w \Vdash \mathcal{E}^\star_\alpha [\![A]\!] (M, N)}$$