

ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

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October 8, 2015

joint work with Darryl McAdams

INTRODUCTION

[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

($\Sigma p \in \textit{Woman}$)

[[*A woman walked in.*]]

∇

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

SatDown(???)

[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

[[A woman walked in. She sat down]]

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(\pi_1(x))$

THE **require** ORACLE: STATICS

require : (0;1)

(operator)

require $x : A$ **in** $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

(notation)

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(notation)

$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)

[[A woman walked in. She sat down]]

∇

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$ **require** $y : \text{Woman}$ **in** $\text{SatDown}(y)$

The meaning of a sentence is a logical proposition.

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What we want:

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{ require } y : \text{Woman in SatDown}(y)$

\sim

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where $M \sim N \stackrel{\text{def}}{=} (M \leq N) \ \& \ (N \leq M)$

$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

[[*The President ran a marathon*]]



[[*The President ran a marathon*]]

▽

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

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\Downarrow

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\Downarrow

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$

A NEGATIVE EXAMPLE

[[*The unicorn ran a marathon*]]



[[*The unicorn ran a marathon*]]

∇

require $x : \text{Unicorn}$ **in** $(\Sigma y \in \text{Marathon}) \text{Ran}(x; y)$

[[*The unicorn ran a marathon*]]

∇

require $x : \textit{Unicorn}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

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(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

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Corollary

The meaning of a judgment \mathcal{J} must be explained in terms of its forcing condition, $w \Vdash \mathcal{J}$, for any stage/world w .

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathcal{J}. (w \Vdash \mathcal{J}) \vee \neg(w \Vdash \mathcal{J}) \quad (\text{Dummett's infelicity})$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

logical consequence \Rightarrow semantic consequence

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logical consequence \Rightarrow **semantic consequence**

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proof conditions \Rightarrow **assertion conditions**

Dummett, Martin-Löf, Sundholm \Rightarrow *Brouwer, Heyting, Van Atten*

global meaning explanation \Rightarrow **local meaning explanation**

Husserl, Dummett, Martin-Löf \Rightarrow *Brouwer, Beth, Kripke,
Grothendieck, Lawvere, Joyal*

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$$|_x \mathcal{J}(x)$$

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$\mathcal{I}_2 (\mathcal{I}_1)$

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$M \Downarrow N$

(evaluation)

assertion acts (judgments) are intensional (local)

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A type	(typehood)

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$A \text{ verif}$	(verification)

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A <i>true</i>	(truth)
$M = N \in A$	(membership)

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(hypothetical judgment)

$$w \Vdash |_x \mathcal{I}(x) \quad \Longleftrightarrow \quad \dots$$

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$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
 w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
 w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
 \end{aligned}$$

where \mathcal{D}_w is the species of constructions that have been effected by stage w

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For a type A , implicit in the explanation of $w \Vdash A \text{ } \textit{verif}$ is a \mathbb{W} -indexed family of PERs $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$ whose members **reflect** the computational content (extension) of verification acts.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$

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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathcal{V} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathcal{B} \text{ bars } w. \forall u \in \mathcal{B}. u \Vdash A \text{ } \textit{verif}$$

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathcal{K}_A(w)$. First, let κ_A be the choice sequence of lists given by enumerating $\mathcal{K}_A(w)$ at each stage w , in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

SPREADS: SETS OF CHOICE SEQUENCES

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$$\mid_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left(\mathfrak{S}(\vec{u} \smallfrown m) \right)$$

3. a neighborhood may always be refined within the spread

$$\mid_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \left(\mathfrak{S}(\vec{u}) \right)$$

THE require ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\langle \rangle)} \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket_w^\alpha (M', N')}{w \Vdash M = N \in_\alpha A}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{M \Downarrow_\alpha M' \quad N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket^\alpha (M', N')}{M = N \in_\alpha A}$$

Deterministic choice for \varkappa_A :

$$\frac{w \Vdash A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_{\alpha} M}$$

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Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_{\alpha} M \quad [M/x]N \Downarrow_{\text{tl}(\alpha)} N'}{\mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_{\alpha} N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our semantics:

$$\frac{A \text{ true}_\alpha \quad x : A \vdash_\alpha N \in B}{\mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B} \text{ require}$$

$$\frac{A \text{ true}_{\alpha} \quad x : A \vdash_{\alpha} N \in B}{\mathbf{require} \ x : A \text{ in } N \in_{\alpha} B} \text{ require}$$

$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{x : A \vdash_\alpha N \in B} \mathcal{E}$$

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$$\frac{\overline{\mathcal{K}_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')} \quad \text{require}$$

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$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D}}{\varkappa_A \ni_\alpha \textcolor{red}{M}} \quad \frac{}{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')} \text{require}$$

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$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\overline{\not\prec_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')}}}{\overline{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}}$$

