ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

INTRODUCTION

$$[\![\textit{A woman walked in.}]\!]$$
$$\triangledown$$

$$\llbracket \textit{A woman walked in.} \rrbracket$$
$$\triangledown$$
$$(\Sigma p \in \textit{Woman})$$

$$\llbracket \text{\textit{A woman walked in.}} \rrbracket$$
$$\bigtriangledown$$
$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$\llbracket \textit{She sat down} \rrbracket$$
$$\triangledown$$

$$⟦\textit{She sat down}⟧$$
$$\triangledown$$
$$\textit{SatDown}(???)$$

$$[\![A\ woman\ walked\ in.\ She\ sat\ down]\!]$$
$$\triangledown$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in \textit{Woman}) \, \textit{WalkedIn}(p))$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
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$$(\Sigma x \in (\Sigma p \in \textit{Woman})\, \textit{WalkedIn}(p))\, \textit{SatDown}(\textit{???})$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in \textit{Woman}) \, \textit{WalkedIn}(p)) \, \textit{SatDown}(\pi_1(x))$$

⟦*Every farmer who owns a donkey beats it.*⟧

$\triangledown$

$(\Pi p \in (\Sigma x \in \text{\textit{Farmer}})\,(\Sigma y \in \text{\textit{Donkey}})\,\text{\textit{Owns}}(x; y))$

⟦*Every farmer who owns a donkey beats it.*⟧

▽

$(\Pi p \in (\Sigma x \in Farmer)\,(\Sigma y \in Donkey)\,Owns(x; y))\,Beats(???; ???)$

⟦*Every farmer who owns a donkey beats it.*⟧

$\triangledown$

$(\Pi p \in (\Sigma x \in Farmer)\,(\Sigma y \in Donkey)\,Owns(x;y))\,Beats(\pi_1(p); \pi_1(\pi_2(p)))$

TWO THINGS TO DEAL WITH

- terms for presuppositions

- terms for presuppositions
- resolution of presuppositions

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- terms for presuppositions (this talk)
- resolution of presuppositions

THE **require** ORACLE: STATICS

$$\textbf{require} : (0;1) \qquad \text{(operator)}$$

$$\textbf{require } x : A \textbf{ in } N \stackrel{\text{def}}{=} \textbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash A \; type \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require } x : A \textbf{ in } N \in B} \qquad (require)$$

$\llbracket$*A woman walked in. She sat down*$\rrbracket$

$\triangledown$

$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$

⟦*A woman walked in. She sat down*⟧

$\bigtriangledown$

$(\Sigma x \in (\Sigma p \in Woman)\ WalkedIn(p))\ \textbf{require}\ y : Woman\ \textbf{in}\ SatDown(y)$

The meaning of a sentence is a logical proposition.

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What we want:

$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\,\mathit{WalkedIn}(p))\;\textbf{require}\;y : \mathit{Woman}\;\textbf{in}\;\mathit{SatDown}(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\,\mathit{WalkedIn}(p))\,\pi_1(x)$$

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The meaning of a sentence is a type-theoretic expression which may <u>evaluate</u> to a canonical proposition.

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$$(\Sigma x \in (\Sigma p \in Woman) \, WalkedIn(p)) \, \textbf{require} \; y : Woman \; \textbf{in} \; SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman) \, WalkedIn(p)) \, \pi_1(x)$$

$$\boxed{\text{where } M \sim N \stackrel{\text{def}}{=} (M \leq N) \wedge (N \leq M)}$$

...but only some of them denote propositions (types)!

$$\frac{M \in A \quad [M/x]\, N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \tag{??}$$

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Can the above be made precise?

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1. impredicativity

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1. impredicativity
2. non-determinism

(HOLD THAT THOUGHT)

⟦ *The President ran a marathon* ⟧
∇

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∇
require x : *President* **in** $(\Sigma y \in Marathon)\, Ran(x; y)$

require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon})\, \textit{Ran}(x; y)$

$$\textbf{require } x : \textit{President } \textbf{in } (\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$$
$$\Downarrow$$
$$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$$

$[\![$ *The unicorn ran a marathon* $]\!]$
∇

$\llbracket$ *The unicorn ran a marathon* $\rrbracket$

$\triangledown$

require $x : Unicorn$ **in** $(\Sigma y \in Marathon)\, Ran(x; y)$

$\llbracket$ *The unicorn ran a marathon* $\rrbracket$

$\triangledown$

require $x : Unicorn$ **in** $(\Sigma y \in Marathon) \, Ran(x; y)$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

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$$w \Vdash \mathscr{J}_\alpha$$

THE CREATING SUBJECT

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Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

...

