

# ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

## INTRODUCTION

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[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

( $\Sigma p \in \textit{Woman}$ )

[[*A woman walked in.*]]

▽

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

*SatDown*(???)



[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

[[*A woman walked in. She sat down*]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(\pi_1(x))$

## THE “DONKEY SENTENCE”

[[*Every farmer who owns a donkey beats it.*]]

∇

$(\Pi p \in (\Sigma x \in \textit{Farmer}) (\Sigma y \in \textit{Donkey}) \textit{Owns}(x; y))$

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## THE “DONKEY SENTENCE”

[[Every farmer who owns a donkey beats it.]]

$\nabla$

$(\Pi p \in (\Sigma x \in \text{Farmer}) (\Sigma y \in \text{Donkey}) \text{Owns}(x; y)) \text{Beats}(\pi_1(p); \pi_1(\pi_2(p)))$

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## THE **require** ORACLE: STATICS

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**require** : (0; 1)

(operator)

**require**  $x : A$  **in**  $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

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(notation)

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)



*[[A woman walked in. She sat down]]*

$\nabla$

*$(\Sigma x \in (\Sigma p \in Woman) WalkedIn(p)) SatDown(???)$*

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$  **require**  $y : \text{Woman}$  **in**  $\text{SatDown}(y)$

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What we want:

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

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$\sim$

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where  $M \sim N \stackrel{\text{def}}{=} (M \leq N) \wedge (N \leq M)$



# EVERY GRAMMATICAL SENTENCE HAS A MEANING

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...but only some of them denote propositions (types)!

$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

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Can the above be made precise? There are two problems:

1. impredicativity
2. non-determinism

(HOLD THAT THOUGHT)



## A POSITIVE EXAMPLE

[[ *The President ran a marathon* ]]



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▽

**require**  $x : \textit{President}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$



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$\Downarrow$

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$

## A NEGATIVE EXAMPLE

[[ *The unicorn ran a marathon* ]]





[[ *The unicorn ran a marathon* ]]

∇

**require**  $x : \textit{Unicorn}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

[[ *The unicorn ran a marathon* ]]

∇

**require**  $x : \text{Unicorn}$  **in**  $(\Sigma y \in \text{Marathon}) \text{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

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$$w \Vdash \mathcal{I}_\alpha$$

## THE CREATING SUBJECT

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1. experiences are never forgotten (**monotonicity, functoriality**)
2. at a point in time, the subject knows whether or not it has experienced a judgment (**decidability**)

## Corollary

*The meaning of a judgment  $\mathcal{J}$  must be explained in terms of its forcing condition,  $w \Vdash \mathcal{J}$ , for any stage/world  $w$ .*

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

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### Remark

*Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:*

$$\forall w. \forall \mathcal{J}. \llbracket w \Vdash \mathcal{J} \rrbracket \vee \neg \llbracket w \Vdash \mathcal{J} \rrbracket \quad (\text{Dummett's infelicity})$$

*The above is impossible in a Beth model.*



logical consequence  $\Rightarrow$  semantic consequence

*Brouwer?, Martin-Löf, Sundholm*  $\Rightarrow$  *Brouwer?, Heyting, Allen, Zeilberger*

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proof conditions  $\Rightarrow$  assertion conditions

*Martin-Löf, Sundholm  $\Rightarrow$  Heyting, Van Atten*

global meaning explanation  $\Rightarrow$  local meaning explanation

*Husserl, Dummett, Martin-Löf  $\Rightarrow$  Brouwer?, Beth, Kripke,  
Grothendieck, Lawvere, Joyal*

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$$|_x \mathcal{J}(x)$$

(general judgment)

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$\mathcal{I}_2 (\mathcal{I}_1)$

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$M \Downarrow N$

(evaluation)

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$A$ <i>type</i>	(typehood)

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$A \text{ verif}$	(verification)

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$A \text{ true}$	(truth)
$M = N \in A$	(membership)

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$w \Vdash  _x \mathcal{J}(x)$	(general judgment)
$w \Vdash \mathcal{J}_2 (\mathcal{J}_1)$	(hypothetical judgment)
$w \Vdash M \Downarrow N$	(evaluation)
$w \Vdash A \text{ type}$	(typehood)
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$w \Vdash |_x \mathcal{I}(x)$

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(hypothetical judgment)

$$w \Vdash |_x \mathcal{I}(x) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
\end{aligned}$$

where  $\mathcal{D}_w$  is the species of constructions that have been effected by stage  $w$

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For a type  $A$ , implicit in the explanation of  $w \Vdash A \text{ } \textit{verif}$  is a PER  $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$  whose members **reflect** the computational content of verification acts.

Truth (**justification**) consists in recognizing the effectiveness of a procedure for **verification**.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$



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$$\mathcal{V}[[A]]_w(M, N) \bowtie w \Vdash A \text{ } \textit{verif}$$

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1. **Canonical membership** reflects **verification**
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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ w \Vdash \mathcal{Z} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathcal{B} \text{ bars } w. \forall u \in \mathcal{B}. u \Vdash A \text{ } \textit{verif}$$

## THE require ORACLE: DYNAMICS

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$



Based on this meaning explanation, we can justify the construct a family of sheaves  $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{W}^{\text{op}}}$ , such that  $\mathcal{K}_A(w)$  is the species of constructions of  $A$  *true* which have been effected at or prior to  $w$ .

Based on this meaning explanation, we can justify the construct a family of sheaves  $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{W}^{\text{op}}}$ , such that  $\mathcal{K}_A(w)$  is the species of constructions of *A true* which have been effected at or prior to  $w$ .

We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of  $\mathcal{K}_A(w)$ . First, let  $\kappa_A$  be the choice sequence of lists given by enumerating  $\mathcal{K}_A(w)$  at each world  $w$ , in order of construction.

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Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

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$$\downarrow_{\vec{u}, m} \mathfrak{S}(\vec{u}) \quad (\mathfrak{S}(\vec{u} \smallfrown m))$$

3. inclusion of neighborhoods is closed under refinement/extension

$$\downarrow_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \quad (\mathfrak{S}(\vec{u}))$$

A spread for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\vec{u}) \quad \frac{\mathfrak{S}(\vec{u}) \quad \mid_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)}} \quad (\text{spread law})$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash A \Downarrow_{\alpha} A' \quad |\mathcal{K}_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = j \quad w \Vdash [\mathcal{K}_{A'}(j)/x] N \Downarrow_{\text{tl}(\alpha)} N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_{\alpha} N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

QUESTIONS?