# ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

Jon Sterling

October 8, 2015

joint work with Darryl McAdams

# INTRODUCTION

$[\![$*A woman walked in.*$]\!]$

$\triangledown$

$$[\![ \textit{A woman walked in.} ]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)$$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$[\![\textit{She sat down}]\!]$$
$$\triangledown$$

$$[\![\textit{She sat down}]\!]$$
$$\triangledown$$
$$\mathit{SatDown}(???)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

# TWO THINGS TO DEAL WITH

# TWO THINGS TO DEAL WITH

- terms for presuppositions

# TWO THINGS TO DEAL WITH

- terms for presuppositions
- resolution of presuppositions

# TWO THINGS TO DEAL WITH

- terms for presuppositions
- resolution of presuppositions

## TWO THINGS TO DEAL WITH

- terms for presuppositions (this talk)
- resolution of presuppositions

# THE **require** ORACLE: STATICS

# require — FORMAL RULES

$$\mathbf{require} : (0; 1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

# **require** — FORMAL RULES

$$\mathbf{require} : (0;1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in B} \qquad (\textit{require})$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\, \mathit{SatDown}(???)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

The meaning of a sentence is a logical proposition.

~~The meaning of a sentence is a logical proposition.~~

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \pi_1(x)$$

~~The meaning of a sentence is a logical proposition.~~

The meaning of a sentence is a type-theoretic expression which may evaluate to a canonical proposition.

What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \pi_1(x)$$

$$\boxed{\text{where } M \sim N \stackrel{\text{def}}{=} (M \preceq N) \wedge (N \preceq M)}$$

# EVERY GRAMMATICAL SENTENCE HAS A MEANING

…but only some of them denote propositions (types)!

# **require**—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \quad (??)$$

# **require**—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \qquad (??)$$

Can the above be made precise?

# **require**—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \tag{??}$$

Can the above be made precise? There are two problems:

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \tag{??}$$

Can the above be made precise? There are two problems:

1. circularity

# **require**—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \qquad (??)$$

Can the above be made precise? There are two problems:

1. circularity
2. non-determinism

(HOLD THAT THOUGHT)

# A POSITIVE EXAMPLE

# A POSITIVE EXAMPLE

$$\llbracket \textit{The President ran a marathon} \rrbracket$$

$$\triangledown$$

$$[\![\textit{ The President ran a marathon } ]\!]$$

$$\triangledown$$

$$\textbf{require } x : \textit{President} \textbf{ in } (\Sigma y \in \textit{Marathon})\, \textit{Ran}(x; y)$$

$$\textbf{require}\ x : \mathit{President}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

# A POSITIVE EXAMPLE

$$\textbf{require } x : \mathit{President} \textbf{ in } (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

$$\Downarrow$$

$$(\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(\mathit{Obama}; y)$$

# A NEGATIVE EXAMPLE

$$[\![\ \textit{The unicorn ran a marathon}\ ]\!]$$
$$\triangledown$$

$$[\![ \textit{The unicorn ran a marathon} ]\!]$$

$$\triangledown$$

$$\textbf{require}\ x : \mathit{Unicorn}\ \textbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

$$[\![\ \textit{The unicorn ran a marathon}\ ]\!]$$

$$\triangledown$$

$$\mathbf{require}\ x : \mathit{Unicorn}\ \mathbf{in}\ (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x;y)$$

(not a proposition)

## Is **require** computationally effective?

Yes, but we need two things:

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge
2. non-determinism must be eliminated

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated

$$\boxed{w \Vdash \mathscr{J}}$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$\boxed{w \Vdash \mathscr{J}}$$

Yes, but we need two things:

1. judgments shall be local / sensitive to knowledge (via forcing)
2. non-determinism must be eliminated (via choice sequences)

$$\boxed{w \Vdash \mathscr{J}_\alpha}$$

# THE CREATING SUBJECT

# THE THEORY OF THE CREATING SUBJECT

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)
2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

1. experiences are never forgotten (monotonicity, functoriality)
2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

**Corollary**

*The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world $w$.*

...

