ORACLES AND CHOICE SEQUENCES FOR TYPE-THEORETIC PRAGMATICS

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October 8, 2015

joint work with Darryl McAdams

INTRODUCTION

$[\![$*A woman walked in.*$]\!]$

$\triangledown$

$$[\![\textit{A woman walked in.}]\!]$$

$$\triangledown$$

$$(\Sigma p \in Woman)$$

$$[\![\textit{A woman walked in.}]\!]$$
$$\triangledown$$
$$(\Sigma p \in Woman)\, WalkedIn(p)$$

$[\![$*She sat down*$]\!]$

$\triangledown$

$$[\![\textit{She sat down}]\!]$$
$$\triangledown$$
$$\mathit{SatDown}(???)$$

$[\![$*A woman walked in. She sat down*$]\!]$

$\triangledown$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\, \mathit{SatDown}(???)$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$
$$\triangledown$$
$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(\pi_1(x))$$

THE "DONKEY SENTENCE"

$$[\![\textit{Every farmer who owns a donkey beats it.}]\!]$$

$$\triangledown$$

$$(\Pi p \in (\Sigma x \in \mathit{Farmer})\,(\Sigma y \in \mathit{Donkey})\,\mathit{Owns}(x; y))$$

$$[\![\textit{Every farmer who owns a donkey beats it.}]\!]$$

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$$(\Pi p \in (\Sigma x \in \mathit{Farmer})\,(\Sigma y \in \mathit{Donkey})\,\mathit{Owns}(x;y))\,\mathit{Beats}(???;???)$$

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$$(\Pi p \in (\Sigma x \in \mathit{Farmer})\,(\Sigma y \in \mathit{Donkey})\,\mathit{Owns}(x; y))\,\mathit{Beats}(\pi_1(p); \pi_1(\pi_2(p)))$$

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- terms for presuppositions

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- resolution of presuppositions

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- terms for presuppositions (this talk)
- resolution of presuppositions

THE **require** ORACLE: STATICS

require — FORMAL RULES

$$\mathbf{require} : (0;1) \qquad \text{(operator)}$$

$$\mathbf{require}\ x : A\ \mathbf{in}\ N \stackrel{\text{def}}{=} \mathbf{require}(A; x.N) \qquad \text{(notation)}$$

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$$\frac{\Gamma \vdash A\ \mathit{type} \quad \Gamma, x : A \vdash N \in B}{\Gamma \vdash \textbf{require}\ x : A\ \textbf{in}\ N \in B} \qquad (\mathit{require})$$

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, SatDown(???)$$

require — EXAMPLES

$$[\![\textit{A woman walked in. She sat down}]\!]$$

$$\triangledown$$

$$(\Sigma x \in (\Sigma p \in \mathit{Woman})\, \mathit{WalkedIn}(p))\ \textbf{require}\ y : \mathit{Woman}\ \textbf{in}\ \mathit{SatDown}(y)$$

The meaning of a sentence is a logical proposition.

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What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

$$\rightsquigarrow$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\, \pi_1(x)$$

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What we want:

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\ \mathbf{require}\ y : Woman\ \mathbf{in}\ SatDown(y)$$

$$\sim$$

$$(\Sigma x \in (\Sigma p \in Woman)\, WalkedIn(p))\ \pi_1(x)$$

$$\boxed{\text{where } M \sim N \stackrel{\text{def}}{=} (M \preceq N) \wedge (N \preceq M)}$$

EVERY GRAMMATICAL SENTENCE HAS A MEANING

...but only some of them denote propositions (types)!

require—NAÏVE DYNAMICS

$$\frac{M \in A \quad [M/x]\,N \Downarrow N'}{\textbf{require}\ x : A\ \textbf{in}\ N \Downarrow N'} \tag{??}$$

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Can the above be made precise? There are two problems:

1. impredicativity
2. non-determinism

(HOLD THAT THOUGHT)

A POSITIVE EXAMPLE

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$$[\![\ \textit{The President ran a marathon}\ ]\!]$$

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$$\Downarrow$$

$$(\Sigma y \in \mathit{Marathon})\,\mathit{Ran}(\mathit{Obama}; y)$$

A NEGATIVE EXAMPLE

$$[\![\textit{The unicorn ran a marathon}]\!]$$
$$\triangledown$$

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$$\triangledown$$

$$\textbf{require } x : \mathit{Unicorn} \textbf{ in } (\Sigma y \in \mathit{Marathon})\, \mathit{Ran}(x; y)$$

$$[\![\ \textit{The unicorn ran a marathon}\ ]\!]$$

$$\triangledown$$

$$\mathbf{require}\ x : \mathit{Unicorn}\ \mathbf{in}\ (\Sigma y \in \mathit{Marathon})\ \mathit{Ran}(x; y)$$

(not a proposition)

Is **require** COMPUTATIONALLY EFFECTIVE?