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$$\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \checkmark \quad \mathcal{F}(M, M) \quad \overline{M \in_\alpha A}}{\overline{\quad}}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \checkmark \quad \mathcal{D} \quad \overline{\not\prec_A \ni_\alpha M} \quad \overline{[M/x]N \Downarrow_\alpha N'} \quad \overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')} \quad \text{require}}{\overline{\text{require } x : A \text{ in } N \in_\alpha B}}$$

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$$\frac{\overline{\frac{\checkmark}{[M/x]N \in_\alpha B \quad (M \in_\alpha A)}{\mathcal{F}(M, M)} \quad \frac{\overline{\mathcal{K}_A \ni_\alpha M}}{M \in_\alpha A}}{\quad}$$

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$$\frac{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \mathcal{F}(M, M) \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\not\prec_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}{\mathcal{D}}}{\mathcal{E}}$$

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$$\frac{}{A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{|y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B} \begin{matrix} \mathcal{F} \\ \mathcal{E} \end{matrix}$$

$$\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \checkmark \mathcal{F}(M,M) \quad \frac{\overline{A \text{ true}_\alpha} \checkmark \mathcal{D} \quad \frac{\not x_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}{[M/x]N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \checkmark \mathcal{D} \quad \frac{\not x_A \ni_\alpha \mathbf{M} \quad \overline{[M/x]N \Downarrow_\alpha \mathbf{N}'}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \quad \overline{\mathcal{V} \llbracket B \rrbracket^\alpha (N', N')}}{\text{require } x : A \text{ in } N \in_\alpha B} \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{\overline{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \quad \mathcal{F}}{x : A \vdash_\alpha N \in B} \quad \mathcal{E}}}$$

$$\frac{\overline{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M)} \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\varkappa_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}{\overline{[M/x]N \in_\alpha B}}$$

$$\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\varkappa_A \ni_\alpha \mathbf{M} \quad \overline{[M/x]N \Downarrow_\alpha N'}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')}} \quad \text{require} \quad \overline{\text{require } x : A \text{ in } N \in_\alpha B}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M)}{\overline{\frac{\overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \mathcal{G} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')}}{\overline{[M/x]N \in_\alpha B}} \quad \mathcal{H}}} \quad \mathcal{D}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \overline{\mathcal{K}_A \ni_\alpha \mathbf{M}} \quad \overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')}}{\overline{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N}' \quad \text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E}} \quad \mathcal{F}$$

$$\frac{\frac{\frac{\checkmark}{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M) \quad \frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\kappa_A \ni_\alpha \mathbf{M}}{M \in_\alpha A}}{\frac{[M/x]N \Downarrow_\alpha \mathbf{N'} \quad \mathcal{G} \quad \mathcal{V}[[B]]^\alpha(N', N') \quad \mathcal{H}}{[M/x]N \in_\alpha B}} \quad \mathcal{H}} \quad \mathcal{G}$$

$$\frac{\frac{\frac{\checkmark}{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\kappa_A \ni_\alpha \mathbf{M}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N'} \quad \mathcal{G}} \quad \frac{\mathcal{V}[[B]]^\alpha(N', N')}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E} \quad \mathcal{F}$$

$$\frac{\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M) \quad \frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\checkmark}{\kappa_A \ni_\alpha \mathbf{M}}}{M \in_\alpha A}}{\frac{\overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \mathcal{G} \quad \overline{\mathcal{V}[[B]]^\alpha(N', N')} \quad \mathcal{H}}{[M/x]N \in_\alpha B} \quad \mathcal{H}$$

$$\frac{\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\checkmark}{\kappa_A \ni_\alpha \mathbf{M}} \quad \frac{\overline{[M/x]N \Downarrow_\alpha \mathbf{N}'} \quad \mathcal{G}}{\text{require } x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \quad \frac{\overline{\mathcal{V}[[B]]^\alpha(N', N')} \quad \mathcal{H}}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \text{require}$$

$$\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{\frac{|_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{x : A \vdash_\alpha N \in B}} \quad \mathcal{E} \quad \mathcal{F}$$

$$\frac{\frac{\overline{[M/x]N \in_\alpha B \ (M \in_\alpha A)} \quad \mathcal{F}(M, M) \quad \frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\checkmark}{\kappa_A \ni_\alpha M} \quad \mathcal{D}}{M \in_\alpha A} \quad \mathcal{D}}{\frac{\overline{[M/x]N \Downarrow_\alpha N'} \quad \mathcal{G} \quad \overline{\mathcal{V}[B]^\alpha(N', N')} \quad \mathcal{H}}{[M/x]N \in_\alpha B} \quad \mathcal{H}$$

$$\frac{\frac{\overline{A \text{ true}_\alpha} \quad \mathcal{D} \quad \frac{\overline{\kappa_A \ni_\alpha M} \quad \mathcal{D}}{\text{require } x : A \text{ in } N \Downarrow_\alpha N'} \quad \mathcal{G} \quad \frac{\overline{\mathcal{V}[B]^\alpha(N', N')} \quad \mathcal{H}}{\text{require } x : A \text{ in } N \in_\alpha B} \quad \mathcal{H} \quad \text{require}$$

□

QUESTIONS?



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