ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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1. Model Construction

We will now ground the intuitive semantics for the extended theory in a concrete model construction, using Kripke logical relations. Let $\mathcal{B}_\Psi$ be the set of abstract binding trees (terms) generated by the signature Ψ. Then, we define a world w as a triple $\langle \mathbf{\Psi}_w, \mathbf{T}_w, \mathbf{K}_w \rangle$, where $\mathbf{T}_w$ is the collection of terms which have been recognized as canonical types so far, and $\mathbf{K}_w\left(A\right)$ is the collection of witnesses of the truth of A that have been constructed so far, for each A in $\mathbf{T}_w$. In other words, the set of worlds $\mathcal{W}$ is defined as follows:

$$\mathcal{W} \triangleq \coprod_{\Psi \in \mathsf{sig}} \coprod_{T \in \mathbb{P}(\mathcal{B}_\Psi)} T \to \mathbb{P}\left(\mathcal{B}_\Psi\right)$$

Let $\mathbf{D}_w$ be the domain of discourse $\mathcal{B}_{\Psi w}$. The accessibility relation $\preceq$ is defined as follows:

$$\begin{aligned} u \preceq v \triangleq\ & \mathbf{\Psi}_u \preceq \mathbf{\Psi}_v \\ & \wedge \mathbf{T}_u \subseteq \mathbf{T}_v \\ & \wedge \forall A \in \mathbf{T}_u.\ \mathbf{K}_u\left(A\right) \subseteq \mathbf{K}_v\left(A\right) \end{aligned}$$

Theorem 1.1. *$\langle \mathcal{W}, \preceq \rangle$ is a Kripke frame.*

Proof. It suffices to show that $\preceq$ is reflexive and transitive, which is immediate from the corresponding properties of signature subsumption and subsethood. □

$$\begin{aligned} \textsc{unit} &\triangleq \cdot, \mathsf{unit} : (), \bullet : () \\ \textsc{prod} &\triangleq \cdot, \times : (0;0), \langle -, - \rangle : (0;0) \\ \textsc{fun} &\triangleq \cdot, \supset : (0;0), \lambda : (1) \end{aligned}$$

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\begin{aligned}
\mathcal{V}^{\alpha}_{w}\llbracket \mathsf{unit} \rrbracket &\triangleq \{(\bullet, \bullet) \mid \mathbf{\Psi}_w \succeq \textsc{unit}\} \\
\mathcal{V}^{\alpha}_{w}\llbracket A \times B \rrbracket &\triangleq \left\{(\langle M, N\rangle, \left\langle M', N'\right\rangle) \mid \mathbf{\Psi}_w \succeq \textsc{prod} \wedge \mathcal{E}^{\alpha}_{w}\llbracket A \rrbracket\left(M, M'\right) \wedge \mathcal{E}^{\alpha}_{w}\llbracket A \rrbracket\left(N, N'\right)\right\} \\
\mathcal{V}^{\alpha}_{w}\llbracket A \supset B \rrbracket &\triangleq \{((\lambda x)E, (\lambda y)F) \mid \mathbf{\Psi}_w \succeq \textsc{fun} \wedge \forall u \succeq w. \forall M, N \in \mathbf{D}_u. \\
&\qquad \mathcal{V}^{\alpha}_{u}\llbracket A \rrbracket (M, N) \Rightarrow \mathcal{E}^{\alpha}_{u}\llbracket B \rrbracket ([M/x]E, [N/y]F)\} \\
\mathcal{E}^{\alpha}_{w}\llbracket A \rrbracket &\triangleq \left\{(M, N) \mid w \Vdash M \Downarrow_{\alpha} M' \wedge w \Vdash N \Downarrow_{\alpha} N' \wedge \mathcal{V}^{\alpha}_{w}\llbracket A \rrbracket\left(M', N'\right)\right\}
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