ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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joint work with Darryl McAdams

INTRODUCTION

$[\![$*A woman walked in.*$]\!]$

$\triangledown$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)$$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$[\![\textit{She sat down}]\!]$$
$$\triangledown$$

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$$\triangledown$$

$$\mathit{SatDown}(???)$$

$[\![$*A woman walked in. She sat down*$]\!]$

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$$[\![\textit{A woman walked in. She sat down}]\!]$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

THE **require** ORACLE: STATICS

require — FORMAL RULES

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$$\mathbf{require} : (0;1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require}\ x : A\ \textbf{in}\ N \in B} \qquad (\textit{require})$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

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$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

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$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\ \mathbf{require}\ y : \mathit{Woman}\ \mathbf{in}\ \mathit{SatDown}(y)$$

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What we want:

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$\boxed{\text{where } M \sim N \overset{\text{def}}{=} (M \preceq N)\,\&\,(N \preceq M)}$

require—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \qquad (??)$$

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$$[\![\textit{The President ran a marathon}]\!]$$

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$$\Downarrow$$

$$(\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(\mathit{Obama}; y)$$

A NEGATIVE EXAMPLE

$$[\![\textit{The unicorn ran a marathon}]\!]$$
$$\triangledown$$

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$$\triangledown$$

$$\textbf{require}\ x : \mathit{Unicorn}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\,\mathit{Ran}(x; y)$$

$$\llbracket \textit{ The unicorn ran a marathon } \rrbracket$$

$$\triangledown$$

$$\textbf{require } x : \mathit{Unicorn} \textbf{ in } (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

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$$\boxed{w \Vdash \mathscr{J}_\alpha}$$

THE CREATING SUBJECT

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Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w.\forall \mathscr{J}.\,(w \Vdash \mathscr{J}) \vee \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

MEANING THEORY WITHOUT UNIFORMITY CONSTRAINTS

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger* logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions ⇒ assertion conditions

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global meaning explanation $\Rightarrow$ **local meaning explanation**

Husserl, Dummett, Martin-Löf $\Rightarrow$ *Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal*

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$M = N \in A$	(membership)

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SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$w \Vdash |_x\ \mathscr{I}(x) \iff \cdots$$
$$w \Vdash \mathscr{I}_2\ (\mathscr{I}_1) \iff \cdots$$

$$
\begin{aligned}
w \Vdash |_x\, \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\
w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) &\iff \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2
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where $\mathscr{D}_w$ is the species of constructions that have been effected by stage w

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For a type A, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$w \Vdash A \ \mathit{true} \iff \exists \mathfrak{B} \ \textbf{bars} \ w. \forall u \in \mathfrak{B}.\ u \Vdash A \ \mathit{verif}$ (due to Dummett)

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$$\bigwedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w(M', N') \end{array} \right\} \rtimes \quad \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A\ \mathit{true} \\ 1 & \neg(i \Vdash A\ \mathit{true}) \end{cases} \tag{KS}$$

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage w, in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

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3. a neighborhood may always be refined within the spread

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

THE **require** ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\, \rho(n) < n \;\; (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha_w(M', N')}{w \Vdash M = N \in_\alpha A}$$

require — DYNAMICS

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash \varkappa_A \ni_\alpha M \quad w \Vdash [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad (\text{for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our semantics:

$$\frac{w \Vdash A \; \mathit{true}_\alpha \qquad w \Vdash x : A \vdash_\alpha N \in B}{w \Vdash \mathbf{require}\; x : A \;\mathbf{in}\; N \in_\alpha B}\;\mathit{require}$$

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$$

$$
\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M) \qquad \dfrac{}{w \Vdash M \in_\alpha A}}{}
$$

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$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M) \quad \dfrac{\overline{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ require$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\ \mathscr{F}}{w \Vdash \mid_{y,z} [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$
\frac{}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}
\qquad
\frac{\dfrac{\dfrac{}{\forall u \geq w. \forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}\,\mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\,\mathscr{E}
$$

$$
\frac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\,\mathscr{F}(w, M, M)
\qquad
\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{w \Vdash [M/x]N \in_\alpha B}
$$

$$
\frac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}
\qquad
\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'}
\qquad
\dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}
$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u \geq w. \forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w,M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\overline{w \Vdash [M/x]N \in_\alpha B}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^{\alpha}_{w}(N',N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ require$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w,M,M) \quad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}\ \mathscr{H}}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\overline{\forall u \geq w. \forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w,M,M) \quad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}\ \mathscr{H}}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \dfrac{\checkmark}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\,\mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\,\mathscr{F}(w, M, M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]_w^\alpha(N', N')}\,\mathscr{H}}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{w \Vdash [M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]_w^\alpha(N', N')}\,\mathscr{H}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\,\mathit{require}$$

$$
\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}
\qquad
\dfrac{\dfrac{\dfrac{}{\forall u \geq w.\forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \ \Rightarrow\ u \Vdash [y/x]N = [z/x]N \in_\alpha B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}
$$

$$
\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M)
\qquad
\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\dfrac{\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G}
\qquad
\dfrac{}{\mathscr{V}[\![B]\!]_w^\alpha(N', N')}\ \mathscr{H}}{w \Vdash [M/x]N \in_\alpha B}}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}
\qquad
\dfrac{\checkmark}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'}
\qquad
\dfrac{\checkmark}{\mathscr{V}[\![B]\!]_w^\alpha(N', N')}\ \mathscr{H}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}
$$

$\square$

Questions?