ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

CMU POP SEMINAR

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joint work with Darryl McAdams

INTRODUCTION

$[\![$*A woman walked in.*$]\!]$
$\triangledown$

$$[\![\text{A woman walked in.}]\!]$$
$$\nabla$$
$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$$⟦\textit{A woman walked in. She sat down}⟧$$
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$$(\Sigma x \in (\Sigma p \in \textit{Woman})\, \textit{WalkedIn}(p))\, \textit{SatDown}(\pi_1(x))$$

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$$[\![\textit{A woman walked in. She sat down}]\!]$$
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$$(\Sigma x \in (\Sigma p \in \textit{Woman}) \; \textit{WalkedIn}(p)) \; \textbf{require} \; y : \textit{Woman} \; \textbf{in} \; \textit{SatDown}(y)$$

THE **require** ORACLE: STATICS

$$\mathbf{require} : (0; 1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \overset{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash M \in A \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require } x : A \textbf{ in } N \in B} \qquad (\textit{require})$$

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What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\ \textbf{require}\ y : Woman\ \textbf{in}\ SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\ SatDown(\pi_1(x))$$

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require } x : A \textbf{ in } N \Downarrow N'} \tag{??}$$

⟦ *The President ran a marathon* ⟧
∇

$$\llbracket \textit{The President ran a marathon} \rrbracket$$
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require $x : \textit{President}$ **in** $(\Sigma y \in \textit{Marathon})\,\textit{Ran}(x;y)$

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$$(\Sigma y \in \textit{Marathon}) \, \textit{Ran}(\textit{Obama}; y)$$

$$⟦ \textit{The unicorn ran a marathon} ⟧$$
$$\nabla$$

$[\![$ *The unicorn ran a marathon* $]\!]$

$\triangledown$

require $x : Unicorn$ **in** $(\Sigma y \in Marathon) \, Ran(x; y)$

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require $x : Unicorn$ **in** $(\Sigma y \in Marathon)\, Ran(x; y)$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

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(Thanks Stefan, Umut, Bill & Bob!)

THE CREATING SUBJECT

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Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$, for any stage/world w.

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...

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Remark

Contra Dummett, I <u>by no means</u> take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathscr{J} . (w \Vdash \mathscr{J}) \lor \neg(w \Vdash \mathscr{J}) \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

LOCAL MEANING THEORY

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ Brouwer?, Heyting, Allen, Zeilberger

logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ Brouwer?, Heyting, Allen, Zeilberger

proof conditions ⇒ assertion conditions

Dummett, Martin-Löf, Sundholm ⇒ Brouwer, Heyting, Van Atten

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Dummett, Martin-Löf, Sundholm ⇒ Brouwer, Heyting, Van Atten

global meaning explanation ⇒ local meaning explanation

Husserl, Dummett, Martin-Löf ⇒ Brouwer, Beth, Kripke, Grothendieck, Lawvere, Joyal

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$$A \ true \qquad \text{(truth)}$$
$$M = N \in A \qquad \text{(membership)}$$

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$$w \Vdash |_x \mathcal{J}(x) \quad \iff \quad \forall u \geq w. \forall x \in \mathcal{D}_u.\ u \Vdash \mathcal{J}(x)$$
$$w \Vdash \mathcal{J}_2\ (\mathcal{J}_1) \quad \iff \quad \forall u \geq w.\ u \Vdash \mathcal{J}_1 \Rightarrow u \Vdash \mathcal{J}_2$$

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where $\mathscr{D}$ is the (pre)sheaf of constructions that have been effected so far

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Intuitionism subsumes constructivism, but goes much further by allowing the observation of non-constructive objects (Fourman)

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THE MEANING OF A PROPOSITION

The meaning of a proposition/type is an intensional (world-indexed) specification of verification acts, i.e. a local meaning explanation for $\boxed{w \Vdash P \ verif}$ (and its synthesis).

For a type A, implicit in the explanation of $w \Vdash A \ verif$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

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In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$w \Vdash A \; true \iff \exists \mathfrak{B} \; \mathbf{bars} \; w. \forall u \in \mathfrak{B}. \; u \Vdash A \; verif$ (due to Dummett)

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$$\mathscr{V}[\![A]\!]_w (M, N) \quad \bowtie \quad w \Vdash A \; \mathit{verif}$$

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\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ \mathscr{V}[\![A]\!]_w (M', N') \end{array} \right\} \quad \bowtie \quad \exists \mathfrak{B} \ \mathbf{bars} \ w. \forall u \in \mathfrak{B}. \ u \Vdash A \ \textit{verif}
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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \ true \\ 1 & \neg(i \Vdash A \ true) \end{cases} \tag{KS}$$

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \quad (\star)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each stage w, in order of time.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

SPREADS: SETS OF CHOICE SEQUENCES

A spread direction $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

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3. a neighborhood may always be refined within the spread

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

THE require ORACLE: DYNAMICS

A spread direction for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n \, \rho(n) < n \ (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \qquad \text{(spread law)}$$

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Reformulate type theory relative to an arbitrary $\alpha \in \mathfrak{S}$! For instance:

$$\frac{w \Vdash M \Downarrow_\alpha M' \quad w \Vdash N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]_w^\alpha (M', N')}{w \Vdash M = N \in_\alpha A}$$

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$$\frac{M \Downarrow_\alpha M' \quad N \Downarrow_\alpha N' \quad \mathscr{V}[\![A]\!]^\alpha (M', N')}{M = N \in_\alpha A}$$