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Remark

Contra Dummett, I <u>by no means</u> take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathcal{J}. \ [\![w \Vdash \mathcal{J}]\!] \lor \neg [\![w \Vdash \mathcal{J}]\!] \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

proof conditions ⇒ assertion conditions

Martin-Löf, Sundholm ⇒ Heyting, Van Atten

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

proof conditions ⇒ assertion conditions

Martin-Löf, Sundholm ⇒ Heyting, Van Atten

global meaning explanation ⇒ local meaning explanation

Husserl, Dummett, Martin-Löf ⇒ *Brouwer?, Beth, Kripke,
Grothendieck, Lawvere, Joyal*

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$M \Downarrow N$	(evaluation)
$A\ \mathit{type}$	(typehood)
$A\ \mathit{verif}$	(verification)

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$A \; type$	(typehood)
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$A \; true$	(truth)

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$$A \; verif \qquad \text{(verification)}$$
$$A \; true \qquad \text{(truth)}$$
$$M = N \in A \qquad \text{(membership)}$$

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$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
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$$w \Vdash |_x \mathscr{J}(x) \iff \cdots$$
$$w \Vdash \mathscr{J}_2 \, (\mathscr{J}_1) \iff \cdots$$

$$w \Vdash |_x \mathscr{J}(x) \quad \Longleftrightarrow \quad \forall u \geq w. \forall x \in \mathscr{D}_u. \ u \Vdash \mathscr{J}(x)$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \quad \Longleftrightarrow \quad \forall u \geq w. \ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2$$

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where $\mathscr{D}_w$ is the species of constructions that have been effected by stage w

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For a type A, implicit in the explanation of $w \Vdash P \; verif$ is a PER $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content of <u>verification acts</u>.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

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In the model, this corresponds to the inevitability of verification (i.e. a <u>bar</u>, in which verification occurs at all nodes):

$w \Vdash A\ \textit{true} \Longleftrightarrow \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \textit{verif}$ (due to Dummett)

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1. **Canonical membership** reflects **verification**

$$\mathscr{V}[\![A]\!]_w(M,N) \ \bowtie \ w \Vdash A \ \mathit{verif}$$

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1. **Canonical membership** reflects **verification**
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$$w \Vdash M = N \in A \ \bowtie \ w \Vdash A \ true$$

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$$w \Vdash M = N \in A \quad \bowtie \quad w \Vdash A \; true$$

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1. **Canonical membership** reflects **verification**
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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ w \Vdash \mathscr{V}[\![A]\!]_w (M', N') \end{array} \right\} \quad \bowtie \quad \exists \mathfrak{B} \ \textbf{bars} \ w. \forall u \in \mathfrak{B}. \ u \Vdash A \ \textit{verif}$$

THE require ORACLE: DYNAMICS

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \ true \\ 1 & \neg(i \Vdash A \ true) \end{cases} \tag{KS}$$

Based on this meaning explanation, we can justify the construct a family of sheaves $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$, such that $\mathscr{K}_A(w)$ is the species of constructions of $A\ true$ which have been effected at or prior to w.

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \qquad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each world w, in order of construction.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

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1. the empty neighborhood shall be admitted

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SPREADS: SETS OF CHOICE SEQUENCES

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3. inclusion of neighborhoods is closed under refinement/extension

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u}\frown m)\ \big(\mathfrak{S}(\vec{u})\big)$$

A spread for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\ \rho(n) < n\ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

require — DYNAMICS

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Deterministic dynamics for **require**:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = j \quad w \Vdash \left[\varkappa_{A'}(j)/x\right]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'}$$

$$\text{(for } \alpha \in \mathfrak{S})$$

QUESTIONS?