

# ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

## INTRODUCTION

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[[*A woman walked in.*]]



[[*A woman walked in.*]]

▽

( $\Sigma p \in \textit{Woman}$ )

[[*A woman walked in.*]]

$\nabla$

$(\Sigma p \in \textit{Woman}) \textit{WalkedIn}(p)$

[[*She sat down*]]



[[*She sat down*]]

▽

*SatDown*(???)



[[*A woman walked in. She sat down*]]



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$

[[*A woman walked in. She sat down*]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(\pi_1(x))$

## THE “DONKEY SENTENCE”

[[*Every farmer who owns a donkey beats it.*]]

∇

$(\Pi p \in (\Sigma x \in \textit{Farmer}) (\Sigma y \in \textit{Donkey}) \textit{Owns}(x; y))$

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## THE “DONKEY SENTENCE”

[[*Every farmer who owns a donkey beats it.*]]

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$(\Pi p \in (\Sigma x \in \text{Farmer}) (\Sigma y \in \text{Donkey}) \text{Owns}(x; y)) \text{Beats}(\pi_1(p); \pi_1(\pi_2(p)))$

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- terms for presuppositions (this talk)
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## THE **require** ORACLE: STATICS

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**require** : (0;1)

(operator)

**require**  $x : A$  **in**  $N \stackrel{\text{def}}{=} \textbf{require}(A; x.N)$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \text{require } x : A \text{ in } N \in B}$$

(require)



[[A woman walked in. She sat down]]

▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \text{SatDown}(???)$

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▽

$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p))$  **require**  $y : \text{Woman}$  **in**  $\text{SatDown}(y)$

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What we want:

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

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$(\Sigma x \in (\Sigma p \in \text{Woman}) \text{WalkedIn}(p)) \pi_1(x)$

where  $M \sim N \stackrel{\text{def}}{=} (M \leq N) \wedge (N \leq M)$



# EVERY GRAMMATICAL SENTENCE HAS A MEANING

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...but only some of them denote propositions (types)!

$$\frac{M \in A \quad [M/x] N \Downarrow N'}{\text{require } x : A \text{ in } N \Downarrow N'} \quad (??)$$

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Can the above be made precise? There are two problems:

1. circularity
2. non-determinism

(HOLD THAT THOUGHT)



## A POSITIVE EXAMPLE

[[ *The President ran a marathon* ]]



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**require**  $x : \textit{President}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

## A POSITIVE EXAMPLE

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$\Downarrow$

$(\Sigma y \in \textit{Marathon}) \textit{Ran}(\textit{Obama}; y)$

## A NEGATIVE EXAMPLE

[[ *The unicorn ran a marathon* ]]





$\llbracket \textit{The unicorn ran a marathon} \rrbracket$

$\nabla$

**require**  $x : \textit{Unicorn}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

[[ *The unicorn ran a marathon* ]]

∇

**require**  $x : \textit{Unicorn}$  **in**  $(\Sigma y \in \textit{Marathon}) \textit{Ran}(x; y)$

(not a proposition)

IS require COMPUTATIONALLY EFFECTIVE?

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$$w \Vdash \mathcal{I}_\alpha$$

## THE CREATING SUBJECT

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1. experiences are never forgotten (**monotonicity, functoriality**)
2. at a point in time, the subject knows whether or not it has experienced a judgment (**decidability**)

## Corollary

*The meaning of a judgment  $\mathcal{J}$  must be explained in terms of its forcing condition,  $w \Vdash \mathcal{J}$ , for any stage/world  $w$ .*

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

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### Remark

*Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:*

$$\forall w. \forall \mathcal{J}. (w \Vdash \mathcal{J}) \vee \neg(w \Vdash \mathcal{J}) \quad (\text{Dummett's infelicity})$$

*The above is impossible in a Beth model.*



logical consequence  $\Rightarrow$  semantic consequence

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*Dummett, Martin-Löf, Sundholm*  $\Rightarrow$  *Brouwer, Heyting, Van Atten*

**global meaning explanation**  $\Rightarrow$  **local meaning explanation**

*Husserl, Dummett, Martin-Löf*  $\Rightarrow$  *Brouwer, Beth, Kripke,  
Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)