Deterministic choice for $\varkappa_A$:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = i \quad \varkappa_{A'}(w)(i) = M}{w \Vdash \varkappa_A \ni_\alpha M}$$

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Deterministic dynamics for **require**:

$$\frac{\varkappa_A \ni_\alpha M \quad [M/x]N \Downarrow_{\mathsf{tl}(\alpha)} N'}{\textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \qquad \text{(for } \alpha \in \mathfrak{S})$$

Theorem

The following rule is valid in our semantics:

$$\frac{A \ true_\alpha \quad x : A \vdash_\alpha N \in B}{\textbf{require } x : A \textbf{ in } N \in_\alpha B} \ require$$

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$$\overline{A \; true_\alpha} \; \mathscr{D} \qquad \overline{x : A \vdash_\alpha N \in B} \; \mathscr{E}$$

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<div style="text-align:center">
</div>

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$$\frac{\dfrac{\dfrac{\checkmark}{A \; true_\alpha} \; \mathscr{D}}{\dfrac{\varkappa_A \ni_\alpha M}{\textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \quad \overline{[M/x]N \Downarrow_\alpha N'}} \qquad \overline{\mathscr{V}[\![B]\!]^\alpha (N', N')}}{\textbf{require } x : A \textbf{ in } N \in_\alpha B} \; require$$

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$$\overline{A \; true_\alpha} \;\; \mathscr{D} \qquad\qquad \cfrac{\overline{\mid_{y,z} [y/x]N = [z/x]N \in_\alpha B \; \big(y = z \in_\alpha A\big)} \;\; \mathscr{F}}{x : A \vdash_\alpha N \in B} \;\; \mathscr{E}$$

$$\cfrac{\cfrac{\overline{[M/x]N \in_\alpha B \; (M \in_\alpha A)}^{\;\checkmark} \;\; \mathscr{F}(M,M) \qquad \cfrac{\cfrac{\overline{A \; true_\alpha}^{\;\checkmark} \;\; \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{[M/x]N \Downarrow_\alpha N'} \;\; \mathscr{G} \qquad \cfrac{}{\mathscr{V} \llbracket B \rrbracket^\alpha (N', N')} \;\; \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\cfrac{\cfrac{\cfrac{\overline{A \; true_\alpha}^{\;\checkmark} \;\; \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \overline{[M/x]N \Downarrow_\alpha N'}}{\textbf{require } x : A \textbf{ in } N \Downarrow_\alpha N'} \qquad \overline{\mathscr{V} \llbracket B \rrbracket^\alpha (N', N')}}{\textbf{require } x : A \textbf{ in } N \in_\alpha B} \;\; require$$

$$\frac{}{A\ true_\alpha}\ \mathscr{D} \qquad \frac{\dfrac{}{\mid_{y,z}[y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}}{M \in_\alpha A}}{\dfrac{\dfrac{}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{}{\mathscr{V}[\![B]\!]^\alpha (N',N')}\ \mathscr{H}}{[M/x]N \in_\alpha B}}$$

$$\frac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{\checkmark}{}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N' \qquad \overline{\mathscr{V}[\![B]\!]^\alpha (N',N')}}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ require$$

$$\frac{}{A\ true_\alpha}\ \mathscr{D} \qquad \frac{\dfrac{}{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \left(y = z \in_\alpha A\right)}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ (M \in_\alpha A)}\ \mathscr{F}(M,M) \qquad \dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\dfrac{\varkappa_A \ni_\alpha M}{M \in_\alpha A}}}{\dfrac{[M/x]N \Downarrow_\alpha N'}{}\ \mathscr{G} \qquad \overline{\mathscr{V}[\![B]\!]^\alpha (N', N')}\ \mathscr{H}}}{[M/x]N \in_\alpha B}$$

$$\frac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M} \qquad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'} \qquad \frac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha (N', N')}\ \mathscr{H}$$
$$\frac{}{\textbf{require}\ x : A\ \textbf{in}\ N \in_\alpha B}\ require$$

$$\overline{A\ true_\alpha}\ \mathscr{D} \qquad \frac{\dfrac{|_{y,z}\ [y/x]N = [z/x]N \in_\alpha B\ \big(y = z \in_\alpha A\big)}{}\ \mathscr{F}}{x : A \vdash_\alpha N \in B}\ \mathscr{E}$$

$$\frac{\dfrac{\dfrac{\checkmark}{[M/x]N \in_\alpha B\ (M \in_\alpha A)}\ \mathscr{F}(M,M)}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G} \qquad \dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\dfrac{\varkappa_A \ni_\alpha M}{M \in_\alpha A}}\quad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha (N',N')}\ \mathscr{H}}{[M/x]N \in_\alpha B}$$

$$\frac{\dfrac{\dfrac{\checkmark}{A\ true_\alpha}\ \mathscr{D}}{\varkappa_A \ni_\alpha M}\quad \dfrac{\checkmark}{[M/x]N \Downarrow_\alpha N'}\ \mathscr{G}}{\mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow_\alpha N'}\quad \dfrac{\checkmark}{\mathscr{V}[\![B]\!]^\alpha (N',N')}\ \mathscr{H}$$
$$\overline{\mathbf{require}\ x : A\ \mathbf{in}\ N \in_\alpha B}\ require$$

$\square$

QUESTIONS?

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