ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

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joint work with Darryl McAdams

INTRODUCTION

NATURAL LANGUAGE SEMANTICS AND ANAPHORA

$[\![$*A woman walked in.*$]\!]$

$\triangledown$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)$$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$[\![$*She sat down*$]\!]$

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$$[\![\textit{She sat down}]\!]$$

$$\triangledown$$

$$\mathit{SatDown}(???)$$

$[\![$*A woman walked in. She sat down*$]\!]$

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$$[\![\textit{A woman walked in. She sat down}]\!]$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

$$\llbracket \textit{A woman walked in. She sat down} \rrbracket$$
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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

INTRODUCING **require**

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

THE **require** ORACLE: STATICS

require — FORMAL RULES

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$$\mathbf{require} : (0; 1) \qquad \text{(operator)}$$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in B} \qquad (\textit{require})$$

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What we want:

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$$\boxed{\text{where } M \sim N \stackrel{\text{def}}{=} (M \preceq N)\ \&\ (N \preceq M)}$$

require—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\; x : A \;\textbf{in}\; N \Downarrow N'} \qquad (??)$$

$$[\![\textit{The President ran a marathon}]\!]$$

$$\triangledown$$

A POSITIVE EXAMPLE

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$$\textbf{require } x : \textit{President} \textbf{ in } (\Sigma y \in \textit{Marathon})\, \textit{Ran}(x; y)$$

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$$(\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(\mathit{Obama}; y)$$

$[\![$ *The unicorn ran a marathon* $]\!]$

$\triangledown$

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$$\triangledown$$

$$\textbf{require } x : Unicorn \textbf{ in } (\Sigma y \in Marathon)\, Ran(x; y)$$

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$$\textbf{require}\ x : \mathit{Unicorn}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

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(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

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Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$*, for any stage/world* w.

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w.\forall \mathscr{J}.\,(w \Vdash \mathscr{J}) \lor \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

MEANING THEORY WITHOUT UNIFORMITY CONSTRAINTS

logical consequence $\Rightarrow$ semantic consequence

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proof conditions $\Rightarrow$ **assertion conditions**

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global meaning explanation ⇒ **local meaning explanation**

Husserl, Dummett, Martin-Löf ⇒ *Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal*

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$A \ \mathit{true}$	(truth)

BETH-KRIPKE SEMANTICS FOR ASSERTIONS

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$M \Downarrow N$	(evaluation)
$A \; \mathit{type}$	(typehood)
$A \; \mathit{verif}$	(verification)
$A \; \mathit{true}$	(truth)
$M = N \in A$	(membership)

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$w \Vdash \vert_x \mathscr{J}(x)$	(general judgment)
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SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$w \Vdash |_x \mathscr{J}(x) \iff \cdots$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \iff \cdots$$

$$
\begin{aligned}
w \Vdash |_x\, \mathscr{J}(x) \quad &\iff \quad \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\
w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) \quad &\iff \quad \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2
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where $\mathscr{D}_w$ is the species of constructions that have been effected by stage w

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For a type A, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A\ \mathit{true} \iff \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif} \qquad \text{(due to Dummett)}$$

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MEMBERSHIP AS REFLECTION ON TRUTH

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1. **Canonical membership** reflects **verification**

$$\boxed{\mathscr{V}[\![A]\!]_w(M,N) \quad \rtimes\!\!\!\ltimes \quad w \Vdash A \ \mathit{verif}}$$

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$$\bigwedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w(M', N') \end{array} \right\} \rtimes \quad \exists \mathfrak{B}\ \mathbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A\ \mathit{true} \\ 1 & \neg(i \Vdash A\ \mathit{true}) \end{cases} \tag{KS}$$

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \qquad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage w, in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

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3. a neighborhood may always be refined within the spread

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

THE **require** ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\ \rho(n) < n \ \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]_w^\alpha(M', N')}{w \Vdash M = N \in_\alpha A}$$

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Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

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require — DYNAMICS

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Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_\alpha M \quad [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{\textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \qquad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our semantics:

$$\frac{A\ \mathit{true}_\alpha \quad x : A \vdash_\alpha N \in B}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

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$$\frac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred} M}} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha {\color{darkred} N'}}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha {\color{darkred} N'}} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\frac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \frac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{[M/x]N \in_\alpha B}$$

$$\frac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N', N')}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{\mid_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\overline{[M/x]N \in_\alpha B}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}\,[\![B]\!]^\alpha\,(N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \quad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}}{[M/x]N \in_\alpha B}\ \mathscr{H}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}\ \mathscr{H}}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\,\mathscr{D}
\qquad
\dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\,\mathscr{F}}{x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\,\mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{}}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}\,\mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha(N',N')}\,\mathscr{H}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\,\mathit{require}$$

$$\dfrac{}{A\ true_\alpha}\,\mathscr{D} \qquad \dfrac{\overline{\left.\right|_{y,z} [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\,\mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}\,\mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha(N',N')}\,\mathscr{H}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$\square$

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