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

…

2. at a point in time, the subject knows whether or not it has experienced a judgment (decidability)

**Remark**

*Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:*

$$\forall w.\forall \mathscr{J}.\,(w \Vdash \mathscr{J}) \vee \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

*The above is impossible in a Beth model.*

logical consequence $\Rightarrow$ semantic consequence

*Brouwer?, Martin-Löf, Sundholm* $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

**logical consequence** $\Rightarrow$ **semantic consequence**

*Brouwer?, Martin-Löf, Sundholm* $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

**proof conditions** $\Rightarrow$ **assertion conditions**

*Dummett, Martin-Löf, Sundholm* $\Rightarrow$ *Brouwer, Heyting, Van Atten*

**logical consequence** ⇒ **semantic consequence**

*Brouwer?, Martin-Löf, Sundholm* ⇒ *Brouwer?, Heyting, Allen, Zeilberger*

**proof conditions** ⇒ **assertion conditions**

*Dummett, Martin-Löf, Sundholm* ⇒ *Brouwer, Heyting, Van Atten*

**global meaning explanation** ⇒ **local meaning explanation**

*Husserl, Dummett, Martin-Löf* ⇒ *Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$

assertion acts (judgments) are intensional (local)

$$|_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$\mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

assertion acts (judgments) are intensional (local)

| | |
|---|---|
| $\vert_x \mathscr{J}(x)$ | (general judgment) |
| $\mathscr{J}_2 \ (\mathscr{J}_1)$ | (hypothetical judgment) |
| $M \Downarrow N$ | (evaluation) |

assertion acts (judgments) are intensional (local)

| | |
|---|---|
| $\mid_x \mathscr{J}(x)$ | (general judgment) |
| $\mathscr{J}_2\ (\mathscr{J}_1)$ | (hypothetical judgment) |
| $M \Downarrow N$ | (evaluation) |
| $A\ \mathit{type}$ | (typehood) |

assertion acts (judgments) are intensional (local)

| | |
|---|---|
| $\vert_x \mathscr{J}(x)$ | (general judgment) |
| $\mathscr{J}_2\ (\mathscr{J}_1)$ | (hypothetical judgment) |
| $M \Downarrow N$ | (evaluation) |
| $A\ \mathit{type}$ | (typehood) |
| $A\ \mathit{verif}$ | (verification) |

assertion acts (judgments) are intensional (local)

| | |
|---|---|
| $\mid_x \mathscr{J}(x)$ | (general judgment) |
| $\mathscr{J}_2\ (\mathscr{J}_1)$ | (hypothetical judgment) |
| $M \Downarrow N$ | (evaluation) |
| $A\ \mathit{type}$ | (typehood) |
| $A\ \mathit{verif}$ | (verification) |
| $A\ \mathit{true}$ | (truth) |

assertion acts (judgments) are intensional (local)

| | |
|---|---|
| $\mid_x \mathscr{J}(x)$ | (general judgment) |
| $\mathscr{J}_2\ (\mathscr{J}_1)$ | (hypothetical judgment) |
| $M \Downarrow N$ | (evaluation) |
| $A\ \mathit{type}$ | (typehood) |
| $A\ \mathit{verif}$ | (verification) |
| $A\ \mathit{true}$ | (truth) |
| $M = N \in A$ | (membership) |

assertion acts (judgments) are intensional (local)

$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$
$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$
$$w \Vdash M \Downarrow N \qquad \text{(evaluation)}$$
$$w \Vdash A \ \mathit{type} \qquad \text{(typehood)}$$
$$w \Vdash A \ \mathit{verif} \qquad \text{(verification)}$$
$$w \Vdash A \ \mathit{true} \qquad \text{(truth)}$$
$$w \Vdash M = N \in A \qquad \text{(membership)}$$

# SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$

$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

# SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$w \Vdash |_x \mathscr{J}(x) \iff \cdots$$
$$w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) \iff \cdots$$

# SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$
\begin{aligned}
w \Vdash \big|_x\, \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\
w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) &\iff \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2
\end{aligned}
$$

$$\begin{aligned} w \Vdash |_x\, \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\ w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) &\iff \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2 \end{aligned}$$

where $\mathscr{D}_w$ is the species of constructions that have been effected by stage $w$

# INTUITIONISTIC VS CONSTRUCTIVE VALIDITY

a statement is intuitionistically valid if it is forced at all nodes

# INTUITIONISTIC VS CONSTRUCTIVE VALIDITY

~~a statement is intuitionistically valid if it is forced at all nodes~~

# INTUITIONISTIC VS CONSTRUCTIVE VALIDITY

~~a statement is intuitionistically valid if it is forced at all nodes~~

a statement is constructively valid if it is forced at all nodes

~~a statement is intuitionistically valid if it is forced at all nodes~~

a statement is constructively valid if it is forced at all nodes

Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects (Fourman)

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $\boxed{w \Vdash P\ \mathit{verif}}$ (and its synthesis).