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$$|_x \mathcal{J}(x)$$

(general judgment)

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$\mathcal{I}_2 (\mathcal{I}_1)$

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$A \text{ true}$	(truth)
$M = N \in A$	(membership)

assertion acts (judgments) are intensional (local)

$w \Vdash  _x \mathcal{J}(x)$	(general judgment)
$w \Vdash \mathcal{J}_2 (\mathcal{J}_1)$	(hypothetical judgment)
$w \Vdash M \Downarrow N$	(evaluation)
$w \Vdash A \text{ type}$	(typehood)
$w \Vdash A \text{ verif}$	(verification)
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$w \Vdash |_x \mathcal{I}(x)$

(general judgment)

$w \Vdash \mathcal{I}_2 (\mathcal{I}_1)$

(hypothetical judgment)

$$w \Vdash |_x \mathcal{I}(x) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \quad \Longleftrightarrow \quad \dots$$

$$w \Vdash |_x \mathcal{I}(x) \iff \forall u \succeq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x)$$

$$w \Vdash \mathcal{I}_2 (\mathcal{I}_1) \iff \forall u \succeq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2$$

$$\begin{aligned}
w \Vdash |_x \mathcal{I}(x) &\iff \forall u \geq w. \forall x \in \mathcal{D}_u. u \Vdash \mathcal{I}(x) \\
w \Vdash \mathcal{I}_2 (\mathcal{I}_1) &\iff \forall u \geq w. u \Vdash \mathcal{I}_1 \Rightarrow u \Vdash \mathcal{I}_2
\end{aligned}$$

where  $\mathcal{D}_w$  is the species of constructions that have been effected by stage  $w$



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Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects (Fourman)

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For a type  $A$ , implicit in the explanation of  $w \Vdash A \text{ } \textit{verif}$  is a  $\mathbb{W}$ -indexed family of PERs  $\mathcal{V}[[A]]_w \subseteq \mathcal{D}_w \times \mathcal{D}_w$  whose members **reflect** the computational content (extension) of verification acts.

Truth (**justification**) consists in recognizing the effectiveness of a procedure for **verification**.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \text{ true} \iff \exists \mathfrak{B} \text{ bars } w. \forall u \in \mathfrak{B}. u \Vdash A \text{ verif} \quad (\text{due to Dummett})$$

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1. **Canonical membership** reflects **verification**

$$\mathcal{V}[[A]]_w(M, N) \bowtie w \Vdash A \text{ } \textit{verif}$$

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1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**

$$w \Vdash M = N \in A \not\bowtie w \Vdash A \text{ true}$$



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1. **Canonical membership** reflects **verification**
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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathcal{V} \llbracket A \rrbracket_w (M', N') \end{array} \right\} \bowtie \exists \mathcal{B} \text{ bars } w. \forall u \in \mathcal{B}. u \Vdash A \text{ } \textit{verif}$$

## THE require ORACLE: DYNAMICS

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \text{ true} \\ 1 & \neg(i \Vdash A \text{ true}) \end{cases} \quad (\text{KS})$$

Let  $\mathcal{K}_A : \mathbf{FinSet}^{\mathbf{Wop}}$  be the sheaf of constructions of  $A$  *true* effected “so far” for each canonical proposition  $A$ .

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathcal{K}_{A'}(w) \quad w \Vdash [M/x] N \Downarrow N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow N'} \quad (*)$$



We need a way to deterministically choose a representative of  $\mathcal{K}_A(w)$ . First, let  $\kappa_A$  be the choice sequence of lists given by enumerating  $\mathcal{K}_A(w)$  at each stage  $w$ , in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of “choosers”.