Yes, but we need two things:

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$$\boxed{w \Vdash \mathscr{J}_\alpha}$$

THE CREATING SUBJECT

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All mathematics is a mental construction performed by an idealized subject, subject to the following observations about knowledge:

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Corollary
The meaning of a judgment $\mathscr{J}$ must be explained in terms of its forcing condition, $\boxed{w \Vdash \mathscr{J}}$*, for any stage/world* w.

REMARK ON DECIDABILITY

...

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Remark

Contra Dummett, I by no means take the above as requiring that the following shall be true in a constructive metatheory, divorced from time:

$$\forall w. \forall \mathscr{J}.\ [\![w \Vdash \mathscr{J}]\!] \vee \neg [\![w \Vdash \mathscr{J}]\!] \qquad \text{(Dummett's infelicity)}$$

The above is impossible in a Beth model.

logical consequence $\Rightarrow$ semantic consequence

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger* logical consequence ⇒ semantic consequence

Brouwer?, Martin-Löf, Sundholm ⇒ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions ⇒ assertion conditions

Martin-Löf, Sundholm ⇒ *Heyting, Van Atten*

MEANING THEORY WITHOUT UNIFORMITY CONSTRAINTS

logical consequence $\Rightarrow$ **semantic consequence**

Brouwer?, Martin-Löf, Sundholm $\Rightarrow$ *Brouwer?, Heyting, Allen, Zeilberger*

proof conditions $\Rightarrow$ **assertion conditions**

Martin-Löf, Sundholm $\Rightarrow$ *Heyting, Van Atten*

global meaning explanation $\Rightarrow$ **local meaning explanation**

Husserl, Dummett, Martin-Löf $\Rightarrow$ *Brouwer?, Beth, Kripke, Grothendieck, Lawvere, Joyal*

assertion acts (judgments) are intensional (local)

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$$\mathscr{J}_2\ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

BETH-KRIPKE SEMANTICS FOR ASSERTIONS

assertion acts (judgments) are intensional (local)

$\vert_x \mathscr{J}(x)$	(general judgment)
$\mathscr{J}_2 \ (\mathscr{J}_1)$	(hypothetical judgment)
$M \Downarrow N$	(evaluation)

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$$A \ \mathit{type} \qquad \text{(typehood)}$$

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$A\ \mathit{type}$	(typehood)
$A\ \mathit{verif}$	(verification)

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$A\ \mathit{verif}$	(verification)
$A\ \mathit{true}$	(truth)
$M = N \in A$	(membership)

assertion acts (judgments) are intensional (local)

$w \Vdash \mid_x \mathscr{J}(x)$	(general judgment)
$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1)$	(hypothetical judgment)
$w \Vdash M \Downarrow N$	(evaluation)
$w \Vdash A \ \mathit{type}$	(typehood)
$w \Vdash A \ \mathit{verif}$	(verification)
$w \Vdash A \ \mathit{true}$	(truth)
$w \Vdash M = N \in A$	(membership)

SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$w \Vdash |_x \mathscr{J}(x) \qquad \text{(general judgment)}$$

$$w \Vdash \mathscr{J}_2 \ (\mathscr{J}_1) \qquad \text{(hypothetical judgment)}$$

$$w \Vdash |_x \mathscr{I}(x) \quad \iff \quad \cdots$$
$$w \Vdash \mathscr{I}_2 \ (\mathscr{I}_1) \quad \iff \quad \cdots$$

SEMANTICS OF HIGHER-ORDER ASSERTIONS

$$
\begin{aligned}
w \Vdash |_x \mathscr{J}(x) &\iff \forall u \geq w. \forall x \in \mathscr{D}_u.\ u \Vdash \mathscr{J}(x) \\
w \Vdash \mathscr{J}_2\ (\mathscr{J}_1) &\iff \forall u \geq w.\ u \Vdash \mathscr{J}_1 \Rightarrow u \Vdash \mathscr{J}_2
\end{aligned}
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where $\mathscr{D}_w$ is the species of constructions that have been effected by stage w

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For a type A, implicit in the explanation of $w \Vdash A\ \mathit{verif}$ is a $\mathbb{W}$-indexed family of PERs $\mathscr{V}[\![A]\!]_w \subseteq \mathscr{D}_w \times \mathscr{D}_w$ whose members reflect the computational content (extension) of verification acts.