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $\boxed{w \Vdash P\ \mathit{verif}}$ (and its synthesis).

For a type $A$, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A\ \mathit{true} \iff \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif} \qquad \text{(due to Dummett)}$$

The analytic judgments of type theory are reflections on mathematical activity.

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**

$$\boxed{\mathscr{V}[\![A]\!]_w(M,N) \quad \rtimes\!\!\!\ltimes \quad w \Vdash A \ \mathit{verif}}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**

$$\boxed{w \Vdash M = N \in A \quad \bowtie \quad w \Vdash A \; \mathit{true}}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a <u>bar</u>**

$$\boxed{w \Vdash M = N \in A \quad \rtimes\!\!\!\ltimes \quad w \Vdash A\ \mathit{true}}$$

The analytic judgments of type theory are reflections on mathematical activity.

1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**
3. **Computation** reflects **the recognition of a bar**

$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w(M', N') \end{array} \right\} \quad \rtimes \quad \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

# THE **require** ORACLE: DYNAMICS

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A\ \mathit{true} \\ 1 & \neg(i \Vdash A\ \mathit{true}) \end{cases} \tag{KS}$$

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of $A\ \mathit{true}$ effected "so far" for each canonical proposition $A$.

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of $A\ \mathit{true}$ effected "so far" for each canonical proposition $A$.

We now can give a precise, but non-deterministic, dynamics to **require**:

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of $A$ *true* effected "so far" for each canonical proposition $A$.

We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \tag{*}$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage $w$, in order of time.

# ELIMINATING NON-DETERMINISM WITH A SPREAD

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage $w$, in order of time.

Idea: reformulate Type Theory relative to a choice sequence of "choosers".

A spread direction $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

A spread direction $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

A spread direction $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$

A spread direction $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

1. the empty neighborhood shall be admitted

$$\mathfrak{S}(\langle\rangle)$$

2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$

3. a neighborhood may always be refined within the spread

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\, \rho(n) < n \;\; (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\, \rho(n) < n \;\; (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha_w(M', N')}{w \Vdash M = N \in_\alpha A}$$

## require — DYNAMICS

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

# **require** — DYNAMICS

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

Deterministic dynamics for **require**:

$$\frac{w \Vdash \varkappa_A \ni_\alpha M \quad w \Vdash [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad (\text{for } \alpha \in \mathfrak{S})$$

**Theorem**

*The following rule is valid in our semantics:*

$$\dfrac{w \Vdash A\ \mathit{true}_\alpha \quad w \Vdash x : A \vdash_\alpha N \in B}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\frac{w \Vdash A\ true_\alpha \quad w \Vdash x : A \vdash_\alpha N \in B}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\frac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\frac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\overline{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \quad \overline{\mathscr{V}[\![B]\!]_w^\alpha (N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{}{w \Vdash \varkappa_A \ni_\alpha M} \quad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\frac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad\qquad \frac{}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^{\alpha}_{w}(N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\overline{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}\,[\![B]\!]^\alpha_w\,(N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u \geq w. \forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M) \quad \dfrac{}{w \Vdash M \in_\alpha A}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ require$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\ge}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\ \mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w,M,M) \quad \dfrac{\dfrac{}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\,\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\,\mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\,\mathscr{F}(w,M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\,\textit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\ \mathscr{D} \qquad \dfrac{\dfrac{\overline{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\ \mathscr{F}}{w \Vdash \big|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w,M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \textit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u\geq w.\forall y,z\in\mathscr{D}_u.\ u\Vdash y=z\in_\alpha A \Rightarrow u\Vdash[y/x]N=[z/x]N\in_\alpha B}\mathscr{F}}{w\Vdash |_{y,z}\ [y/x]N=[z/x]N\in_\alpha B\ \big(y=z\in_\alpha A\big)}}{w\Vdash x:A\vdash_\alpha N\in B}\mathscr{E}$$