## SPREADS: SETS OF CHOICE SEQUENCES

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$$|_{\vec{u}, m} \mathfrak{S}(\vec{u}) \left( \mathfrak{S}(\vec{u} \smallfrown m) \right)$$

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$$\mid_{\vec{u}, m} \mathfrak{S}(\vec{u}) \quad (\mathfrak{S}(\vec{u} \smallfrown m))$$

3. a neighborhood may always be refined within the spread

$$\mid_{\vec{u}} \mathfrak{S}(\vec{u} \smallfrown m) \quad (\mathfrak{S}(\vec{u}))$$

A spread direction for index-choosers:

$$\frac{\overline{\mathfrak{S}(\langle \rangle)}}{\mathfrak{S}(\langle \rangle)} \quad \frac{\mathfrak{S}(\vec{u}) \quad \bigwedge_n \rho(n) < n \quad (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \smallfrown \rho)} \quad (\text{spread law})$$

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Reformulate type theory relative to an arbitrary  $\alpha \in \mathfrak{S}$ ! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathcal{V} \llbracket A \rrbracket_w^\alpha (M', N')}{w \Vdash M = N \in_\alpha A}$$



Deterministic choice for  $\varkappa_A$ :

$$\frac{w \Vdash A \Downarrow_{\alpha} A' \quad |\varkappa_{A'}(w)| = \ell \quad \text{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_{\alpha} M}$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash \varkappa_A \ni_{\alpha} M \quad w \Vdash [M/x]N \Downarrow_{\text{tl}(\alpha)} N'}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_{\alpha} N'} \quad (\text{for } \alpha \in \mathfrak{S})$$

## Theorem

*The following rule is valid in our semantics:*

$$\frac{w \Vdash A \text{ true}_\alpha \quad w \Vdash x : A \vdash_\alpha N \in B}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B} \text{ require}$$

$$\frac{w \Vdash A \text{ true}_{\alpha} \quad w \Vdash x : A \vdash_{\alpha} N \in B}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_{\alpha} B} \text{ require}$$

$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

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$$\frac{\frac{}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \textcolor{red}{N'}}{\quad} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')}}{\quad} \text{require}$$

$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{w \Vdash \varkappa_A \ni_\alpha M \quad [M/x]N \Downarrow_\alpha N'}{w \Vdash \mathbf{require} \, x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \mathbf{require} \, x : A \text{ in } N \in_\alpha B} \text{require}$$

$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{}{w \Vdash \neg_A \ni_\alpha M} \quad \frac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')} \text{require}$$

$$\frac{}{w \Vdash \mathbf{require} \ x : A \ \mathbf{in} \ N \in_\alpha B}$$



$$\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \qquad \frac{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}$$

$$\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{w \Vdash \not\prec_A \ni_\alpha M \quad [M/x]N \Downarrow_\alpha N'}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha N'} \quad \frac{}{\mathcal{V}[[B]]_w^\alpha(N', N')} \text{require}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y=z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha \textcolor{red}{M}} \quad \frac{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha \textcolor{red}{N'}} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \textit{require}
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\frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \checkmark \quad \mathcal{F}(w, M, M) \quad \overline{w \Vdash M \in_\alpha A}}{}$$

$$\begin{array}{c}
\frac{\overline{w \Vdash A \text{ true}_\alpha} \checkmark \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M} \quad [M/x]N \Downarrow_\alpha \mathbf{N}'} \\
\frac{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}' \quad \overline{\mathcal{V}[\![B]\!]_w^\alpha(N', N')}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \text{require}
\end{array}$$

$$\frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{\frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \checkmark \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash \not\prec_A \ni_\alpha M} \quad \overline{w \Vdash \not\prec_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}$$

$$\frac{\frac{\overline{w \Vdash A \text{ true}_\alpha} \checkmark \mathcal{D}}{w \Vdash \not\prec_A \ni_\alpha M} \quad \frac{\overline{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \quad \overline{\mathcal{V}[[B]_w^\alpha(N', N')}}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \text{require}$$

$$\begin{array}{c}
\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{\mathcal{F}} \\
\frac{w \Vdash A \text{ true}_\alpha \quad \mathcal{D} \quad \frac{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)}{\mathcal{E}}}{w \Vdash x : A \vdash_\alpha N \in B}
\end{array}$$

$$\begin{array}{c}
\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\checkmark}{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}} \mathcal{D} \\
\frac{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B \quad \mathcal{F}(w, M, M) \quad \frac{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}{w \Vdash M \in_\alpha A}}{}
\end{array}$$

