

ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

INTRODUCTION

[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

($\Sigma p \in \textit{Woman}$)

[[*A woman walked in.*]]

▽

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

SatDown(???)

[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(\pi_1(x))$

THE “DONKEY SENTENCE”

[[*Every farmer who owns a donkey beats it.*]]

∇

$(\Pi p \in (\Sigma x \in \textit{Farmer}) (\Sigma y \in \textit{Donkey}) \textit{Owns}(x; y))$

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$(\Pi p \in (\Sigma x \in \text{Farmer}) (\Sigma y \in \text{Donkey}) \text{Owns}(x; y)) \text{Beats}(\pi_1(p); \pi_1(\pi_2(p)))$

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- resolution of presuppositions

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- terms for presuppositions (this talk)
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THE **require** ORACLE: STATICS

require : (0; 1)

(operator)

require $x : A$ **in** $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

(notation)

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require $x : A$ **in** $N \stackrel{\text{def}}{=} \text{require}(A; x.N)$

(notation)

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)

[[A woman walked in. She sat down]]

∇

$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$ **require** $y : \text{Woman}$ **in** $\text{SatDown}(y)$

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What we want:

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where $M \sim N \stackrel{\text{def}}{=} (M \leq N) \wedge (N \leq M)$

EVERY GRAMMATICAL SENTENCE HAS A MEANING

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...but only some of them denote propositions (types)!

$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

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1. impredicativity

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Can the above be made precise? There are two problems:

1. impredicativity
2. non-determinism

(HOLD THAT THOUGHT)

A POSITIVE EXAMPLE

[[*The President ran a marathon*]]



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▽

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

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\Downarrow

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$

A NEGATIVE EXAMPLE

[[*The unicorn ran a marathon*]]



[[*The unicorn ran a marathon*]]

∇

require $x : \textit{Unicorn}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

[[*The unicorn ran a marathon*]]

∇

require $x : \textit{Unicorn}$ **in** $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

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$$w \Vdash \mathcal{I}_\alpha$$

THE CREATING SUBJECT

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All mathematics is a mental construction performed by an idealized subject, **subject to the following observations about knowledge**:

1. experiences are never forgotten (**monotonicity, functoriality**)
2. at a point in time, the subject knows whether or not it has experienced a judgment (**decidability**)

Corollary

The meaning of a judgment \mathcal{J} must be explained in terms of its forcing condition, $w \Vdash \mathcal{J}$, for any stage/world w .

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathcal{J}. \llbracket w \Vdash \mathcal{J} \rrbracket \vee \neg \llbracket w \Vdash \mathcal{J} \rrbracket \quad (\text{Dummett's infelicity})$$

The above is impossible in a Beth model.

logical consequence \Rightarrow semantic consequence

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Brouwer?, Martin-Löf, Sundholm \Rightarrow Brouwer?, Heyting, Allen, Zeilberger

proof conditions \Rightarrow assertion conditions

Martin-Löf, Sundholm \Rightarrow Heyting, Van Atten

global meaning explanation \Rightarrow local meaning explanation

*Husserl, Dummett, Martin-Löf \Rightarrow Brouwer?, Beth, Kripke,
Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)

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$$|_x \mathcal{J}(x)$$

(general judgment)

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$\mathcal{I}_2 (\mathcal{I}_1)$

(hypothetical judgment)

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$A \text{ verif}$	(verification)

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$M = N \in A$	(membership)

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$w \Vdash |_x \mathcal{I}(x)$

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(hypothetical judgment)

$$w \Vdash |_x \mathcal{I}(x) \quad \Longleftrightarrow \quad \dots$$

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$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
 w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
 w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
 \end{aligned}$$

where \mathcal{D}_w is the species of constructions that have been effected by stage w

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For a type A , implicit in the explanation of $w \Vdash A \text{ } \textit{verif}$ is a \mathbb{W} -indexed family of PERs $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$ whose members **reflect** the computational content (extension) of verification acts.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$

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$$\mathcal{V}[[A]]_w(M, N) \bowtie w \Vdash A \text{ verif}$$

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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ w \Vdash \mathcal{Z} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ } \textit{verif}$$

THE require ORACLE: DYNAMICS

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$

Based on this meaning explanation, we can justify the construct a family of sheaves $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{W}^{\text{op}}}$, such that $\mathcal{K}_A(w)$ is the species of constructions of A *true* which have been effected at or prior to w .

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathcal{K}_A(w)$. First, let κ_A be the choice sequence of lists given by enumerating $\mathcal{K}_A(w)$ at each world w , in order of construction.

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Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

SPREADS: SETS OF CHOICE SEQUENCES

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$$\downarrow_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left(\mathfrak{S}(\vec{u} \smallfrown m) \right)$$

3. inclusion of neighborhoods is closed under refinement/extension

$$\downarrow_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \left(\mathfrak{S}(\vec{u}) \right)$$

A spread for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\vec{u}) \quad \frac{\mathfrak{S}(\vec{u}) \quad |_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)}} \quad (\text{spread law})$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\mathcal{K}_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = j \quad w \Vdash [\mathcal{K}_{A'}(j)/x] N \Downarrow_{\text{tl}(\alpha)} N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_\alpha N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

QUESTIONS?