$$\dfrac{\dfrac{\checkmark}{w\Vdash M\in_\alpha A\Rightarrow w\Vdash[M/x]N\in_\alpha B}\mathscr{F}(w,M,M) \quad \dfrac{\dfrac{\dfrac{\checkmark}{w\Vdash A\ true_\alpha}\mathscr{D}}{w\Vdash \varkappa_A \ni_\alpha M}}{w\Vdash M\in_\alpha A}}{\dfrac{}{w\Vdash[M/x]N\in_\alpha B}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w\Vdash A\ true_\alpha}\mathscr{D}}{w\Vdash\varkappa_A\ni_\alpha M} \qquad \dfrac{}{[M/x]N\Downarrow_\alpha N'}}{w\Vdash \textbf{require}\ x:A\ \textbf{in}\ N\Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}}{w\Vdash\textbf{require}\ x:A\ \textbf{in}\ N\in_\alpha B}\textit{require}$$

$$
\cfrac{}{w \Vdash A \ \mathit{true}_\alpha}\ \mathscr{D}
\qquad
\cfrac{\cfrac{\cfrac{}{\forall u {\geq} w. \forall y,z {\in} \mathscr{D}_u.\ u {\Vdash} y {=} z {\in_\alpha} A \Rightarrow u {\Vdash} [y/x]N {=} [z/x]N {\in_\alpha} B}\ \mathscr{F}}{w \Vdash \mathop{|}_{y,z} [y/x]N = [z/x]N \in_\alpha B \ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\ \mathscr{E}
$$

$$
\cfrac{\cfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\ \mathscr{F}(w, M, M)
\quad
\cfrac{\cfrac{\cfrac{\checkmark}{w \Vdash A \ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\cfrac{\cfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \quad \cfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}\ \mathscr{H}}{w \Vdash [M/x]N \in_\alpha B}}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\checkmark}{w \Vdash A \ \mathit{true}_\alpha}\ \mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \cfrac{}{[M/x]N \Downarrow_\alpha N'}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \cfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}
$$

$$\dfrac{}{w \Vdash A \; true_\alpha}\;\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\; u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\;\mathscr{F}}{w \Vdash |_{y,z}\, [y/x]N = [z/x]N \in_\alpha B \;\left(y = z \in_\alpha A\right)}}{w \Vdash x : A \vdash_\alpha N \in B}\;\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\;\mathscr{F}(w, M, M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A \; true_\alpha}\;\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{\dfrac{}{w \Vdash [M/x]N \Downarrow_\alpha N'}\;\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]_w^\alpha (N', N')}\;\mathscr{H}}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A \; true_\alpha}\;\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\;\mathscr{G}}{w \Vdash \textbf{require}\; x : A \;\textbf{in}\; N \Downarrow_\alpha N'} \qquad \dfrac{}{\mathscr{V}[\![B]\!]_w^\alpha (N', N')}}{w \Vdash \textbf{require}\; x : A \;\textbf{in}\; N \in_\alpha B}\;\mathit{require}$$

$$\dfrac{}{w \Vdash A\ true_\alpha}\,\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{}{\forall u{\geq}w.\forall y,z{\in}\mathscr{D}_u.\ u{\Vdash}y{=}z{\in_\alpha}A \Rightarrow u{\Vdash}[y/x]N{=}[z/x]N{\in_\alpha}B}\,\mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\,\mathscr{E}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\,\mathscr{F}(w,M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{w \Vdash [M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}\,\mathscr{H}}{w \Vdash [M/x]N \in_\alpha B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ true_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha_w(N',N')}\,\mathscr{H}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ \mathit{require}$$

$$
\dfrac{}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}
\qquad
\dfrac{\dfrac{\dfrac{}{\forall u \geq w. \forall y,z \in \mathscr{D}_u.\ u \Vdash y = z \in_\alpha A \Rightarrow u \Vdash [y/x]N = [z/x]N \in_\alpha B}\,\mathscr{F}}{w \Vdash |_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}}{w \Vdash x : A \vdash_\alpha N \in B}\,\mathscr{E}
$$

$$
\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash M \in_\alpha A \Rightarrow w \Vdash [M/x]N \in_\alpha B}\,\mathscr{F}(w,M,M) \quad \dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M}}{w \Vdash M \in_\alpha A}}{w \Vdash [M/x]N \Downarrow_\alpha N'}\,\mathscr{G} \quad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}\,\mathscr{H}}{w \Vdash [M/x]N \in_\alpha B}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{w \Vdash A\ \mathit{true}_\alpha}\,\mathscr{D}}{w \Vdash \varkappa_A \ni_\alpha M} \quad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\,\mathscr{G}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \quad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha_w (N', N')}\,\mathscr{H}}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\,\mathit{require}
$$

□

# Questions?