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INTUITIONISTIC SEMANTICS OF TRUTH

Truth (justification) consists in recognizing the effectiveness of a procedure for verification.

In the model, this corresponds to the inevitability of verification (i.e. a bar, in which verification occurs at all nodes):

$$w \Vdash A\ \mathit{true} \iff \exists \mathfrak{B}\ \textbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif} \qquad \text{(due to Dummett)}$$

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1. **Canonical membership** reflects **verification**

$$\boxed{\mathscr{V}[\![A]\!]_w(M,N) \quad \rtimes\!\!\ltimes \quad w \Vdash A\ \mathit{verif}}$$

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1. **Canonical membership** reflects **verification**
2. **Membership** reflects **justification**

$$\boxed{w \Vdash M = N \in A \quad \bowtie \quad w \Vdash A \; \mathit{true}}$$

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3. **Computation** reflects **the recognition of a <u>bar</u>**

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1. **Canonical membership** reflects **verification**
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$$\wedge \left\{ \begin{array}{l} w \Vdash M \Downarrow M' \\ w \Vdash N \Downarrow N' \\ w \Vdash \mathscr{V}\,[\![A]\!]_w\,(M', N') \end{array} \right\} \quad \rtimes \quad \exists \mathfrak{B}\ \mathbf{bars}\ w. \forall u \in \mathfrak{B}.\ u \Vdash A\ \mathit{verif}$$

THE **require** ORACLE: DYNAMICS

choice sequences (streams of objects) may be propounded over time based on the previous experience of the creating subject.

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Example:

$$\alpha(i) = \begin{cases} 0 & i \Vdash A \; \mathit{true} \\ 1 & \neg(i \Vdash A \; \mathit{true}) \end{cases} \tag{KS}$$

Let $\mathscr{K}_A : \mathbf{FinSet}^{\mathbb{W}^{\mathrm{op}}}$ be the sheaf of constructions of A true effected prior to w for each canonical proposition A.

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We now can give a precise, but non-deterministic, dynamics to **require**:

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We now can give a precise, but non-deterministic, dynamics to **require**:

$$\frac{w \Vdash A \Downarrow A' \quad M \in \mathscr{K}_{A'}(w) \quad w \Vdash [M/x]\,N \Downarrow N'}{w \Vdash \mathbf{require}\ x : A\ \mathbf{in}\ N \Downarrow N'} \quad (*)$$

We need a way to deterministically choose a representative of $\mathscr{K}_A(w)$. First, let $\varkappa_A$ be the choice sequence of lists given by enumerating $\mathscr{K}_A(w)$ at each world w, in order of construction.

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Idea: reformulate Type Theory relative to a choice sequence of "choosers".

A spread $\mathfrak{S}$ is a restriction on choice sequences which is defined by a condition on their finite approximations (prefixes, neighborhoods), subject to the following laws:

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1. the empty neighborhood shall be admitted

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SPREADS: SETS OF CHOICE SEQUENCES

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2. if a neighborhood is admitted, so shall all its subneighborhoods

$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$

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$$|_{\vec{u},m}\ \mathfrak{S}(\vec{u})\ \left(\mathfrak{S}(\vec{u} \frown m)\right)$$

3. inclusion of neighborhoods is closed under refinement/extension

$$|_{\vec{u}}\ \mathfrak{S}(\vec{u} \frown m)\ \left(\mathfrak{S}(\vec{u})\right)$$

A spread for index-choosers:

$$\frac{}{\mathfrak{S}(\langle\rangle)} \qquad \frac{\mathfrak{S}(\vec{u}) \quad |_n\, \rho(n) < n \;\; (n \in \mathbb{N}^+)}{\mathfrak{S}(\vec{u} \frown \rho)} \qquad \text{(spread law)}$$

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Deterministic dynamics for **require**:

$$\frac{w \Vdash A \Downarrow_\alpha A' \quad |\varkappa_{A'}(w)| = \ell \quad \mathsf{hd}(\alpha)(\ell) = j \quad w \Vdash \left[\varkappa_{A'}(j)/x\right] N \Downarrow_{\mathsf{tl}(\alpha)} N'}{w \Vdash \textbf{require}\ x : A\ \textbf{in}\ N \Downarrow_\alpha N'}$$

(for $\alpha \in \mathfrak{S}$)

QUESTIONS?