ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

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joint work with Darryl McAdams

INTRODUCTION

$[\![$*A woman walked in.*$]\!]$

$\bigtriangledown$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

INTRODUCING require

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\ \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

THE **require** ORACLE: STATICS

require — FORMAL RULES

$$\mathbf{require} : (0;1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in B} \qquad (\textit{require})$$

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What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \textbf{require}\ y : Woman\ \textbf{in}\ SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

require—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \quad (??)$$

$[\![$ *The President ran a marathon* $]\!]$

$\triangledown$

A POSITIVE EXAMPLE

$$[\![\textit{The President ran a marathon}]\!]$$

$$\triangledown$$

$$\textbf{require } x : \textit{President} \textbf{ in } (\Sigma y \in \textit{Marathon})\, \textit{Ran}(x; y)$$

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$$\Downarrow$$
$$(\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(\mathit{Obama}; y)$$

A NEGATIVE EXAMPLE

$$\llbracket \textit{The unicorn ran a marathon} \rrbracket$$

$$\bigtriangledown$$

A NEGATIVE EXAMPLE

$$[\![\textit{The unicorn ran a marathon}]\!]$$

$$\triangledown$$

$$\textbf{require } x : Unicorn \textbf{ in } (\Sigma y \in Marathon)\, Ran(x; y)$$

$$[\![\textit{The unicorn ran a marathon}]\!]$$

$$\triangledown$$

$$\textbf{require}\ x : \mathit{Unicorn}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

(not a proposition)

IS **require** COMPUTATIONALLY EFFECTIVE?

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(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

THE THEORY OF THE CREATING SUBJECT

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Corollary

The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w.\forall \mathscr{J}.\,(w \Vdash \mathscr{J}) \vee \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger* logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions $\Rightarrow$ assertion conditions

Dummett, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer, Heyting, Van Atten*

MEANING THEORY WITHOUT UNIFORMITY CONSTRAINTS

logical consequence ⇒ **semantic consequence**

Brouwer?, Martin-Löf, Sundholm ⇒ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions ⇒ **assertion conditions**

Dummett, Martin-Löf, Sundholm ⇒ *Brouwer, Heyting, Van Atten*

global meaning explanation ⇒ **local meaning explanation**

Husserl, Dummett, Martin-Löf ⇒ *Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal*

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$M = N \in A$	(membership)

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$$\begin{aligned} w \Vdash |_x\, \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\ w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) &\iff \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2 \end{aligned}$$

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$$

where $\mathscr{D}$ is the (pre)sheaf of constructions that have been effected so far

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INTUITIONISTIC VS CONSTRUCTIVE VALIDITY

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Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects **(Fourman)**

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For a type A, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification <u>acts</u>.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A \; \mathit{true} \iff \exists \mathfrak{B} \; \textbf{bars} \; w. \forall u \in \mathfrak{B}.\; u \Vdash A \; \mathit{verif} \qquad \text{(due to Dummett)}$$

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1. **Canonical membership** reflects **verification**

$$\mathscr{V}[\![A]\!]_w(M,N) \quad \rtimes\!\!\!\ltimes \quad w \Vdash A \; \mathit{verif}$$

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$$\bigwedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w(M', N') \end{array} \right\} \bowtie \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A\ \mathit{true} \\ 1 & \neg(i \Vdash A\ \mathit{true}) \end{cases} \tag{KS}$$

THE JUSTIFICATIONS PRESHEAF

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage w, in order of time.

ELIMINATING NON-DETERMINISM WITH A SPREAD

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

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$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$
3. a neighborhood may always be refined within the spread
$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

THE **require** ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\ \rho(n) < n \ \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha_w (M', N')}{w \Vdash M = N \in_\alpha A}$$

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$$\frac{M \Downarrow_\alpha M' \quad N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha (M', N')}{M = N \in_\alpha A}$$

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

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Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_\alpha M \quad [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our intuitive semantics:

$$\frac{A\ true_\alpha \quad x : A \vdash_\alpha N \in B}{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \textit{require}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}\,[\![B]\!]^\alpha\,(N', N')}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathsf{require}$$

$$\dfrac{}{A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{\mid_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}\ \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\,\mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\,\mathscr{F}}{x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\,\mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}\,\mathscr{H}}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\,\mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha(N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\mathscr{V}[\![B]\!]^\alpha(N', N')}\ \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha(N', N')}\ \mathscr{H}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$
\frac{}{A\ \mathit{true}_\alpha}\ \mathscr{D}
\qquad\qquad
\cfrac{\cfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}
$$

$$
\cfrac{\cfrac{\cfrac{\checkmark}{[M/x]N \in_\alpha B\ \ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \cfrac{\cfrac{\cfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred}M}}}{M \in_\alpha A}}{\cfrac{}{[M/x]N \Downarrow_\alpha {\color{darkred}N'}}\ \mathscr{G} \qquad \cfrac{}{\mathscr{V}[\![B]\!]^\alpha (N', N')}\ \mathscr{H}}}{[M/x]N \in_\alpha B}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\checkmark}{A\ \mathit{true}_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha {\color{darkred}M}} \qquad \cfrac{\checkmark}{[M/x]N \Downarrow_\alpha {\color{darkred}N'}}\ \mathscr{G}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha {\color{darkred}N'}} \qquad \cfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha (N', N')}\ \mathscr{H}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}
$$

$\square$

Questions?

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