$$\begin{array}{c}
\frac{\checkmark}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \\
\frac{w \Vdash \varkappa_A \ni_\alpha \mathbf{M} \quad \overline{[M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \\
\frac{\quad \overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (\mathbf{N}', \mathbf{N}')} \quad \text{require}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B}
\end{array}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{w \Vdash x : A \vdash_\alpha N \in B} \quad \mathcal{E}}
\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B} \quad \mathcal{F}}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)} \quad \mathcal{E}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \quad \mathcal{F}(w, M, M)}{w \Vdash [M/x]N \in_\alpha B}
\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \mathcal{D}
\frac{w \Vdash \varkappa_A \ni_\alpha M}{w \Vdash M \in_\alpha A}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \frac{\overline{[M/x]N \Downarrow_\alpha N'}}{\overline{w \Vdash \mathbf{require} \, x : A \text{ in } N \Downarrow_\alpha N'}} \quad \frac{\overline{\mathcal{V}[[B]]_w^\alpha(N', N')}}{w \Vdash \mathbf{require} \, x : A \text{ in } N \in_\alpha B} \quad \text{require}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{w \Vdash x : A \vdash_\alpha N \in B} \quad \mathcal{E}}
\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B} \quad \mathcal{F}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \quad (y = z \in_\alpha A)}$$

$$\frac{\overline{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \quad \mathcal{F}(w, M, M)}{\overline{w \Vdash [M/x]N \in_\alpha B}}
\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{w \Vdash \not\prec_A \ni_\alpha \textcolor{red}{M}} \quad \mathcal{D}}
\frac{\overline{w \Vdash M \in_\alpha A}}{\overline{w \Vdash \not\prec_A \ni_\alpha \textcolor{red}{M}}}$$

$$\frac{\overline{w \Vdash A \text{ true}_\alpha} \quad \mathcal{D}}{\overline{w \Vdash \not\prec_A \ni_\alpha \textcolor{red}{M}} \quad \mathcal{D}}
\frac{\overline{[M/x]N \Downarrow_\alpha \textcolor{red}{N'}}}{\overline{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha \textcolor{red}{N'}}}
\frac{\overline{\mathcal{V}[\![B]\!]_w^\alpha(N', N')}}{\overline{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \quad \textit{require}}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \mathcal{H}}
\end{array}$$



$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}
\end{array}$$

$$\frac{\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \mathcal{F} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \text{require}}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \mathcal{F}(w, M, M) \quad \frac{\overline{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M) \\
\frac{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{w \Vdash \textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \mathcal{G} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \mathcal{H} \\
\frac{}{w \Vdash \textbf{require } x : A \textbf{ in } N \in_\alpha B} \text{require}
\end{array}$$

$$\begin{array}{c}
\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\frac{\overline{\forall u \geq w. \forall y, z \in \mathcal{D}_u. u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}}{w \Vdash |_{y,z} [y/x]N = [z/x]N \in_\alpha B \ (y = z \in_\alpha A)} \mathcal{F}}{w \Vdash x : A \vdash_\alpha N \in B} \mathcal{E}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A} \mathcal{F}(w, M, M)}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B} \mathcal{F}(w, M, M) \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash M \in_\alpha A} \mathcal{D}}{\frac{\frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{E} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{\mathcal{H}}}{w \Vdash [M/x]N \in_\alpha B} \mathcal{H}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{}{w \Vdash A \text{ true}_\alpha} \mathcal{D} \quad \frac{\overline{w \Vdash \varkappa_A \ni_\alpha \mathbf{M}}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{F} \quad \frac{\overline{w \Vdash [M/x]N \Downarrow_\alpha \mathbf{N}'}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{E}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \Downarrow_\alpha \mathbf{N}'} \mathcal{H} \quad \frac{\overline{\mathcal{V} \llbracket B \rrbracket_w^\alpha (N', N')}}{\mathcal{H}} \text{ require}}{w \Vdash \mathbf{require} \ x : A \text{ in } N \in_\alpha B} \mathcal{H}
\end{array}$$

□

QUESTIONS?